LOGISTIC EQUATIONS WITH DIFFUSION, DELAY AND IMPULSES

by

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Contents

Summary

This thesis contains a discussion of logistic equations with diffusion, impulses and time delays both discrete and continuous type. The boundary conditions used in these problems are Dirichlet, Neumann and Robin boundary conditions. Both single and multi species logistic equations are investigated. The impulse times employed here are the fixed ones. Some results on the problems are:

(1) Single species logistic equation with diffusion, impulses, discrete delay, Dirichlet and Robin boundary conditions.
  - Existence and uniqueness of solution:
    - Dirichlet boundary case(Corollary 3.1).
    - Robin boundary case(Corollary 3.2).
  - Conditions for the existence of zero attractor (Theorem 3.9).
  - Conditions for the existence of positive attractor (Theorem 3.11).

(2) Logistic equation with diffusion, impulses, continuous delay and Neumann boundary condition.
  - Single species: existence and uniqueness of solution (Theorem 4.1).
  - Single species: conditions for the existence of zero attractors (Theorem 4.2).
  - Single species: conditions for the existence of positive attractor (Theorem 4.3).
  - Multi species: conditions for the existence of positive attractor (Theorem 4.4).

This thesis is organised as follows: in Chapter 2, the background of these problems is presented. Chapter 3 is concerned with the existence and uniqueness of solution, zero and positive attractor of logistic equations with diffusion, impulses, discrete time delay, and Dirichlet and Robin boundary conditions. We discuss the existence and uniqueness of solution of diffusive logistic equations with distributed delay, impulses and Neumann boundary condition, zero and positive attractors in Chapter 4. Some conditions to obtain a positive attractor for multi species logistic equation with diffusion, distributed delay, impulses and Neumann boundary condition are presented in the last section of Chapter 4.

Certification

I certify that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge and belief it does not contain any material previously published or written by another person except where due reference is made in the text.

Jalina Widjaja

I believe that this thesis is properly presented, conforms to the specifications for the thesis and this is of sufficient standard to be, *prima facie*, worthy of examination.

Dr. Murk J. Bottema, Principal Supervisor

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I dedicate this thesis to my dear mother and father.

CHAPTER 1

Introduction

Logistic equations are commonly used to model population dynamics. For example, a fish population in a lake has dynamics that can be expressed as a system of ordinary differential equations as follows:

$$\frac{du}{dt}(t) = u(t)(a - bu(t)) \tag{1.1}$$

$$u(0) = u_o \tag{1.2}$$

where $u(t)$ is the number of fish at time t, u_o is positive, and a and b are positive constants which represent the growth rate of population and carrying capacity of the environment. The solution of this system is

$$u(t) = \frac{au_o}{bu_o(1 - \exp(-at)) + a\exp(-at)}.$$

The solutions of this system will converge over time to $\frac{a}{b}$. If a number of fish are harvested at certain times then the population, in general, will not converge to $\frac{a}{b}$ anymore. This phenomenon is described in the following example.

Example 1:

Assume $\frac{a}{b} > 1000$ and $u_o = 1000$, at time $t_k = k \in \mathbf{Z}^+$, the number of fish taken at t_k is $\left(1 - \frac{a-1000b}{a\exp(a)}\right) u(k^-)$. The proof that the population does not converge to $\frac{a}{b}$ is as follows:

The solution of the system is

$$\begin{aligned} u(t) &= \frac{au(k-1)}{bu(k-1)(1 - \exp(-a(t-k+1)) + a\exp(-a(t-k+1))} \\ &\leq \frac{au(k-1)}{-bu(k-1)\exp(-a(t-k+1)) + a\exp(-a(t-k+1))} \\ &\leq \frac{au(k-1)}{(a - bu(k-1))\exp(-a(t-k+1))} \qquad \text{for } t \in (k-1, k). \end{aligned}$$

The solution, u, is an increasing function with respect to time t as long as the initial condition is less than $\frac{a}{b}$. From the condition on the harvest sizes, we have $u(k) < u(k^-)$. Let $w(t) = \frac{aw(k-1)}{(a-bw(k-1))\exp(-a(t-k+1))}$ for $t \in (k-1,k)$, $w(0) = u_o$, and $w(k) = \frac{a-1000b}{a\exp(a)}w(k^-)$. So $u(t) \le w(t)$ for $t \in [0,\infty)$.

$$w(1^-) = \frac{1000a}{(a-1000b)\exp(-a)} \tag{1.3}$$

$$w(1) = \frac{a-1000b}{a\exp(a)}\frac{1000a}{(a-1000b)\exp(-a)} \tag{1.4}$$

$$w(1) = 1000 \tag{1.5}$$

Because the harvests occur at times $t_k = k$, the distance between every two consecutive harvest times is 1. From Equation (1.5), w on time interval $[k-1,k)$ for $k = 2,3,4,...$ is the same as w on time interval $[0,1)$. There exists $\epsilon_o = \frac{\frac{a}{b}-1000}{2} > 0$ such that for every positive constant M, we can find $t = k > M$ that satisfies $\frac{a}{b} - w(t) > \epsilon_o$. Because $u(t) \le w(t)$ then this holds for u as well. Hence u does not converge to $\frac{a}{b}$.

Even if the size of the harvest decreases over time or the time interval goes to infinity, the population does not necessarily converge to $\frac{a}{b}$. For instance, if in Example 1, the number of fish taken at time $t_k = k$ is $\left(1 - \frac{a-1000b}{ak\exp(a)}\right)u(k^-)$, then the initial condition on every time interval is smaller than 1000. Thus u does not converge to $\frac{a}{b}$.

As another example, if the number of fish taken in Example 1 is $u(t_k^-) - \frac{a-1000b}{a\exp(ak)}u(k^-)$ and the harvests occur at time $t_k = \sum_{i=1}^k i$ for $k = 1,2,3,...$, then initial condition on every time interval between two harvest times is 1000. Hence the population does not converge to $\frac{a}{b}$.

However, combinations of harvest sizes and time intervals between two harvest times do exist such that the population converges to $\frac{a}{b}$. One of the combinations is as follows:

- $u_o < \frac{a}{b}$, $a > 1$;
- the harvest times are $t_k = \sum_{i=1}^k i = \frac{k^2+k}{2}$, so that $t_k - t_{k-1} = k$;
- the number of fish taken at time t_k is less than or equal to

$$\frac{au_o}{bu_o(1-\exp(-a)) + a\exp(-a)} - \frac{au_o}{bu_o + 0.5a} \qquad \text{for } k = 1$$

$$\frac{au_o}{bu_o + \frac{a\exp(-ak)}{1+t_{k-1}}} - \frac{au_o}{bu_o + \frac{a}{1+t_k}} \qquad \text{for } k = 2,3,4,\ldots.$$

By this combination, u, the solution of this system, satisfies the following relation:

$$u(t) \geq \frac{au_o}{bu_o + \frac{a}{1+t}}$$

and

$$u(t) \leq \frac{au_o}{bu_o(1-\exp(-at)) + a\exp(-at)}.$$

Hence u converges to $\frac{a}{b}$.

If some of the fish do not remain in the region where the harvest takes place, then the logistic equation must include a diffusion term. For this case, finding the steady-state and the conditions on harvest sizes and harvest times for which the solution converges to the steady-state is much more difficult because closed form descriptions of the solution are not available.

More realistic descriptions of fish population include time delay in order to model gestation period, for example, as well as diffusion and impulses. This model can be presented as follows:

$$\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = u(t,x)(a - bu(t-\tau,x)) \qquad \text{for } t \neq t_k \tag{1.6}$$

$$u(t,x) = 0 \qquad \text{on } \partial\Omega \tag{1.7}$$

$$u(t,x) = \eta(t,x) \qquad \text{for } t \in [\tau, 0] \tag{1.8}$$

$$u(t_k, x) = I_k(u({t_k}^-, x)) \tag{1.9}$$

or

$$\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = u(t,x)(a - bu(t,x) - \int_{-\infty}^{t} f(t-s)u(s,x)ds) \quad \text{for } t \neq t_k \tag{1.10}$$

$$\frac{\partial u}{\partial \nu}(t,x) = 0 \quad \text{on } \partial\Omega \tag{1.11}$$

$$u(t,x) = u_o(t,x) \quad \text{for } t \in (-\infty, 0] \tag{1.12}$$

$$u(t_k, x) = I_k(u({t_k}^-, x)). \tag{1.13}$$

The objective of this thesis is to find conditions on the size and the timing of the impulses so that the solution of problem (1.6)–(1.9) and the solution of problem (1.10)–(1.13) converge to the steady-state of the corresponding problem without impulses.

CHAPTER 2

Background

Many processes both natural and man-made in various areas such as biology, medicine, chemistry, physics, engineering, and economics, involve time delays. One of these processes, for example, is reforestation. A cut forest, after re-planting, will take at least 20 years before reaching maturity. For certain species of trees (e.g. redwood), it would be much longer. Hence, any mathematical model for forest harvesting and regeneration clearly must have time delays built into it. Another example is animal activities and responses mechanism. Animals must take time to digest their food before reaping the benefit of the energy and nutrition to do their activities [**23**].

Researchers have investigated these phenomena and modelled them using delay differential equations. Ordinary delay differential equations have been studied in detail ([**18**],[**3**], [**10**]) and are not considered further in this thesis. Parabolic partial differential equations with time delays have been given considerable attention and various methods have been constructed to study the existence and stability. Results in this area will be discussed more detail in Section 3.

Many of the processes mentioned above are characterized by the fact that the system parameters are subject to short-term perturbations in time. Consider, for example, the problem of modelling a fish population in a hatchery. Here the natural growth of the fish population is disturbed by harvesting at certain time intervals and by adding fresh breed. This problem therefore involves impulses [**34**].

An adequate apparatus for mathematical simulation of such processes and phenomena is impulsive differential equations. Impulsive differential equations have been studied mostly in the ordinary case [**25**]. In the last ten years, the theory of impulsive partial differential equations has undergone rapid development. Further discussions of some results in impulsive partial differential system are presented in Section 2.

If the process involves both time delays and impulses, it is modelled by impulsive differential equations with time delays. Impulsive ordinary differential equations with time delays have been investigated by many researchers. Some results of these problems are discussed in Section 1.

Diffusion also has important and interesting effects. For example, in reaction diffusion models, patterns of finite wavelength were only obtained in the situations where chemical equilibrium was stable in the absence of diffusion. Thus, diffusion, normally regarded as an influence that tends to erase zones of relativity high chemical concentration, can act to promote instability and inhomogeneity [**54**].

Segel and Jackson [**48**] showed how diffusion can cause too rapid decay of the chemical that stabilizes an interaction, thereby permitting a continual increase of the destabilizer and a consequent instability. In the same paper, they also showed how such diffusive instabilities can arise from the effects of random dispersal on models of predator-prey interactions. Levin [**29**] independently treated the same problem for discrete intercommunicating 'patches' of species. In [**49**], the ecological results in [**48**] and the work in [**29**] were extended by taking nonlinear effects into account.

In this chapter, we present some results on differential equations with diffusion and or impulse and or time delay that have been reported in the literature. A discussion of these results is presented in three sections as follows:

(1) Impulsive ordinary differential equations with time delays.
(2) Impulsive nonlinear parabolic systems.
(3) Nonlinear parabolic systems with time delays.

Most of the following results can be applied to logistic equations.

1. Impulsive Ordinary Differential Equations with Time Delays

The original motivation for studying delay differential equations (DDEs) came mainly from their applications in feedback control theory (Minorsky [**32**]). The great interest in such kinds of problems has certainly contributed significantly to the rapid development of the theory of differential equations with dependence on the past state. Some applications were mentioned at the beginning of this chapter. Examples of references that contain these applications are books by Kolmanovskii and Nosov [**22**] for various topics in engineering, physics, biology and economics, Pielou [**39**] for ecological system, Cushing [**6**] and Gopalsamy [**18**] for population dynamics. The results of DDEs in ordinary case are mostly in the existence, uniqueness, and stability of solutions. Recently, the oscillations of nonlinear DDEs have become of interest ([**18**], [**58**]).

There are two types of time delay : discrete and continuous. Both delays can be finite or infinite. Some results for problems with discreet delay can be found in [**35**], [**18**], [**23**]. Discussions in problems with continuous delay are in [**55**], [**46**], [**17**].

Meanwhile impulsive differential equations are used as natural descriptions of observed evolution phenomena of several real world problems. The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effect. For example, initial value problems of such equations do not necessarily, in general, possess any solutions even when the corresponding differential equation is smooth. Fundamental properties such as continuous dependence relative to initial data may be violated, and qualitative properties such as stability need new interpretations. Moreover, a simple impulsive differential equation may exhibit several new phenomena such as rhythmical beating, merging of solution, and non-continuability of solutions [**25**].

There are two kinds of impulse times: fixed ones and variable ones. Here variable impulse time means that the impulse time is not known initially but it is given, for example, in the form of a function of the solution. These kinds of impulses are motivated, for instance, in the case where harvest times are determined by a population reaching a specific threshold.

A comprehensive reference for ordinary impulsive differential equations can be found in [**25**]. In this book, the method of upper and lower solution is employed to find the solution of impulsive differential equations. Stability of impulsive differential systems is investigated by means of discontinuous Liapunov functional. The theory of stability in terms of two measures is used to unify several known stability concepts in impulsive differential equations. Further discussion on the method of upper and lower solution will be presented in Chapter 3. A discontinuous Liapunov functional will be studied in Chapter 4.

The general form of impulsive ordinary differential equations with discrete time delay is:

$$\dot{x}(t) = f(t, x(t-\tau)), \qquad t \neq t_k,\ t > t_o \tag{2.1}$$

$$\Delta x(t_k) = x(t_k + 0) - x(t_k) = I_k(x(t_k)), \qquad t_k > t_o,\ k = 1, 2, ..., \tag{2.2}$$

$$x(t) = \phi_o(t), \qquad t \in [t_o - \tau, t_o] \tag{2.3}$$

where $\tau > 0, t_o < t_1 < t_2 < ... < t_k < ..., \lim_{k\to\infty} t_k = \infty$.

When the right hand side function of (2.1) was bounded by a linear function of x, multiplied by a continuous function of t, Yu and Zhang [**60**] chose the impulse function to be positive if the value of x is negative and vice versa. Hence, they proved the stability of zero solution using a result in [**59**] on the well-known $\frac{3}{2}$ stability of one dimension delay differential equations.

The stability of the zero solution of system (2.1)-(2.3) had been investigated by Bainov and Stamova [**8**] by virtue of piecewise continuous functions which are analogues of classical Liapunov function in conjunction with Razumikhin's technique. They set f to be a continuous function and to satisfy the Lipschitz condition with respect to the second variable. The impulse function I_k is also continuous and implicitly non-increasing. The Liapunov-like function must be positive definite and its derivative is non-positive to make the zero solution stable. To achieve the uniform stability of the zero solution, the Liapunov-like function is required to be positive definite, decrescent and its derivative is non-positive. Moreover, if the derivative is negative definite then the zero solution is uniformly asymptotically stable. Applications of these results to population dynamics are presented at the last section of the paper.

In [**50**], Shen et al. used Liapunov's direct method for finding sufficient conditions for the zero solution to be uniformly and asymptotically stable. They used Liapunov like functionals which are continuous on every time interval formed by two consecutive impulse times. These functionals satisfy local Lipschitz conditions, are positive definite and decrescent. The impulse functions were chosen such that the Liapunov functionals values after the jump would be less than the values before the jump.

The impulsive differential equations with continuous delay usually are in the following format:

$$\dot{x} = f(t, x, Tx, Sx) \qquad , t \neq t_k \tag{2.4}$$

$$x(t_k^+) - x(t_k^-) = I_k(x) \qquad , k = 1, 2, 3, ... \tag{2.5}$$

$$x(t_o) = x_o \tag{2.6}$$

where $Tx(t) = \int_{t_o}^{t} k(t,s)x(s)ds$, $Sx(t) = \int_{t_o}^{a} h(t,s)x(s)ds$, k and h are continuous functions, and a is a positive constant.

Guo and Liu [**19**] proved the existence of solution of system (2.4)–(2.6) using the method of upper and lower solution. They required the impulse function I_k to be nondecreasing for every k and f satisfies a kind of Lipschitz condition.

In [**40**], system (2.4)–(2.6) with f a function of t, x and Tx only, is discussed. If k is bounded and the norm of x after a jump is less than the norm before the jump, then the zero solution is uniformly stable. This is proved by using a Liapunov functional which is formed by the fundamental matrix solution of the corresponding linear system.

Bainov and Kulev [**24**] employed the theory of stability in terms of two measures to find the stability of the zero solution of system (2.4)–(2.6) with

f as a function of x, t and Tx only. They used a system without continuous delay for a comparison, so that the stability of this system corresponds with the stability of system (2.4)–(2.6). A Liapunov-like functional was used to prove the result.

2. Impulsive Nonlinear Parabolic Systems

There are two main approaches to the study of impulsive parabolic equations. The first is based on the concept of impulsive evolution systems in Banach space, and was introduced by Rogovchenko and Trofimchuk [**42**],[**43**]. The second approach is based on the concept of upper and lower solutions, and was exploited for the first time by Erbe et al. [**11**] and developed further in [**21**] and [**5**].

The general form of the impulsive nonlinear parabolic system is:

$$u_t = f(t, x, u, u_x, u_{xx}) \qquad , (t, x) \in ((0, T] \setminus \{t_k\}_{k=1}^{p} \times \Omega \in R^n) \tag{2.7}$$

$$u(0, x) = u_o(x) \qquad \text{on } \bar{\Omega} \tag{2.8}$$

$$Bu(t, x) = p(t, x)u(t, x) + q(t, x)\frac{\partial u(t, x)}{\partial \nu} = h(t, x)$$
$$, (t, x) \in ((0, T] \setminus \{t_k\}_{k=1}^{p} \times \partial\Omega) \tag{2.9}$$

$$u(t^{+}{}_k, x) - u(t_k, x) = g_k(t_k, x) \qquad , 1 \le k \le p,\ x \in \bar{\Omega} \tag{2.10}$$

Erbe et al. [**11**] established some maximum and comparison principles relative to upper and lower solutions of nonlinear scalar parabolic partial differential equations in (2.7)-(2.10). For this, they assigned Ω to be a smooth bounded domain, $g_k : R \to R$ to be continuous and the mapping $z + g_k(z)$ to be increasing for $z \in R$. By considering the problem in several cases and using the maximum principle for parabolic differential equations without impulses, they showed that if the impulse is not too drastic or if both reaction rate and impulse satisfy the global Lipschitz condition, then the classical maximum and comparison principles for parabolic equations without impulses still hold for the above system. They also presented sufficient conditions for the asymptotic stability of the steady-state solution of reaction-diffusion equations with homogeneous Neumann boundary conditions.

Using a similar method, Kirane and Rogovchenko [**21**] obtained the same results for nonlinear vector impulsive parabolic partial differential equations.

Rogovchenko in [**44**] and [**45**] presented sufficient conditions for the existence and uniqueness of mild and classical solutions of linear and nonlinear (periodic) impulsive evolution systems. He also gave an estimate of the solution with dependence on initial data, and the stability of the stationary solution. The proofs of the theorems used the concept of impulsive evolution systems in Banach space. Some of these results were applied to population models.

In [**9**], Drici et al. extended the generalized method of quasilinearization to impulsive parabolic equations with impulses at fixed moments. They obtained sufficient conditions for the existence of monotone sequences which converge uniformly to a unique solution of initial boundary value problem (2.7)–(2.9) with $u({t^+}_k, x) = I(u(t_k, x))$ where I is nondecreasing in u for fixed x, and showed that the convergence is quadratic. This result was extended in [**26**] to the discussion of other qualitative properties of the solution such as positivity and boundedness.

Bainov et al. [**1**] presented sufficient conditions for the stability of the trivial solution of a more general class of initial boundary value problems than problem (2.7)–(2.10) with delay by comparing them to suitable ordinary impulsive differential systems.

Estimates of solutions of impulsive parabolic equations with Dirichlet boundary conditions are discussed in [**2**]. The estimates are the minimal and maximal solutions of the corresponding impulsive ordinary differential equations. An application to population dynamic is presented. The estimates of population density for a certain impulsive single species model are found.

Struk and Tlachenko [**52**] investigated a two-dimensional Lotka-Volterra system with diffusion and impulses. Using the comparison theorems, they found conditions for the permanence of the system. One of these conditions is that the impulse functions are bounded by some exponential functions. They also proved the existence of a unique periodic solution that is globally asymptotically stable, strictly positive and piecewise continuous via a Liapunov functional if the system is permanent and the impulse functions and their derivatives are bounded.

3. Nonlinear Parabolic Systems with Time Delays

Nonlinear parabolic systems with time delays are generally in the following form:

$$\frac{\partial u_i}{\partial t}(t,x) - Lu_i(t,x) = f_i(t,x,u_1(t,x),...,u_n(t,x),u_1(t-\tau_1,x),...,u_n(t-\tau_n),$$
$$k(t,u_1(t-s,x)),...,u_n(t-s,x))$$
$$,(t,x) \in (0,\infty) \setminus t_{k}{}_{k=1}^{\infty} \times \Omega \in R^n,\ i=1,2,...,n. \tag{2.11}$$

$$B_i u_i(t,x) = \left(\alpha_i \frac{\partial}{\partial \nu} + \beta_i\right) u_i(t,x) = h_i(t,x)$$
$$,(t,x) \in (0,\infty) \setminus t_{k}{}_{k=1}^{\infty} \times \partial\Omega, i=1,2,...,n. \tag{2.12}$$
$$u_i(t,x) = \eta_i(t,x) \quad \text{in } [\max\{\tau_i : i=1,..,n\},0] \times \bar{\Omega},\ i=1,2,...,n. \tag{2.13}$$

where L is an elliptic operator.

Parabolic systems in (2.11)-(2.13) have been investigated by many researchers. Most of the discussions are devoted to the existence and asymptotic behaviour of the solution. In earlier works [**30**], [**53**] the system was formulated as an evolution equation of functional type, and the the existence and dynamic of solution was investigated by the semigroup approach. Recently, the method of upper and lower solution and its associated monotone iteration have been used to study the existence and dynamics of (2.11)-(2.13). An advantage of the monotone method is that it leads to a method for computing numerical solutions.

From the semigroup approach, the existence and uniqueness of a non-continuable mild solution of system (2.11)-(2.13) were proved in case $f_i =$

$g_i(t, x, \mathbf{u}(t-\tau_i, x))$ and $L = d_i\Delta$. The techniques used to obtain these results were differential inequalities, invariant sets, and Liapunov functions [**30**].

The existence of upper and lower sequence which converge monotonically to a unique solution of system (2.11)-(2.13) if there is a pair of upper and lower solution, was presented in [**35**], in case $i = 1, f = p(t, x, u) + q(t, x, u_t)$ where q is decreasing or increasing and p, q are Holder continuous.

The same result was proved in [**36**], with more general conditions on each f_i. It was shown that each f_i is Holder continuous in (t, x), locally Lipschitz continuous in $\mathbf{u}(t, x)$ and $\mathbf{u}(t-\tau, x)$, and has the mixed quasimonotone property in a sector between upper and lower solutions. Also f may depend on functional values of $\mathbf{u}(t-\tau, x)$ such as $\int_0^{\tau_i} u_i(t-s, x)ds$ and $\int_{\tau_i}^t u(t-s, x)ds$. The existence and uniqueness of the solution of system (2.11)-(2.13) with $f_i = p_i(t, x, \mathbf{u}(t, x), \mathbf{u}(t-\tau, x)) + \int_\Omega q_i(t, x, \mathbf{u}(t, x), \mathbf{u}(t-\tau, x))$ was also obtained.

In another paper [**37**], the dynamics of system (2.11)–(2.13) was investigated using the results in [**36**]. It was shown that if the elliptic system corresponding to (2.11)–(2.13) has a pair of coupled lower and upper solutions then there is a pair of quasisolutions of the elliptic system. Also the sector formed by the quasisolutions is an attractor of the delayed parabolic system (2.11)–(2.13).

The global existence and the dynamics of system (2.11)–(2.13) where the nonlinear 'reaction function', f, may depend on both continuous delay (finite or infinite), and/or discrete delays are presented in [**38**].

Using the method of upper and lower solution, Feng and Lu in [**12**] proposed the existence-uniqueness theorem of the positive steady-state of system (2.11)–(2.13) with $i = 1, h(t, x) = 0$, and

$$f(t, x) = r(x)u(t, x)\frac{K(x) - au(t, x) - bu(t-\tau, x)}{K(x) - ac(x)u(t, x) - bc(x)u(t-\tau, x)}.$$

They also showed that this unique steady-state solution is asymptotically stable via monotone convergence results in [**35**] and the comparison argument for parabolic systems.

Davidson and Gourley [**7**] investigated the existence and uniqueness of the non-negative solution of system (2.11)–(2.13) with $L = \Delta,\ i = 1,\ h(t,x) = 0,\ \alpha = 0,\ \beta = 1,\ f = \lambda u(t,x)\frac{1-u(t-\tau,x)}{1+cu(t-\tau,x)}$ by applying the method of upper and lower solution. The local and global stability of the trivial solution were discussed in the case of homogeneous Dirichlet boundary condition and $\lambda < \lambda_1$, where λ_1 is the principal eigenvalue of the $-\Delta$. The conditions for achieving global stability of the trivial solution were found by taking L_2 inner product of the differential equation with the solution to obtain an ordinary differential inequality in the term of L_2 norm of the solution. They also presented theorems on existence and uniqueness of solution and on stability of the non-negative steady-state of this system. By investigating the corresponding linearised eigenvalue problem, the unique positive steady-state was proved to be locally stable under a condition on the time delay.

The system (2.11)–(2.13) with $L = \Delta,\ i = 1$ and homogeneous Neumann or Dirichlet boundary condition was investigated in [**16**]. The function on the right hand side of equation (2.11), $f(u(t), u(t-\tau))$, was assumed to be locally Lipschitz and to satisfy one sided growth estimates. It was proved that bounded solutions will be attracted to an equilibrium set if the time delay satisfies a certain condition. Obtaining the decay estimates on u by using Liapunov function for the undelayed counterpart of (2.11) is the main part of the proof.

Freitas [**14**] discussed the stability and bifurcation of stationary solutions of problems (2.11)-(2.13) with homogeneous Dirichlet or Neumann boundary condition and $L = \Delta,\ i = 1, f = f(x, u(t,x), u(t-\tau,x))$. The main tool used here is the linearization around stationary solutions and the comparison of these linear equations with those obtained from the problem without time delay. Further discussion of this method will be presented in Chapter 3.

An investigation on single species semilinear Volterra diffusion equations with homogeneous Neumann boundary condition was presented in [**57**]. In terms of system (2.11)–(2.13), this problem had $L = \Delta,\ i = 1,\ f = u(t)(a - bu(t) - \int_0^t k(t-s)u(s)ds$ where a and b are non-negative constants. Through the semigroup approach, the existence and uniqueness of solution were proved using a contraction mapping and a standard method based on Banach's fixed point theorem. The behaviour of solutions of the problem with and without non-delay logistic terms were obtained under smoothness and positivity conditions on the initial function, the coefficients and the kernel in equation (2.11). The proof was based on the energy method with the use of an appropriate Liapunov functional.

Redlinger [**41**] extended the results in [**31**] on the asymptotic behaviour of solutions of Volterra's population equations to the case with diffusion and homogeneous Neumann boundary condition. This is the case of equation (2.11)–(2.13) with $L = \Delta,\ i = 1,\ f = u(t)\left(a - bu(t) - \int_r^t k(t-s)u(s)ds\right)$, $r = 0$ or $r = -\infty$. The conditions assumed for k are continuity and integrability on $(0, \infty)$. These conditions are weaker than the ones in [**46**] (k is non-negative and decreasing) and in [**57**](k is non-negative, $k \in C^1(0, \infty)$ and $tk \in L_1(0, \infty)$). The proof used recursively defined sequences of pairs of upper and lower solution. The existence and uniqueness of the solution were also presented.

A similar problem with homogeneous Dirichlet boundary condition was discussed by Schiaffino and Tesei in [**47**]. They showed the existence, uniqueness and non-negativity of solutions using the semigroup approach under Holder continuity and measurability conditions on the coefficients. The existence and uniqueness of the equilibrium solution of the problem which is globally attractive in a Banach space of continuous real functions on the domain with respect to non-negative solutions were proved using convergent upper and lower sequences.

Shi and Chen [**51**] investigated Volterra's population equations with diffusion and homogeneous Neumann boundary condition. Here the coefficients in the equations were bounded functions. A priori bounds were obtained by making the use of the method of upper and lower solutions and the technique of monotone iteration. By using the Liapunov method, the stability characteristics of solutions of the problem were discussed.

Izsac [**20**] discussed a partial integrodifferential equation in the following form:

$$\dot{u} = -Au(t) + \int_{-\infty}^{t} k(t-s)g(s,u(s))ds$$
$$u = u_o(t) \qquad \text{in}(-\infty, 0]$$

where A is an M-accretive operator. The existence of solutions of this problem was proved using the Schauder's fixed point theorem. An application to an n-species Lotka-Volterra competitive system was presented.

The global existence of solutions for semilinear Volterra functional integrodifferential equations in a Banach space described by

$$\dot{u} = A(t)u(t) + f(t, u_t, \int_0^t k(t,s)g(s,u_s)ds$$
$$u_o = \phi$$

where A is a linear closed densely defined operator, was studied in [**33**] by using the Leray-Schauder Alternative.

The results mentioned in these three sections induced the investigation of logistic equations with diffusion, delay and impulses that are presented in this thesis. Diffusive logistic equations with discreet time delay and fixed time impulses under Robin boundary conditions are studied. The existence and uniqueness of solution and conditions for the existence of zero and positive attractors for this problem are obtained. Similar results are also gained for diffusive logistic equations with continuous time delay and fixed time impulses under Neumann boundary conditions.

CHAPTER 3

Diffusive Logistic Equations with Discrete Time Delay and Impulses

1. Introduction

In this chapter, the existence and uniqueness of the solution of diffusive logistic equations with impulses and time delay are presented by using the method of upper and lower solutions and its associated monotone iteration as in [**7**], [**35**], and [**36**]. The boundary conditions in this problem are Dirichlet and Robin (including Neumann) boundary conditions. The zero attractor and the positive attractor are also discussed. The techniques in [**7**], [**12**], and [**37**] are used to derive some conditions for the zero and positive steady-states of the problem without impulses to be attractors of the problem with impulses.

This chapter is organised as follows: in Section 2, the existence of solution of this system and its uniqueness in a sector under two different boundary conditions are presented. In Section 3 attractors of this system will be discussed. Some conditions under which the zero function is an attractor, are presented in Subsection 3.1. We investigate conditions on the impulses to obtain a positive attractor, in Subsection 3.2.

2. The Existence of Solution and Its Uniqueness in a Sector

2.1. Dirichlet Boundary Condition. We start by proving a existence and uniqueness theorem for the logistic equation with diffusion, delay, and jumps.

Let Ω be a bounded open domain in R^n, let $\partial\Omega$ be the boundary of Ω, and let $D = (0, \infty) \times \Omega$, $S = (0, \infty) \times \partial\Omega$. To accommodate delays, initial conditions are specified on an interval of length τ. Thus we introduce $D_{-\tau} = [-\tau, 0] \times \overline{\Omega}$ and $E = [-\tau, \infty) \times \overline{\Omega}$. The times at which jumps appear are denoted by $0 < t_1 < t_2 < t_3 < ...$, where $\lim_{k\to\infty} t_k = \infty$. Because of the delay, impulses at times t_k influence the behaviour of the solution at times $t_k + r\tau$ for $k, r \in \mathbf{Z}^+$. Hence it will be necessary to consider the partition $\bar{t}_0 = 0 < \bar{t}_1 < \bar{t}_2 < ...$ where $\bar{t}_i = r\tau$ or $\bar{t}_i = t_k + r\tau$ for some positive integer k and r, $i = 1, 2, 3, ...$

To specify the initial boundary value problem (IBVP), we introduce $M_i = \{(\bar{t}_i, x) |\ \bar{t}_i \in (0, \infty), x \in \Omega\}$, $M = \cup_{i=1}^{\infty} M_i$, $N_i = \{(\bar{t}_i, x) | \bar{t}_i \in (0, \infty), x \in \partial\Omega\}$, $N = \cup_{i=1}^{\infty} N_i$, non-negative constant a, and positive constant b.

The IBVP may now be stated as follows.

$$\frac{\partial u}{\partial t}(t, x) - \Delta u(t, x) = u(t, x)(a - bu(t - \tau, x)) \qquad \text{in } D \setminus M \tag{3.1}$$

$$u(t, x) = 0 \qquad \text{on } S \setminus N \tag{3.2}$$

$$u(t, x) = \eta(t, x) \qquad \text{in } D_{-\tau} \tag{3.3}$$

$$u(t_k, x) = I_k(u({t_k}^-, x)) \qquad \text{in } \overline{\Omega},\ k = 1, 2, ... \tag{3.4}$$

where $\eta : D_{-\tau} \to R^n$ is uniformly Holder continuous with exponent α, $\eta(t, x) \geq 0$ in $D_{-\tau}$, and $\eta(0, x) \not\equiv 0$.

Let $L = \frac{\partial}{\partial t} - \Delta$, $u = u(t, x)$ and $u_{-\tau} = u(t - \tau, x)$.

This system describes the dynamics of a population which has natural growth rate a and depends on the state in the past. The carrying capacity of the environment is $\frac{a}{b}$. At fixed times, t_k, the population undergoes instantaneous change (e.g. harvesting).

To solve this problem, two assumptions are made. The following definition is used in the assumptions.

Definition 3.1. *Let Ω be a bounded domain in $\mathbf{R}^n$ with boundary $\partial\Omega$. Then $\partial\Omega$ belongs to class $C^{m+\alpha}$for some non-negative integer m and some positive number $\alpha \in (0, 1)$ if in neighbourhoods of each point of $\partial\Omega$, there exists a local representation of $\partial\Omega$ having the form*

$$x_i = h_i (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$$

where the function h_i belongs to class $C^{m+\alpha}$.

Assumptions

A1 $\partial\Omega \in C^{2+\alpha}$.

A2 For each k, $I_k \in C(\mathbf{R}, \mathbf{R})$ is a non-decreasing function and $I_k(0) = 0$.

We will prove the existence and uniqueness of the solution of IBVP (3.1)–(3.4) by applying a similar method used in [**35**] and [**36**].

First, we consider the problem in a finite time interval $[0, T]$ for some positive real number T. In other words, we solve the following problem:

$$u_t(t,x) - \Delta u(t,x) = u(t,x)(a - bu(t-\tau, x)) \qquad \text{in } D_T \setminus M_T \quad (3.5)$$

$$u(t,x) = 0 \qquad \text{on } S_T \setminus N_T \quad (3.6)$$

$$u(t,x) = \eta(t,x) \qquad \text{in } D_{-\tau} \quad (3.7)$$

$$u(t_k, x) = I_k(u(t_k{}^-, x)) \qquad \text{in } \overline{\Omega},\ k = 1, 2, ..., p_T \quad (3.8)$$

where $D_T = (0,T] \times \Omega$, $S_T = (0,T] \times \partial\Omega$, $E_T = [-\tau, T] \times \overline{\Omega}$, $M_T = \cup_{i=1}^{p'_T} M_i$ and $N_T = \cup_{i=1}^{p'_T} N_i$ with $p_T = \min\{k | t_k \geq T\}$ and $p'_T = \min\{j | \bar{t}_j \geq T\}$.

We seek a solution of IBVP (3.5)–(3.8) in $\mathcal{B}_T$, that is, the set of functions $u(t,x), u : E_T \to R^n$ that satisfy

(1) $u(t,x) \in C^\alpha(E_T \setminus (M_T \cup N_T) \cap C^{1,1}(\overline{D_T} \setminus (M_T \cup N_T))$.
(2) $\frac{\partial^2 u}{\partial x_i \partial x_j}$ exists $\forall i, j = 1, 2, ..., n$, and is uniformly Holder continuous with exponent α, $0 < \alpha < 1$ in $D_T \setminus M_T$.
(3) $\lim_{t \to \bar{t}_i{}^+} v(t,x) = v(\bar{t}_i, x)$ and $\lim_{t \to \bar{t}_i{}^-} v(t,x) = v(\bar{t}_i{}^-, x)$ exist for $x \in \overline{\Omega}$ where $v(t,x) = \left(u(t,x), \frac{\partial u}{\partial t}(t,x), \frac{\partial u}{\partial x_i}(t,x), \frac{\partial^2 u}{\partial x_i \partial x_j}\right)$, $i, j = 1, 2, ..., n$.

We use the method of upper and lower solutions as in [**35**] for solving the IBVP (3.1)–(3.4).

Definition 3.2. *A pair of functions $\tilde{u}, \hat{u} \in \mathcal{B}_T$ with $\tilde{u}(t,x) \geq \hat{u}(t,x)$ is called an ordered upper and lower solution of IBVP (3.1)–(3.4) if it satisfies*

$$\hat{u} \leq \eta(t,x) \leq \tilde{u} \qquad \textit{in } D_{-\tau} \quad (3.9)$$

$$L\tilde{u} \geq \tilde{u}(a - b\hat{u}_{-\tau}) \qquad \textit{in } D_T \setminus M_T \quad (3.10)$$

$$L\hat{u} \leq \hat{u}(a - b\tilde{u}_{-\tau}) \qquad \textit{in } D_T \setminus M_T \quad (3.11)$$

$$\hat{u} \leq 0 \leq \tilde{u} \qquad \textit{on } S_T \setminus N_T \quad (3.12)$$

$$\tilde{u}(t_k, x) \geq I_k(\tilde{u}(t_k{}^-, x)) \qquad \textit{in } \overline{\Omega} \quad (3.13)$$

$$\hat{u}(t_k, x) \le I_k(\hat{u}(t_k^-, x)) \qquad \textit{in } \overline{\Omega}. \tag{3.14}$$

For any ordered upper and lower solution, $\tilde{u}, \hat{u}$, the sector $\langle \hat{u}, \tilde{u} \rangle$ is defined as the functional interval $\langle \hat{u}, \tilde{u} \rangle = \{v \in \mathcal{B}_T | \hat{u} \le v \le \tilde{u}\}$. We seek a solution of IBVP (3.1)–(3.4) in this sector by constructing upper and lower sequences then showing that these sequences converge to a unique function, which is the solution.

By the local Lipschitz property of the function on the right hand side of equation (3.1), there exists a positive constant K such that

$$|u(a - bw) - v(a - bz)| \le K(|u - v| + |w - z|) \text{ for } u, v, w, z \in \langle \hat{u}, \tilde{u} \rangle. \tag{3.15}$$

Let $\mathcal{L} = \frac{\partial}{\partial t} + \Delta + K$. The upper sequence $\{\overline{u}^{(m)}\}$ and lower sequence $\{\underline{u}^{(m)}\}$ are constructed as follows

$$\overline{u}^{(0)} = \tilde{u} \qquad \text{in } D_T \tag{3.16}$$

$$\underline{u}^{(0)} = \hat{u} \qquad \text{in } D_T \tag{3.17}$$

$$\underline{u}^{(m)} = \eta = \overline{u}^{(m)} \qquad \text{in } D_{-\tau} \tag{3.18}$$

$$\mathcal{L}\overline{u}^{(m)} = \overline{u}^{(m-1)}(K + a - b\underline{u}_{-\tau}^{(m-1)}) \qquad \text{in } D_T \setminus M_T \tag{3.19}$$

$$\mathcal{L}\underline{u}^{(m)} = \underline{u}^{(m-1)}(K + a - b\overline{u}_{-\tau}^{(m-1)}) \qquad \text{in } D_T \setminus M_T \tag{3.20}$$

$$\underline{u}^{(m)} = 0 = \overline{u}^{(m)} \qquad \text{on } S_T \setminus N_T \tag{3.21}$$

$$\overline{u}^{(m)}(t_k, x) = I_k(\overline{u}^{(m)}(t_k^-, x)) \qquad \text{in } \overline{\Omega} \tag{3.22}$$

$$\underline{u}^{(m)}(t_k, x) = I_k(\underline{u}^{(m)}(t_k^-, x)) \qquad \text{in } \overline{\Omega}. \tag{3.23}$$

We will use the following theorem on the existence of solution of the initial boundary value problem given by

$$\begin{aligned} \tilde{L}u &= \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \, \partial x_j} - \sum_{i=1}^{n} b_i(t, x) \frac{\partial u}{\partial x_i} \\ &= f(t, x, u) \qquad \text{in } (0, T] \times \Omega \end{aligned} \tag{3.24}$$

$$Bu = \alpha_1(t,x)\frac{\partial u}{\partial \nu}(t,x) + \beta_0(t,x)u = h(t,x) \qquad \text{on } (0,T] \times \partial\Omega \tag{3.25}$$

$$u(0,x) = u_0(x) \qquad \text{in } \Omega \tag{3.26}$$

where $\tilde{L}$ is a uniformly parabolic operator to prove the well-definedness of upper and lower sequences.

THEOREM 3.1 (Theorem 2.1.1 in [**37**]). *Let $\alpha_0 = 0$, $f(t,x,u) = q(t,x) - c(t,x)u$ with q locally Holder continuous in $x \in \Omega$, uniformly in t. Then for any continuous function h and u_0 which satisfy compatibility condition $\beta_0(0,x)u_0(x) = h(0,x)$ on $\partial\Omega$, problem (3.24)–(3.26) has a unique solution u. Moreover u can be represented by the formula*

$$\begin{aligned} u(t,x) =& J^{(1)}(t,x) + \int_0^t d\tau \int_\Omega G(t,x;\tau,\xi)q(\tau,\xi)d\xi \\ &+ \int_0^t d\tau \int_{\partial\Omega} \frac{\partial \Gamma}{\partial \nu_\xi}(t,x;\tau,\xi)\psi(\tau,\xi)d\xi \end{aligned} \tag{3.27}$$

where

- *Γ is the fundamental solution of parabolic operator*

$$\left(\frac{\partial}{\partial t} - \sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(t,x)\frac{\partial}{\partial x_i} + c(t,x)\right);$$

- *G is the Green's function of $\left(\frac{\partial}{\partial t} - \sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(t,x)\frac{\partial}{\partial x_i}\right)$ $+c(t,x)$ with boundary operator B;*
- *$\psi(t,x) = 2\int_0^t d\tau \int_{\partial\Omega} \frac{\partial\Gamma}{\partial\nu_\xi}(t,x;\tau,\xi)\psi(\tau,\xi)d\xi - \frac{2h(t,x)}{\beta(t,x)}$*
- *$J^{(1)}(t,x) = \int_\Omega G(t,x;0,\xi)u_0(\xi)d\xi$.*

This theorem can be applied to problem (3.16)–(3.23) and gives that the upper and lower sequences are well-defined and form a subset of $\mathcal{B}_T$.

Before we proceed to prove the monotone property of upper and lower sequence, let us consider a lemma that has an important role in the proof.

LEMMA 3.1 (Positivity Lemma in [**37**]). *If* $w \in C(\overline{(0,T] \times \Omega}) \cap C^{1,2}((0,T] \times \Omega)$ *and satisfies the relation*

$$\begin{aligned} w_t - d\nabla_2 w + cw &\geq 0 && \text{in } (0,T] \times \Omega \\ \alpha_1 \frac{\partial w}{\partial \nu}(t,x) + \beta_1 w &\geq 0 && \text{on } (0,T] \times \partial\Omega \\ w(0,x) &\geq 0 && \text{in } \Omega \end{aligned}$$

where α_1, β_1 *are non-negative constants,* d *is positive constant, and* $c \equiv c(t,x)$ *is any bounded function in* $(0,T] \times \Omega$*, then* $w \geq 0$ *in* $\overline{(0,T] \times \Omega}$*.*

Using this lemma, we prove the following theorem.

THEOREM 3.2. $\{\overline{u}^{(m)}\}$ *and* $\{\underline{u}^{(m)}\}$ *possess the monotone property* $\hat{u} \leq \underline{u}^{(m)} \leq \underline{u}^{(m+1)} \leq \overline{u}^{(m+1)} \leq \overline{u}^{(m)} \leq \tilde{u}$ *in* $[-\tau, T] \times \overline{\Omega}$*.*

PROOF. From the definition of ordered upper and lower solution, it follows that $\hat{u} \leq \underline{u}^{(m)} \leq \underline{u}^{(m+1)} \leq \overline{u}^{(m+1)} \leq \overline{u}^{(m)} \leq \tilde{u}$ in $D_{-\tau}$. Consider the problem in $[0, \overline{t}_1) \times \overline{\Omega}$. Let $w^{(1)} = \overline{u}^{(0)} - \overline{u}^{(1)} = \tilde{u} - \overline{u}^{(1)}$.

$$\mathcal{L}w^{(1)} = \mathcal{L}\tilde{u} - \mathcal{L}\overline{u}^{(1)} = L\tilde{u} - \tilde{u}(a - b\eta) \geq 0 \text{ in } (0, t_1) \times \Omega.$$

From the boundary condition and initial function we have

$$\begin{aligned} w^{(1)} &\geq 0 && \text{on } (0, t_1) \times \partial\Omega \\ w^{(1)} &\geq 0 && \text{in } (-\tau, 0] \times \overline{\Omega}. \end{aligned}$$

It follows from Lemma 3.1 that $\tilde{u} \geq \overline{u}^{(1)}$ in $(t,x) \in [0, t_1) \times \Omega$. Similarly, it can be shown that $\hat{u} \leq \underline{u}^{(1)}$ in $[0, t_1) \times \overline{\Omega}$. If $\overline{t}_1 = t_1$, then by the assumption on I_k, $\tilde{u}(t_1, x) \geq \overline{u}^{(1)}(t_1, x)$ and $\hat{u}(t_1, x) \leq \underline{u}^{(1)}(t_1, x)$. If $\overline{t}_1 = \tau < t_1$, then $\overline{u}(\overline{t}_1, x) = \overline{u}(\overline{t}_1^-, x)$ and $\underline{u}(\overline{t}_1, x) = \underline{u}(\overline{t}_1^-, x)$.

Let $w_{(1)} = \overline{u}^{(1)} - \underline{u}^{(1)}$. Then

$$\begin{aligned} \mathcal{L}w_{(1)} = \mathcal{L}\overline{u}^{(1)} - \mathcal{L}\underline{u}^{(1)} = (K + a - b\eta)(\tilde{u} - \hat{u}) &\geq 0 && \text{in } (0, t_1) \times \Omega \\ w_{(1)} &\geq 0 && \text{on } (0, t_1) \times \partial\Omega \\ w_{(1)} &\geq 0 && \text{in } (-\tau, 0] \times \overline{\Omega}. \end{aligned}$$

It follows from Lemma 3.1 that $\overline{u}^{(1)} \geq \underline{u}^{(1)}$ in $(0, \bar{t}_1)$. If $\bar{t}_1 = t_1$, then by the assumption A2, $\overline{u}^{(1)}(\bar{t}_1, x) \geq \underline{u}^{(1)}(\bar{t}_1, x)$. If $\bar{t}_1 = \tau < t_1$, then from the continuity of u, we have $\overline{u}^{(1)}(\bar{t}_1, x) \geq \underline{u}^{(1)}(\bar{t}_1, x)$.

Suppose $\overline{u}^{(m)} \leq \overline{u}^{(m-1)}$ and $\underline{u}^{(m)} \geq \overline{u}^{(m)}$ for some $m \in \{2, 3, ...\}$.

$$\begin{aligned}\mathcal{L}w^{(m)} &= \mathcal{L}\overline{u}^{(m)} - \mathcal{L}\overline{u}^{(m+1)} \\ &= K(\overline{u}^{(m-1)} - \overline{u}^{(m)}) + \overline{u}^{(m-1)}(a - b\eta) - \overline{u}^{(m)}(a - b\eta) \\ &= (K + a - b\eta)(\overline{u}^{(m-1)} - \overline{u}^{(m)}) \\ &\geq 0 \qquad \text{in } (0, \bar{t}_1) \times \Omega\end{aligned}$$

$$\begin{aligned}w^{(m)} &\geq 0 \qquad \text{on } (0, \bar{t}_1) \times \partial\Omega \\ w^{(m)} &\geq 0 \qquad \text{in } (-\tau, 0] \times \overline{\Omega}.\end{aligned}$$

By application of Lemma 3.1, we have $\overline{u}^{(m)} \geq \overline{u}^{(m+1)}$ in $[0, \bar{t}_1) \times \Omega$ and if $\bar{t}_1 = t_1$, then by the monotone non-decreasing property of I_k, $\overline{u}^{(m)}(\bar{t}_1, x) \geq \overline{u}^{(m+1)}(\bar{t}_1, x)$. The same result is obtained if $\bar{t}_1 = \tau < t_1$ by the continuity of u. Similarly, $\underline{u}^{(m)} \leq \underline{u}^{(m+1)}$ in $[0, \bar{t}_1) \times \Omega$ and $\overline{u}^{(m)}(\bar{t}_1, x) \geq \overline{u}^{(m+1)}(\bar{t}_1, x)$.

Now assume $\overline{u}^{(m)} \geq \underline{u}^{(m)}$ for some $m \in \{2, 3, ...\}$.

$$\begin{aligned}\mathcal{L}w_{(m+1)} &= \mathcal{L}\overline{u}^{(m+1)} - \mathcal{L}\underline{u}^{(m+1)} \\ &= K(\overline{u}^{(m)} - \underline{u}^{(m)}) + \overline{u}^{(m)}(a - b\eta) - \underline{u}^{(m)}(a - b\eta) \\ &\geq K(\overline{u}^{(m)} - \underline{u}^{(m)}) + \overline{u}^{(m)}(a - b\overline{u}^{(m)}_{-\tau}) - \underline{u}^{(m)}(a - b\overline{u}^{(m)}_{-\tau}) \\ &= (K + a - b\overline{u}^{(m)}_{-\tau})(\overline{u}^{(m)} - \underline{u}^{(m)}) \\ &\geq 0 \qquad\qquad \text{in } (0, \bar{t}_1) \times \Omega \\ w^{(m)} &\geq 0 \qquad\qquad \text{on } (0, \bar{t}_1) \times \partial\Omega \\ w^{(m)} &\geq 0 \qquad\qquad \text{in } (-\tau, 0] \times \overline{\Omega}.\end{aligned}$$

By the same steps as before, we obtain $\overline{u}^{(m+1)} \geq \underline{u}^{(m+1)}$ in $[0, \bar{t}_1) \times \Omega$ and $\overline{u}^{(m+1)}(\bar{t}_1, x) \geq \underline{u}^{(m+1)}(\bar{t}_1, x)$. These results show that the monotone property $\hat{u} \leq \underline{u}^{(m)} \leq \underline{u}^{(m+1)} \leq \overline{u}^{(m+1)} \leq \overline{u}^{(m)} \leq \tilde{u}$ for $t \in [-\tau, \bar{t}_1]$ and $m = 1, 2, ...$ holds.

Suppose the monotone property holds in the time interval $[-\tau, \bar{t}_j]$ for some $j \in \{2, 3, ...{p'}_T\}$ and consider upper and lower sequences in the time interval

$[\bar{t}_j, \bar{t}_{j+1})$.

$$\begin{aligned}
\mathcal{L}w^{(1)} &= \mathcal{L}\tilde{u} - \mathcal{L}\overline{u}^{(1)} \\
&= L\tilde{u} - \tilde{u}(a - b\hat{u}_{-\tau}) \\
&\geq 0 && \text{in } (\bar{t}_j, t_{j+1}) \times \Omega \\
w^{(1)} &\geq 0 && \text{on } (\bar{t}_j, \bar{t}_{j+1}) \times \partial\Omega \\
w^{(1)} &\geq 0 && \text{in } (-\tau, \bar{t}_j] \times \overline{\Omega}.
\end{aligned}$$

Hence $\tilde{u} \geq \overline{u}^{(1)}$ in $[0, \bar{t}_{j+1}) \times \Omega$. From the continuity of the solution and the monotone non-decreasing property of I_k, it follows that $\tilde{u}(\bar{t}_{j+1}, x) \geq \overline{u}^{(1)}(\bar{t}_{j+1}, x)$. Similarly we can prove $\hat{u} \leq \underline{u}$ in $[0, \bar{t}_{j+1}] \times \Omega$.

$$\begin{aligned}
\mathcal{L}w_{(1)} &= \mathcal{L}\overline{u}^{(1)} - \mathcal{L}\underline{u}^{(1)} \\
&= K(\tilde{u} - \hat{u}) + \tilde{u}(a - b\hat{u}_{-\tau}) - \hat{u}(a - b\tilde{u}_{-\tau}) \\
&\geq K(\tilde{u} - \hat{u}) + \tilde{u}(a - b\tilde{u}_{-\tau}) - \hat{u}(a - b\tilde{u}_{-\tau}) \\
&= (K + a - b\tilde{u}_{-\tau})(\tilde{u} - \hat{u}) \\
&\geq 0 && \text{in } (\bar{t}_j, \bar{t}_{j+1}) \times \Omega \\
w_{(1)} &\geq 0 && \text{on } (\bar{t}_j, \bar{t}_{j+1}) \times \partial\Omega \\
w_{(1)} &\geq 0 && \text{in } (-\tau, \bar{t}_j] \times \overline{\Omega}.
\end{aligned}$$

These imply $\overline{u}^{(1)} \geq \underline{u}^{(1)}$ in $(t, x) \in [\bar{t}_j, \bar{t}_{j+1}] \times \Omega$.

Suppose $\overline{u}^{(m)} \leq \overline{u}^{(m-1)}$, $\underline{u}^{(m)} \geq \underline{u}^{(m-1)}$ and $\overline{u}^{(m)} \geq \underline{u}^{(m)}$ for some $m \in \{2, 3, ...\}$.

$$\begin{aligned}
\mathcal{L}w^{(m+1)} &= \mathcal{L}\overline{u}^{(m)} - \mathcal{L}\overline{u}^{(m+1)} \\
&= K(\overline{u}^{(m-1)} - \overline{u}^{(m)}) + \overline{u}^{(m-1)}(a - b\underline{u}_{-\tau}^{(m-1)}) - \overline{u}^{(m)}(a - b\underline{u}_{-\tau}^{(m)}) \\
&\geq K(\overline{u}^{(m-1)} - \overline{u}^{(m)}) + \overline{u}^{(m-1)}(a - b\underline{u}_{-\tau}^{(m)}) - \overline{u}^{(m)}(a - b\underline{u}_{-\tau}^{(m)}) \\
&= (K + a - b\underline{u}_{-\tau}^{(m)})(\overline{u}^{(m-1)} - \overline{u}^{(m)}) \\
&\geq 0 \qquad \text{in } (\bar{t}_j, \bar{t}_{j+1}) \times \Omega
\end{aligned}$$

$$w^{(m)} \geq 0 \qquad \text{on } (\bar{t}_j, \bar{t}_{j+1}) \times \partial\Omega$$
$$w^{(m)} \geq 0 \qquad \text{in } (-\tau, t_k] \times \overline{\Omega}.$$

By application of Lemma 3.1, the continuity of the solution, and the monotone decreasing property of I_k, we have $\overline{u}^{(m)} \geq \overline{u}^{(m+1)}$ in $[0, \bar{t}_{j+1}] \times \Omega$. Also $\underline{u}^{(m)} \leq \underline{u}^{(m+1)}$.

$$\begin{aligned} \mathcal{L}w_{(m)} &= \mathcal{L}\overline{u}^{(m+1)} - \mathcal{L}\underline{u}^{(m+1)} \\ &= K(\overline{u}^{(m)} - \underline{u}^{(m)}) + \overline{u}^{(m)}(a - b\underline{u}^{(m)}_{-\tau}) - \underline{u}^{(m)}(a - b\overline{u}^{(m)}_{-\tau}) \\ &\geq K(\overline{u}^{(m)} - \underline{u}^{(m)}) + \overline{u}^{(m)}(a - b\overline{u}^{(m)}_{-\tau}) - \underline{u}^{(m)}(a - b\overline{u}^{(m)}_{-\tau}) \end{aligned}$$

$$\begin{aligned} \mathcal{L}w_{(m)} &\geq (K + a - b\overline{u}^{(m)}_{-\tau})(\overline{u}^{(m)} - \underline{u}^{(m)}) \\ &\geq 0 \qquad \text{in } (\bar{t}_j, \bar{t}_{j+1}) \times \Omega \\ w^{(m)} &\geq 0 \qquad \text{on } (\bar{t}_j, \bar{t}_{j+1}) \times \partial\Omega \\ w^{(m)} &\geq 0 \qquad \text{in } (-\tau, \bar{t}_j] \times \overline{\Omega}. \end{aligned}$$

It follows from the same argument as before that $\overline{u}^{(m+1)} \geq \underline{u}^{(m+1)}$ in $[0, \bar{t}_j + 1] \times \Omega$. By these results, it is concluded that $\hat{u} \leq \underline{u}^{(m)} \leq \underline{u}^{(m+1)} \leq \overline{u}^{(m+1)} \leq \overline{u}^{(m)} \leq \tilde{u}$ in E_T for all positive integer m. □

From this result, we conclude that the limits, $\overline{u} = \lim\limits_{m\to\infty} \overline{u}^{(m)}$ exist and $\underline{u} = \lim\limits_{m\to\infty} \underline{u}^{(m)}$, and these limits satisfy

$$\underline{u} \leq \overline{u}.$$

We will use the integral representation of solution in Theorem 3.1 to prove that the limit of the upper and lower sequence, $\overline{u}$ and $\underline{u}$, satisfy the following equations:

$$\underline{u} = \eta = \overline{u} \qquad \text{in } D_{-\tau} \tag{3.28}$$
$$\mathcal{L}\overline{u} = \overline{u}(K + a - b\underline{u}_{-\tau}) \qquad \text{in } D_T \setminus M_T \tag{3.29}$$
$$\mathcal{L}\underline{u} = \underline{u}(K + a - b\overline{u}_{-\tau}) \qquad \text{in } D_T \setminus M_T \tag{3.30}$$

$$\underline{u} = 0 = \overline{u} \qquad \text{on } S_T \setminus N_T \tag{3.31}$$

$$\overline{u}(t_k, x) = I_k(\overline{u}(t_k{}^-, x)) \qquad \text{in } \overline{\Omega} \tag{3.32}$$

$$\underline{u}(t_k, x) = I_k(\underline{u}(t_k{}^-, x)) \qquad \text{in } \overline{\Omega}. \tag{3.33}$$

LEMMA 3.2. *The limit of upper and lower sequence, $\overline{u}$ and $\underline{u}$, satisfy equations (3.28)–(3.33).*

PROOF. From the construction of upper and lower sequences, $\overline{u}$ and $\underline{u}$ satisfy equations (3.28) and (3.31)–(3.33).

Consider the initial boundary value problem formed by equation (3.18),(3.19) and (3.21) on interval $(0, \overline{t}_1)$. By Theorem 3.1 the integral representation of solution of this problem is

$$\overline{u}^{(m)}(t,x) = J^{(1)}(t,x) + \int_0^t d\kappa \int_\Omega G(t,x;\kappa,\xi)\overline{u}^{(m-1)}(\kappa,\xi) \times (K + a - b\underline{u}^{(m-1)}(\kappa - \tau, \xi))d\xi \tag{3.34}$$

with $u_0(\xi)$ in $J^{(1)}$ replaced by $\eta(0,\xi)$. By taking the limit of this equation as $m \to \infty$ and applying dominated convergence theorem, we obtain the following equation

$$\overline{u}(t,x) = J^{(1)}(t,x) + \int_0^t d\kappa \int_\Omega G(t,x;\kappa,\xi)\overline{u}(\kappa,\xi)(K + a - b\underline{u}(\kappa - \tau, \xi))d\xi. \tag{3.35}$$

This equation is the integral representation of the solution of the initial boundary value problem formed by equations (3.28), (3.29) and (3.31).

The integral representation of the solution of the problem formed by equations (3.18),(3.20) and (3.21) on the interval time $(0, \overline{t}_1)$ is as follows

$$\underline{u}^{(m)}(t,x) = J^{(1)}(t,x) + \int_0^t d\kappa \int_\Omega G(t,x;\kappa,\xi)\underline{u}^{(m-1)}(\kappa,\xi) \times (K + a - b\overline{u}^{(m-1)}(\kappa - \tau, \xi))d\xi \tag{3.36}$$

with $u_0(\xi)$ in $J^{(1)}$ replaced by $\eta(0,\xi)$. By applying steps analogous to those above, we obtain the integral representation of the solution of the initial boundary value problem formed by equations (3.28), (3.30) and (3.31) which is given by

$$\underline{u}(t,x) = J^{(1)}(t,x) + \int_0^t d\kappa \int_\Omega G(t,x;\kappa,\xi)\underline{u}(\kappa,\xi)(K+a-b\overline{u}(\kappa-\tau,\xi))d\xi. \tag{3.37}$$

By these results, we conclude that $\overline{u}$ and $\underline{u}$ satisfy equations (3.28)–(3.33) on interval time $[-\tau, \overline{t}_1)$.

For the other time intervals, $(\overline{t}_j, \overline{t}_{j+1})$ for $j = 1, 2, 3, ..., p'_T - 1$, the integral representation of the solution of the problem formed by equation (3.19), (3.21) and initial condition $\overline{u}^{(m)}(\overline{t}_j, x) = \overline{u}^{(m)}(\overline{t}_j^-, x)$ if $\overline{t}_j \neq t_k$ or $\overline{u}^{(m)}(\overline{t}_j, x) = I_k(\overline{u}^{(m)}(\overline{t}_j^-, x))$ if $\overline{t}_j = t_k$, is

$$\begin{aligned}\overline{u}^{(m)}(t+\overline{t}_j, x) = J^{(1)}(t,x) + &\int_0^t d\kappa \int_\Omega G(t,x;\kappa,\xi)\overline{u}^{(m-1)}(\kappa+\overline{t}_j,\xi) \\ &\times (K+a-b\underline{u}^{(m-1)}(\kappa-\tau+\overline{t}_j,\xi))d\xi,\end{aligned} \tag{3.38}$$

with $u_0(\xi)$ in $J^{(1)}$ replaced by the suitable initial condition in this interval. If we take the limit of equation (3.38) as $m \to \infty$ and apply dominated convergence theorem, we will obtain the following equation

$$\begin{aligned}\overline{u}(t+\overline{t}_j, x) = J^{(1)}(t,x) + &\int_0^t d\kappa \int_\Omega G(t,x;\kappa,\xi)\overline{u}(\kappa+\overline{t}_j,\xi) \\ &\times (K+a-b\underline{u}(\kappa-\tau+\overline{t}_j,\xi))d\xi,\end{aligned} \tag{3.39}$$

which is the integral presentation of the solution of the problem formed by equations (3.29), (3.31) and initial condition $\overline{u}(\overline{t}_j, x) = \overline{u}(\overline{t}_j^-, x)$ if $\overline{t}_j \neq t_k$ or $\overline{u}(\overline{t}_j, x) = I_k(\overline{u}(\overline{t}_j^-, x))$ if $\overline{t}_j = t_k$.

The integral representation of the solution of the problem formed by equations (3.20), (3.21) with initial condition $\underline{u}^{(m)}(\overline{t}_j, x) = \underline{u}^{(m)}(\overline{t}_j^-, x)$ if $\overline{t}_j \neq t_k$ or $\underline{u}^{(m)}(\overline{t}_j, x) = I_k(\underline{u}^{(m)}(\overline{t}_j^-, x))$ if $\overline{t}_j = t_k$, is given by the following equation.

$$\begin{aligned}\underline{u}^{(m)}(t+\overline{t}_j, x) = J^{(1)}(t,x) + &\int_0^t d\kappa \int_\Omega G(t,x;\kappa,\xi)\underline{u}^{(m-1)}(\kappa+\overline{t}_j,\xi) \\ &\times (K+a-b\overline{u}^{(m-1)}(\kappa-\tau+\overline{t}_j,\xi))d\xi\end{aligned} \tag{3.40}$$

with $u_0(\xi)$ in $J^{(1)}$ replaced by the suitable initial condition. By applying the previous steps we have the integral representation of the initial boundary value problem formed by equations (3.30), (3.31) and initial condition $\underline{u}(\bar{t}_j, x) = \underline{u}(\bar{t}_j^-, x)$ if $\bar{t}_j \neq t_k$ or $\underline{u}(\bar{t}_j, x) = I_k(\underline{u}(\bar{t}_j^-, x))$ if $\bar{t}_j = t_k$ as follows

$$\underline{u}(t + \bar{t}_j, x) = J^{(1)}(t, x) + \int_0^t d\kappa \int_\Omega G(t, x; \kappa, \xi)\underline{u}(\kappa + \bar{t}_j, \xi) \\ \times (K + a - b\overline{u}(\kappa - \tau + \bar{t}_j, \xi))d\xi. \tag{3.41}$$

Thus $\overline{u}$ and $\underline{u}$ satisfy equations (3.28)–(3.33) on interval time $[-\tau, \bar{t}_{j+1})$ for $j = 1, 2, 3, ..., p'_T - 1$. □

Next we will prove that the limit of the upper and lower sequence is the unique solution of problem (3.5)–(3.8). For this purpose we use estimates of the fundamental function and its outward normal derivative and Green's function in [**35**] as follows.

$$|\Gamma(t, x; \kappa, \xi)| \leq \frac{K_0}{(t-\kappa)^\mu} \frac{1}{|x-\xi|^{n-2+\mu}} \qquad , 0 < \mu < 1 \tag{3.42}$$

$$\left|\frac{\partial \Gamma}{\partial \nu_x}\right| \leq \frac{K_0}{(t-\kappa)^\mu} \frac{1}{|x-\xi|^{n+1-2\mu-\gamma}} \qquad , 1 - \frac{\gamma}{2} < \mu < 1. \tag{3.43}$$

$$|G(t, x; \kappa, \xi)| \leq \frac{K_1}{(t-\kappa)^\mu} \frac{1}{|x-\xi|^{n-2+\mu}} \qquad , 0 < \mu < 1. \tag{3.44}$$

Let $\|v\|_t = \max\{v(s, x) : s \in [0, t], x \in \overline{\Omega}\}$.

THEOREM 3.3. *The upper and lower sequence,* $\{\overline{u}^{(m)}\}$ *and* $\{\underline{u}^{(m)}\}$, *converge monotonically to a unique solution* u *of IBVP (3.5)–(3.8).*

PROOF. First we will show that $\overline{u} = \underline{u}$ in $[-\tau, \bar{t}_1] \times \overline{\Omega}$. From (3.35), (3.37), (3.15), (3.42) and (3.43) we have the following relation:

$$(\overline{u}(t, x) - \underline{u}(t, x)) \leq \int_0^t d\kappa \int_\Omega G(t, x; \kappa, \xi)((K + a)(\overline{u}(\kappa, \xi) - \underline{u}(\kappa, \xi)) \\ - b(\overline{u}(\kappa, \xi)\underline{u}(\kappa - \tau, \xi) - \underline{u}(\kappa, \xi)\overline{u}(\kappa - \tau, \xi))) \quad , t \in (0, \bar{t}_1) d\xi$$

$$|\overline{u}(t, x) - \underline{u}(t, x)| \leq 4KK_1 \int_0^t (t-\kappa)^{-\mu} d\kappa \int_\Omega |x - \xi|^{-n+2-\mu} d\xi \|\overline{u} - \underline{u}\|_t \ , t \in (0, \bar{t}_1)$$

$$\|\overline{u} - \underline{u}\|_t \leq Mt^{1-\mu}\|\overline{u} - \underline{u}\|_t \qquad , t \in (0, \bar{t}_1) \tag{3.45}$$

for some positive constant M. If $t < M^{\frac{-1}{1-\mu}}$ then inequality (3.45) implies $\|\overline{u} - \underline{u}\|_t = 0$. This means that $\overline{u}(t, x) = \underline{u}(t, x)$ for $t \in [0, \min\{\bar{t}_1, M^{\frac{-1}{1-\mu}}\})$.

Let $\{0 = \bar{t}_1^{j_1}, \bar{t}_1^{j_2}, ..., \bar{t}_1^{j_{n1}} = \bar{t}_1\}$ be a partition on $[0, t_1]$ with $\bar{t}_1^{j_k} - \bar{t}_1^{j_l} < M^{\frac{-1}{1-\mu}}$, $k, l \in \{1, 2, ..., n1\}$. With suitable initial conditions (note that the initial conditions for $\overline{u}$ and $\underline{u}$ in an interval are the same), we obtain the same result on time interval $[\bar{t}_1^{j_i}, \bar{t}_1^{j_{i+1}})$, $i = 1, ..., n1-1$ by repeating the steps above. Thus $\overline{u}(t, x) = \underline{u}(t, x)$ in $[-\tau, t_1) \times \overline{\Omega}$.

By continuing this process on time interval $[\bar{t}_i, \bar{t}_{i+1})$ for $i = 1, ..., p'_T - 1$, we obtain that $\overline{u}(t, x) = \underline{u}(t, x)$ in $[-\tau, T] \times \overline{\Omega}$ and it is the solution of IBVP (3.5)–(3.8).

The uniqueness of the solution can be proved by using the same method.

□

Lemma 3.3. *There exists a pair of ordered, lower and upper solutions of IBVP (3.5)–(3.8).*

Proof. The lower solution is the zero function and the upper solution be given by v with

$$v(t, x) = 0 \qquad \text{on } S_T \setminus N_T$$
$$v(t, x) = v_j(t, x) \text{ in } [\bar{t}_j, \bar{t}_{j+1}) \times \Omega$$
$$v(\bar{t}_j, x) = I_k(v_{j-1}(\bar{t}_j^-, x)), \text{ if } \bar{t}_j = t_k \text{ for some } k \in \{1, 2, ..., p_T\}, \ x \in \overline{\Omega} \text{ or}$$
$$v(\bar{t}_j, x) = v_{j-1}(\bar{t}_j^-, x), \text{ if } \bar{t}_j \neq t_k \text{ for every } k \in \{1, 2, ..., p_T\}, \ x \in \overline{\Omega}$$

where

$$v_{-1}(t, x) = \eta(t, x) \qquad \text{in } D_{-\tau}$$
$$v_0(t, x) = h_0(t, x) \exp{(a + 1)t} \qquad t \in [0, \bar{t}_1), \ x \in \overline{\Omega},$$

$h_0(t, x)$ is the solution of heat equation

$$Lh_0 = 0, \quad t \in [0, \bar{t}_1), x \in \Omega$$

with boundary condition $h_0 = 0$, $x \in \overline{\Omega}$ and initial condition $h_0(0, x) = \eta(0, x)$, $x \in \overline{\Omega}$

$$v_j(t, x) = h_j(t, x) \exp{\left[(a + 1)(t - \bar{t}_j)\right]}, \ j = 1, 2, 3, ..., p'_T,$$

$h_j(t,x)$ is the solution of heat equation

$$Lh_j = 0,\ t \in [\bar{t}_j, \bar{t}_{j+1}), x \in \Omega$$

with boundary condition $h_j = 0,\ x \in \overline{\Omega}$ and initial condition $h_j(\bar{t}_j, x) = I_k(v_{j-1}(\bar{t}_j^{\,-}, x))$ for every $x \in \overline{\Omega}$ if $t_j = t_k$ for some $k \in \{1, 2, ..., p_T\}$, or $h_j(\bar{t}_j, x) = v_{j-1}(\bar{t}_j^{\,-}, x),\ x \in \overline{\Omega}$ if $t_j \neq t_k$ for every $k \in \{1, 2, ..., p_T\}$.

As $\eta(0,x) \geq 0$ and $\eta(0,x) \not\equiv 0$ in $\overline{\Omega}$, it follows from Lemma 3.1 that $h_0 > 0,\ x \in \Omega$, and by the monotone non-decreasing property of I_k and the continuity of the solution, we also have $h_j > 0, x \in \Omega,\ j = 1, 2, ..., p_T$. Hence $v > 0$ in D_T.

By definition, the function v satisfies the following IBVP

$$\begin{aligned}
Lv = (a+1)v \geq av \qquad & \text{in } D_T \setminus (M_T \cup N_T) \\
v = \eta \qquad & \text{in } D_{-\tau} \\
v = 0 \qquad & \text{on } S_T \setminus N_T \\
v(t_k, x) = I_k(v(t_k^{\,-}, x)) \qquad & \text{in } \overline{\Omega}.
\end{aligned}$$

□

Corollary 3.1. *There exists a unique solution of IBVP (3.5)–(3.8) in* $\langle 0, v \rangle$.

Remark 3.1. *The existence and uniqueness of a global solution of IBVP (3.1)–(3.4) follows from the arbitrariness of* T.

2.2. Neumann and Robin Boundary Condition. In this subsection we discuss the existence and uniqueness of solution of problem (3.1)–(3.4) with the boundary condition (3.2) is replaced by

$$\frac{\partial u}{\partial \nu}(t,x) + \beta u(t,x) = 0 \text{ on } S \setminus N \tag{3.46}$$

where β is a non-negative constant.

Three assumptions are used to solve the problem: **A1**, **A2**, and

A3 The initial function, η, is continuously differentiable in $\overline{\Omega}$.

We use the same method to solve the problem as was used for Dirichlet boundary condition case. The theorems and their proofs for this initial boundary value problem are similar to the ones for the previous case. Here we just write the differences that have a significant role in these theorems and proofs.

First we construct upper and lower sequence whose terms are defined by inequalities (3.16)–(3.23), with (3.21) replaced by

$$\frac{\partial \underline{u}^{(m)}}{\partial \nu}(t,x) + \beta \underline{u}^{(m)}(t,x) \leq 0 \leq \frac{\partial \overline{u}^{(m)}}{\partial \nu}(t,x) + \beta \overline{u}^{(m)}(t,x) \text{ on } S_T \setminus N_T$$

To prove the well-definedness of the upper and lower sequences, we need the following theorem.

THEOREM 3.4 (Theorem 2.1.1 in [**37**]). *Let $a_{ij}(0,x)$ and $u_0(x)$ be continuously differentiable in a neighbourhood of $\partial\Omega$ and let $\alpha_0 = 1$, $f(t,x,u) = q(t,x) - c(t,x)u$ with q locally Holder continuous in x, uniformly in $[0,T] \times \overline{\Omega}$. Then for any continuous function h on $[0,T] \times \partial\Omega$ and u_0 in $\overline{\Omega}$, problem (3.24)–(3.26) has a unique solution u which is Holder continuous. Moreover u can be represented by the formula*

$$\begin{aligned} u(t,x) =& J^{(0)}(t,x) + \int_0^t d\tau \int_\Omega \Gamma(t,x;\tau,\xi) q(\tau,\xi) d\xi \\ &+ \int_0^t d\tau \int_{\partial\Omega} \frac{\partial \Gamma}{\partial \nu_\xi}(t,x;\tau,\xi) \psi(\tau,\xi) d\xi \end{aligned} \tag{3.47}$$

where Γ are as in Theorem 3.1 and

- $\psi(t,x) = 2\int_0^t d\tau \int_{\partial\Omega} \left[\frac{\partial\Gamma}{\partial\nu_\xi}(t,x;\tau,\xi) + \beta(t,x)\Gamma(t,x;\tau,\xi)\right] \psi(\tau,\xi) d\xi + 2H(t,x)$ *or*

$$\psi(t,x) = 2H(t,x) + 2\sum_{j=1}^{\infty} \int_0^t d\tau \int_{\partial\Omega} Q_j(t,x;\tau,\xi) H(\tau,\xi) d\xi, \tag{3.48}$$

- $J^{(0)}(t,x) = \int_\Omega \Gamma(t,x;0,\xi) u_0(\xi) d\xi,$
- $H(t,x) = J^{(2)}(t,x) + h(t,x) - \int_0^t d\tau \int_\Omega Q(t,x;\tau,\xi) q(\tau,\xi) d\xi,$
- $Q(t,x;\tau,\xi) = \frac{\partial\Gamma}{\partial\nu}(t,x;\tau,\xi) + \beta(t,x)\Gamma(t,x;\tau,\xi).$
- $Q_{j+1}(t,x;\tau,\xi) = \int_0^t ds \int_\Omega Q(t,x;s,y) Q_j(s,y;\tau,\xi) dy.$
- $J^{(2)}(t,x) = \int_\Omega Q(t,x;0,\xi) u_0(\xi) d\xi.$

The infinite series $\sum_{j=1}^{\infty} Q_j(t,x;\tau,\xi)$ converges uniformly and absolutely on S_T. From [**15**] Q, H, and ψ have the following estimates.

$$|Q(t,x;\kappa,\xi)| \le \frac{K_2}{(t-\kappa)^{\mu}} \frac{1}{|x-\xi|^{n+1-2\mu-\gamma}} \qquad , 1-\frac{\gamma}{2} < \mu < 1 \tag{3.49}$$

$$|H(t,x)| \le \frac{K_3}{t^{\mu}} \qquad , \frac{1}{2} < \mu < 1 \tag{3.50}$$

$$|\psi(t,x)| \le \frac{K_4}{t^{\mu}} \qquad , \frac{1}{2} < \mu < 1. \tag{3.51}$$

Using Lemma 3.1 and the same technique as in Dirichlet boundary condition case, the monotone property of the upper and lower sequence is obtained.

$$\hat{u} \le \underline{u}^{(m)} \le \underline{u}^{(m+1)} \le \overline{u}^{(m+1)} \le \overline{u}^{(m)} \le \tilde{u} \tag{3.52}$$

in $[-\tau, T] \times \overline{\Omega}$.

By this property we conclude that the upper and lower sequence are convergent and the limits satisfy the following relation.

$$\lim_{m\to\infty} \underline{u}^{(m)}(t,x) = \underline{u}(t,x) \le \overline{u}(t,x) = \lim_{m\to\infty} \overline{u}^{(m)}(t,x)$$

for $(t,x) \in [-\tau, T] \times \overline{\Omega}$.

Before we prove that $\overline{u} = \underline{u}$ and that is a solution of the problem, we need this lemma.

LEMMA 3.4. *The limit of upper and lower sequence, $\overline{u}$ and $\underline{u}$, satisfy the following equations.*

$$\underline{u} = \eta = \overline{u} \qquad \textit{in } D_{-\tau} \tag{3.53}$$

$$\mathcal{L}\overline{u} = \overline{u}(K + a - b\underline{u}_{-\tau}) \qquad \textit{in } D_T \setminus M_T \tag{3.54}$$

$$\mathcal{L}\underline{u} = \underline{u}(K + a - b\overline{u}_{-\tau}) \qquad \textit{in } D_T \setminus M_T \tag{3.55}$$

$$\frac{\partial \underline{u}}{\partial \nu}(t,x) + \beta\underline{u}(t,x) = 0 = \frac{\partial \overline{u}}{\partial \nu}(t,x) + \beta\overline{u}(t,x) \qquad \textit{on } S_T \setminus N_T \tag{3.56}$$

$$\overline{u}(t_k,x) = I_k(\overline{u}(t_k{}^-,x)) \qquad \textit{in } \overline{\Omega} \tag{3.57}$$

$$\underline{u}(t_k,x) = I_k(\underline{u}(t_k{}^-,x)) \qquad \textit{in } \overline{\Omega}. \tag{3.58}$$

Proof. The proof is similar to the proof of Lemma 3.2. From Theorem 3.4 we can express $\overline{u}^{(m)}$ and $\underline{u}^{(m)}$ as follows.

$$\overline{u}^{(m)}(t,x) = \int_0^t d\kappa \int_\Omega \Gamma(t,x;\kappa,\xi)\overline{u}^{(m-1)}(\kappa,\xi)(K+a-b\underline{u}^{(m-1)}(\kappa-\tau,\xi))d\xi$$
$$+ J^{(0)}(t,x) + \int_0^t d\kappa \int_{\partial\Omega} \frac{\partial\Gamma}{\partial\nu_\xi}(t,x;\kappa,\xi)\psi_1^{(m-1)}(\kappa,\xi)d\xi$$
$$,(t,x)\in(0,\overline{t}_1) \tag{3.59}$$
$$\underline{u}^{(m)}(t,x) = \int_0^t d\tau \int_\Omega \Gamma(t,x;\kappa,\xi)\underline{u}^{(m-1)}(\kappa,\xi)(K+a-b\overline{u}^{(m-1)}(\kappa-\tau,\xi))d\xi$$
$$+ J^{(0)}(t,x) + \int_0^t d\tau \int_{\partial\Omega} \frac{\partial\Gamma}{\partial\nu_\xi}(t,x;\kappa,\xi)\psi_2^{(m-1)}(\kappa,\xi)d\xi$$
$$,(t,x)\in(0,\overline{t}_1) \tag{3.60}$$

where

$$\psi_i^{(m)}(t,x) = 2\int_0^t d\kappa \int_{\partial\Omega} \left[\frac{\partial\Gamma}{\partial\nu_\xi}(t,x;\kappa,\xi) + \beta(t,x)\Gamma(t,x;\kappa,\xi)\right]\psi_i^{(m)}(\kappa,\xi)d\xi$$
$$+ 2H_i^{(m)}(t,x) \qquad ,(t,x)\in(0,\overline{t}_1),\ i=1,2$$

and

$$H_1^{(m)}(t,x) = -\int_0^t d\kappa \int_\Omega Q(t,x;\kappa,\xi)\overline{u}^{(m)}(\kappa,\xi)(K+a-b\underline{u}^{(m)}(\kappa-\tau,\xi))d\xi$$
$$+ J^{(2)}(t,x) \qquad ,(t,x)\in(0,\overline{t}_1), \tag{3.61}$$
$$H_2^{(m)}(t,x) = -\int_0^t d\kappa \int_\Omega Q(t,x;\kappa,\xi)\underline{u}^{(m)}(\kappa,\xi)(K+a-b\overline{u}^{(m)}(\kappa-\tau,\xi))d\xi$$
$$+ J^{(2)}(t,x) \qquad ,(t,x)\in(0,\overline{t}_1). \tag{3.62}$$

Here $u_0(x)$ in $J^{(0)}$ and $J^{(2)}$ is replaced by $\eta(0,x)$.

Taking limits of $H_1^{(m)}$ and $H_2^{(m)}$ with respect to m and applying dominated convergence theorem we have

$$H_1(t,x) = \lim_{m\to\infty} H_1^{(m)}(t,x)$$

$$
\begin{aligned}
H_1(t,x) &= -\lim_{m\to\infty}\int_0^t d\kappa \int_\Omega Q(t,x;\kappa,\xi)\overline{u}^{(m)}(\kappa,\xi)(K+a-b\underline{u}^{(m)}(\kappa-\tau,\xi)) \\
&\quad + \lim_{m\to\infty} J^{(2)}(t,x) \\
&= J^{(2)}(t,x) - \int_0^t d\kappa \int_\Omega Q(t,x;\kappa,\xi)\overline{u}(\kappa,\xi)(K+a-b\underline{u}(\kappa-\tau,\xi))d\xi, \\
&\qquad , (t,x) \in (0,\bar{t}_1), \qquad (3.63)
\end{aligned}
$$

$$
\begin{aligned}
H_2(t,x) &= \lim_{m\to\infty} H_2^{(m)}(t,x) \\
&= -\lim_{m\to\infty}\int_0^t d\kappa \int_\Omega Q(t,x;\kappa,\xi)\underline{u}^{(m)}(\kappa,\xi)(K+a-b\overline{u}^{(m)}(\kappa-\tau,\xi))d\xi \\
&\quad + \lim_{m\to\infty} J^{(2)}(t,x) \\
&= J^{(2)}(t,x) - \int_0^t d\kappa \int_\Omega Q(t,x;\kappa,\xi)\underline{u}(\kappa,\xi)(K+a-b\overline{u}(\kappa-\tau,\xi))d\xi \\
&\qquad , (t,x) \in (0,\bar{t}_1). \qquad (3.64)
\end{aligned}
$$

The boundedness of $\overline{u}(K+a-b\underline{u}_{-\tau})$ and $\underline{u}(K+a-b\overline{u}_\tau)$ and the continuity of $\eta(0,x)$ ensure that H_1 and H_2 are continuous on $[0,t_1)\times\partial\Omega$. So we can obtain the limits of $\psi_1^{(m)}$ and $\psi_2^{(m)}$ as follows.

$$
\begin{aligned}
\psi_i(t,x) &= \lim_{m\to\infty} \psi_i^{(m)}(t,x) \\
&= 2\int_0^t d\kappa \int_{\partial\Omega}\left[\frac{\partial\Gamma}{\partial\nu_\xi}(t,x;\kappa,\xi) + \beta(t,x)\Gamma(t,x;\kappa,\xi)\right]\psi_i(\kappa,\xi)d\xi \\
&\quad + 2H_i(t,x) \qquad , (t,x)\in(0,\bar{t}_1),\ i=1,2.
\end{aligned}
$$

From these results and by applying the dominated convergence theorem, we get the following equations.

$$
\begin{aligned}
\overline{u}(t,x) &= \lim_{m\to\infty} \overline{u}^{(m)}(t,x) \\
&= \lim_{m\to\infty}\int_0^t d\kappa \int_\Omega \Gamma(t,x;\kappa,\xi)\overline{u}^{(m-1)}(\kappa,\xi)(K+a-b\underline{u}^{(m-1)}(\kappa-\tau,\xi))d\xi \\
&\quad + \lim_{m\to\infty}\left(J^{(0)}(t,x) + \int_0^t d\kappa \int_{\partial\Omega}\frac{\partial\Gamma}{\partial\nu_\xi}(t,x;\kappa,\xi)\psi_1^{(m-1)}(\kappa,\xi)d\xi\right)
\end{aligned}
$$

$$\overline{u}(t,x) = J^{(0)}(t,x) + \int_0^t d\kappa \int_\Omega \Gamma(t,x;\kappa,\xi)\overline{u}(\kappa,\xi)(K+a-b\underline{u}(\kappa-\tau,\xi))d\xi$$
$$+ \int_0^t d\kappa \int_{\partial\Omega} \frac{\partial\Gamma}{\partial\nu_\xi}(t,x;\kappa,\xi)\psi_1(\kappa,\xi)d\xi \quad ,(t,x)\in(0,\overline{t}_1) \tag{3.65}$$

$$\begin{aligned}\underline{u}(t,x) &= \lim_{m\to\infty}\underline{u}^{(m)}(t,x)\\ &= \lim_{m\to\infty}\int_0^t d\tau \int_\Omega \Gamma(t,x;\kappa,\xi)\underline{u}^{(m-1)}(\kappa,\xi)(K+a-b\overline{u}^{(m-1)}(\kappa-\tau,\xi))d\xi\\ &\quad + \lim_{m\to\infty}\left(J^{(0)}(t,x) + \int_0^t d\kappa \int_{\partial\Omega}\frac{\partial\Gamma}{\partial\nu_\xi}(t,x;\kappa,\xi)\psi_2^{(m)}(\kappa,\xi)d\xi\right)\\ &= J^{(0)}(t,x) + \int_0^t d\tau \int_\Omega \Gamma(t,x;\kappa,\xi)\underline{u}(\kappa,\xi)(K+a-b\overline{u}(\kappa-\tau,\xi))d\xi\\ &\quad + \int_0^t d\kappa \int_{\partial\Omega}\frac{\partial\Gamma}{\partial\nu_\xi}(t,x;\kappa,\xi)\psi_2(\kappa,\xi)d\xi \quad ,(t,x)\in(0,\overline{t}_1).\end{aligned} \tag{3.66}$$

By applying these integral representations to the solution on every time interval with some adjustments on initial conditions, the proof follows in the same pattern as in the proof of Lemma 3.1. □

THEOREM 3.5. *The upper and lower sequence,* $\{\overline{u}^{(m)}\}$ *and* $\{\overline{u}^{(m)}\}$*, converge monotonically to a unique solution* u *of IBVP (3.5)–(3.8), with equation (3.6) replaced by*

$$\frac{\partial u}{\partial \nu}(t,x) + \beta u(t,x) = 0 \ \textit{on}\ S_T \setminus N_T. \tag{3.67}$$

PROOF. Sequentially, we will show that $\overline{u} = \underline{u}$ in $[-\tau,\overline{t}_1)\times\overline{\Omega}$, $[\overline{t}_1,\overline{t}_2)\times\overline{\Omega}$,...,$[\overline{t}_{p'_T-1},T]\times\overline{\Omega}$. From (3.65) and (3.66) we have the following relation.

$$\begin{aligned}(\overline{u}(t,x)-\underline{u}(t,x)) \le \int_0^t d\kappa \int_\Omega &\Gamma(t,x;\kappa,\xi)((K+a)(\overline{u}(\kappa,\xi)-\underline{u}(\kappa,\xi))\\ &- b(\overline{u}(\kappa,\xi)\underline{u}(\kappa-\tau,\xi)-\underline{u}(\kappa,\xi)\overline{u}(\kappa-\tau,\xi))) \quad ,t\in(0,\overline{t}_1)d\xi\\ &+ \int_0^t d\kappa \int_{\partial\Omega}\frac{\partial\Gamma}{\partial\nu_\xi}(t,x,\kappa,\xi)(\psi_1(\kappa,\xi)-\psi_2(\kappa,\xi))d\xi.\end{aligned}$$

From equations (3.63), (3.64), (3.48), (3.50) and (3.51) we obtain

$$
\begin{aligned}
|\psi_1(t,x) - \psi_2(t,x)| \leq 2|H_1(t,x) - H_2(t,x)| \\
&+ 2\left| \int_0^t d\kappa \int_\Omega \sum_{j=1}^{\infty} Q_j(t,x;\kappa,\xi)(H_1(\kappa,\xi) - H_2(\kappa,\xi))d\xi \right| \\
&\leq \frac{M_1}{t^\mu}\|\overline{u} - \underline{u}\|_t, \quad \frac{1}{2} < \mu < 1 \qquad (3.68)
\end{aligned}
$$

for some positive constant M_1.

It follows from estimates (3.42), (3.43), and (3.68) that

$$
|\overline{u}(t,x) - \underline{u}(t,x)| \leq M_2 t^{1-\mu}\|\overline{u} - \underline{u}\|_t, \quad \frac{1}{2} < \mu < 1 \qquad (3.69)
$$

for some positive constant M_2.

If $t < M_2^{\frac{-1}{1-\mu}}$ then inequality (3.69) implies $\|\overline{u} - \underline{u}\|_t = 0$. This means that $\overline{u}(t,x) = \underline{u}(t,x)$ for $t \in [0, \min\{\overline{t}_1, M^{\frac{-1}{1-\mu}}\})$.

Let $\{0 = \overline{t}_1^{j_1}, \overline{t}_1^{j_2}, ..., \overline{t}_1^{j_{n1}} = \overline{t}_1\}$ be a partition on $[0, t_1]$ with $\overline{t}_1^{j_k} - \overline{t}_1^{j_l} < M^{\frac{-1}{1-\mu}}$, $k, l \in \{1, 2, ..., n1\}$. With suitable initial conditions (note that the initial conditions for $\overline{u}$ and $\underline{u}$ in an interval are the same), we obtain the same result on time interval $[\overline{t}_1^{j_i}, \overline{t}_1^{j_{i+1}})$, $i = 1, ..., n1-1$ by repeating the steps above. Thus $\overline{u}(t,x) = \underline{u}(t,x)$ in $[-\tau, t_1) \times \overline{\Omega}$.

By continuing this process on time interval $[\overline{t}_i, \overline{t}_{i+1})$ for $i = 1, ..., p'_T - 1$, we obtain that $\overline{u}(t,x) = \underline{u}(t,x)$ in $[-\tau, T] \times \overline{\Omega}$ and it is the solution of the Robin boundary problem.

The uniqueness of the solution can be proved using the same method. □

LEMMA 3.5. *There exists a pair of ordered, lower and upper solutions of IBVP (3.5), (3.67), (3.7), and (3.8).*

The proof is similar to the proof of Lemma 3.3. In this case the lower solution is the zero function and the upper solution be given by v with

$$v(t,x) = 0 \qquad \text{on } S_T \setminus N_T$$
$$v(t,x) = v_j(t,x) \text{ in } [\bar{t}_j, \bar{t}_{j+1}) \times \Omega$$
$$v(\bar{t}_j, x) = I_k(v_{j-1}(\bar{t}_j^-, x)), \quad \text{if } \bar{t}_j = t_k \text{ for some } k \in \{1,2,...,p_T\},\ x \in \overline{\Omega} \text{ or}$$
$$v(\bar{t}_j, x) = v_{j-1}(\bar{t}_j^-, x), \quad \text{if } \bar{t}_j \neq t_k \text{ for every } k \in \{1,2,...,p_T\},\ x \in \overline{\Omega}$$

where

$$v_{-1}(t,x) = \eta(t,x) \qquad \text{in } D_{-\tau}$$
$$v_0(t,x) = h_0(t,x) \exp{(a+1)t} \qquad t \in [0, \bar{t}_1),\ x \in \overline{\Omega},$$

$h_0(t,x)$ is the solution of heat equation

$$Lh_0 = 0, \quad t \in [0, \bar{t}_1), x \in \Omega$$

with boundary condition $\frac{\partial h_0}{\partial \nu} + \beta h_0 = 0,\ x \in \overline{\Omega}$ and initial condition $h_0(0,x) = \eta(0,x),\ x \in \overline{\Omega}$

$$v_j(t,x) = h_j(t,x) \exp\left[(a+1)(t - \bar{t}_j)\right],\ j = 1,2,3,...,p'_T,$$

$h_j(t,x)$ is the solution of heat equation

$$Lh_j = 0,\ t \in [\bar{t}_j, \bar{t}_{j+1}), x \in \Omega$$

with boundary condition $\frac{\partial h_j}{\partial \nu} + \beta h_j = 0,\ x \in \overline{\Omega}$ and initial condition $h_j(\bar{t}_j, x) = I_k(v_{j-1}(\bar{t}_j{}^-, x))$ for every $x \in \overline{\Omega}$ if $t_j = t_k$ for some $k \in \{1,2,...,p_T\}$, or $h_j(\bar{t}_j, x) = v_{j-1}(\bar{t}_j{}^-, x),\ x \in \overline{\Omega}$ if $t_j \neq t_k$ for every $k \in \{1,2,...,p_T\}$.

COROLLARY 3.2. *There exists a unique solution of IBVP (3.5), (3.67), (3.7), and (3.8) in* $\langle 0, v \rangle$.

REMARK 3.2. *The existence and uniqueness of a global solution of IBVP (3.1), (3.46), (3.3), and(3.4) follows from the arbitrariness of* T.

3. Attractors

In this section we will discuss two attractors of the following IBVP

$$u_t(t,x) - \Delta u(t,x) = u(t,x)(a - bu(t-\tau,x)) \quad \text{in } D \setminus M \quad (3.70)$$

$$\alpha_2 \frac{\partial u}{\partial \nu}(t,x) + \beta_2 u(t,x) = 0 \quad \text{on } S \setminus N \quad (3.71)$$

$$u(t,x) = \eta(t,x) \quad \text{in } D_{-\tau} \quad (3.72)$$

$$u(t_k,x) = I_k(u(t_k{}^-,x)) \quad \text{in } \overline{\Omega},\ k = 1,2,... \quad (3.73)$$

where $\alpha_2 = 0$ or 1 and β_2 is a non-negative constant. The assumptions of the coefficients and functions in (3.70)–(3.73) are the same as the ones in the previous section. The attractors considered here are the zero attractor and the positive attractor which are the steady-states of IBVP (3.70)–(3.72) without impulses. Some conditions are required to ensure that these steady-states are attractors for the problem with impulses.

The steady-states of IBVP (3.70)–(3.72) without impulses satisfy the boundary value problem :

$$-\Delta u(x) = u(x)(a - bu(x)) \quad \text{in } \Omega \quad (3.74)$$

$$\alpha_2 \frac{\partial u}{\partial \nu}(x) + \beta_2 u(x) = 0 \quad \text{on } \partial\Omega \quad (3.75)$$

The existence and stability of the steady-states of problem (3.70)–(3.73) can be obtained by applying the following theorems and lemma.

Let $\mathcal{D}(x,D) = \sum_{i,j=1}^n a_{ij}(x) D_i D_j + \sum_{i=1}^n a_i(x) D_i + a_0(x)$ and $\tilde{B}(x,D) = \alpha_0 \frac{\partial w}{\partial \nu} + \beta_0(x) w$ where a_{ij}, a_i, $a_0 \in C^{\mu}(\overline{\Omega})$, $i = 1,2,...,n$, $\alpha_0 = 0$ or 1, $\beta_0 \in C^{1+\mu}(\partial\Omega)$ for some $\mu \in (0,1)$. Let $X = \{w \in C^{2+\mu}(\overline{\Omega}) | \tilde{B}(x,D)w = 0\}$. Consider the parabolic problem given by

$$\frac{\partial w}{\partial t} - \mathcal{D}(x,D)w = m(x)w - c(x)wh(x,w) \quad \text{in } \Omega \quad (3.76)$$

$$\tilde{B}(x,D)w = 0 \quad \text{on } \partial\Omega. \quad (3.77)$$

$$w(x,0) = \psi_0 \quad \text{on } \overline{\Omega} \quad (3.78)$$

where ψ_0 is a non-negative function in X^+ and

(B1) $c, m \in C^{\mu}(\overline{\Omega})$, $c \geq 0 (\neq 0)$, (B2) $h \in C^{\mu,1+\mu}(\overline{\Omega} \times [0,\infty), \mathbf{R})$, $h(x,0) = 0$, $h(x,w)$ and $\frac{\partial h}{\partial w}(x,w)$ is positive for all positive w, $\lim_{\xi\to\infty} h(x,\xi) = \infty$ for each $x \in \Omega$.

LEMMA 3.6 (Lemma 2.2 in [**13**]). *The following statements are equivalent.*

(a) *The principal eigenvalue of $\mathcal{D} - m$ with boundary condition* (3.77) *is positive.*

(b) *There exists a function $\overline{w} \in C^{2+\mu}(\overline{\Omega})$ such that*

$$\mathcal{D}(x,D)\overline{w} + m(x)\overline{w} \geq 0 \qquad \text{in } \Omega$$

$$\tilde{B}(x,D)\overline{w} \geq 0 \qquad \text{on } \partial\Omega.$$

with at least one of these inequalities strict. In other words, $\overline{w}$ is a positive strict super solution of

$$\mathcal{D}(x,D)w + m(x)w = 0 \qquad \text{in } \Omega \tag{3.79}$$

$$\tilde{B}(x,D)w = 0 \qquad \text{on } \partial\Omega. \tag{3.80}$$

LEMMA 3.7 (Remark 2.1 in [**13**]). *Let $\sigma^{\Omega}(\mathcal{D}+m,\tilde{B})$ be the principal eigenvalue of eigenvalue problem (3.79)–(3.80).*

(a) $\sigma^{\Omega}(\mathcal{D}+m_1,\tilde{B}) < \sigma^{\Omega}(\mathcal{D}+m_2,\tilde{B})$ *whenever $m_1 < m_2$.*

(b) *The mapping $\lambda \to \sigma^{\Omega}(\mathcal{D}+\lambda m,\tilde{B}) : \mathbf{R} \to \mathbf{R}$ is concave and analytic.*

(c) $\sigma^{\Omega}(\mathcal{D}+m,\tilde{B}) < \sigma^{\Omega}(\mathcal{D}+m_2,\tilde{D})$ *where $\tilde{D}$ is homogeneous Dirichlet boundary operator.*

LEMMA 3.8 (Lemma 3.2 in [**13**]). *If the principal eigenvalue of $\mathcal{D}-m$ with boundary condition* (3.77) *is non-negative then problem (3.76)–(3.77) does not admit a positive steady-state.*

THEOREM 3.6 (Theorem 3.7 in [**13**]). *Suppose (B1) and (B2) hold. Then the following assertions are true.*

(a) *If the principal eigenvalue of $\mathcal{D}-m$ with boundary condition* (3.77) *is non-negative then the zero solution of problem (3.76)–(3.78) is globally asymptotically stable.*

(b) *If the principal eigenvalue of $\mathcal{D}-m$ with boundary condition* (3.77) *is negative and there exists a positive steady-state, w_0, of problem (3.76)–(3.78), then w_0 is globally asymptotically stable.*

(c) *If the principal eigenvalue of* $\mathcal{D}-m$ *with boundary condition* (3.77) *is negative and problem (3.76)–(3.78) does not admit a positive steady-state then* $\lim_{t\to\infty}\|w(\cdot,t,w_0\|_{C(\overline{\Omega})}=\infty$ *for each* $\psi_0\in X^+$.

The existence and uniqueness of the solution of problem (3.76)–(3.78) is given by the following theorem.

Theorem 3.7 (Theorem 3.5 and Lemma 3.1 in [**13**]). *Problem (3.76)–(3.78) admits at most one positive solution. Moreover, suppose conditions (B1), (B2) and*

(B3) *Let* Ω_0 *be a possibly void* $C^{2+\mu}$*-sub domain of* Ω *such that* $\overline{\Omega}_0=\{x\in\overline{\Omega}|c(x)=0\}$. *For Robin boundary case,* $\overline{\Omega}_0\subset\Omega$. *In case of Dirichlet boundary conditions, it is only required that* $\Omega_0\subset\Omega$. *Hence* c *is allowed to vanish on* $\partial\Omega$,

hold. Then problem (3.76)–(3.78) admits a positive solution if and only if

$$\sigma_1<0<\zeta_1$$

where σ_1 *is the principal eigenvalue of* $\mathcal{D}-m$ *with homogeneous Robin boundary condition,* ζ_1 *is the principal eigenvalue of* $\mathcal{D}-m$ *with homogeneous Dirichlet boundary condition, and* $\zeta_1=\infty$ *if* $\Omega_0=\emptyset$.

Let λ_1 be the principal eigenvalue of $-\Delta$ with homogeneous Dirichlet or Robin boundary condition. If $a\le\lambda_1$, then the zero function is the only non-negative steady-state of IBVP (3.70)–(3.72) without impulses according to Lemma 3.8. This steady-state is stable at any time delay. It follows from Theorem 3.7 that if $a>\lambda_1$, then there exists a unique positive steady-state. By applying Theorem 3.6 we obtain that the positive steady-state of problem (3.70)–(3.72) without impulses and delay is stable while the zero steady-state is unstable.

3.1. The Zero Attractor. We will present some conditions under which the zero function becomes an attractor in the cases $a=\lambda_1$, $a<\lambda_1$ and $a>\lambda_1$. For proving these results, the technique in [**7**] will be used.

If $a = \lambda_1$ and I_k is a contraction mapping, then the zero function is an attractor for IBVP (3.70)–(3.73). In fact these conditions will make the convergence of the solution faster than in the case without impulses.

In the following theorem, we present some sufficient conditions so that the zero function is an attractor in case $a < \lambda_1$.

THEOREM 3.8. *Suppose $a < \lambda_1$ where λ_1 is the principal eigenvalue of $-\Delta$ with homogeneous Dirichlet or Robin boundary conditions. If there exists positive numbers ϑ_k which satisfy :*

$$0 \leq I_k(u) \leq \vartheta_k u \ \ \textit{for every positive integer } k$$

and either $\vartheta < 1$ or

$$\vartheta_k \leq \exp(-2(a - \lambda_1)(t_k - t_{k-1}))$$

for every k greater than some non-negative integer N_1, then any non-negative solution, $u_1(t,x)$, of IBVP (3.70)-(3.73) satisfies $\|u_1(t,\cdot)\|_{L^2(\Omega)} \to 0$ as $t \to \infty$ for any value of $\tau \geq 0$.

PROOF. If we add impulses to the system, then the behaviour of the solution of IBVP (3.70)–(3.73) depends on the size of jumps and the frequency of the impulses.

Let $u_1(t,x)$ denote the solution of IBVP (3.70)–(3.73), and $u(t,x)$ denote the solution of IBVP (3.70)–(3.72) without impulses.

If $\vartheta_k \leq 1$ then it follows from Lemma 3.1 that $u_1(t,x) \leq u(t,x), t \in (t_k, t_{k+1})$, $k \geq N1$. From the convergence of $u(t,x)$, we obtain

$$\lim_{t\to\infty} \|u_1(t,\cdot)\|_{L^2(\Omega)} = 0.$$

Consider IBVP (3.70)-(3.72) without impulses. Taking the standard (L^2) inner product of equation (3.1) with $u(t,x)$, using integration by parts, and recalling the boundary condition, we obtain the following results.

$$\int_\Omega \frac{\partial u}{\partial t}(t,x)u(t,x)dx - \int_\Omega u(t,x)\Delta u(t,x)dx = \int_\Omega u^2(t,x)(a - bu(t-\tau,x))dx$$
$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u^2(t,x)dx - u(t,x)\nabla u(t,x)|_{\partial\Omega} + \int_\Omega |\nabla u(t,x)|^2 dx$$
$$= \int_\Omega u^2(t,x)(a - bu(t-\tau,x))dx.$$

Since $u(t,x)\nabla u(t,x)|_{\partial\Omega} = 0$ for Dirichlet and Neumann boundary conditions, and $-u(t,x)\nabla u(t,x)|_{\partial\Omega} = \beta u^2(t,x) \geq 0$ for Robin boundary condition,

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u^2(t,x)dx + \int_\Omega |\nabla u(t,x)|^2 dx \leq \int_\Omega a u^2(t,x)dx.$$

By Poincare's inequality and eigenvalue property (c) in Lemma 3.7, for any sufficiently smooth function u,

$$\lambda_1 \int_\Omega u^2 dx \leq \int_\Omega |\nabla u|^2 dx.$$

Using this inequality

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u^2(t,x)dx \leq \int_\Omega a u^2(t,x)dx - \lambda_1 \int_\Omega u^2(t,x)dx$$
$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u^2(t,x)dx \leq (a-\lambda_1)\int_\Omega u^2(t,x)dx$$
$$\frac{1}{2}\frac{\partial}{\partial t}\|u(t,\cdot)\|_{L^2(\Omega)} \leq (a-\lambda_1)\|u(t,\cdot)\|_{L^2(\Omega)}$$
$$\|u(t,\cdot)\|_{L^2(\Omega)} \leq \|u(0,\cdot)\|_{L^2(\Omega)} \exp\{2(a-\lambda_1)t\}. \tag{3.81}$$

Consider IBVP (3.70)–(3.73) in the time interval $[t_{k-1}, t_k)$. There is no impulse in this interval. Hence

$$\|u_1(t_k^-,\cdot)\|_{L^2(\Omega)} \leq \|u_1(t_{k-1},\cdot)\|_{L^2(\Omega)} \exp\{2(a-\lambda_1)(t_k - t_{k-1})\}$$

$$\|u_1(t_k,x)\|_{L^2(\Omega)} = \|I_k(u_1(t_k^-,x))\|_{L^2(\Omega)} \leq \vartheta_k \|u(t_k^-,x)\|_{L^2(\Omega)}.$$

Let $\gamma_k = \vartheta_k \exp\{2(a-\lambda_1)(t_k - t_{k-1}\}$. By combining this result with the monotone non-decreasing (with respect to u) property of I_k we have

$$\begin{aligned}
\|u_1(t_k,x)\|_{L^2(\Omega)} &< \vartheta_k \|u_1(t_{k-1},x)\|_{L^2(\Omega)} \exp\{2(a-\lambda_1)(t_k - t_{k-1})\} \\
&= \gamma_k \|u_1(t_{k-1},x)\|_{L^2(\Omega)} \\
&< \gamma_k \gamma_{k-1} \|u_1(t_{k-2},x)\|_{L^2(\Omega)} \\
&\vdots \\
&< \left(\prod_{j=1}^{k} \gamma_j\right) \|u_1(0,x)\|_{L^2(\Omega)} \text{ for every } k.
\end{aligned}$$

It follows from

$$\lim_{k\to\infty} \prod_{j=1}^{k} \gamma_j = 0 \tag{3.82}$$

that $\lim_{k\to\infty} \|u_1(t_k,x)\| = 0$. Hence $\lim_{t\to\infty} \|u_1(t,x)\| = 0$.

□

When $a > \lambda_1$, the zero solution of the logistic equation $u_t = u(a - bu)$ is not stable and remains unstable if diffusion and/or delay are introduced. Clearly the zero solution is an attractor if, at any time t_k , there is an impulse with $I_k(u) = 0$. The question then remains if the zero function can be an attractor when none of the impulses are identically zero. The following theorem demonstrates that the answer is affirmative and that, for a particular form of impulses, both the timing and the size of the impulses determine whether the zero function is an attractor.

THEOREM 3.9. *Suppose $I_k(u) \le \vartheta_k u$ for some constant $\vartheta_k > 0$ and $a > \lambda_1$ where λ_1 is the principal eigenvalue of $-\Delta$ with homogeneous Dirichlet or Robin boundary conditions. Let γ_k be as defined below*

$$\gamma_k = \vartheta_k \exp\{2(a-\lambda_1)(t_k - t_{k-1})\}.$$

If there exists a non-negative integer N_1 such that

$$\gamma_k < 1 \textit{ for all } k > N_1 \tag{3.83}$$

then any non-negative solution $u_1(t,x)$ *of IBVP (3.1)–(3.4) satisfies*

$$\lim_{t\to\infty} \|u_1(t,\cdot)\|_{L^2(\Omega)} = 0$$

for any delay τ*.*

The proof is the same as the one for Theorem 3.8.

REMARK 3.3. *According to the condition on* I_k *in (3.83) and its implication (3.82), the zero function is an attractor even if the spacing between impulses grows without bound, as long as the impulses bring the solution back down close enough to zero. On the other hand, as long as* $w_k < 1$ *(except for a finite number of* k*) the zero function is an attractor if the impulses are sufficiently frequent.*

3.2. The Positive Attractor. For this section we assume $a > \lambda_1$ where λ_1 is principal eigenvalue of $-\Delta$ with homogeneous Robin boundary condition. This assumption implies the existence and uniqueness of positive steady of IBVP (3.70)–(3.72) without impulses.

For the logistic equation $u_t = u(a - bu)$ both with and without diffusion, the positive steady-state is stable [**13**]. If delay is introduced, the stability can change (Wright's conjecture in [**56**]). The following lemma gives conditions on delay such that the stability of steady-state preserved.

LEMMA 3.9 (Corollary 2.7 in [**14**]). *Let* w_0 *be a stable hyperbolic stationary solution of IBVP given by*

$$\begin{aligned}
\frac{\partial w}{\partial t}(t,x) &= \Delta w(t,x) + f(x, w(t,x), w(t,x)) && ,x \in \Omega \\
w(t,x) &= 0 && ,x \in \partial\Omega \\
w(t,x) &= \psi(0,x) && ,x \in \Omega,
\end{aligned}$$

where f *is a function that is at least continuous respect to the first variable, continuously differentiable respect to the second and third variable. Let* $d(x) =$

$D_3 f(x, w, w)$. *Then* w_0 *is also a stable hyperbolic stationary solution of IBVP*

$$\begin{aligned} &\frac{\partial w}{\partial t}(t,x) = \Delta w(t,x) + f(x, w(t,x), w(t-\tau,x)) && , x \in \Omega \\ &w(t,x) = 0 && , x \in \partial\Omega \\ &w(t,x) = \psi(t,x) && , (t,x) \in [-\tau, 0] \times \Omega \end{aligned}$$

if $\tau \in [0, \tau^*)$ *where* $\tau^* = \left(\max_{x \in \overline{\Omega}} |d(x)|\right)^{-1}$.

The proof of this lemma is based on the fact that zero is an eigenvalue of the linearized problem with delay if and and only if it is an eigenvalue of the linearized problem without delay around a stationary solution. To make the stability unchanged, all non-real eigenvalues of the linearized problem with delay must have negative real parts. The condition on time delay is obtained by simplifying the following equation

$$\int_\Omega (\overline{u}\Delta u - u\Delta\overline{u})dx + \left(e^{-\lambda\tau} - e^{-\overline{\lambda}\tau}\right) \int_\Omega d(x)|u|^2 dx = (\lambda - \overline{\lambda}) \int_\Omega |u|^2 dx \quad (3.84)$$

to get

$$e^{\alpha\tau} = \tau|\mathrm{sinc}(\beta\tau)| \left| \int_\Omega d(x)|u|^2 dx \right|$$

where $\lambda = \alpha + i\beta$ is a non-real eigenvalue of the linearized problem with delay, $\overline{\lambda}$ is the conjugate of λ, u is the corresponding eigenfunction, $\overline{u}$ is the conjugate of u, and

$$\mathrm{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & , x \neq 0 \\ 1 & , x = 0. \end{cases}$$

Since α has to be negative and $|\mathrm{sinc}(x)| \leq 1$, $\tau \left| \int_\Omega d(x)|u|^2 dx \right| < 1$.

The proof of Lemma 3.9 requires that

$$\int_\Omega (\overline{u}\Delta u - u\Delta\overline{u})dx = \left(\overline{u}\frac{\partial u}{\partial \nu} - u\frac{\partial \overline{u}}{\partial \nu}\right)\Big|_{\partial\Omega} - \int_\Omega (\nabla\overline{u}\cdot\nabla u - \nabla u \cdot \nabla\overline{u})\, dx = 0.$$

This holds for the problem with homogeneous Dirichlet boundary condition but also for the one with homogeneous Robin boundary condition because

$$\left(\overline{u}\frac{\partial u}{\partial \nu} - u\frac{\partial \overline{u}}{\partial \nu}\right)\Big|_{\partial\Omega} = (\overline{u}(-\beta u) - u(-\beta\overline{u}))\Big|_{\partial\Omega} = 0.$$

Since the argument regarding the eigenvalues for Dirichlet case also holds for Robin case, Lemma 3.9 is true for the problem with homogeneous Robin boundary conditions.

From this lemma we can obtain a result on IBVP (3.70)–(3.72) without impulses as follows.

LEMMA 3.10. *Suppose* $a > \lambda_1$ *where* λ_1 *is the principal eigenvalue of* $-\Delta$ *with homogeneous Robin boundary condition. Let* ϕ *denote the positive steady-state of IBVP (3.70)–(3.72) without impulses and let* $B_M = \max_{x\in\Omega} b\phi(x)$. *If* $\tau B_M < 1$, *then* ϕ *is globally asymptotically stable.*

PROOF. It follows from Theorem 3.6 that ϕ is globally asymptotically stable for IBVP (3.70)–(3.72) without delays and impulses. By applying Lemma 3.9, we conclude that the stability of ϕ does not change for IBVP (3.70)–(3.73). □

In most cases, when impulses are introduced to the diffusive delay logistic problem, the steady-state of the problem without impulses is no longer an attractor to the problem with impulses. Even if the size of jumps tends to zero or the distance between impulse times is large, the steady-state of IBVP (3.70)–(3.72) without impulses may not be an attractor. The following problem is an example of this case.

$$u_t(t,x) - \Delta u(t,x) = u(t,x)\left(1 - u(t-\tau,x)\right) \qquad \text{in } D \setminus M \tag{3.85}$$

$$\frac{\partial u}{\partial \nu}(t,x) = 0 \qquad \text{on } S \setminus N \tag{3.86}$$

$$u(t,x) = \eta(t,x) \qquad \text{in } D_{-\frac{1}{2}} \tag{3.87}$$

$$u(t_k,x) = \beta_k u(t_k^-,x) \qquad k = 1,2,... \tag{3.88}$$

where the impulse times are $t_k = \sum_{i=1}^{k} \frac{1}{i}$, $0 \le \tau < 1$, Ω, D, S, M, N and η are as IBVP (3.1)–(3.4), and $\beta_1 = \frac{1}{e}$, $\beta_k = \exp\left(-\frac{1}{k-1}\right)$, $k = 2,3....$ The constant function $\phi(x) = 1$ is a stable steady-state for IBVP (3.85)–(3.87) without impulses, but it is not an attractor for IBVP (3.85)–(3.88). We show this result by applying the comparison theorem in [**2**] to obtain this relation

$$u(t,x) \le w(t) \qquad (t,x) \in [0,\infty) \times \overline{\Omega}$$

where u is any solution of problem (3.85)–(3.88) and w is a solution of impulsive differential equation

$$\dot{w}(t) = w(t), \quad t \in (0,\infty) \setminus \{t_k\}_{k=1}^{n} \tag{3.89}$$

$$w(0) = w_0 = \max_{x \in \overline{\Omega}} u(0,x) \tag{3.90}$$

$$w(t_k) = \beta_k w(t_k^-). \tag{3.91}$$

Let $w_0 < 1$. The solution of problem (3.89)–(3.91) is given by

$$w(t) = \begin{cases} w_0 e^t & , t \in [0, t_1) \\ w_0 \exp\left(\frac{-k+1}{k}\right) \exp(t - t_k) & , t \in [t_k, t_{k+1}),\ k = 1,2,3,... \end{cases}$$

$$\begin{aligned} \lim_{t\to\infty} w(t) &= \lim_{t\to\infty} w_0 \exp\left(\frac{-k+1}{k}\right) \exp(t - t_k) \\ &\le \lim_{k\to\infty} w_0 \exp\left(\frac{-k+1}{k}\right) \exp\left(\frac{1}{k+1}\right) \\ &= \frac{w_0}{e}. \end{aligned}$$

Thus $\lim_{t\to\infty} u(t,x) \le \frac{w_0}{e} < 1$.

To find conditions on the impulses so that the positive steady-state of IBVP (3.70)–(3.72) without impulses is also an attractor for IBVP (3.70)–(3.73), we need the following theorem.

THEOREM 3.10 (Theorem 5.3.3 in [**37**]). *Let u_s be a steady-state solution of problem (3.24)–(3.26) where f and h are independence from t, and let f be a C^1-function on a neighbourhood of u_s. Let real number μ_0 and positive function $\psi(x)$ be the principal eigenvalue and eigenfunction of the eigenvalue problem*

$$\sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial^2 \psi}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} b_i(t,x) \frac{\partial \psi}{\partial x_i} + (\mu + f_u(x, u_s(x)))\,\psi = 0 \qquad \text{in } \Omega$$

$$\alpha(t,x) \frac{\partial u}{\partial \nu} + \beta(t,x) u = h(x) \qquad \text{on } \partial\Omega$$

with $\max\{\psi(x) : x \in \Omega\} = 1$. *If* μ_0 *is positive then there exists positive constants* ρ, ζ *such that*

$$|u(t,x) - u_s(x)| \le \rho e^{-\zeta t}\psi(x) \qquad t > 0,\, x \in \overline{\Omega}$$

whenever it holds at $t = 0$.

LEMMA 3.11. *Let* a *and* λ_1 *be as in Lemma (3.10) and* $w(t,x)$ *be the solution of IBVP (3.1)–(3.3) without impulses and delay, and let* $\phi(x)$ *be the steady-state of the same problem. If* $|w(0,x) - \phi(x)| < \frac{\mu_0}{2b}\psi(x)$ *there exists positive numbers* ρ *and* ζ *such that*

$$|w(t,x) - \phi(x)| \le \rho e^{-\zeta t}\psi(x) \quad t > 0, x \in \overline{\Omega} \tag{3.92}$$

where μ_0 *and* $\psi(x)$ *are the principal eigenvalue and eigenfunction of*

$$-\Delta v + (2b\phi - a)v = \mu v \qquad \text{in } \Omega \tag{3.93}$$

$$\alpha(t,x)\frac{\partial v}{\partial \nu} + \beta(t,x)v = 0 \qquad \text{on } \partial\Omega. \tag{3.94}$$

PROOF. Because $\phi(x)$ is a solution of BVP (3.74)–(3.75), it is also a strict positive upper solution of the following problem

$$-\Delta v + (2b\phi - a)v = 0 \qquad \text{in } \Omega$$

$$\alpha(t,x)\frac{\partial v}{\partial \nu} + \beta(t,x)v = 0 \qquad \text{on } \partial\Omega.$$

By applying Lemma 3.6, we know that μ_0, the principal eigenvalue of (3.93)–(3.94) is positive. Hence we can use Theorem 3.10 to guarantee the existence of positive numbers ρ and ζ such that

$$\rho \le \frac{\mu_0 - \zeta}{2b} \tag{3.95}$$

and the inequality in (3.92) holds. □

The following theorem provides conditions on impulses for which ϕ is an attractor for IBVP (3.70)–(3.73).

THEOREM 3.11. *Let* λ_1, ϕ *and* B_M *be as in Lemma 3.10. Assume* $\tau B_M < 1$ *and* $a > \lambda_1$. *If the spacing between impulses,* $t_k - t_{k-1}$, *and the impulse function* I_k *satisfy*

S0: *$I_k : \overline{\Omega} \times R \to R$ is a non-decreasing function with respect to the second variable, and $I_k(0) = 0$ for every positive integer k.*

S1: *$|I_k(x,z) - z| \leq \rho_{k-1} e^{-\zeta t_{k-1}} \delta_k \left(1 - e^{-\zeta(t_k - t_{k-1})}\right) \psi(x)$ for every k, where*

$$\rho_k = \rho \prod_{i=1}^{k} \left[1 + \delta_i \left(e^{\zeta(t_i - t_{i-1})} - 1\right)\right].$$

S2: *$0 \leq \delta_k < 1$ for every k.*

S3: *$\prod_{k=1}^{\infty} \left[1 + \delta_k (e^{\zeta(t_k - t_{k-1})} - 1)\right] < \infty$.*

where ρ, ζ and $\psi(x)$ as in Lemma 3.11, then ϕ is an attractor of IBVP (3.70)–(3.73).

Proof. From S0, the existence of solution of IBVP (3.70)–(3.73) is guaranteed by Corollary 3.2. By applying Lemma 3.10 we know that the delay does not effect the stability of ϕ.

Consider the problem in time interval $(0, t_1)$. For any solution of IBVP (3.70)–(3.73), $u(t,x)$, whose value at initial point $|u(0,x) - \phi(x)| < \frac{\mu_0}{2b} \psi(x)$, Lemma 3.11 implies the existence of positive numbers ρ and ζ such that

$$|u(t,x) - \phi(x)| \leq \rho e^{-\zeta t} \psi(x).$$

It follows from S1 that

$$\begin{aligned}
|u(t_1,x) - \phi(x)| &\leq |u(t_1,x) - u(t_1^-,x)| + |u(t_1^-,x) - \phi(x)| \\
&\leq \rho e^{-\zeta t_1} \psi(x) + \rho \delta_1 \left(1 - e^{-\zeta t_1}\right) \psi(x) \\
&= \rho e^{-\zeta t_1} \psi(x) \left[1 + \delta_1 \left(e^{\zeta t_1} - 1\right)\right] \\
&= \rho_1 e^{-\zeta t_1} \psi(x)
\end{aligned}$$

with $\rho_1 = \rho \left[1 + \delta_1 \left(e^{\zeta t_1} - 1\right)\right]$. From (S2) we know that $\rho_1 e^{-\zeta t_1} \leq \rho$. This number can be used as a new bound in time interval (t_1, t_2).

$$|u(t,x) - \phi(x)| \leq \rho_1 e^{-\zeta t} \psi(x), \quad t \in (t_1, t_2).$$

$$\begin{aligned}|u(t_2,x)-\phi(x)| &\le \rho_1 e^{-\zeta t_2}\psi(x)+\rho_1\delta_2 e^{-\zeta t_1}\left(1-e^{-\zeta(t_2-t_1)}\right)\psi(x)\\ &= \rho e^{-\zeta t_2}\psi(x)\left[1+\delta_1\left(e^{\zeta(t_2-t_1)}-1\right)\right]\\ &= \rho_2 e^{-\zeta t_2}\psi(x)\end{aligned}$$

with $\rho_2=\rho_1\left[1+\delta_1\left(e^{\zeta(t_2-t_1)}-1\right)\right]$. It follows from S2 that $\rho_2 e^{-\zeta t_2}\le\rho_1$. By continuing this process we have

$$|u(t,x)-\phi(x)|\le\rho_k e^{-\zeta t}\psi(x),\ \text{for } t\in[t_k,t_{k+1})$$

with $\rho_k=\rho\prod_{i=1}^{k}\left[1+\delta_i\left(e^{\zeta(t_i-t_{i-1})}-1\right)\right]$. Using condition S3, we obtain

$$|u(t,x)-\phi(x)|\le\rho\prod_{k=1}^{\infty}\left[1+\delta_k(e^{\zeta(t_k-t_{k-1})}-1)\right]e^{-\zeta t}\psi(x)$$

$$\lim_{t\to\infty}|u(t,x)-\phi(x)|=0.$$

□

Remark 3.4. *The linearised problem of IBVP (3.70)–(3.71) around zero, given by*

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x)-\Delta u(t,x)&=au(t,x) && \text{in } \Omega,\ t\ne t_k\\ \alpha_2\frac{\partial u}{\partial\nu}(t,x)+\beta_2 u(t,x)&=0 && \text{on } \partial\Omega\\ u(0,x)&=\eta(0,x) && \text{in } \Omega\\ u(t_k,x)&=I_k(u(t_k{}^-,x)) && \text{in } \overline{\Omega},\ k=1,2,...,\end{aligned}$$

requires similar conditions on impulse functions to the conditions in Theorem 3.8 and in Theorem 3.9 in order to have zero function as an attractor.

By applying similar techniques to those used in Theorem 3.8, we obtain that the solution of the linearized problem of IBVP (3.70)–(3.73) around 'equilibrium' ϕ,

$$\begin{aligned}\frac{\partial}{\partial t}(u-\phi)(t,x)-\Delta w(t,x)=&(a-b\phi(x))(u(t,x)-\phi(x))\\ &-b\phi(x)(u(t-\tau,x)-\phi(x))\quad \text{in } \Omega,\ t\ne t_k\end{aligned}$$

$$\alpha_2 \frac{\partial}{\partial \nu}(u(t,x) - \phi(x)) + \beta_2(u(t,x) - \phi(x)) = 0 \qquad \textit{on } \partial\Omega$$

$$u(t,x) = \eta(t,x) \qquad \textit{in } [-\tau, 0] \times \Omega$$

$$u(t_k, x) = I_k(u({t_k}^-, x)) \quad \textit{in } \overline{\Omega},\ k = 1, 2, ...,$$

converges to ϕ if the impulse functions and time delay satisfy the following conditions:

(a) $I_k(x, z) = \phi(x) + \gamma_o(z - \phi(x)) \exp(-2(a - \lambda_1)(t_k - t_{k-1}))$ *with* $0 \leq \gamma_o < 1$,

(b) $\tau < (\max\{bu^*(x) : x \in \Omega\})^{-1}$.

CHAPTER 4

Diffusive Logistic Equations with Continuous Time Delay and Impulses

In this chapter, a system of logistic equations with three phenomena, diffusion, continuous infinite time delay, and impulses which occur at fixed times is discussed. This system is considered in a bounded domain and infinite interval of time. This work is motivated by some results found in [**46**], [**57**], [**55**], [**17**].

Schiaffino [**46**] investigated the system of a single species with diffusion and continuous delay. The asymptotic behaviour of the solution was shown by using a prior estimate. It was assumed that the initial conditions are non-negative, the delay kernel is a decreasing function and the hereditary term is dominated by non-delay logistic term. Yamada [**57**] found some sufficient conditions for the global asymptotic stability of positive equilibrium in terms of the Laplace transformation of the delay kernel. The proof was based on the energy method with use of a certain Liapunov functional.

Worz-Busekros [**55**] obtained sufficient conditions for the global asymptotic behaviour of the solution of a multi species logistic system with infinite delay by assuming the delay kernels are a convex combination of exponential functions. Gopalsamy [**17**] discussed a similar problem and showed the asymptotic behaviour of the solution using a continuous Liapunov-like (non-negative and non-differentiable) function. In [**4**] Bereketoglu and Gyori used the method based on finding a positive bounded function that satisfied the system with a certain perturbation.

The stability of logistic equations with impulses can be found e.g. in [**27**], [**25**], [**28**]. The stability of steady-states are usually obtained using the Liapunov functional. Some papers use the notion of stability in terms of two measures [**25**],[**24**].

The existence of solution of diffusive logistic equation with continuous delay and impulses is investigated in Section 1. Some conditions are discussed in Section 2 under which the zero function become an attractor for this system. In Section 3, the positive steady-state of the system without impulses is considered. An extension to multi species system is obtained in the last section.

1. Existence of Solution

Let Ω, $\partial\Omega$, T, E, S, t_k, M_i, M N_i, N, a, b be as in Chapter 3. The problem which will be investigated is the following initial boundary value problem

$$\frac{\partial u}{\partial t}(t,x)-\Delta u(t,x) = u(t,x)(a-bu(t,x)-\int_{-\infty}^{t} f(t-s)u(s,x)ds) \text{ in } E\backslash M \quad (4.1)$$

$$\frac{\partial u}{\partial \nu}(t,x) = 0 \qquad \text{on } S \setminus N \qquad (4.2)$$

$$u(t,x) = u_o(t,x) \qquad \text{in } (-\infty,0]\times\bar{\Omega} \qquad (4.3)$$

$$u(t_k,x) = I_k(u(t_k{}^{-},x)) \qquad \text{in } \bar{\Omega},\ k=1,2,\dots. \qquad (4.4)$$

To solve this problem, several assumptions are made:

C1 $u_o \geq 0(\not\equiv 0),\ u_o \in C^1((-\infty,0];D(-\Delta))\cap L^1((-\infty,0];L^p(\Omega)),\ p\geq n.$
C2 $f(t)\geq 0(\not\equiv 0),\ f\in C^1([0,\infty))\cap L^1([0,\infty)),\ tf\in L^1([0,\infty)).$
C3 $I_k \in C^2(\mathbf{R})$ and $I_k \geq 0.$

To find the solution, we consider the problem in every time interval formed by two consecutive impulse times, as follows:

$$(u_k)_t(t,x)-\Delta u_k(t,x) = u_k(t,x)\left(a_k(t,x)-bu_k(t,x)-\int_0^t f(t-s)u(s,x)ds\right)$$
$$\text{in } t\in(0,t_k-t_{k-1})\times\Omega \qquad (4.5)$$

$$\frac{\partial u_k}{\partial \nu}(t,x)=0 \qquad \text{on } (0,t_k-t_{k-1})\times\partial\Omega \qquad (4.6)$$

$$u_k(0,x) = \begin{cases} u_o(x) & \text{for } k=1 \\ I_{k-1}(u_{k-1}((t_{k-1}-t_{k-2})^{-},x)) & \text{for } k=2,3,\dots \end{cases} \quad \text{in } \bar{\Omega} \qquad (4.7)$$

with $a_1 = a$,

$$a_k(t,x) = a - \int_{-\infty}^{0} f(t-s)u_o(s,x)ds - \sum_{i=1}^{k-1} \int_{0}^{t_i - t_{i-1}} f_i(t-s)u_i(s,x)ds$$

for $k = 2, 3, ...$, $t_o = 0$ and $f_k(t) = f(t + t_{k-1})$ for $k = 1, 2, ...$

The existence and uniqueness of the solution of problem (4.5)–(4.7) follow from Yamada's results in [**57**].

LEMMA 4.1 (Proposition 3.2 and Remark 2.2 in [**57**]). *If* $u_o \in D(A) = \{u \in W^{2,p}; \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$, $g \in C^1[0,T]$, *and* $a_1 \in C^\alpha([0,T]; C(\overline{\Omega}))$ *with* $\alpha \in (0,1)$, *then there exists a positive constant* $T_o \leq T$ *such that the initial boundary value problem given by*

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + u(t,x)\left(a_1(t,x) - b_1 u(t,x) - \int_0^t g(t-s)u(s)ds\right) \quad , x \in \Omega,\ t \geq 0 \tag{4.8}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad , x \in \partial\Omega,\ t \geq 0 \tag{4.9}$$

$$u(0,x) = u_o(x) \geq 0 \quad , x \in \Omega \tag{4.10}$$

has a unique solution $u \in C([0,T_o]; L^p(\Omega)) \cap C([0,T_o]; D(A))$. *Moreover* u *has the following properties:*

(i) $u(t,x) \geq 0$ *for* $x \in \overline{\Omega}$ *and* $t \in [0,T_o]$ *if* $u_o \geq 0$ *for* $x \in \overline{\Omega}$.

(ii) *If* $u_o(\geq 0)$ *is not identically zero then* $u(t,x)$ *is positive for* $x \in \overline{\Omega}$ *and* $t \in (0,T_o]$.

Let v *be a maximal solution of equations (4.9), (4.10), and*

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + u(t,x)\left(a_2 - b_1 u(t,x) - \int_0^t g(t-s)u(s)ds\right) \quad x \in \Omega,\ t \geq 0 \tag{4.11}$$

where $a_2 = a_1(0,x)$). *In other words there is no solution of problem (4.11),(4.9), and(4.10) on* $[0,T']$ *if* $T' > T$. *If* $\|v(t)\|_\infty$ *is bounded on* $[0,T) \cap [0,\theta]$ *for any* $\theta > 0$, *then it can be shown that* $T = \infty$ *by using a translation argument.*

Hence by solving problem (4.5)–(4.7) for $k = 1, 2, ...$ sequentially, the existence and uniqueness of the solution of problem (4.1)–(4.4) are obtained.

THEOREM 4.1. *There exists a unique solution of problem (4.1)–(4.4), this solution is given by*

$$u(t,x) = u_k(t - t_{k-1}, x), \quad \textit{in } [t_{k-1}, t_k),\ k = 1, 2, ..$$

where u_k is solution of problem (4.5)–(4.7).

2. Zero Attractor

To specify the conditions under which zero function will become an attractor, we use similar techniques to those in Subsection 3.1. First we find a bound of the solution of the problem without impulses.

$$\int_\Omega \frac{\partial u}{\partial t}(t,x)u(t,x)dx - \int_\Omega u(t,x)\Delta u(t,x)dx = \int_\Omega u^2(t,x)\left(a - bu(t,x) - \int_0^\infty f(s)u(t-s,x)ds\right)dx.$$

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u^2(t,x)dx + \int_\Omega |\nabla u(t,x)|^2 dx = \int_\Omega u^2(t,x)\left(a - bu(t,x) - \int_0^\infty f(s)u(t-s,x)ds\right)dx.$$

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u^2(t,x)dx + \int_\Omega |\nabla u(t,x)|^2 dx \le \int_\Omega au^2(t,x)dx.$$

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u^2(t,x)dx \le \int_\Omega au^2(t,x)dx.$$

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u^2(t,x)dx \le a\int_\Omega u^2(t,x)dx.$$

$$\frac{1}{2}\frac{\partial}{\partial t}\|u(t,\cdot)\|_{L^2(\Omega)} \le a\|u(t,\cdot)\|_{L^2(\Omega)}.$$

$$\|u(t,\cdot)\|_{L^2(\Omega)} \le \|u(0,\cdot)\|_{L^2(\Omega)}\exp\{2at\}.$$

Since $a_k(t,x) \le a$, we can use this estimate on every time interval

$$\|u(t_k^-,\cdot)\|_{L^2(\Omega)} \le \|u(t_{k-1},\cdot)\|_{L^2(\Omega)}\exp\{2a(t_k - t_{k-1})\}.$$

$$\|u(t_k,x)\|_{L^2(\Omega)} = \|I_k(u(t_k^-,x))\|_{L^2(\Omega)}.$$

Let $I_k(z) = w_k z$ where $0 < w_k < \exp(-2a(t_k - t_{k-1}))$ for every $k = 1, 2, 3....$ Then we have

$$\begin{aligned} \|u(t_k,x)\|_{L^2(\Omega)} &< w_k \|u(t_{k-1},x)\|_{L^2(\Omega)} \exp\{2a(t_k - t_{k-1})\} \\ &= \gamma_k \|u(t_{k-1},x)\|_{L^2(\Omega)} \\ &< \gamma_k \gamma_{k-1} \|u(t_{k-2},x)\|_{L^2(\Omega)} \qquad (4.12) \\ &\vdots \\ \|u(t_k,x)\|_{L^2(\Omega)} &< \left(\prod_{j=1}^{k} \gamma_j\right) \|u(0,x)\|_{L^2(\Omega)} \text{ for every } k \end{aligned}$$

where $\gamma_k = w_k \exp(2a(t_k - t_{k-1})) < 1$. It follows from

$$\lim_{k\to\infty} \prod_{j=1}^{k} \gamma_j = 0 \qquad (4.13)$$

that $\lim_{k\to\infty} \|u(t_k,x)\| = 0$. Hence $\lim_{t\to\infty} \|u(t,x)\| = 0$.

Now we can state this result as follows

THEOREM 4.2. *Let $I_k(z) = w_k z$ where $0 < w_k < \exp(-2a(t_k - t_{k-1}))$ for every $k = 1, 2, 3....$ Then 0 is an attractor of system (4.1)–(4.4).*

3. Positive Attractor

In this section, conditions will be found under which $u^* = \frac{a}{b + \int_{-\infty}^{t} f(s)ds}$ is a positive attractor of problem (4.1)–(4.4). The motivation of choosing u^* as a positive attractor and the technique used for proving this, come from Theorem 3.2 in [**57**] and Theorem 2.1 in [**17**]. Yamada [**57**] investigated the stability of positive 'equilibrium' of logistic equation with diffusion and continuous time delay. The time delay in this system represents an effect from the past state started at 0. Gopalsamy [**17**] discussed the asymptotic behaviour of n-species Volterra system with the interval of integration $(-\infty, t)$.

Yamada [**57**] show the stability of system (4.8)–(4.10) in the following lemma.

Lemma 4.2 (Theorem 3.2 in [**57**]). *Let* $b_1 + inf\{Re\left(\int_0^\infty e^{-i\varsigma t} g(t)dt; \varsigma \in R\right) > 0\}$. *Then the solution* u *of problem (4.8)–(4.10) satisfies*

$$\lim_{t\to\infty} u(t,x) = \frac{a_1}{b_1 + \int_0^\infty g(s)ds} \quad \text{on } \overline{\Omega}.$$

In [**17**], Gopalsamy investigated the asymptotic behaviour of n-species Volterra system given by

$$\frac{dx_i}{dt} = x_i(t)\left(r_i + \sum_{j=1}^n a_{ij}x_j(t) + \sum_{j=1}^n b_{ij}\int_{-\infty}^t k_{ij}(t-s)x_j(s)ds\right)$$
$$,t > 0,\ i = 1,2,...,n \tag{4.14}$$

$$x_i(s) = \phi_i(s) \geq 0, \qquad s \in (-\infty, 0] \tag{4.15}$$

where $\sup_{s\leq 0} \phi_i(s) < \infty$, $i = 1,2,...,n$, ϕ_i is a bounded nonnegative integrable function on $(-\infty, 0]$ with a possible jump discontinuity at $s = 0$ so that $\phi_i(0) > 0$, $i = 1,2,...,n$, and

(G1) the delay kernels k_{ij}, $i,j = 1,2,..,n$ are defined on $[0,\infty)$, bounded, integrable and normalised

$$\int_0^\infty k_{ij}(s)ds = 1; \quad \int_0^\infty |k_{ij}(s)|ds < \infty, \quad \int_0^\infty s|k_{ij}(s)|ds < \infty,$$

(G2) r_i, a_{ij}, b_{ij} are real constants satisfying

$$a_{ii} < 0, \quad |a_{ii}| > \sum_{j=1}^n |b_{ij}|| \int_0^\infty |k_{ij}(s)|ds + \sum_{\substack{j=1\\ j\neq i}}^n a_{ij},\ i = 1,2,...,n$$

such that there exists a solution $x^* = (x_1^*, x_2^*, ..., x_n^*)$ with $x_i^* > 0$ of the linear system

$$\sum_{j=1}^n (a_{ij} + b_{ij})x_j^* = r_i, \quad i = 1,2,...,n.$$

Lemma 4.3 (Theorem 2.1 in [**17**]). *Assume that hypotheses (G1)-(G2) hold for system (4.14)–(4.15). Then all the solutions of (4.14)–(4.15) satisfy*

$$\lim_{t\to\infty} x_i(t) = x_i^* \quad ,i = 1,2,...,n.$$

THEOREM 4.3. *Let* $b \geq 2\int_0^\infty f(s)ds$ *and the impulse function* $I_k(z) = (1+\beta_k(z))z$ *with* β_k *satisfy the following relation*

$$|\beta_k(z)| \leq 1 - \exp\left(\frac{1-\exp\left(\frac{1}{k^2}\right)}{z+u^*}\left[z - u^* - u^*\ln\left(\frac{z}{u^*}\right)\right]\right)$$

where $u^* = \frac{a}{b+\int_0^\infty f(s)ds}$. *Then* $\lim_{t\to\infty}(u(t,x) - u^*) = 0$.

PROOF. Let $V(t,u)$ be the Liapunov functional for system (4.1)–(4.4).

$$V(t,u) = \int_\Omega \left[u(t,x) - u^* - u^*\ln\left(\frac{u(t,x)}{u^*}\right)\right] + \int_\Omega\int_0^\infty f(s)\int_{t-s}^t (u(z,x)-u^*)^2 dzdsdx.$$

Let $\vartheta(t,x) = u(t,x) - u^*$.

By applying the equation and inequality given by

$$u(t,x)\left(a - bu(t,x) - \int_{-\infty}^t f(t-s)u(s,x)ds\right) = -u(t,x)\left(b\vartheta(t,x) + \int_0^\infty f(s)\vartheta(t-s,x)ds\right) \tag{4.16}$$

$$-\vartheta(t,x)\vartheta(t-s,x) \leq \vartheta^2(t,x) + \vartheta^2(t-s,x), \tag{4.17}$$

we obtain a bound for the derivative of V with respect to t along the solution of IBVP (4.1)–(4.4) as follows

$$\begin{aligned}
\frac{dV}{dt} &= \int_\Omega \frac{\partial u}{\partial t}(t,x)\left(1 - \frac{u^*}{u(t,x)}\right) + \int_\Omega\int_0^\infty f(s)(\vartheta^2(t,x) - \vartheta^2(t-s,x))dsdx \\
&= \int_\Omega \left(\Delta u - u(t,x)\left(b\vartheta(t,x) + \int_0^\infty f(s)\vartheta(t-s,x)ds\right)\right)\left(1 - \frac{u^*}{u(t,x)}\right)dx \\
&\quad + \int_\Omega\int_0^\infty f(s)(\vartheta^2(t,x) - \vartheta^2(t-s,x)dsdx \\
&= \int_\Omega \left(-u^*\frac{(\nabla u(t,x))^2}{u^2(t,x)} - bu(t,x)\vartheta(t,x)\left(1 - \frac{u^*}{u(t,x)}\right)\right)dx \\
&\quad - \int_\Omega u(t,x)\left(1 - \frac{u^*}{u(t,x)}\right)\int_0^\infty f(s)\vartheta(t-s,x)dsdx \\
&\quad + \int_\Omega\int_0^\infty f(s)(\vartheta^2(t,x) - \vartheta^2(t-s,x))dsdx
\end{aligned}$$

$$\begin{aligned}
\frac{dV}{dt} \leq & -b\int_\Omega \vartheta^2(t,x)dx + \int_\Omega\int_0^\infty f(s)(\vartheta^2(t,x)-\vartheta^2(t-s,x))dsdx \\
& -\int_\Omega\int_0^\infty f(s)\vartheta(t,x)\vartheta(t-s,x)dsdx \\
\leq & -b\int_\Omega \vartheta^2(t,x)dx + \int_\Omega\left(2\int_0^\infty f(s)\vartheta^2(t,x)ds\right)dx \\
\frac{dV}{dt} \leq & \int_\Omega \big((-b+2\varpi)\vartheta^2(t,x)\big)\,dx \qquad (4.18)\\
\leq & 0
\end{aligned}$$

where $\varpi = \int_0^\infty f(s)ds$. Thus V is non-increasing on every time interval $[t_k, t_{k+1})$.

Let $\psi_k(z) = \frac{\exp\left(\frac{1}{k^2}\right)-1}{z+u^*}\left[z - u^* - u^*\ln\left(\frac{z}{u^*}\right)\right] \geq 0$.

From the condition on β_k we have

$$|\beta_k(z)| \leq 1 - \exp(-\psi_k(z)) \leq \psi_k(z).$$

$$\begin{aligned}
&|\beta_k(u(t_k^-,x))|(u(t_k^-,x)+u^*) \\
&\qquad \leq \left(\exp\left(\frac{1}{k^2}\right)-1\right)\left[u(t_k^-,x) - u^* - u^*\ln\left(\frac{u(t_k^-,x)}{u^*}\right)\right].
\end{aligned} \tag{4.19}$$

For $\beta_k(z) < 0$ we have $\beta_k(z) \geq \exp(-\psi_k(z)) - 1$. Thus $\ln(1+\beta_k(z)) \geq -\psi_k(z)$. And for $\beta_k(z) \geq 0$, $\ln(1+\beta_k(z)) \leq \beta_k(z) \leq \psi_k(z)$. Hence $|\ln(1+\beta_k(z))| \leq \psi_k(z)$.

$$\begin{aligned}
&|\ln(1+\beta_k(u(t_k^-,x)))|(u(t_k^-,x)+u^*) \\
&\qquad \leq \left(\exp\left(\frac{1}{k^2}\right)-1\right)\left[\vartheta(t_k^-,x) - u^*\ln\left(\frac{u(t_k^-,x)}{u^*}\right)\right].
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
V(t_k, u(t_k,x)) = & \int_\Omega \big(u(t_k^-,x)(1+\beta_k(\vartheta(t_k^-,x)))\big)\,dx \\
& - \int_\Omega u^*\ln\left(\frac{u(t_k^-,x)(1+\beta_k(u(t_k^-,x)))}{u^*}\right)dx \\
& + \int_\Omega\int_0^\infty f(s)\int_{t_k-s}^{t_k}\vartheta^2(z,x)dzdsdx.
\end{aligned}$$

$$
\begin{aligned}
V(t_k,u(t_k,x)) = & \int_\Omega \left(\vartheta(t_k^-,x) - u^*\ln\left(\frac{u(t_k^-,x)}{u^*}\right)\right)dx \\
& + \int_\Omega \left(\beta_k(u(t_k^-,x))u(t_k^-,x) - u^*\ln(1+\beta_k(u(t_k^-,x)))\right)dx \\
& + \int_\Omega\int_0^\infty f(s)\int_{t_k-s}^{t_k}\vartheta^2(z,x)dzdsdx \\
\leq & \int_\Omega \left(\vartheta(t_k^-,x) - u^*\ln\left(\frac{u(t_k^-,x)}{u^*}\right)\right)dx \\
& + \int_\Omega |\beta_k(u(t_k^-,x))u(t_k^-,x) - u^*\ln(1+\beta_k(u(t_k^-,x)))|dx \\
& + \int_\Omega\int_0^\infty f(s)\int_{t_k-s}^{t_k}\vartheta^2(z,x)dzdsdx \\
\leq & \int_\Omega \left(\vartheta(t_k^-,x) - u^*\ln\left(\frac{u(t_k^-,x)}{u^*}\right)\right)dx \\
& + \int_\Omega \leq \left(|\beta_k(u(t_k^-,x))u(t_k^-,x)| + |u^*\ln(1+\beta_k(u(t_k^-,x)))|\right)dx \\
& + \int_\Omega\int_0^\infty f(s)\int_{t_k-s}^{t_k}\vartheta^2(z,x)dzdsdx \\
\leq & \int_\Omega \left(\vartheta(t_k^-,x) - u^*\ln\left(\frac{u(t_k^-,x)}{u^*}\right)\right)dx \\
& + \int_\Omega \max\{|\beta_k(u(t_k^-,x))|, |\ln(1+\beta_k(u(t_k^-,x)))|\}(u(t_k^-,x)+u^*)dx \\
& + \int_\Omega\int_0^\infty f(s)\int_{t_k-s}^{t_k}\vartheta^2(z,x)dzdsdx.
\end{aligned}
$$

$$
V(t_k,u(t_k,x)) \leq V(t_k^-,u(t_k^-,x))\exp\left(\frac{1}{k^2}\right).
$$

$$
V(t,u(t,x))) \leq V(0,u(0,x))\prod_{k=1}^{p}\exp\left(\frac{1}{k^2}\right) \text{ for } t\in(t_p,t_{p+1}),\ p\geq 1. \tag{4.21}
$$

By integrating the inequality in 4.18 we obtain

$$
\int_0^t \frac{dV}{ds}(s,u(s,x))ds \leq \int_\Omega\int_0^t \left((-b+2\alpha)(u(s,x)-u^*)^2\right)dtdx. \tag{4.22}
$$

Expanding the hand side of inequality 4.22 we have

$$
\int_0^t \frac{dV}{ds}(s,u(s,x))ds = \int_{t_p}^t \frac{dV}{ds}(s,u(s,x))ds + \sum_{k=1}^{p}\int_{t_{k-1}}^{t_k}\frac{dV}{ds}(s,u(s,x))ds
$$

for $t \in (t_p, t_{p+1}]$.

$$\begin{aligned}\int_0^t \frac{dV}{ds}(s, u(s,x))ds =& V(t, u(t,x)) - V(t_p, u(t_p, x)) \\ &+ \sum_{k=1}^{p} \left(V(t_k^-, u(t_k^-, x)) - V(t_{k-1}, u(t_{k-1}, x))) \right) \\ =& V(t, u(t,x)) - V(0, u(0,x)) \\ &+ \sum_{k=1}^{p} \left(V(t_k^-, u(t_k^-, x)) - V(t_k, u(t_k, x))) \right). \end{aligned} \tag{4.23}$$

Since V is non-increasing,

$$V(0, u(0,x)) + \sum_{k=1}^{p} \left(V(t_k, u(t_k, x)) - V(t_k^-, u(t_k^-, x)) \right) \geq 0.$$

From (4.19), (4.20), (4.21) we have

$$\begin{aligned} V(t_k, u(t_k,x)) -& V(t_k^-, u(t_k^-, x))) \\ &= (\beta_k(u(t_k^-, x))u(t_k^-, x) - u^* \ln(1 + \beta_k(u(t_k^-, x)))). \\ |V(t_k, u(t_k,x)) -& V(t_k^-, u(t_k^-, x)))| \\ &= |\beta_k(u(t_k^-, x))u(t_k^-, x) - u^* \ln(1 + \beta_k(u(t_k^-, x)))|. \\ |V(t_k, u(t_k,x)) -& V(t_k^-, u(t_k^-, x)))| \\ &\leq |\beta_k(u(t_k^-, x))|u(t_k^-, x) + u^*| \ln(1 + \beta_k(u(t_k^-, x)))| \\ &\leq \left(\exp\left(\frac{1}{k^2}\right) - 1 \right) V(t_k^-, u(t_k^-, x))) \\ &\leq \left(\exp\left(\frac{1}{k^2}\right) - 1 \right) V(0, u(0,x)) \prod_{j=1}^{k-1} \exp\left(\frac{1}{j^2}\right). \end{aligned}$$

Using the comparison test of infinite series(to series $\sum_{k=1}^{\infty} \frac{1}{k^{1.5}}$), we find that

$$\sum_{k=1}^{\infty} \left(V(t_k, u(t_k, x)) - V(t_k^-, u(t_k^-, x)) \right)$$

is convergent. Thus $V(0, u(0,x)) + \sum_{k=1}^{\infty} \left(V(t_k, u(t_k, x)) - V(t_k^-, u(t_k^-, x)) \right)$ is finite and non-negative. Then we do the same action to the right hand side of equation (4.18).

$$\int_\Omega \int_0^t (-b + 2\varpi)\vartheta^2(s,x) ds dx$$
$$= \int_\Omega \int_{t_p}^t \big((-b + 2\varpi)\vartheta^2(s,x)\big) ds dx$$
$$+ \sum_{k=1}^{p} \int_\Omega \int_{t_{k-1}}^{t_k} \big((-b + 2\varpi)\vartheta^2(s,x)\big) ds dx \tag{4.24}$$

for $t \in (t_p, t_{p+1}]$.

From equation (4.18) and (4.23), we obtain the following inequality.

$$V(t,u(t,x)) + \int_\Omega \int_0^t \big((b - 2\varpi)(\vartheta^2(s,x)\big) ds dx$$
$$\leq V(0,u(0,x)) + \sum_{k=1}^{p} \big(V(t_k,u(t_k,x)) - V(t_k^-,u(t_k^-,x))\big). \tag{4.25}$$

Since V is positive and the right hand side (4.25) is finite and positive,

$$\int_\Omega \int_0^t \big((-b + 2\varpi)\vartheta^2(s,x)\big) ds dx$$

is finite for all t. From the assumption of impulse time, $t_k \to \infty$ as $k \to \infty$, and the solution u is bounded, differentiable and uniformly continuous in every time interval (t_k, t_{k+1}), we have

$$\lim_{t\to\infty} \vartheta(t,x) = \lim_{t\to\infty} (u(t,x) - u^*) = 0.$$

□

4. Multi Species System

An extension to multi species system is discussed in this section. We make some assumptions to obtain a positive attractor for this system. Some of these assumptions are similar to C1-C3. The proof uses a similar method to the one for the single species system.

The multi species system is given by

$$\frac{u_i}{dt}(t,x) - \Delta u_i(t,x) = u_i(t,x)\left(a_i + \sum_{j=1}^{m} b_{ij}u_j(t,x)\right) + u_i(t,x)\sum_{j=1}^{m} c_{ij}\int_{-\infty}^{t} f_j(t-s)u_j(s,x)ds \text{ in } E\setminus M \quad (4.26)$$

$$\frac{\partial u_i}{\partial \nu}(t,x) = 0 \qquad \text{on } S\setminus N \qquad (4.27)$$

$$u_i(t,x) = \eta_i(t,x) \qquad \text{in } (-\infty,0]\times\bar{\Omega} \qquad (4.28)$$

$$u_i(t_k,x) = (1+\beta_{ik}(u_i(t_k^-,x)))(u_i({t_k}^-,x)) \qquad \text{in } \bar{\Omega},\ k=1,2,... \qquad (4.29)$$

for $i = 1, 2, ...m$ where a_i, b_{ij}, c_{ij} are constants.

Some assumptions are made as follows:

P1. $\eta_i \geq 0(\not\equiv 0),\ \eta_i \in C^1((-\infty,0];D(-\Delta)) \cap L^1((-\infty,0];L^p(\Omega)),\ p \geq n.$

P2. $f_i(t) \geq 0(\not\equiv 0),\ f_i \in C^1([0,\infty)) \cap L^1([0,\infty)),\ tf_i \in L^1([0,\infty)).$

P3. $a_i > 0,\ b_{ii} < 0.$

P4. There exists $u_i^* > 0,\ i = 1,...,m$ such that $a_i + \sum_{j=1}^{m}(b_{ij} + c_{ij}\varpi_j)u_j^* = 0$, $\varpi_j = \int_0^\infty f_j(s)ds.$

P5. $b_{ii} + \alpha_i|c_{ii}| + \frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{m}(|b_{ij}| + |b_{ji}| + \alpha_i|c_{ji}| + \alpha_j|c_{ij}|) < 0.$

P6. β_{ik} is a function on $\mathbf{R}^+$ and satisfies this relation:

$$|\beta_{ik}(z)| \leq 1 - \exp\left(\frac{1-\exp\left(\frac{1}{k^2}\right)}{z+u_i^*}\left[z - u_i^* - u_i^*\ln\left(\frac{z}{u_i^*}\right)\right]\right)$$

for every $i = 1, ..m,\ k = 1, 2, 3, ...$

By substituting $u_1^*, u_2^*, ..., u_m^*$ in condition P4 to equation (4.26), this equation now can be written as:

$$\frac{du_i}{dt}(t,x) = \Delta u_i(t,x) + u_i(t,x)\left(\sum_{j=1}^{m} b_{ij}(u_j(t,x) - u_j^*)\right) + u_i(t,x)\sum_{j=1}^{m} c_{ij}\int_0^\infty f_j(s))(u_j(t-s,x) - u_j^*)ds.$$

THEOREM 4.4. *Assume that P1–P6 hold. The function* $\mathbf{u}^* = (u_1^*, ..., u_m^*)$ *is a positive attractor for the system (4.26)–(4.29).*

PROOF. To show that $\mathbf{u}^*$ is an attractor for system (4.26)–(4.29), we choose the Liapunov functional as follows

$$V(t,u_1,..,u_m) = \sum_{i=1}^{m}\int_\Omega \left(u_i(t,x) - u_i^* - u_i^*\ln\left(\frac{u_i(t,x)}{u_i^*}\right)\right)dx$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{|c_{ij}|}{2}\int_\Omega\int_0^\infty f_j(s)\int_{t-s}^{t}(u_j(z,x)-u_j^*)^2dzdsdx.$$

The derivative of V along the solution of system (4.26)–(4.29) is given by

$$\frac{dV}{dt} = \sum_{i=1}^{m}\int_\Omega \frac{u_i(t,x)-u_i^*}{u_i(t,x)}\Delta u_i(t,x)dx$$
$$+\sum_{i=1}^{m}\int_\Omega (u_i(t,x)-u_i^*)\left(\sum_{j=1}^{m} b_{ij}(u_j(t,x)-u_j^*)\right)dx$$
$$+\sum_{i=1}^{m}\int_\Omega (u_i(t,x)-u_i^*)\sum_{j=1}^{m} c_{ij}\int_0^\infty f_j(s)(u_j(t-s,x)-u_j^*)dsdx$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{|c_{ij}|}{2}\int_\Omega\int_0^\infty f_j(s)((u_j(t,x)-u_j^*)^2-(u_j(t-s,x)-u_j^*)^2)dsdx.$$

We apply Green's theorem to the first term of the right hand side and use the condition on the boundary in equation (4.27) to obtain

$$\frac{dV}{dt} = \sum_{i=1}^{m} -u_i^*\int_\Omega \frac{(\nabla u_i(t,x))^2}{u_i^2(t,x)}dx + \sum_{i=1}^{m}\sum_{j=1}^{m}\int_\Omega b_{ij}(u_i(t,x)-u_i^*)(u_j(t,x)-u_j^*)dx$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\int_\Omega\int_0^\infty f_j(s)(u_i(t,x)-u_i^*)(u_j(t-s,x)-u_j^*)dsdx$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{|c_{ij}|}{2}\int_\Omega\int_0^\infty f_j(s)((u_j(t,x)-u_j^*)^2-(u_j(t-s,x)-u_j^*)^2)dsdx.$$

Since the first term in the right hand side is negative and from the fact that $2rs \leq r^2+s^2$ for every real number r and s, we obtain the following inequalities:

$$\begin{aligned}\frac{dV}{dt} &\leq \sum_{i=1}^{m}\int_\Omega \left(b_{ii}(u_i(t,x)-u_i^*)^2 + \sum_{\substack{j=1\\j\neq i}}^{m} b_{ij}(u_i(t,x)-u_i^*)(u_j(t,x)-u_j^*)\right) dx \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{|c_{ij}|}{2}\int_\Omega\int_0^\infty f_j(s)((u_i(t,x)-u_i^*)^2 + (u_j(t-s,x)-u_j^*)^2)dsdx \\ &+ \sum_{i=1}^{m}\int_\Omega \left(\varpi_i \frac{|c_{ii}|}{2}(u_i(t,x)-u_i^*)^2 + \sum_{\substack{j=1\\j\neq i}}^{m} \varpi_j \frac{|c_{ij}|}{2}(u_j(t,x)-u_j^*)^2 \right) dx \\ &- \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{|c_{ij}|}{2}\int_\Omega\int_0^\infty f_j(s)(u_j(t-s,x)-u_j^*)^2 dsdx.\end{aligned}$$

$$\begin{aligned}\frac{dV}{dt} &\leq \int_\Omega \left(b_{ii}(u_i(t,x)-u_i^*)^2 + \sum_{\substack{j=1\\j\neq i}}^{m} \frac{|b_{ij}|}{2}((u_i(t,x)-u_i^*)^2 + (u_j(t,x)-u_j^*)^2)\right) dx \\ &+ \sum_{i=1}^{m}\int_\Omega \left(\varpi_i \frac{|c_{ii}|}{2} + \sum_{\substack{j=1\\j\neq i}}^{m} \varpi_j \frac{|c_{ij}|}{2}\right)(u_i(t,x)-u_i^*)^2 dx \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{m}\int_\Omega\int_0^\infty f_j(s)(u_j(t-s,x)-u_j^*)^2 dsdx \\ &+ \sum_{i=1}^{m}\int_\Omega \left(\varpi_i \frac{|c_{ii}|}{2}(u_i(t,x)-u_i^*)^2 + \sum_{\substack{j=1\\j\neq i}}^{m} \varpi_j \frac{|c_{ij}|}{2}(u_j(t,x)-u_j^*)^2 \right) dx \\ &- \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{|c_{ij}|}{2}\int_\Omega\int_0^\infty f_j(s)(u_j(t-s,x)-u_j^*)^2 dsdx.\end{aligned}$$

$$\begin{aligned}\frac{dV}{dt} &\leq \sum_{i=1}^{m}\int_\Omega \left[b_{ii} + \varpi_i|c_{ii}| + \sum_{\substack{j=1\\j\neq i}}^{m}\left(\frac{|b_{ij}|}{2} + \varpi_j\frac{|c_{ij}|}{2}\right)\right](u_i(t,x)-u_i^*)^2 dx \\ &+ \sum_{i=1}^{m}\sum_{\substack{j=1\\j\neq i}}^{m}\int_\Omega\left(\frac{|b_{ij}|}{2} + \varpi_j\frac{|c_{ij}|}{2}\right)(u_j(t,x)-u_j^*)^2 dx.\end{aligned} \tag{4.30}$$

By exchanging the indices i and j, inequality (4.30) becomes

$$\frac{dV}{dt} \leq \sum_{i=1}^{m} \int_{\Omega} [b_{ii} + \varpi_i |c_{ii}| +] (u_i(t,x) - u_i^*)^2 dx$$
$$\sum_{i=1}^{m} \int_{\Omega} \sum_{\substack{j=1 \\ j\neq i}}^{m} \left(\frac{|b_{ij}|}{2} + \frac{|b_{ji}|}{2} + \varpi_j \frac{|c_{ij}|}{2} + \varpi_i \frac{|c_{ji}|}{2} \right) (u_i(t,x) - u_i^*)^2 dx \quad (4.31)$$
$$\leq 0.$$

Thus V is non-increasing on $[0, t_1)$ and $[t_k, t_{k+1})$ for $k = 1, 2, 3, ...$

Let $\psi_{ik}(z) = \frac{\exp\left(\frac{1}{k^2}\right) - 1}{z + u_i^*} \left[z - u_i^* - u_i^* \ln\left(\frac{z}{u_i^*}\right) \right] \geq 0.$

From the condition on β_{ik} in C6 we have

$$|\beta_{ik}(z)| \leq 1 - \exp(-\psi_{ik}(z)) \leq \psi_{ik}(z)$$
$$|\beta_{ik}(u_i(t_k^-, x))| (u_i(t_k^-, x) + u_i^*) \leq \left(\exp\left(\frac{1}{k^2}\right) - 1 \right) \times \left[u_i(t_k^-, x) - u_i^* - u_i^* \ln\left(\frac{u_i(t_k^-, x)}{u_i^*} \right) \right]. \quad (4.32)$$

For $\beta_{ik}(z) < 0$ we have $\beta_{ik}(z) \geq \exp(-\psi_{ik}(z)) - 1$ so that $-\ln(1 + \beta_{ik}(z)) \leq \psi_{ik}(z)$, and for $\beta_{ik}(z) \geq 0$, $\ln(1 + \beta_{ik}(z)) \leq \beta_{ik}(z) \leq \psi_{ik}(z)$. Thus

$$|\ln(1 + \beta_{ik}(u_i(t_k^-, x)))| (u_i(t_k^-, x) + u_i^*) \leq \left(\exp\left(\frac{1}{k^2}\right) - 1 \right) \left[u_i(t_k^-, x) - u_i^* - u_i^* \ln\left(\frac{u_i(t_k^-, x)}{u_i^*} \right) \right]. \quad (4.33)$$

Using these properties, we obtain the following results.

$$V(t_k, u_1(t_k, x), ..., u_m(t_k, x)) = \sum_{i=1}^{m} \int_{\Omega} u_i(t_k^-, x) \left(1 + \beta_{ik}(u_i(t_k^-, x))) - u_i^*\right) dx$$
$$- \sum_{i=1}^{m} \int_{\Omega} u_i(t_k^-, x) u_i^* \ln\left(\frac{u_i(t_k^-, x)(1 + \beta_{ik}(u_i(t_k^-, x)))}{u_i^*} \right) dx$$
$$+ \sum_{i=1}^{m} \int_{\Omega} \int_0^{\infty} f(s) \int_{t_k - s}^{t_k} (u_i(z, x) - u_i^*)^2 dz ds dx.$$

$$
\begin{aligned}
V(t_k, u_1(t_k,x),..., u_m(t_k,x))& \\
\le \sum_{i=1}^{m} \int_\Omega & \left(u_i(t_k^-,x) - u_i^* - u_i^* \ln\left(\frac{u_i(t_k^-,x)}{u_i^*}\right)\right) dx \\
&+ \sum_{i=1}^{m} \int_\Omega |\beta_{ik}(u_i(t_k^-,x))u_i(t_k^-,x) - u_i^* \ln(1+\beta_{ik}(u_i(t_k^-,x)))| dx \\
&+ \sum_{i=1}^{m} \int_\Omega \int_0^\infty f(s) \int_{t_k-s}^{t_k} (u_i(z,x)-u_i^*)^2 dz ds dx.
\end{aligned}
$$

Using the triangle inequality, (4.32) and (4.33) we have

$$
\begin{aligned}
V(t_k,& u_1(t_k,x), ..., u_m(t_k,x)) \\
&\le \sum_{i=1}^{m} \int_\Omega \left(u_i(t_k^-,x) - u_i^* - u_i^* \ln\left(\frac{u_i(t_k^-,x)}{u_i^*}\right)\right) dx \\
&\quad + \sum_{i=1}^{m} \int_\Omega |\beta_{ik}(u_i(t_k^-,x))u_i(t_k^-,x)| + |u_i^* \ln(1+\beta_{ik}(u_i(t_k^-,x)))| dx \\
&\quad + \sum_{i=1}^{m} \int_\Omega \int_0^\infty f(s) \int_{t_k-s}^{t_k} (u_i(z,x)-u_i^*)^2 dz ds dx \\
&\le \int_\Omega \sum_{i=1}^{m} \left(u_i(t_k^-,x) - u_i^* - u_i^* \ln\left(\frac{u_i(t_k^-,x)}{u_i^*}\right)\right) dx \\
&\quad + \int_\Omega \sum_{i=1}^{m} \left(\max\{|\beta_{ik}(u_i(t_k^-,x))|, |\ln(1+\beta_{ik}(u_i(t_k^-,x)))|\}(u_i(t_k^-,x)+u_i^*)\right) dx \\
&\quad + \int_\Omega \sum_{i=1}^{m} \int_0^\infty f(s) \int_{t_k-s}^{t_k} (u_i(z,x)-u_i^*)^2 dz ds dx.
\end{aligned}
$$

$$
V(t_k, u_1(t_k,x), ..., u_m(t_k,x)) \le V(t_k^-, u_1(t_k^-,x), ..., u_m(t_k^-,x)) \exp\left(\frac{1}{k^2}\right). \tag{4.34}
$$

By applying the relation in (4.34) sequentially we have a bound for V as follows:

$$
V(t, u_1(t,x), ..., u_m(t,x)) \le V(0, u_1(0,x), ..., u_m(0,x)) \prod_{k=1}^{p} \exp\left(\frac{1}{k^2}\right) \tag{4.35}
$$

for $t \in (t_p, t_{p+1}),\ p \ge 1$.

Let $K_i = b_{ii} + \varpi_i |c_{ii}| + \sum_{\substack{j=1 \\ j\neq i}}^{m} \left(\frac{|b_{ij}|}{2} + \frac{|b_{ji}|}{2} + \varpi_j \frac{|c_{ij}|}{2} + \varpi_i \frac{|c_{ji}|}{2} \right)$. By integrating both sides of (4.31) we obtain

$$\int_0^t \frac{dV}{ds}(s, u_1(s,x), ..., u_m(s,x))ds \leq \int_\Omega \int_0^t \sum_{i=1}^{m} \left(K_i (u_i(s,x) - u_i^*)^2 \right) dtdx. \tag{4.36}$$

By expanding the left hand side term of (4.36) we have

$$\begin{aligned}
&\int_0^t \frac{dV}{ds}(s, u_1(s,x), ..., u_m(s,x))ds \\
&\qquad = \int_{t_p}^t \frac{dV}{ds}(s, u_1(s,x), ..., u_m(s,x))ds \\
&\qquad\quad + \sum_{k=1}^{p} \int_{t_{k-1}}^{t_k} \frac{dV}{ds}(s, u_1(s,x), ..., u_m(s,x))ds, \qquad t \in (t_p, t_{p+1}] \\
&\qquad = V(t, u_1(t,x), ..., u_m(s,x)) - V(t_p, u_1(t_p,x), ..., u_m(t_p,x)) \\
&\qquad\quad + \sum_{k=1}^{p} V(t_k^-, u_1(t_k^-,x), ..., u_m(t_k^-,x)) \\
&\qquad\quad - \sum_{k=1}^{p} V(t_{k-1}, u_1(t_{k-1},x), ..., u_m(t_{k-1},x)).
\end{aligned}$$

By rearranging the sums we obtain

$$\begin{aligned}
&\int_0^t \frac{dV}{ds}(s, u_1(s,x), ..., u_m(s,x))ds \\
&\qquad = V(t, u_1(t,x), ..., u_m(t,x)) - V(0, u_1(0,x), ..., u_m(0,x)) \\
&\qquad\quad + \sum_{k=1}^{p} V(t_k^-, u_1(t_k^-,x), ..., u_m(t_k^-,x)) \\
&\qquad\quad - \sum_{k=1}^{p} V(t_k, u_1(t_k,x), ..., u_m(t_k,x)).
\end{aligned} \tag{4.37}$$

Since V is non-increasing on every interval $[t_k, t_{k+1}),\ k = 1, 2, ...$ and $[0, t_1)$,

$$\begin{aligned}
V(0, u_1(0,x), ..., u_m(0,x)) &+ \sum_{k=1}^{p} V(t_k, u_1(t_k,x), ..., u_m(t_k,x)) \\
&- \sum_{k=1}^{p} V(t_k^-, u_1(t_k^-,x), ..., u_m(t_k^-,x))
\end{aligned}$$

is non-negative.

From (4.32), (4.33), and (4.35) we have

$$
\begin{aligned}
&V(t_k, u_1(t_k, x), ..., u_m(t_k, x)) - V(t_k^-, u_1(t_k^-, x), ..., u_m(t_k^-, x)) \\
&\qquad = \sum_{i=1}^{m} (\beta_{ik}(u_i(t_k^-, x))u_i(t_k^-, x) - u_i^* \ln(1 + \beta_{ik}(u_i(t_k^-, x)))). \\
&|V(t_k, u_1(t_k, x), ..., u_m(t_k, x)) - V(t_k^-, u_1(t_k^-, x), ..., u_m(t_k^-, x))| \\
&\qquad = \sum_{i=1}^{m} |\beta_{ik}(u_i(t_k^-, x))u_i(t_k^-, x) - u_i^* \ln(1 + \beta_{ik}(u_i(t_k^-, x)))| \\
&\qquad \le \sum_{i=1}^{m} |\beta_{ik}(u_i(t_k^-, x))|u_i(t_k^-, x) + u_i^*| \ln(1 + \beta_{ik}(u_i(t_k^-, x)))| \\
&\qquad \le \left(\exp\left(\frac{1}{k^2}\right) - 1\right) V(t_k^-, u_1(t_k^-, x), ...u_m(t_k^-, x)) \\
&\qquad \le \left(\exp\left(\frac{1}{k^2}\right) - 1\right) V(0, u_1(0, x), ..., u_m(0, x)) \prod_{j=1}^{k-1} \exp\left(\frac{1}{j^2}\right).
\end{aligned}
$$

From the fact that

$$
\sum_{k=1}^{\infty} \big(V(t_k, u_1(t_k, x), ..., u_m(t_k, x)) - V(t_k^-, u_1(t_k^-, x), ..., u_m(t_k^-, x))\big)
$$

is convergent, we have

$$
\begin{aligned}
0 \le V(0, u_1(0, x), ..., u_m(0, x)) + \sum_{k=1}^{\infty} & V(t_k, u_1(t_k, x), ..., u_m(t_k, x)) \\
& - V(t_k^-, u_1(t_k^-, x), ..., u_m(t_k^-, x)) < \infty.
\end{aligned}
\tag{4.38}
$$

By using (4.36) and (4.37) we obtain the following relation:

$$
\begin{aligned}
V(t, u_1(t, x), ..., u_m(t, x)) - \int_\Omega \int_0^t \sum_{i=1}^{m} \big(K_i(u_i(s, x) - u_i^*)^2\big)\, dsdx \qquad& \\
\le V(0, u_1(0, x), ..., u_m(0, x))& \\
+ \sum_{k=1}^{p} V(t_k, u_1(t_k, x), ..., u_m(t_k, x))& \\
- \sum_{k=1}^{p} V(t_k^-, u(t_k^-, x), ..., u_(t_k^-, x)).&
\end{aligned}
\tag{4.39}
$$

From (4.38) the limit of the right hand side of (4.39) is finite and positive as $p \to \infty$. Since V is positive,

$$\int_\Omega \sum_{i=1}^{m} \int_0^t \left(-K_i(u_i(s,x) - u_i^*)^2\right) ds dx$$

is bounded independently from t. From the assumption $t_k \to \infty$ as $k \to \infty$ and the fact that the solution u is bounded, differentiable and uniformly continuous in every time interval (t_k, t_{k+1}), we have

$$\lim_{t\to\infty} (u_i(t,x) - u_i^*) = 0.$$

□

REMARK 4.1. *Conditions on the impulse function in Theorem 4.3 and Theorem 4.4 do not depend explicitly on the distance between two consecutive impulse times. The effect of the choice of impulse times is included in the size of jumps which depend on the difference between the value of the solution and u^*. Hence, if the distance between two consecutive impulse times is larger, then the solution will be closer to the equilibrium and larger jump can be tolerated.*

REMARK 4.2. *The linearized problem of IBVP (4.1)–(4.4) around* 0 *is given by*

$$\begin{aligned}
\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) &= au(t,x) && \text{in } \Omega\ , t \neq t_k \\
\frac{\partial u}{\partial \nu}(t,x) &= 0 && \text{on } \partial\Omega \\
u(0,x) &= u_o(0,x) && \text{in } \bar{\Omega} \\
u(t_k,x) &= I_k(u(t_k{}^-,x)) && \text{in } \bar{\Omega},\ k = 1,2,\ldots.
\end{aligned}$$

Conditions for the existence of zero attractor for this problem are the same as the conditions in Theorem 4.2.

The solution of the linearized problem of IBVP (4.1)–(4.4) around u^*

$$\frac{\partial}{\partial t}(u-u^*)(t,x) - \Delta(u-u^*)(t,x)$$
$$= -bu^*(u(t,x) - u^*) + u^* \int_0^\infty f(s)(u(t-s,x) - u^*) ds$$
$$\text{in } \Omega,\ t \neq t_k$$

$$\frac{\partial}{\partial \nu}(u-u^*)(t,x) = 0 \qquad\qquad on\ \partial\Omega$$

$$u(t,x) = u_o(t,x) \qquad\qquad in\ (-\infty,0]\times\bar{\Omega}$$

$$u(t_k,x) = I_k(u({t_k}^-,x)) \qquad\qquad in\ \bar{\Omega},\ k=1,2,...,$$

is attracted to u^* *if* $\int_0^\infty f(s)ds < b$ *and the impulse functions are in the following form*

$$I_k(z) = (1-\sigma_o)z + \sigma_o u^*$$

with $0<\sigma_o<1$. *This can be proved using Liapunov functional as follows*

$$V(t,u) = \int_\Omega (u(t,x)-u^*)^2 dx + u^* \int_\Omega \int_0^\infty f(s) \int_{t-s}^t (u(w,x)-u^*)^2 dz ds dx.$$

Let $\vartheta(t,x) = u(t,x)-u^*$ *and* $\varpi = \int_0^\infty f(s)ds$.
The derivative of the Liapunov functional is given by

$$\begin{aligned}
\frac{dV}{dt} =& \int_\Omega 2\vartheta(t,x)\frac{\partial\vartheta}{\partial t}dx + u^* \int_\Omega\int_0^\infty f(s)\left(\vartheta^2(t,x)-\vartheta^2(t-s,x)\right)dsdx\\
=& \int_\Omega 2\vartheta(t,x)\left(\Delta\vartheta(t,x) - bu^*\vartheta(t,x) + u^*\int_0^\infty f(s)\vartheta(t-s,x)ds\right)dx\\
&+ u^*\int_\Omega\int_0^\infty f(s)\left(\vartheta^2(t,x)-\vartheta^2(t-s,x)\right)dsdx\\
=& -2\int_\Omega |\nabla\vartheta(t,x)|^2dx - 2bu^*\int_\Omega \vartheta^2(t,x)dx\\
&+ 2u^*\int_\Omega\int_0^\infty f(s)\vartheta(t,x)\vartheta(t-s,x)dsdx\\
&+ \varpi u^*\int_\Omega \vartheta^2(t,x)dx - u^*\int_\Omega\int_0^\infty f(s)\vartheta^2(t-s,x)dsdx\\
\le& -2\int_\Omega |\nabla\vartheta(t,x)|^2dx + (\varpi-2b)u^*\int_\Omega\vartheta^2(t,x)dx\\
&+ u^*\int_\Omega\int_0^\infty f(s)\left(\vartheta^2(t,x)+\vartheta^2(t-s,x)\right)dsdx\\
&- u^*\int_\Omega\int_0^\infty f(s)\vartheta^2(t-s,x)dsdx\\
\le& 2(\varpi-b)u^*\int_\Omega\vartheta^2(t,x)dx\\
\le& 0.
\end{aligned}$$

Hence V *is decreasing on every interval* $[t_k,t_{k+1}),\ k=1,2,....$

When impulses occur at times t_k for $k = 1, 2, 3, ...,$ the Liapunov function will be in the following form:

$$\begin{aligned}
V(t_k, u) &= \int_\Omega (u(t_k, x) - u^*)^2 dx + u^* \int_\Omega \int_0^\infty f(s) \int_{t_k - s}^{t_k} (u(w, x) - u^*)^2 dw ds dx \\
&= \int_\Omega ((1 - \sigma_o) u(t_k^-, x) + \sigma_o u^* - u^*)^2 dx \\
&\quad + u^* \int_\Omega \int_0^\infty f(s) \int_{t_k - s}^{t_k} (u(w, x) - u^*)^2 dw ds dx \\
&= \int_\Omega (1 - \sigma_o)^2 (u(t_k^-, x) - u^*)^2 dx \\
&\quad + u^* \int_\Omega \int_0^\infty f(s) \int_{t_k - s}^{t_k} (u(w, x) - u^*)^2 dw ds dx.
\end{aligned}$$

Because $1 - \sigma_o < 1$, we have

$$V(t_k, u) < \int_\Omega (u(t_k^-, x) - u^*)^2 dx + u^* \int_\Omega \int_0^\infty f(s) \int_{t_k - s}^{t_k} (u(w, x) - u^*)^2 dw ds dx$$

$$V(t_k, u) < V(t_k^-, u).$$

Thus V is decreasing on interval $[0, \infty)$ and the solution of the linearized problem, $u(t, x)$, converges to the positive attractor, u^*.

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