A TOPOLOGICAL APPROACH TO SPECTRAL FLOW

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Contents

Declaration

I certify that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge and belief it does not contain any material previously published or written by another person except where due reference is made in the text.

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Abstract

It is a well-known result of T. Kato that given a continuous path of square matrices of a fixed dimension, the eigenvalues of the path can be chosen continuously. In this paper, we give an infinite-dimensional analogue of this result, which naturally arises in the context of unitary spectral flow. This provides a new approach to spectral flow, which seems to be missing from the literature. It is the purpose of this paper to fill in this gap.

1. Introduction

By "operators" we always mean bounded linear operators on a separable Hilbert space $\mathcal{H}$.

1.1. Motivation.

1.1.1. *T. Kato's finite-dimensional continuous enumeration.* The task of *continuous enumeration* is akin to tracking the individual movements of, for example, a swarm of bees. Our "bees" are utterly identical, they pass through one another, and they can make instant changes of direction infinitely many times per second (since we consider merely continuous paths), so that we cannot know which is which after a collision. However, it still seems intuitive that we should be able to assign (although not uniquely) a finite number of continuous functions which completely describe the movement of the "swarm".

Now, we give a rigorous formulation of finite-dimensional continuous enumeration due to T. Kato. The following exposition is directly taken from [Bha, §VI.1]. Let $\mathbb{C}^n_{\mathrm{sym}}$ be the quotient topological space obtained from $\mathbb{C}^n$ via the equivalence relation which identifies two n-tuples of complex numbers, if they are permutations of each other. That is, $\mathbb{C}^n_{\mathrm{sym}}$ can be viewed as the space of "unordered n-tuples" of complex numbers. Given an n-tuple $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, we denote its equivalence class in $\mathbb{C}^n_{\mathrm{sym}}$ by $(\lambda_1, \dots, \lambda_n)^*$. The topological space $\mathbb{C}^n_{\mathrm{sym}}$ thus defined inherits a metric

$$\operatorname{dist}((\lambda_1, \dots, \lambda_n)^*, (\mu_1, \dots, \mu_n)^*) := \min_{\pi} \max_{1 \le i \le n} |\lambda_i - \mu_{\pi_i}|,$$

where the minimum is taken over all permutations π. The following result is Kato's selection theorem ([Kat2, Theorem II.5.2]):

Theorem 1.1. *Let $\lambda(\cdot)$ be a continuous mapping from an interval I of $\mathbb{R}$ into the space $\mathbb{C}^n_{\mathrm{sym}}$. Then there exist n continuous complex-valued functions $\lambda_1(\cdot), \dots, \lambda_n(\cdot)$ on I, such that $\lambda(t) = (\lambda_1(t), \dots, \lambda_n(t))^*$ for all $t \in I$.*

As is typical, although seemingly obvious, an existence theorem of this kind is not altogether straightforward to prove. Furthermore, the following example shows that the domain I cannot be replaced by a general metric space:

Example 1.2. Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices of complex entries equipped with the ordinary uniform norm. In [Bha, §VI.1], it is proved that the mapping

$$M_n(\mathbb{C}) \ni A \longmapsto (\lambda_1(A), \dots, \lambda_n(A))^* \in \mathbb{C}^n_{\mathrm{sym}}, \tag{1}$$

where $\lambda_1(A), \dots, \lambda_n(A)$ are the eigenvalues of A repeated according to their algebraic multiplicities, is continuous. Let us consider the case $n = 2$, and set $A(z) := \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$ for all $z \in \mathbb{C}$. The mapping $A(\cdot)$ is continuous on any open subset I of $\mathbb{C}$ and the eigenvalues of $A(z)$ are $\pm z^{1/2}$. Continuity of the mapping (1) implies that $I \ni z \longmapsto (\lambda_1(A(z)), \lambda_2(A(z)))^* \in \mathbb{C}^n_{\mathrm{sym}}$ is continuous. However, if the domain I contains the origin, then this continuous mapping cannot be represented by constituent continuous functions.

Given a square matrix $A \in M_n(\mathbb{C})$, we may identify the spectrum $\sigma(A)$ of A with the unordered tuple as in (1). The following result is an immediate consequence of Theorem 1.1 and the continuity of the mapping $M_n(\mathbb{C}) \ni A \longmapsto \sigma(A) \in \mathbb{C}^n_{\mathrm{sym}}$:

Theorem 1.3 (Kato's finite-dimensional continuous enumeration)**.** *If $A(\cdot)$ is a continuous path of square complex matrices of a fixed dimension n, then there exist continuous paths $\lambda_1(\cdot), \dots, \lambda_n(\cdot)$ in $\mathbb{C}$, s.t. $\sigma(A(\cdot)) = (\lambda_1(\cdot), \dots, \lambda_n(\cdot))^*$.*

In this paper, we give a certain infinite-dimensional analogue of Kato's continuous enumeration of eigenvalues, which naturally arises in the context of the so-called *unitary spectral flow*. This provides a new approach to spectral flow, which seems to be missing from the literature. It is the purpose of this paper to fill in this gap.

1.1.2. *Self-adjoint Fredholm spectral flow.* The origin of spectral flow goes back to Atiyah-Patodi-Singer [APS]. Spectral flow has since found many connections, famously for example to the Fredholm index (see [RS]). Given a continuous one-parameter family $\{F(t)\}_{t\in[0,1]}$ of self-adjoint Fredholm operators, we naively understand the spectral flow of the continuous path F to be the number of eigenvalues of $F(t)$ that cross 0 rightward minus the number that cross 0 leftward as t monotonically increases from 0 to 1. The usual way of making this idea rigorous involves the notion of intersection number: we precisely define the spectral flow of the path F to be the intersection number of the graph $\bigcup_{t\in[0,1]}\sigma(F(t))$ with the line $\lambda=-\epsilon$, where ϵ is any sufficiently small positive number. Spectral flow turns out to be a homotopy invariant.

1.1.3. *Unitary spectral flow.* The notion of unitary spectral flow is discussed in [Pus]. Let $\mathfrak{U}_p(\mathcal{H},I)$ be the set of all unitary operators U such that $U-I$ is in the p-Schatten class $\mathfrak{S}_p(\mathcal{H})$ (see below for definition), where I denotes the identity operator. Throughout this paper, we let p be a fixed number in $[1,\infty]$. The collection $\mathfrak{U}_p(\mathcal{H},I)$ thus defined admits a natural complete metric

$$\mathrm{dist}\,(U,U') := \|U-U'\|_{\mathfrak{S}_p}, \quad \forall U,U'\in\mathfrak{U}_p(\mathcal{H},I),$$

where $\|\cdot\|_{\mathfrak{S}_p}$ is the norm on $\mathfrak{S}_p(\mathcal{H})$. It follows from Weyl's theorem on the stability of essential spectrum that the essential spectrum[1] $\sigma_{\mathrm{ess}}(U)$ of any unitary operator $U\in\mathfrak{U}_p(\mathcal{H},I)$ is $\{1\}$. We can then understand the spectral flow of a continuous path $U(\cdot)$ of unitary operators in $\mathfrak{U}_p(\mathcal{H},I)$ to be the integer-valued function $\mathrm{sf}\,(-;U):(0,2\pi)\to\mathbb{Z}$ given by

$$\begin{aligned}\mathrm{sf}\,(\theta;U) := &\langle\text{the number of eigenvalues of } U(t) \text{ that cross } e^{i\theta} \text{ anticlockwise}\rangle \\ &-\langle\text{the number of eigenvalues of } U(t) \text{ that cross } e^{i\theta} \text{ clockwise}\rangle\end{aligned} \tag{2}$$

as t monotonically increases from 0 to 1.

1.1.4. *Unitary spectral flow and spectral shift function.* In [Pus] the naive definition (2) is made precise and is used to express the spectral shift function (SSF) as the averaged spectral flow of a path of unitary operators. This path of unitary operators is obtained from the scattering matrix by analytic continuation of the spectral parameter (energy) into the complex plane: see [Pus, (4.9)] for details. Let us briefly recall the definition of SSF. If H,H_0 are two self-adjoint operators with a trace-class difference $H-H_0\in\mathfrak{S}_1(\mathcal{H})$, then the SSF $\xi(-;H,H_0)$ of this pair, introduced by [Lif] and [Kre] (see also [GM], [Yaf], [Sim1]), is a unique real-valued integrable function satisfying

$$\mathrm{Tr}\,(\phi(H)-\phi(H_0)) = \int_{\mathbb{R}}\phi'(\lambda)\xi(\lambda;H,H_0)d\lambda$$

for all compactly supported smooth functions ϕ on $\mathbb{R}$.

[1] Recall that given a normal operator N, the *discrete spectrum* $\sigma_{\mathrm{dis}}(N)$ is the set of all those eigenvalues of N which are isolated points of the spectrum $\sigma(N)$ and have finite multiplicities. The complement of the discrete spectrum in the spectrum is the *essential spectrum* $\sigma_{\mathrm{ess}}(N)$.

1.1.5. *Calculating unitary spectral flow via continuous enumeration.* Suppose for simplicity that $U(\cdot)$ is a loop in $\mathfrak{U}_p(\mathcal{H}, I)$ based at I. According to the naive definition (2), the spectral flow $\mathrm{sf}\,(-;U)$ in this case assumes some constant value $N \in \mathbb{Z}$ independent of the angle θ : the number N represents the net number of windings that the eigenvalues of $U(\cdot)$ make in the anti-clockwise direction. Perhaps, it should be possible to continuously enumerate the eigenvalues of $U(\cdot)$ as in the finite-dimensional setting. At this point, we recall the notion of extended enumeration due to Kato:

Definition 1.4. Given a normal operator N, a sequence $(\lambda_i)_{i\in\mathbb{N}}$ of complex numbers is called an *extended enumeration* of the discrete spectrum $\sigma_{\mathrm{dis}}(N)$, if $(\lambda_i)_{i\in\mathbb{N}}$ contains all eigenvalues of N in $\sigma_{\mathrm{dis}}(N)$ taking into account their multiplicities, and in addition, may contain some boundary points of the essential spectrum $\sigma_{\mathrm{ess}}(N)$ repeated arbitrarily often.

We propose the possibility of selecting a sequence $(\lambda_j(\cdot))_{j\in\mathbb{N}}$ of loops in $\mathbb{T}$ based at the boundary point 1 of the common essential spectrum, such that for each $t \in [0,1]$ the sequence $(\lambda_j(t))_{j\in\mathbb{N}}$ is an extended enumeration of $\sigma_{\mathrm{dis}}(U(t))$. It is necessary to consider extended enumerations by allowing λ_j's to take the boundary value 1. If such an enumeration is possible, an intuitive understanding of the number $\mathrm{sf}\,U := N$ would be the formal sum

$$\mathrm{sf}\,U = [\lambda_1]_{\pi_1} + [\lambda_2]_{\pi_1} + \dots, \tag{3}$$

where each $[\lambda_i]_{\pi_1}$ is the homotopy class in the fundamental group $\pi_1(\mathbb{T}, 1) \cong \mathbb{Z}$, representing the net number of windings that λ_i makes in the anti-clockwise direction.

1.2. **Infinite-dimensional continuous enumeration.** The infinite analogue of a finite unordered tuple is often called a *multiset*. Given a nonempty set X, a multiset in X is understood naively as a subset of X, whose elements can be repeated more than once. For instance, the multiset $\{x, x\}^*$ in X, where we are using * to distinguish multisets from ordinary subsets of X, is considered to be different from $\{x\}^*$ or $\{x, x, x\}^*$. Given any unitary operator $U \in \mathfrak{U}_p(\mathcal{H}, I)$, we may identify its spectrum $\sigma(U)$ with the following multiset in $\mathbb{T}$:

$$\sigma(U) = \sigma_{\mathrm{dis}}(U) \cup \{1\} \equiv \{z_1, z_2, z_3, \dots, 1, 1, 1, \dots\}^*, \tag{4}$$

where z_i's are the eigenvalues in $\sigma_{\mathrm{dis}}(U)$ taking multiplicities into account and 1's are repeated infinitely many times. The question which needs to be addressed next is the following: is there a natural topology in the set of multisets which makes the mapping $\mathfrak{U}_p(\mathcal{H}, I) \ni U \longmapsto \sigma(U)$ continuous? The answer is affirmative, and it is based upon the following estimates.

1.2.1. *The Hoffman-Wielandt inequality.* Hoffman-Wielandt proved the following well-known matrix inequality (see [Bha, Theorem VI.4.1] for details):

Theorem 1.5 (Hoffman-Wielandt). *If N, N' are two $n \times n$ normal matrices, then we can enumerate the eigenvalues of N, N' as $(\lambda_1, \dots, \lambda_n)$, $(\lambda'_1, \dots, \lambda'_n)$ respectively, so that*

$$\left[\sum_{i=1}^{n} |\lambda_i - \lambda'_i|^2\right]^{\frac{1}{2}} \le \|N - N'\|_{\mathfrak{S}_2}.$$

We are interested in infinite-dimensional analogues of the Hoffman-Wielandt inequality: given a pair N, N' of normal operators with $N - N' \in \mathfrak{S}_\Phi$, can we choose a pair $(\lambda_i), (\lambda'_i)$ of extended enumerations of the discrete spectra of N, N' respectively, such that

$$\left[\sum_{i=1}^{\infty} |\lambda_i - \lambda'_i|^p\right]^{\frac{1}{p}} \le C\,\|N - N'\|_{\mathfrak{S}_p}, \tag{5}$$

where C is a positive constant which does not depend on N, N'? Kato ([Kat1, Theorem II]) proved (5) under the assumption that N, N' are self-adjoint operators and $C = 1$. Kato's result was extended to unitary N, N' with $C = \pi/2$ by Bhatia-Sinha ([BS]). Bhatia-Davis ([BD, Corollary 2.3]) proved (5) under the assumption that $N, N', N-N'$ are normal operators and $C = 1$.

1.2.2. *Summable multisets.* Formally, a *multiset* in $\mathbb{T}$ is a mapping $S\colon \mathbb{T} \to \{0, 1, 2, \dots, \infty\}$, which assigns to each point $z \in \mathbb{T}$ a unique nonnegative integer or an infinity $S(z)$ which is understood as the *multiplicity* of the point z. A *countable multiset* in $(\mathbb{T}, 1)$ is a multiset S in $\mathbb{T}$ with the following properties:

1. The fixed point 1 is the only point having infinite multiplicity in S.
2. The *support* of S given by $\operatorname{supp} S := \{z \in \mathbb{T} \mid S(z) > 0\}$ is countable.

We shall make use of the trivial multiset $O_1 := \{1, 1, 1, \dots\}^*$. A sequence $(z_i)_{i\in\mathbb{N}}$ in $\mathbb{T}$ is called an *enumeration* of a countable multiset S, if it contains each point of $\mathbb{T}$ according to its multiplicity in S. Evidently, S admits a representation $S = \{z_1, z_2, \dots\}^*$. Given countable multisets $S = \{z_1, z_2, \dots\}^*$ and $T = \{w_1, w_2, \dots\}^*$ in $(\mathbb{T}, 1)$, we define their *p-distance* by

$$d_p(S, T) := \inf_{\pi} \left[\sum_{i=1}^{\infty} |z_i - w_{\pi_i}|^p \right]^{\frac{1}{p}}, \tag{6}$$

where the infimum is taken over all permutations π. A countable multiset S in $(\mathbb{T}, 1)$ is said to be *p-summable*, if $d_p(S, O_1) < \infty$. In this paper it is shown that the set of all p-summable multisets in $(\mathbb{T}, 1)$, denoted by $\mathcal{S}_p(\mathbb{T}, 1)$, forms a complete metric space with the metric d_p. In fact, we have chosen the metric d_p so that Bhatia-Sinha's result ([BS]) immediately implies:

1. The spectrum of each unitary operator $U \in \mathcal{U}_p(\mathcal{H}, I)$ can be viewed as a member of $\mathcal{S}_p(\mathbb{T}, 1)$ through (4). That is, $\sigma_{\text{dis}}(U)$ can be shown to be p-summable.
2. The mapping $\mathcal{U}_p(\mathcal{H}, I) \ni U \longmapsto \sigma(U) \in \mathcal{S}_p(\mathbb{T}, 1)$ is continuous.

Indeed, we have

$$d_p(\sigma(U), \sigma(U')) \le \frac{\pi}{2} \|U - U'\|_{\mathfrak{S}_p} \qquad \forall U, U' \in \mathcal{U}_p(\mathcal{H}, I), \tag{7}$$

and setting $U' := I$ ensures the p-summability of each $\sigma(U)$ since $\sigma(I) = \{1, 1, 1, \dots\}^*$.

1.2.3. *Continuous enumeration in the setting of unitary spectral flow.* In this paper, it is shown that any continuous path of the form $S : [0, 1] \to \mathcal{S}_p(\mathbb{T}, 1)$ admits a *continuous enumeration* $(\lambda_i(\cdot))_{i\in\mathbb{N}}$ in the sense that each λ_i is a continuous path in $\mathbb{T}$ with the property that for each $t \in [0, 1]$ the sequence $(\lambda_i(t))_{i\in\mathbb{N}}$ is an enumeration of the multiset $S(t)$. An immediate consequence of this result and (7) is the following unitary analogue of Kato's continuous enumeration:

Theorem 1.6. *Let $\mathcal{H}$ be a separable Hilbert space. If $U(\cdot)$ is a continuous path in $\mathcal{U}_p(\mathcal{H}, I)$, then there exists a sequence $(\lambda_j(\cdot))_{j\in\mathbb{N}}$ of continuous paths in $\mathbb{T}$, such that*

1. $\sigma(U(\cdot)) = \{\lambda_1(\cdot), \lambda_2(\cdot), \dots\}^*$.
2. $(\lambda_j(\cdot))_{j\in\mathbb{N}}$ *is an extended enumeration of* $\sigma_{\text{dis}}(U(\cdot))$ *pointwise.*

In fact, we obtain this result as a special case. More precisely, we generalise this setting by replacing the identity operator I by any fixed unitary operator U_0. Details are summarised below.

1.3. **Main results.**

1.3.1. *Generalisation to symmetric norms.* We have only considered the p-Schatten classes $\mathfrak{S}_p(\mathcal{H})$ so far, but they are only special types of the general Schatten-class $\mathfrak{S}_\Phi(\mathcal{H})$, where Φ is a so-called *symmetric norm* (see below for definition). In fact, the previously mentioned theorems by Bhatia-Shinha and Bhatia-Davis are concerned with symmetric norms:

Theorem 1.7 ([BS])**.** *Let $\mathcal{H}$ be a separable Hilbert space, and let Φ be a symmetric norm. For any pair U, U' of unitary operators on $\mathcal{H}$ with $U - U' \in \mathfrak{S}_\Phi(\mathcal{H})$, there exists a pair $(\lambda_i), (\lambda'_i)$ of extended enumerations of the discrete spectra of U, U' respectively, s.t.*

$$\Phi(|\lambda_1 - \lambda'_1|, |\lambda_2 - \lambda'_2|, \dots) \leq \frac{\pi}{2}\|U - U'\|_{\mathfrak{S}_\Phi}.$$

Theorem 1.8 ([BD, Corollary 2.3])**.** *Let $\mathcal{H}$ be a separable Hilbert space, and let Φ be a symmetric norm. For any pair N, N' of normal operators on $\mathcal{H}$ with $N - N'$ being normal Φ-Schatten class, there exists a pair $(\lambda_i), (\lambda'_i)$ of extended enumerations of the discrete spectra of U, U' respectively, s.t.*

$$\Phi(|\lambda_1 - \lambda'_1|, |\lambda_2 - \lambda'_2|, \dots) \leq \|N - N'\|_{\mathfrak{S}_\Phi}.$$

In this paper, we work with the general Schatten class $\mathfrak{S}_\Phi(\mathcal{H})$ for completeness.

1.3.2. *General multiset theory.* Sections §3-6 are devoted to general multiset theory about a metric space X and a fixed point $x_0 \in X$. Given a symmetric norm Φ, the definition of $\mathcal{S}_\Phi(X, x_0)$ requires the obvious modification (see §3.2 and §3.3 for details). As before, we make use of the multiset $O_{x_0} := \{x_0, x_0, x_0, \dots\}^*$. The following are our main results:

1. Theorem 3.7 asserts that $\mathcal{S}_\Phi(X, x_0)$ is metric space. In addition, it is shown in Theorem 3.20 that if X is complete and if Φ is a *regular symmetric norm* (see below for definition), then $\mathcal{S}_\Phi(X, x_0)$ is complete.
2. Theorem 5.1 asserts that any continuous path in $\mathcal{S}_\Phi(X, x_0)$ has a continuous enumeration.
3. In §6, under the assumption that Φ is a regular symmetric norm, and that X is a path-connected, locally simply connected metric space, we construct a group isomorphism

$$\Psi_\Phi : \pi_1(\mathcal{S}_\Phi(X, x_0), O_{x_0}) \simeq H_1(X), \tag{8}$$

   where $\pi_1(\mathcal{S}_\Phi(X, x_0), O_{x_0})$ is the fundamental group and $H_1(X)$ is the first singular homology group. The formal sum (3) is used to define Ψ_Φ.

1.3.3. *Infinite-dimensional analogues of Kato's continuous enumeration.* One of our main results is Theorem 1.6 with the identity operator I replaced by a fixed operator U_0. To state this result, we consider the metric space $\mathcal{U}_\Phi(\mathcal{H}, U_0)$ whose definition is obvious (see §7.2 for details). Since the essential spectrum $K := \sigma_{\text{ess}}(U_0)$ is no longer a point-set, we need to form the quotient space $\mathbb{T}/K = \{[z]_K\}_{z \in \mathbb{T}}$ via the equivalence relation which identifies points of K and leaves other points as they are. Let $\mathcal{K}$ denote the equivalence class represented by points of K. The topological space $X := \mathbb{T}/K$ is a metrizable space (see Theorem 7.1) with a fixed point $x_0 := \mathcal{K}$, and so we may consider

$$\mathcal{S}_\Phi(\mathbb{T}, K) := \mathcal{S}_\Phi(\mathbb{T}/K, \mathcal{K}).$$

As before, we can view the spectrum of each unitary operator $U \in \mathcal{U}_\Phi(\mathcal{H}, U_0)$ as a multiset in $\mathbb{T}/K$ through (39). With the notations introduced above in mind, Theorem 7.5 is our main theorem, which is an infinite-dimensional version of Kato's continuous enumeration. We also give an analogous result for self-adjoint perturbations (see Theorem 7.8).

1.3.4. *Unitary Spectral Flow.* In §8, we give an alternative approach to the unitary spectral flow. Note first that if we set $(X, x_0) := (\mathbb{T}, 1)$, then the isomorphism (8) is of the form

$$\Psi_\Phi : \pi_1(\mathcal{S}_\Phi(\mathbb{T},1), O_1) \simeq \mathbb{Z}.$$

If $U(\cdot)$ is a loop of unitary operators in $\mathcal{U}_\Phi(\mathbb{T},1)$ based at the identity operator I, then $\sigma(U(\cdot))$ is a loop in $\mathcal{S}_\Phi(\mathbb{T},1)$ based at O_1. We define the *spectral flow* of the path $U(\cdot)$ to be

$$\operatorname{sf} U := \Psi_\Phi([\sigma(U)]_{\pi_1}) \in \mathbb{Z},$$

where $[\,\cdot\,]_{\pi_1}$ denotes the homotopy class in $\pi_1(\mathcal{S}_\Phi(\mathbb{T},1), O_1)$. This definition is indeed consistent with the naive one (3), where the existence of each loop $\lambda_i(\cdot)$ in $(\mathbb{T},1)$ is asserted in Theorem 1.6.

2. Preliminaries

Here, we briefly recall standard facts about symmetric norms and Schatten class operators. Details can be found in [GK] and [Sim2].

2.1. **Symmetric norms.** Let c_0 be the set of all real-valued sequences converging to 0, and let c_{00} be the set of all real-valued sequences with a finite number of non-zero terms. Evidently, c_0 and c_{00} can be both viewed as vector spaces over $\mathbb{R}$.

Definition 2.1. A norm Φ on c_{00}, which assigns to each sequence $\xi = (\xi_i)_{i\in\mathbb{N}}$ in c_{00} a unique non-negative number $\Phi(\xi) = \Phi(\xi_1, \xi_2, \ldots)$, is called a *symmetric norm*, if the following conditions are satisfied:

1. $\Phi(1, 0, 0, \ldots) = 1$.
2. $\Phi(\xi_1, \xi_2, \ldots) = \Phi(|\xi_{\pi_1}|, |\xi_{\pi_2}|, \ldots)$ for any $\xi \in c_{00}$ and any permutation π.

Let Φ be a symmetric norm. A sequence $\xi \in c_0$ is said to be Φ-*summable*, if the limit

$$\Phi(\xi) := \lim_{n\to\infty} \Phi(\xi_1, \ldots, \xi_n, 0, 0, \ldots)$$

is finite. The vector space of all Φ-*summable* sequences, denoted by ℓ_Φ, is called the *natural domain* of the symmetric norm Φ. The pair (ℓ_Φ, Φ) turns out to be a Banach space (see [Sim2, Theorem 1.16 (d)]). The symmetric norm Φ is said to be *regular*, if

$$\lim_{n\to\infty} \Phi(\xi_{n+1}, \xi_{n+2}, \ldots) = 0 \iff \lim_{n\to\infty} (\xi_1, \ldots, \xi_n, 0, 0, \ldots) = \xi \qquad \forall \xi \in \ell_\Phi.$$

Let ℓ_Φ^+ be the set of all those sequences in ℓ_Φ with non-negative terms.

Example 2.2. Given a fixed number $p \in [1, \infty]$, we define the regular symmetric norm Φ_p by

$$\Phi_p(\xi) = \begin{cases} \left(\sum_{i=1}^\infty |\xi_i|^p\right)^{1/p}, & \text{if } p < \infty, \\ \sup_{i\in\mathbb{N}} |\xi_i|, & \text{if } p = \infty, \end{cases} \tag{9}$$

where $\xi \in c_{00}$. The natural domain $\ell_p := \ell_{\Phi_p}$ is known as the set of *p-summable sequences* in $\mathbb{R}$. Evidently, $\ell_\infty = c_0$. See [GK, §III. 7] for more details.

Given a sequence $\xi = (\xi_i)_{i\in\mathbb{N}}$ of non-negative terms in c_0, we define the sequence $\xi^\downarrow = (\xi_i^\downarrow)_{i\in\mathbb{N}}$ to be the non-increasing rearrangement of $\xi_1, \xi_2, \ldots$. That is, we define $\xi^\downarrow$ through

$$\xi_1^\downarrow = \max_{i\in\mathbb{N}} \xi_i, \;\; \xi_1^\downarrow + \xi_2^\downarrow = \max_{i\neq j} (\xi_i + \xi_j), \;\; \ldots$$

The non-increasing rearrangement of a finite sequence of non-negative terms can be defined analogously.

2.2. **Schatten class operators.** Let Φ be a symmetric norm, and let $\mathcal{H}$ be a separable Hilbert space. The *singular numbers* of a compact operator A on $\mathcal{H}$, denoted by $s_1(A), s_2(A), \ldots$, are the eigenvalues of the positive operator $|A| := \sqrt{A^*A}$, that are repeated according to their multiplicities and arranged in the non-increasing order. The operator A is said to be Φ-*summable*, if $(s_i(A))_{i\in\mathbb{N}} \in \ell_\Phi$: that is,

$$\|A\|_{\mathfrak{S}_\Phi} := \lim_{n\to\infty} \Phi(s_1(A), \ldots, s_n(A), 0, 0, \ldots) < \infty. \tag{10}$$

The set $\mathfrak{S}_\Phi(\mathcal{H})$ of all Φ-summable operators, known as the Φ-*Schatten class*, forms a Banach space with the norm (10). Details can be found in [GK, §III.4]. The p-*Schatten class* is the Banach space $\mathfrak{S}_p(\mathcal{H}) := \mathfrak{S}_{\Phi_p}(\mathcal{H})$.

2.3. **Majorisation and inequalities.** Here, we introduce the notion of majorisation which allows us to develop useful inequalities involving symmetric norms. Let $\mathbb{R}^n_+$ be the set of all finite sequences of length n whose terms are non-negative real numbers. Given two finite sequences $\xi, \eta \in \mathbb{R}^n_+$, we say that ξ is *weakly majorized* by η, written $\xi \prec_w \eta$, if

$$\sum_{j=1}^k \xi_j^\downarrow \leq \sum_{j=1}^k \eta_j^\downarrow \quad \forall k = 1, \ldots, n,$$

A norm Φ on $\mathbb{R}^n$ is referred to as a *finite symmetric norm*, if the two conditions specified in Definition 2.1 are satisfied. It is a well-known fact (see [Bha, Example II.3.13]) that a finite symmetric norm Φ on $\mathbb{R}^n$ respects weak majorization. That is,

$$\xi \prec_w \eta \Rightarrow \Phi(\xi) \leq \Phi(\eta) \qquad \forall \xi, \eta \in \mathbb{R}^n_+.$$

We will make use of the following obvious lemma throughout this subsection:

Lemma 2.3. *If Φ is a symmetric norm, then the following is a finite symmetric norm:*

$$\mathbb{R}^n \in (\xi_1, \ldots, \xi_n) \longmapsto \Phi(\xi_1, \ldots, \xi_n, 0, 0, \ldots) \in \mathbb{R}$$

To begin we consider the following standard facts (see [GK, §III.3] for details), which will be freely used throughout the paper without any further comment:

Lemma 2.4. *Let Φ be a symmetric norm, and let $\xi, \eta \in \ell_\Phi^+$:*

1. *$\Phi(\xi_1, \xi_2, \ldots) = \Phi(\xi_{\pi_1}, \xi_{\pi_2}, \ldots)$ for any permutation π. In particular, $\Phi(\xi) = \Phi(\xi^\downarrow)$.*
2. *If $\xi_i \leq \eta_i$ for each $i \in \mathbb{N}$, then $\Phi(\xi) \leq \Phi(\eta)$.*
3. *$\xi_1^\downarrow \leq \Phi(\xi) \leq \sum_{i=1}^\infty \xi_i^\downarrow$.*

Note that the last assertion implies $\ell_1 \subseteq \ell_\Phi \subseteq \ell_\infty$.

Proof. For the first assertion, observe that for each $n \in \mathbb{N}$ there exists a large enough index N_n, s.t. $\xi_{\pi_1}, \ldots, \xi_{\pi_n}$ is among $\xi_1, \ldots, \xi_{N_n}$. Since a finite symmetric norm respects weak majorisation, we have $\Phi(\xi_{\pi_1}, \ldots, \xi_{\pi_n}, 0, 0, \ldots) \leq \Phi(\xi_1, \ldots, \xi_{N_n}, 0, 0, \ldots)$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ establishes $\Phi(\xi_\pi) \leq \Phi(\xi)$. A similar argument shows $\Phi(\xi) \leq \Phi(\xi_\pi)$, and the firs assertion follows. The second assertion follows from $(\xi_1, \ldots, \xi_n) \prec_w (\eta_1, \ldots, \eta_n)$ for all $n \in \mathbb{N}$. The last assertion follows from $\Phi(\xi) = \Phi(\xi^\downarrow)$ and

$$(\xi_1^\downarrow, 0, \ldots, 0) \prec_w (\xi_1^\downarrow, \xi_2^\downarrow, \ldots, \xi_n^\downarrow) \prec_w \left(\sum_{i=1}^n \xi_i^\downarrow, 0, \ldots, 0\right) \qquad \forall n \in \mathbb{N}.$$

□

We will conclude this section by obtaining an infinite analogue of the following inequality:

Lemma 2.5. *For any finite symmetric norm* Φ *on* $\mathbb{R}^n$*, we have*

$$\Phi(|\xi_1^\downarrow - \eta_1^\downarrow|, \ldots, |\xi_n^\downarrow - \eta_n^\downarrow|) \leq \Phi(|\xi_1 - \eta_1|, \ldots, |\xi_n - \eta_n|) \qquad \forall \xi, \eta \in \mathbb{R}_+^n. \tag{11}$$

Proof. The claim follows from the following non-trivial majorization:

$$(|\xi_1^\downarrow - \eta_1^\downarrow|, \ldots, |\xi_n^\downarrow - \eta_n^\downarrow|) \prec_w (|\xi_1 - \eta_1|, \ldots, |\xi_n - \eta_n|) \qquad \forall \xi, \eta \in \mathbb{R}_+^n.$$

See [MO, Theorem 6.A.2.a] for details. □

We believe that an infinite analogue of this inequality must be a standard result, but were unable to find an appropriate reference. Here, we will present our own proof for which we do not claim the originality. We prove the following lemma first.

Lemma 2.6. *Let* Φ *be a regular symmetric norm, and let* $\xi \in \ell_\Phi^+$*. If we let* $\xi^{(n)} := (\xi_1, \ldots, \xi_n, 0, 0, \ldots)$ *for each* $n \in \mathbb{N}$*, then* $(\xi^{(n)})^\downarrow \to \xi^\downarrow$ *as* $n \to \infty$*.*

Note that $(\xi^{(n)})^\downarrow \neq (\xi^\downarrow)^{(n)}$ in general (otherwise this claim would be trivial).

Proof. Here, we consider the non-trivial case where ξ is a sequence with infinitely many non-zero terms. For each $n \in \mathbb{N}$, we set $\xi_{(n)} := (\xi_{n+1}, \xi_{n+2}, \ldots)$. It follows from the regularity of Φ that for any $\epsilon > 0$ there exists an index n_0 s.t. $\Phi(\xi_{(n_0)}) < \epsilon/2$ and $\Phi[(\xi^\downarrow)_{(n_0)}] < \epsilon/2$. Furthermore, there exists an index $N > n_0$ s.t. for all $n > N$ we have $\Phi(\xi_{(n)}) < \xi^\downarrow_{n_0+1}$. It follows that for all $n > N$ the numbers $\xi_{n+1}, \xi_{n+2}, \ldots$ are all strictly less than $\xi^\downarrow_{n_0}$: that is, the first n_0 terms of $\xi^\downarrow, (\xi^{(n)})^\downarrow$ are identical. For all $n > N$ we have

$$\begin{aligned}
\Phi(\xi^\downarrow - (\xi^{(n)})^\downarrow) &= \Phi(0, \ldots, 0, \xi^\downarrow_{n_0+1} - (\xi^{(n)})^\downarrow_{n_0+1}, \ldots) \\
&\leq \Phi[(\xi^\downarrow)_{(n_0)}] + \Phi[((\xi^{(n)})^\downarrow)_{(n_0)}] \\
&< \frac{\epsilon}{2} + \Phi[((\xi^{(n)})^\downarrow)_{(n_0)}].
\end{aligned}$$

It remains to prove $\Phi[((\xi^{(n)})^\downarrow)_{(n_0)}] < \epsilon/2$ for all $n > N$. Let $n > N$ be fixed. Then there exists a permutation π of $\{1, \ldots, n\}$ s.t. $\xi_{\pi_1} \geq \ldots \geq \xi_{\pi_n}$. It is easy to observe that

$$((\xi^{(n)})^\downarrow)_{(n_0)} = (\xi_{\pi_{n_0+1}}, \ldots, \xi_{\pi_n}, 0, 0, \ldots).$$

Since $\xi_{\pi_{n_0+1}}, \ldots, \xi_{\pi_n}$ are the smallest $n - n_0$ terms of ξ^n, we have

$$\begin{aligned}
\Phi[((\xi^{(n)})^\downarrow)_{(n_0)}] &= \Phi(\xi_{\pi_{n_0+1}}, \ldots, \xi_{\pi_n}, 0, 0, \ldots) \\
&\leq \Phi(\xi_{n_0+1}, \ldots, \xi_n, 0, 0, \ldots) \\
&\leq \Phi(\xi_{n_0+1}, \ldots, \xi_n, \xi_{n+1}, \xi_{n+2}, \ldots) \\
&= \Phi(\xi_{(n_0)}) < \frac{\epsilon}{2}.
\end{aligned}$$

The proof is complete. □

We are now in a position to prove the following result:

Theorem 2.7. *Given a regular symmetric norm* Φ*, we have*

$$\Phi(|\xi_1^\downarrow - \eta_1^\downarrow|, |\xi_2^\downarrow - \eta_2^\downarrow|, \ldots) \leq \Phi(|\xi_1 - \eta_1|, |\xi_2 - \eta_2|, \ldots) \qquad \forall \xi, \eta \in \ell_\Phi^+. \tag{12}$$

That is, $\ell_\Phi^+ \ni \xi \longmapsto \xi^\downarrow \in \ell_\Phi$ *is* 1*-Lipschitz continuous.*

Proof. Let Φ be a regular symmetric norm. It follows from Inequality (11) that for any $\xi, \eta \in \ell_\Phi^+$

$$\Phi(|(\xi^{(n)})^\downarrow - (\eta^{(n)})^\downarrow|) \leq \Phi(|\xi^{(n)} - \eta^{(n)}|) \qquad \forall n \in \mathbb{N}.$$

By Lemma 2.6, taking the limit as $n \to \infty$ completes the proof. □

3. Summable Multisets

3.1. **Countable multisets.** Let X be a nonempty set with a fixed point $x_0 \in X$. A *multiset* in X is understood naively as a subset of X, whose elements can be repeated more than once. For instance, the multiset $\{x, x\}^*$, where we use notation $\{\dots\}^*$ to distinguish it from ordinary subsets of X, is considered to be different from $\{x\}^*$. We shall make use of the following multiset throughout the paper:

$$O_{x_0} := \{x_0, x_0, x_0, \dots\}^*,$$

where x_0 is repeated infinitely many times. Formally, we define a multiset in X to be any mapping $S : X \to \{0, 1, 2, \dots, \infty\}$ assigning to each point $x \in X$ a unique non-negative integer of infinity, $S(x)$, which is understood as the multiplicity of x in S.

Definition 3.1. A *countable multiset in* (X, x_0) is a multiset S in X s.t.

1. The fixed point x_0 is the only point in S having the infinite multiplicity.
2. The *support* of S, defined by $\operatorname{supp} S := \{x \in X \mid S(x) > 0\}$, is a countable subset of X.

Evidently, O_{x_0} is a trivial example of a countable multiset in (X, x_0). Throughout this paper, we will only consider multisets of this kind, and freely make use of the following convention without any further comment. Given a finite or infinite sequence $(s_1, s_2, \dots)$ in X, we assume that the multiset $\{s_1, s_2, \dots\}^*$ contains the fixed point x_0 infinitely many times, so that it can always be viewed as a countable multiset in (X, x_0).

Example 3.2. With the above convention in mind, the correct interpretation of the multiset $S := \{x_1, x_1\}^*$, where $x_1 \neq x_0$, is the mapping $S : X \to \{0, 1, 2, \dots, \infty\}$ given by $S(x_1) = 2, S(x_0) = \infty$, and $S(x) = 0$ whenever $x \neq x_0$ and $x \neq x_1$.

Let us introduce the following terminology:

Definition 3.3. Let S be a countable multiset in (X, x_0):

1. A sequence $(s_i)_{i \in \mathbb{N}}$ is called an *enumeration* of S, if the representation $S = \{s_1, s_2, s_3, \dots\}^*$ holds. If the enumeration $(s_i)_{i \in \mathbb{N}}$ contains the fixed point x_0 infinitely many times, it is called a *proper enumeration* of S.
2. The *rank* of S, denoted by $\operatorname{rank} S$, is the sum of the multiplicities of all points in $\operatorname{supp} S$ except the fixed point x_0.

Remark 3.4. Let S be a countable multiset in (X, x_0). Any two proper enumerations of S are identical up to a permutation. Furthermore, given an enumeration $(s_i)_{i \in \mathbb{N}}$ of S, the sequence $(s_1, x_0, s_2, x_0, \dots)$ is a proper enumeration of S.

Given two countable multisets S, T in (X, x_0), we agree to write $T \leq S$ if $T(x) \leq S(x)$ for all $x \in X$. We define the *sum* $S + T$, and *difference* $S - T$ in case $T \leq S$, by

$$(S \pm T)(x) = \begin{cases} \infty, & \text{if } x = x_0, \\ S(x) \pm T(x), & \text{otherwise.} \end{cases}$$