Mathematics of the Crystal Model

By

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Declaration

I declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree. Except where it states otherwise by reference or acknowledgment, the work presented is entirely my own.

Hatem Alwardi

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Summary

In this thesis we investigate a new geometrical model of elementary particles derived in a unified way from the fundamental electromagnetic and spin fields. In basic physical phenomenology it thus has the characteristic features of a compound-particle model, but unlike most of the existing theories, dynamical treatment of the interaction makes up an essential part of the model. The scheme is motivated by the observation of an interesting analogy between the properties of particles. Maxwell's equations, together with the Lorentz force law, form the foundation of classical electromagnetism, classical optics, and electric circuits. Departing from this point we establish a steady-state model to describe most experimentally established subatomic particles, but also enigmas such as dark matter and the Higgs boson. In addition, we use a fixed crystalline structure as a scaffold for the elements' atoms including their electrons. This novel crystal theory aims to explain quantum mechanics and phenomena such as dark matter and teleportation, basing it on the known properties of atoms, their electronic shells, and nuclei. The crystalline symmetries, rotations, reflections, translations, and changes of colour are calculated and explained. They relate to transitions in the physics, such as matter to anti-matter, protons to neutrons, magnetic to electric, or invariant properties related to time and motion. The classical "standard model" concepts or reasoning via fields or forces, or less intuitive classes of particles such as quarks and gluons are replaced by a kind of simplified structure in a fixed "Parmenidean" crystal comprised of congruent tetrahedra in colours red, yellow, and black which represent axiomatically the positive, negative, and neutral charges. Basic physical ideas such as Pauli's exclusion principle, Hund's rule in atomic chemistry, the rules for sizes of electron shells and the need for excess neutrons in many stable nuclides are given reasons within the crystal model. Finally, a theorem about the spin of a particle is proved, that it can be calculated from the surface properties.

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1. Introduction

There are many observable effects in the quantum world which cannot be understood within the framework of classical mechanics or classical electrodynamics. Two characteristic examples show very clearly that the description within classical physics is incomplete and must be supplemented by some fundamental principles: the quantisation of atomic bound states for an electron in the field of a positive point charge, and the electromagnetic radiation emitted by an electron bound in an atom which, in a purely classical framework, would render atomic quantum states unstable. However, at this conjuncture quantum physics needs a very cogent model where atoms can be formed, and particles can interact with all the important properties being evident to us. Planck with Heisenberg, Bohr, Davisson, De Broglie, Einstein, and Schrödinger throughout history tried to discover the shape of the atoms, yet there are very ambiguous notations and assumptions hindering quantum physics to reach a way of describing elements, their atoms and their nuclei without contradictions nor leaps of imagination [1]. Despite that many experiments in the early 19^{th} century until now showed us many useful discoveries such as electricity and magnetism, followed by the realisation of spin and wave equations, yet the full understanding of atoms, particles, and their properties, especially the particles' actual interactions have been absent. Here, in a geometrical way, it is possible to develop a model that combines many discovered quantum properties of matter, as well as ideas about facts that are not definitive, such as dark matter and teleportation. Thus, we strive toward a foundation for the most valuable concepts of quantum physics and basic theories by relating it to a certain mathematical geometry, a crystalline structure or honeycomb with a special colouring of its tetrahedra scaffolded by the body-centred cubic lattice, a well-known regular kind of packing found in 3-dimensional space that is common in nature. The colours RBY in the crystal correspond to the charges of particles. It is known that the particles have certain quantum mechanical spin. Both charge and spin are additive

so that they can be calculated from the sum of the charges and spins of all the component particles [3]. Using the crystal, we develop this further and in particular prove a spin theorem that calculates the spin of a particle from the spins of triangles on its surface.

2. Basic Tetrahedron

A fundamental geometrical shape used in this thesis is one of the famous polyhedra, that is a three-dimensional solid with four flat triangular faces, called the tetrahedron. We start out to create a special kind of tetrahedron that can be put together with others, joining at common faces. We make all the faces congruent to a special kind of isosceles triangle. The tetrahedron has two long separated edges of length 2, being orthogonal in 3-d space to each-other, joined with four short edges of length $\sqrt{3}$. The vertices of the tetrahedra are all to be found in a certain body-centred cubic lattice (BCC), so that the eight nearest vertices from a fixed vertex form cube in space. The BCC is the union of two cubic lattices C_1 and C_2, where a vertex of C_2 is the central point of a cube of C_1 and vice-versa. Therefore, each tetrahedron has one long edge from C_1 and one long edge from C_2. Another property that follows is that there are 24 tetrahedra attached at each vertex of the BCC lattice. The 24 faces of these tetrahedra away from the central point form a rhombic dodecahedron, with the 24 triangles of the tetrahedra being paired into 12 rhombuses, which has one diagonal long edge and four short external edges. The tetrahedra themselves fill space with no holes forming what is commonly called a tetrahedral “honeycomb”. This can also be extracted from a rhombic dodecahedral honeycomb by subdividing each solid dodecahedron into 24 tetrahedra through the central point of each solid.

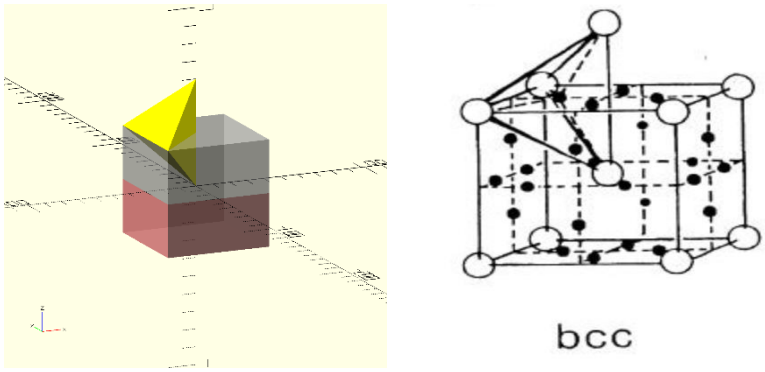

Figure 1 The BCC containing the tetrahedron drawn by OpenSCAD software.

It is important to note that here we only describe particles and single atoms but not molecules, for the molecules do not constitute a single entity expressing what is inside and outside the nucleus, and what is in the electron shells. Colours of tetrahedra in the crystal are the way to differentiate between the basic constituents of particles. R as red (positive charge), Y as yellow (negative charge), and B as black (neutral) have been chosen. In common research, atoms are based on protons, neutrons, and electrons, but evidently the proton and neutron are not elementary particles, so we must go deeper and utilise dark matter B and the anti-particle of electron Y named positron R [2]. Therefore, we will choose R, Y and B tetrahedra as the elementary particles of this nominated study to form a proper quantum state for each atom, which is a nucleus and electron cloud combined.

3. Numbered Vertices, Spin and Colouring

A rotation in 3-d space describes the circular movement of a particle around an axis. There is also the angular momentum carried by elementary particles, composite particles, and atomic nuclei which is called spin [3]. However, with quantum spin, we think not of actual movement but of a state or property related to orientation. A fixed labelling 0, 1, 2, 3 of the BCC's vertices induces that spin of any tetrahedron 0, 1, 2, 3 within it to obtain a spin which is either even (+ ½) or odd (- ½). With the following labelling of the BCC the important cubic sub-

lattices are the 0, 2 vertices of C_1 and the 1, 3 vertices of C_2. With cartesian coordinates (a, b, c) for points in 3-d space, C_1 is the set {(a, b, c) | a, b, c in $2\mathbb{Z}$ (even integers)}, and C_2 = {(a, b, c) | a, b, c in $2\mathbb{Z}+1$ (odd integers)}.

Definition 3.1: The labelling of BCC is:

0: $a + b + c \equiv 0 \pmod 4$; has all a, b, c even, e.g (0, 2, 2) has label 0.

1: $a + b + c \equiv 1 \pmod 4$; has all a, b, c odd, e.g. (-1, 3, 3) has label 1.

2: $a + b + c \equiv 2 \pmod 4$; has all a, b, c even, e.g. (2, -2, 2) has label 2.

3: $a + b + c \equiv 3 \pmod 4$; has all a, b, c odd, e.g. (3, -5, 1) has label 3.

There are two related coordinate systems that are used to describe positions in the crystal. The starting system has the (a, b, c) coordinates of the BCC lattice: that is, all the a, b, c are even integers, or all are odd. But this does not give the horizontal planes of the shells (layers of the AGP's). To achieve this, we define an (x, y, z) coordinate system that is given by following transformation:
$(x, y, z) = (\frac{a-c}{2}, \frac{b}{2}, \frac{a+c}{2})$, with the a, b, c in the BCC's three orthogonal directions.
Thus,
x represents the magnetism or what been called spin motion;
y represents the time direction while time goes in a positive way;
z represents the energy, and this increases up continuously as it represents the shells (more details in Chapter 8).

Theorem 3.2: The vertices of each fundamental tetrahedron has a labelling 0, 1, 2, 3 induced by the BCC.

Proof: The distances between the vertices of a tetrahedron are either $\sqrt{3}$ or 2, but the distance between vertices with the same label in the BCC is at least $2\sqrt{2}$, since

the minimal differences are (2, 2, 0), (2, -2, 0) and so on. Thus, the four vertices of the tetrahedron are labelled differently.

Definition 3.3: The property of having even or odd spin is given in the following Fig. 2 and depends upon the orientation in space of the labelled tetrahedron.

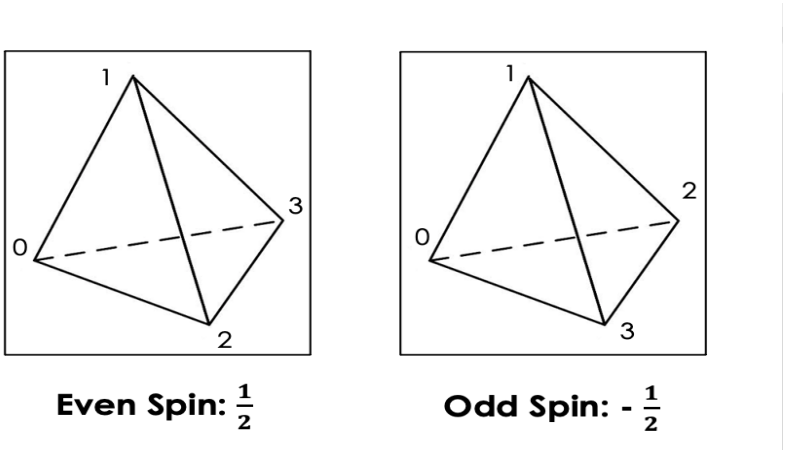

Figure 2 Two tetrahedra extracted from the labelling of the BCC to indicate the odd and even spins.

Note that the orientation of any face of the tetrahedron determines its spin. E.g., the face 1,2,3 is orientated from the outside either clockwise for odd spin or anticlockwise for even spin.

Theorem 3.4:

1. Mirror images of a tetrahedron have opposite spin.
2. Tetrahedra that join at a common face have opposite spin.
3. Tetrahedra that are translated or rotated via a mapping such as (x, y, z) to (x, y, z) + (u, v, w) have the same spin.

Proof: A mirror image reverses orientation and therefore reverses spin. Two tetrahedra with a common face are mirror images through that face. Translations and rotations preserve orientation and therefore spin.

Definition 3.5: The connection graph of the tetrahedra is the graph with vertices which are the tetrahedra and has two of these joined by an edge whenever the tetrahedra have a common triangular face.

Theorem 3.6: The connection graph is bipartite, with the bipartition of the vertices being induced by the spin.
Proof: From Theorem 3.4 (2) adjoining tetrahedra have opposite spin.

Note that in graph theory, a bipartite graph is one that has a 2-colouring, or equivalently, every cycle in the graph has an even number of vertices (or edges). We can see that in the connection graph there are minimal cycles (circuits) of size 4 and 6. These are obtained by rotations about the long and short edges respectively, by 90 degrees in the long case, by 60 degrees in the short case.

In addition to a 2-colouring, the connection graph has an essentially unique (regular) 3-colouring (up to the 6 permutations of the 3 colours).

Definition 3.7: A regular 3-colouring of the crystal into R, Y, B tetrahedra satisfies the following properties:

1. It must be a way of assigning the three colours to the tetrahedra of the crystal so that no two tetrahedra of the same colour have a common face.
2. The colouring of the four tetrahedra around a long edge alternate in two of the colours RYRY, RBRB or YBYB around that edge.
3. The colouring of the six tetrahedra around a short edge goes either in the sequence RBYRBY or RYBRYB depending on the rotation clockwise or anticlockwise.

Theorem 3.8: The crystal has a regular 3-colouring (which we consider fixed from now on).

Proof: Consider a tetrahedron, colour it Y. There are four adjoining tetrahedra paired into those that have a common long edge. These pairs must have the same colour from Def. 3.7 (2). Hence, we may assume that we colour these pairs with the remaining two colours. Once we have done this the rotations about the short edges given by Def. 3.7 (3) will determine the entire colouring of the crystal uniquely, e.g. by first colouring the 24 tetrahedra of a rhombic dodecahedron (Chapter 2) and then translating this into the adjoining position of the honeycomb.

4. Atomic Graph Paper

After the tetrahedron in question was described in the model used in our study and the numbering of the tetrahedron's vertices was considered, here we must go deeper and explain how the model works to illustrate the physical and chemical properties accurately.

The main rules described in the atomic graph paper are:

1- A shell of any atom is given by one piece of graph paper.
2- Subshells s, p, d, f, g, and h where each subshell contains a number of electron pairs (more details in Chapter 9).
3- Orbitals describe the location and wave-like behaviour of an electron in an atom.
4- Two directions on the paper indicate time direction y (top left to bottom right) and spin direction x (bottom left to top right).
5- The subatomic particles that make up every atom with calculated spin, charge, and parity.
6- The tubes (separated by the blank squares) in the y direction alternate in magnetic spin.
7- The subshells alternate in parity as they go out, starting with positive parity in the central s-subshell.

4.1 AGP1 for odd shells

Bohr model shells developed in the old quantum theory by Arnold Sommerfeld modified the orbits to have quantum numbers. Then subshells appeared to give the electron subshells named sharp, principle, diffuse, fundamental, and g as the 5th shell and higher (theoretically) [4].

In the crystal theory, time and spin are the two dimensions corresponding to the atomic shells and their subshells. We draw on a graph paper the three mentioned coloured particles R, Y, B to demonstrate the main atomic particles, proton, neutron, and electron [2]. Squares labelled R, Y or B represent coloured tetrahedra in a certain horizontal layer of the crystal.

Y-electrons in protons are also labelled by integers starting from zero to indicate the subshell location in the atomic graph paper as indicated in Fig. 3. The protons are formed by the joined straight squares YBR in the APG. They are coloured orange in the AGP1, blue (or aquamarine) in the even AGP2. In the first shell and middle red s-subshell square, we get the proton as the simplest atom given by straight YBR labelled by zero as a starting point of the first free electron Y_0. The neutron next to the proton is the bent YBR formed by a 90° rotation of the proton in the crystal. Using the three particles that form every atom, proton, and neutron we can build the shells, see their subshells and their orbitals. We must consider the odd shells and the even shells based on the quantum number n that will be explained later in Chapter 10 in more detail, so we begin with the odd shell as the lowest energy level of every atom, adding layers of shells in a vertical direction to this frozen core.

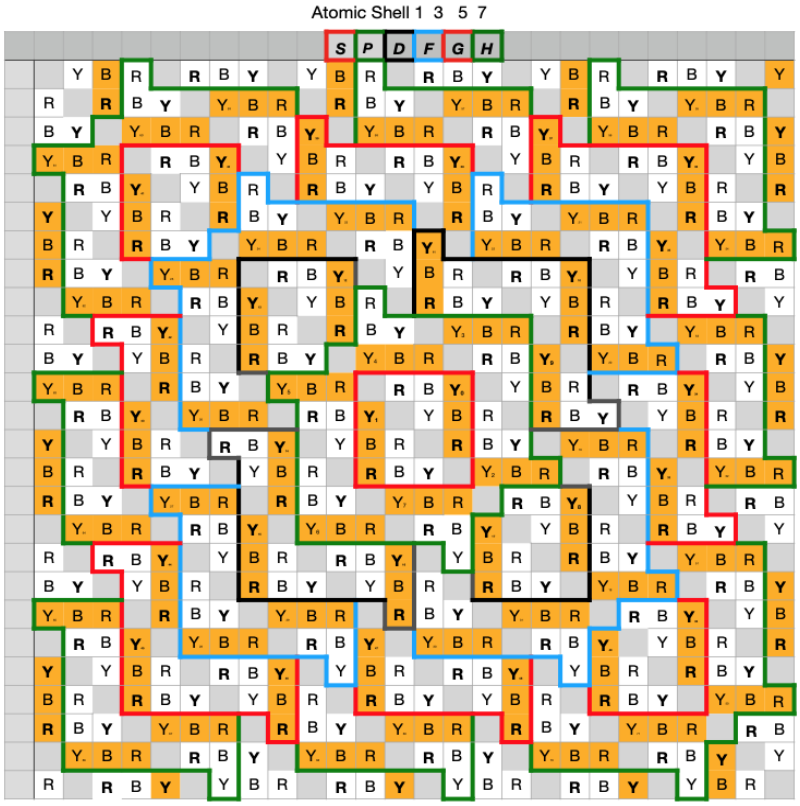

Figure 3 the atomic graph paper of odd shells starting from 1.

4.2 AGP2 for even shells

In the even shells, protons and neutrons take slightly different positions to the odd shells, yet the interactions between them are preserved. This second layer (not coloured) in the crystal is a reflection in the horizontal direction of the first layer. More precisely, it is a translation of the first layer by one step in the time direction. After that, the shells alternate appropriately using the AGP1 or AGP2. Thus, any atom or nuclide (the nucleus of an atom) can be described using a number of AGP's, one for each electron shell.

The facial connections between consecutive shells only go through a bold **R** or **Y** connected to a plain Y or R (resp.) directly above it. Thus "boldness" of an R or Y character in the AGP denotes a horizontal flat upper surface face of the coloured tetrahedron, while a non-bold (plain) Y or R has a horizontal lower face.

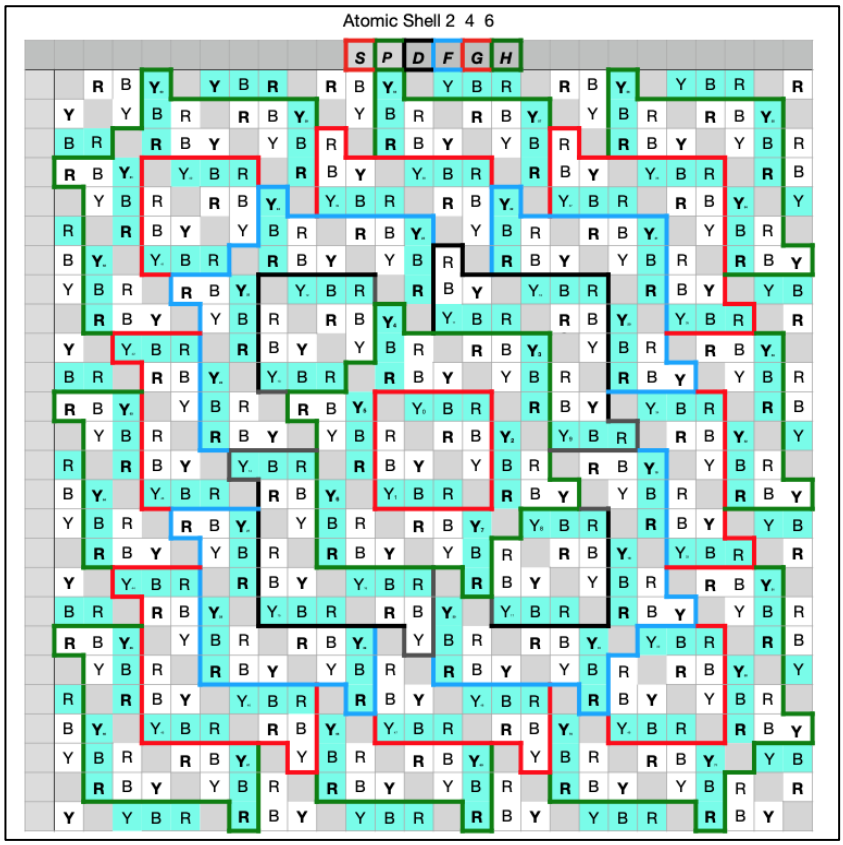

Figure 4 the atomic graph paper of even shells starting from 2.

4.3 Left and right-handed particles

Handedness or chirality is a property of orientation of particles that describes the relationship of a mirror image to the particle itself. In classical physics, if the helicity of a particle's spin is a positive direction, the spin direction is defined to be the same as its motion, and it is called a right-handed. If the direction and the motion of the particle are opposite, then the particle is called left-handed. The importance of handedness is to determine whether the particle transformation is a right or left representation. By Figure 2, the positive and negative orientations of a tetrahedron are indicating the particle's spin-direction from a basic (static) property of the crystal, as opposed to a definition from movement. The mirror image changes the matter but not the mass to become an anti-particle. For instance, the mirror image of an electron Y is an anti-electron R called "positron". A reflection also takes positive time to negative time, and the positive charge to

the negative charge. The general rule says that any left-handed particle can have a reflection or a mirror image that is a right-handed particle and vice versa.

In the crystal the particle to anti-particle operation corresponds to taking mirror images and switching the red to yellow colours. The left-handedness of most known particles is taken to account by the AGP. For example, an RB adjacent pair can be completed in two ways to a straight or a bent YBR. The straight protons and the bent neutrons in the AGP turn out to be all left-handed (related by a translation/rotation in the crystal), while the bent and straight counterparts are right-handed, and we don't normally use them in the AGP.

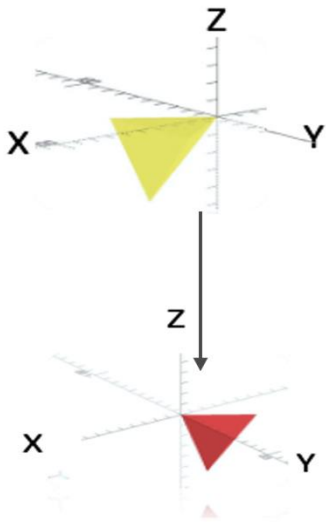

Figure 5 The reflection (mirror image) of an electron to a positron drawn with OpenSCAD.

5. Connections Described Using Graph Paper

One of the more important concepts that must be addressed is the one that holds the cohesion (or binding) force between particles within the crystal model. What was clarified in the previous are the basic concepts through which we can understand the nature of each particle by colouring the main three tetrahedra, but in the mathematical perspective there are links between geometric shapes

centered through the point connections between the connected tetrahedra or longitudinal sides of equal congruent edges or even more strongly through adjoining congruent faces. The different kinds of connections here can induce us to define in graph theory the specific physical forms that can allow the investigation of binding forces inside or between particles.

5.1 Horizontal connection in the same layer

In the (face) connection graph G any two tetrahedra of the crystal with adjoining common faces combine by joining of isosceles triangles. This graph represents the connections between the coloured particles. In particular, the horizontal connections are within the chemical shell or crystal layer, where horizontal is defined by the same value of z in the (x, y, z) coordinate system specified in Section 3.

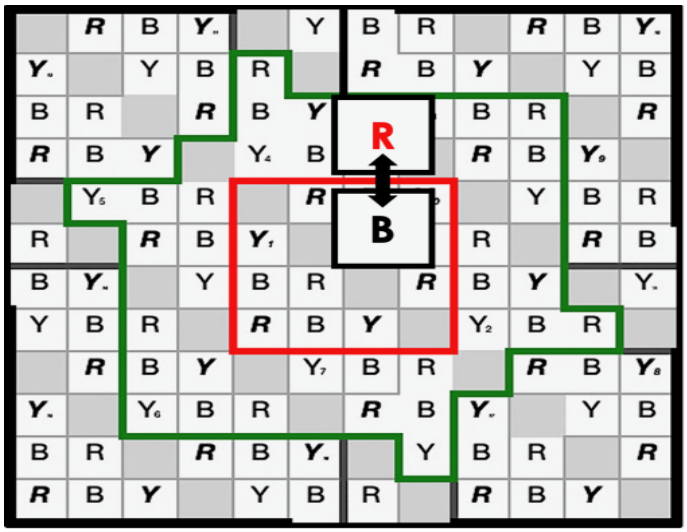

Figure 6 Horizontal connection with R and B in the same shell between subshells S and P.

5.2 Corner connection in graph paper

These face connections between tetrahedra are represented in the AGP by squares R and Y that are connected in a single corner diagonally, bottom left to top right. The corresponding electrons and positrons will have opposite spin as they are in different tubes going top left to bottom right in the AGP.

5.3 Vertical connection

The contact between shells is represented by the vertical connections between faces, and this concerns the change in energy of the shells around the nucleus. A bold **R** or **Y** in a shell is joined to the plain Y or R in the shell above it. Associated with this we notice the connection of one proton with a neutron from the same sub-shell directly above or below it. In this way protons are kept apart from one-another which would be expected as they have a positive charge which repels. See Fig. 7.

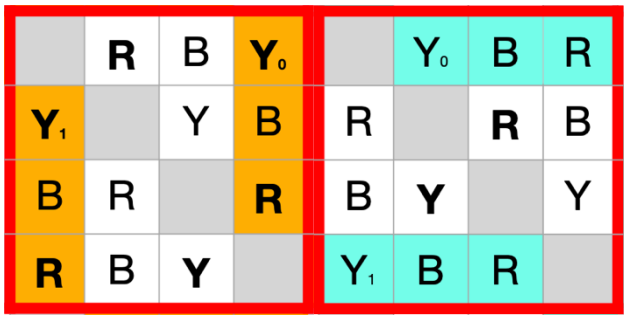

Figure 7 Shells 1 and 2 with s-subshells directly above one-another illustrate the straight proton positions relative to the bent neutrons.

5.4 Other possible connections between tetrahedra

Further connections here are based on the joining of a point or two points or three between the different tetrahedra. There are many cases that are possible:

- Connection by faces: $R \overset{3}{\leftrightarrow} Y$, $B \overset{3}{\leftrightarrow} Y$, and $R \overset{3}{\leftrightarrow} B$.
- Connection by edges: $R \overset{2}{\leftrightarrow} R$, $B \overset{2}{\leftrightarrow} B$, $Y \overset{2}{\leftrightarrow} Y$, $R \overset{2}{\leftrightarrow} Y$, $B \overset{2}{\leftrightarrow} Y$, and $R \overset{2}{\leftrightarrow} B$.
- Connection by points: $R \overset{1}{\leftrightarrow} R$, $B \overset{1}{\leftrightarrow} B$, $Y \overset{1}{\leftrightarrow} Y$, $R \overset{1}{\leftrightarrow} Y$, $B \overset{1}{\leftrightarrow} Y$, and $R \overset{1}{\leftrightarrow} B$.

These possibilities maybe be important for further investigations into binding energy and the origin of mass in particles. See Fig. 8.

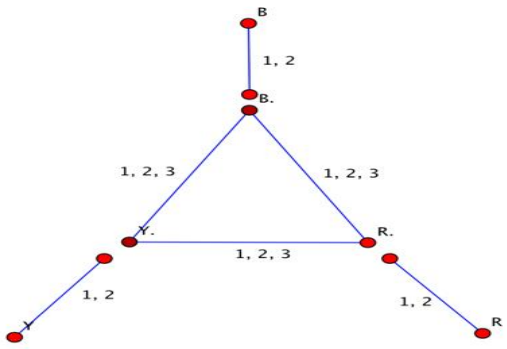

Figure 8 The possibility of joining points between the coloured particles (drawn by Cinderella software).

6. Hexagonal Lattice of a Shell

The hexagonal lattice here is used to represent the symmetry of the crystal's planar connection graph within a shell or AGP. It can be made to look like a hexagonal plane lattice. The red, yellow, and black coloured points are the tetrahedra involved via the AGP graph. From Fig. 9, when a red connects with yellow it means the two tetrahedra are joined at a common face. But if a black connects with a black it means that there is a single black tetrahedron with a self-connection. Thereafter, the black tetrahedron can join to two further reds and two further yellows within the same shell. This hexagonal structure illustrates very easily the three kinds of hexagonal circuits in the planar graph. The first type contains 2 R's and 4 B's which in fact corresponds to a circuit of tetrahedra RBRB about a long edge in the crystal. The second type contains 2 Y's and 4 B's which corresponds to a circuit YBYB about a long edge. The third type has 2 R's, 2 Y's and 2 B's that are not joined (on opposite sides) in the hexagonal lattice, and so it corresponds to a circuit RYBRYB about a short edge in the crystal.

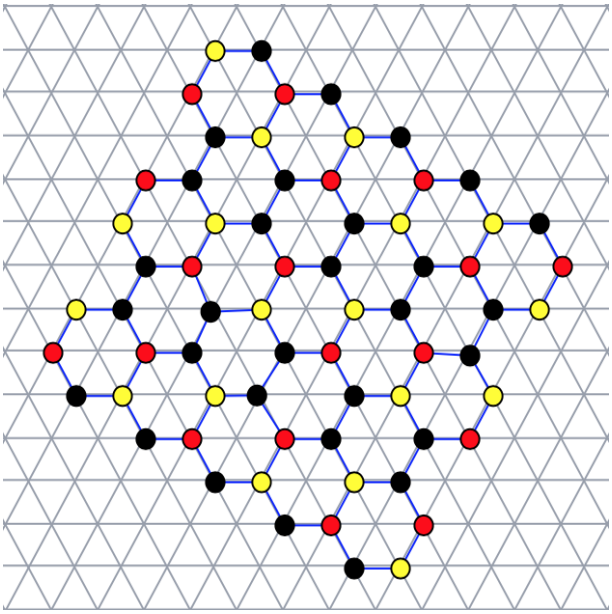

Figure 9 The hexagonal lattice of the AGP graph.

7. Affine Coxeter Group of the Crystal

The infinite group of symmetries falls within the Coxeter group system. First, the tetrahedra in the crystal form a tetragonal disphenoid honeycomb or oblate tetrahedrille which has vertices from the BCC lattice, otherwise called the $A_3{}^*/D_3{}^*$ lattice. From [6] we find clarifications of definitions for the affine Coxeter groups. Basically, they are generated by reflections subject to some relations between them, also containing translations in space. Now we go directly to describe the groups of symmetries produced by the coloured crystal. The Coxeter group of the BCC lattice is $\widetilde{B_3}$. This is the group of the whole crystal without colouring the tetrahedra. It is closely related to the group of the rhombic dodecahedral honeycomb that packs 3-dimentional space. Inside each rhombic dodecahedron there are 24 tetrahedra. Each tetrahedron is coloured R, B, Y. There are six permutations of three colours RYB. This leads to a normal subgroup N of index 6 in $\widetilde{B_3}$ that fixes all the colours. Then there is the point subgroup that fixes a point of the BCC lattice. This is a finite group B_3 of order 48. There is the group fixing that point but also the three colours, and that is a normal subgroup of this group isomorphic to the elementary abelian group of order 8. Otherwise, there

are subgroups such as the one fixing a single colour and possibly permuting the other two, also the subgroups that consist entirely of rotations/translations and not reflections. These will conserve the spin, orientation, handedness of the particles.

First, we note a certain lemma that tells us about normal subgroups.

Lemma 7.1 (Felix Klein, folklore): If there is a group A acting on a set of objects T as a permutation group, and if there is a subset R of T that is fixed (not elementwise by the group). Then the subgroup B of A that fixes each element of R is a normal subgroup of A.

Proof: Clearly B is a subgroup of A because it is closed under products and inverses. There is a homomorphism h from A to the symmetric group S_R (set of permutations of R) that restricts a permutation of A to act only on R. Thus, the kernel B of h is a normal subgroup of A by the classical homomorphism theorem.

We clarify these results about symmetry groups in several theorems below.

Theorem 7.2: The point group of the BCC lattice is a group of order 48 isomorphic to the group of the cube (or octahedron) in 3-d space.

Proof: The fundamental solid made from 24 tetrahedra around a central point of the BCC lattice is the rhombic dodecahedron. This can be formed from a cube by putting a pyramid onto each of the 6 faces of the cube. The 6 tops of these pyramids form an octahedron. Thus, we see that the symmetries of the cube are in 1-1 correspondence with those of the fundamental solid.

Note that the group of the cube is isomorphic to the wreath product $\mathbb{Z}_2$ Wr S_3 of the cyclic group $\mathbb{Z}_2$ with the symmetric group S_3, or equivalently it is the group of signed permutations on three letters, e.g., R, B, Y. It may be represented using 3 by 3 matrices which have a single ±1 in each row and column and 0's elsewhere.

Theorem 7.3: The subgroup of the point group that fixes the three colours is the elementary abelian group of order 8, isomorphic to the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ made from 3 independent generator involutions. This group contains the identity element together with 3 rotations about 180 degrees around the three orthogonal directions of the BCC lattice together with 4 reflections so that every non-identity element is an involution.

Proof: There are 24 triangles on the outside of the rhombic dodecahedron, split into 8 of each colour. Given two of these triangles one can find the unique rotation taking one to the other. However, rotations preserve spin and four of the red triangles and their tetrahedra have positive spin and four negative spin. That means the number of rotations preserving the colours is four. In addition, there are four reflections that change the spins. In total there are eight elements of this group. These are easily seen as the transformations that fix the three orthogonal coordinate a, b, c axes.

Theorem 7.4: The full group of the BCC lattice is $\widetilde{B_3}$, which is generated by the point group together with the translations that move one vertex of the BCC to another. They move all the other points the same direction and distance.

Proof: This is classically a well-known result [6].

Note that the translation group of the BCC is a free (torsion free) abelian group with 3 generators. Using (a, b, c) coordinates we define a translation T_{rst} to be

the mapping (a, b, c) to (a + r, b + s, c + t), moving everything by the vector (r, s, t). Then one set of generators is $\{T_{200}, T_{020}, T_{111}\}$.

Theorem 7.5: The subgroup N of the full group that fixes the colours is the group of the coloured crystal. Thus

$$N \lhd \widetilde{B_3}, \text{ and } \widetilde{B_3}/N \cong S_3.$$

Proof: This is a normal subgroup N of $\widetilde{B_3}$ of index 6 (with a factor group S_3) by Theorem 7.1 generated by the elementary abelian point group of order 8 of Theorem 7.2 and the translations that always fix the colours.

Theorem 7.6: The subgroup of the point group that fixes a single colour, denoted P_R, P_B or P_Y, but perhaps permuting the other colours, is a subgroup of index 3 in the point group P. These three groups are conjugate (non-normal) subgroups of order 48/3 = 16.

Proof: The point group is a permutation group of order 48 that permutes the three colours, so that the stabilizer of a colour (the subgroup fixing the colour) is a subgroup of index 3.

Note that the P_B subgroup, together with the translations, is the group that can be used to determine the colouring of the black tetrahedra, and as it contains reflections it can also take particles to antiparticles, such as a positron R to an electron Y. For any two black tetrahedra containing a common vertex there will be two elements of this group moving one to the other and fixing that point. One will be a rotation and the other a reflection.

Here we analyse more of the symmetry operations in the crystal and relate it to the four generator reflections of the Coxeter group. One can move throughout the crystal using reflections at common faces of the tetrahedra.

Every tetrahedron in the crystal has its vertices labelled 0, 1, 2, 3 as we saw in Fig. 2. Hence, the three coloured tetrahedra Y, R, B reflect to each-other through common faces by using the operators x_0, x_1, x_2, x_3, where x_i denotes reflecting in the face $\{0, 1, 2, 3\} \setminus \{i\}$ of the tetrahedron.

This picture shows the connections within a deuteron and the corresponding generators of the Coxeter group.

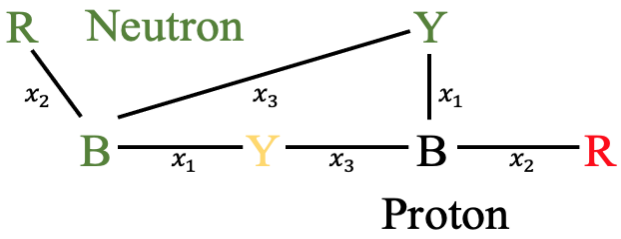

Figure 10 A deuteron and the corresponding generators of the Coxeter group.

The following diagram shows a sequence of alternating protons and neutrons connected in the time y direction and how the generators of the Coxeter group change. This sequence repeats every four times along a tube of the crystal.

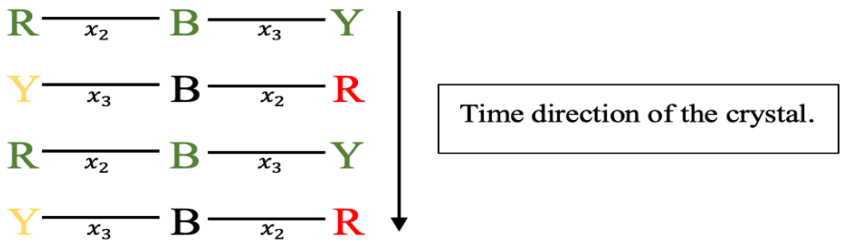

Figure 11 Protons and neutrons connected in the time y direction with the generators of the Coxeter group illustrated.

The four operations x_i above related to the tetrahedra of the crystal have an analogy with the following Coxeter group, satisfying similar relations. The group $\tilde{A}_3$ has a set S of four Coxeter generators $\{x_0, x_1, x_2, x_3\}$, where x_i are reflections in 3-d space. The presentation of $\tilde{A}_3$ via generators and relations can be written

$$\langle x_0, x_1, x_2, x_3 \mid {x_i}^2 = (x_0x_2)^2 = (x_1x_3)^2 = (x_0x_1)^3 = (x_0x_3)^3 = (x_1x_2)^3 = (x_2x_3)^3 = 1 \rangle.$$

This group is infinite and includes translations. Note that the product of two reflections is a rotation.

Consequently, the Coxeter-Dynkin diagram is the square graph; see Fig. 12.

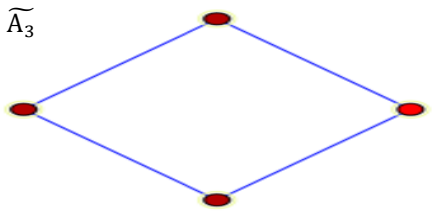

Figure 12 The Coxeter-Dynkin diagram $\widetilde{A_3}$.

Now we consider the finite point group in more detail.

There are four deuterons obtained from 4 protons and 4 neutrons found in the AGP1 and AGP2 s-subshells. These make the element Be-8 [18] from the rhombic dodecahedron in the crystal; see Fig. 13. We label the vertices of the rhombic dodecahedron V = 1, 2, 3, …, 14. There are 24 faces of the planar graph on the surface of the rhombic dodecahedron including the outer face, which is made from the three vertices 2, 9, 10; see Fig. 14.

Figure 13 Beryllium-8 with 4 deuterons in the crystal drawn with OpenSCAD.

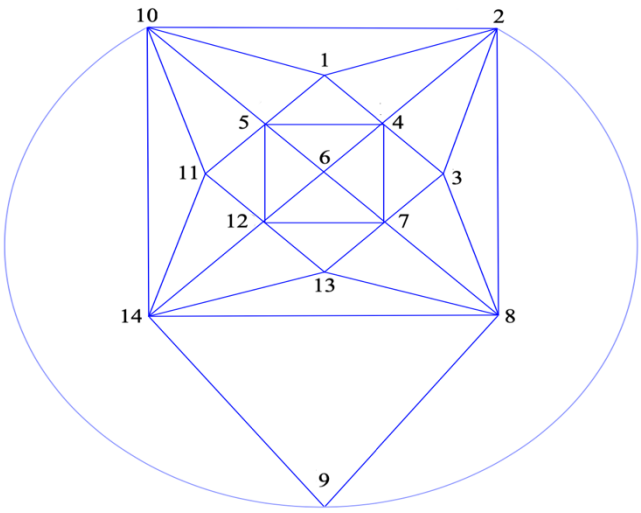

Figure 14 The rhombic dodecahedron with labelled vertices V = 1, 2, 3, …, 14.

The Coxeter group B_3 that is the point group of the crystal can be considered as the subgroup of S_{14} as it acts on the 14 vertices of the rhombic dodecahedron. Again, there are four interesting elements of this permutation group that have similar relations to the previous group elements with the same labels. Let us keep these labels as a matter of interest.

We define them as follows.

The operators x_0, x_1, x_2, x_3 acting on the vertices are,

i- x_0 = reflection in horizontal plane.

= (1 11) (2 14) (4 12) (3 13)

ii- x_1 = reflection in diagonal plane.

= (4 10) (3 9) (6 11) (7 14)

iii- x_2 = reflection in vertical plane.

= (1 3) (8 10) (5 7) (11 13)

iv- x_3 = reflection in diagonal plane.

= (2 5) (3 6) (8 12) (9 11)

Hence,

$x_0\, x_1$ = (1 11) (2 14) (4 12) (3 13) (4 10) (3 9) (6 11) (7 14)

= (1 11 6) (2 14 7) (3 9 13) (4 10 12) (5) (8)

$\Rightarrow\ (x_0\, x_1)^3 = 1.$

$x_0\, x_1$ is a 120° rotation (of order 3) about the fixed line through vertices 5 and 8. This verifies one of the generator relations in the definition of the Coxeter group. Also, all the other relations of $\tilde{A}_3$ are easily verified. The reason why these latter operators do not generate an infinite group is that many additional relations are satisfied.

Inside this rhombic dodecahedron graph, see Fig. 14, there are two important subsets. First, there is the cube of 8 vertices 2, 4, 5, 7, 8, 10, 12, 14. Also, there is the dual octahedron 1, 3, 6, 9, 11, 13. The group acts on these elements separately forming two orbits.

The group generated by these elements x_i is of order 24 and isomorphic to S_4 (e.g. by a Magma calculation).

Interestingly, there is another subgroup of the group B_3 isomorphic to S_4 that is the group of rotations. We can list these here and see that the different types correspond to the conjugacy classes of S_4. Note that the rotations are about lines connecting opposite points on the rhombic dodecahedron.

- Identity, e
- 6 rotations at 6 vertices of octahedron by 90 degrees, (1234)
- 3 rotations at 6 vertices of octahedron by 180 degrees, (12)(34)
- 8 rotations at 8 vertices of cube by 120 degrees, (123)
- 6 rotations at the 12 centres of the rhombic faces, (12)

The sizes of the conjugacy classes match: conjugate permutations are those with the same cycle structure e.g. the permutations (12), (13), (14), (23), (24), (34) form the conjugacy class containing (12). Since 1+6+3+8+6 =24 we have the entire group S_4 represented by the collection of rotations.

8. Shells in the Crystal

In this chapter we summarize how the shells work in the crystal using the coordinate system (x, y, z). The shells represent the electron layers going out from the nucleus, but in the crystal model they also contain the layers of protons and neutrons that are associated with the electrons. The energy level is represented by the value of z which is also related to the shell number.

- Shell 1: z in $[i - 1, i]$, the lowest energy innermost shell.
- Shell 2: z in $[i, i + 1]$.
- Shell 3: z in $[i + 1, i + 2]$.
- Shell k: z in $[i + k - 2, i + k - 1]$, top outermost shell.

The top shell is directly observable but below that the atom is not directly observable. This gives a reason for the existence of unseen connections such as teleportation and entanglement in quantum mechanics.

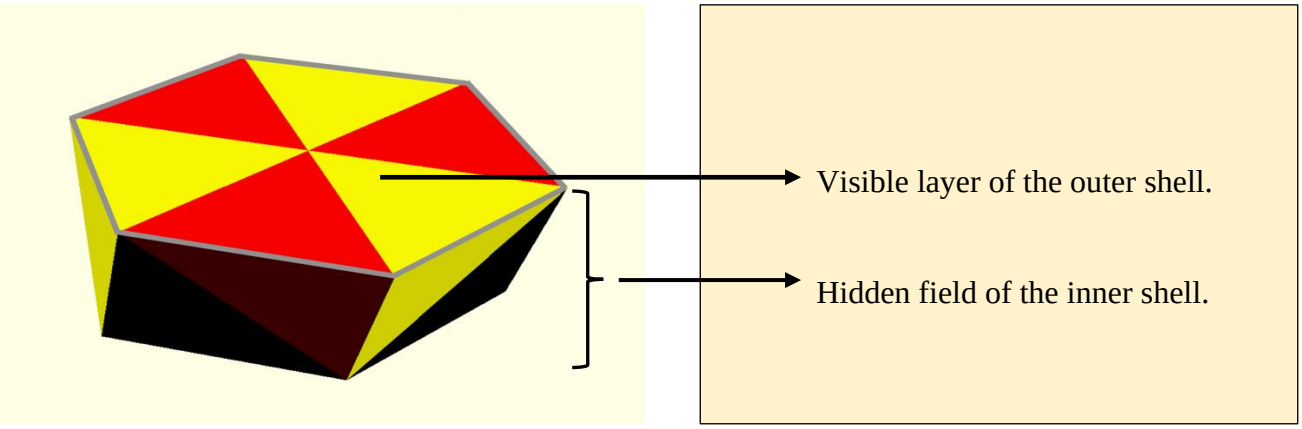

Figure 15 He-4 with grey highlighted surface-edges indicates the outer seen world of the atom drawn with OpenSCAD.

If an electron Y transfers via a wave function or some outside stimulus (an energetic photon), this does not restructure the crystal which remains invariant. We could detect the electromagnetic field via the RY structures within the crystal, but there are other forces such as the strong force which are hidden away in the depths of the crystal.

9. Subshells

In classical quantum physics when atoms eject an electron that must generate a shell vacancy, and the new vacancies may occur in shell 1 or shell 2 or even higher shells. Subsequently, the ejection of an electron initiates a change of all vacancies up to the outermost subshell [15, 16]. Based on electron ejection or photon emission, scholars discovered subshells contained inside the shells as follows:

i. The s-subshell of any shell consists of two electrons maximally.
ii. The p-subshell of any shell consists of six electrons maximally.
iii. The d-subshell of any shell consists of ten electrons maximally.

iv. The f-subshell of any shell consists of fourteen electrons maximally.

v. All subshells split into pairs of electrons of opposite spin.

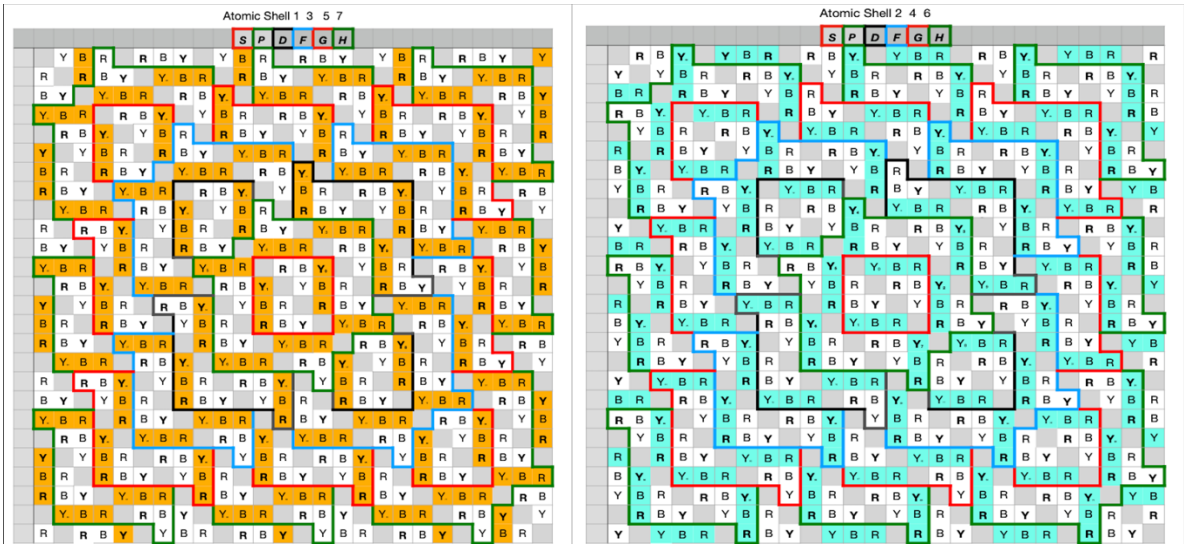

Figure 16 The two atomic graph papers illustrate the shells and the subshells and how many electrons Y are contained in the various subshells.

In the crystal model the subshells in a certain shell (one AGP) surround the previous one, going out from the centre. Lines of various colours denote these subshells:

- red ↦ s.
- green ↦ p.
- black ↦ d.
- blue ↦ f.
- red ↦ g.
- green ↦ h.

Each electron in a proton is numbered Y_i starting with i = 0 from the middle red s-subshell going in anticlockwise spiral outwards. This gives us also the numbering of the proton that contains the corresponding electron inside each deuteron. This numbers each deuteron from the electron-proton inside. The

electrons in the neutrons are therefore not specifically numbered but inherit specific numbers from the neutrons that contain them.

10. Quantum Numbers Explained by the Crystal

The location and energy of every electron in an atom is determined by a set of four quantum numbers that describe the different atomic orbitals and spins. Classically, orbitals are a region of probability (given by a wave function) where an electron can be found in space. There are subshells (of types s, p, d, f, …) with different shapes. In addition, the more electrons in an atom, the more orbitals are needed to accommodate them all.

10.1 Principal Quantum Number

The first quantum number is the principal quantum number n which can have any positive integer value. A higher energy level of this number usually indicates a further distance away from the nucleus [17]. In the atomic graph paper, the quantum number is indicated by the value of z (the vertical direction in the crystal).

10.2 Orbital/Azimuthal Angular Momentum Quantum Number

The next quantum number is the orbital angular momentum quantum number l with value (from zero to n $-$ 1). These are also labelled s, p, d, f, g… For instance, if there is a quantum number n = 3 then

$l = 0, \ldots, \mathrm{n} - 1,$

$\Rightarrow l = 0, 1, 2$ subshells called s, p, d.

In standard physics the value of l corresponds to an orbital topological shape. Thus, the s-subshell is a spherical shape, and the other subshells are split into several pieces. In the crystal the s subshell is topologically a sphere (the He-4 atom), while the higher subshell p is a torus, and further out they split into pieces.

10.3 Magnetic Quantum Number

The third quantum number is the magnetic quantum number M_l. This number M_l can have an integer value anywhere from negative l to positive l. For instance, if we have $l = 2$, then it can take five different values:

$M_l = -2, -1, 0, 1, 2.$

This quantum number basically determines how many pairs of electrons of opposite spin there can be per energy level. In the crystal graph paper AGP these pairs are formed by electrons symmetrically situated on opposite sides of the central point.

10.4 Spin Quantum Number

The last quantum number that an electron possesses is the spin quantum number M_s. This M_s has a value either - ½ or + ½ [17].

In conclusion, in quantum theory, every electron in an atom has a unique set of quantum numbers, so no two electrons can have precisely the same quantum numbers as stated by the Pauli Exclusion Principle [2]. In the crystal model, an electron is joined to a specific proton. The spin of a proton and its daughter electron are the same and the Pauli principle is evident from the unique positioning of electrons and other particles in the crystal.

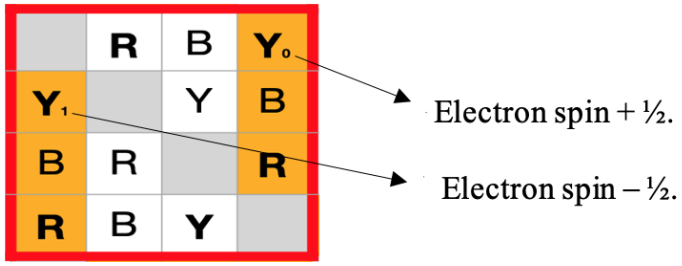

Figure 17 Two electrons of opposite spins and protons in the s-subshell from AGP1.

11. Spin Theorem

In this section we prove the spin theorem.

We can show the theorem about spin of a particle, calculating it from the external surface properties.

The spin of the triangular face depends on the numbering of its vertices in the crystal. Consider Fig. 2 and Def. 3.3. A tetrahedron of even spin has four kinds of triangular faces, 012, 031, 023, 132 looking from the outside and going clockwise around each face. On the other hand, a tetrahedron of odd spin has triangular faces, 021, 013, 032, 123 going clockwise. Thus, the spin of a tetrahedron can be calculated from the orientation of just one face e.g. 012. Using this observation we obtain the spin theorem that follows.

Theorem 11.1: The spin of a particle can be calculated from the spins of the triangles on its surface.

Proof: The spin of a particle is the sum of the spins of its constituent tetrahedra; see Chapter 1. In a particle there are triangular faces that are internal to the structure. These spins cancel out because spins are opposite on each side of the face. Therefore, the total spin can be calculated from the 012 triangles on the outside surface. Similarly, the other triangles 013, 023 or 123 could have been used to calculate the spin.

Note that [18, Sec. 7.2] contains a similar theorem about charge: the triangular faces can be categorised by their colours and how they connected to another colour. For example, a Ry triangle is a red face of a tetrahedron that is connected to a yellow face of the adjoining tetrahedron. Similarly, a Yr triangle is a yellow

face of a tetrahedron that is connected to a red face of the adjoining tetrahedron. Then the formula for charge of a particle is

$$C = R – Y = (Ry – Yr) / 2,$$

where Ry and Yr are the numbers of those types of triangles on the surface of the particle. This is very similar to the spin theorem we have just proved.

12. Conclusion

We interpreted many properties such as spin, charge and antiparticles using the crystal. Accurate scientific principles are combined with physical data into the crystal model. Symmetry groups of the crystal are calculated, including the point group and the full group of the body-centred cubic lattice. We established a spin theorem about particles that shows the property of spin is a surface property, calculated in the crystal model from the spins of the surface triangles. The model leads to a more detailed and general understanding of the crystal model of quantum physics and chemistry.

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