Spectral shift function in von Neumann algebras

Nurulla AZAMOV

A Thesis presented for the Degree of Doctor of Philosophy

in

School of Informatics and Engineering
Faculty of Science and Engineering
Flinders University

January 14, 2008

Declaration

I certify that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge and belief it does not contain any material previously published or written by another person except where due reference is made in the text.

Nurulla Azamov, Candidate

Acknowledgements

I would like to thank my supervisors Dr. Fyodor Sukochev and Professor Peter Dodds for their guidance and advice.

I also would like to thank Flinders University and the School of Informatics and Engineering for the opportunity to be a postgraduate student.

Abstract

The main subsect of this thesis is the theory of Lifshits-Kreĭn spectral shift function in semifinite von Neumann algebras and its connection with the theory of spectral flow. Main results are an analogue of the Kreĭn trace formula for semifinite von Neumann algebras, the semifinite analogue of the Birman-Solomyak spectral averaging formula, a connection between the spectral shift function and the spectral flow and a Lidskii's type formula for Dixmier traces. In particular, it is established that in the case of operators with compact resolvent, the spectral shift function and the spectral flow are identical notions.

Contents

Introduction

Let H_0 and H_1 be two self-adjoint operators. Then the *spectral shift function* for the pair H_0, H_1 can be defined as any function ξ such that, for any compactly supported smooth function f, the equality

$$\operatorname{Tr}(f(H_1) - f(H_0)) = \int_{-\infty}^{\infty} f'(\lambda)\xi(\lambda)\, d\lambda \tag{1}$$

holds, provided that the difference $f(H_1) - f(H_0)$ is trace class. This formula is called the *trace formula* of I. M. Lifshits-M. G. Kreĭn.

The notion of the spectral shift function was introduced in 1952 by the physicist I. M. Lifshits [Lif]. Lifshits considered the perturbation of a self-adjoint operator by a one-dimensional perturbation. He defined the spectral shift function $\xi(\lambda)$ by the formula

$$\xi(\lambda) = \operatorname{Tr}(E_\lambda^{H_1} - E_\lambda^{H_0}),$$

where E_λ^H is the spectral projection of a self-adjoint operator H, corresponding to the half-line $(-\infty, \lambda)$. Acting formally, one can recover the trace formula (1) from this definition in the following way:

$$\begin{aligned}\operatorname{Tr}(f(H_1) - f(H_0)) &= \operatorname{Tr}\left(\int f(\lambda)\, dE_\lambda^{H_1} - \int f(\lambda)\, dE_\lambda^{H_0}\right) \\ &= \operatorname{Tr}\left(\int f(\lambda)\, d(E_\lambda^{H_1} - E_\lambda^{H_0})\right) = \int f(\lambda)\, d\operatorname{Tr}\left(E_\lambda^{H_1} - E_\lambda^{H_0}\right) \\ &= \int f'(\lambda)\operatorname{Tr}\left(E_\lambda^{H_1} - E_\lambda^{H_0}\right)\, d\lambda = \int f'(\lambda)\xi(\lambda)\, d\lambda.\end{aligned}$$

The difficulty with this argument is that, apart of its formality, the difference

$$E_\lambda^{H_1} - E_\lambda^{H_0} \tag{2}$$

is not necessarily trace class even for one-dimensional perturbations. This was shown by M. G. Kreĭn [Kr]. Kreĭn considered integral operators

$$H_0 f(x) = \int_0^\infty k_0(x,y) f(y)\, dy,$$
$$H_1 f(x) = \int_0^\infty k_1(x,y) f(y)\, dy,$$

on the Hilbert space $L^2(0,\infty)$ with kernels

$$k_0(x,y) = \begin{cases} \frac{1}{2}\left(e^{x-y} - e^{-x-y}\right), & \text{if } 0 \leqslant x \leqslant y, \\ \frac{1}{2}\left(e^{-x+y} - e^{-x-y}\right), & \text{if } 0 \leqslant y \leqslant x, \end{cases}$$
$$k_1(x,y) = \begin{cases} \frac{1}{2}\left(e^{x-y} + e^{-x-y}\right), & \text{if } 0 \leqslant x \leqslant y, \\ \frac{1}{2}\left(e^{-x+y} + e^{-x-y}\right), & \text{if } 0 \leqslant y \leqslant x, \end{cases}$$

respectively. In this case the perturbation $V = H_1 - H_0$ is a one-dimensional operator $\langle \cdot, \varphi \rangle \varphi$, where $\varphi(x) = e^{-x}$. Kreĭn showed that (2) is an integral operator with kernel

$$-\frac{2}{\pi} \frac{\sin \sqrt{\lambda}(x+y)}{x+y},$$

and that it is not a compact operator.

Recently, V. Kostrykin and K. A. Makarov [KM] showed that, in this case, for all $\lambda \in (0,1)$, the spectrum of (2) is purely absolutely continuous and is equal to $[-1,1]$. The general nature of the difference (2) was established in [Pu]. A. B. Pushnitski proved that the essential spectrum of (2) is equal to $[-a,a]$, where $a = \frac{1}{2}\left\| S(\lambda; H_1, H_0) - 1 \right\|$ and $S(\lambda; H_1, H_0)$ is the scattering matrix of the pair H_0, H_1. As was noted by Krein, the operators H_0 and H_1 are actually the resolvents of the Dirichlet and Neumann one-dimensional Laplacian $\frac{d^2}{dx^2}$ at the spectral point -1. A free one-dimensional particle on $(0,\infty)$ undergoes a phase shift equal to π at 0 when one changes the Dirichlet boundary condition to the Neumann boundary condition. So, in this case $S(\lambda) = e^{i\pi} = -1$ and the result of Kostrykin-Makarov immediately follows from Pushnitski's result.

In [Kr] M. G. Kreĭn created the mathematical theory of the spectral shift function. He proved that if the perturbation $V = H_1 - H_0$ is trace class, then there exists a unique (up to a set of Lebesgue measure zero) summable function $\xi(\cdot)$ such that, for a class of admissible functions, which includes compactly supported functions $f \in C^2(\mathbb{R})$, the trace formula (1) holds. Surprisingly, for $f \in C_c^1(\mathbb{R})$, the difference $f(H_1) - f(H_0)$ is not necessarily trace class [Far].

The method of proof which Kreĭn used was to establish the trace formula first for one-dimensional perturbations, after that, for finite-dimensional perturbations, and finally to use an approximation argument for general trace class perturbations. The major step of this proof was the first step, i.e. the case of one-dimensional perturbation. Kreĭn showed that for one-dimensional perturbations the perturbation determinant

$$\det(1 + V(H_0 - z)^{-1})$$

satisfies the conditions of a theorem from complex analysis, more precisely, that it is a Herglotz function and behaves like $\frac{1}{y}$ for large values of $y = \operatorname{Im} z$.

One of the aims of this thesis is to establish the analogue of the Lifshits-Kreĭn theory for general semifinite von Neumann algebras $\mathcal{N}$ with a faithful semifinite normal trace τ. In the case of a bounded self-adjoint operator $H_0 \in \mathcal{N}$ and τ-trace class perturbation V, this problem has been solved by R. W. Carey and J. D. Pincus in [CP]. The novelty of our approach [ADS] is to consider the case of an (unbounded) operator H_0 affiliated with $\mathcal{N}$. The main result of Section 3.1 is Theorem 3.1.13, which is a semifinite analogue of classical result of M. G. Kreĭn.

The main difficulty here is that there is no proper analogue of the classical Fredholm determinant in semifinite von Neumann algebras (recall that Fuglede-Kadison determinant takes only non-negative values; in the type I case, the Fuglede-Kadison determinant is just the absolute value of Fredholm determinant). This difficulty is overcome with the use of the Brown measure [Brn]. The Brown measure together with the semifinite analogue of the Lidskii theorem [Brn] (see [Lid], [Si, Section 3] for the classical Lidskii theorem) allows us to prove the conditions of the above mentioned theorem from complex analysis in case of τ-finite perturbations. Generalization to an arbitrary relatively trace class perturbations follows the lines of the classical case $\mathcal{N} = \mathcal{B}(\mathcal{H})$.

Note that, if $\tau(1) < \infty$, then the spectral shift formula (3.25) may be derived directly by the argument given in [Kr_2], provided f is absolutely continuous and $f' \in L^1(\mathbb{R})$. This argument yields the formula

$$\xi_{H+V,H}(\lambda) = \tau\left(E_\lambda^H\right) - \tau\left(E_\lambda^{H+V}\right), \quad \text{a.e. } \lambda \in \mathbb{R}. \tag{3}$$

This formula goes back to Lifshits [Lif] and reduces the calculation of the spectral shift function to computation of the spectral distributions of the operators $H + V, H$. In the setting given by Theorem 3.1.13, again in the special case of finite trace, the formula (3) may be derived from (3.25) by a standard argument.

Another problem considered in this thesis, is the semifinite version of Birman-Solomyak formula for the spectral shift function. In 1975, Birman and Solomyak established the beautiful formula for the spectral shift function

$$\xi(\lambda) = \frac{d}{d\lambda} \int_0^1 \operatorname{Tr}(V E_\lambda^{H_r})\, dr, \tag{4}$$

where $H_r = H_0 + rV$, $V \in \mathcal{L}^1(\mathcal{H})$. This formula is called the *spectral averaging* formula. Birman-Solomyak proved this formula using double operator integrals. This formula was established by V. A. Javrjan in [Jav] four years earlier. Javrjan considered the spectral averaging of Sturm-Liouville operator on a half-line with respect to boundary condition. This corresponds to one-dimensional perturbation.

In 1998, B. Simon [Si_2] found a short and simple proof of (4). As B. Simon notes in [Si_2], the formula (4) was rediscovered by many authors, who were not aware of V. A. Javrjan and Birman-Solomyak's papers; among them Kotani [Ko], who, in development of the celebrated result of Goldshtein-Molchanov-Pastur [GMP], used spectral averaging to show that the spectrum of certain random one-dimensional Schrödinger operators is a purely point spectrum with probability 1 and that the corresponding eigenfunctions decay exponentially.

The main result of Section 3.4 (Theorem 3.4.2) establishes a semifinite analogue of Birman-Solomyak's spectral averaging formula. The proof follows essentially the original proof of Birman-Solomyak. Since Birman-Solomyak's proof uses double operator integrals, it was necessary to develop the theory of the double (and in general multiple) operator integral to von Neumann algebras. The double operator integral theory developed in [dPSW] and [dPS] is not applicable in this situation, since the unperturbed operator H_0 is not in general τ-measurable, as required in [dPSW, dPS]. In the type I case, τ-measurability is equivalent to boundedness. Consequently, it is first necessary to develop the theory of double (multiple) operator integrals in von Neumann algebras, that will cover the situation that H_0 is unbounded.

Multiple operator integrals were first introduced in the celebrated work of Yu. L. Daletskiĭ and S. G. Kreĭn [DK]. A multiple operator integral is an expression of the form

$$T_\varphi^{H_0,H_1,\ldots,H_n}(V_1,\ldots,V_n) := \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi(\lambda_0,\ldots,\lambda_n)\,dE_{\lambda_0}^{H_0}V_1\,dE_{\lambda_1}^{H_1}V_2\,dE_{\lambda_2}^{H_2}\ldots V_n\,dE_{\lambda_n}^{H_n},$$

where $V_1,\ldots,V_n$ are bounded operators on $\mathcal{H}$, $H_0,\ldots,H_n$ are self-adjoint operators on $\mathcal{H}$, and φ is a function of $n+1$ variables. The initial approach of [DK] to the definition of multiple operator integrals is to consider them as repeated integrals

$$\int_{-\infty}^{\infty}\left(\cdots\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\varphi\,dE_{\lambda_0}^{H_0}\right)V_1\,dE_{\lambda_1}^{H_1}\right)V_2\ldots\right)V_n\,dE_{\lambda_n}^{H_n},$$

for which purpose they define first the spectral integrals of operator valued functions

$$\int_{-\infty}^{\infty}F(\lambda)\,dE_\lambda^H.$$

Another approach to the theory of the multiple operator integral was given in [Pa]. In this work, B. S. Pavlov considers the multiple operator integral as an integral over the vector-valued measure

$$\Delta_0\times\ldots\times\Delta_n\in\mathcal{B}(\mathbb{R}^{n+1})\mapsto E_{\Delta_0}^{H_0}V_1\,E_{\Delta_1}^{H_1}V_2\,E_{\Delta_2}^{H_2}\ldots V_n\,E_{\Delta_n}^{H_n}.$$

Pavlov proves that, if $V_1,\ldots,V_n\in\mathcal{L}^2(\mathcal{H})$, then this measure is countably additive and has bounded weak variation, so that for any bounded measurable function φ the multiple operator integral can be considered as integral over this vector-valued measure. He then extends this definition to arbitrary bounded V_j's under some additional conditions on φ.

For the purpose of generalizing the theory of multiple operator integrals to von Neumann algebras, it is convenient [ACDS] to define the multiple operator integral as follows. If one can write the function φ in the form (see (3.27))

$$\varphi(\lambda_0, \lambda_1, \dots, \lambda_n) = \int_S \alpha_0(\lambda_0, \sigma) \dots \alpha_n(\lambda_n, \sigma)\, d\nu(\sigma), \tag{5}$$

then one can see that formally

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \varphi(\lambda_0, \dots, \lambda_n)\, dE_{\lambda_0}^{H_0} V_1\, dE_{\lambda_1}^{H_1} V_2\, dE_{\lambda_2}^{H_2} \dots V_n\, dE_{\lambda_n}^{H_n}$$
$$= \int_S \alpha_0(H_0, \sigma) V_1 \alpha_1(H_1, \sigma) \dots V_n \alpha_n(H_n, \sigma)\, d\nu(\sigma).$$

We call the representations of the form (3.27) BS-representations. The idea is to define the multiple operator integral by the right hand side of this equality. One has to prove that this definition is well-defined, i.e. that it does not depend on the representation (5) of the function φ. This is done in Theorem 3.2.8. This idea is taken from the work of Solomyak and Sten'kin [SS], who actually used implicitly this definition of multiple operator integral. The difference was that they considered series of the form

$$\varphi(\lambda_0, \lambda_1, \dots, \lambda_n) = \sum_{k=1}^{\infty} \alpha_{0,k}(\lambda_0) \dots \alpha_{n,k}(\lambda_n).$$

This same idea had been used earlier to define multiple operator integrals independently by V. V. Peller [Pel].

An advantage of our new approach to the definition of multiple operator integrals, is that once some BS representation for φ is found, one can work with the multiple operator integral as the usual integral of operator-valued functions, consequently using the well-developed and the well-known theory of such integrals. Another advantage is that sometimes different BS representations for the same function φ turn out to be better suited for a particular problem. For example, it is known that the difference $f(A) - f(B)$ can be represented as

$$f(A) - f(B) = T_{f^{[1]}}^{A,B}(A - B),$$

where $f^{[1]}(\lambda, \mu) = \frac{f(\lambda) - f(\mu)}{\lambda - \mu}$ is the first divided difference of the function f. Examples of usage of different BS-representations of $f^{[1]}$ can be found in Sections 3.3 and 4.1.

The last chapter is devoted to the notion of spectral flow and its connection with the theory of spectral shift function. The notion of spectral flow was introduced by M. Atiyah, V. Patodi and I. M. Singer in [APS] as the net number of eigenvalues which cross zero from the left to the right. E. Getzler [Ge, Theorem 2.6] established the following formula

$$\mathrm{sf}(D_0, D_1) = \sqrt{\frac{\varepsilon}{\pi}} \int_0^1 \mathrm{Tr}\left(\dot{D}_u e^{-\varepsilon D_u^2}\right) du + \frac{1}{2}\eta_\varepsilon(D_1) - \frac{1}{2}\eta_\varepsilon(D_0), \tag{6}$$

where

$$\eta_\varepsilon(D_0) := \frac{1}{\sqrt{\pi}} \int_\varepsilon^\infty \tau\left(D_0 e^{-tD_0^2}\right) t^{-1/2}\, dt$$

is η- invariant of D_0, and $\{D_u\}$ is a piecewise smooth path connecting D_0 and D_1. The integral (6) is interpreted as an integral of the one-form

$$\alpha_\varepsilon(X) = \sqrt{\frac{\varepsilon}{\pi}}\, \mathrm{Tr}(Xe^{-\varepsilon D^2}), \tag{7}$$

where $X \in \mathcal{B}_{sa}(\mathcal{H})$, the real Banach space of bounded self-adjoint operators on the Hilbert space $\mathcal{H}$. J. Phillips [Ph, Ph_2] gave a definition of spectral flow different from the original definition of [APS]. This definition interprets spectral flow as Fredholm index and as such it can be generalized also to the case of semifinite von Neumann algebras. In [CP, CP_2] A. L. Carey and J. Phillips generalized the integral formulas for spectral flow to the semifinite case, establishing integral formulas for the θ-summable and p-summable cases (i.e. $(1+D^2)^{-p/2}$ has finite τ-trace). In particular they establish the formula ($p > 1$)

$$\mathrm{sf}(D_0, D_1) = \tilde{C}_p^{-1} \int_0^1 \mathrm{Tr}\left(\dot{D}_t(1+D_t^2)^{-p}\right) dt + \beta_p(D_1) - \beta_p(D_0),$$

where $\beta_p(D)$ is an analogue of the η-invariant for the p-summable case [CP]. In the case of a p-summable spectral triple $(\mathcal{A}, D_0, \mathcal{N})$ and perturbation $V = u[D_0, u^*]$, the operators D_0 and $D_1 = uD_0u^*$ are unitarily equivalent, so that the last formula takes the form

$$\mathrm{sf}(D_0, uD_0u^*) = \tilde{C}_p^{-1} \int_0^1 \mathrm{Tr}\left(u[D_0, u^*](1 + (D_0 + tu[D_0, u^*])^2)^{-p/2}\right) dt. \tag{8}$$

In [CPRS], this formula is the starting point for a proof of the Local Index Theorem of Connes-Moscovici in non-commutative geometry. One of the ideas of the proof is to consider p as complex variable, i.e. to consider analytical continuation of the last integral as a function of p. In [CPS], using the zeta-function representation for the Dixmier trace due to A. Connes [Co], it was shown that when $p \to 1^+$, the spectral flow becomes the Dixmier trace [CPS, Theorem 6.2]. At the noncommutative geometry workshop at Banff in 2005, it was observed that when $p \to \infty$, the last integral formula for spectral flow becomes the Birman-Solomyak formula for the spectral shift function. This key observation was developed in [ACS]. One of the main results of [ACS] states that if D_0 is an operator with compact resolvent and D_1 its perturbation by a bounded self-adjoint operator, then

$$\mathrm{sf}(\lambda; D_0, D_1) = \xi_{D_1, D_0}(\lambda) + \frac{1}{2}\tau\left(\mathrm{N}_{D_1 - \lambda}\right) - \frac{1}{2}\tau\left(\mathrm{N}_{D_0 - \lambda}\right).$$

Combined with the Lifshits-Kreĭn trace formula (1), this formula also implies that when the "endpoints" D_0 and D_1 are unitarily equivalent, the spectral flow (= the spectral shift) function is constant. This sheds some light on integral formulas for spectral flow like (8), since it is otherwise difficult to understand why one should take into account eigenvalues very far from 0 to compute the spectral flow at 0. The point is that, in the case of unitarily equivalent endpoints $D_0 \sim D_1$, the spectral flow at all points is the same ("the law of conservation of spectrum"), so that one can compute "parts" of spectral flow anywhere on the spectral line.

The well-known Lidskii theorem (in its general semifinite form given in [Brn]) asserts that if $\mathcal{N}$ is a semifinite von Neumann factor with a faithful normal semifinite trace τ, then the trace $\tau(T)$ of an arbitrary operator $T \in L^1(\mathcal{N}, \tau)$ is given by

$$\tau(T) = \int_{\sigma(T)\setminus\{0\}} \lambda \, d\mu_T(\lambda),$$

where μ_T is a Borel measure (the so-called Brown measure of T) on the non-zero spectrum of T. In the case when $\mathcal{N}$ is a type I factor, the measure μ_T is the counting measure on the set of all eigenvalues of T. In Section 2.2.2, we present an analogue of such a formula for Dixmier traces.

In the case of a standard (normal) trace, the assertion of the Lidskii theorem for self-adjoint operators is immediate due to the absolute convergence of the series $\sum_{n\geqslant 1} \lambda_n(T)$ of any $T = T^*$ from the trace class. This is not the case any longer for Dixmier (non-normal) traces, since the latter series diverges for any $T = T^* \in \mathcal{L}^{1,\infty}(\mathcal{N}, \tau)$ which does not belong to the trace class.

The main result of Section 2.1 is Theorem 2.2.11. The Lidskii type formula given there holds for all operators $T \in \mathcal{L}^{1,\mathrm{w}}$. The ideal $\mathcal{L}^{1,\mathrm{w}}$ usually arises in geometric applications. In particular, if $\mathcal{N}$ is the algebra of all bounded operators on $L^2(M)$ where M is a compact Riemannian n-manifold (respectively, if $\mathcal{N}$ is the II_∞ factor $L^\infty(\mathbb{R}^n) \rtimes \mathbb{R}^n_{discr}$ [CMS, Sh]), the ideal $\mathcal{L}^{1,\mathrm{w}}$ contains all pseudodifferential operators (respectively, all almost periodic pseudodifferential operators) of order $-n$.

The Lidskii formula for Dixmier traces τ_ω, where ω is an arbitrary dilation invariant state on $L^\infty(0,\infty)$, takes an especially simple form for the case of measurable operators T (by definition, an operator $T \in \mathcal{L}^{1,\infty}(\mathcal{N}, \tau)$ is measurable if $\tau_\omega(T)$ does not depend on ω). In this case, $\tau_\omega(T)$ coincides with the true limit

$$\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_{\lambda \notin \frac{1}{t}G} \lambda \, d\mu_T(\lambda).$$

The proof of Theorem 2.2.11 depends crucially on the recent characterization of positive measurable operators from $\mathcal{L}^{1,\infty}(\mathcal{N}, \tau)$ as those for which the limit

$$\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T) \, ds$$

exists [LSS, Theorem 6.6], and the spectral characterization of sums of commutators in type II factors [DK_2, Fac]. The spectral characterization of sums of commutators is a very deep result, obtained independently by T. Fack [Fac, Theorem 3], N. J. Kalton [Kal, DK] and K. J. Dykema and N. J. Kalton [DK_2, Theorem 6.8, Corollary 6.10] (though the main idea of the proof seems to be the same in all these papers). This result implies that the Dixmier trace of any operator T from the Dixmier ideal depends only on the Brown measure of the operator T. However, Theorem 2.2.11 goes much further and gives an explicit formula for the Dixmier trace $\tau_\omega(T)$ in terms of the Brown measure μ_T.

Now we give a brief description of the sections of Chapter 1 (Preliminaries). In Section 1.5 (Theory of τ-Fredholm operators) we give an exposition of the theory of τ-Fredholm operators. We follow mainly the original Breuer's works [Br, Br_2] and [PR, Appendix B]. Breuer proved his results for semifinite factors, but as shown in [PR, Appendix B] the difference between the factor case and the non-factor case is not significant. In Section 1.6 (Spectral flow in von Neumann algebras) an exposition is given of J. Phillips' theory of spectral flow in semifinite von Neumann algebras. Here I follow the papers of J. Phillips [Ph, Ph_2].

Section 1.7 (Fuglede-Kadison determinant) contains an exposition of the Fuglede-Kadison determinant [FKa], following L. G. Brown's paper [Brn]. B. Fuglede and R. V. Kadison introduced this determinant in the case of type II_1 factors, while L. G. Brown considers semifinite factors. We give this theory for semifinite von Neumann algebras, not necessarily factors. In Section 1.8 (The Brown measure), an exposition of the Brown measure is given, following the original work of Brown [Brn].

The main results of this thesis are Theorem 3.2.8 [ACDS] (new approach to multiple operator integrals), Theorem 2.2.11 [AS] (Lidskii theorem for Dixmier traces), Theorem 3.1.13 [ADS] (Krein's formula for spectral shift function in semifinite von Neumann algebras), Theorem 3.3.3 [ACDS], Theorem 3.3.6 [ACDS] (high order Fréchet derivative of functions $f(H)$ of self-adjoint operators H with suitable restrictions on the function f), Theorem 3.4.2 [ACDS] (semifinite Birman-Solomyak spectral averaging formula), Theorem 4.1.17 [ACDS] (formula for Fréchet derivative in terms of double operator integrals), Theorem 4.2.5 [ACS] (trace formula for operators with compact resolvent), Theorem 4.3.13 [ACS], Theorem 4.3.18 [ACS] (connection between spectral flow and spectral shift function), Theorem 4.3.21 [ACS] (infinitesimal spectral flow), Theorem 4.3.24 [ACS] (spectral flow for $\mathcal{I}$-summable spectral triples), Theorem 4.3.31 [ACS] (Carey–Phillips formula with new proof).