Spectral shift function in von Neumann algebras

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Declaration

I certify that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge and belief it does not contain any material previously published or written by another person except where due reference is made in the text.

Nurulla Azamov, Candidate

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Abstract

The main subsect of this thesis is the theory of Lifshits-Kreĭn spectral shift function in semifinite von Neumann algebras and its connection with the theory of spectral flow. Main results are an analogue of the Kreĭn trace formula for semifinite von Neumann algebras, the semifinite analogue of the Birman-Solomyak spectral averaging formula, a connection between the spectral shift function and the spectral flow and a Lidskii's type formula for Dixmier traces. In particular, it is established that in the case of operators with compact resolvent, the spectral shift function and the spectral flow are identical notions.

Contents

Introduction

Let H_0 and H_1 be two self-adjoint operators. Then the *spectral shift function* for the pair H_0, H_1 can be defined as any function ξ such that, for any compactly supported smooth function f, the equality

$$\operatorname{Tr}(f(H_1) - f(H_0)) = \int_{-\infty}^{\infty} f'(\lambda)\xi(\lambda)\,d\lambda \tag{1}$$

holds, provided that the difference $f(H_1) - f(H_0)$ is trace class. This formula is called the *trace formula* of I. M. Lifshits-M. G. Kreĭn.

The notion of the spectral shift function was introduced in 1952 by the physicist I. M. Lifshits [Lif]. Lifshits considered the perturbation of a self-adjoint operator by a one-dimensional perturbation. He defined the spectral shift function $\xi(\lambda)$ by the formula

$$\xi(\lambda) = \operatorname{Tr}(E_\lambda^{H_1} - E_\lambda^{H_0}),$$

where E_λ^H is the spectral projection of a self-adjoint operator H, corresponding to the half-line $(-\infty, \lambda)$. Acting formally, one can recover the trace formula (1) from this definition in the following way:

$$\begin{aligned}
\operatorname{Tr}(f(H_1) - f(H_0)) &= \operatorname{Tr}\left(\int f(\lambda)\,dE_\lambda^{H_1} - \int f(\lambda)\,dE_\lambda^{H_0}\right) \\
&= \operatorname{Tr}\left(\int f(\lambda)\,d(E_\lambda^{H_1} - E_\lambda^{H_0})\right) = \int f(\lambda)\,d\operatorname{Tr}\left(E_\lambda^{H_1} - E_\lambda^{H_0}\right) \\
&= \int f'(\lambda)\operatorname{Tr}\left(E_\lambda^{H_1} - E_\lambda^{H_0}\right)\,d\lambda = \int f'(\lambda)\xi(\lambda)\,d\lambda.
\end{aligned}$$

The difficulty with this argument is that, apart of its formality, the difference

$$E_\lambda^{H_1} - E_\lambda^{H_0} \tag{2}$$

is not necessarily trace class even for one-dimensional perturbations. This was shown by M. G. Kreĭn [Kr]. Kreĭn considered integral operators

$$H_0 f(x) = \int_0^\infty k_0(x,y) f(y)\, dy,$$
$$H_1 f(x) = \int_0^\infty k_1(x,y) f(y)\, dy,$$

on the Hilbert space $L^2(0,\infty)$ with kernels

$$k_0(x,y) = \begin{cases} \frac{1}{2}\left(e^{x-y} - e^{-x-y}\right), & \text{if } \ 0 \leqslant x \leqslant y, \\ \frac{1}{2}\left(e^{-x+y} - e^{-x-y}\right), & \text{if } \ 0 \leqslant y \leqslant x, \end{cases}$$
$$k_1(x,y) = \begin{cases} \frac{1}{2}\left(e^{x-y} + e^{-x-y}\right), & \text{if } \ 0 \leqslant x \leqslant y, \\ \frac{1}{2}\left(e^{-x+y} + e^{-x-y}\right), & \text{if } \ 0 \leqslant y \leqslant x, \end{cases}$$

respectively. In this case the perturbation $V = H_1 - H_0$ is a one-dimensional operator $\langle \cdot, \varphi \rangle \varphi$, where $\varphi(x) = e^{-x}$. Kreĭn showed that (2) is an integral operator with kernel

$$-\frac{2}{\pi} \frac{\sin \sqrt{\lambda}(x+y)}{x+y},$$

and that it is not a compact operator.

Recently, V. Kostrykin and K. A. Makarov [KM] showed that, in this case, for all $\lambda \in (0,1)$, the spectrum of (2) is purely absolutely continuous and is equal to $[-1,1]$. The general nature of the difference (2) was established in [Pu]. A. B. Pushnitski proved that the essential spectrum of (2) is equal to $[-a,a]$, where $a = \frac{1}{2}\|S(\lambda; H_1, H_0) - 1\|$ and $S(\lambda; H_1, H_0)$ is the scattering matrix of the pair H_0, H_1. As was noted by Krein, the operators H_0 and H_1 are actually the resolvents of the Dirichlet and Neumann one-dimensional Laplacian $\frac{d^2}{dx^2}$ at the spectral point -1. A free one-dimensional particle on $(0,\infty)$ undergoes a phase shift equal to π at 0 when one changes the Dirichlet boundary condition to the Neumann boundary condition. So, in this case $S(\lambda) = e^{i\pi} = -1$ and the result of Kostrykin-Makarov immediately follows from Pushnitski's result.

In [Kr] M. G. Kreĭn created the mathematical theory of the spectral shift function. He proved that if the perturbation $V = H_1 - H_0$ is trace class, then there exists a unique (up to a set of Lebesgue measure zero) summable function $\xi(\cdot)$ such that, for a class of admissible functions, which includes compactly supported functions $f \in C^2(\mathbb{R})$, the trace formula (1) holds. Surprisingly, for $f \in C^1_c(\mathbb{R})$, the difference $f(H_1) - f(H_0)$ is not necessarily trace class [Far].

The method of proof which Kreĭn used was to establish the trace formula first for one-dimensional perturbations, after that, for finite-dimensional perturbations, and finally to use an approximation argument for general trace class perturbations. The major step of this proof was the first step, i.e. the case of one-dimensional perturbation. Kreĭn showed that for one-dimensional perturbations the perturbation determinant

$$\det(1 + V(H_0 - z)^{-1})$$

satisfies the conditions of a theorem from complex analysis, more precisely, that it is a Herglotz function and behaves like $\frac{1}{y}$ for large values of $y = \operatorname{Im} z$.

One of the aims of this thesis is to establish the analogue of the Lifshits-Kreĭn theory for general semifinite von Neumann algebras $\mathcal{N}$ with a faithful semifinite normal trace τ. In the case of a bounded self-adjoint operator $H_0 \in \mathcal{N}$ and τ-trace class perturbation V, this problem has been solved by R. W. Carey and J. D. Pincus in [CP]. The novelty of our approach [ADS] is to consider the case of an (unbounded) operator H_0 affiliated with $\mathcal{N}$. The main result of Section 3.1 is Theorem 3.1.13, which is a semifinite analogue of classical result of M. G. Kreĭn.

The main difficulty here is that there is no proper analogue of the classical Fredholm determinant in semifinite von Neumann algebras (recall that Fuglede-Kadison determinant takes only non-negative values; in the type I case, the Fuglede-Kadison determinant is just the absolute value of Fredholm determinant). This difficulty is overcome with the use of the Brown measure [Brn]. The Brown measure together with the semifinite analogue of the Lidskii theorem [Brn] (see [Lid], [Si, Section 3] for the classical Lidskii theorem) allows us to prove the conditions of the above mentioned theorem from complex analysis in case of τ-finite perturbations. Generalization to an arbitrary relatively trace class perturbations follows the lines of the classical case $\mathcal{N} = \mathcal{B}(\mathcal{H})$.

Note that, if $\tau(1) < \infty$, then the spectral shift formula (3.25) may be derived directly by the argument given in [Kr_2], provided f is absolutely continuous and $f' \in L^1(\mathbb{R})$. This argument yields the formula

$$\xi_{H+V,H}(\lambda) = \tau\left(E_\lambda^H\right) - \tau\left(E_\lambda^{H+V}\right), \quad \text{a. e. } \lambda \in \mathbb{R}. \tag{3}$$

This formula goes back to Lifshits [Lif] and reduces the calculation of the spectral shift function to computation of the spectral distributions of the operators $H + V, H$. In the setting given by Theorem 3.1.13, again in the special case of finite trace, the formula (3) may be derived from (3.25) by a standard argument.

Another problem considered in this thesis, is the semifinite version of Birman-Solomyak formula for the spectral shift function. In 1975, Birman and Solomyak established the beautiful formula for the spectral shift function

$$\xi(\lambda) = \frac{d}{d\lambda} \int_0^1 \operatorname{Tr}(V E_\lambda^{H_r})\, dr, \tag{4}$$

where $H_r = H_0 + rV$, $V \in \mathcal{L}^1(\mathcal{H})$. This formula is called the *spectral averaging* formula. Birman-Solomyak proved this formula using double operator integrals. This formula was established by V. A. Javrjan in [Jav] four years earlier. Javrjan considered the spectral averaging of Sturm-Liouville operator on a half-line with respect to boundary condition. This corresponds to one-dimensional perturbation.

In 1998, B. Simon [Si_2] found a short and simple proof of (4). As B. Simon notes in [Si_2], the formula (4) was rediscovered by many authors, who were not aware of V. A. Javrjan and Birman-Solomyak's papers; among them Kotani [Ko], who, in development of the celebrated result of Goldshtein-Molchanov-Pastur [GMP], used spectral averaging to show that the spectrum of certain random one-dimensional Schrödinger operators is a purely point spectrum with probability 1 and that the corresponding eigenfunctions decay exponentially.

The main result of Section 3.4 (Theorem 3.4.2) establishes a semifinite analogue of Birman-Solomyak's spectral averaging formula. The proof follows essentially the original proof of Birman-Solomyak. Since Birman-Solomyak's proof uses double operator integrals, it was necessary to develop the theory of the double (and in general multiple) operator integral to von Neumann algebras. The double operator integral theory developed in [dPSW] and [dPS] is not applicable in this situation, since the unperturbed operator H_0 is not in general τ-measurable, as required in [dPSW, dPS]. In the type I case, τ-measurability is equivalent to boundedness. Consequently, it is first necessary to develop the theory of double (multiple) operator integrals in von Neumann algebras, that will cover the situation that H_0 is unbounded.

Multiple operator integrals were first introduced in the celebrated work of Yu. L. Daletskiĭ and S. G. Kreĭn [DK]. A multiple operator integral is an expression of the form

$$T_\varphi^{H_0,H_1,\ldots,H_n}(V_1,\ldots,V_n) := \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi(\lambda_0,\ldots,\lambda_n)\,dE_{\lambda_0}^{H_0}V_1\,dE_{\lambda_1}^{H_1}V_2\,dE_{\lambda_2}^{H_2}\ldots V_n\,dE_{\lambda_n}^{H_n},$$

where $V_1,\ldots,V_n$ are bounded operators on $\mathcal{H}$, $H_0,\ldots,H_n$ are self-adjoint operators on $\mathcal{H}$, and φ is a function of $n+1$ variables. The initial approach of [DK] to the definition of multiple operator integrals is to consider them as repeated integrals

$$\int_{-\infty}^{\infty}\left(\cdots\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\varphi\,dE_{\lambda_0}^{H_0}\right)V_1\,dE_{\lambda_1}^{H_1}\right)V_2\ldots\right)V_n\,dE_{\lambda_n}^{H_n},$$

for which purpose they define first the spectral integrals of operator valued functions

$$\int_{-\infty}^{\infty}F(\lambda)\,dE_\lambda^H.$$

Another approach to the theory of the multiple operator integral was given in [Pa]. In this work, B. S. Pavlov considers the multiple operator integral as an integral over the vector-valued measure

$$\Delta_0\times\ldots\times\Delta_n\in\mathcal{B}(\mathbb{R}^{n+1})\mapsto E_{\Delta_0}^{H_0}V_1\,E_{\Delta_1}^{H_1}V_2\,E_{\Delta_2}^{H_2}\ldots V_n\,E_{\Delta_n}^{H_n}.$$

Pavlov proves that, if $V_1,\ldots,V_n\in\mathcal{L}^2(\mathcal{H})$, then this measure is countably additive and has bounded weak variation, so that for any bounded measurable function φ the multiple operator integral can be considered as integral over this vector-valued measure. He then extends this definition to arbitrary bounded V_j's under some additional conditions on φ.

For the purpose of generalizing the theory of multiple operator integrals to von Neumann algebras, it is convenient [ACDS] to define the multiple operator integral as follows. If one can write the function φ in the form (see (3.27))

$$\varphi(\lambda_0, \lambda_1, \dots, \lambda_n) = \int_S \alpha_0(\lambda_0, \sigma) \dots \alpha_n(\lambda_n, \sigma)\, d\nu(\sigma), \tag{5}$$

then one can see that formally

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \varphi(\lambda_0, \dots, \lambda_n)\, dE_{\lambda_0}^{H_0} V_1\, dE_{\lambda_1}^{H_1} V_2\, dE_{\lambda_2}^{H_2} \dots V_n\, dE_{\lambda_n}^{H_n}$$
$$= \int_S \alpha_0(H_0, \sigma) V_1 \alpha_1(H_1, \sigma) \dots V_n \alpha_n(H_n, \sigma)\, d\nu(\sigma).$$

We call the representations of the form (3.27) BS-representations. The idea is to define the multiple operator integral by the right hand side of this equality. One has to prove that this definition is well-defined, i.e. that it does not depend on the representation (5) of the function φ. This is done in Theorem 3.2.8. This idea is taken from the work of Solomyak and Sten'kin [SS], who actually used implicitly this definition of multiple operator integral. The difference was that they considered series of the form

$$\varphi(\lambda_0, \lambda_1, \dots, \lambda_n) = \sum_{k=1}^{\infty} \alpha_{0,k}(\lambda_0) \dots \alpha_{n,k}(\lambda_n).$$

This same idea had been used earlier to define multiple operator integrals independently by V. V. Peller [Pel].

An advantage of our new approach to the definition of multiple operator integrals, is that once some BS representation for φ is found, one can work with the multiple operator integral as the usual integral of operator-valued functions, consequently using the well-developed and the well-known theory of such integrals. Another advantage is that sometimes different BS representations for the same function φ turn out to be better suited for a particular problem. For example, it is known that the difference $f(A) - f(B)$ can be represented as

$$f(A) - f(B) = T_{f^{[1]}}^{A,B}(A - B),$$

where $f^{[1]}(\lambda, \mu) = \frac{f(\lambda) - f(\mu)}{\lambda - \mu}$ is the first divided difference of the function f. Examples of usage of different BS-representations of $f^{[1]}$ can be found in Sections 3.3 and 4.1.

The last chapter is devoted to the notion of spectral flow and its connection with the theory of spectral shift function. The notion of spectral flow was introduced by M. Atiyah, V. Patodi and I. M. Singer in [APS] as the net number of eigenvalues which cross zero from the left to the right. E. Getzler [Ge, Theorem 2.6] established the following formula

$$\mathrm{sf}(D_0, D_1) = \sqrt{\frac{\varepsilon}{\pi}} \int_0^1 \mathrm{Tr}\left(\dot{D}_u e^{-\varepsilon D_u^2}\right) du + \frac{1}{2}\eta_\varepsilon(D_1) - \frac{1}{2}\eta_\varepsilon(D_0), \tag{6}$$

where

$$\eta_\varepsilon(D_0) := \frac{1}{\sqrt{\pi}} \int_\varepsilon^\infty \tau\left(D_0 e^{-tD_0^2}\right) t^{-1/2}\, dt$$

is η- invariant of D_0, and $\{D_u\}$ is a piecewise smooth path connecting D_0 and D_1. The integral (6) is interpreted as an integral of the one-form

$$\alpha_\varepsilon(X) = \sqrt{\frac{\varepsilon}{\pi}}\, \mathrm{Tr}(Xe^{-\varepsilon D^2}), \tag{7}$$

where $X \in \mathcal{B}_{sa}(\mathcal{H})$, the real Banach space of bounded self-adjoint operators on the Hilbert space $\mathcal{H}$. J. Phillips [Ph, Ph_2] gave a definition of spectral flow different from the original definition of [APS]. This definition interprets spectral flow as Fredholm index and as such it can be generalized also to the case of semifinite von Neumann algebras. In [CP, CP_2] A. L. Carey and J. Phillips generalized the integral formulas for spectral flow to the semifinite case, establishing integral formulas for the θ-summable and p-summable cases (i.e. $(1 + D^2)^{-p/2}$ has finite τ-trace). In particular they establish the formula ($p > 1$)

$$\mathrm{sf}(D_0, D_1) = \tilde{C}_p^{-1} \int_0^1 \mathrm{Tr}\left(\dot{D}_t(1 + D_t^2)^{-p}\right) dt + \beta_p(D_1) - \beta_p(D_0),$$

where $\beta_p(D)$ is an analogue of the η-invariant for the p-summable case [CP]. In the case of a p-summable spectral triple $(\mathcal{A}, D_0, \mathcal{N})$ and perturbation $V = u[D_0, u^*]$, the operators D_0 and $D_1 = uD_0u^*$ are unitarily equivalent, so that the last formula takes the form

$$\mathrm{sf}(D_0, uD_0u^*) = \tilde{C}_p^{-1} \int_0^1 \mathrm{Tr}\left(u[D_0, u^*](1 + (D_0 + tu[D_0, u^*])^2)^{-p/2}\right) dt. \tag{8}$$

In [CPRS], this formula is the starting point for a proof of the Local Index Theorem of Connes-Moscovici in non-commutative geometry. One of the ideas of the proof is to consider p as complex variable, i.e. to consider analytical continuation of the last integral as a function of p. In [CPS], using the zeta-function representation for the Dixmier trace due to A. Connes [Co], it was shown that when $p \to 1^+$, the spectral flow becomes the Dixmier trace [CPS, Theorem 6.2]. At the noncommutative geometry workshop at Banff in 2005, it was observed that when $p \to \infty$, the last integral formula for spectral flow becomes the Birman-Solomyak formula for the spectral shift function. This key observation was developed in [ACS]. One of the main results of [ACS] states that if D_0 is an operator with compact resolvent and D_1 its perturbation by a bounded self-adjoint operator, then

$$\mathrm{sf}(\lambda; D_0, D_1) = \xi_{D_1, D_0}(\lambda) + \frac{1}{2}\tau\left(\mathrm{N}_{D_1 - \lambda}\right) - \frac{1}{2}\tau\left(\mathrm{N}_{D_0 - \lambda}\right).$$

Combined with the Lifshits-Kreĭn trace formula (1), this formula also implies that when the "endpoints" D_0 and D_1 are unitarily equivalent, the spectral flow (= the spectral shift) function is constant. This sheds some light on integral formulas for spectral flow like (8), since it is otherwise difficult to understand why one should take into account eigenvalues very far from 0 to compute the spectral flow at 0. The point is that, in the case of unitarily equivalent endpoints $D_0 \sim D_1$, the spectral flow at all points is the same ("the law of conservation of spectrum"), so that one can compute "parts" of spectral flow anywhere on the spectral line.

The well-known Lidskii theorem (in its general semifinite form given in [Brn]) asserts that if $\mathcal{N}$ is a semifinite von Neumann factor with a faithful normal semifinite trace τ, then the trace $\tau(T)$ of an arbitrary operator $T \in L^1(\mathcal{N}, \tau)$ is given by

$$\tau(T) = \int_{\sigma(T)\setminus\{0\}} \lambda \, d\mu_T(\lambda),$$

where μ_T is a Borel measure (the so-called Brown measure of T) on the non-zero spectrum of T. In the case when $\mathcal{N}$ is a type I factor, the measure μ_T is the counting measure on the set of all eigenvalues of T. In Section 2.2.2, we present an analogue of such a formula for Dixmier traces.

In the case of a standard (normal) trace, the assertion of the Lidskii theorem for self-adjoint operators is immediate due to the absolute convergence of the series $\sum_{n\geqslant 1} \lambda_n(T)$ of any $T = T^*$ from the trace class. This is not the case any longer for Dixmier (non-normal) traces, since the latter series diverges for any $T = T^* \in \mathcal{L}^{1,\infty}(\mathcal{N}, \tau)$ which does not belong to the trace class.

The main result of Section 2.1 is Theorem 2.2.11. The Lidskii type formula given there holds for all operators $T \in \mathcal{L}^{1,\mathrm{w}}$. The ideal $\mathcal{L}^{1,\mathrm{w}}$ usually arises in geometric applications. In particular, if $\mathcal{N}$ is the algebra of all bounded operators on $L^2(M)$ where M is a compact Riemannian n-manifold (respectively, if $\mathcal{N}$ is the II_∞ factor $L^\infty(\mathbb{R}^n) \rtimes \mathbb{R}^n_{discr}$ [CMS, Sh]), the ideal $\mathcal{L}^{1,\mathrm{w}}$ contains all pseudodifferential operators (respectively, all almost periodic pseudodifferential operators) of order $-n$.

The Lidskii formula for Dixmier traces τ_ω, where ω is an arbitrary dilation invariant state on $L^\infty(0,\infty)$, takes an especially simple form for the case of measurable operators T (by definition, an operator $T \in \mathcal{L}^{1,\infty}(\mathcal{N}, \tau)$ is measurable if $\tau_\omega(T)$ does not depend on ω). In this case, $\tau_\omega(T)$ coincides with the true limit

$$\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_{\lambda \notin \frac{1}{t} G} \lambda \, d\mu_T(\lambda).$$

The proof of Theorem 2.2.11 depends crucially on the recent characterization of positive measurable operators from $\mathcal{L}^{1,\infty}(\mathcal{N}, \tau)$ as those for which the limit

$$\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T) \, ds$$

exists [LSS, Theorem 6.6], and the spectral characterization of sums of commutators in type II factors [DK_2, Fac]. The spectral characterization of sums of commutators is a very deep result, obtained independently by T. Fack [Fac, Theorem 3], N. J. Kalton [Kal, DK] and K. J. Dykema and N. J. Kalton [DK_2, Theorem 6.8, Corollary 6.10] (though the main idea of the proof seems to be the same in all these papers). This result implies that the Dixmier trace of any operator T from the Dixmier ideal depends only on the Brown measure of the operator T. However, Theorem 2.2.11 goes much further and gives an explicit formula for the Dixmier trace $\tau_\omega(T)$ in terms of the Brown measure μ_T.

Now we give a brief description of the sections of Chapter 1 (Preliminaries). In Section 1.5 (Theory of τ-Fredholm operators) we give an exposition of the theory of τ-Fredholm operators. We follow mainly the original Breuer's works [Br, Br_2] and [PR, Appendix B]. Breuer proved his results for semifinite factors, but as shown in [PR, Appendix B] the difference between the factor case and the non-factor case is not significant. In Section 1.6 (Spectral flow in von Neumann algebras) an exposition is given of J. Phillips' theory of spectral flow in semifinite von Neumann algebras. Here I follow the papers of J. Phillips [Ph, Ph_2].

Section 1.7 (Fuglede-Kadison determinant) contains an exposition of the Fuglede-Kadison determinant [FKa], following L. G. Brown's paper [Brn]. B. Fuglede and R. V. Kadison introduced this determinant in the case of type II_1 factors, while L. G. Brown considers semifinite factors. We give this theory for semifinite von Neumann algebras, not necessarily factors. In Section 1.8 (The Brown measure), an exposition of the Brown measure is given, following the original work of Brown [Brn].

The main results of this thesis are Theorem 3.2.8 [ACDS] (new approach to multiple operator integrals), Theorem 2.2.11 [AS] (Lidskii theorem for Dixmier traces), Theorem 3.1.13 [ADS] (Krein's formula for spectral shift function in semifinite von Neumann algebras), Theorem 3.3.3 [ACDS], Theorem 3.3.6 [ACDS] (high order Fréchet derivative of functions $f(H)$ of self-adjoint operators H with suitable restrictions on the function f), Theorem 3.4.2 [ACDS] (semifinite Birman-Solomyak spectral averaging formula), Theorem 4.1.17 [ACDS] (formula for Fréchet derivative in terms of double operator integrals), Theorem 4.2.5 [ACS] (trace formula for operators with compact resolvent), Theorem 4.3.13 [ACS], Theorem 4.3.18 [ACS] (connection between spectral flow and spectral shift function), Theorem 4.3.21 [ACS] (infinitesimal spectral flow), Theorem 4.3.24 [ACS] (spectral flow for $\mathcal{I}$-summable spectral triples), Theorem 4.3.31 [ACS] (Carey–Phillips formula with new proof).

Chapter 1

Preliminaries

1.1 Operators in Hilbert space

1.1.1 Notation

We denote by $\mathbb{R}$ the field of all real numbers, and by $\mathbb{C}$ the field of all complex numbers. We denote by $L^1(\mathbb{R})$ the Banach space of 1-summable functions on $\mathbb{R}$ with the norm $\|\cdot\|_1$. By $\mathcal{H}$ we denote a complex separable (if not stated otherwise) Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$, anti-linear in the first variable, and the norm $\|\xi\| = \sqrt{\langle \xi, \xi \rangle}$.

If $\Omega \subset \mathbb{R}^n$ is an open set then we write $C_c^\infty(\Omega)$ for the set of all compactly supported C^∞-smooth functions on Ω, and $B(\mathbb{R}^n)$ (respectively, $B_c(\mathbb{R}^n)$) for the set of all bounded Borel functions on $\mathbb{R}^n$ (respectively, compactly supported bounded Borel functions on $\mathbb{R}^n$).

Suppose that T is a closed linear operator in $\mathcal{H}$, with dense domain $\mathcal{D}(T) \subseteq \mathcal{H}$. The **resolvent set** ρ_T is the set of those complex numbers λ for which $\lambda - T \colon \mathcal{D}(T) \to \mathcal{H}$ has a bounded inverse with domain dense in $\mathcal{H}$. Since T is closed, it follows from [HPh, Theorem 2.16.3] that $\lambda \in \rho_T$ if and only if $T - \lambda$ is injective and surjective. The closed graph theorem then implies that the **resolvent**

$$R_\lambda(T) := (\lambda - T)^{-1}, \quad \lambda \in \rho_T,$$

is a bounded linear operator on $\mathcal{H}$. The **spectrum** of a closed linear operator T is the set

$$\sigma_T := \mathbb{C} \setminus \rho_T.$$

We will use the **first resolvent identity**

$$R_\lambda(T) - R_\mu(T) = (\mu - \lambda) R_\lambda(T) R_\mu(T), \quad \lambda, \mu \in \rho_T, \tag{1.1}$$

and the **second resolvent identity**

$$R_\lambda(S) - R_\lambda(T) = R_\lambda(S)(S-T)R_\lambda(T), \quad \lambda \in \rho_S \cap \rho_T. \tag{1.2}$$

Also,

$$\frac{d}{dz} R_z(T) = -R_z(T)^2, \tag{1.3}$$

where derivative taken in $\|\cdot\|$-topology.

If T is a self-adjoint operator (not necessarily bounded) then

$$\|R_\lambda(T)\| \leqslant |\lambda|^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{1.4}$$

1.1.2 Topologies of $\mathcal{B}(\mathcal{H})$

By $\mathcal{B}(\mathcal{H})$, we denote the set of all bounded linear operators on the Hilbert space $\mathcal{H}$.

A set of operators $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is said to be a (complex) *algebra*, if for any complex numbers $\alpha, \beta \in \mathbb{C}$ and any operator $S, T \in \mathcal{A}$ the operators $\alpha S + \beta T$ and ST also belong to $\mathcal{A}$. An algebra $\mathcal{A}$ of operators on $\mathcal{H}$ is said to be *involutive* if $T \in \mathcal{A}$ implies that $T^* \in \mathcal{A}$. An involutive algebra $\mathcal{A}$ of operators is said to be a $*$-*ideal* of an algebra of operators $\mathcal{N}$, if for any $A \in \mathcal{A}$ and $S \in \mathcal{N}$ we have AS, $SA \in \mathcal{A}$.

On the algebra $\mathcal{B}(\mathcal{H})$ there exist several natural topologies. The *uniform topology* is the topology of the norm

$$\|T\| = \sup_{x \in \mathcal{H}, \|x\|=1} \|Tx\|, \quad T \in \mathcal{B}(\mathcal{H}).$$

The *strong operator topology* (or *so*-topology) is a locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the system $\{p_\xi(\cdot),\ \xi \in \mathcal{H}\}$ of seminorms

$$p_\xi(T) = \|T\xi\|, \quad T \in \mathcal{B}(\mathcal{H}).$$

The *strong* operator topology* (or *so**-topology) is a locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the system $\left\{p_\xi(\cdot), p_\xi^*(\cdot),\ \xi \in \mathcal{H}\right\}$ of seminorms

$$p_\xi(T) = \|T\xi\|, \ \ p_\xi^*(T) = \|T^*\xi\|, \quad T \in \mathcal{B}(\mathcal{H}).$$

The *weak operator topology* (or *wo*-topology) is a locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the system $\{p_{\xi,\eta}(\cdot),\ \xi, \eta \in \mathcal{H}\}$ of seminorms

$$p_{\xi,\eta}(T) = |\langle T\xi, \eta\rangle|, \quad T \in \mathcal{B}(\mathcal{H}).$$

The σ-*weak topology* is the topology generated by seminorms

$$A \in \mathcal{B}(\mathcal{H}) \mapsto p_{\bar\xi,\bar\eta}(A) = \left(\sum_{k=1}^\infty |\langle A\xi_k, \eta_k\rangle|\right)^{-1/2},$$

the *σ-strong topology* is the topology generated by seminorms

$$A \in \mathcal{B}(\mathcal{H}) \mapsto p_{\bar{\xi}}(A) = \left(\sum_{k=1}^{\infty} \|A\xi_k\|^2 \right)^{-1/2},$$

the *σ-strong* topology* is the topology generated by seminorms

$$p_{\bar{\xi}}(A), \quad p_{\bar{\xi}}(A^*,)$$

where the sequences $\bar{\xi} = (\xi_1, \xi_2, \ldots)$ and $\bar{\eta} = (\eta_1, \eta_2, \ldots)$ are such that

$$\sum_{k=1}^{\infty} \|\xi_k\|^2 < \infty, \quad \sum_{k=1}^{\infty} \|\eta_k\|^2 < \infty.$$

Theorem 1.1.1 [BR, Proposition 2.4.1] *The σ-strong topology is finer than the strong operator topology, but the two topologies coincide on the unit ball $\mathcal{B}_1(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$. The unit ball $\mathcal{B}_1(\mathcal{H})$ is complete in the uniform structure defined by these topologies. Multiplication $(A, B) \mapsto AB$ is continuous as a map $\mathcal{B}_1(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ in these topologies.*

Recall that $\mathcal{H}$ is a *separable* Hilbert space.

Proposition 1.1.2 [Di, Proposition I.3.1] *The unit ball $\mathcal{B}_1(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$ endowed with the strong operator topology is a metrisable space.*

It is not difficult to see that this proposition is true also for strong* operator topology.

1.1.3 Self-adjoint operators

In this section we collect some theorems about unbounded operators. Their proofs can be found in [RS].

By $\mathcal{B}(\mathbb{R})$, we denote the σ-algebra of all Borel subsets of $\mathbb{R}$. For a self-adjoint operator T let E^T_Δ be the spectral projection of T corresponding to $\Delta \in \mathcal{B}(\mathbb{R})$, and let E^T_λ be the spectral projection of T corresponding to $(-\infty, \lambda]$. This means, in particular, that for any $\lambda \in [-\infty, \infty)$

$$E^T_\lambda = \inf_{\mu > \lambda} E^T_\mu. \tag{1.5}$$

The following lemma is [CP, Appendix B, Lemma 1].

Lemma 1.1.3 *If A and B are (possibly unbounded) self-adjoint operators with $\operatorname{dom}(A) = \operatorname{dom}(B)$ and $0 < c \leqslant A \leqslant B$ on their common domain, then $0 \leqslant B^{-1} \leqslant A^{-1} \leqslant c^{-1}$ on all of $\mathcal{H}$. Here c is a scalar operator.*

Proof. For $\xi \in \operatorname{dom}(B)$, the operator $K(B^{\frac{1}{2}}\xi) = A^{\frac{1}{2}}\xi$ is well-defined and $\|K\| \leqslant 1$, since $\left\|A^{\frac{1}{2}}\xi\right\| \leqslant \left\|B^{\frac{1}{2}}\xi\right\|$. Since the closure of $B^{\frac{1}{2}}|_{\operatorname{dom}(B)}$ is $B^{\frac{1}{2}}$, one checks that $K(B^{\frac{1}{2}}\xi) = A^{\frac{1}{2}}\xi$ makes sense for all $\xi \in \operatorname{dom} B^{\frac{1}{2}} \subseteq \operatorname{dom} A^{\frac{1}{2}}$ and so $KB^{\frac{1}{2}} = A^{\frac{1}{2}}$. Since $B^{\frac{1}{2}} \geqslant c^{\frac{1}{2}}$, it follows that $\operatorname{ran} B^{\frac{1}{2}} = \mathcal{H}$ and hence K is everywhere defined and injective. Since $\operatorname{ran} K \supseteq \operatorname{ran} A^{\frac{1}{2}} = \mathcal{H}$, it follows that K is invertible. Thus, $B^{\frac{1}{2}} = K^{-1}A^{\frac{1}{2}}$ and so $B^{-\frac{1}{2}} = A^{-\frac{1}{2}}K$ and, taking adjoint of this equality, $B^{-\frac{1}{2}} = K^*A^{-\frac{1}{2}}$. So, for any $\xi \in \mathcal{H}$

$$\begin{aligned}\left\langle B^{-1}\xi, \xi\right\rangle &= \left\langle K^*A^{-\frac{1}{2}}\xi, K^*A^{-\frac{1}{2}}\xi\right\rangle = \left\|K^*A^{-\frac{1}{2}}\xi\right\|^2 \\ &\leqslant \|K^*\|^2 \cdot \left\|A^{-\frac{1}{2}}\xi\right\|^2 \leqslant \left\|A^{-\frac{1}{2}}\xi\right\|^2 = \left\langle A^{-1}\xi, \xi\right\rangle .\end{aligned}$$

□

Theorem 1.1.4 [RS, Section VIII.3] *Spectral resolution* $\left\{E^T_\Delta,\ \Delta \in \mathcal{B}(\mathbb{R})\right\}$ *of a self-adjoint operator* T *on Hilbert space* $\mathcal{H}$ *is* σ*-additive in the strong operator topology.*

Theorem 1.1.5 [RS, Theorem VIII.7] *If* H *is a (possibly unbounded) self-adjoint operator on* $\mathcal{H}$*, then the function* $\mathbb{R} \ni t \mapsto e^{itH} \in \mathcal{B}(\mathcal{H})$ *is so***-continuous.*

Actually, [RS, Theorem VIII.7] says that the function $\mathbb{R} \ni t \mapsto e^{itH} \in \mathcal{B}(\mathcal{H})$ is continuous in *so*-topology, but for e^{itH} this evidently implies continuity in *so**-topology.

By definition, a sequence of self-adjoint operators A_n *resolvent strongly* (respectively, *resolvent uniformly*) converges to self-adjoint operator A if the sequence of resolvents of A_n converges to the resolvent of A in *so*-topology (respectively, uniformly).

Theorem 1.1.6 [RS, Theorem VIII.20(b)] *Let* A *and* $A_1, A_2, \ldots$ *be self-adjoint operators on* $\mathcal{H}$*. If the sequence* A_n *resolvent strongly converges to* A *and* f *is a bounded Borel function on* $\mathbb{R}$ *then the sequence* $f(A_n)$ *converges to* $f(A)$ *in so-topology.*

We note that, for any spectral resolution E_λ and any $\xi, \eta \in \mathcal{H}$, the measure $\Delta \mapsto \langle E_\Delta \xi, \eta\rangle$ has finite total variation (this easily follows from polar decomposition and the fact that the total variation of non-negative measure $\langle E_\Delta \xi, \xi\rangle$ is $\|\xi\|^2$).

Lemma 1.1.7 *If* A *is a self-adjoint (possibly unbounded) operator on a Hilbert space* $\mathcal{H}$ *and if* f *is a function on* $\mathbb{R}$ *which is the Fourier transform of a finite*

Borel measure m on $\mathbb{R}$ then

$$f(A) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{isA}\, dm(s),$$

where the integral is taken in so-topology.

Proof. For any ξ, $\eta \in \mathcal{H}$ we have

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \left\langle \int_{\mathbb{R}} e^{isA}\, dm(s)\, \xi, \eta \right\rangle &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left\langle e^{isA}\xi, \eta \right\rangle\, dm(s) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{is\lambda} \left\langle dE_\lambda\, \xi, \eta \right\rangle \right) dm(s) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{is\lambda}\, dm(s) \right) \left\langle dE_\lambda\, \xi, \eta \right\rangle \\
&= \int_{\mathbb{R}} f(\lambda)\, \left\langle dE_\lambda\, \xi, \eta \right\rangle \\
&= \left\langle \int_{\mathbb{R}} f(\lambda)\, dE_\lambda\, \xi, \eta \right\rangle = \left\langle f(A)\xi, \eta \right\rangle,
\end{aligned}$$

where the interchange of integrals by Fubini's theorem is possible, since both measures have finite total variation. ☐

Lemma 1.1.8 *(Duhamel's formula). If B is an unbounded self-adjoint operator on a Hilbert space $\mathcal{H}$, if V is a bounded self-adjoint operator on $\mathcal{H}$ and if $A = B + V$, then*

$$e^{isA} - e^{isB} = \int_0^s e^{i(s-t)A} iV e^{itB}\, dt, \tag{1.6}$$

where the integral converges in so-topology.

Proof. Let $F(t) = e^{-itA}e^{itB}$. Taking derivative of $F(t)$ in the *so*-topology gives

$$F'(t) = -iAe^{-itA}e^{itB} + e^{-itA}(iB)e^{itB} = -e^{-itA}i(A-B)e^{itB}.$$

So,

$$-\int_0^s e^{-itA}i(A-B)e^{itB}\, dt = F(s) - F(0) = e^{-isA}e^{isB} - 1.$$

Multiplying the last equality by e^{isA} from the left gives (1.6). ☐

1.1.4 Numerical range

The **numerical range** $\mathrm{W}(T)$ of an operator $T \in \mathcal{B}(\mathcal{H})$ is the set

$$\mathrm{W}(T) := \{\langle T\eta, \eta\rangle : \eta \in \mathcal{H},\ \|\eta\| = 1\}$$

of complex numbers. Numerical range has the following properties.

Theorem 1.1.9 *(Toeplitz-Hausdorff theorem) If $T \in \mathcal{B}(\mathcal{H})$ then* $\mathrm{W}(T)$ *is a convex subset of* $\mathbb{C}$.

Theorem 1.1.10 *If $T \in \mathcal{B}(\mathcal{H})$ then*

$$\sigma_T \subseteq \overline{\mathrm{W}(T)}.$$

Theorem 1.1.11 *If an operator $T \in \mathcal{B}(\mathcal{H})$ is normal then*

$$\overline{\mathrm{W}(T)} = \operatorname{conv} \mathrm{W}(T),$$

where conv *denotes the convex hull.*

Proofs of these theorems can be found in [Hal, Chapter 22], Problems 210, 214 and 216 respectively.

1.1.5 The Bochner integral

In this section we collect the properties of the Bochner integral which will be used in the subsequent text.

Let (S, ν) be a measure space and let $\mathcal{X}$ be a Banach space. A function $f\colon S \to \mathcal{X}$ is said to be **Bochner integrable**, if there exists a sequence of *simple functions* (i.e. finitely-valued) $f_n\colon S \to \mathcal{X}$ norm converging a.e., such that

$$\lim_{n\to\infty} \int_S \|f(s) - f_n(s)\| \, d\nu(s) = 0.$$

In this case the **Bochner integral** of the function f is defined as

$$\int_S f(s)\, d\nu(s) = \lim_{n\to\infty} \int_S f_n(s)\, d\nu(s).$$

It can be shown that this definition is well defined in the sense that it does not depend on a choice of the sequence $\{f_n\}$ [Y, V.5].

When it is necessary, we shall use the terms ν-Bochner integrable and ν-the Bochner integral.

Lemma 1.1.12 [Y, Corollary V.5.2] *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and let $T\colon \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator. If a function $f\colon S \to \mathcal{X}$ is Bochner integrable, then the function $Tf\colon S \to \mathcal{Y}$ is also Bochner integrable, and*

$$\int_S Tf(s)\, d\nu(s) = T \int_S f(s)\, d\nu(s).$$

Theorem 1.1.13 [DS, Theorem III.6.16] *(Lebesgue Dominated Convergence Theorem for the Bochner integral) Let (S, ν) be a measure space and let $\mathcal{X}$ be a Banach space. Let $f_1, f_2, \ldots$ be a sequence of Bochner-integrable functions $S\colon \to \mathcal{X}$ converging ν-almost everywhere to a function f. Suppose that there exists a Bochner-integrable function $g\colon S \to \mathcal{X}$ such that for all $n = 1, 2, \ldots$ $\|f_n(s)\| \leqslant \|g(s)\|$ ν-almost everywhere. Then f is Bochner integrable and*

$$\int_S \|f_n(s) - f(s)\| \, d\nu(s) \quad \to \quad 0.$$

Lemma 1.1.14 [Y, Corollary V.5.1] *If $f\colon (S, \nu) \to \mathcal{X}$ is a Bochner integrable function, then*

$$\left\| \int_S f(s) \, d\nu(s) \right\| \leqslant \int_S \|f(s)\| \cdot |d\nu(s)| \, .$$

Theorem 1.1.15 [DS, Theorem III.11.13] *Let (S, ν) and (T, μ) be two finite measure spaces. Let $\mathcal{X}$ be a Banach space and let $f\colon S \times T \to \mathcal{X}$ be a $\nu \times \mu$-Bochner integrable function. Then, for ν-almost all $s \in S$, the function $f(s, \cdot)$ is μ-Bochner integrable on T and the function $\int_T f(\cdot, t) \, d\mu(t)$ is μ-Bochner integrable on S. Moreover,*

$$\int_S \left(\int_T f(s,t) \, d\mu(t) \right) \, d\nu(s) = \int_{S \times T} f(s,t) \, d\nu \times \mu(s,t).$$

1.2 Fréchet derivative

Definition 1.2.1 *Let $\mathcal{X}_1$ be a topological vector space, $\mathcal{X}_2$ be a locally convex topological vector space, and let $\mathcal{E}_1, \mathcal{E}_2$ be normed spaces embedded in $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively. Let $X_0 \in \mathcal{X}_1$ and*

$$f\colon X_0 + \mathcal{E}_1 \to f(X_0) + \mathcal{E}_2.$$

*The function f is called **Fréchet differentiable** at $X_0 \in \mathcal{X}_1$ along $\mathcal{E}_1$ if there exists a (necessarily unique) bounded (linear) operator $L\colon \mathcal{E}_1 \to \mathcal{E}_2$ such that*

$$f(X_0 + V) - f(X_0) = L(V) + r(X_0, V),$$

where $\|r(X_0, V)\|_{\mathcal{E}_2} = o(\|V\|_{\mathcal{E}_1})$. We write $L = \mathcal{D}_{\mathcal{E}_1, \mathcal{E}_2} f(X_0)$. In case $\mathcal{E}_1 = \mathcal{E}_2 =:$ $\mathcal{E}$ we write $\mathcal{D}_{\mathcal{E}} f(X_0)$.

Theorem 1.2.2 *Let*

$$f\colon X_0 + \mathcal{E}_1 \to f(X_0) + \mathcal{E}_2.$$

be Fréchet differentiable along $\mathcal{E}_1$ and let $X_t = X_0 + tV$, $t \in [a,b]$, where $a, b \in \mathbb{R}$ and $V \in \mathcal{E}_1$. If the function $[a,b] \ni t \to \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)$ is piecewise continuous, then

$$\int_a^b \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)(V)\,dt = f(X_b) - f(X_a),$$

where the integral is the Bochner integral.

Proof. The additivity of the integral allows us to assume that the Fréchet derivative $\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X)$ is continuous. Let $l \in \mathcal{X}_2^*$. Let $g(t) = l(f(X_0 + tV))$.

We have

$$\begin{aligned} g'(t) &= \lim_{s\to t} \frac{l(f(X_s)) - l(f(X_t))}{s-t} \\ &= \lim_{s\to t} \frac{(s-t)l(\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)(V)) + l(r(X_t,(s-t)V))}{s-t} = l(\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)(V)). \end{aligned}$$

Hence,

$$l\left(\int_a^b \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)(V)\,dt\right) = \int_a^b l\left(\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)(V)\right)\,dt = l\left(f(X_b) - f(X_a)\right),$$

where the integral and the l functional can be interchanged since $\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2}$ is continuous. Since $\mathcal{X}_2$ is locally convex, the proof is complete. □

Theorem 1.2.3 *Let*

$$f\colon X_0 + \mathcal{E}_1 \to f(X_0) + \mathcal{E}_2$$

be Fréchet differentiable and let $\{X_t\}_{t\in[a,b]}$ be a smooth path in $X_0 + \mathcal{E}_1$. If the function $[a,b] \ni t \to \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)$ is piecewise continuous, then

$$\int_a^b \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)(\dot{X}_t)\,dt = f(X_b) - f(X_a).$$

Proof. The additivity of the integral allows us to assume that the Fréchet derivative $\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X)$ is continuous. Further, Theorem 1.2.2 implies that it is enough to show that for given $\varepsilon > 0$ there exists a piecewise linear path $\{Y_t\}_{t\in[a,b]}$ such that $Y_a = X_a$, $Y_b = X_b$ and

$$\left\| \int_a^b \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)(\dot{X}_t)\,dt - \int_a^b \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(Y_t)(\dot{Y}_t)\,dt \right\|_{\mathcal{E}_2} < \varepsilon.$$

Dividing the segment $[a,b]$ into n equal parts, we see that it is enough to show that

$$(E) := \left\| \int_a^b \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)(\dot{X}_t)\,dt - \int_a^b \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(Y_t)(\dot{Y}_t)\,dt \right\|_{\mathcal{E}_2} = o(b-a)$$

as $b \to a$, where $\{Y_t\}_{t\in[a,b]}$ is already a straight line path with $Y_a = X_a$, $Y_b = X_b$.

Let $V = \frac{X_b - X_a}{b-a}$. By Theorem 1.2.2 we have

$$\begin{aligned}(E) &= \left\| f(X_b) - f(X_a) - \int_a^b \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)(\dot{X}_t)\, dt \right\|_{\mathcal{E}_2} \\ &= \left\| (b-a)\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_a)(V) + (b-a)\, o(\|V\|_{\mathcal{E}_1}) - \int_a^b \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)(\dot{X}_t)\, dt \right\|_{\mathcal{E}_2} \\ &\leqslant \left\| \int_a^b \Big(\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)(\dot{X}_t) - \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_a)(V) \Big)\, dt \right\|_{\mathcal{E}_2} + o(b-a).\end{aligned}$$

Now we write

$$\begin{aligned}&\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)(\dot{X}_t) - \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_a)(V) \\ &\qquad = [\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_t)(\dot{X}_t) - \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_a)(\dot{X}_t)] \\ &\qquad\qquad + [\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_a)(\dot{X}_t) - \mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X_a)(V)]\end{aligned}$$

and use continuity of $\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2} f(X)$ and $\dot{X}_t$. □

1.3 von Neumann algebras

1.3.1 Basic properties of von Neumann algebras

An involutive algebra $\mathcal{N}$ of operators on a Hilbert space $\mathcal{H}$ is said to be a **von Neumann algebra**, if it contains the identity operator and is closed in the weak operator topology. If $\mathcal{A}$ is any subset of $\mathcal{B}(\mathcal{H})$ then by $\mathcal{A}'$ we denote its **commutant**, which is by definition

$$\mathcal{A}' = \{S \in \mathcal{B}(\mathcal{H})\colon ST = TS \ \text{ for any } \ T \in \mathcal{A}\}\,.$$

A von Neumann algebra $\mathcal{N}$ is called a **factor**, if its **center** $\mathcal{N} \cap \mathcal{N}'$ is equal to $\mathbb{C}1$.

We state some well known properties of von Neumann algebras.

Theorem 1.3.1 (von Neumann's bicommutant theorem) *An involutive algebra $\mathcal{A}$ of operators with identity operator is a von Neumann algebra if and only if it coincides with its second commutant: $\mathcal{A}'' = \mathcal{A}$.*

Theorem 1.3.2 [BR, Theorem 2.4.23] *Let $\mathcal{N}$ and $\mathcal{M}$ be two von Neumann algebras. If φ is a $*$-homomorphism from $\mathcal{M}$ onto $\mathcal{N}$ then φ is σ-weakly and σ-strongly continuous.*

This theorem implies that the σ-weak and σ-strong topologies of a von Neumann algebra $\mathcal{N}$ do not depend on a representation of $\mathcal{N}$.

Let $\mathcal{N}$ be a von Neumann algebra in Hilbert space $\mathcal{H}$ and let $\mathcal{K}$ be another Hilbert space.

If $U\colon \mathcal{H} \to \mathcal{K}$ is an isomorphism of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ then the mapping $T \in \mathcal{N} \mapsto UTU^{-1} \in \mathcal{B}(\mathcal{K})$ is an isomorphism of $\mathcal{N}$ onto another von Neumann algebra $\mathcal{M} = U\mathcal{N}U^{-1}$ in $\mathcal{K}$. This isomorphism is called **spatial isomorphism**.

The mapping $T \in \mathcal{N} \mapsto T \otimes 1 \in \mathcal{N} \otimes \mathbb{C}$ is an isomorphism of $\mathcal{N}$ onto a von Neumann algebra $\mathcal{N} \otimes \mathbb{C}$ in $\mathcal{H} \otimes \mathcal{K}$. This isomorphism is called an **ampliation** of $\mathcal{N}$.

If E' is a projection from commutant $\mathcal{N}'$ of $\mathcal{N}$ with **central support** (i.e. the infimum of all projections F' from the center of $\mathcal{N}'$, such that $E' \leqslant F'$) equal to 1, then the mapping $T \in \mathcal{N} \mapsto E'TE'$ is an isomorphism of $\mathcal{N}$ onto von Neumann algebra $E'\mathcal{N}E'$ on Hilbert space $E'\mathcal{H}$. This isomorphism is called an **induction** of $\mathcal{N}$ via E'.

Theorem 1.3.3 [Di, I.4.4] *Every $*$-isomorphism of two von Neumann algebras can be realized as combination of a spatial isomorphism, an ampliation and an induction.*

If E is a projection from $\mathcal{N}$ then the set $E\mathcal{N}E$ is a von Neumann algebra on the Hilbert space $E\mathcal{H}$. This von Neumann algebra is called **reduced von Neumann algebra**.

1.3.2 Projections in von Neumann algebras

We recall some well-known facts of geometry of projections in von Neumann algebras. For details, see [Di], [SZ, Chapter 4].

Equivalence of projections

Let $\mathcal{N}$ be a von Neumann algebra in a Hilbert space $\mathcal{H}$. We denote by $\mathcal{P}(\mathcal{N}) = \{P \in \mathcal{N} \colon P^2 = P = P^*\}$ the set of all projections of $\mathcal{N}$.

The projections E and F from a von Neumann algebra $\mathcal{N}$ are said to be *equivalent*, if there exists an operator $u \in \mathcal{N}$ such that $u^*u = E$ and $uu^* = F$. In this case one writes $E \sim F$. The relation $\sim$ is an equivalence relation. We write $E \prec F$, if there exists a projection P such that $E \sim P$ and $P \leqslant F$.

If $\{E_\alpha\}_{\alpha \in I}$ is a set of projections from $\mathcal{B}(\mathcal{H})$, then by definition $\bigvee E_\alpha$ is the smallest projection E such that $E_\alpha \leqslant E$ for all $\alpha \in I$, and $\bigwedge E_\alpha$ is the largest projection E such that $E \leqslant E_\alpha$ for all $\alpha \in I$. If $\{E_\alpha\}_{\alpha \in I} \subset \mathcal{P}(\mathcal{N})$, then $\bigvee E_\alpha \in \mathcal{P}(\mathcal{N})$ and $\bigwedge E_\alpha \in \mathcal{P}(\mathcal{N})$ [Di]. Evidently,

$$\bigvee E_\alpha = [\operatorname{span} \bigcup E_\alpha \mathcal{H}], \qquad \bigwedge E_\alpha = [\bigcap E_\alpha \mathcal{H}], \tag{1.7}$$

where, if $\mathcal{K}$ is a linear subspace of $\mathcal{H}$, then $[\mathcal{K}]$ denotes the projection onto the closure of $\mathcal{K}$. By $E^\perp$ we denote the orthogonal complement of projection E, so that $E^\perp := 1 - E$.

Kernel and range projections

For $T \in \mathcal{B}(\mathcal{H})$ we denote by N_T the (orthogonal) projection $[\ker T]$ onto the kernel of T, and we denote by R_T the (orthogonal) projection $[\operatorname{ran} T]$ onto the closure of the range of T. If $T \in \mathcal{B}(\mathcal{H})$, then the **right support projection** $\operatorname{supp}_r(T)$ of T, is the smallest projection $P \in \mathcal{B}(\mathcal{H})$ such that $TP = T$. Similarly, the **left support projection** $\operatorname{supp}_l(T)$ of T is the smallest projection $Q \in \mathcal{B}(\mathcal{H})$ such that $QT = T$. If $T = T^*$, then $\operatorname{supp}_r(T) = \operatorname{supp}_l(T)$ and in this case the projection $\operatorname{supp}(T) := \operatorname{supp}_r(T) = \operatorname{supp}_l(T)$ is called the **support projection** of T. For any $T \in \mathcal{B}(\mathcal{H})$, one has $\operatorname{supp}_r(T) = \operatorname{supp}(|T|)$.

Note that

$$\mathrm{R}_T = \operatorname{supp}_l(T) \quad \text{and} \quad \mathrm{R}_{T^*} = \operatorname{supp}_r(T),$$

so that R_T and R_{T^*} are projections which are minimal with respect to the properties

$$\mathrm{R}_T T = T \quad \text{and} \quad T\mathrm{R}_{T^*} = T. \tag{1.8}$$

If $T \in \mathcal{N}$, then N_T and R_T also belongs to $\mathcal{N}$.

Note that for any two operators $S, T \in \mathcal{N}$, one has the evident relations

$$\mathrm{N}_T \leqslant \mathrm{N}_{ST}, \tag{1.9}$$

$$\mathrm{R}_{ST} \leqslant \mathrm{R}_S. \tag{1.10}$$

Lemma 1.3.4 [Di, III.1.1, Proposition 2] *For any $T \in \mathcal{N}$ we have*

$$\mathrm{R}_T \sim \mathrm{R}_{T^*}.$$

Lemma 1.3.5 *Let $T \in \mathcal{N}$. The following equalities hold true*

$$\mathrm{N}_T = \mathrm{R}_{T^*}^\perp, \tag{1.11}$$

$$\mathrm{N}_{T^*} = \mathrm{R}_T^\perp. \tag{1.12}$$

Proof. The second equality follows from the first one by replacing T by T^*, so that we will prove only the first equality.

Let $\eta \in \mathcal{H}$ be such that $\mathrm{N}_T\eta = \eta$. This is equivalent to $T\eta = 0$. Hence, for any $\xi \in \mathcal{H}$ we have

$$0 = \langle T\eta, \xi\rangle = \langle \eta, T^*\xi\rangle .$$

This means that $\eta \perp \operatorname{ran} T^*$, which implies that $\mathrm{R}_{T^*}\eta = 0$, i.e. $(1 - \mathrm{R}_{T^*})\eta = \eta$. Now, all implications are true also in reverse order, so that we are done. □

Lemma 1.3.6 (The parallelogram rule [SZ, Corollary 4.4]) *For any two projections E and F in a von Neumann algebra $\mathcal{N}$ one has the relations*

$$E \vee F - F \ \sim \ E - E \wedge F, \tag{1.13}$$

$$E - E \wedge F^{\perp} \ \sim \ F - F \wedge E^{\perp}. \tag{1.14}$$

It follows from the first equality in (1.7) that for any $S, T \in \mathcal{B}(\mathcal{H})$ the following relation holds

$$\mathrm{R}_{S+T} \leqslant \mathrm{R}_S \vee \mathrm{R}_T. \tag{1.15}$$

Lemma 1.3.7 [Br] *Let $B \in \mathcal{N}$ and let $S \in \mathcal{N}$ be an invertible operator. Then*

$$\mathrm{N}_B = \mathrm{N}_{SB}, \tag{1.16}$$

$$\mathrm{N}_B \sim \mathrm{N}_{BS}. \tag{1.17}$$

Proof. Since S is invertible, we have

$$\ker(SB) = \{\xi \in \mathcal{H} \colon SB\xi = 0\} = \{\xi \in \mathcal{H} \colon B\xi = 0\} = \ker B,$$

so that (1.16) follows.

We observe, that for any projection $F \in \mathcal{N}$, it follows from (1.16) that

$$\mathrm{N}_{SF} = \mathrm{N}_F = F^{\perp}.$$

It follows from Lemma 1.3.4, (1.11) and the previous equality that

$$\mathrm{R}_{SF} \ \sim \ \mathrm{R}_{(SF)^*} = 1 - \mathrm{N}_{SF} = 1 - F^{\perp} = F. \tag{1.18}$$

Now, letting $F = \mathrm{N}_{BS}$ we have $BSF = BS\mathrm{N}_{BS} = 0$, so that $B\mathrm{R}_{SF} = BSF = 0$. The last equality implies that $\mathrm{R}_{SF} \leqslant \mathrm{N}_B$, which together with (1.18) implies that $\mathrm{N}_{BS} = F \sim \mathrm{R}_{SF} \leqslant \mathrm{N}_B$, i.e.

$$\mathrm{N}_{BS} \prec \mathrm{N}_B.$$

Now, since B is an arbitrary operator from $\mathcal{N}$ and S is an arbitrary invertible operator from $\mathcal{N}$, replacing in the last equality B by BS and replacing S by S^{-1} gives $\mathrm{N}_B \prec \mathrm{N}_{BS}$, so that $\mathrm{N}_B \sim \mathrm{N}_{BS}$. □

1.3.3 Semifinite von Neumann algebras

Here we give necessary information on semifinite von Neumann algebras from [Di]. We denote by $\mathcal{A}_+$ the non-negative part $\{T \in \mathcal{A} \colon T \geqslant 0\}$ of a $*$-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. A **trace** τ on a von Neumann algebra $\mathcal{N}$ is a map $\tau \colon \mathcal{N}_+ \to [0, +\infty]$, such that for all $S, T \in \mathcal{N}_+$, $\alpha \in [0, +\infty)$ and for any unitary $U \in \mathcal{N}$,

$$\tau(S+T) = \tau(S) + \tau(T), \ \tau(\alpha T) = \alpha\tau(T) \quad \text{and} \quad \tau(UTU^*) = \tau(T). \quad (1.19)$$

A trace τ on a von Neumann algebra $\mathcal{N}$ is called *faithful* if for any non-negative operator $T \in \mathcal{N}$ the equality $\tau(T) = 0$ implies that $T = 0$. A trace τ on a von Neumann algebra $\mathcal{N}$ is called *normal* if for any bounded increasing net of non-negative operators $\{T_\alpha\}_{\alpha \in I}$ the equality $\tau\left(\sup\limits_{\alpha \in I} T_\alpha\right) = \sup\limits_{\alpha \in I} \tau(T_\alpha)$ holds. A trace τ on a von Neumann algebra $\mathcal{N}$ is called *semifinite* if, for each $S \in \mathcal{N}_+$, $\tau(S)$ is the supremum of the numbers $\tau(T)$ for those $T \in \mathcal{N}_+$ such that $T \leqslant S$ and $\tau(T) < +\infty$. A trace τ extends by linearity to linear combinations of elements $T \in \mathcal{N}_+$ such that $\tau(T) < \infty$, and all properties of trace (1.19) still hold for this continuation.

Any two equivalent projections have the same trace by the last equality in (1.19).

A von Neumann algebra $\mathcal{N}$ is called *semifinite* if for any non-zero $T \in \mathcal{N}_+$ there exists a normal semifinite trace τ on $\mathcal{N}_+$ such that $0 < \tau(T) < +\infty$.

If $\mathcal{N}$ is a semifinite von Neumann algebra with a faithful normal semifinite trace τ, then a projection from $\mathcal{N}$ is said to be *τ-finite* if its τ-trace is finite. An operator $T \in \mathcal{N}$ is said to be *τ-finite* if R_T is a τ-finite projection. Evidently, if Q is a τ-finite projection and P is a projection such that $P \leqslant Q$ then P is also τ-finite, and hence if at least one of the projections P or Q is τ-finite then $P \wedge Q$ is also τ-finite. If a projection P is equivalent to a τ-finite projection then P is also τ-finite.

Lemma 1.3.8 *The set of τ-finite operators is a two-sided $*$-ideal of $\mathcal{N}$.*

Proof. Let $S, T \in \mathcal{N}$ be two τ-finite operators, and let $A \in \mathcal{N}$. We have to check that R_{S+T}, R_{S^*}, R_{SA} and R_{AS} are τ-finite projections. That R_{SA} is τ-finite follows from (1.10). That R_{S^*} is τ-finite follows from Lemma 1.3.4. Since by Lemma 1.3.4 and (1.10) $\mathrm{R}_{AS} \sim \mathrm{R}_{S^*A^*} \leqslant \mathrm{R}_{S^*}$, it also follows that R_{AS} is τ-finite.

That R_{S+T} is τ-finite follows from (1.15) and the parallelogram rule (1.13).
$\square$

1.3.4 Operators affiliated with a von Neumann algebra

Let $\mathcal{N}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let T be a closed operator with dense domain $\mathcal{D}(T) \subset \mathcal{H}$. The operator T is said to be *affiliated* with $\mathcal{N}$ if and only if for all unitary operators $U \in \mathcal{N}'$ we have $U\mathcal{D}(T) \subset \mathcal{D}(T)$ and $TU = UT$ on $\mathcal{D}(T)$. In this case, one writes $T\eta\mathcal{N}$.

If T is an operator affiliated with $\mathcal{N}$, then T can be represented in a unique way in the form

$$T = V\,|T|\,,$$

where $|T| = \sqrt{T^*T}$ and V is a partial isometry whose left support coincides with the left support of T, i.e. $\mathrm{R}_T = \mathrm{R}_V$ [Di]. This representation is called the **polar decomposition** of T. The operator V belongs to $\mathcal{N}$ and $|T|$ is affiliated with $\mathcal{N}$ [Di]. Also, all spectral projections of $|T|$ belong to $\mathcal{N}$.

1.3.5 Generalized s-numbers

In this subsection, we collect those properties of generalized s-numbers of τ-measurable operators from [DDP_2, FK], which will be used later. For details see [DDP_2, FK].

Let $\mathcal{N}$ be a semifinite von Neumann algebra and let τ be a faithful normal semifinite trace on $\mathcal{N}$.

Definition 1.3.9 *Let $T\eta\mathcal{N}$. An operator T is* τ-measurable *if and only if for every $\varepsilon > 0$ there exists a projection $E \in \mathcal{P}(\mathcal{N})$ such that $E\mathcal{H} \subset D(T)$ and $\tau\left(E^{\perp}\right) \leqslant \varepsilon$.*

Let $\tilde{\mathcal{N}}$ be the set of all τ-measurable operators.

Definition 1.3.10 *Let $T \in \tilde{\mathcal{N}}$ be a τ-measurable operator. The* ***generalized s-number*** *$\mu_t(T),\ t > 0$, of the operator T is the number*

$$\mu_t(T) = \inf\left\{\|TE\| : E \in \mathcal{P}(\mathcal{N}) \text{ and } \tau\left(E^{\perp}\right) \leqslant t\right\}.$$

The function $\mu_t(T),\ t > 0$, is called the **generalized singular value function**.

Definition 1.3.11 *Let $\mathcal{N}$ be a von Neumann algebra with a faithful normal semifinite trace τ. An operator $T \in \mathcal{N}$ is said to be* τ-compact, *if it belongs to the norm closure of the set of τ-finite operators from $\mathcal{N}$. The set of all τ-compact operators from $\mathcal{N}$ is denoted by $\mathcal{K}(\mathcal{N}, \tau)$.*

Lemma 1.3.12 *The set $\mathcal{K}(\mathcal{N}, \tau)$ is a norm closed two-sided $*$-ideal.*

This follows from the norm continuity of the maps $A \mapsto BA$, $A \mapsto AB$ and $A \mapsto A^*$.

Lemma 1.3.13 *An operator $T \in \mathcal{N}$ is τ-compact if and only if $\lim_{t\to\infty} \mu_t(T) = 0$.*

We denote by $\mathcal{K}(\tilde{\mathcal{N}}, \tau) := \left\{ T \in \tilde{\mathcal{N}} : \lim_{\lambda\to\infty} \mu_t(T) = 0 \right\}$ the set of all (possibly unbounded) τ-compact operators.

Lemma 1.3.14 *If T is a τ-measurable operator then the map $t \in (0, \infty) \mapsto \mu_t(T)$ is non-increasing and continuous from the right. Moreover,*

$$\mu_0(T) := \lim_{t\to 0^+} \mu_t(T) = \|T\| \in [0, \infty]. \tag{1.20}$$

Lemma 1.3.15 *If T is a τ-measurable operator then, for any $t > 0$ and $\alpha \in \mathbb{C}$, (i) $\mu_t(T) = \mu_t(|T|) = \mu_t(T^*)$ and (ii) $\mu_t(\alpha T) = |\alpha|\, \mu_t(T)$.*

Lemma 1.3.16 *If S, T are τ-measurable operators and $0 \leqslant S \leqslant T$, then, for any $t > 0$, $\mu_t(S) \leqslant \mu_t(T)$.*

Lemma 1.3.17 *If T is a τ-measurable operator and f is a continuous increasing function on $[0, \infty)$ with $f(0) \geqslant 0$, then, for any $t > 0$, $\mu_t(f(|T|)) = f(\mu_t(|T|))$.*

Lemma 1.3.18 *If S, T are τ-measurable operators, then $S + T$ is also τ-measurable and, for any $s, t > 0$, $\mu_{s+t}(S + T) \leqslant \mu_s(S) + \mu_t(T)$.*

Lemma 1.3.19 *If T is a τ-measurable operator and R, $S \in \mathcal{N}$, then STR is also τ-measurable and, for any $t > 0$, $\mu_t(STR) \leqslant \|S\|\,\|R\|\, \mu_t(T)$.*

Lemma 1.3.20 *If S, T are τ-measurable operators then ST is also τ-measurable and, for any $s, t > 0$, $\mu_{s+t}(ST) \leqslant \mu_s(S)\mu_t(T)$.*

It is shown in [DDP$_2$] that the trace τ extends uniquely to the positive cone of the $*$-algebra of all τ-measurable operators as a positive extended-real function which is positively homogeneous, additive, normal and unitarily invariant.

Proposition 1.3.21 *If f is a continuous and increasing function on $[0; +\infty)$ and $f(0) = 0$ then for any τ-measurable operator T*

$$\tau\left(f(|T|)\right) = \int_0^\infty f(\mu_t(T))\, dt.$$

In particular, for any $p \in (0,\infty)$

$$\|T\|_p := \tau\left(|T|^p\right)^{1/p} = \left(\int_0^\infty \mu_t(T)^p\,dt\right)^{1/p}. \tag{1.21}$$

Proposition 1.3.22 [FK] *If* $T \in \tilde{\mathcal{N}}$, *then*
1) if $|T| = \int \lambda\, dE_\lambda$ *then* $\mu_t(T) = \inf\left\{\lambda \geqslant 0\colon \tau\left(E_\lambda^\perp\right) \leqslant t\right\}$;
2) $\mu_t(T) = \inf\left\{\|S - T\| : S \in \tilde{\mathcal{N}},\ \tau\left(\operatorname{supp}(|S|)\right) \leqslant t\right\}$;
3) if f *is a non-decreasing right-continuous function on* $[0,+\infty)$ *and* $f(0) \geq 0$ *then* $\mu_t(f(|T|)) = f(\mu_t(|T|)) \quad \forall t > 0$;
4) if $T \in \mathcal{K}(\mathcal{N},\tau)$ *and* f *is a non-negative Borel function on* $[0;+\infty)$ *and* $f(0) = 0$ *then* $\tau\left(f(|T|)\right) = \int\limits_0^{+\infty} f(\mu_t(T))\,dt$;

Proposition 1.3.23 *For all* $\varepsilon, \delta > 0$ *the following sets, denoted by* $V(\varepsilon,\delta)$, *coincide and they form the base of zero neighborhoods of a topology on* $\tilde{\mathcal{N}}$:

$$\begin{aligned} V(\varepsilon,\delta) &:= \left\{T \in \tilde{\mathcal{N}}\colon \exists E \in \mathcal{P}(\mathcal{N})\ \ \|TE\| \leqslant \varepsilon \textit{ and } \tau\left(E^\perp\right) \leqslant \delta\right\} \\ &= \left\{T \in \tilde{\mathcal{N}}\colon \mu_\delta(T) \leqslant \varepsilon\right\}. \end{aligned}$$

This topology is said to be **topology of convergence in measure**, and it makes $\tilde{\mathcal{N}}$ a complete topological *-algebra.

The **distribution function** of $T \in \mathcal{N}$ is defined by

$$\lambda_t(T) := \tau\left(\chi_{(t,\infty)}(|T|)\right) = \tau\left(1 - E_t^{|T|}\right),\ t > 0,$$

where χ_B denotes the **indicator function** for the set B. The distribution function $\lambda_t(T)$ is a non-increasing right-continuous function. The singular value function $\mu_t(T)$ is the non-increasing, right-continuous inverse of the distribution function $\lambda_t(T)$.

Proposition 1.3.24 *Let* $T \in \tilde{\mathcal{N}}$ *be a* τ*-measurable operator. The following statements are equivalent:*
(i) T *is* τ*-compact;*
(ii) $\lambda_\varepsilon(T) < +\infty$ *for all* $\varepsilon > 0$;
(iii) There exist a sequence $\{T_n\}$, $n = 1, 2, \ldots,$ *of* τ*-measurable operators (bounded if wished) such that* $\tau\left(\operatorname{supp}(|T_n|)\right) < +\infty$ *for all* $n = 1, 2, \ldots,$ *and* $T_n \xrightarrow{\mu} T$.

If $E \in \mathcal{N}$ is a τ-finite projection, then

$$\mu_t(E) = \chi_{[0,\tau(E))}(t),\ t \geqslant 0. \tag{1.22}$$

By Proposition 1.3.22(2), $\mu_s(T) = \inf\{t \geqslant 0 : \lambda_t(T) \leqslant s\}$ and for any $s, t > 0$,

$$s \geqslant \lambda_t(T) \text{ if and only if } \mu_s(T) \leqslant t. \tag{1.23}$$

Furthermore,

$$\int_0^{\lambda_t(T)} \mu_s(T)\, ds = \tau\left(|T|\chi_{(t,\infty)}(|T|)\right), \ \forall\ t > 0. \tag{1.24}$$

Following [FK], we write

$$\Phi_t(T) = \int_0^t \mu_s(T)\, ds, \quad t > 0.$$

Let $f, g \colon (0,\infty) \to [0,\infty)$ be two non-increasing functions. We write $f \lll g$ if for all $t > 0$ (see e.g. [DDP$_2$])

$$\int_0^t f(s)\, ds \leqslant \int_0^t g(s)\, ds.$$

We also write $S \lll T$ [DDP$_2$], if $\mu(S) \lll \mu(T)$, i.e. if, for all $t > 0$,

$$\Phi_t(S) \leqslant \Phi_t(T).$$

Lemma 1.3.25 [FK, Theorem 4.4 (ii)] *If S, T are τ-measurable operators then $\mu(S+T) \lll \mu(S) + \mu(T)$, i.e. for all $t > 0$*

$$\Phi_t(S+T) \leqslant \Phi_t(S) + \Phi_t(T).$$

Proposition 1.3.26 [FK, Lemma 4.1] *Assume that $\mathcal{N}$ has no minimal projection. For any τ-measurable operator T, we have*

$$\Phi_t(T) = \sup\left\{\tau\left(E\,|T|\,E\right) : E \in \mathcal{P}(\mathcal{N}) \text{ with } \tau(E) \leqslant t\right\}. \tag{1.25}$$

Lemma 1.3.27 [CDS, Lemma 2.3] *Let $0 \leqslant S, T, S', T' \in \tilde{\mathcal{N}}$. If $S' \lll S$, $T' \lll T$ and if $S'T' = 0$, then $S' + T' \lll S + T$.*

Lemma 1.3.28 *If $0 \leqslant S, T \in \mathcal{N}$, then*

$$\Phi_t(S) + \Phi_t(T) \leqslant \Phi_{2t}(S+T).$$

Proof. Using the argument of the proof of [CDS, Lemma 2.3], one can assume that $\mathcal{N}$ has no minimal projections. By 1.3.27, we may replace the algebra $\mathcal{N}$ by $\mathcal{N} \oplus \mathcal{N}$, and the operators S and T by the operators $S \oplus 0$ and $0 \oplus T$. Hence, the formula (1.25) applied to $S+T$, S and T, yields the claim. $\square$

For $T \in \mathcal{N}$ and $t > 0$ let

$$\Lambda_t(T) = \int_0^t \log \mu_s(T)\, ds.$$

Proposition 1.3.29 [FK, Theorem 4.2 (ii)] *For any T_1, $T_2 \in \mathcal{N}$ and any $t > 0$*

$$\Lambda_t(T_1 T_2) \leqslant \Lambda_t(T_1) + \Lambda_t(T_2). \tag{1.26}$$

1.3.6 Non-commutative L^p-spaces

In this subsection, we recall some basic properties of non-commutative L^p-spaces, following [DDP$_2$].

Let $\mathcal{N}$ be a semifinite von Neumann algebra with a faithful normal semifinite trace τ. An operator $T \in \mathcal{N}$ is said to be *p-summable* , where $p \in (0, \infty)$, if $\tau(|T|^p)$ is finite, where $|T| := \sqrt{T^*T}$. If $p \geqslant 1$, then the set $\mathcal{L}^p(\mathcal{N}, \tau)$ of all p-summable operators from $\mathcal{N}$ is a normed space with the norm $\|T\|_p$, given by (1.21). The completion of $\mathcal{L}^p(\mathcal{N}, \tau)$ in the norm $\|\cdot\|_p$ is denoted by $L^p(\mathcal{N}, \tau)$. The relation $L^p(\mathcal{N}, \tau) \subset \tilde{\mathcal{N}}$ holds [DDP$_2$], and, for all $A \in L^p(\mathcal{N}, \tau)$, $\|A\|_p = \left(\int_0^\infty \mu_t(A)^p\, dt\right)^{1/p} = \tau\left(|A|^p\right)^{1/p}$. The norms $\|\cdot\|_p$, $p \geqslant 1$, are **order continuous** [DDP$_2$, p. 730], i.e. $0 \leqslant T_\alpha \downarrow_\alpha 0$ in $L^p(\mathcal{N}, \tau)$ implies that $\|T_\alpha\|_p \downarrow_\alpha 0$.

Elements of the space $L^1(\mathcal{N}, \tau)$ are called *τ-trace class* operators, elements of the space $\mathcal{L}^1(\mathcal{N}, \tau) = L^1(\mathcal{N}, \tau) \cap \mathcal{N}$ are the bounded τ-trace class operators. The trace τ extends uniquely to $L^1(\mathcal{N}, \tau)$ as a normal unitarily-invariant linear function [DDP$_2$].

The space $L^2(\mathcal{N}, \tau)$ is a Hilbert space with the scalar product $\langle S, T\rangle = \tau(S^*T)$. The representation π_l of $\mathcal{N}$ on $L^2(\mathcal{N}, \tau)$ given by formula $\pi_l(A)B = AB$ is called **left regular representation** of $\mathcal{N}$.

The space $\mathcal{L}^p(\mathcal{N}, \tau)$, equipped with the norm

$$\|\cdot\|_{\mathcal{L}^p} := \|\cdot\|_p + \|\cdot\|,$$

is a Banach space and is a $*$-ideal of the algebra $\mathcal{N}$. In particular, this implies that if $V \in \mathcal{L}^1(\mathcal{N}, \tau)$ then $\mathrm{Re}(V)$, $\mathrm{Im}(V) \in \mathcal{L}^p(\mathcal{N}, \tau)$ and if $V = V^* \in \mathcal{L}^p(\mathcal{N}, \tau)$ then V_+, $V_- \in \mathcal{L}^p(\mathcal{N}, \tau)$. The same if true for $L^p(\mathcal{N}, \tau)$.

For any $A, B \in \mathcal{N}$ and $T \in L^p(\mathcal{N}, \tau)$, one has $ATB \in L^p(\mathcal{N}, \tau)$ and

$$\|ATB\|_p \leqslant \|A\|\, \|T\|_p\, \|B\|,$$

and for any $S \in \mathcal{L}^p(\mathcal{N}, \tau)$ one has $ASB \in \mathcal{L}^p(\mathcal{N}, \tau)$ and

$$\|ASB\|_{\mathcal{L}^p} \leqslant \|A\|\, \|S\|_{\mathcal{L}^p}\, \|B\|.$$

We will use these inequalities without further reference.

Lemma 1.3.30 [ACDS] *Let $(\mathcal{N}, \tau)$ be a semifinite von Neumann algebra. If $A_\alpha \in \mathcal{N}$, $\alpha \in I$, is a uniformly bounded net converging in the so-topology to an operator $A \in \mathcal{N}$ and if $V \in L^1(\mathcal{N}, \tau)$, then the net $\{A_\alpha V\}_{\alpha \in I}$ converges to AV in $L^1(\mathcal{N}, \tau)$.*

Proof. (A) Replacing the net $\{A_\alpha\}$ with $\{A_\alpha - A\}$, we can assume that $A = 0$. Since the net $\{A_\alpha\}_{\alpha \in I}$ is uniformly bounded, by Theorem 1.1.1 it follows that $A_\alpha \to 0$ in the σ-strong operator topology. Since the σ-strong topology does not depend on representation by Theorem 1.3.2, it can be assumed that $\mathcal{N}$ acts on $L^2(\mathcal{N}, \tau)$ in the left regular representation, in particular $\|A_\alpha Y\|_2 \to 0$ for every $Y \in L^2(\mathcal{N}, \tau)$.

(B) Assume first that $V \geqslant 0$. Let $Y = V^{1/2} \in L^2(\mathcal{N}, \tau)$. Then

$$\tau(|A_\alpha V|) = \tau\left(U_\alpha A_\alpha Y^2\right) = \tau\left(A_\alpha Y (U_\alpha^* Y)^*\right) \leqslant \|A_\alpha Y\|_2 \cdot \|U_\alpha^* Y\|_2 \to 0,$$

where U_α^* is the partial isometry from the polar decomposition of $A_\alpha V$.

(C) Now, if V is self-adjoint and $V = V_+ - V_-$ with $V_+, V_- \geqslant 0$ then by (B) we have that $A_\alpha V_+ \to AV_+$ and $A_\alpha V_- \to AV_-$ in $L^1(\mathcal{N}, \tau)$. Hence, $A_\alpha V \to AV$ in $L^1(\mathcal{N}, \tau)$.

(D) For an arbitrary $V \in L^1(\mathcal{N}, \tau)$ we have by (C) $A_\alpha \operatorname{Re}(V) \to A \operatorname{Re}(V)$ and $A_\alpha \operatorname{Im}(V) \to A \operatorname{Im}(V)$ in $L^1(\mathcal{N}, \tau)$. Hence, $A_\alpha V \to AV$ in $L^1(\mathcal{N}, \tau)$. □

Lemma 1.3.31 [DDP$_2$] *If $X \in \mathcal{N}$ and $Y \in L^1(\mathcal{N}, \tau)$, then*

$$\tau(XY) = \tau(YX).$$

Proof. (A) Using the decomposition $T = \operatorname{Re}(T) + i \operatorname{Im}(T)$ for X and Y, we reduce to the case of self-adjoint X and Y. Using $T = T_+ - T_-$ for self-adjoint X and Y, we reduce to the case of positive X and Y.

Any bounded operator from $\mathcal{N}$ is a linear combination of no more than four unitary operators from $\mathcal{N}$, see e.g. [RS, §VI.6]. This and (1.19) imply that if Y is bounded, then $\tau(XY) = \tau(YX)$.

(B) Let $Y_n = E^Y_{[0,n]} Y$. Then $\|XY - XY_n\|_1 \leqslant \|X\| \, \|Y - Y_n\|_1 = \|X\| \, \tau(Y - Y_n) \to 0$. So, using (A)

$$\tau(XY) = \lim_{n \to \infty} \tau(XY_n) = \lim_{n \to \infty} \tau(Y_n X) = \tau(YX).$$

□

Lemma 1.3.32 [BK, Theorem 17], [DDP$_2$] *Let $X, Y \in \mathcal{N}$ be such that XY and YX belong to $L^1(\mathcal{N}, \tau)$. Then*

$$\tau(XY) = \tau(YX).$$

Proof. Let $P = \mathrm{supp}_l(X)$, $Q = \mathrm{supp}_r(X)$, and

$$P_n = E^{XX^*}_{[n^{-1},n]}, \qquad Q_n = E^{X^*X}_{[n^{-1},n]}, \quad n = 1, 2, \ldots .$$

We observe that

$$\lim_{n\to\infty} P_n = P, \qquad \lim_{n\to\infty} Q_n = Q$$

in *so*-topology by Theorem 1.1.4, and

$$P_n X = P_n X Q_n = X Q_n. \tag{1.27}$$

We claim that $\tau(P_n XY) = \tau(YXQ_n)$, $n = 1, 2, \ldots$. In fact, from $n^{-1} Q_n \leqslant X^*X$, we get $Y^* Q_n Y \leqslant n Y^* X^* X Y$; hence, $Q_n Y \in L^1(\mathcal{N}, \tau)$ follows from $XY \in L^1(\mathcal{N}, \tau)$.

Since $Q_n Y$, $YXQ_n \in L^1(\mathcal{N}, \tau)$ and $P_n X$, $Q_n \in \mathcal{N}$, using (1.27) Lemma 1.3.31 twice we compute

$$\tau(P_n XY) = \tau(P_n X Q_n Y) = \tau(Q_n Y P_n X) = \tau(Q_n Y X Q_n) = \tau(YXQ_n).$$

Since XY and YX are in $L^1(\mathcal{N}, \tau)$, using Lemma 1.3.30 twice, together with the claim, we conclude that

$$\begin{aligned} \tau(XY) = \tau(PXY) &= \lim_{n\to\infty} \tau(P_n XY) \\ &= \lim_{n\to\infty} \tau(YXQ_n) = \tau(YXQ) = \tau(YX). \end{aligned}$$

□

Lemma 1.3.33 *Let A, $B \in \mathcal{N}$ and suppose that one of these operators is τ-trace class. Let $T = T^*$ be affiliated with $\mathcal{N}$. Then the measure $\mu(\Delta) := \tau\left(AE^T_\Delta B\right)$ is countably additive and has finite variation.*

Proof. It follows directly from Theorem 1.1.4 and Lemma 1.3.30. □

1.3.7 Holomorphic functional calculus

Let $T \in \mathcal{B}(\mathcal{H})$ and f is a function holomorphic in an open set G containing the spectrum σ_T of T. Then $f(T)$ is defined by the Cauchy formula

$$f(T) = \frac{1}{2\pi i} \int_\gamma f(\lambda) R_\lambda(T)\, d\lambda, \tag{1.28}$$

where γ is any piecewise smooth contour in G containing σ_T and the integral converges in norm [DS, Chapter III.14].

The following theorem can be found in [GK, Brn]. The proof is taken from [GK, Chapter IV].

Theorem 1.3.34 *Let $(\mathcal{N}, \tau)$ be a semifinite von Neumann algebra with faithful normal semifinite trace τ. Let $U \subset \mathbb{R}$ be an interval and let $A\colon U \to \mathcal{L}^1(\mathcal{N}, \tau)$ be a function that is continuously differentiable in $\mathcal{L}^1(\mathcal{N}, \tau)$-norm. Let $\sigma := \overline{\bigcup_{t\in U} \sigma_{A(t)}}$ be a bounded set. If f is a function holomorphic in a neighbourhood of σ, then the function $U \ni t \mapsto f(A(t)) \in \mathcal{L}^1(\mathcal{N}, \tau)$ is $\mathcal{L}^1(\mathcal{N}, \tau)$ differentiable and*

$$\frac{d}{dt}\tau\left(f(A(t))\right) = \tau\left(f'(A(t))A'(t)\right).$$

Proof. We write $A_t = A(t)$ for clarity. For s, $t \in U$ we have by (1.28) and the resolvent identity (1.2)

$$\begin{aligned}\frac{\tau\left(f(A_s) - f(A_t)\right)}{s-t} &= \frac{1}{2\pi i}\,\tau\left(\int_\gamma f(z)\frac{R_z(A_s) - R_z(A_t)}{s-t}\,dz\right)\\ &= \frac{1}{2\pi i}\,\tau\left(\int_\gamma f(z)R_z(A_s)\frac{A_s - A_t}{s-t}R_z(A_t)\,dz\right),\end{aligned}$$

where γ is an anticlockwise oriented contour around σ, lying in the domain of analyticity of f. Since A_t is $\mathcal{L}^1(\mathcal{N}, \tau)$-differentiable, the integrand, and hence, the last integral converges in $\mathcal{L}^1(\mathcal{N}, \tau)$ when $s \to t$. Hence the trace can be interchanged with $\frac{d}{dt}$ and the integral, so that

$$\begin{aligned}(E) \;&:= \frac{d}{dt}\tau\left(f(A_t)\right) = \frac{1}{2\pi i}\,\tau\left(\int_\gamma f(z)R_z(A_t)A_t'R_z(A_t)\,dz\right)\\ &= \frac{1}{2\pi i}\int_\gamma f(z)\tau\left(A_t'R_z(A_t)^2\right)\,dz = -\frac{1}{2\pi i}\int_\gamma f(z)\tau\left(A_t'\frac{d}{dz}R_z(A_t)\right)\,dz,\end{aligned}$$

the last equality by (1.3). Hence, integrating by parts,

$$\begin{aligned}(E) \;&= \frac{1}{2\pi i}\int_\gamma f'(z)\tau\left(A_t'R_z(A_t)\right)\,dz\\ &= \frac{1}{2\pi i}\,\tau\left(A_t'\int_\gamma f'(z)R_z(A_t)\,dz\right) = \tau\left(A_t'f'(A_t)\right).\end{aligned}$$

□

Corollary 1.3.35 *If $A \in C^1([a,b], 1 + \mathcal{L}^1(\mathcal{N}, \tau))$, and if the closure of the union of spectra of $A(t)$, $t \in [a,b]$ is a subset of a branch of* Log*, then* $\mathrm{Log}(A(\cdot)) \in C^1(\mathbb{R}, \mathcal{L}^1(\mathcal{N}, \tau))$ *and*

$$\frac{d}{dt}\tau\big(\mathrm{Log}(A(t))\big) = \tau\left(\frac{d}{dt}\mathrm{Log}(A(t))\right) = \tau\big(A(t)^{-1}A'(t)\big).$$

For a proof, apply the previous theorem to $A(\cdot) - 1$.

1.3.8 Invariant operator ideals in semifinite von Neumann algebras

Here we follow [ACDS].

Definition 1.3.36 *If $\mathcal{E}$ is a $*$-ideal in a von Neumann algebra $\mathcal{N}$ which is complete in some norm $\|\cdot\|_{\mathcal{E}}$, then we will call $\mathcal{E}$ an* ***invariant operator ideal*** *if*
(1) $\|S\|_{\mathcal{E}} \geqslant \|S\|$ for all $S \in \mathcal{E}$,
(2) $\|S^*\|_{\mathcal{E}} = \|S\|_{\mathcal{E}}$ for all $S \in \mathcal{E}$,
(3) $\|ASB\|_{\mathcal{E}} \leqslant \|A\| \, \|S\|_{\mathcal{E}} \, \|B\|$ for all $S \in \mathcal{E}$ and $A,\, B \in \mathcal{N}$.

Definition 1.3.37 *We say that an invariant operator ideal $\mathcal{E}$ has* property (F) *if, for all nets $\{A_\alpha\} \subset \mathcal{E}$ such that there exists $A \in \mathcal{N}$ for which $A_\alpha \to A$ in the so^*-topology and $\|A_\alpha\|_{\mathcal{E}} \leqslant 1$ for all α, it follows that $A \in \mathcal{E}$ and $\|A\|_{\mathcal{E}} \leqslant 1$.*

Lemma 1.3.38 *An invariant operator ideal $\mathcal{E}$ has property (F) if and only if the unit ball of $\mathcal{E}$ endowed with so^*-topology is a complete separable metrisable space.*

Proof. The "if" part is evident. Since $\mathcal{H}$ is separable, the unit ball $(\mathcal{B}_1(\mathcal{H}), so^*)$ of $\mathcal{B}(\mathcal{H})$ is a metrisable space by Proposition 1.1.2. Hence, the unit ball $(\mathcal{E}_1, so^*)$ of $\mathcal{E}$ is also metrisable. Since $\mathcal{H}$ is separable the unit ball $(\mathcal{B}_1(\mathcal{H}), so^*)$ is also separable. Thus, every subset of $(\mathcal{B}_1(\mathcal{H}), so^*)$ is separable [DS, I.6.12], and in particular $\mathcal{E}_1$. Since by Theorem 1.1.1 the unit ball $(\mathcal{B}_1(\mathcal{H}), so^*)$ is complete, the property (F) of $\mathcal{E}$ implies that $(\mathcal{E}_1, so^*)$ is also complete. □

Every von Neumann algebra with the uniform norm is an invariant operator ideal with property (F). The ideal $\mathcal{K}(\mathcal{N}, \tau)$ endowed with $\|\cdot\|$-norm, is an invariant operator ideal, though $\mathcal{K}(\mathcal{N}, \tau)$ does not have the property (F).

Lemma 1.3.39 [DDPS, Proposition 1.6] *For $1 \leqslant p < \infty$ the space $\mathcal{L}^p(\mathcal{N}, \tau)$ with norm $\|\cdot\|_{\mathcal{L}^p}$ is an invariant operator ideal with property (F).*

1.4 Integration of operator-valued functions

This section is based on [ACDS] and [dPS, §5]. Unlike [dPS], we consider von Neumann algebras on separable Hilbert spaces, instead of σ-finite von Neumann algebras on arbitrary Hilbert spaces. This allows to simplify some proofs.

Let (S, Σ) be a measurable space, let X be a metric space. A function $\xi\colon S \to X$ is called *simple* (respectively, *elementary*), if the set $\xi(S)$ is finite (respectively, countable), and if $\xi^{-1}(\{x\}) \in \Sigma$ for every $x \in X$.

Proposition 1.4.1 [VTCh, Proposition I.1.9] *For any function $\xi\colon S \to X$ the following assertions are equivalent.*
(a) ξ is measurable and the set $\xi(S)$ is separable.
(b) There exists a sequence $\xi_1, \xi_2, \ldots, \xi_n, \ldots\colon S \to X$ of elementary functions, such that $\xi_n(s)$ converges to $\xi(s)$ uniformly with respect to $s \in S$.
(c) There exists a sequence $\eta_1, \eta_2, \ldots, \eta_n, \ldots\colon S \to X$ of simple functions, such that $\eta_n(s)$ converges to $\xi(s)$ for every $s \in S$.

Proposition 1.4.2 [VTCh, Proposition I.1.10] *Let (S, Σ) be a measurable space, let X be a complete metric space, let Γ be a family of real-valued continuous functions on X, separating the points of X, and let $\xi\colon S \to X$ be a function, such that $\xi(S)$ is separable. The following assertions are equivalent.*
(a) ξ is measurable.
(b) For every $f \in \Gamma$ the function $f \circ \xi$ is measurable.

Let (S, Σ, ν) be a finite measure space, let $(\mathcal{N}, \tau)$ be a semifinite von Neumann algebra with faithful normal semifinite trace τ and let $\mathcal{E}$ be an invariant operator ideal of $\mathcal{N}$.

Definition 1.4.3 *A $\|\cdot\|$-bounded function $f\colon (S, \nu) \to \mathcal{E}$ will be called*

(*i*) weakly measurable *if, for any $\xi,\ \eta \in \mathcal{H}$, the function $\langle f(\cdot)\xi, \eta\rangle$ is measurable;*

(*ii*) $*$- measurable *if, for all $\eta \in \mathcal{H}$, the functions $f(\cdot)\eta,\ f(\cdot)^*\eta\colon (S, \nu) \to \mathcal{H}$ are Bochner measurable from S into $\mathcal{H}$;*

(*iii*) *so**-measurable *if there exists a sequence of simple measurable functions $f_n\colon S \to \mathcal{E}$ such that $f_n(\sigma) \to f(\sigma)$ in the so*-topology for a. e. $\sigma \in S$.*

Proposition 1.4.4 *If $\mathcal{E}$ has property (F), then, for any $\mathcal{E}$-bounded function $f\colon (S, \nu) \to \mathcal{E}$, the following conditions are equivalent.*

(i) f is weakly measurable,

(ii) f is $*$- measurable,

(iii) f is so*-measurable.

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are evident (and do not depend on property (F)). That (i) $\Rightarrow$ (iii) follows from Lemma 1.3.38 and Propositions 1.4.1 and 1.4.2. $\square$

We denote the set of all $\|\cdot\|$-bounded $*$- measurable functions $f\colon S \to \mathcal{E}$ by $\mathcal{L}_\infty^{so^*}(S, \nu; \mathcal{E})$. Examples of such functions are the bounded $\|\cdot\|$-Bochner-measurable functions and, in the case that S is a locally compact space, all *so**-continuous bounded functions.

The following lemma is a simple consequence of the previous proposition (see also [dPS, Lemmas 5.5, 5.6]).

Lemma 1.4.5 [dPS] *(i) The set $\mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$ is a $*$-algebra;*
(ii) if $\varphi \in B_{\mathbb{R}}(\mathbb{R})$, $f \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{B}_{sa}(\mathcal{H}))$, then $\varphi(f) \in \mathcal{L}_\infty^{so^*}(S,\nu)$.

Proof. (i) Let $f, g \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$. Then $f+g$ and f^* belong to $\mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$, since f and g are weakly measurable. Now, by Definition 1.4.3(iii), let $\{f_n\}$ and $\{g_n\}$ be sequences of simple functions, such that $f_n \to f$ and $g_n \to g$ in the so^*-topology. Then, by Theorem 1.1.1 (more exactly, its so^* analogue), we have $f_n g_n \to fg$ in the so^*-topology. Hence, fg also belong to $\mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$.

(ii) By Definition 1.4.3(iii), let $\{f_n\}$ be a sequence of simple functions converging to f in so^*-topology. By Theorem 1.1.6, the sequence $\varphi(f_n)$ of simple functions converges to $\varphi(f)$ in so^*-topology. Hence, $\varphi(f) \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$. $\square$

Definition 1.4.6 *For any bounded function $f \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$, we define the integral $\int_S f(\sigma)\,d\nu(\sigma)$ by the formula*

$$\left(\int_S f(\sigma)\,d\nu(\sigma)\right)\eta = \int_S f(\sigma)\eta\,d\nu(\sigma), \tag{1.29}$$

where the last integral is the Bochner integral.

We will call this integral the *so^*-integral* of f with respect to ν. Evidently, such an integral exists and it is a bounded linear operator with (uniform) norm less or equal to $|\nu|\,\|f\|_\infty$.

Lemma 1.4.7 *If $\mathcal{E}$ has property (F), and if the sequence $f_n \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$, $n = 1,2,\dots$ is $\mathcal{E}$-bounded and ν-a. e. converges to $f\colon S \to \mathcal{B}(\mathcal{H})$ in the so^*-topology, then $f \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$.*

Proof. We have that, for any $\eta \in \mathcal{H}$, the sequence $f_n(\sigma)\eta$ converges to $f(\sigma)\eta$ for ν-a.e. $\sigma \in S$. Since the $\mathcal{H}$-valued functions $f_n(\cdot)\eta$ are Bochner measurable and since the pointwise limit of a sequence of Bochner measurable functions is a Bochner measurable function, we have that $f \in \mathcal{L}_\infty^{so^*}(S,\nu)$. That $f(\sigma) \in \mathcal{E}$ for a. e. $\sigma \in S$ follows from property (F). $\square$

Lemma 1.4.8 *If $\mathcal{E}$ has property (F), $f \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$ and if f is uniformly $\mathcal{E}$-bounded, then $\int_S f\,d\nu \in \mathcal{E}$.*

Proof. By Proposition 1.4.4, we can choose a sequence of simple functions $f_n \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{E})$ converging a. e. in so^*-topology to f. Evidently, $A_n := \int_S f_n\,d\nu \in \mathcal{E}$

for all $n \in \mathbb{N}$. By the definition (1.29) of operator-valued integral, the sequence $\{A_n\}_{n=1}^{\infty}$ converges to $\int_S f\,d\nu$ in the so^*-topology by the Lebesgue Dominated Convergence Theorem for the Bochner integral. That $\int_S f\,d\nu \in \mathcal{E}$ now follows from the property (F) of $\mathcal{E}$. □

Under the assumptions of Lemma 1.4.7, we have that

$$\int_S f_n\,d\nu \to \int_S f\,d\nu$$

in the so^*-topology. This follows directly from the definition of the so^*-integral and the Dominated Convergence Theorem for the Bochner integral (Theorem 1.1.13).

Lemma 1.4.9 *For any $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{L}_{\infty}^{so^*}(S, \nu; \mathcal{E})$*

$$A \int_S B(\sigma)\,d\nu(\sigma) = \int_S AB(\sigma)\,d\nu(\sigma)$$

The lemma follows directly from Lemma 1.1.12.

Lemma 1.4.10 *If (S_i, Σ_i, ν_i), $i = 1, 2$ are two finite measure spaces and if $f \in \mathcal{L}_{\infty}^{so^*}(S_1 \times S_2, \nu_1 \times \nu_2)$, then $f(\cdot, t) \in \mathcal{L}_{\infty}^{so^*}(S_1, \nu_1)$ for almost all $t \in S_2$ and*

$$\int_{S_2} \int_{S_1} f(s,t)\,d\nu_1(s)\,d\nu_2(t) = \int_{S_1 \times S_2} f(s,t)\,d(\nu_1 \times \nu_2)(s,t). \tag{1.30}$$

Proof. Since $f(\cdot, \cdot)$ is integrable, for any $\eta \in \mathcal{H}$, there exists a ν_2-null set $A_\eta \subset S_2$ such that, for all $t \notin A_\eta$, the function $f(\cdot, t)\eta$ is Bochner integrable (see Theorem 1.1.15). If $\{\xi_j\}_{j=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}$ and $A = \bigcup_{j=1}^{\infty} A_{\xi_j}$, then $\nu_2(A) = 0$ and, for any $\eta \in \mathcal{H}$ and $t \notin A$, we have

$$f(\cdot, t)\eta = \sum_{j=1}^{\infty} c_n f(\cdot, t)\xi_n,$$

where $\eta = \sum_{j=1}^{\infty} c_n \xi_n$. Since linear combinations and uniformly bounded pointwise limits of sequences of Bochner integrable functions on the measure space (S, ν) are Bochner integrable (by the Lebesgue Dominated Convergence Theorem), it follows that $f(\cdot, t)\eta$ is integrable for $t \notin A$. Similarly, there exists a ν_2-null set A' such that $f(\cdot, t)^*\eta$ is integrable for all $\eta \in \mathcal{H}$ and $t \notin A'$. Hence, $f(\cdot, t)$ is integrable for all $t \notin A \cup A'$ and the operator-valued function $g(t) := \int_{S_1} f(s,t)\,d\nu_1(s)$ is well-defined. Now, the integral $g(t)\eta = \int_{S_1} f(s,t)\eta\,d\nu_1(s)$ exists and is equal to $\int_{S_1 \times S_2} f(s,t)\eta\,d(\nu_1 \times \nu_2)(s,t)$ by Fubini's theorem for the Bochner integral of $\mathcal{H}$-valued functions (Theorem 1.1.15). The latter means that the equality (1.30) holds. □

Lemma 1.4.11 *If $f \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{N})$, then*
(i) $X := \int_S f(\sigma)\,d\nu(\sigma)$ belongs to $\mathcal{N}$;
(ii) X as an element of the W^*-algebra $\mathcal{N}$ does not depend on any representation of $\mathcal{N}$.

Proof. (i) Let $A' \in \mathcal{N}'$. Then by Lemma 1.4.9

$$A'X\eta = \int_S A'f(\sigma)\eta\,d\nu(\sigma) = \int_S f(\sigma)A'\eta\,d\nu(\sigma) = \int_S f(\sigma)\,d\nu(\sigma)A'\eta = XA'\eta$$

for any $\eta \in \mathcal{H}$. Hence, $X \in \mathcal{N}$.
(ii) This follows from the fact that two representations of a von Neumann algebra can be obtained from each other by ampliation, reduction and spatial isomorphism (Theorem 1.3.3), since for each of these isomorphisms the claim is evident. $\square$

Lemma 1.4.12 [dPS] *If $f \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{L}^1(\mathcal{N},\tau))$ and $f \geqslant 0$ then $\tau(f(\sigma))$ is a measurable function.*

Proof. Let by Definition 1.4.3(iii) $\{f_n\}$ be a sequence of simple functions taking values in $\mathcal{L}^1(\mathcal{N},\tau)$ and converging for a.e. $\sigma \in S$ to f in so^*-topology. Let $E \in \mathcal{N}$ be a τ-finite projection. By Lemma 1.3.30, the sequence $f_n(\sigma)E$ converges to $f(\sigma)E$ for a.e. $\sigma \in S$ in $L^1(\mathcal{N},\tau)$-topology. Hence, $\tau(f_n(\sigma)E)$ converges to $\tau(f(\sigma)E)$ for a.e. $\sigma \in S$. Since the functions $\tau(f_n(\sigma)E)$ are simple and so measurable, so is $\tau(f(\sigma)E) = \tau(\sqrt{f(\sigma)}E\sqrt{f(\sigma)})$. Hence, $\tau(f(\sigma)) = \sup_{E:\,\tau(E)<\infty}\tau(\sqrt{f(\sigma)}E\sqrt{f(\sigma)})$ is also measurable. $\square$

Lemma 1.4.13 *If $(\mathcal{N},\tau)$ is a semifinite von Neumann algebra, if $f \in \mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{L}^1(\mathcal{N},\tau))$ and if f is uniformly $\mathcal{L}^1(\mathcal{N},\tau)$-bounded, then $X := \int_S f(\sigma)\,d\nu(\sigma) \in \mathcal{L}^1(\mathcal{N},\tau)$, the function $\tau\left(f(\cdot)\right)$ is measurable and*

$$\tau\left(\int_S f(\sigma)\,d\nu(\sigma)\right) = \int_S \tau\left(f(\sigma)\right)\,d\nu(\sigma).$$

Proof. Lemma 1.4.8 implies that $X \in \mathcal{L}^1(\mathcal{N},\tau)$, so that the left hand side of the equality above makes sense. By linearity and by Lemma 1.4.5(i), we can assume that $f(\cdot) \geqslant 0$. By Lemma 1.4.12 the function $\tau\left(f(\cdot)\right)$ is measurable. By Lemma 1.4.11(ii), we can assume that $\mathcal{N}$ acts on $L^2(\mathcal{N},\tau)$ in the left regular representation. Let E be an arbitrary τ-finite projection from $\mathcal{N}$. Then $E \in L^2(\mathcal{N},\tau)$ and by the definition (1.29) of the operator-valued integral

$$XE = \int_S f(\sigma)E\,d\nu(\sigma),$$

where the right hand side is the Bochner integral in $L^2(\mathcal{N},\tau)$. Since E is τ-finite, the convergence in $L^2(\mathcal{N},\tau)$ of the Bochner integral implies convergence in $L^1(\mathcal{N},\tau)$, so that we have

$$\tau\left(XE\right)=\int_S \tau\left(f(\sigma)E\right)\,d\nu(\sigma).$$

Now, normality of the trace τ and the dominated convergence theorem imply that

$$\tau\left(X\right)=\int_S \tau\left(f(\sigma)\right)\,d\nu(\sigma).$$

□

Lemma 1.4.14 *If (S,ν) is a finite measure space and if $f\in\mathcal{L}_\infty^{so^*}(S,\nu;\mathcal{L}^1(\mathcal{N},\tau))$ is uniformly $\mathcal{L}^1(\mathcal{N},\tau)$-bounded, then*

$$\left\|\int f(\sigma)\,d\nu(\sigma)\right\|_*\leqslant\int\left\|f(\sigma)\right\|_*\,d\left|\nu\right|(\sigma),$$

where $\left\|\cdot\right\|_*$ is any of the norms $\left\|\cdot\right\|$, $\left\|\cdot\right\|_1$ or $\left\|\cdot\right\|_{\mathcal{L}^1}$.

Proof. By definition, for any $\eta\in\mathcal{H}$, the function $\sigma\mapsto f(\sigma)\eta$ is Bochner measurable. Hence, the function $\sigma\mapsto\left\|f(\sigma)\right\|=\sup_{\eta\in\mathcal{H}\colon\left\|\eta\right\|\leqslant1}\left\|f(\sigma)\eta\right\|$ is also measurable. Similarly, since the function $\sigma\mapsto\tau\left(f(\sigma)B\right)$ is measurable, the function $\sigma\mapsto\left\|f(\sigma)\right\|_1=\sup_{B\in\mathcal{N}\colon\left\|B\right\|\leqslant1}\left|\tau\left(f(\sigma)B\right)\right|$ is also measurable. Hence, the right hand side of the last equality is well-defined.

For $\eta\in\mathcal{H}$ with $\left\|\eta\right\|\leqslant1$, by definition of the so^*-integral and Lemma 1.1.14, we have

$$\begin{aligned}\left\|\int f(\sigma)\,d\nu(\sigma)\eta\right\|&=\left\|\int f(\sigma)\eta\,d\nu(\sigma)\right\|\\&\leqslant\int\left\|f(\sigma)\eta\right\|\,d\left|\nu\right|(\sigma)\leqslant\int\left\|f(\sigma)\right\|\,d\left|\nu\right|(\sigma).\end{aligned}\tag{1.31}$$

Hence, the inequality is true for the operator norm $\left\|\cdot\right\|$. Since

$$\left\|A\right\|_1=\sup_{B\in\mathcal{N}\colon\left\|B\right\|\leqslant1}\left|\tau\left(AB\right)\right|,$$

it follows that

$$\begin{aligned}\left\|\int f(\sigma)\,d\nu(\sigma)\right\|_1&=\sup_{B\in\mathcal{N}\colon\left\|B\right\|\leqslant1}\left|\tau\left(\int f(\sigma)\,d\nu(\sigma)B\right)\right|\\&=\sup_{B\in\mathcal{N}\colon\left\|B\right\|\leqslant1}\left|\int\tau\left(f(\sigma)B\right)\,d\nu(\sigma)\right|\\&\leqslant\sup_{B\in\mathcal{N}\colon\left\|B\right\|\leqslant1}\int\left|\tau\left(f(\sigma)B\right)\right|\,d\nu(\sigma)\leqslant\int\left\|f(\sigma)\right\|_1\,d\left|\nu\right|(\sigma),\end{aligned}\tag{1.32}$$

where the second equality follows from the definition of the so^*-integral and Lemma 1.4.13. Combining (1.31) and (1.32) we get the inequality for norm $\|\cdot\|_{\mathcal{L}^1}$. □

1.5 Theory of τ-Fredholm operators

In this section we give an exposition of Breuer's theory of operators relatively Fredholm with respect to a semifinite von Neumann algebra. We follow the works [Br, Br$_2$] and [PR, Appendix B].

1.5.1 Definition and elementary properties of τ-Fredholm operators

Definition 1.5.1 *An operator $T \in \mathcal{N}$ is said to be* τ-Fredholm *if and only if*
(BF1) the projection N_T is τ-finite;
(BF2) there exists a τ-finite projection E such that $\mathrm{ran}(E^\perp) \subseteq \mathrm{ran}(T)$.

The set of τ-Fredholm operators will be denoted by $\mathcal{F}(\mathcal{N}, \tau)$.

Remark 1.5.2 *We note that one can define another notion, that of* Breuer-Fredholm operator. *We recall that a projection E in a von Neumann algebra $\mathcal{N}$ is said to be* finite *(relative to $\mathcal{N}$) if it is not equivalent to any projection $F < E$. An operator T is said to be* Breuer-Fredholm *if the projection N_T is finite and there exists a finite projection $E \in \mathcal{N}$ such that* $\mathrm{ran}(E^\perp) \subseteq \mathrm{ran}(T)$. *This notion does not depend on trace τ. Any τ-finite projection is necessarily finite, so that a τ-Fredholm operator is Breuer-Fredholm. But the converse is not true. In case when $\mathcal{N}$ is a semifinite factor, finite projections are the same as τ-finite projections for any faithful normal semifinite trace τ on $\mathcal{N}$, which is actually unique (up to a constant) in this case. Hence, for factors Breuer-Fredholm operators and τ-Fredholm operators are the same.*

We do not use the notion of Breuer-Fredholm operator.

Lemma 1.5.3 *If an operator T from $\mathcal{N}$ is τ-Fredholm then the projection N_{T^*} is τ-finite.*

Proof. (BF2) implies that

$$E^\perp \leqslant \mathrm{R}_T.$$

This and (1.12) implies that

$$E \geqslant \mathrm{R}_T^\perp = \mathrm{N}_{T^*}.$$

Since E is τ-finite, the projection N_{T^*} is also τ-finite. □

Definition 1.5.4 *If T is τ-Fredholm then the τ-**index** of T is the real number*

$$\tau\text{-}\operatorname{ind}(T) = \tau\left(\mathrm{N}_T\right) - \tau\left(\mathrm{N}_{T^*}\right). \tag{1.33}$$

Definition 1.5.5 *Let P, Q be two projections in $\mathcal{N}$. If $T \in \mathcal{N}$, $T_{11} \in P\mathcal{N}Q$, $T_{12} \in P\mathcal{N}Q^\perp$, $T_{21} \in P^\perp\mathcal{N}Q$, $T_{22} \in P^\perp\mathcal{N}Q^\perp$ and*

$$T = T_{11} + T_{12} + T_{21} + T_{22},$$

then we write

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}_{[P,Q]}.$$

Evidently, for any two fixed projections P and Q from $\mathcal{N}$ every operator $T \in \mathcal{N}$ can be represented in this form:

$$T = \begin{pmatrix} PTQ & PTQ^\perp \\ P^\perp TQ & P^\perp TQ^\perp \end{pmatrix}_{[P,Q]}.$$

and this representation is unique.

Lemma 1.5.6 *Let P, Q, R be three projections and let $T, S \in \mathcal{N}$ be such that*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}_{[P,Q]}, \qquad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}_{[Q,R]}.$$

Then

$$TS = \begin{pmatrix} T_{11}S_{11} + T_{12}S_{21} & T_{11}S_{11} + T_{12}S_{21} \\ T_{21}S_{11} + T_{22}S_{21} & T_{21}S_{12} + T_{22}S_{22} \end{pmatrix}_{[P,R]}.$$

Proof. Direct calculation. □

1.5.2 The semifinite Fredholm alternative

The following theorem due to Breuer [Br, Theorem 1] is a generalized Fredholm alternative and is very important in the theory of τ-Fredholm operators. Breuer proved this theorem for semifinite factors, but as is shown in [PR, Appendix B], the difference between the factor case and non-factor case is very small.

Theorem 1.5.7 *If K is a τ-compact operator then $T := 1 - K$ is a τ-Fredholm operator, and the projections N_T and N_{T^*} are equivalent (and τ-finite), so that the τ-index of T is zero.*

Proof. (A) Claim: if K is τ-finite then $T = 1 - K$ is τ-Fredholm.

Suppose that R_K is a τ-finite projection, i.e. suppose that K is a τ-finite operator. Then, since $\mathrm{R}_K \sim \mathrm{R}_{K^*}$ by Lemma 1.3.4, and $\tau(\mathrm{R}_K \vee \mathrm{R}_{K^*}) = \tau(\mathrm{R}_K) + \tau(\mathrm{R}_{K^*}) - \tau(\mathrm{R}_K \wedge \mathrm{R}_{K^*})$, by the parallelogram rule Lemma 1.3.6, it follows that the projection

$$E := \mathrm{R}_K \vee \mathrm{R}_{K^*} \tag{1.34}$$

is also τ-finite. The projection E satisfies the relations $EK = KE = K$, so that

$$E^\perp T = E^\perp, \tag{1.35}$$

$$TE^\perp = E^\perp. \tag{1.36}$$

(1.36) implies that $E^\perp \mathcal{H} = \operatorname{ran} TE^\perp \subseteq \operatorname{ran} T$, so that (BF2) of Definition 1.5.1 holds. For $\xi \in \mathcal{H}$, if $\mathrm{N}_T \xi = \xi$, then $T\xi = 0$, and hence by (1.35) $E^\perp \xi = E^\perp T \xi = 0$, so that $E\xi = \xi$, which implies that $\mathrm{N}_T \leqslant E$, so that (BF1) also holds. Hence, T is τ-Fredholm.

(B) Claim: if K is τ-finite and $T = 1 - K$ then the projections N_T and N_{T^*} are equivalent (and τ-finite).

If $\mathrm{N}_{1-K^*}\xi = \xi$ then $(1 - K^*)\xi = 0$, $K^*\xi = \xi$, $\mathrm{R}_{K^*}\xi = \xi$, so that $\mathrm{R}_{K^*} \geqslant \mathrm{N}_{1-K^*}$, and hence by (1.34) we have

$$E \geqslant \mathrm{N}_{1-K^*}. \tag{1.37}$$

Now, the formula (1.12) applied to the operator $1 - K$, gives

$$\mathrm{R}_{1-K} = \mathrm{N}_{1-K^*}^\perp.$$

The formula (1.37) implies $E\mathrm{N}_{1-K^*} = \mathrm{N}_{1-K^*}$, so that multiplying the last formula by E from the left we get

$$E\mathrm{R}_{1-K} = E - \mathrm{N}_{1-K^*}. \tag{1.38}$$

Let $\xi \in \operatorname{ran}(E\mathrm{R}_{1-K})$. Then there exists $\eta \in \mathcal{H}$, such that $\xi = E\mathrm{R}_{1-K}\eta$. Hence, since for any $\varepsilon > 0$ there exists $\eta' \in \mathcal{H}$, such that

$$\|\mathrm{R}_{1-K}\eta - (1-K)\eta'\| < \varepsilon,$$

it follows that

$$\begin{aligned} \|\xi - (E-K)\eta'\| &= \|\xi - (E - EK)\eta'\| \\ &= \|E\mathrm{R}_{1-K}\eta - E(1-K)\eta'\| \leqslant \|\mathrm{R}_{1-K}\eta - (1-K)\eta'\| < \varepsilon. \end{aligned} \tag{1.39}$$

Consequently, $\xi \in \overline{\operatorname{ran}(E-K)}$.

Now, let $\xi \in \operatorname{ran}(E-K)$. Then there exists $\eta \in \mathcal{H}$, such that

$$\xi = (E-K)\eta = (E-EK)\eta = E(1-K)\eta = E\mathrm{R}_{1-K}\eta'$$

for some $\eta' \in \mathcal{H}$, which means that $\xi \in \operatorname{ran}(E\mathrm{R}_{1-K})$. Hence, $\overline{\operatorname{ran}(E\mathrm{R}_{1-K})} = \overline{\operatorname{ran}(E-K)}$. This means that

$$E\mathrm{R}_{1-K} = \mathrm{R}_{E-K}, \tag{1.40}$$

since by (1.38) the LHS of (1.40) is also a projection, and the ranges of projections on each side of (1.40) coincide.

Now, it follows from (1.38) and (1.40) that

$$\mathrm{R}_{E-K} = E - \mathrm{N}_{1-K^*},$$

and

$$\mathrm{R}_{E-K^*} = E - \mathrm{N}_{1-K}.$$

Since by Lemma 1.3.4 $\mathrm{R}_{E-K^*} \sim \mathrm{R}_{E-K}$ and the projection E is τ-finite, it follows that the projections N_{1-K} and N_{1-K^*} are equivalent and τ-finite, and hence

$$\tau\text{-}\operatorname{ind}(1-K) = \tau(\mathrm{N}_{1-K}) - \tau(\mathrm{N}_{1-K^*}) = 0.$$

(C) Here we prove that $1-K$ is τ-Fredholm and that $\mathrm{N}_{1-K} \sim \mathrm{N}_{1-K^*}$ in the general case of τ-compact K.

Since the ideal $\mathcal{K}(\mathcal{N},\tau)$ of τ-compact operators coincide with norm closure of the ideal of finite operators (Lemma 1.3.12), for $\varepsilon = \frac{1}{2}$ there exists a τ-finite operator K_0, such that $\|K-K_0\| < \varepsilon$, so that the operator

$$S = 1 - (K - K_0)$$

is invertible. We have

$$1 - K = S - K_0 = (1 - K_0 S^{-1})S$$

and

$$1 - K^* = S^* - K_0^* = S^*\left(1 - (K_0 S^{-1})^*\right),$$

so that it follows from (1.17) and (1.16) that

$$\mathrm{N}_{1-K} \sim \mathrm{N}_{1-K_0S^{-1}},$$
$$\mathrm{N}_{1-K^*} = \mathrm{N}_{1-(K_0S^{-1})^*}.$$

Since K_0S^{-1} is a τ-finite operator (by Lemma 1.3.8), it follows from parts (A) and (B) that the projections N_{1-K} and N_{1-K^*} are τ-finite and equivalent. Hence, it is left to prove the axiom (BF2) of Definition 1.5.1.

(D) Since K_0 is τ-finite, it follows from Lemma 1.3.8 and the parallelogram rule (Lemma 1.3.6) that the projection

$$F := \mathrm{R}_{K_0 S^{-1}} \vee \mathrm{R}_{(S^*)^{-1} K_0}$$

is τ-finite. Now, the relation

$$(1-K)(1-S^{-1}FS)S^{-1} = 1-F$$

implies that $\operatorname{ran}(1-F) \subseteq \operatorname{ran}(1-K)$, proving (BF2) of Definition 1.5.1. $\square$

1.5.3 The semifinite Atkinson theorem

Definition 1.5.8 *Let $T \in \mathcal{N}$. An operator $S \in \mathcal{N}$ is said to be a τ-**parametrix** of the operator T if the operators $1 - ST$ and $1 - TS$ are τ-compact.*

The aim of this subsection is to prove Theorem 1.5.14, which is a generalization of Atkinson's theorem due to Breuer [Br$_2$, Theorem 1].

We recall that if $\mathcal{K}_1$ is a subspace of a subspace $\mathcal{K}_2$ of $\mathcal{H}$ then $\mathcal{K}_2 \ominus \mathcal{K}_1$ is the orthogonal complement of $\mathcal{K}_1$ in $\mathcal{K}_2$, i.e. $\mathcal{K}_2 \ominus \mathcal{K}_1 = \mathcal{K}_2 \cap \mathcal{K}_1^{\perp}$.

Lemma 1.5.9 *Let $S, T \in \mathcal{N}$. The restriction of T to the subspace $\ker ST \ominus \ker T$ is a bijective map onto $\operatorname{ran} T \cap \ker S$.*

Proof. (A) ($T|_{\ker ST \ominus \ker T}$ is injective). Let $\xi \in \ker ST \ominus \ker T$. This means that $ST\xi = 0$ and $\xi \perp \ker T$. If $T\xi = 0$ then $\xi \perp \xi$ so that $\xi = 0$. Hence $T|_{\ker ST \ominus \ker T}$ is injective.

(B) ($T|_{\ker ST \ominus \ker T}$ is surjective). Let $\eta \in \operatorname{ran} T \cap \ker S$. This means that $S\eta = 0$ and that there exists $\xi \in \mathcal{H}$, such that $\eta = T\xi$. Let $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \ker T$, $\xi_2 \perp \ker T$. Then $ST\xi_2 = ST\xi - ST\xi_1 = S\eta - 0 = 0$, which means that $\xi_2 \in \ker ST \ominus \ker T$. Also, $T\xi_2 = T\xi - T\xi_1 = \eta - 0 = \eta$. Hence, $T|_{\ker ST \ominus \ker T}$ is surjective. $\square$

Lemma 1.5.10 *Let $E_1 \leqslant E_2 \leqslant \dots$ be a non-decreasing sequence of projections in $\mathcal{N}$. If the projection $E_\infty = \bigvee_{n=1}^{\infty} E_n$ is τ-finite, then for any projection $F \in \mathcal{N}$*

$$E_\infty \wedge F = \bigvee_{n=1}^{\infty} (E_n \wedge F).$$

Proof. The parallelogram rule (1.14) implies

$$F - F \wedge E_n^{\perp} \sim E_n - E_n \wedge F^{\perp} \tag{1.41}$$

$$F - F \wedge E_\infty^{\perp} \sim E_\infty - E_\infty \wedge F^{\perp}. \tag{1.42}$$

Further, $E_n \leqslant E_\infty$ implies

$$F - F \wedge E_n^\perp \ \leqslant\ F - F \wedge E_\infty^\perp.$$

This, together with (1.41) and (1.42), implies

$$E_n - E_n \wedge F^\perp \ \prec\ E_\infty - E_\infty \wedge F^\perp.$$

Taking traces, we get

$$\tau\left(E_n\right) - \tau\left(E_n \wedge F^\perp\right) \ \leqslant\ \tau\left(E_\infty\right) - \tau\left(E_\infty \wedge F^\perp\right)$$

or

$$\tau\left(E_\infty \wedge F^\perp\right) - \tau\left(E_n \wedge F^\perp\right) \ \leqslant\ \tau\left(E_\infty - E_n\right).$$

By normality of τ, the left hand side tends to 0 when $n \to \infty$. Hence,

$$\lim_{n\to\infty} \tau\left(E_n \wedge F^\perp\right) = \tau\left(E_\infty \wedge F^\perp\right).$$

Again, by normality of τ, we have

$$\lim_{n\to\infty} \tau\left(E_n \wedge F^\perp\right) = \tau\left(\bigvee (E_n \wedge F^\perp)\right),$$

so that

$$\tau\left(\bigvee (E_n \wedge F^\perp)\right) = \tau\left(E_\infty \wedge F^\perp\right)$$

Since τ is faithful, it follows that

$$\bigvee (E_n \wedge F^\perp) = E_\infty \wedge F^\perp.$$

□

The following lemma is a combination of [Br, Lemma 13] and its Corollary.

Lemma 1.5.11 *If $T \in \mathcal{N}$ is a τ-Fredholm operator, then there exists a non-decreasing sequence of projections $E_1 \leqslant E_2 \leqslant \ldots$ in $\mathcal{N}$, such that for all $n = 1, 2, \ldots$ the projection $E_n^\perp$ is τ-finite,* $\operatorname{ran} E_n \subseteq \operatorname{ran} T$ *and* $\bigvee\limits_{n=1}^{\infty} E_n = \mathrm{R}_T$.

Proof. (A) First, let $A \in \mathcal{N}$ and $A \geqslant 0$. Let $F_n = 1 - E^A_{[0,1/n]}$, $n = 1, 2, \ldots$, so that $F_1 \leqslant F_2 \leqslant \ldots$ Since by (1.5)

$$\bigwedge_{n=1}^{\infty} E^A_{[0,1/n]} = E^A_{\{0\}} = \mathrm{N}_A,$$

we have

$$\bigvee_{n=1}^{\infty} F_n = \mathrm{N}_A^\perp = \mathrm{R}_A.$$

The pair $(F_n\mathcal{H}, F_n^\perp\mathcal{H})$ of subspaces of $\mathcal{H}$ reduces A. By the spectral theorem, the restriction of A to the subspace $F_n\mathcal{H}$ is invertible, and hence the range of this restriction is $F_n\mathcal{H} = \operatorname{ran} F_n$. Hence, $\operatorname{ran} F_n \subset \operatorname{ran} A$.

(B) Let $B \in \mathcal{N}$. If $B = V\,|B|$ is the polar decomposition of B, then let $P_n = VF_nV^*$, where $\{F_n\}$ is the sequence constructed in (A) for $|B|$. Then $P_1 \leqslant P_2 \leqslant \ldots$ and $\bigvee\limits_{n=1}^{\infty} P_n = \mathrm{R}_B$.

(C) By (BF2), let E be a projection such that $\operatorname{ran} E \subset \operatorname{ran} T$ and $E^\perp$ is τ-finite. If $F = \mathrm{R}_T - E$, then the sequence $E_n = E + P_n$, where P_n is the sequence constructed in (B) for $B = FT$, satisfies the conditions of the lemma. □

Lemma 1.5.12 *If S, $T \in \mathcal{N}$ are τ-Fredholm operators, then*

$$\mathrm{N}_{ST} - \mathrm{N}_T \;\sim\; \mathrm{R}_T \wedge \mathrm{N}_S. \tag{1.43}$$

Proof.

Since the range of the projection $\mathrm{N}_{ST} - \mathrm{N}_T$ is $\ker ST \ominus \ker T$, it follows from Lemma 1.5.9 that the range of the operator $T(\mathrm{N}_{ST} - \mathrm{N}_T)$ is the subspace $\operatorname{ran} T \cap \ker S$. Hence,

$$\mathrm{R}_{T(\mathrm{N}_{ST}-\mathrm{N}_T)} \leqslant \mathrm{R}_T \wedge \mathrm{N}_S. \tag{1.44}$$

Now, the difficulty to overcome is that $\operatorname{ran} T$ is not necessarily closed.

Since by Lemma 1.5.9 T is bijective on the range $\ker ST \ominus \ker T$ of the projection $\mathrm{N}_{ST} - \mathrm{N}_T$, it follows that $\ker\left(T(\mathrm{N}_{ST} - \mathrm{N}_T)\right) = \ker\left(\mathrm{N}_{ST} - \mathrm{N}_T\right)$, so that $\mathrm{N}_{T(\mathrm{N}_{ST}-\mathrm{N}_T)} = \mathrm{N}_{\mathrm{N}_{ST}-\mathrm{N}_T}$. Hence, by (1.11) we have

$$\mathrm{R}_{(\mathrm{N}_{ST}-\mathrm{N}_T)T^*} = \mathrm{N}^\perp_{T(\mathrm{N}_{ST}-\mathrm{N}_T)} = \mathrm{N}^\perp_{\mathrm{N}_{ST}-\mathrm{N}_T} = \mathrm{N}_{ST} - \mathrm{N}_T. \tag{1.45}$$

By Lemma 1.5.11 there exists a sequence $E_1 \leqslant E_2 \leqslant \ldots$ of projections of $\mathcal{N}$, such that $E_1^\perp$ is τ-finite, $\operatorname{ran} E_n \subseteq \operatorname{ran} T$ and $\bigvee_{n\in\mathbb{N}} E_n = \mathrm{R}_T$. Then Lemma 1.5.9 implies that

$$E_n \wedge \mathrm{N}_S \leqslant \mathrm{R}_{T(\mathrm{N}_{ST}-\mathrm{N}_T)}. \tag{1.46}$$

Define $E_0 = E_1 \wedge \mathrm{N}_S^\perp$. Since S is τ-Fredholm, the projection N_S is τ-finite, so that by the parallelogram rule (1.14) the projection

$$E_1 - E_1 \wedge \mathrm{N}_S^\perp \;\sim\; \mathrm{N}_S - \mathrm{N}_S \wedge E_1^\perp$$

is τ-finite. Hence, the projection $E_0^\perp = E_1^\perp + \left[E_1 - E_1 \wedge \mathrm{N}_S^\perp\right]$ is also τ-finite.

The relations $E_0\mathrm{N}_S = 0$ and $E_0 \leqslant E_n$ imply

$$E_n \wedge \mathrm{N}_S = (E_n - E_0) \wedge \mathrm{N}_S, \quad \mathrm{R}_T \wedge \mathrm{N}_S = (\mathrm{R}_T - E_0) \wedge \mathrm{N}_S. \tag{1.47}$$

Using Lemma 1.5.10 it follows from previous equalities that

$$\bigvee_{n=1}^{\infty} (E_n \wedge \mathrm{N}_S) = \mathrm{R}_T \wedge \mathrm{N}_S. \tag{1.48}$$

The relations (1.46) and (1.48) imply that

$$\mathrm{R}_{T(\mathrm{N}_{ST}-\mathrm{N}_T)} \geqslant \mathrm{R}_T \wedge \mathrm{N}_S.$$

Combining it with (1.44), we get

$$\mathrm{R}_{T(\mathrm{N}_{ST}-\mathrm{N}_T)} = \mathrm{R}_T \wedge \mathrm{N}_S. \tag{1.49}$$

The relations (1.45), (1.49) and Lemma 1.3.4 now imply

$$\mathrm{N}_{ST} - \mathrm{N}_T = \mathrm{R}_{(\mathrm{N}_{ST}-\mathrm{N}_T)T^*} \ \sim \ \mathrm{R}_{T(\mathrm{N}_{ST}-\mathrm{N}_T)} = \mathrm{R}_T \wedge \mathrm{N}_S,$$

which is (1.43). □

Lemma 1.5.13 *If $T \in \mathcal{N}$, P is a projection in $\mathcal{N}$ such that* $\operatorname{ran} P^\perp \subset \operatorname{ran} T$ *and if $Q := \mathrm{N}_{P^\perp T}$ then the map $P^\perp T Q^\perp \colon Q^\perp \mathcal{H} \to P^\perp \mathcal{H}$ is bijective.*

Proof. (Injective) Let $\xi \in Q^\perp \mathcal{H}$ and $P^\perp T Q^\perp \xi = 0$. Then $P^\perp T \xi = 0$, $\mathrm{N}_{P^\perp T}\xi = \xi$, $Q\xi = \xi$ or $Q^\perp \xi = 0$. Since $\xi \in Q^\perp \mathcal{H}$, it follows that $\xi = 0$.

(Surjective) Let $\eta \in P^\perp \mathcal{H}$. Since $\operatorname{ran} P^\perp \subset \operatorname{ran} T$, there exists $\xi \in \mathcal{H}$ such that $\eta = T\xi$. Hence, $\eta = P^\perp \eta = P^\perp T \xi$. Now, since $P^\perp T Q = P^\perp T \mathrm{N}_{P^\perp T} = 0$, we have $P^\perp T = P^\perp T Q^\perp$, so that $\eta = P^\perp T Q^\perp \xi$. □

The **Calkin algebra** $\mathcal{Q}(\mathcal{N}, \tau)$ is by definition the factor-algebra $\mathcal{N}/\mathcal{K}(\mathcal{N}, \tau)$. Let

$$\pi \colon T \in \mathcal{N} \mapsto T + \mathcal{K}(\mathcal{N}, \tau) \in \mathcal{Q}(\mathcal{N}, \tau).$$

Since $\mathcal{K}(\mathcal{N}, \tau)$ is norm-closed ideal of $\mathcal{N}$, the algebra $\mathcal{Q}(\mathcal{N}, \tau)$ is a Banach algebra with the norm

$$\|\pi(T)\|_{\mathcal{Q}(\mathcal{N},\tau)} = \inf_{K \in \mathcal{K}(\mathcal{N},\tau)} \|T + K\|.$$

Theorem 1.5.14 *Let $T \in \mathcal{N}$. Then the following conditions are equivalent.*
(i) T is τ-Fredholm;
(ii) $\pi(T)$ is an invertible element of $\mathcal{Q}(\mathcal{N}, \tau)$;
(iii) T has a τ-parametrix.

Proof. $(iii) \Rightarrow (i)$. Let S be a τ-parametrix of T, i.e. there exist τ-compact operators $K, L \in \mathcal{N}$ such that

$$ST = 1 - K, \tag{1.50}$$
$$TS = 1 - L. \tag{1.51}$$

Since $1-K$ is τ-Fredholm by Theorem 1.5.7, the projection N_{1-K} is τ-finite. Since (1.50) with (1.9) imply $\mathrm{N}_T \leqslant \mathrm{N}_{1-K}$, it follows that the projection N_T is also τ-finite, so that (BF1) is satisfied.

The equality (1.51) implies

$$\operatorname{ran}(1-L) \subseteq \operatorname{ran} T.$$

Since $1-L$ is τ-Fredholm by Theorem 1.5.7, by axiom (BF2) there is a τ-finite projection $E \in \mathcal{N}$ such that

$$\operatorname{ran}(1-E) \subseteq \operatorname{ran}(1-L)$$

and consequently

$$\operatorname{ran}(1-E) \subseteq \operatorname{ran} T,$$

so that the axiom (BF2) of Definition 1.5.1 is also satisfied for operator T.

(i) $\Rightarrow$ *(ii)*. Suppose that T is τ-Fredholm.

(A) There exists a τ-finite projection P such that

$$\operatorname{ran} P^{\perp} \subset \operatorname{ran} T.$$

Lemma 1.5.12 implies

$$\mathrm{N}_{P^{\perp}T} - \mathrm{N}_T \ \sim\ \mathrm{R}_T \wedge P.$$

Hence, the projection $Q := \mathrm{N}_{P^{\perp}T}$ is τ-finite.

By Lemma 1.5.13, the operator

$$P^{\perp}TQ^{\perp} \colon Q^{\perp}\mathcal{H} \to P^{\perp}\mathcal{H}$$

is bijective and hence is invertible by Banach's inverse mapping theorem. This means that

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}_{[P,Q]}$$

with invertible T_{22}. If

$$S = \begin{pmatrix} 0 & 0 \\ 0 & T_{22}^{-1} \end{pmatrix}_{[Q,P]},$$

then by Lemma 1.5.6 we will have

$$ST = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix}_{[Q,Q]}, \quad TS = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix}_{[P,P]} \quad \in \quad 1 + \mathcal{K}(\mathcal{N}, \tau),$$

where the last inclusion is true since P and Q are τ-finite.

(ii) $\Rightarrow$ *(iii)*. Let S and S' be such that $ST,\ TS' \in 1 + \mathcal{K}(\mathcal{N}, \tau)$. Then $\pi(S)\pi(T) = \pi(T)\pi(S') = 1$, so that

$$\pi(S) = \pi(S)\pi(T)\pi(S') = \pi(S'),$$

which means that there exists $K \in \mathcal{K}(\mathcal{N}, \tau)$ such that $S = S' + K$. Hence, $TS = T(S'+K) = TS' + TK \in 1 + \mathcal{K}(\mathcal{N}, \tau)$. It follows that S is a τ-parametrix for T. $\square$

1.5.4 Properties of τ-Fredholm operators

Proposition 1.5.15 *If T is τ-Fredholm then T^* is also τ-Fredholm and in this case*

$$\tau\text{-}\operatorname{ind}(T^*) = -\tau\text{-}\operatorname{ind}(T).$$

Proof. If S is a τ-parametrix for T then S^* is a τ-parametrix for T^*. Hence, T^* is τ-Fredholm by Theorem 1.5.14. The formula follows directly from definition (1.33) of the index. □

Proposition 1.5.16 *If S, T are τ-Fredholm then ST is also τ-Fredholm and in this case*

$$\tau\text{-}\operatorname{ind}(ST) = \tau\text{-}\operatorname{ind}(S) + \tau\text{-}\operatorname{ind}(T). \tag{1.52}$$

Proof. If S' is a τ-parametrix for S and T' is a τ-parametrix for T then $S'T'$ is a τ-parametrix for ST. Hence, ST is τ-Fredholm by Theorem 1.5.14.

To prove (1.52), we note that Lemma 1.5.12 implies

$$\mathrm{N}_{ST} - \mathrm{N}_T \;\sim\; \mathrm{R}_T \wedge \mathrm{N}_S, \tag{1.53}$$

$$\mathrm{N}_{(ST)^*} - \mathrm{N}_{S^*} \;\sim\; \mathrm{R}_{S^*} \wedge \mathrm{N}_{T^*}. \tag{1.54}$$

One has according to the parallelogram rule (Lemma 1.3.6)

$$\mathrm{N}_S - \mathrm{N}_{T^*}^{\perp} \wedge \mathrm{N}_S \;\sim\; \mathrm{N}_{T^*} - \mathrm{N}_S^{\perp} \wedge \mathrm{N}_{T^*}.$$

According to Lemma 1.3.5, we have $\mathrm{N}_{T^*}^{\perp} = \mathrm{R}_T$ and $\mathrm{N}_S^{\perp} = \mathrm{R}_{S^*}$, so that

$$\mathrm{N}_S - \mathrm{R}_T \wedge \mathrm{N}_S \;\sim\; \mathrm{N}_{T^*} - \mathrm{R}_{S^*} \wedge \mathrm{N}_{T^*}. \tag{1.55}$$

Combining (1.53), (1.54) and (1.55), we have

$$\mathrm{N}_S - (\mathrm{N}_{ST} - \mathrm{N}_T) \;\sim\; \mathrm{N}_{T^*} - (\mathrm{N}_{(ST)^*} - \mathrm{N}_{S^*}).$$

Taking traces gives

$$\tau(\mathrm{N}_S) - \tau(\mathrm{N}_{ST}) + \tau(\mathrm{N}_T) = \tau(\mathrm{N}_{T^*}) - \tau\left(\mathrm{N}_{(ST)^*}\right) - \tau(\mathrm{N}_{S^*}),$$

or

$$\tau(\mathrm{N}_{ST}) - \tau\left(\mathrm{N}_{(ST)^*}\right) = \tau(\mathrm{N}_S) - \tau(\mathrm{N}_{S^*}) + \tau(\mathrm{N}_T) - \tau(\mathrm{N}_{T^*}),$$

so that (1.52) follows. □

Proposition 1.5.17 *If $T \in \mathcal{N}$ is τ-Fredholm and if $K \in \mathcal{N}$ is τ-compact then $T + K$ is also τ-Fredholm and*

$$\tau\text{-}\operatorname{ind}(T + K) = \tau\text{-}\operatorname{ind}(T).$$

Proof. If S is a τ-parametrix for T then S is also τ-parametrix for $T+K$, so that $T+K$ is τ-Fredholm by Theorem 1.5.14.

So, let $ST = 1 - L_1$ and $S(T+K) = 1 - L_2$, where L_1, L_2 are τ-compact operators. By Theorem 1.5.7 and Proposition 1.5.16 we have

$$0 = \tau\text{-}\operatorname{ind}(1 - L_1) = \tau\text{-}\operatorname{ind}(ST) = \tau\text{-}\operatorname{ind}(S) + \tau\text{-}\operatorname{ind}(T),$$

$$0 = \tau\text{-}\operatorname{ind}(1 - L_2) = \tau\text{-}\operatorname{ind}(S(T+K)) = \tau\text{-}\operatorname{ind}(S) + \tau\text{-}\operatorname{ind}(T+K).$$

Hence, $\tau\text{-}\operatorname{ind}(T+K) = \tau\text{-}\operatorname{ind}(T)$. $\square$

Proposition 1.5.18 *The set $\mathcal{F}(\mathcal{N},\tau)$ of τ-Fredholm operators is open in the norm topology of $\mathcal{N}$ and the index $\tau\text{-}\operatorname{ind}$ is a locally constant function on $\mathcal{F}(\mathcal{N},\tau)$.*

Proof. (A) Let T be a τ-Fredholm operator. By Theorem 1.5.14, there exists $S \in \mathcal{N}$ such that $ST = 1 + K_1$ and $TS = 1 + K_2$, where $K_1, K_2 \in \mathcal{K}(\mathcal{N},\tau)$. If $0 < \varepsilon < \|S\|^{-1}$, then, for any $A \in \mathcal{N}$ with $\|A\| < \varepsilon$, we have $\|AS\| < 1$ and $\|SA\| < 1$, so that the operators $1 + SA$ and $1 + AS$ are invertible. We have

$$\begin{aligned}(1+SA)^{-1}S(T+A) &= (1+SA)^{-1}(ST+SA) \\ &= (1+SA)^{-1}(1+K_1+SA) = 1 + (1+SA)^{-1}K_1,\end{aligned} \tag{1.56}$$

and

$$\begin{aligned}(T+A)S(1+AS)^{-1} &= (TS+AS)(1+AS)^{-1} \\ &= (1+K_2+AS)(1+AS)^{-1} = 1 + K_2(1+AS)^{-1}.\end{aligned} \tag{1.57}$$

Since the operators $(1+SA)^{-1}K_1$ and $K_2(1+AS)^{-1}$ are τ-compact, the operator $T+A$ is invertible in $\mathcal{Q}(\mathcal{N},\tau)$. By Theorem 1.5.14, $T+A$ is τ-Fredholm.

(B) Taking τ-indices of (1.56), by Proposition 1.5.16 and Theorem 1.5.7, we see that

$$\tau\text{-}\operatorname{ind}((1+SA)^{-1}) + \tau\text{-}\operatorname{ind}(S) + \tau\text{-}\operatorname{ind}(T+A) = 0.$$

Since $(1+SA)^{-1}$ is invertible, we have $\tau\text{-}\operatorname{ind}((1+SA)^{-1}) = 0$. Hence,

$$\tau\text{-}\operatorname{ind}(T+A) = -\tau\text{-}\operatorname{ind}(S) = \tau\text{-}\operatorname{ind}(T),$$

where the last equality follows from $ST = 1 + K_1$ after taking τ-index. $\square$

1.5.5 Skew-corner τ-Fredholm operators

The aim of this subsection is to give an exposition of the theory of skew corner τ-Fredholm operators. Here we follow [CPRS$_2$, Chapter 3], with some improvements.

In the case of skew-corner Fredholm operators, an operator $T \in P\mathcal{N}Q$ is considered as a map from $Q\mathcal{H}$ to $P\mathcal{H}$, where P and Q are some projections from $\mathcal{N}$. This notion will be necessary in the theory of spectral flow of J. Phillips (Section 1.6).

If $T \in \mathcal{N}$ and Q is a projection in $\mathcal{N}$ then we denote by N_T^Q the projection onto $\ker(T) \cap Q\mathcal{H}$, i.e.

$$\mathrm{N}_T^Q = \mathrm{N}_T \wedge Q.$$

Lemma 1.5.19 *Let P and Q be two projections in $\mathcal{N}$ and let $T \in P\mathcal{N}Q$. Then*

$$\mathrm{N}_T^Q = \mathrm{N}_T Q = Q\mathrm{N}_T = Q - \mathrm{R}_{T^*} \tag{1.58}$$

and

$$\mathrm{N}_{T^*}^P = \mathrm{N}_{T^*} P = P\mathrm{N}_{T^*} = P - \mathrm{R}_T. \tag{1.59}$$

Proof. Since $T = PTQ$, the relation $Q\xi = 0$ implies $T\xi = 0$, i.e. $\mathrm{N}_T\xi = \xi$, so that $\mathrm{N}_T \geqslant Q^\perp$. Hence, $\mathrm{N}_T Q = \mathrm{N}_T(1 - Q^\perp) = \mathrm{N}_T - Q^\perp = \mathrm{N}_T - Q^\perp \mathrm{N}_T = Q\mathrm{N}_T$. This yields $\mathrm{N}_T^Q = \mathrm{N}_T Q = Q\mathrm{N}_T$. Further, $\mathrm{N}_T Q = \mathrm{N}_T - Q^\perp = Q - (1 - \mathrm{N}_T) = Q - \mathrm{R}_{T^*}$ by (1.11).

Since, $T^* \in Q\mathcal{N}P$, (1.59) follows from (1.58) applied to P and T^* instead of Q and T. □

Definition 1.5.20 *Let $\mathcal{N}$ be a semifinite von Neumann algebra, and let τ be a normal semifinite faithful trace in $\mathcal{N}$. Let P and Q be two projections from $\mathcal{N}$ and let $T \in P\mathcal{N}Q$. Then T is called* $(P \cdot Q)$ τ-Fredholm *if and only if*
(BF1') the projection N_T^Q is τ-finite;
(BF2') the projection $\mathrm{N}_{T^*}^P$ is τ-finite;
(BF3') there exists a τ-finite projection $E \leqslant P$ such that $\operatorname{ran}(P - E) \subseteq \operatorname{ran}(T)$.

If T is $(P \cdot Q)$ τ-Fredholm then the skew corner τ-index *of T is*

$$\tau\text{-}\mathrm{ind}_{P\text{-}Q}(T) = \tau\big(\mathrm{N}_T^Q\big) - \tau\big(\mathrm{N}_{T^*}^P\big).$$

This index can be considered as the τ-index of T as an operator from $Q\mathcal{H}$ to $P\mathcal{H}$, in which case N_T^Q is exactly the kernel of $T\colon Q\mathcal{H} \to P\mathcal{H}$ and $\mathrm{N}_{T^*}^P$ is the kernel of $T^*\colon P\mathcal{H} \to Q\mathcal{H}$.

The set of $(P \cdot Q)$ τ-Fredholm operators will be denoted by $\mathcal{F}_{P\text{-}Q}(\mathcal{N}, \tau)$.

Lemma 1.5.21 *Let P and Q be two projections in $\mathcal{N}$. If T is $(P \cdot Q)$ τ-Fredholm then T is $(\mathrm{R}_T \cdot \mathrm{R}_{T^*})$ τ-Fredholm, and one has the relation*

$$\tau\text{-ind}_{\mathrm{R}_T\text{-}\mathrm{R}_{T^*}} T = 0.$$

Proof. By axiom (BF3'), let $E \leqslant P$ be a τ-finite projection such that

$$\operatorname{ran}(P - E) \subseteq \operatorname{ran} T. \tag{1.60}$$

By Lemma 1.3.5, both projections $\mathrm{N}_T^{\mathrm{R}_{T^*}}$ and $\mathrm{N}_{T^*}^{\mathrm{R}_T}$ are zero, so that the axioms (BF1') and (BF2') hold true. We claim that the projection $F := \mathrm{R}_T \wedge (P - E)$ satisfies the axiom (BF3') with respect to R_T. The properties $F \leqslant \mathrm{R}_T$ and $\operatorname{ran}(\mathrm{R}_T - F) \subseteq \operatorname{ran} T$ are evident. By the parallelogram rule (Lemma 1.3.6) applied to von Neumann algebra $P\mathcal{N}P$, we have

$$\mathrm{R}_T - F = \mathrm{R}_T - \mathrm{R}_T \wedge (P - E) = E - E \wedge (P - \mathrm{R}_T),$$

so that the projection $\mathrm{R}_T - F$ is τ-finite. Hence, T is $(\mathrm{R}_T \cdot \mathrm{R}_{T^*})$ τ-Fredholm. The equality is evident. $\square$

Lemma 1.5.22 *Let P and Q be two projections in $\mathcal{N}$. If T is $(P \cdot Q)$ τ-Fredholm then the projections $P - \mathrm{R}_T$ and $Q - \mathrm{R}_{T^*}$ are τ-finite and one has the relation*

$$\tau\text{-ind}_{P\text{-}Q} T = \tau\left(Q - \mathrm{R}_{T^*}\right) - \tau\left(P - \mathrm{R}_T\right).$$

Proof. This follows directly from Lemma 1.5.19 and Definition 1.5.20. $\square$

Lemma 1.5.23 *Let P, Q, R be projections in $\mathcal{N}$, let $T \in P\mathcal{N}Q$ be $(P \cdot Q)$ τ-Fredholm and $S \in R\mathcal{N}P$ be $(R \cdot P)$ τ-Fredholm. Then*

$$\mathrm{N}_{ST}^Q - \mathrm{N}_T^Q \;\sim\; \mathrm{R}_T \wedge \mathrm{N}_S^P.$$

Proof. (A) Let $T^* = V\,|T^*|$ and $S = U\,|S|$ be the polar decompositions of T^* and S. Then $U^*S,\ TV \in P\mathcal{N}P$. Hence, by Lemma 1.5.12 applied to the operators U^*S and TV in the von Neumann algebra $P\mathcal{N}P$, we have

$$\mathrm{N}_{U^*STV}^P - \mathrm{N}_{TV}^P \;\sim\; \mathrm{R}_{TV} \wedge \mathrm{N}_{U^*S}^P. \tag{1.61}$$

(B) It is not difficult to see that the restriction of V to $(\mathrm{N}_{STV}^P - \mathrm{N}_{TV}^P)\mathcal{H}$ gives the equivalence

$$\mathrm{N}_{ST}^Q - \mathrm{N}_T^Q \;\sim\; \mathrm{N}_{STV}^P - \mathrm{N}_{TV}^P. \tag{1.62}$$

Indeed, since this restriction is also an isometry, and since evidently $V(\mathrm{N}_{STV}^P - \mathrm{N}_{TV}^P)\mathcal{H} \subset (\mathrm{N}_{ST}^Q - \mathrm{N}_T^Q)\mathcal{H}$, one needs only to show that for any $\eta \in Q\mathcal{H} \cap \ker ST$

which is orthogonal to $Q\mathcal{H} \cap \ker T$, there exists (a necessarily unique) $\eta_0 \in P\mathcal{H} \cap \ker STV$, orthogonal to $P\mathcal{H} \cap \ker TV$, such that $V\eta_0 = \eta$. One may take $\eta_0 = V^*\eta$.

(C) It is clear that

$$\mathrm{N}_S^P = \mathrm{N}_{U^*S}^P, \qquad \mathrm{N}_{U^*STV}^P = \mathrm{N}_{STV}^P$$

and

$$\mathrm{R}_T = \mathrm{R}_{TV}.$$

Combining these equalities with (1.61) and (1.62) completes the proof. □

The statement of the following lemma is used in the proof of the skew-corner Atkinson theorem in [CPRS_2, Lemma 3.4] without a proof, though it seems to be not so evident.

Lemma 1.5.24 *Let $P, Q, P_0 \in \mathcal{N}$ be projections, let $P_0 \leqslant P$, $T \in P\mathcal{N}Q$ and $(P - P_0)\mathcal{H} \subseteq T\mathcal{H}$. Let*

$$Q_0 := \mathrm{N}^Q_{(P-P_0)T}.$$

Then the map

$$(P - P_0)T(Q - Q_0) \colon (Q - Q_0)\mathcal{H} \to (P - P_0)\mathcal{H}$$

is bijective.

Proof. (Injective) Let $\xi \in (Q - Q_0)\mathcal{H}$ and $(P - P_0)T(Q - Q_0)\xi = 0$. Then $(P - P_0)T\xi = 0$, so that $\mathrm{N}_{(P-P_0)T}\xi = \xi$, and, since we also have $Q\xi = \xi$, it follows that $Q_0\xi = \mathrm{N}^Q_{(P-P_0)T}\xi = \xi$. So, $\xi = 0$.

(Surjective) Let $\eta \in (P - P_0)\mathcal{H}$. Since $(P - P_0)\mathcal{H} \subseteq T\mathcal{H}$, there exists $\xi \in \mathcal{H}$ such that $\eta = T\xi$, and so, $\eta = (P - P_0)T\xi$. Now, since

$$(P - P_0)TQ_0 = (P - P_0)T\mathrm{N}^Q_{(P-P_0)T} = 0,$$

we have $(P - P_0)T = (P - P_0)TQ = (P - P_0)T(Q - Q_0)$, so that $\eta = (P - P_0)T(Q - Q_0)\xi$. □

Definition 1.5.25 *If $T \in P\mathcal{N}Q$ then a* skew corner parametrix *of T is any operator $S \in Q\mathcal{N}P$ such that $ST = Q + K_1$ and $TS = P + K_2$, where K_1 is a τ-compact operator from $Q\mathcal{N}Q$ and K_2 is a τ-compact operator from $P\mathcal{N}P$.*

Theorem 1.5.26 *An operator $T \in P\mathcal{N}Q$ is $(P \cdot Q)$ τ-Fredholm if and only if it has a skew corner parametrix.*

Proof. (A) Let S be a parametrix for T. Then there exists $K_2 \in \mathcal{K}_{P\mathcal{N}P}$ such that $TS = P+K_2$, so that TS is τ-Fredholm in $P\mathcal{N}P$ by Theorem 1.5.7. Hence there exists a τ-finite projection $E \leqslant P$ such that $\operatorname{ran}(P-E) \subseteq \operatorname{ran}(TS) \subseteq \operatorname{ran}(T)$. So, the axiom (BF3') holds and $\mathrm{N}^P_{T^*} = P - \mathrm{R}_T \leqslant E$ is τ-finite, and hence (BF2') also holds. On the other hand, $T^*S^* = (ST)^* = Q + K_1$, where $K_1 \in \mathcal{K}_{Q\mathcal{N}Q}$, is also τ-Fredholm in $Q\mathcal{N}Q$ by the same Theorem 1.5.7, so that, by the same argument, N^Q_T is also τ-finite, that is, T is $(P \cdot Q)$ τ-Fredholm.

(B) Now, let $T \in P\mathcal{N}Q$ be $(P \cdot Q)$ τ-Fredholm. Then there exists a τ-finite projection P_0 such that $(P-P_0)\mathcal{H} \subseteq T\mathcal{H}$. By Lemma 1.5.23

$$\mathrm{N}^Q_{(P-P_0)T} - \mathrm{N}^Q_T \;\sim\; \mathrm{R}_T \wedge \mathrm{N}^P_{P-P_0} = \mathrm{R}_T \wedge P_0,$$

and so, since N^Q_T and P_0 are τ-finite, the projection

$$Q_0 := \mathrm{N}^Q_{(P-P_0)T}$$

is also τ-finite. By Lemma 1.5.24 and Banach's inverse mapping theorem, the map $(P-P_0)T(Q-Q_0)\colon (Q-Q_0)\mathcal{H} \to (P-P_0)\mathcal{H}$ has bounded inverse, say $S\colon (P-P_0)\mathcal{H} \to (Q-Q_0)\mathcal{H}$. Since P_0 and Q_0 are τ-finite, S is a parametrix for T. $\square$

Lemma 1.5.27 *If T is a $(P \cdot Q)$ τ-Fredholm operator, then T^* is a $(Q \cdot P)$ τ-Fredholm operator and*

$$\tau\text{-ind}_{Q\text{-}P}(T^*) = -\tau\text{-ind}_{P\text{-}Q}(T);$$

Proof. If S is a parametrix for T i.e. $P - TS \in \mathcal{K}_{P\mathcal{N}P}$ and $Q - ST \in \mathcal{K}_{Q\mathcal{N}Q}$ then $P - S^*T^* \in \mathcal{K}_{P\mathcal{N}P}$ and $Q - T^*S^* \in \mathcal{K}_{Q\mathcal{N}Q}$, i.e. S^* is a parametrix for T^*. Hence T^* is $(Q \cdot P)$ τ-Fredholm. The equality is evident. $\square$

Proposition 1.5.28 *Let S be $(R \cdot P)$ τ-Fredholm and T be $(P \cdot Q)$ τ-Fredholm. Then ST is $(R \cdot Q)$ τ-Fredholm and*

$$\tau\text{-ind}_{R\text{-}Q}(ST) = \tau\text{-ind}_{R\text{-}P}(S) + \tau\text{-ind}_{P\text{-}Q}(T).$$

Proof. (A) By Theorem 1.5.26, there exist a parametrix S' of S and a parametrix T' of T. Hence, for some $K \in \mathcal{K}_{P\mathcal{N}P}$, we have $R - STT'S' = R - S(P+K)S' = R - SS' + SKS' \in \mathcal{K}_{R\mathcal{N}R}$. Analogously, $Q - T'S'ST \in \mathcal{K}_{Q\mathcal{N}Q}$, i.e., $T'S'$ is a parametrix for ST, so that ST is $(R \cdot Q)$ τ-Fredholm by Theorem 1.5.26.

(B) By (A) and Lemma 1.5.27, the operators S^*, T^*, ST and T^*S^* are all skew τ-Fredholm, so that by Lemma 1.5.23, we have

$$\begin{aligned}\mathrm{N}^Q_{ST} - \mathrm{N}^Q_T \;&\sim\; \mathrm{R}_T \wedge \mathrm{N}^P_S \\ \mathrm{N}^R_{T^*S^*} - \mathrm{N}^R_{S^*} \;&\sim\; \mathrm{R}_{S^*} \wedge \mathrm{N}^P_{T^*}.\end{aligned}$$

By the parallelogram rule (Lemma 1.3.6) applied to the von Neumann algebra $P\mathcal{N}P$, we have

$$\mathrm{N}_S^P - (P - \mathrm{N}_{T^*}^P) \wedge \mathrm{N}_S^P \ \sim\ \mathrm{N}_{T^*}^P - (P - \mathrm{N}_S^P) \wedge \mathrm{N}_{T^*}^P.$$

By Lemma 1.5.19 $P - \mathrm{N}_{T^*}^P = \mathrm{R}_T$ and $P - \mathrm{N}_S^P = \mathrm{R}_{S^*}$, so that

$$\mathrm{N}_S^P - \mathrm{R}_T \wedge \mathrm{N}_S^P \ \sim\ \mathrm{N}_{T^*}^P - \mathrm{R}_{S^*} \wedge \mathrm{N}_{T^*}^P.$$

Using these similarities, we calculate

$$\begin{aligned}
\tau\text{-ind}_{R\text{-}Q}(ST) &= \tau\big(\mathrm{N}_{ST}^Q\big) - \tau\big(\mathrm{N}_{T^*S^*}^R\big)\\
&= \tau\big(\mathrm{N}_{ST}^Q - \mathrm{N}_T^Q\big) - \tau\big(\mathrm{N}_{T^*S^*}^R - \mathrm{N}_{S^*}^R\big) + \tau\big(\mathrm{N}_T^Q\big) - \tau\big(\mathrm{N}_{S^*}^R\big)\\
&= \tau\big(\mathrm{R}_T \wedge \mathrm{N}_S^P\big) - \tau\big(\mathrm{R}_{S^*} \wedge \mathrm{N}_{T^*}^P\big) + \tau\big(\mathrm{N}_T^Q\big) - \tau\big(\mathrm{N}_{S^*}^R\big)\\
&= \tau\big(\mathrm{N}_S^P\big) - \tau\big(\mathrm{N}_{T^*}^P\big) + \tau\big(\mathrm{N}_T^Q\big) - \tau\big(\mathrm{N}_{S^*}^R\big)\\
&= \tau\text{-ind}_{R\text{-}P}(S) + \tau\text{-ind}_{P\text{-}Q}(T).
\end{aligned}$$

□

Proposition 1.5.29 *If T is $(P \cdot Q)$ τ-Fredholm and $K \in \mathcal{K}_{P\mathcal{N}Q}$, then $T + K$ is also $(P \cdot Q)$ τ-Fredholm and*

$$\tau\text{-ind}_{P\text{-}Q}(T + K) = \tau\text{-ind}_{P\text{-}Q}(T).$$

Proof. Evidently, if S is a parametrix for T then S is also a parametrix for $T + K$.

So, let $ST = Q + K_1$ and $S(T + K) = Q + K_2$, where $K_1, K_2 \in \mathcal{K}_{Q\mathcal{N}Q}$. Then, by Theorem 1.5.7 applied to von Neumann algebra $Q\mathcal{N}Q$ and by Proposition 1.5.28, we have

$$0 = \tau\text{-ind}_{Q\text{-}Q}(Q + K_1) = \tau\text{-ind}_{Q\text{-}Q}(ST) = \tau\text{-ind}_{Q\text{-}P}(S) + \tau\text{-ind}_{P\text{-}Q}(T)$$

and

$$\begin{aligned}
0 = \tau\text{-ind}_{Q\text{-}Q}(Q + K_2) &= \tau\text{-ind}_{Q\text{-}Q}(S(T + K))\\
&= \tau\text{-ind}_{Q\text{-}P}(S) + \tau\text{-ind}_{P\text{-}Q}(T + K).
\end{aligned}$$

Hence, $\tau\text{-ind}_{P\text{-}Q}(T + K) = \tau\text{-ind}_{P\text{-}Q}(T)$. □

Proposition 1.5.30 *Let P and Q be projections in $\mathcal{N}$. The set $\mathcal{F}_{P\text{-}Q}(\mathcal{N}, \tau)$ of $(P \cdot Q)$ τ-Fredholm operators is open in the norm topology in $P\mathcal{N}Q$, i.e. for any $(P \cdot Q)$ τ-Fredholm operator T there exists $\varepsilon > 0$ such that for any $A \in P\mathcal{N}Q$ with $\|A\| < \varepsilon$ the operator $T + A$ is a $(P \cdot Q)$ τ-Fredholm operator, and, moreover,*

$$\tau\text{-ind}_{P\text{-}Q}(T + A) = \tau\text{-ind}_{P\text{-}Q}(T).$$

Proof. The proof follows verbatim the proof of Proposition 1.5.18 with references to Theorem 1.5.26 and Proposition 1.5.28 instead of Theorem 1.5.14 and Proposition 1.5.16. □

1.5.6 Essential codimension of two projections

In this subsection, we give an exposition of the notion of essential codimension of two projections, due to J. E. Avron, R. Seiler and B. Simon [ASS]. Here we follow [Ph$_2$] and [BCPRSW].

Definition 1.5.31 *A pair (P, Q) of two projections in $\mathcal{N}$ is said to be a* Fredholm pair *if* $\|\pi(P) - \pi(Q)\| < 1$.

Proposition 1.5.32 *If P, Q is a Fredholm pair of projections in $\mathcal{N}$, then PQ is a $(P \cdot Q)$ τ-Fredholm operator.*

Proof. Since

$$\|\pi(PQP) - \pi(P)\| = \big\|\pi(P)\big[\pi(Q) - \pi(P)\big]\pi(P)\big\| \leqslant \|\pi(Q) - \pi(P)\| < 1,$$

it follows that PQP is invertible in $P\mathcal{N}P$ modulo τ-compact operators in $P\mathcal{N}P$, and hence PQP is a τ-Fredholm operator in $P\mathcal{N}P$ by Theorem 1.5.14. Hence, the projection N^P_{PQP} is τ-finite. Since $\mathrm{N}^P_{QP} \leqslant \mathrm{N}^P_{PQP}$ (see (1.9)), it follows that N^P_{QP} is also τ-finite. Similarly, the projection $\mathrm{N}^Q_{(QP)^*} = \mathrm{N}^Q_{PQ}$ is also τ-finite. Now, since PQP is a τ-Fredholm operator in $P\mathcal{N}P$, by (BF3') there exists a τ-finite projection $E \leqslant P$ such that $\operatorname{ran}(P - E) \subseteq \operatorname{ran} PQP$. But since $\operatorname{ran} PQP \subseteq \operatorname{ran} PQ$, it follows that $\operatorname{ran}(P - E) \subseteq \operatorname{ran} PQ$. Hence, PQ is a $(P \cdot Q)$ τ-Fredholm operator. $\square$

Definition 1.5.33 *If (P, Q) is a Fredholm pair of projections in $\mathcal{N}$, then the* essential codimension $\mathrm{ec}(P, Q)$ *of the pair (P, Q) is the number*

$$\mathrm{ec}(P, Q) := \tau\text{-}\mathrm{ind}_{P\text{-}Q}(PQ).$$

Let (P, Q) be a Fredholm pair. If P and Q commute then

$$\mathrm{ec}(P, Q) = \tau\left(Q - PQ\right) - \tau\left(P - PQ\right). \tag{1.63}$$

Really,

$$\begin{aligned} \mathrm{ec}(P, Q) = \tau\text{-}\mathrm{ind}_{P\text{-}Q}(PQ) &= \tau\left(\mathrm{N}^Q_{PQ}\right) - \tau\left(\mathrm{N}^P_{PQ}\right) \\ &= \tau\left((1 - PQ)Q\right) - \tau\left((1 - PQ)P\right) = \tau\left(Q - PQ\right) - \tau\left(P - PQ\right). \end{aligned}$$

In particular, if $P \leq Q$ then $\mathrm{ec}(P, Q) = \tau\left(Q - P\right)$.

Lemma 1.5.34 *If (P, Q) is a Fredholm pair then*

$$\mathrm{ec}(P, Q) = -\,\mathrm{ec}(Q, P).$$

Proof. The last equality means

$$\tau\text{-ind}_{P\text{-}Q}(PQ) = -\tau\text{-ind}_{Q\text{-}P}(QP),$$

which follows from Lemma 1.5.27. □

Proposition 1.5.35 *If P_1, P_2, P_3 are projections in $\mathcal{N}$ such that*

$$\|\pi(P_1) - \pi(P_2)\|_{\mathcal{Q}(\mathcal{N},\tau)} < \frac{1}{2} \quad \textit{and} \quad \|\pi(P_2) - \pi(P_3)\|_{\mathcal{Q}(\mathcal{N},\tau)} < \frac{1}{2},$$

then

$$\mathrm{ec}(P_1, P_3) = \mathrm{ec}(P_1, P_2) + \mathrm{ec}(P_2, P_3). \tag{1.64}$$

Proof. Since

$$\|\pi(P_1) - \pi(P_3)\|_{\mathcal{Q}(\mathcal{N},\tau)} \leqslant \|\pi(P_1) - \pi(P_2)\|_{\mathcal{Q}(\mathcal{N},\tau)} + \|\pi(P_2) - \pi(P_3)\|_{\mathcal{Q}(\mathcal{N},\tau)} < 1,$$

it follows from Proposition 1.5.32 that the terms in equality (1.64) are well-defined. By Lemma 1.5.34, the equality (1.64) is equivalent to

$$\mathrm{ec}(P_1, P_2) + \mathrm{ec}(P_2, P_3) + \mathrm{ec}(P_3, P_1) = 0$$

which by definition of ec means

$$\tau\text{-ind}_{P_1\text{-}P_2}(P_1P_2) + \tau\text{-ind}_{P_2\text{-}P_3}(P_2P_3) + \tau\text{-ind}_{P_3\text{-}P_1}(P_3P_1) = 0.$$

So, by Proposition 1.5.28, we need to show that

$$0 = \tau\text{-ind}_{P_1\text{-}P_1}(P_1P_2 \cdot P_2P_3 \cdot P_3P_1),$$

or

$$0 = \tau\text{-ind}_{P_1\text{-}P_1}(P_1P_2P_3P_1). \tag{1.65}$$

We have

$$\begin{aligned}
&\|\pi(P_1P_2P_3P_1) - \pi(P_1)\|_{\mathcal{Q}(\mathcal{N},\tau)} \\
&\qquad = \|\pi(P_1)[\pi(P_2P_3) - \pi(P_1)]\pi(P_1)\|_{\mathcal{Q}(\mathcal{N},\tau)} \\
&\qquad \leqslant \|\pi(P_2P_3) - \pi(P_1)\|_{\mathcal{Q}(\mathcal{N},\tau)} \\
&\qquad \leqslant \|\pi(P_2P_3) - \pi(P_2)\|_{\mathcal{Q}(\mathcal{N},\tau)} + \|\pi(P_2) - \pi(P_1)\|_{\mathcal{Q}(\mathcal{N},\tau)} \\
&\qquad \leqslant \|\pi(P_3) - \pi(P_2)\|_{\mathcal{Q}(\mathcal{N},\tau)} + \|\pi(P_2) - \pi(P_1)\|_{\mathcal{Q}(\mathcal{N},\tau)} < 1.
\end{aligned}$$

Thus, there is a τ-compact operator K in the reduced von Neumann algebra $P_1\mathcal{N}P_1$ with

$$\|P_1P_2P_3P_1 - P_1 + K\| < 1,$$

which means that the operator $P_1P_2P_3P_1 + K$ is invertible in $P_1\mathcal{N}P_1$. Hence,

$$\tau\text{-ind}_{P_1\text{-}P_1}(P_1P_2P_3P_1 + K) = 0,$$

and Proposition 1.5.29 now implies (1.65). □

1.5.7 The Carey-Phillips theorem

The aim of this subsection is to prove the Carey-Phillips theorem (Theorem 1.5.37).

Let P and Q be projections in $\mathcal{B}(\mathcal{H})$. Since $\mathrm{N}_P^Q = \mathrm{N}_P \wedge Q \leqslant Q$ and $\mathrm{N}_Q^P = \mathrm{N}_Q \wedge P \leqslant \mathrm{N}_Q = Q^\perp$, the projections N_P^Q and N_Q^P are orthogonal. Moreover

$$\begin{aligned} Q\mathrm{N}_Q^P &= \mathrm{N}_Q^P Q = 0, \\ P\mathrm{N}_Q^P &= \mathrm{N}_Q^P P = \mathrm{N}_Q^P \end{aligned} \tag{1.66}$$

and similarly,

$$\begin{aligned} P\mathrm{N}_P^Q &= \mathrm{N}_P^Q P = 0, \\ Q\mathrm{N}_P^Q &= \mathrm{N}_P^Q Q = \mathrm{N}_P^Q. \end{aligned} \tag{1.67}$$

If E is the orthogonal complement of $\mathrm{N}_Q^P + \mathrm{N}_P^Q$, so that

$$E \oplus \mathrm{N}_Q^P \oplus \mathrm{N}_P^Q = 1, \tag{1.68}$$

then it also follows that $PE = EP$ and $QE = EQ$. Hence, $P_1 := PE$ and $Q_1 = QE$ are projections in $\mathcal{H}_E = E\mathcal{H}$. Also,

$$\mathrm{N}_{Q_1}^{P_1} = Q_1^\perp \wedge P_1 = (Q^\perp \wedge P)E = 0,$$

and by symmetricity,

$$\mathrm{N}_{P_1}^{Q_1} = P_1^\perp \wedge Q_1 = 0.$$

This means that

$$\ker P_1 \cap \operatorname{ran} Q_1 = \{0\} \tag{1.69}$$

$$\operatorname{ran} P_1 \cap \ker Q_1 = \{0\}. \tag{1.70}$$

Lemma 1.5.36 *There exists a self-adjoint unitary U in $\mathcal{B}(\mathcal{H}_E)$ such that $U(P_1 - Q_1)U^* = Q_1 - P_1$.*

Proof. (A) Let $B = 1 - (P_1 + Q_1)$ and let $B = U\,|B|$ be the polar decomposition of B. Then B anticommutes with $P_1 - Q_1$ and so B^2 commutes with $P_1 - Q_1$, and hence any continuous function of B^2 commutes with $P_1 - Q_1$. In particular, $|B|$ commutes with $P_1 - Q_1$. Hence, we have

$$U(P_1 - Q_1)\,|B| = U\,|B|\,(P_1 - Q_1) = B(P_1 - Q_1) = (Q_1 - P_1)B = (Q_1 - P_1)U\,|B|\,.$$

That is, the operator $U(P_1 - Q_1)$ is equal to $(Q_1 - P_1)U$ on $|B|\,\mathcal{H}$.

(B) Let $\xi \in \ker B$. This means that

$$\eta := P_1^\perp \xi = Q_1 \xi. \tag{1.71}$$

Since $P_1\eta = 0$ and $\eta \in \operatorname{ran} Q_1$, it follows from (1.69) that $\eta = 0$. Now, (1.71) and (1.70) imply that $\xi = 0$.

Hence $\ker B = \{0\}$. This implies that the range of $|B|$ is dense in $\mathcal{H}_E$ (see (1.11)). Since the range of $|B|$ is dense in $\mathcal{H}_E$, by (A) the operators $U(P_1 - Q_1)$ and $(Q_1 - P_1)U$ coincide on $\mathcal{H}_E$. Since B is self-adjoint, U is also self-adjoint. □

Theorem 1.5.37 [CP$_2$] *Let $f \in C([-1,1],\mathbb{R})$ be an odd function. Let P, Q be two projections in $\mathcal{N}$ such that their difference $P - Q$ is τ-compact operator and $f(P-Q) \in \mathcal{L}^1(\mathcal{N},\tau)$. Then the pair (P,Q) is a Fredholm pair and*

$$\tau[f(P-Q)] = f(1)\operatorname{ec}(Q,P).$$

Proof. By (1.68), (1.66) and (1.67) we have

$$P = P_1 \oplus 1 \oplus 0$$

and

$$Q = Q_1 \oplus 0 \oplus 1.$$

So,

$$f(P-Q) = f(P_1 - Q_1) \oplus f(1) \oplus -f(1). \tag{1.72}$$

Since f is odd, we have by Lemma 1.5.36

$$Uf(P_1 - Q_1)U^* = -f(Q_1 - P_1).$$

Since $f(P_1 - Q_1)$ is τ-trace class, it follows that $\tau\left(f(P_1 - Q_1)\right) = 0$. Hence, taking the trace of (1.72) and noting that $\mathrm{N}_Q^P = \mathrm{N}_{QP}^P$, $\mathrm{N}_P^Q = \mathrm{N}_{PQ}^Q$, we have

$$\begin{aligned}\tau\left(f(P-Q)\right) &= f(1)\left(\tau\left(\mathrm{N}_Q^P\right) - \tau\left(\mathrm{N}_P^Q\right)\right)\\ &= f(1)\left(\tau\left(\mathrm{N}_{QP}^P\right) - \tau\left(\mathrm{N}_{(QP)^*}^Q\right)\right) = f(1)\,\tau\text{-}\mathrm{ind}_{Q\text{-}P}(QP).\end{aligned}$$

□

1.6 Spectral flow in semifinite von Neumann algebras

In this section, we give an exposition of the spectral flow theory of J. Phillips [Ph, Ph$_2$].

In the type I case, the spectral flow of a path from a self-adjoint operator D_1 to a self-adjoint operator D_2 measures the net number of eigenvalues crossing zero. We first define the notion of spectral flow and establish its properties, as it was done by J. Phillips in [Ph, Ph_2].

We denote by sign the function defined as $\operatorname{sign}(x) = -1$ for $x < 0$ and $\operatorname{sign}(x) = 1$ for $x \geqslant 0$.

Proposition 1.6.1 *Let $[a, b] \subset \mathbb{R}$ and $F\colon [a, b] \to \mathcal{F}_{sa}(\mathcal{N}, \tau)$ be a norm continuous path of self-adjoint τ-Fredholm operators. Let $\chi(\cdot) = \chi_{[0,\infty)}(\cdot)$ and let $P_t = \chi(F_t)$, $t \in [a, b]$. Then the path of projections*

$$t \in [a, b] \mapsto \pi(P_t) \in \mathcal{Q}(\mathcal{N}, \tau)$$

is $\|\cdot\|_{\mathcal{Q}(\mathcal{N},\tau)}$-continuous, and, hence, is also uniformly continuous.

Proof. (A) If $T \in \mathcal{N}$ is a self-adjoint operator and $f \in C(\mathbb{R})$, then

$$f(\pi(T)) = \pi(f(T)),$$

where the left hand side is understood in the sense of the continuous functional calculus in C^*-algebras (see e.g. [BR, Theorem 2.1.11B]). Indeed, this equality is evidently true for $f(x) = x^n$, $n = 0, 1, 2, \dots$. Hence, it is true for polynomials. So, by Stone-Weierstrass theorem, it is true for any continuous function f.

(B) Here we prove that if T is a self-adjoint τ-Fredholm operator then $\chi(\pi(T)) = \pi(\chi(T))$.

By Theorem 1.5.14 the operator T is an invertible element of C^*-algebra $\mathcal{Q}(\mathcal{N}, \tau)$, and hence $0 \notin \sigma_{\pi(T)}$ in $\mathcal{Q}(\mathcal{N}, \tau)$. Hence, there exist two continuous functions f_1 and f_2 on $\mathbb{R}$, which coincide with χ on $\sigma_{\pi(T)}$ and such that $f_1 \geqslant \chi \geqslant f_2$. Thus, using (A) we have

$$\begin{aligned} \chi(\pi(T)) = f_1(\pi(T)) = \pi(f_1(T)) \\ \geqslant \pi(\chi(T)) \geqslant \pi(f_2(T)) = f_2(\pi(T)) = \chi(\pi(T)). \end{aligned}$$

Hence, $\chi(\pi(T)) = \pi(\chi(T))$.

(C) Since F_t is norm-continuous, $\pi(F_t)$ is $\mathcal{Q}(\mathcal{N}, \tau)$-continuous. Since all $\pi(F_t)$ are invertible in $\mathcal{Q}(\mathcal{N}, \tau)$, their spectra are bounded away from 0. Hence, χ is continuous function on their spectra, and, hence, $\chi(\pi(F_t))$ is continuous. Thus, by (B), the path $\pi(\chi(F_t))$ is continuous. □

Remark 1.6.2 *Note that the map $t \in [a, b] \mapsto P_t$ itself is usually discontinuous in the norm topology.*

Definition 1.6.3 *Let $[a, b] \subset \mathbb{R}$ and let $F\colon [a, b] \to \mathcal{F}_{sa}(\mathcal{N}, \tau)$ be a norm continuous path of self-adjoint τ-Fredholm operators. Let $t \in [a, b] \mapsto P_t := \chi(F_t)$*

be the corresponding path of projections. By Proposition 1.6.1 we can choose a partition $t_0 = a < t_1 < ... < t_n = b$ of the segment $[a, b]$ such that $\|\pi(P_{t_{j-1}}) - \pi(P_{t_j})\| < \frac{1}{2}$ for all $j = 1, \ldots, n$. The ***spectral flow*** *of the path $\{F_t\}_{t\in[a,b]}$ is the number*

$$\mathrm{sf}(\{F_t\}) := \sum_{i=1}^{n} \mathrm{ec}(P_{t_{i-1}}, P_{t_i}).$$

Remark 1.6.4 *If the path $\{F_t\}$ lies entirely in $F_0 + \mathcal{K}(\mathcal{N}, \tau)$ then $\pi(P_t) = \mathrm{const}$ and*

$$\mathrm{sf}(\{F_t\}) := \mathrm{ec}(P_0 P_1) = \tau\text{-}\mathrm{ind}_{P_0\text{-}P_1}(P_0 P_1).$$

We note, that a topology weaker that the norm-topology, e.g. the strong operator topology, does not suffice. The reason is that the spectrum of a self-adjoint operator changes continuously under small perturbations for the norm topology, but not under small perturbations for the strong operator topology. A trivial example is given by the spectral projections E_n which converge to 1 in the strong operator topology. Here the spectrum of E_n is $\{0, 1\}$ while the spectrum of 1 is $\{1\}$, so that spectrum "jumps" from 0 to 1 as $n \to \infty$.

Theorem 1.6.5 *The spectral flow is well-defined, i.e. it is independent on a choice of the partition.*

Proof. In order to show that the spectral flow is independent of the partition, it is enough to show that it does not change when we add one more point to the partition. This follows from Proposition 1.5.35. □

Proposition 1.6.6 *Properties of spectral flow.*
(1) $\mathrm{sf}(F, F) = 0$;
(2) $\mathrm{sf}(F_1, F_2) + \mathrm{sf}(F_2, F_3) = \mathrm{sf}(F_1, F_3)$;
(3) $\mathrm{sf}(F_1, F_2) = -\,\mathrm{sf}(F_2, F_1)$;
(4) $\mathrm{sf}(\alpha F_1, \alpha F_2) = \mathrm{sf}(F_1, F_2)$ *for any* $\alpha > 0$.

Proof. (1) and (2) follow directly from the definition of spectral flow. (3) follows from Lemma 1.5.34. (4) follows from the fact that the projections $\chi(F)$ and $\chi(\alpha F)$ coincide. □

Proposition 1.6.7 *Let $[a, b] \subset \mathbb{R}$, and let $F, G\colon [a, b] \to \mathcal{F}_{sa}(\mathcal{N}, \tau)$ be two norm continuous paths of self-adjoint τ-Fredholm operators such that $F(a) = G(a)$ and $F(b) = G(b)$. If these paths are norm homotopic via a homotopy leaving the end-points fixed, then their spectral flows coincide.*

Proof. (A) For $T \in \mathcal{F}_{sa}(\mathcal{N},\tau)$ let

$$N(T) = \left\{ S \in \mathcal{F}_{sa}(\mathcal{N},\tau) \colon \; \|\pi(\chi(S)) - \pi(\chi(T))\| < \frac{1}{4} \right\}.$$

Then $N(T)$ is open in $\mathcal{F}_{sa}(\mathcal{N},\tau)$ since $S \mapsto \pi(\chi(S)) = \chi(\pi(S))$ is continuous on $\mathcal{F}_{sa}(\mathcal{N},\tau)$. Moreover, if S_1, $S_2 \in N(T)$, then by the definition of spectral flow, all paths from S_1 to S_2 lying entirely in $N(T)$ have the same spectral flow, namely, $\mathrm{ec}(\chi(S_1), \chi(S_2))$.

(B) Let $H \colon [a,b] \times [a,b] \to \mathcal{F}_{sa}(\mathcal{N},\tau)$ be a homotopy from $\{F(t)\}$ to $\{G(t)\}$. That is, H is continuous, $H(t,a) = F(t)$ for all $t \in [a,b]$, $H(t,b) = G(t)$ for all $t \in [a,b]$, $H(a,s) = F(a) = G(a)$ for all $s \in [a,b]$ and $H(b,s) = F(b) = G(b)$ for all $s \in [a,b]$. The image of H in $\mathcal{F}_{sa}(\mathcal{N},\tau)$ is compact, so that there exists a finite cover by open sets of the form $N(T)$, say $\{N_1, \ldots, N_k\}$. Then the finite family $\left\{H^{-1}(N_1), \ldots, H^{-1}(N_k)\right\}$ forms a finite cover of $[a,b] \times [a,b]$. Thus, there exists $\varepsilon_0 > 0$ (the Lebesgue number of the cover) so that any subset of $[a,b] \times [a,b]$ of diameter less than ε_0 is contained in some element of this finite cover of $[a,b] \times [a,b]$. Thus, if we partition $[a,b] \times [a,b]$ into a grid of squares of diameter less than ε_0, then the image of each square will lie entirely within some N_i.

Now, it is clear that we can construct a finite sequence of paths $\gamma_0, \ldots, \gamma_N$ from $F(a)$ to $G(a)$ such that $\gamma_0 = \{F_t\}$, $\gamma_N = \{G_t\}$ and each two successive paths differ only by a small lasso (i.e. by the boundary of a small square of the grid just built). Since the spectral flow along any such lasso is zero by (A), the spectral flows of each two successive paths coincide by Proposition 1.6.6(3). Hence, the spectral flows of the paths $\gamma_0 = \{F_t\}$ and $\gamma_N = \{G_t\}$ also coincide. □

This proposition allows one to write the spectral flow in the form $\mathrm{sf}(F_a, F_b)$.

For a self-adjoint operator D, let

$$F_D := D(1 + D^2)^{-1/2}.$$

Lemma 1.6.8 *If $D = D^* \eta \mathcal{N}$ has τ-compact resolvent and $V = V^* \in \mathcal{N}$, then $D + V$ also has τ-compact resolvent.*

Proof. This follows from the second resolvent identity (1.2) applied to the pair $D + V$ and D. □

The following lemma and its proof is taken from [CP] (Lemma 2.7).

Lemma 1.6.9 *If D_0 is a self-adjoint operator with τ-compact resolvent, $V \in \mathcal{N}_{sa}$ and $D_1 = D_0 + V$, then the operator $F_{D_1} - F_{D_0}$ is τ-compact.*

Proof. (A) Let $g(x) := \frac{x}{1+x^2}$. Using the argument of the proof of Lemma 4.1.9(i), one can see that the measure $|\xi| \, \mathcal{F}(g)(\xi) \, d\xi$ has a finite variation C. By Lemma

1.1.7 and (1.6),

$$\begin{aligned} g(D_1) - g(D_0) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} [e^{isD_1} - e^{isD_0}] \mathcal{F}(g)(s)\, ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_0^s e^{i(s-t)D_1} iV e^{itD_0}\, dt \mathcal{F}(g)(s)\, ds, \end{aligned}$$

so that

$$\|g(D_1) - g(D_0)\| \leqslant \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \|V\| \, |s\mathcal{F}(g)(s)| \; ds \leqslant \frac{1}{\sqrt{2\pi}} C \, \|V\| \, .$$

Applying this estimate to the operators

$$D_i(1 + D_i^2 + \lambda)^{-1} = \frac{1}{\sqrt{1+\lambda}}\, g\left(\frac{D_i}{\sqrt{1+\lambda}}\right), \qquad i = 0, 1,$$

one finds that

$$\left\| D_1(1 + D_1^2 + \lambda)^{-1} - D_0(1 + D_0^2 + \lambda)^{-1} \right\| \leqslant \frac{C_1}{1+\lambda} \, \|V\| \, , \tag{1.73}$$

for some constant C_1.

(B) By [CP, Appendix A, Lemma 4], for all $\xi \in \mathrm{dom}(D_0) = \mathrm{dom}(D_1)$

$$F_{D_1}\xi - F_{D_0}\xi = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} [D_1(1 + D_1^2 + \lambda)^{-1} - D_0(1 + D_0^2 + \lambda)^{-1}] \xi \, d\lambda,$$

where the integral is norm convergent in $\mathcal{H}$. But, by (1.73), the last integral (without ξ) converges in the operator norm. The operator

$$D_0(1 + D_0^2 + \lambda)^{-1} = \frac{1 + D_0^2}{1 + D_0^2 + \lambda} \cdot \frac{D_0}{D_0 + i} \cdot \frac{1}{D_0 - i}$$

is τ-compact, and since D_1 also has τ-compact resolvent by Lemma 1.6.8, the operator $D_1(1 + D_1^2 + \lambda)^{-1}$ is also τ-compact. So, the claim follows from the closedness of $\mathcal{K}(\mathcal{N}, \tau)$ for the operator norm (Lemma 1.3.12). □

Theorem 1.6.10 [CP] *Let $(\mathcal{A}, \mathcal{N}, D)$ be a semifinite spectral triple, and let $u \in \mathcal{A}$ be a unitary. Then the spectral flow from D to uDu^* is*

$$\mathrm{sf}(D, u^*Du) = \tau\text{-}\mathrm{ind}_{P\text{-}P}(PuP),$$

where $P := \chi(D)$.

Proof. By definition

$$\mathrm{sf}(D, uDu^*) := \mathrm{sf}(F_D, F_{uDu^*}).$$

Let $F_D = \widetilde{F}_D\,|F_D|$ be the polar decomposition of F_D. Let

$$\widetilde{F}_D = 2P - 1, \quad \widetilde{F}_{uDu^*} = 2Q - 1 = 2(uPu^*) - 1.$$

Since $uDu^* = D + [u, D]u^*$ and $[u, D]$ is bounded by definition of spectral triple, it follows from Lemma 1.6.9 that $F_D - F_{uDu^*}$ is τ-compact. Since D has τ-compact resolvent, it follows that

$$\begin{aligned}\widetilde{F}_D - F_D = \widetilde{F}_D(1 - |F_D|) &= \widetilde{F}_D(1 - |F_D|^2)(1 + |F_D|)^{-1} \\ &= \widetilde{F}_D(1 + D^2)^{-1}(1 + |F_D|)^{-1}\end{aligned}$$

is also τ-compact. Hence, $2(P-Q) = \widetilde{F}_D - \widetilde{F}_{uDu^*} = (\widetilde{F}_D - F_D) + (F_D - F_{uDu^*}) + (F_{uDu^*} - \widetilde{F}_{uDu^*})$ is also τ-compact, so that the pair (P, Q) is Fredholm (since $\|P - Q\|_{\mathcal{Q}(\mathcal{N},\tau)} = 0 < 1$). By Proposition 1.5.32, it follows that PQ is a $(P \cdot Q)$ τ-Fredholm operator. Hence, by Definition 1.6.3 of spectral flow, we have

$$\mathrm{sf}(F_D, F_{uDu^*}) = \tau\text{-}\mathrm{ind}_{P\text{-}Q}(PQ).$$

So,

$$\begin{aligned}\mathrm{sf}(D, uDu^*) = \tau\text{-}\mathrm{ind}_{P\text{-}Q}(PQ) &= \tau\text{-}\mathrm{ind}_{P\text{-}Q}(PuPu^*) \\ &= \tau\text{-}\mathrm{ind}_{P\text{-}P}(PuP) + \tau\text{-}\mathrm{ind}_{P\text{-}Q}(u^*) = \tau\text{-}\mathrm{ind}_{P\text{-}P}(PuP),\end{aligned}$$

where the third equality follows from Proposition 1.5.28 and the last equality follows from

$$\tau\text{-}\mathrm{ind}_{P\text{-}Q}(u^*) = \tau\left(\mathrm{N}^Q_{u^*}\right) - \tau\left(\mathrm{N}^P_u\right) = 0,$$

since u and u^* are invertible, so that $\mathrm{N}^Q_{u^*} = \mathrm{N}^P_u = 0$. □

The following example is due to J. Phillips.

Example 1.6.11 [BCPRSW] *If $\mathcal{N}$ is a II_∞ factor, then $\mathcal{N}$ contains an abelian von Neumann subalgebra $\mathcal{A}$ isomorphic to $L^\infty(\mathbb{R}, dx)$. Let $B_0 \in \mathcal{A}$ be the continuous function:*

$$\begin{cases} B_0(x) = -1, & \text{if } x \in (-\infty, -1], \\ B_0(x) = x, & \text{if } x \in [-1, 1], \\ B_0(x) = 1, & \text{if } x \in [1, \infty). \end{cases}$$

Let $r \in \mathbb{R}$, and $B_t(x) = B_0(x + tr)$ for all $x \in \mathbb{R}$. Then $\{B_t\}$ is a continuous path in $\mathcal{F}^{sa}_*$. Let $\chi = \chi_{[0,+\infty)}$. Then $\chi(B_t) = \chi_{[-tr,+\infty)}$ and thus

$$\pi(\chi(B_t)) = \mathrm{const}$$

in $\mathcal{Q}(\mathcal{N}, \tau)$. Hence,

$$P_0 = \chi(B_0) = \chi_{[0,+\infty)}, \quad P_1 = \chi(B_1) = \chi_{[-r,\infty)}$$

and, using (1.63),

$$\begin{aligned}\mathrm{sf}\,\{B_t\} &= \mathrm{ec}(P_0, P_1) = \tau\text{-}\mathrm{ind}_{P_0\text{-}P_1}(P_0P_1)\\ &= \tau\,(P_1 - P_0P_1) - \tau\,(P_0 - P_0P_1) = \int_{-r}^{0} dx - 0 = r.\end{aligned}$$

In this example, the spectral picture is constant. That is, $\sigma_{B_t} = [-1, 1]$ *and* $\sigma_{\pi(B_t)} = \{-1, 1\}$ *for all* $t \in [0, 1]$. *Thus, one cannot tell from the spectrum alone (even knowing the multiplicities) what the spectral flow will be.*

We note, that this is characteristic for the type II case. In the case of $\mathcal{N} = \mathcal{B}(\mathcal{H})$, this is not possible.

1.7 Fuglede-Kadison's determinant in semifinite von Neumann algebras

In this section, we give an exposition of the theory of Fuglede-Kadison determinant in semifinite von Neumann algebras, following L. G. Brown's paper [Brn]. As usual, $\mathcal{N}$ denotes a semifinite von Neumann algebra, and τ denotes a faithful normal semifinite trace on $\mathcal{N}$. Let $GL(\mathcal{N})$ be the group of invertible elements of $\mathcal{N}$.

1.7.1 de la Harpe-Scandalis determinant

The classical Fredholm determinant is defined for operators of the form $1 + T$, where $T \in \mathcal{L}^1(\mathcal{H})$, i.e. T is a trace-class operator, and it follows from Lidskii's theorem that this determinant can be given by the formula

$$\det(1 + T) = \prod_{j=1}^{\infty} (1 + \lambda_j(T)), \tag{1.74}$$

where $\lambda_1(T), \lambda_2(T), \ldots$ is the list of eigenvalues of T, counting multiplicities.

Our aim in this subsection is to introduce and to study the complex valued determinant in semifinite von Neumann algebras. The formula (1.74) cannot be generalized to semifinite von Neumann algebras directly. But if the spectrum of the operator T does not intersect the half-line $(-\infty, -1]$, then one can rewrite the last formula as

$$\det(1 + T) = e^{\sum\limits_{j=1}^{\infty} \log\left(1+\lambda_j(T)\right)} = e^{\mathrm{Tr}(\log(1+T)))},$$

where log denotes that branch of Log defined on the set $\mathbb{C} \setminus (-\infty, 0]$ such that $\log 1 = 0$. $\log(1 + T)$ is defined by holomorphic functional calculus.

This trivial observation is the rationale for introducing the next definition.

Definition 1.7.1 *Let $\mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ be the set of τ-trace class operators from $\mathcal{N}$, whose spectrum does not intersect the half-line $(-\infty,-1]$, and let $T \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$. Then the τ-**determinant** of $1+T$ is*

$$\det{}_\tau(1+T) = e^{\tau(\log(1+T))} \in \mathbb{C}. \tag{1.75}$$

By an $\mathcal{L}^1$-smooth path $\{A_t\}_{t\in[0,1]}$, we mean a map $A\colon t \in [0,1] \mapsto A_t \in \mathcal{L}^1(\mathcal{N},\tau)$ such that the following limit

$$\lim_{h\to 0} \frac{A_{t+h} - A_t}{h} =: A'_t$$

exists for all $t \in [0,1]$ in the norm of $\mathcal{L}^1(\mathcal{N},\tau)$, and $t \mapsto A'_t$ is $\mathcal{L}^1(\mathcal{N},\tau)$-continuous.

One of the ideas of [CFM] is the following lemma.

Lemma 1.7.2 *Let $A \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ and $A\colon t \in [0,1] \mapsto A_t \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ is an $\mathcal{L}^1$-smooth path such that $A_0 = 0$ and suppose that $A_1 = A$. Then*

$$\det{}_\tau(1+A) = \exp\left\{\int_0^1 \tau\left((1+A_t)^{-1}A'_t\right)\, dt\right\}. \tag{1.76}$$

Remarks. 1) The condition $A_t \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ ensures that $-1 \notin \sigma_{A_t}$, i.e. $1 + A_t$ is invertible for all $t \in [0,1]$, so that $(1 + A_t)^{-1}$ makes sense.

2) One can note that the right hand side of (1.76) makes sense for all $\mathcal{L}^1$-smooth paths $\{A_t\}_{t\in[0,1]}$ with $A_0 = 0$ and $A_1 = A$ and such that $-1 \notin \sigma_{A_t}$ (to ensure invertibility of $1 + A_t$). Nevertheless, the condition $A_t \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ is essential, since if two different paths $\{A_t\}_{t\in[0,1]}$ and $\{B_t\}_{t\in[0,1]}$ are not homotopic, then the right hand side of (1.76) gives different numbers. In the case $\mathcal{N} = \mathcal{B}(\mathcal{H})$, it does not make a difference since these different values of the integral in (1.76) for different curves $\{A_t\}_{t\in[0,1]}$ differ by $2n\pi i$, where $n \in \mathbb{Z}$, and after exponentiation in (1.76) all differences will play no role.

3) We note that the spectrum of an operator is upper semi-continuous with respect to the uniform topology. This means that the spectrum of A_t cannot jump over the half-line $(-\infty,-1]$, and this is essential. Hence, the continuity of the path in the uniform topology is essential, a weaker topology is not appropriate for our purposes. For details see [Kat, IV-§3.1].

Proof. We have, by Theorem 1.3.34,

$$\begin{aligned}\int_0^1 \tau\left((1+A_t)^{-1}A'_t\right)\, dt &= \int_0^1 \left\{\tau\big(\log(1+A_t)\big)\right\}'\, dt \\ &= \tau\big(\log(1+A_1)\big) - \tau\big(\log(1+A_0)\big) = \tau\big(\log(1+A)\big),\end{aligned}$$

which proves (1.76). $\square$

Remark 1 *This proof actually shows that the determinant, defined by the right hand side of (1.76), does not depend on the choice of the path* $\{A_t\}_{t\in[0,1]}$ *, provided* $A_t \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ *for all* t*. This happens since all such paths are homotopic.*

Proposition 1.7.3 *Let* $A, B \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$, $\|A\| < \sqrt{2}-1$, $\|B\| < \sqrt{2}-1$. *Then*

$$\det_\tau\big((1+A)(1+B)\big) = \det_\tau(1+A)\det_\tau(1+B). \tag{1.77}$$

Proof. First, we note that $\|A+B+AB\| \leqslant \|A\|+\|B\|+\|A\|\,\|B\| < 1$, so that $A+B+AB \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ and the determinant in left hand side of (1.77) is well-defined. Now, if $A_t = tA$ and $B_t = tB$, then $\{A_t\}_{t\in[0,1]}$ and $\{B_t\}_{t\in[0,1]}$ are two $\mathcal{L}^1$-smooth paths, such that $A_0 = B_0 = 0$ and $A_1 = A$, $B_1 = B$ and $\|A_t\| < \sqrt{2}-1$, $\|B_t\| < \sqrt{2}-1$. If $C_t = A_t + B_t + A_tB_t$ then $C := C_1 = A_1 + B_1 + A_1B_1 = A + B + AB$. Now,

$$\begin{aligned}
\tau\left((1+C_t)^{-1}C_t'\right) &= \tau\Big(\{(1+A_t)(1+B_t)\}^{-1}\left(A_t+B_t+A_tB_t\right)'\Big)\\
&= \tau\left((1+B_t)^{-1}(1+A_t)^{-1}\left\{A_t'+B_t'+A_t'B_t+A_tB_t'\right\}\right)\\
&= \tau\big((1+B_t)^{-1}(1+A_t)^{-1}A_t'(1+B_t)\\
&\qquad + (1+B_t)^{-1}(1+A_t)^{-1}(1+A_t)B_t'\big)\\
&= \tau\left((1+B_t)^{-1}B_t'\right) + \tau\left((1+A_t)^{-1}A_t'\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\det_\tau\left((1+A)(1+B)\right)\\
&\quad= \det_\tau(1+C) = \exp\left\{\int_0^1 \tau\left((1+C_t)^{-1}C_t'\right)\,dt\right\}\\
&\quad= \exp\left\{\int_0^1 \tau\left((1+A_t)^{-1}A_t'\right)\,dt + \int_0^1 \tau\left((1+B_t)^{-1}B_t'\right)\,dt\right\}\\
&\quad= \det_\tau(1+A)\det_\tau(1+B).
\end{aligned}$$

$\square$

Lemma 1.7.4 *The product property (1.77) of the determinant (1.75) holds for any pair* A *and* B *of operators from* $\mathcal{L}^{1,\pi}(\mathcal{N},\tau)$*, if there exists two* $\mathcal{L}^1$*-smooth paths* $\{A_t\}_{t\in[0,1]}$ *and* $\{B_t\}_{t\in[0,1]}$ *from* $\mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ *such that the path* $\{A_t+B_t+A_tB_t\}_{t\in[0,1]}$ *also lies in* $\mathcal{L}^{1,\pi}(\mathcal{N},\tau)$.

The proof of this lemma is the same as that of Proposition 1.7.3.

Lemma 1.7.5 *Every operator $A \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ can be connected with the zero operator* 0 *by a $\mathcal{L}^1$-smooth path lying entirely in $\mathcal{L}^{1,\pi}(\mathcal{N},\tau)$.*

Proof. For example, $A_t = tA$. □

This lemma means that the τ-determinant is defined for all operators from $\mathcal{L}^{1,\pi}(\mathcal{N},\tau)$.

We note that the definition of complex-valued determinant (1.7.1) coincides with the path-dependent determinant of de la Harpe-Scandalis [HS], if one chooses a path lying in $\mathcal{L}^{1,\pi}(\mathcal{N},\tau)$.

1.7.2 Technical lemmas

Lemma 1.7.6 *If $T \in 1 + \mathcal{L}^1(\mathcal{N},\tau)$ then $|T| \in 1 + \mathcal{L}^1(\mathcal{N},\tau)$.*

Proof. We have $T^* \in 1 + \mathcal{L}^1(\mathcal{N},\tau)$, and so $|T|^2 = T^*T \in 1 + \mathcal{L}^1(\mathcal{N},\tau)$. Hence, $|T|^2 - 1 = (|T|-1)(|T|+1) \in \mathcal{L}^1(\mathcal{N},\tau)$. Since $|T|+1$ is invertible, it follows that $|T| - 1 \in \mathcal{L}^1(\mathcal{N},\tau)$. □

Lemma 1.7.7 *If $T \in 1+\mathcal{L}^1(\mathcal{N},\tau)$, then there exists a unitary operator U such that $T = U\,|T|$.*

Proof. Since $T \in 1+\mathcal{K}(\mathcal{N},\tau)$, it follows from Theorem 1.5.7 that the projections N_T and N_{T^*} are equivalent. So, if U_1 is an isometry with initial projection N_T and final projection N_{T^*} and if $T = U_2\,|T|$ is the polar decomposition of T, then

$$U_1 = \begin{pmatrix} U_1 & 0 \\ 0 & 0 \end{pmatrix}_{[\mathrm{N}_{T^*},\mathrm{N}_T]}, \quad U_2 = \begin{pmatrix} 0 & 0 \\ 0 & U_2 \end{pmatrix}_{[\mathrm{N}_{T^*},\mathrm{N}_T]},$$

and

$$T = \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}_{[\mathrm{N}_{T^*},\mathrm{N}_T]},$$

so that $U = U_1 + U_2$ is a unitary operator and $T = U\,|T|$, since

$$\begin{aligned} U^*U &= \begin{pmatrix} U_1^* & 0 \\ 0 & U_2^* \end{pmatrix}_{[\mathrm{N}_T,\mathrm{N}_{T^*}]} \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}_{[\mathrm{N}_{T^*},\mathrm{N}_T]} \\ &= \begin{pmatrix} \mathrm{N}_T & 0 \\ 0 & \mathrm{N}_T^{\perp} \end{pmatrix}_{[\mathrm{N}_T,\mathrm{N}_T]} = 1 \end{aligned}$$

and, similarly,

$$\begin{aligned} UU^* &= \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}_{[\mathrm{N}_{T^*},\mathrm{N}_T]} \begin{pmatrix} U_1^* & 0 \\ 0 & U_2^* \end{pmatrix}_{[\mathrm{N}_T,\mathrm{N}_{T^*}]} \\ &= \begin{pmatrix} \mathrm{N}_{T^*} & 0 \\ 0 & \mathrm{N}_{T^*}^{\perp} \end{pmatrix}_{[\mathrm{N}_{T^*},\mathrm{N}_{T^*}]} = 1, \end{aligned}$$

and

$$\begin{aligned} U_2\,|T| &= \begin{pmatrix} 0 & 0 \\ 0 & U_2 \end{pmatrix}_{[\mathrm{N}_{T^*},\mathrm{N}_T]} \begin{pmatrix} 0 & 0 \\ 0 & |T| \end{pmatrix}_{[\mathrm{N}_T,\mathrm{N}_T]} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & U_2\,|T| \end{pmatrix}_{[\mathrm{N}_{T^*},\mathrm{N}_T]} = T, \end{aligned}$$

where the last equality follows from (1.11), (1.12) and (1.8). □

Lemma 1.7.8 *If $T \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$ is invertible and $T = U\,|T|$ is the polar decomposition, then $U \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$ and there exist self-adjoint operators $S_1,\ S_2 \in \mathcal{L}^1(\mathcal{N}, \tau)$, such that $U = e^{iS_1}$ and $|T| = e^{S_2}$.*

Proof. Since U is invertible, so is $|T| = U^*T$. Since $|T| \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$ by Lemma 1.7.6, $U = T\,|T|^{-1} \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$. If

$$U = \int_0^{2\pi} e^{i\lambda}\, dE_\lambda^U$$

is the spectral integral, then let

$$S_1 = \int_0^{2\pi} \lambda\, dE_\lambda^U.$$

Further, since $|T|$ is a positive invertible operator, which belongs to the Banach algebra $\mathbb{C}1 + \mathcal{L}^1(\mathcal{N}, \tau)$, its logarithm is well-defined in this Banach algebra by the holomorphic functional calculus. □

1.7.3 Definition of Fuglede-Kadison determinant and its properties

Definition 1.7.9 *The **Fuglede-Kadison determinant** is the following function:*

$$T \in 1 + \mathcal{L}^1(\mathcal{N}, \tau) \mapsto \Delta(T) := e^{\tau(\log |T|)} = \exp\left[\tau\left(\log |T|\right)\right] \in [0, +\infty),$$

where if $\ker T \neq \{0\}$ *or* $\log_- |T| \notin \mathcal{L}^1(\mathcal{N}, \tau)$ *then we set, by definition,* $\Delta(T) := 0$. *Here* $\log_- = \min(0, \log)$.

We note that the definition implies

$$\Delta(T) = \Delta(|T|).$$

Lemma 1.7.10 *If $T \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$, then*

$$\Delta(T)^2 = \Delta\left(|T|^2\right).$$

Proof. Taking logarithms, it suffices to show

$$2\tau\left(\log|T|\right) = \tau\left(\log(|T|^2)\right).$$

It suffices further to prove that, for any non-negative operator A,

$$2\log(A) = \log(A^2).$$

This readily follows from the spectral theorem in the form [RS, Theorem VII.3]. □

Lemma 1.7.11 *If $T \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$, then*

$$\Delta(T) = \Delta(T^*).$$

Proof. By Lemma 1.7.7, there exists a unitary operator U such that $T = U|T|$. Hence,

$$\log(U|T|^2 U^*) = U\log(|T|^2)U^*,$$

so that

$$\tau[\log(U|T|^2 U^*)] = \tau[\log(|T|^2)] \in [-\infty, +\infty).$$

It follows that

$$\Delta\left(U|T|^2 U^*\right) = \Delta\left(|T|^2\right) \in [0, \infty).$$

Since $|T^*|^2 = TT^* = U|T|^2 U^*$, it follows from Lemma 1.7.10 that

$$\Delta(T^*) = \Delta(T).$$

□

Proposition 1.7.12 *If $S \in \mathcal{L}^1(\mathcal{N}, \tau)$, then*

$$\Delta\left(e^S\right) = e^{\mathrm{Re}(\tau(S))};$$

Proof. By Lemma 1.7.10, we have $\Delta(A)^2 = \Delta\left(|A|^2\right)$. Hence, for $t \in \mathbb{R}$,

$$\begin{aligned} 2\log\Delta\left(e^{tS}\right) = 2\log\Delta\left(|e^{tS}|\right) &= \log\Delta\left(|e^{tS}|^2\right) \\ &= \log\Delta\left(e^{tS^*}e^{tS}\right) = \tau\left(\log[e^{tS^*}e^{tS}]\right), \end{aligned}$$

so that by Corollary 1.3.35 (the argument of log is a positive invertible operator, so that log is holomorphic in a neighbourhood of its spectrum)

$$\begin{aligned} \frac{d}{dt}(2\log\Delta\left(e^{tS}\right)) &= \frac{d}{dt}\tau\left(\log(e^{tS^*}e^{tS})\right) \\ &= \tau\left((e^{tS^*}e^{tS})^{-1}\cdot\frac{d}{dt}(e^{tS^*}e^{tS})\right) \\ &= \tau\left(e^{-tS}e^{-tS^*}\cdot(S^*e^{tS^*}e^{tS} + e^{tS^*}e^{tS}S)\right) \\ &= \tau(S^* + S) = 2\,\mathrm{Re}(\tau(S)). \end{aligned}$$

This means that $\log\Delta\left(e^{tS}\right) = t\,\mathrm{Re}(\tau\left(S\right))$, and taking $t=1$ and exponentiating this equality we get the claim. □

Lemma 1.7.13 *The absolute value of the τ-determinant is equal to the Fuglede-Kadison determinant.*

Proof. If $T\in\mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ then

$$|\det_\tau(1+T)| = e^{\mathrm{Re}\,\tau\log(1+T)},$$

which by Proposition 1.7.12 is equal to $\Delta\left(1+T\right)$. □

Proposition 1.7.14 *For any two invertible operators $A,B\in 1+\mathcal{L}^1(\mathcal{N},\tau)$ the following equality holds true*

$$\Delta\left(AB\right) = \Delta\left(A\right)\Delta\left(B\right).$$

Proof. (A) Claim: for any two invertible positive operators $A,B\in 1+\mathcal{L}^1(\mathcal{N},\tau)$ we have

$$\tau\left(\log(BAB)\right) = 2\tau\left(\log B\right)+\tau\left(\log A\right). \tag{1.78}$$

To prove this equality we can replace A by e^{tS}, where $S=S^*$ and $t>0$, and calculate derivatives of both sides, using Corollary 1.3.35

$$\begin{aligned}\frac{d}{dt}\left(\tau\left[\log(Be^{tS}B\right]\right) &= \tau\left((Be^{tS}B)^{-1}\frac{d}{dt}(Be^{tS}B)\right)\\ &= \tau\left(B^{-1}e^{-tS}B^{-1}\cdot BSe^{tS}B\right) = \tau\left(S\right),\end{aligned}$$

and

$$\frac{d}{dt}\left(2\tau\left(\log B\right)+\tau\left(\log e^{tS}\right)\right) = \tau\left(e^{-tS}\cdot Se^{tS}\right) = \tau\left(S\right).$$

Since with $t=0$ the equality (1.78) is true, it has been proven.

(B) Claim: for any two invertible positive operators $A,B\in 1+\mathcal{L}^1(\mathcal{N},\tau)$ we have

$$\Delta\left(AB\right) = \Delta\left(A\right)\Delta\left(B\right).$$

It follows from (1.78) that

$$\log\Delta\left(BA^2B\right) = 2\log\Delta\left(B\right)+\log\Delta\left(A^2\right),$$

so that

$$\Delta\left(BA^2B\right) = \Delta\left(B\right)^2\Delta\left(A^2\right).$$

Since by Lemma 1.7.10 we have

$$\Delta\left(AB\right)^2 = \Delta\left(|AB|^2\right) = \Delta\left(BA^2B\right),$$

it follows that

$$\Delta(AB)^2 = \Delta(B)^2 \Delta\left(A^2\right).$$

Now, Lemma 1.7.10 completes the proof of (B).

(C) Now, let A and B be two invertible operators. Since $\Delta(A) = \Delta(|A|)$, it follows that

$$\begin{aligned}\Delta(AB) = \Delta(|AB|) = \Delta\left(\sqrt{B^* A^* AB}\right)\\ = \Delta\left(\sqrt{B^*|A||A|B}\right) = \Delta(||A|B|) = \Delta(|A|B).\end{aligned}$$

Further, using Lemma 1.7.11, applying the above equality to the pair B^* and $|A|$ and using (B), we have

$$\begin{aligned}\Delta(|A|B) = \Delta(B^*|A|) = \Delta(|B^*||A|)\\ = \Delta(|B^*|)\Delta(|A|) = \Delta(B)\Delta(A).\end{aligned}$$

□

Proposition 1.7.15 $\log\Delta(\cdot)$ *is real-analytic in the* $\mathcal{L}^1(\mathcal{N},\tau)$*-topology when restricted to invertible elements of* $1+\mathcal{L}^1(\mathcal{N},\tau)$. *Also, if* $A(\cdot)$ *is an* $\mathcal{L}^1(\mathcal{N},\tau)$*-holomorphic function of a complex variable with invertible values in* $1+\mathcal{L}^1(\mathcal{N},\tau)$ *then* $\log\Delta(A(\cdot))$ *is harmonic.*

Proof. (A) Claim: the proposition is true for the open set $\left\{A \in 1+\mathcal{L}^1(\mathcal{N},\tau)\colon \|A-1\| < 1\right\}$.

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n}(A-1)^n$$

converges in $\mathcal{L}^1(\mathcal{N},\tau)$-norm in the open ball $\|A-1\|_{\mathcal{L}^1} < 1$. It follows from Proposition 1.7.12 that in the open ball $\|A-1\|_{\mathcal{L}^1} < 1$

$$\log\Delta(A) = \operatorname{Re}\tau(\log A) = \operatorname{Re}\sum_{k=1}^{\infty}\frac{(-1)^{n-1}}{n}\tau[(A-1)^n].$$

Since $\operatorname{Re}\tau$ is an $\mathbb{R}$-linear $\mathcal{L}^1(\mathcal{N},\tau)$-bounded functional, the claim is proved.

(B) That the proposition is true also for $GL(\mathcal{N})\cap 1+\mathcal{L}^1(\mathcal{N},\tau)$ follows from Lemma 1.7.6, Lemma 1.7.8 and Proposition 1.7.14. □

Lemma 1.7.16 *If* $A \in \mathcal{L}^1(\mathcal{N},\tau)$, $1+A \geqslant 0$ *and* $\mathrm{N}_{1+A} = \{0\}$ *then*

$$\Delta(1+A) = \exp\left(\int_0^1 \tau\left((1+tA)^{-1}A\right)\,dt\right).$$

Proof. (A) If $1+A$ is invertible, then $A \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ and the equality under claim follows from Lemmas 1.7.2 and 1.7.13 by taking the path $A_t = tA$, $t \in [0,1]$, which connects 0 with A (or it also follows directly from the proof of Lemmas 1.7.2).

(B) If $1+A$ is not invertible then let $g_\delta(x) = \max(\delta, x)$, $\delta > -1$, and $A_\delta = g_\delta(A)$. Then since $\log(1+A_\delta)$ is decreasing as $\delta \to -1^+$, by normality of τ we have

$$\Delta(1+A) = \lim_{\delta\to-1^+} \Delta(1+A_\delta).$$

Since $1+A_\delta$ is invertible, we have by (A)

$$\Delta(1+A) = \lim_{\delta\to-1^+} \exp\left(\int_0^1 \tau\left((1+tA_\delta)^{-1}A_\delta\right) dt\right).$$

Since

$$\tau\left((1+tA_\delta)^{-1}A_\delta\right) = \frac{1}{t}\tau\left(1-(1+tA_\delta)^{-1}\right)$$

decreases as $\delta \to -1^+$, by monotone convergence theorem and normality of τ we have

$$\lim_{\delta\to-1^+} \int_0^1 \tau\left((1+tA_\delta)^{-1}A_\delta\right) dt = \int_0^1 \tau\left((1+tA)^{-1}A\right) dt.$$

The proof is complete. □

Lemma 1.7.17 *If $A, B \in 1+\mathcal{L}^1(\mathcal{N},\tau)$ and if $|A| \leqslant |B|$ then $\Delta(A) \leqslant \Delta(B)$.*

Proof. Since $\Delta(A) = \Delta(|A|)$ and $|A|, |B| \in 1+\mathcal{L}^1(\mathcal{N},\tau)$ by Lemma 1.7.6, we can assume that $0 \leqslant A \leqslant B$. Let $A = 1+S$ and $B = 1+T$, so that $S, T \in \mathcal{L}^1(\mathcal{N},\tau)$ and $-1 \leqslant S \leqslant T$. If $\mathrm{N}_{1+S} \neq \{0\}$, then by definition $\Delta(1+S) = -\infty$, so there is nothing to prove. Otherwise, by Lemma 1.7.16 we have

$$\Delta(1+S) = \exp\left(\int_0^1 \tau\left((1+tS)^{-1}S\right)\right) dt$$

and

$$\Delta(1+T) = \exp\left(\int_0^1 \tau\left((1+tT)^{-1}T\right)\right) dt.$$

So, it is enough to prove that $\tau\left((1+tS)^{-1}S\right) \leqslant \tau\left((1+tT)^{-1}T\right)$. This is the same as

$$\frac{1}{t}\tau\left(1-(1+tS)^{-1}\right) \leqslant \frac{1}{t}\tau\left(1-(1+tT)^{-1}\right).$$

So, it is enough to prove that $(1+tS)^{-1} \geqslant (1+tT)^{-1}$, $t \in (0,1)$. But since for $t \in (0,1)$ there exists $c > 0$ such that $0 < c \leqslant 1+tS \leqslant 1+tT$, this follows from Lemma 1.1.3. □

Remark 2 *Actually, as L. G. Brown notes in [Brn], this lemma is a simple consequence of the fact that* log *is an operator monotone function. We gave a proof which does not use this result. L. G. Brown says also that this lemma follows very easily from spectral dominance arguments.*

Proposition 1.7.18 $\log \Delta(\cdot)$ *is upper semi-continuous in the* $\mathcal{L}^1(\mathcal{N}, \tau)$*-topology, i.e., if* $\{A_\alpha\}$ *is a net of operators from* $1 + \mathcal{L}^1(\mathcal{N}, \tau)$ *which converges to* A *in* $\mathcal{L}^1(\mathcal{N}, \tau)$ *topology then*

$$\lim_\alpha \log \Delta(A_\alpha) \leqslant \log \Delta(A).$$

If the net $\{|A_\alpha|\}$, *in addition, is non-increasing then equality holds.*

Proof. (A) For $\varepsilon > 0$ and $T = A - 1$, set

$$f_\varepsilon(A) = \frac{1}{2} \log \Delta \left(|A|^2 + \varepsilon |T|^2 \right).$$

Since

$$\begin{aligned}
\varepsilon |T|^2 + |A|^2 &= \varepsilon |T|^2 + (A^* - 1)(A - 1) + A^* + A - 1 \\
&= \varepsilon T^* T + T^* T + T^* + T + 1 \\
&= \left((1+\varepsilon)^{1/2} T^* + (1+\varepsilon)^{-1/2} \right) \left((1+\varepsilon)^{1/2} T + (1+\varepsilon)^{-1/2} \right) \\
&\qquad + \frac{\varepsilon}{1+\varepsilon} \\
&\geqslant \frac{\varepsilon}{1+\varepsilon},
\end{aligned} \tag{1.79}$$

it follows that $|A|^2 + \varepsilon |T|^2$ is invertible. Hence, by Proposition 1.7.15 the function $\varepsilon \mapsto f_\varepsilon(A)$ is continuous. It also follows from the last equality that the map $A \in 1 + \mathcal{L}^1(\mathcal{N}, \tau) \mapsto f_\varepsilon(A)$ is $\mathcal{L}^1(\mathcal{N}, \tau)$-continuous. Further, by Lemma 1.7.17 we have that $f_\varepsilon(A) \geqslant \log \Delta(A)$ and $f_\varepsilon(A)$ is non-decreasing in ε.

(B) Claim:

$$\lim_{\varepsilon \to 0^+} f_\varepsilon(A) = \log \Delta(A).$$

Let $A_\delta = g_\delta(A)$, where $\delta \in (0, 1/2)$ and $g_\delta(\lambda) = \max(\delta, \lambda)$. A_δ is invertible and, since $A \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$, the operator A_δ also belongs to $\mathcal{L}^1(\mathcal{N}, \tau)$. Hence, since by Lemma 1.7.17

$$\frac{1}{2} \log \Delta \left(A_\delta^2 + \varepsilon |T|^2 \right) \geqslant f_\varepsilon(A)$$

and by Proposition 1.7.15

$$\lim_{\varepsilon \to 0^+} \frac{1}{2} \log \Delta \left(A_\delta^2 + \varepsilon |T|^2 \right) = \log \Delta(A_\delta),$$

it follows that

$$f_\varepsilon(A) \leqslant \log \Delta\left(A_\delta\right).$$

Since $\log \Delta\left(A_\delta\right) = \tau\left(\log(A_\delta)\right)$, by normality of trace τ we have

$$f_\varepsilon(A) \leqslant \log \Delta\left(A\right).$$

(C) Let $a_n(\varepsilon)$ be a sequence of non-decreasing functions such that the point-wise limit $\lim\limits_{n\to\infty} a_n(\varepsilon)$ exists. Then

$$X := \lim_{n\to\infty} \lim_{\varepsilon\to 0^+} a_n(\varepsilon) \leqslant \lim_{\varepsilon\to 0^+} \lim_{n\to\infty} a_n(\varepsilon) =: Y.$$

For sake of completeness, we give a proof of (C). Let $\delta > 0$. There exists $\varepsilon_0 > 0$ such that $\lim\limits_{n\to\infty} a_n(\varepsilon_0) < Y + \delta$. By monotonicity of a_n we have $\lim_{\varepsilon\to 0^+} a_n(\varepsilon) \leqslant a_n(\varepsilon_0)$. Hence,

$$X = \lim_{n\to\infty} \lim_{\varepsilon\to 0^+} a_n(\varepsilon) \leqslant \lim_{n\to\infty} a_n(\varepsilon_0) < Y + \delta,$$

so that $X \leqslant Y$.

(D) Using (B), (C), (A) and again (B), we have

$$\lim_\alpha \log \Delta\left(A_\alpha\right) = \lim_\alpha \lim_{\varepsilon\to 0^+} f_\varepsilon(A_\alpha) \leqslant \lim_{\varepsilon\to 0^+} \lim_\alpha f_\varepsilon(A_\alpha) = \lim_{\varepsilon\to 0^+} f_\varepsilon(A) = \log \Delta\left(A\right).$$

□

Proposition 1.7.19 *For any $S, T \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$ one has the relation*

$$\Delta\left(ST\right) = \Delta\left(S\right)\Delta\left(T\right). \tag{1.80}$$

Proof. (A) Assume first that $S, T \geqslant 0$ and that T is invertible. Let $S_n = g_n(S)$, where $g_n(\lambda) = \max(\frac{1}{n}, \lambda)$, $n = 2, 3, \ldots$. Since, by Lemma 1.7.10, $\Delta\left(ST\right)^2 = \Delta\left(|ST|^2\right) = \Delta\left(TS^2T\right)$, we see, by Lemma 1.7.17, that

$$\Delta\left(S_nT\right) = \sqrt{\Delta\left(TS_n^2T\right)} \geqslant \sqrt{\Delta\left(TS^2T\right)} = \Delta\left(ST\right).$$

Thus Proposition 1.7.18 implies that

$$\lim_{n\to\infty} \Delta\left(S_nT\right) = \lim_{n\to\infty} \sqrt{\Delta\left(TS_n^2T\right)} = \sqrt{\Delta\left(TS^2T\right)} = \Delta\left(ST\right). \tag{1.81}$$

Taking $T = 1$ implies also that $\lim\limits_{n\to\infty} \Delta\left(S_n\right) = \Delta\left(S\right)$. Since S_n is invertible, Proposition 1.7.14 and (1.81) imply that $\Delta\left(ST\right) = \Delta\left(S\right)\Delta\left(T\right)$.

(B) Now assume that $S, T \geqslant 0$ are not necessarily invertible. Let $T_n = g_n(T)$, $n = 2, 3, \ldots$. Since, by Lemmas 1.7.11 and 1.7.10, $\Delta\left(ST\right)^2 = \Delta\left(|(ST)^*|^2\right) = \Delta\left(ST^2S\right)$, it follows from Lemma 1.7.17 that

$$\Delta\left(ST_n\right) = \sqrt{\Delta\left(ST_n^2S\right)} \geqslant \sqrt{\Delta\left(ST^2S\right)} = \Delta\left(ST\right).$$

So, by Proposition 1.7.18,

$$\lim_{n\to\infty} \Delta\left(ST_n\right) = \lim_{n\to\infty} \sqrt{\Delta\left(ST_n^2 S\right)} = \sqrt{\Delta\left(ST^2 S\right)} = \Delta\left(ST\right). \tag{1.82}$$

Since T_n is invertible, by part (A)

$$\Delta\left(ST\right) = \lim_{n\to\infty} \Delta\left(S\right)\Delta\left(T_n\right) = \Delta\left(S\right)\Delta\left(T\right).$$

(C) Now, exactly the same argument, as in the proof of part (C) of Proposition 1.7.14, shows that (1.80) holds for arbitrary $S, T \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$. □

Proposition 1.7.20 *If E is a projection from $\mathcal{N}$ such that $TE = ETE$ then*

$$\Delta\left(T\right) = \Delta_{E\mathcal{N}E}\left(ETE\right) \cdot \Delta_{E^\perp\mathcal{N}E^\perp}\left(E^\perp TE^\perp\right).$$

Proof. The equality $TE = ETE$ implies $E^\perp TE = E^\perp ETE = 0$, so that

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}_{[E,E]}.$$

Since

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}_{[E,E]} = \begin{pmatrix} 1 & 0 \\ 0 & T_{22} \end{pmatrix}_{[E,E]} \begin{pmatrix} 1 & T_{12} \\ 0 & 1 \end{pmatrix}_{[E,E]} \begin{pmatrix} T_{11} & 0 \\ 0 & 1 \end{pmatrix}_{[E,E]}, \tag{1.83}$$

by Proposition 1.7.19 we have

$$\begin{aligned} \Delta\left(T\right) &= \Delta \begin{pmatrix} 1 & 0 \\ 0 & T_{22} \end{pmatrix}_{[E,E]} \Delta \begin{pmatrix} 1 & T_{12} \\ 0 & 1 \end{pmatrix}_{[E,E]} \Delta \begin{pmatrix} T_{11} & 0 \\ 0 & 1 \end{pmatrix}_{[E,E]} \\ &= \Delta_{E^\perp\mathcal{N}E^\perp}\left(T_{22}\right) \Delta\left(\exp \begin{pmatrix} 0 & T_{12} \\ 0 & 0 \end{pmatrix}_{[E,E]}\right) \Delta_{E\mathcal{N}E}\left(T_{11}\right) \\ &= \Delta_{E^\perp\mathcal{N}E^\perp}\left(T_{22}\right) \exp \operatorname{Re} \tau \begin{pmatrix} 0 & T_{12} \\ 0 & 0 \end{pmatrix}_{[E,E]} \Delta_{E\mathcal{N}E}\left(T_{11}\right) \\ &= \Delta_{E^\perp\mathcal{N}E^\perp}\left(T_{22}\right) \Delta_{E\mathcal{N}E}\left(T_{11}\right), \end{aligned}$$

where the third equality follows from Proposition 1.7.12. □

Proposition 1.7.21 *If $T \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$ and B is an invertible operator from $\mathcal{N}$, then*

$$\Delta\left(B^{-1}TB\right) = \Delta\left(T\right).$$

Proof. (A) Suppose first that T is invertible. In this case, in the polar decomposition $T = U\left|T\right|$, the operators $U, |T| \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$ by Lemmas 1.7.6 and 1.7.8. Hence, Proposition 1.7.19 implies

$$\Delta\left(B^{-1}TB\right) = \Delta\left(B^{-1}UB \cdot B^{-1}\left|T\right|B\right) = \Delta\left(B^{-1}UB\right)\Delta\left(B^{-1}\left|T\right|B\right).$$

Since by Lemma 1.7.8 $U = e^{S_1}$ and $|T| = e^{S_2}$ for some $S_1, S_2 \in \mathcal{L}^1(\mathcal{N}, \tau)$, by Proposition 1.7.14

$$\Delta\left(B^{-1}UB\right) = \Delta\left([B^{-1}e^{S_1/2}][e^{S_1/2}B]\right) = \Delta\left([e^{S_1/2}B][B^{-1}e^{S_1/2}]\right) = \Delta\left(U\right),$$

and similarly, $\Delta\left(B^{-1}\left|T\right|B\right) = \Delta\left(\left|T\right|\right)$. Hence,

$$\Delta\left(B^{-1}TB\right) = \Delta\left(U\right)\Delta\left(\left|T\right|\right)$$

and by Proposition 1.7.19

$$\Delta\left(B^{-1}TB\right) = \Delta\left(U\left|T\right|\right) = \Delta\left(T\right).$$

(B) For general T, let $T_n = Ug_n(|T|)$, where $g_n(\lambda) = \max(\frac{1}{n}, \lambda)$, $n = 2, 3, \ldots$. Then, since $|T| \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$ (Lemma 1.7.6), we have that T_n, $n \geqslant 2$, is invertible, $T_n \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$, $T_n \to T$ in $\mathcal{L}^1(\mathcal{N}, \tau)$ and $\Delta\left(T_n\right) \to \Delta\left(T\right)$. So, by (A)

$$\Delta\left(T\right) = \lim_{n\to\infty} \Delta\left(T_n\right) = \lim_{n\to\infty} \Delta\left(B^{-1}T_nB\right).$$

Then, since $B^{-1}T_nB \to B^{-1}TB$ in $\mathcal{L}^1(\mathcal{N}, \tau)$, Proposition 1.7.18 implies that $\Delta\left(T\right) \leqslant \Delta\left(B^{-1}TB\right)$. By symmetry, also $\Delta\left(B^{-1}TB\right) \leqslant \Delta\left(T\right)$. $\square$

1.8 The Brown measure

In this section we give an exposition of the theory of the Brown measure for τ-trace class operators. We follow here [Brn].

The Brown measure of an operator from a semifinite von Neumann algebra $\mathcal{N}$ is a measure on the spectrum of the operator. It describes spectral properties of the operator and it is a generalization of both the spectral measure of a normal operator and the counting measure of a compact operator.

1.8.1 Weyl functions

For purposes of exposition, it will be convenient to adopt the following terminology.

Definition 1.8.1 *We call a function $\varphi\colon [0,\infty) \to [0,\infty)$ a **Weyl function** if φ is non-decreasing, $\varphi(0) = 0$ and the function $\mathbb{R} \ni t \mapsto \varphi(e^t)$ is convex.*

Lemma 1.8.2 *A function $\varphi\colon [0,\infty) \to [0,\infty)$, which is continuous at 0, is a Weyl function if and only if it is the limit of an increasing sequence of linear combinations of functions $\log_+(rx)$, $r > 0$ with non-negative coefficients.*

Proof. (If) If $\varphi(x) = \log_+(rx)$ then $\varphi(e^t) = \max(0, a + bt)$, for some $a \in \mathbb{R}$ and $b > 0$. Hence, the functions $\log_+(rx)$, $r > 0$, are Weyl functions. Also, a positive linear combination of Weyl functions is a Weyl function. It is also clear that the limit of an increasing sequence of Weyl functions is also a Weyl function.

(Only if) This assertion follows from the well-known fact that any increasing convex function f on $\mathbb{R}$, with $\lim_{t\to-\infty} = 0$, is the limit of an increasing sequence of linear combinations of a constant function and functions of the form $\max(0, a + bt)$, $b > 0$. □

Any Weyl function is the sum of a Weyl function, which is continuous at 0, and a function φ with $\varphi(0) = 0$ and $\varphi(t) = \text{const}$ for $t > 0$.

Note that, since $(\log(1 + e^t))'' = \frac{e^t}{(1+e^t)^2} \geqslant 0$, the function $\log(1 + x)$ is a Weyl function.

Proposition 1.8.3 *Let $s_1, s_2 \colon (0, \infty) \to [0, \infty)$ be non-increasing, and assume that*

$$\int_0^1 \log_+ s_j(x)\, dx < \infty, \qquad j = 1, 2.$$

Then the following are equivalent:
(i)

$$\int_0^t \log s_1(x)\, dx \leqslant \int_0^t \log s_2(x)\, dx \quad \text{for all} \quad t \in (0, \infty).$$

(ii)

$$\int_0^\infty \log_+(rs_1(x))\, dx \leqslant \int_0^\infty \log_+(rs_2(x))\, dx \quad \text{for all} \quad r \in (0, \infty).$$

(iii)

$$\int_0^t \varphi(s_1(x))\, dx \leqslant \int_0^t \varphi(s_2(x))\, dx \quad \text{for all} \quad t \in (0, \infty],$$

where φ is an arbitrary Weyl function.

Proof. The hypothesis implies that for any $t > 0$ and $r > 0$

$$\int_0^t \log_+(rs_j(x)) \ < \infty.$$

(i) ⇒ (ii). Note that, for $j = 1, 2$,

$$\begin{aligned} \int_0^\infty \log_+(rs_j(x))\, dx &= \sup_{t>0} \left\{ \int_0^t \log(rs_j(x))\, dx \right\} \\ &= \sup_{t>0} \left\{ t \log r + \int_0^t \log s_j(x)\, dx \right\}. \end{aligned} \tag{1.84}$$

(ii) $\Rightarrow$ (i). First assume that $s_2(t) > 0$. Then for $r = \frac{1}{s_2(t)}$,

$$\int_0^t \log(rs_1(x))\,dx \leqslant \int_0^t \log_+(rs_1(x))\,dx \leqslant \int_0^\infty \log_+(rs_1(x))\,dx$$
$$\leqslant \int_0^\infty \log_+(rs_2(x))\,dx = \int_0^t \log(rs_2(x))\,dx,$$

which gives the result for t. Now let $t^* = \inf\{t\colon s_2(t) = 0\}$. Since

$$\int_0^{t^*} \log s_1(x)\,dx = \lim_{t\to t^*-} \int_0^t \log s_1(x)\,dx$$
$$\leqslant \lim_{t\to t^*-} \int_0^t \log s_2(x)\,dx = \int_0^{t^*} \log s_2(x)\,dx,$$

it suffices to show that

$$\text{if } t^* < t_2, \text{then } s_1(t_2) = 0. \tag{1.85}$$

For any $r > 1$ by (1.84) we have

$$\int_0^\infty \log_+(rs_1(x))\,dx \leqslant t_2 \log r + \int_0^{t_2} \log s_1(x)\,dx$$

and

$$\int_0^\infty \log_+(rs_2(x))\,dx \leqslant t^* \log r + \int_0^{t^*} \log_+ s_2(x)\,dx.$$

So, for large enough r, the inequality $s_1(t_2) > 0$ would contradict (ii).

(ii) $\Rightarrow$ (iii). If $t < \infty$ then we may change $s_1(x)$ and $s_2(x)$ to 0 for $x > t$, since (i) will remain true. Thus we are reduced to the case $t = \infty$. If φ is continuous at 0, then it is enough to apply Lemma 1.8.2 and the monotone convergence theorem. Otherwise, it is enough to consider the function $\varphi(0) = 0$, and $\varphi(t) = 1$, if $t > 0$. For this case the claim follows from (1.85).

(iii) $\Rightarrow$ (ii). Trivial, since $\log_+(rx)$ is a Weyl function. $\square$

Lemma 1.8.4 *For any $T \in \mathcal{N}$ and for any Weyl function φ*

$$\tau\left(\varphi(|T^n|)\right) \leqslant \tau\left(\varphi(|T|^n)\right).$$

Proof. By Proposition 1.3.29, we have $\Lambda_t(T^n) \leqslant n\Lambda_t(T)$, so that

$$\begin{aligned}\int_0^t \log(\mu_s(T^n))\,ds &\leqslant n\int_0^t \log(\mu_s(T))\,ds \\ &= \int_0^t \log(\mu_s(T)^n)\,ds \\ &= \int_0^t \log(\mu_s(|T|^n))\,ds,\end{aligned}$$

the last equality by Lemma 1.3.17. Hence, equivalence of (i) and (iii) in Proposition 1.8.3 gives

$$\int_0^\infty \varphi(\mu_t(T^n))\,dt \leqslant \int_0^\infty \varphi(\mu_t(|T|^n))\,dt,$$

which by Proposition 1.3.21 is the same as

$$\tau\left(\varphi(|T^n|)\right) \leqslant \tau\left(\varphi(|T|^n)\right).$$

□

1.8.2 The Weierstrass function

Let $g_k\colon \mathbb{C} \to \mathbb{C},\ k \geqslant 1$, be an entire function, defined by

$$g_k(z) = (1-z)e^{z+\frac{1}{2}z^2+\ldots+\frac{1}{k-1}z^{k-1}}, \quad z \in \mathbb{C}.$$

Lemma 1.8.5 *The following is true*

$$\log|g_k(z)| = O(|z|^k), \quad \text{as} \quad z \to 0; \tag{1.86}$$

for any $\varepsilon > 0$

$$\log|g_k(z)| = O(|z|^{k-1+\varepsilon}), \quad \text{as} \quad z \to \infty; \tag{1.87}$$

and for some $C > 0$

$$\log|g_k(z)| \leqslant C\,|z|^k, \quad \forall z \in \mathbb{C}. \tag{1.88}$$

Proof. We have,

$$\begin{aligned}\log|g_k(z)| &= \operatorname{Re}\log\left((1-z)e^{z+\frac{1}{2}z^2+\ldots+\frac{1}{k-1}z^{k-1}}\right)\\ &= -\operatorname{Re}\left(\frac{1}{k}z^k + \frac{1}{k+1}z^{k+1} + \ldots + \frac{1}{n}z^n + \ldots\right).\end{aligned}$$

Here it does not matter which branch of logarithm is taken, since the real parts will be the same. Hence, (1.86) follows.

The assertion (1.87) follows from

$$\log|g_k(z)| = \log|1-z| + \operatorname{Re}\left(z + \frac{1}{2}z^2 + \ldots + \frac{1}{k-1}z^{k-1}\right).$$

The inequality (1.88) follows from (1.86) and (1.87), since at the discontinuity point $z = 1$, the function is $-\infty$. □

In (1.87), ε is necessary for the case $k = 1$.

1.8.3 Subharmonic functions

We give first the definition of a subharmonic function and collect some properties of subharmonic functions. These are necessary for the exposition of the theory of the Brown measure, given later.

Let G be a domain i.e. an open connected subset of $\mathbb{C}$.

Definition 1.8.6 *An upper semicontinuous function $u\colon G \to [-\infty, \infty)$ is said to be* subharmonic *in G if, for any $z \in G$, there exists $\varepsilon > 0$ such that for any $0 < r < \varepsilon$*

$$u(z) \leqslant \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, d\theta. \tag{1.89}$$

The proof of the following facts can be found in, e.g. [HK] or [Vl].

Proposition 1.8.7 [HK] *(i) If u is subharmonic in G then u is subharmonic in any open subset U of G.*
(ii) If $u_1, \dots, u_k$ are subharmonic in G and $t_1, \dots, t_k \geqslant 0$, then $t_1 u_1 + \dots + t_k u_k$ is subharmonic in G.
(iii) If $u_1, \dots, u_k$ are subharmonic in G, then $\max u_1, \dots, u_k$ is subharmonic in G.
(iv) A non-constant subharmonic function cannot have a local maximum.

Lemma 1.8.8 [HK] *Decreasing sequence $u_n(z)$ of subharmonic functions converges to a subharmonic function.*

Lemma 1.8.9 [HK] *If u is subharmonic in G then it is locally summable in G.*

Proposition 1.8.10 [HK] *If u is subharmonic in G, then $\Delta u \geq 0$ in G (in distribution sense). Conversely, if $u \in \mathcal{D}'(G)$ and $\Delta u \geq 0$, then u is a measurable function which is almost everywhere equal to a subharmonic function.*

If u is locally integrable, then Δu, computed in distributional sense, is a measure which is finite on compact sets. The measure $\frac{1}{2\pi}\Delta u$ is called the **Riesz measure** of the subharmonic function u. The Riesz measure of a subharmonic function is finite on compact subsets.

Let $E(z, w)$ be one of the functions $\log|z - w|$, $\log\left|g_k\left(\frac{z}{w}\right)\right|$, $k \geqslant 1$. The function $E(z, w)$ satisfies the equation $\Delta_z E(z, w) = 2\pi\delta(z - w)$, where δ is the Dirac δ-function.

If μ_0 is a non-negative measure in $\mathbb{C}$ with compact support and K is a compact subset of $\mathbb{C}$, then the function

$$v_K(z) = \int_K E(z,w)\, d\mu_0(w)$$

is subharmonic and $\Delta v_K(z) = \mu_0\big|_K$. Hence, if μ_0 is the Riesz measure of a subharmonic function u, then $u = 2\pi v_K + h$, where h is a subharmonic function, harmonic on the interior of K.

Lemma 1.8.11 [HK] *Let u be subharmonic in $\mathbb{C}$, and let μ_0 be its Riesz measure. If $E(z,w)$ is a Green's function for the Laplace operator Δ, such that for some $R > 0$*

$$\int_{\{w:|w|>R\}} |E(z,w)|\ d\mu_0(w) < \infty \tag{1.90}$$

for all $z \in \mathbb{C}$, then

$$u(z) = \int_{\mathbb{C}} E(z,w)\, d\mu_0(w) + h(z),$$

where $h(z)$ is an entire harmonic function.

This lemma follows from Riesz representation theorem [HK, Theorem 3.9] and the estimate (1.90).

Lemma 1.8.12 [HK, Theorem 3.14] *Assume that u is subharmonic on $\mathbb{C}$, and that u is harmonic in a neighbourhood of 0, and $u(0) = 0$. Let μ_0 be the Riesz measure of u. Then for all $r > 0$*

$$\int \log_+ \frac{r}{|w|}\, d\mu_0(w) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\, d\theta.$$

If u is a subharmonic function in $\mathbb{C}$, then we set

$$S_n u(z) := \sum_{j=0}^{n-1} u(\rho^j z),$$

where $\rho = e^{\frac{2\pi i}{n}}$. It follows directly from the definition of the subharmonic function (1.89), that the function $S_n u$ is subharmonic in $\mathbb{C}$. It is easy to check that

$$\int_0^{2\pi} S_n u(re^{i\theta})\, d\theta = n \int_0^{2\pi} u(re^{i\theta})\, d\theta. \tag{1.91}$$

Let

$$S_{n+}u(z) = \max\left(0, S_n u(z)\right).$$

By Proposition 1.8.7(iii), the function $S_{n+}u$ is also subharmonic in $\mathbb{C}$.

Proposition 1.8.13 *Let $u\colon \mathbb{C} \to \mathbb{R}$ be subharmonic in $\mathbb{C}$, and let u be harmonic in a neighborhood of 0. Let μ_0 be the Riesz measure of u. Let $p > 0$ be such that*

$$\int_{\mathbb{C}} |w|^{-p}\, d\mu_0(w) < \infty \tag{1.92}$$

and let $k \geqslant p$ be an integer. Assume that u vanishes to order at least k at zero and

$$S_{n+}u(z) = o(|z|^n) \tag{1.93}$$

as $z \to \infty$ for all $n \geqslant k$. Then for all $z \in \mathbb{C}$

$$u(z) = \int_{\mathbb{C}} \log \left| g_k\left(\frac{z}{w}\right)\right| \, d\mu_0(w).$$

Proof. (A) The estimates (1.86) and (1.92) imply that the last integral converges when $\frac{z}{w}$ is very small. Hence, if $v(z)$ is the integral, then by Lemma 1.8.11,

$$u = v + h, \tag{1.94}$$

where h is an entire harmonic function.

We have to show that $h = 0$.

Since μ_0 is zero in a neighbourhood of 0, (1.92) implies that

$$\int_{\mathbb{C}} |w|^{-k}\, d\mu_0(w) < \infty$$

for any $k \geqslant p$. Since by (1.88)

$$|v(z)| \leqslant C\, |z|^k \int |w|^{-k}\ d\mu_0(w) \leqslant C_1\, |z|^k\,,$$

v vanishes to order k at 0, and since u also vanishes to order k at zero, so does $h = u - v$.

(B) Let $\varepsilon > 0$ and let $R = R(\varepsilon) > 0$ be such that

$$\int_{\{w\colon |w| > R\}} |w|^{-k}\ d\mu_0(w) < \varepsilon. \tag{1.95}$$

By (1.87),

$$\begin{aligned} \Big| \int_{\{w\colon |w|\leqslant R\}} & \log \left| g_k\left(\frac{z}{w}\right)\right| d\mu_0(w) \Big| \\ & \leqslant C \int_{\{w\colon |w|\leqslant R\}} \left|\frac{z}{w}\right|^{k-1/2} d\mu_0(w) \leqslant C_1(R)\, |z|^{k-\frac{1}{2}}\,. \end{aligned}$$

By (1.88) and (1.95),

$$\left| \int_{\{w:\ |w|>R\}} \log \left| g_k \left(\frac{z}{w} \right) \right| \, d\mu_0(w) \right| \leqslant C \int_{\{w:\ |w|>R\}} \left| \frac{z}{w} \right|^k \, d\mu_0(w) < C\varepsilon \, |z|^k .$$

Hence,

$$v_+(z) = o(|z|^k) \tag{1.96}$$

as $z \to \infty$.

(C) Since

$$\int_0^{2\pi} v(re^{i\theta}) \, d\theta = \int_0^{2\pi} v_+(re^{i\theta}) \, d\theta - \int_0^{2\pi} v_-(re^{i\theta}) \, d\theta \geqslant v(0) = 0,$$

it follows from (1.96) that

$$\int_0^{2\pi} \left| v(re^{i\theta}) \right| \, d\theta = o(r^k) \qquad \text{as} \ \ r \to \infty.$$

Hence, for any $n \geqslant k$,

$$\int_0^{2\pi} \left| S_n v(re^{i\theta}) \right| \, d\theta = o(r^n) \qquad \text{as} \ \ r \to \infty,$$

since $r^k \leqslant r^n$ for $r \geqslant 1$.

Similarly, by (1.93), for any $n \geqslant k$,

$$\int_0^{2\pi} \left| S_n u(re^{i\theta}) \right| \, d\theta = o(r^n) \qquad \text{as} \ \ r \to \infty.$$

(D) Any entire harmonic function $\tilde{h}$ has an expansion

$$\tilde{h}(re^{i\theta}) = \sum_{m=0}^{\infty} a_m r^m \cos m\theta + \sum_{m=1}^{\infty} b_m r^m \sin m\theta, \quad a_m, b_m \in \mathbb{R},$$

where for $m > 0$,

$$a_m = \frac{1}{\pi r^m} \int_0^{2\pi} \tilde{h}(re^{i\theta}) \cos m\theta \, d\theta \tag{1.97}$$

and

$$b_m = \frac{1}{\pi r^m} \int_0^{2\pi} \tilde{h}(re^{i\theta}) \sin m\theta \, d\theta. \tag{1.98}$$

Indeed, any entire harmonic function $\tilde{h}(z)$ is the real part of an entire analytic function $f(z) = \sum_{m=0}^{\infty} c_m z^m$. Hence, letting $c_m = a_m + ib_m$ and $z = re^{i\theta}$,

$$\tilde{h}(z) = \sum_{m=0}^{\infty} a_m r^m \cos m\theta - b_m r^m \sin m\theta.$$

The formulas (1.97) and (1.98) for the coefficients a_m and b_m can be found by multiplying this equality by $\cos m\theta$ for a_m and by $\sin m\theta$ for b_m and integrating both sides over $[0, 2\pi]$.

(E) For m divisible by n,

$$a_m(S_n h) = n a_m(h) \quad \text{and} \quad b_m(S_n h) = n b_m(h). \tag{1.99}$$

The equality (1.94) implies $S_n u = S_n v + S_n h$. So, it follows from (C) that for any $n \geqslant k$

$$\int_0^{2\pi} \left|S_n h(re^{i\theta})\right| \, d\theta = o(r^n), \quad \text{as} \quad r \to \infty.$$

By (1.99) and (1.97), this implies $a_n(h) = \frac{1}{n} a_n(S_n h) = 0$, and similarly, $b_n(h) = 0$ for any $n \geqslant k$. Since, by (A), h vanishes to order k at 0, $h = 0$. □

1.8.4 Technical results

Lemma 1.8.14 *Let $A_1, ..., A_n, B_1, ..., B_n \in \mathcal{B}(\mathcal{H})$ be such that the operator $\sum_{k=1}^{n} |A_k|^2 \in \mathcal{B}(\mathcal{H})$ is invertible. Then*

$$\left(\sum_{k=1}^{n} B_k^* A_k\right) \left(\sum_{k=1}^{n} |A_k|^2\right)^{-1} \left(\sum_{k=1}^{n} A_k^* B_k\right) \leqslant \sum_{k=1}^{n} |B_k|^2.$$

Proof. Let $\bar{A}_j = A_j \left(\sum_{k=1}^{n} |A_k|^2\right)^{-1/2}$ and let

$$\bar{A} \colon \mathcal{H} \to \mathcal{H} \oplus \ldots \oplus \mathcal{H}, \qquad B \colon \mathcal{H} \to \mathcal{H} \oplus \ldots \oplus \mathcal{H}$$

be given by

$$A\xi = \bar{A}_1 \xi \oplus \bar{A}_2 \xi \oplus \ldots \oplus \bar{A}_n \xi, \qquad B\xi = B_1 \xi \oplus B_2 \xi \oplus \ldots \oplus B_n \xi.$$

Then $\bar{A}^* \bar{A} = 1$ and hence $\bar{A} \bar{A}^* \leqslant 1$. Therefore,

$$(B^* \bar{A})(\bar{A}^* B) = B^* (\bar{A} \bar{A}^*) B \leqslant B^* B,$$

as desired. □

Lemma 1.8.15 *Let $A_1(z), ..., A_n(z)$ be holomorphic functions of a complex variable, relative to the topology of $\mathcal{L}^1(\mathcal{N}, \tau)$, such that $A_1(z) - 1, A_2(z), ..., A_n(z) \in \mathcal{L}^1(\mathcal{N}, \tau)$. If $\sum_{k=1}^{n} |A_k(z)|^2$ is invertible for all $z \in \mathbb{C}$, then*

$$u(z) := \log \Delta \left(\sum_{k=1}^{n} |A_k(z)|^2\right)$$

is subharmonic.

Proof. By Proposition 1.7.15, u is a C^∞-function. By Proposition 1.8.10 it is enough to show $\Delta u \geqslant 0$. We have

$$\frac{\partial u}{\partial z} = \tau\left[\left(\sum_{j=1}^{n} A_j(z)^* A_j(z)\right)^{-1} \sum_{j=1}^{n} A_j(z)^* A_j'(z)\right],$$

$$\frac{1}{4}\Delta u = \frac{\partial^2 u}{\partial \bar{z}\partial z} = \tau\left[\left(\sum_{j=1}^{n} |A_j(z)|^2\right)^{-1} \sum_{j=1}^{n} \left|A_j'(z)\right|^2\right.$$
$$\left. - \left(\sum_{j=1}^{n} |A_j(z)|^2\right)^{-1} \sum_{j=1}^{n} A_j'(z)^* A_j(z) \left(\sum_{j=1}^{n} |A_j(z)|^2\right)^{-1} \sum_{j=1}^{n} A_j(z)^* A_j'(z)\right].$$

Since, by Theorem 1.3.32, $\tau[X^{-1}Y] = \tau[X^{-1/2}YX^{-1/2}]$, in order to show that $\Delta u \geqslant 0$, it is sufficient to show that

$$\sum_{j=1}^{n} \left|A_j'(z)\right|^2 \geqslant \sum_{j=1}^{n} A_j'(z)^* A_j(z) \left(\sum_{j=1}^{n} |A_j(z)|^2\right)^{-1} \sum_{j=1}^{n} A_j(z)^* A_j'(z).$$

This last follows from Lemma 1.8.14 with $B_j = A_j'(z)$. $\square$

Theorem 1.8.16 *Let $A_1(z), ..., A_n(z)$ be holomorphic functions of a complex variable in the topology of $\mathcal{L}^1(\mathcal{N}, \tau)$, such that $A_1(z) - 1, A_2(z), ..., A_n(z) \in \mathcal{L}^1(\mathcal{N}, \tau)$. Then*

$$u(z) := \log \Delta \left(\sum_{k=1}^{n} |A_k(z)|^2\right)$$

is subharmonic.

Proof. Let $T(z) = A_1(z) - 1$, and for $\varepsilon > 0$ let

$$u_\varepsilon(z) = \log \Delta \left(\varepsilon\,|T(z)| + \sum_{k=1}^{n} |A_k(z)|^2\right).$$

Then $\varepsilon\,|T(z)| + \sum_{k=1}^{n} |A_k(z)|^2$ is invertible for any z by the argument of (1.79), so that by Lemma 1.8.15 the function u_ε is subharmonic. Since by Lemma 1.7.17 $u_\varepsilon \downarrow u(z)$ as $\varepsilon \downarrow 0$, by Lemma 1.8.8 u is also subharmonic. $\square$

1.8.5 The Brown measure

As usual, $\mathcal{N}$ is a semifinite von Neumann algebra, τ is a faithful normal semifinite trace on $\mathcal{N}$.

Let $T \in \mathcal{L}^k(\mathcal{N}, \tau)$, where k is a positive integer and set

$$u_T(z) := \log \Delta\left(g_k(zT)\right) = \tau\left(\log |g_k(zT)|\right), \quad z \in \mathbb{C}.$$

Theorem 1.8.17 *The function $u_T(z)$ is subharmonic in $\mathbb{C}$ and is harmonic in $\left\{z \in \mathbb{C}\colon z^{-1} \notin \sigma_T\right\}$. In particular, $u_T(z)$ is harmonic in a neighbourhood of 0 and vanishes to order k at 0.*

Proof. Since $g_k(z)-1$ vanishes to order k at 0 by (1.86), it follows that $g_k(zT) \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$. Combined with the case $n = 1$ of Theorem 1.8.16 and Lemma 1.7.10, this implies that $u_T(z)$ is subharmonic. Moreover, $g_k(zT)$ is invertible whenever $\frac{1}{z} \notin \sigma_T$, and in this region u is harmonic by Proposition 1.7.15. In particular, u is harmonic in a neighbourhood of 0 and vanishes to order k at 0. □

This enables one to introduce a measure in $\mathbb{C}$, which characterizes spectral properties of an operator T.

Definition 1.8.18 *The **Brown measure** μ_T of an operator $T \in \mathcal{L}^1$ is a non-negative measure in $\mathbb{C} \setminus \{0\}$, defined by the formula*

$$d\mu_T(\lambda) := d\mu_T^*(\lambda^{-1}),$$

where

$$d\mu_T^* = \frac{1}{2\pi}\Delta u_T(z)$$

is the Riesz measure of the subharmonic function $u_T(z)$.

The Brown measure is a non-negative measure, which is finite on compact subsets of $\mathbb{C} \setminus \{0\}$.

The **support of a measure** μ in $\mathbb{C}$ is defined as the complement of the union of all open sets of μ-measure zero.

Corollary 1.8.19 *Let T be a τ-trace class operator from $\mathcal{N}$. The support $\operatorname{supp}(\mu_T)$ of the Brown measure of T is a subset of $\sigma_T^* := \sigma_T \setminus \{0\}$.*

Proof. Directly follows from definition of the Brown measure and Theorem 1.8.17. □

1.8.6 The Lidskii theorem for the Brown measure

The aim of this subsection is to prove the Lidskii theorem for the Brown measure (Theorem 1.8.27). The main concern of the proof is to check the conditions of Proposition 1.8.13 for the function $u_T(z)$.

Lemma 1.8.20 *If $T \in \mathcal{L}^p(\mathcal{N}, \tau)$, where p is a positive integer, then for any integer $n \geqslant p$*

$$S_n u_T(z) = \log \Delta \left(1 - z^n T^n\right).$$

Proof. Using the product property of the Fuglede-Kadison determinant (Proposition 1.7.19) we see that

$$S_n u_T(z) = \sum_{j=0}^{n-1} \log \Delta \left(g_k(\rho^j zT)\right) = \log \Delta \left(\prod_{j=0}^{n-1} g_k(\rho^j zT) \right).$$

Hence, it suffices to show that $\prod_{j=0}^{n-1} g_k(\rho^j zT) = 1 - z^n T^n$. We have

$$\begin{aligned}
\prod_{j=0}^{n-1} g_k(\rho^j zT) &= \prod_{j=0}^{n-1} (1 - \rho^j zT) e^{\rho^j zT + \frac{1}{2}(\rho^j zT)^2 + \ldots + \frac{1}{k-1}(\rho^j zT)^{k-1}} \\
&= e^{\sum\limits_{j=0}^{n-1} \rho^j zT + \frac{1}{2}\sum\limits_{j=0}^{n-1} \rho^{2j}(zT)^2 + \ldots + \frac{1}{k-1}\sum\limits_{j=0}^{n-1} \rho^{(k-1)j}(zT)^{k-1}} \cdot \prod_{j=0}^{n-1} (1 - \rho^j zT) \\
&= \prod_{j=0}^{n-1} (1 - \rho^j zT) = 1 - z^n T^n,
\end{aligned}$$

where the third equality follows from the fact that $\sum\limits_{j=0}^{n-1} \rho^{mj} = 0$ for all $m = 1, 2, \ldots, p-1$. $\square$

Lemma 1.8.21 *If $T \in \mathcal{L}^1(\mathcal{N}, \tau)$ then*

$$\tau\left(\log|1+T|\right) \leqslant \tau\left(\log(1+|T|)\right).$$

Proof. We have $\tau\left(\log|1+T|\right) \leqslant \tau\left(\log_+|1+T|\right)$ and, by Proposition 1.3.21,

$$\tau\left(\log_+|1+T|\right) = \int_0^\infty \log_+ \mu_t(1+T)\, dt,$$

so that

$$\tau\left(\log|1+T|\right) \leqslant \int_0^\infty \log_+ \mu_t(1+T)\, dt.$$

Hence, since $\mu_t(1+T) \leqslant \mu_0(1) + \mu_t(T) = 1 + \mu_t(T)$ by Lemma 1.3.18, we have

$$\begin{aligned}\tau\left(\log|1+T|\right) &\leqslant \int_0^\infty \log_+\big(1+\mu_t(T)\big)\,dt \\ &= \int_0^\infty \log\big(1+\mu_t(T)\big)\,dt = \tau\left(\log(1+|T|)\right),\end{aligned}$$

where the last equality follows from Proposition 1.3.21. □

Lemma 1.8.22 *If $T \in \mathcal{L}^k(\mathcal{N},\tau)$ then for any $n \geqslant k$*

$$S_{n+}u_T(z) = o(|z|^n) \quad as \quad z \to \infty.$$

Proof. By Lemma 1.8.20 we have

$$S_n u_T(z) = \log\Delta\left(1 - z^n T^n\right).$$

Since $T^n \in \mathcal{L}^1(\mathcal{N},\tau)$, we may suppose that $n = k = 1$. Then, by Lemma 1.8.21,

$$u_T(z) = \tau\left(\log|1 - zT|\right) \leqslant \tau\left(\log(1 + |z|\,|T|)\right).$$

So, it suffices to show that

$$(E) := \lim_{r\to\infty} \frac{\tau\left(\log(1 + r\,|T|)\right)}{r} = 0.$$

l'Hôpital's rule and Theorem 1.3.34 imply

$$(E) = \lim_{r\to\infty} \tau\left((1 + r\,|T|)^{-1}\,|T|\right) = \lim_{r\to\infty}\int_{\mathbb{R}} \frac{\lambda}{1+r\lambda}\,\tau\left(dE_\lambda^{|T|}\right).$$

Since $|T| \in \mathcal{L}^1(\mathcal{N},\tau)$, we have $\int_{\mathbb{R}} \lambda\,\tau\left(dE_\lambda^{|T|}\right) = \tau\left(|T|\right) < \infty$, so that the integrand is majorized by a summable function and converges to 0. By the dominated convergence theorem $(E) = 0$. □

Lemma 1.8.23 *If $T \in \mathcal{L}^k(\mathcal{N},\tau)$, then for any $r > 0$*

$$\frac{1}{2\pi}\int_0^{2\pi} u_T(re^{i\theta})\,d\theta \leqslant \tau\left(\log_+(r\,|T|)\right),$$

where $\log_+(x) = \max(0, \log(x))$.

Proof. Using successively (1.91), Lemma 1.8.20, Lemma 1.8.21, Lemma 1.8.4 applied to the Weyl function $\varphi(x) = \log(1+x)$ and the spectral theorem, we have for $n \geqslant k$

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} u_T(re^{i\theta})\,d\theta &= \frac{1}{2\pi n}\int_0^{2\pi} S_n u_T(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi n}\int_0^{2\pi} \tau\left(\log\left|1 - r^n e^{in\theta} T^n\right|\right)\,d\theta \\
&\leqslant \frac{1}{n}\tau\left(\log(1 + r^n\left|T^n\right|)\right) \\
&\leqslant \frac{1}{n}\tau\left(\log(1 + r^n\left|T\right|^n)\right) \\
&= \frac{1}{n}\int_0^{\infty} \log(1 + r^n\lambda^n)\,d\nu(\lambda),
\end{aligned}$$

where $\nu(F) = \tau\left(E_F^{|T|}\right)$. For $\lambda < \frac{1}{r}$, we have $\log(1 + r^n\lambda^n) \leqslant (r\lambda)^n \leqslant (r\lambda)^p$, where the last function is ν-summable. Thus, by the dominated convergence theorem

$$\lim_{n\to\infty}\int_0^{1/r} \log(1 + r^n\lambda^n)\,d\nu(\lambda) = 0.$$

For $\lambda \geqslant \frac{1}{r}$, we have $\log(1 + r^n\lambda^n) \leqslant \log(2r^n\lambda^n) \leqslant \log 2 + n\log(r\lambda)$. Hence,

$$\begin{aligned}
\frac{1}{n}\int_{1/r}^{\infty} \log(1 + r^n\lambda^n)\,d\nu(\lambda) &\leqslant \frac{\log 2}{n}\nu\left([1/r,\infty)\right) + \int_{1/r}^{\infty}\log(r\lambda)\,d\nu(\lambda) \\
&\to 0 + \tau\left(\log_+(r\left|T\right|)\right) = \tau\left(\log_+(r\left|T\right|)\right),
\end{aligned}$$

as $n \to \infty$. $\square$

We recall (see e.g. [SW, p. 202, (V.3.18)]) that, if ν is a non-negative measure on a measurable space S, f is a non-negative measurable function on S and f^* is the non-increasing rearrangement of f relative to ν, then for all non-negative measurable functions $\varphi\colon [0,\infty) \to [0,\infty)$

$$\int_0^{\infty} \varphi(f^*(s))\,ds = \int_S \varphi(f(w))\,d\nu(w). \tag{1.100}$$

Theorem 1.8.24 *If φ is a Weyl function, then*

$$\int_{\mathbb{C}} \varphi(|w|)\,d\mu_T(w) \leqslant \tau\left(\varphi(|T|)\right).$$

Also, for all $t > 0$

$$\int_0^t \varphi(\mu_1(s))\,ds \leqslant \int_0^t \varphi(\mu_s(T))\,ds,$$

where μ_1 is the non-increasing rearrangement of $|w|$ relative to $d\mu_T$.

Proof. We have

$$\begin{aligned}
\int_0^\infty \log_+(r\mu_1(s))\,ds &= \int_{\mathbb{C}} \log_+(r\,|w|)\,d\mu_T(w) \\
&= \int_{\mathbb{C}} \log_+\left(\frac{r}{|w|}\right)\,d\mu_T^*(w) \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta \\
&\leqslant \tau\left(\log_+(r\,|T|)\right) \\
&= \int_0^\infty \log_+(r\mu_s(T))\,ds,
\end{aligned}$$

where the first equality follows from (1.100), the third equality follows from Lemma 1.8.12, the fourth inequality follows from Lemma 1.8.23 and the fifth equality follows from Proposition 1.3.21. Hence, implication (ii) $\Rightarrow$ (iii) of Proposition 1.8.3 gives, for all $t \in (0,\infty]$,

$$\int_0^t \varphi(\mu_1(s))\,ds \leqslant \int_0^t \varphi(\mu_s(T))\,ds.$$

Now, applying once more (1.100) and Proposition 1.3.21, one gets from the last equality with $t = \infty$,

$$\int_{\mathbb{C}} \varphi(|w|)\,d\mu_T(w) \leqslant \tau\left(\varphi(|T|)\right).$$

□

Corollary 1.8.25 *(i) For any $q > 0$,*

$$\int_{\mathbb{C}} |w|^q \; d\mu_T(w) < \|T\|_q^q .$$

(ii)

$$\mu_T(\sigma_T \setminus \{0\}) \leqslant \tau\left(\mathrm{supp}_r(T)\right).$$

Proof. For (i), take $\varphi(x) = x^q$ in Theorem 1.8.24; for (ii), take $\varphi(0) = 0$ and $\varphi(x) = 1$ for $x > 0$. □

Theorem 1.8.26 *If k is a positive integer and if $T \in \mathcal{L}^k(\mathcal{N},\tau)$, then, for all $z \in \mathbb{C}$,*

$$\tau\left(\log|g_k(zT)|\right) = \int_{\mathbb{C}} \log|g_k(zw)| \; d\mu_T(w). \tag{1.101}$$

Proof. We can apply Proposition 1.8.13 to the function $u_T(z)$ since all its conditions are fulfilled by Theorem 1.8.17, Lemma 1.8.22 and Corollary 1.8.25(i). □

Theorem 1.8.27 *(The Lidskii theorem for the Brown measure) Let k be a positive integer, let $T \in \mathcal{L}^k(\mathcal{N}, \tau)$ and let $f(z)$ be a function holomorphic in a neighborhood of $\sigma_T \cup \{0\}$ which vanishes to order k at 0. Then*

$$\tau\left(f(T)\right) = \int_{\sigma_T^*} f(w)\, d\mu_T(w).$$

Proof. A calculation gives

$$(\log g_k(x))' = -\frac{x^{k-1}}{1-x}, \quad x \in \mathbb{R}.$$

So, differentiating (1.101) with respect to $z \in \mathbb{R}$, by Theorem 1.3.34 we obtain, for any $z \in \mathbb{C} \setminus \{0\}$,

$$\tau\left(\frac{T^k}{1-zT}\right) = \int_{\mathbb{C}} \frac{w^k}{1-zw}\, d\mu_T(w),$$

after cancelling z^{k-1} from both sides. (Note that we can differentiate under the integral by Corollary 1.8.25(i) and Lebesgue Dominated Convergence theorem). Thus, we have the theorem for functions of the form

$$g(w) = \frac{w^k}{a-w},$$

where $a \notin \sigma_T \cup \{0\}$. Now, write $f(w) = w^k \tilde{f}(w)$, where $\tilde{f}$ is a uniform limit of linear combinations $\tilde{f}_n$ of functions $\frac{1}{\cdot - w}$ in a neighbourhood of $\sigma_T \cup \{0\}$ (in order to get this representation, just take Riemannian sums of the Cauchy integral for f). Then we have

$$\tau\left(T^k \tilde{f}_n(T)\right) = \int_{\mathbb{C}} w^k \tilde{f}_n(w)\, d\mu_T(w).$$

In order to complete the proof, it remains to be shown that

$$\lim_{n\to\infty} \int_{\mathbb{C}} w^k \tilde{f}_n(w)\, d\mu_T(w) = \int_{\mathbb{C}} w^k \tilde{f}(w)\, d\mu_T(w)$$

and

$$\lim_{n\to\infty} \tau\left(T^k \tilde{f}_n(T)\right) = \tau\left(T^k \tilde{f}(T)\right).$$

The first equality follows from Corollary 1.8.25(i) and the Lebesgue Dominated Convergence theorem. The second equality follows from the fact that $T^k \in \mathcal{L}^1(\mathcal{N}, \tau)$ and $\tilde{f}_n(T)$ converges to $\tilde{f}(T)$ uniformly. □

1.8.7 Additional properties of the Brown measure

Let $f_* \mu_T$ be an inverse image of the Brown measure μ_T, i.e. $f_* \mu_T(B) := \mu_T(f^{-1}(B))$ for any Borel subset B of the complex plane $\mathbb{C}$.

Proposition 1.8.28 [Brn, Theorem 4.1] *Let $T \in \mathcal{L}^p(\mathcal{N}, \tau)$, $p \in (0, \infty)$. If a function $f(z)$ is holomorphic in a neighborhood of the spectrum of T and if $f(0) = 0$ in case of $\tau(1) = \infty$, then*

$$\mu_{f(T)} = f_* \mu_T.$$

We omit the proof of this proposition.

Proposition 1.8.29 *If $T \in \mathcal{L}^p(\mathcal{N}, \tau)$, where $p \in (0, \infty)$, is normal, then for any Borel subset B of $\mathbb{C}$ one has the relation*

$$\mu_T(B) = \tau(\chi_B(T)),$$

where χ_B is the indicator function of the set B.

Proof. Let f be a continuous function on σ_T, vanishing in a neighbourhood of 0. By the Stone-Weierstrass theorem, the function f can be uniformly approximated on σ_T by a sequence of polynomials $\{f_n\}$ vanishing at 0 to order p. For each polynomial f_n, we have by the Brown-Lidskii Theorem 1.8.27

$$\tau(f_n(T)) = \int_{\sigma_T \setminus \{0\}} f_n(w)\, d\mu_T(w),$$

and taking the limit $n \to \infty$ we get $\tau(f(T)) = \int f(w)\, d\mu_T(w)$. This and a standard measure theory argument complete the proof. $\square$

Lemma 1.8.30 *Let $\bar{A} \in 1 + \mathcal{L}^1(\mathcal{N}, \tau)$, let E be a projection from $\mathcal{N}$ and suppose that*

$$\bar{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{[E,E]},$$

where A is invertible in $E\mathcal{N}E$. Then

$$\Delta(\bar{A}) = \Delta(A)\, \Delta(D - CA^{-1}B).$$

Proof. This follows from Proposition 1.7.20 and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{[E,E]} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix}_{[E,E]} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}_{[E,E]}.$$

$\square$

Theorem 1.8.31 *If $S, T \in \mathcal{N}$ and $ST, TS \in \mathcal{L}^p(\mathcal{N}, \tau)$ for some $p \in (0, \infty)$, then*

$$\mu_{ST} = \mu_{TS}.$$

Proof. (A) Let for $\varepsilon > 0$

$$P = E^{|S^*|}_{[0,\varepsilon]}, \qquad Q = E^{|S|}_{[0,\varepsilon]},$$

and let

$$\bar{A} = g_k(ST) = \left(\begin{array}{cc} A & B \\ C & D \end{array} \right)_{[P,P]} \qquad \bar{A}' = g_k(TS) = \left(\begin{array}{cc} A' & B' \\ C' & D' \end{array} \right)_{[Q,Q]}.$$

We need to show that $\Delta\left(g_k(zST)\right) = \Delta\left(g_k(zTS)\right).$ We may absorb z into T, so that it is enough to show

$$\Delta\left(g_k(ST)\right) = \Delta\left(g_k(TS)\right).$$

By Lemma 1.8.30, the last equality will be proved if we show that

$$\Delta(A) = \Delta(A'), \tag{1.102}$$
$$\Delta\left(D - CA^{-1}B\right) = \Delta\left(D' - C'(A')^{-1}B'\right). \tag{1.103}$$

(B) Claim: there exists a sufficiently small $\varepsilon > 0$ such that the operators

$$A = Pg_k(ST)P \qquad \text{and} \qquad A' = Qg_k(TS)Q$$

are invertible in $P\mathcal{N}P$ and $Q\mathcal{N}Q$ respectively.

We can write $g_k(w) = 1 + \sum_{n=k}^{\infty} a_n w^n$, where the power series has radius of convergence ∞. Since $\|PS\|,\ \|SQ\| \leqslant \varepsilon$, for the invertibility of A and A', it is sufficient to choose $\varepsilon > 0$ such that

$$\varepsilon \sum_{n=k}^{\infty} |a_n|\, \|S\|^{n-1}\, \|T\|^n < 1.$$

(C) This will imply $\|A - 1\| < 1$ and hence

$$\log \Delta(A) = \operatorname{Re} \tau\left(\log(A)\right) = \operatorname{Re} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \tau\left((A-1)^m\right).$$

Moreover, the series for g_k can be used to expand $\tau\left((A-1)^m\right).$ Since a similar expansion holds for A', (1.102) will follow from

$$\tau\left(P(ST)^{n_1}P(ST)^{n_2}\dots P(ST)^{n_m}P\right) = \tau\left(Q(TS)^{n_1}Q(TS)^{n_2}\dots Q(TS)^{n_m}Q\right), \tag{1.104}$$

for $n_1, n_2, \dots, n_m \geqslant k$. To prove (1.104) we use Lemma 1.3.32. Here

$$X = PS,$$
$$Y = QT(ST)^{n_1-1}P(ST)^{n_2}\dots P(ST)^{n_m}P,$$

and in order to show that $\tau(XY) =$ the LHS of (1.104) and $\tau(YX) =$ the RHS of (1.104), we use the obvious fact

$$PS = SQ. \tag{1.105}$$

(D) (1.103) will follow from

$$(D - CA^{-1}B)S = S(D' - C'(A')^{-1}B'). \tag{1.106}$$

Indeed, the operator $P^{\perp}S = SQ^{\perp}$ is invertible from $Q^{\perp}\mathcal{H}$ to $P^{\perp}\mathcal{H}$. Hence, (1.106) implies that $(D - CA^{-1}B)$ is similar to $(D' - C'(A')^{-1}B')$, and Proposition 1.7.21 completes the proof of (1.103). Now we must prove (1.106), and we note that it is equivalent to

$$\begin{aligned} &P^{\perp}g_k(ST)P^{\perp}S - P^{\perp}g_k(ST)(Pg_k(ST)P)^{-1}g_k(ST)P^{\perp}S \\ &\quad = SQ^{\perp}g_k(TS)Q^{\perp} - SQ^{\perp}g_k(TS)(Qg_k(TS)Q)^{-1}g_k(TS)Q^{\perp}. \end{aligned} \tag{1.107}$$

Here the inverses are taken relative to $P\mathcal{N}P$ and $Q\mathcal{N}Q$, respectively.

Finally, (1.107) follows from repeated application of (1.105) and

$$g_k(ST)S = Sg_k(TS). \tag{1.108}$$

In particular, we note that (1.105) and (1.108) imply $(Pg_k(ST)P)^{-1}S = S(Qg_k(TS)Q)^{-1}$. $\square$

Lemma 1.8.32 *One has*

$$d\mu_{T^*}(w) = d\mu_T(\bar{w}).$$

Proof. We have $g_k(zT^*) = g_k(\bar{z}T)^*$. This and Lemma 1.7.11 imply

$$\begin{aligned} u_{T^*}(z) &= \log\Delta\left(g_k(zT^*)\right) = \log\Delta\left(g_k(\bar{z}T)^*\right) \\ &= \log\Delta\left(g_k(\bar{z}T)\right) = u_T(\bar{z}). \end{aligned}$$

Hence, $d\mu_{T^*}(w) = d\mu_T(\bar{w})$. $\square$

Proposition 1.8.33 *If $T \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ then $T^* \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ and*

$$\det\nolimits_\tau(1 + T^*) = \overline{\det\nolimits_\tau(1 + T)}.$$

Proof. Theorem 1.8.27 and Lemma 1.8.32 imply that

$$\begin{aligned} \tau\left(\log(1 + T^*)\right) &= \int_{\mathbb{C}} \log(1+\lambda)\, d\mu_{T^*}(\lambda) \\ &= \int_{\mathbb{C}} \log(1+\lambda)\, d\mu_T(\bar{\lambda}) = \int_{\mathbb{C}} \log(1+\bar{\lambda})\, d\mu_T(\lambda) \\ &= \overline{\int_{\mathbb{C}} \log(1+\lambda)\, d\mu_T(\lambda)} = \overline{\tau\left(\log(1+T)\right)}, \end{aligned}$$

so that

$$\begin{aligned}\det_\tau(1+T^*) &= e^{\tau(\log(1+T^*))} = e^{\overline{\tau(\log(1+T))}}\\ &= \overline{e^{\tau(\log(1+T))}} = \overline{\det_\tau(1+T)}.\end{aligned}$$

□

Proposition 1.8.34 *If* $T \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ *and* $S \in GL(\mathcal{N})$ *then* $S^{-1}TS \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ *and*

$$\det_\tau(1+S^{-1}TS) = \det_\tau(1+T).$$

Proof. Theorems 1.8.27 and 1.8.31 imply

$$\begin{aligned}\tau\left(\log(1+S^{-1}TS)\right) &= \int_{\mathbb{C}} \log(1+\lambda)\, d\mu_{S^{-1}TS}(\lambda)\\ &= \int_{\mathbb{C}} \log(1+\lambda)\, d\mu_T(\lambda) = \tau\left(\log(1+T)\right).\end{aligned}$$

□

The last two lemmas can also be proved using Lemma 1.7.2.

Chapter 2

Spectrality of Dixmier trace

2.1 The Dixmier trace in semifinite von Neumann algebras

The Dixmier trace is a specifically constructed example of a non-normal trace on $\mathcal{B}(\mathcal{H})$, which was first discovered by Dixmier in 1966 [Di$_2$]. In particular, the Dixmier trace vanishes on all finite-rank operators. The Dixmier trace has further found various applications. Alain Connes [Co] interpreted the Dixmier trace as a non-commutative integral.

2.1.1 The Dixmier traces in semifinite von Neumann algebras

In this subsection, we follow [Co].

Definition 2.1.1 *The* ***Dixmier ideal*** $\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ *is the set of all operators* $T \in \mathcal{N}$ *such that*

$$\|T\|_{(1,\infty)} := \|T\| + \sup_{t \geqslant 0} \frac{1}{\log(2+t)} \Phi_t(T) < \infty. \tag{2.1}$$

It is clear that if $\|T\|_{(1,\infty)} < \infty$, then $\lim_{t\to\infty} \mu_t(T) = 0$. So, by Lemma 1.3.13,

$$\mathcal{L}^{1,\infty}(\mathcal{N},\tau) \subset \mathcal{K}(\mathcal{N},\tau). \tag{2.2}$$

Proposition 2.1.2 *The set* $\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ *is an invariant operator ideal of* $\mathcal{N}$.

Proof. (A) $(\mathcal{L}^{1,\infty}(\mathcal{N},\tau), \|\cdot\|_{(1,\infty)})$ is a normed linear space.

If $\|T\|_{(1,\infty)} = 0$, then, evidently, $T = 0$. It follows from Lemma 1.3.15(ii), that $\|\alpha T\|_{(1,\infty)} = \alpha \|T\|_{(1,\infty)}$. Finally, by Lemma 1.3.25, $\|S+T\|_{(1,\infty)} \leqslant \|S\|_{(1,\infty)} + \|S\|_{(1,\infty)}$.

(B) It follows from Lemma 1.3.15(i), that if $T \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$, then $T^* \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ and $\|T^*\|_{(1,\infty)} = \|T\|_{(1,\infty)}$.

It follows from Lemma 1.3.19, that if $S, R \in \mathcal{N}$ and $T \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$, then $STR \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$, and, moreover,

$$\|STR\|_{(1,\infty)} \leqslant \|S\| \, \|T\|_{(1,\infty)} \, \|R\| \, .$$

Hence, $\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ is a $*$-ideal.

(C) $(\mathcal{L}^{1,\infty}(\mathcal{N},\tau), \|\cdot\|_{(1,\infty)})$ is complete.

Let $T_1, T_2, \ldots \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ be a Cauchy sequence in the norm $\|\cdot\|_{(1,\infty)}$. Since, $\{T_n\}$ is also a Cauchy sequence in the operator norm, it follows from (2.2) and Lemma 1.3.12 that this sequence converges in the operator norm to a τ-compact operator $T \in \mathcal{N}$.

If in Lemma 1.3.18 we let $s \to 0$, then, by (1.20), for all $t > 0$,

$$|\mu_t(S) - \mu_t(T)| \leqslant \|S - T\| \, .$$

Hence, for all $t > 0$,

$$\limsup_{k\to\infty} |\mu_t(T_k) - \mu_t(T)| \leqslant \limsup_{k\to\infty} \|T_k - T\| = 0,$$

and so, for all $t > 0$, $\lim_{k\to\infty} \mu_t(T_k) = \mu_t(T)$. This implies that, for all $t > 0$,

$$\lim_{k\to\infty} \Phi_t(T_k) = \Phi_t(T). \tag{2.3}$$

If $M \geq 1$, then, for any $k = 1, 2, \ldots,$

$$\sup_{0\leqslant t\leqslant M} \frac{1}{\log(2+t)} \Phi_t(T_k) \leqslant \sup_{n\geq 1} \|T_n\|_{(1,\infty)} < +\infty.$$

Passing to the limit $k \to \infty$ and using (2.3), we obtain

$$\sup_{0\leqslant t\leqslant M} \frac{1}{\log(2+t)} \Phi_t(T) \leqslant \sup_{n\geq 1} \|T_n\|_{(1,\infty)} \, .$$

Since the inequality above holds for every $M \geq 1$, we obtain

$$\sup_{t\geqslant 0} \frac{1}{\log(2+t)} \Phi_t(T) \leqslant \sup_{n\geq 1} \|T_n\|_{(1,\infty)} \, .$$

It follows that $T \in \mathcal{L}^{1,\infty}(\mathcal{H})$.

Now, we show that T_k converges to T in $\|\cdot\|_{(1,\infty)}$ norm. Let $\epsilon > 0$. Let $M \in \mathbb{N}$ be such that for all $k, m \geqslant M$

$$\|T_k - T_m\|_{(1,\infty)} < \frac{\varepsilon}{2}.$$

It follows that, for all $k, m \geqslant M$,

$$\sup_{t \geqslant 0} \frac{1}{\log(2+t)} \Phi_t(T_k - T_m) \leqslant \|T_k - T_m\|_{(1,\infty)} < \frac{\varepsilon}{2}, \tag{2.4}$$

Since, for all $t > 0$,

$$\lim_{m \to \infty} \mu_t(T_k - T_m) = \mu_t(T_k - T),$$

it follows from (2.4), that for every fixed $t > 0$,

$$\frac{1}{\log(2+t)} \Phi_t(T_k - T) = \lim_{m \to \infty} \frac{1}{\log(2+t)} \Phi_t(T_k - T_m) \leqslant \frac{\varepsilon}{2}. \tag{2.5}$$

Therefore,

$$\sup_{t \geqslant 0} \frac{1}{\log(2+t)} \Phi_t(T_k - T) \leqslant \frac{\varepsilon}{2}.$$

If M is large enough, so that for all $k \geqslant M$ $\|T_k - T\| < \frac{\varepsilon}{2}$, then $\|T_k - T\|_{(1,\infty)} < \varepsilon$. $\square$

Remark 2.1.3 *In the case of a general semifinite von Neumann algebra $\mathcal{N}$, the summand $\|T\|$ in the definition (2.1) of the Dixmier norm is necessary to ensure the completeness of the normed space $(\mathcal{L}^{1,\infty}(\mathcal{N}, \tau), \|\cdot\|_{(1,\infty)})$. The norm on the ideal $\mathcal{L}^{1,\infty}(\mathcal{N}, \tau)$, as defined in [CPS, p. 75], is not complete. In the case $\mathcal{N} = \mathcal{B}(\mathcal{H})$ the summand $\|T\|$ in (2.1) can be removed.*

The **dilation operator** D_α, $\alpha \in \mathbb{R}$, is defined on $L^\infty[0, \infty)$ by the formula

$$D_\alpha f(t) = f(e^\alpha t).$$

We recall that a **state** on $L^\infty[0, \infty)$ is a normalized positive linear functional on $L^\infty[0, \infty)$. Let ω be a **dilation-invariant state** on $L^\infty[0, \infty)$, i.e. a state on $L^\infty[0, \infty)$, such that $\omega(D_\alpha f) = \omega(f)$. The value of a dilation-invariant state ω on a bounded function f will also be denoted by

$$\omega(f) = \omega\text{-}\lim_{t \to \infty} f(t).$$

Proposition 2.1.4 *Let ω be a dilation-invariant state on $L^\infty[0, \infty)$ and let $f \in L^\infty[0, \infty)$. If the limit $\lim_{t \to \infty} f(t)$ exists, then it is equal to $\omega(f)$.*

Proof. (A) If $\chi_{[0,a]}$ is the indicator of the interval $[0,a]$, $a > 0$, and if $\alpha \in \mathbb{R}$, then $\omega(\chi_{[0,a]}) = \omega(D_\alpha \chi_{[0,a]})) = \omega(\chi_{[0,e^{-\alpha}a]})$. If e.g. $\alpha > 0$, it follows that $\omega(\chi_{(e^{-\alpha}a,a]}) = 0$. This implies that ω vanishes on compactly supported functions.

(B) Since $\omega(1) = 1$, we may assume that $\lim_{t\to\infty} f(t) = 0$. Let $\varepsilon > 0$. Using (A) and changing f to 0 on a sufficiently large interval $[0,a]$, we may assume that $\|f\|_\infty < \varepsilon$. Positivity of ω implies, that $\omega(f) < \varepsilon$. Hence, $\omega(f) = 0$. □

Corollary 2.1.5 *Any dilation-invariant state vanishes on compactly supported functions.*

Definition 2.1.6 *Let ω be a dilation invariant state on $L^\infty[0,\infty)$. The **Dixmier trace** $\tau_\omega(T)$ of a non-negative operator $T \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ is the number*

$$\tau_\omega(T) := \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(2+t)} \Phi_t(T).$$

It will be shown later, that the Dixmier trace τ_ω can be extended to the whole ideal $\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ as a unitarily invariant non-normal linear functional.

Remark 2.1.7 *Usually, in the definition of the Dixmier ideal and the Dixmier trace, one uses $\frac{1}{\log(1+t)}$ instead of $\frac{1}{\log(2+t)}$. The choice of $\frac{1}{\log(2+t)}$ simplifies some proofs. If T is τ-measurable, but not necessarily bounded, then one can also consider the so-called Dixmier traces at 0 [DPSS, DPSSS, DPSSS$_2$], in which case the usual Dixmier trace is called the Dixmier trace at ∞. For the Dixmier traces at 0, the choice of $\frac{1}{\log(1+t)}$ is essential. The Dixmier traces at 0 have not yet found any applications.*

Proposition 2.1.8 *If $0 \leqslant T \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ and if $\alpha > 0$, then*

$$\tau_\omega(\alpha T) = \alpha \tau_\omega(T).$$

Proof. This follows from Lemma 1.3.15. □

Proposition 2.1.9 *If $0 \leqslant S, T \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$, then*

$$\tau_\omega(S + T) = \tau_\omega(S) + \tau_\omega(T).$$

Proof. Lemma 1.3.25 implies that $\tau_\omega(S + T) \leqslant \tau_\omega(S) + \tau_\omega(T)$.

By Lemma 1.3.28,

$$\frac{1}{\log(2+t)}\Phi_t(S)+\frac{1}{\log(2+t)}\Phi_t(T) \leqslant \frac{\log(2+2t)}{\log(2+t)}\left(\frac{1}{\log(2+2t)}\Phi_{2t}(S+T)\right).$$

Since $\frac{\log(2+2t)}{\log(2+t)}=1+o(1)$, by Proposition 2.1.4 and dilation invariance of ω, it follows that

$$\tau_\omega(S)+\tau_\omega(T)\leqslant\tau_\omega(S+T).$$

□

Definition 2.1.10 *The Dixmier trace of a self-adjoint operator $T\in\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ is $\tau_\omega(T):=\tau_\omega(T_+)-\tau_\omega(T_-)$. The Dixmier trace of an arbitrary operator $T\in\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ is $\tau_\omega(T):=\tau_\omega(\mathrm{Re}(T))+i\tau_\omega(\mathrm{Im}(T))$.*

Evidently, the Dixmier trace, thus defined, is a linear functional on $\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$.

Proposition 2.1.11 *The Dixmier trace τ_ω is a trace on $\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$, i.e. τ_ω is a linear functional such that, for any $T\in\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ and any unitary operator $U\in\mathcal{N}$,*

$$\tau_\omega(UTU^{-1})=\tau_\omega(T).$$

Proof. The linearity has been already proved. The unitary invariance of the Dixmier trace follows immediately from the definition of the Dixmier trace and unitary invariance of generalized singular values $\mu_t(T)$. □

Proposition 2.1.12 *The Dixmier trace τ_ω has the following properties.*
1) For any $T\in\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ and for any $S\in\mathcal{N}$, $\quad\tau_\omega(ST)=\tau_\omega(TS)$.
2) For any $T\in\mathcal{L}^{1}(\mathcal{N},\tau)$, $\quad\tau_\omega(T)=0$.

Proof. 1) Since every operator $S\in\mathcal{N}$ is a linear combination of four unitary operators from $\mathcal{N}$ (see e.g. [RS, VI.6]), it is sufficient to prove the equality $\mathrm{Tr}_\omega(UT)=\mathrm{Tr}_\omega(TU)$ for a unitary U. By Proposition 2.1.11, $\mathrm{Tr}_\omega(UT)=\mathrm{Tr}_\omega(UTUU^{-1})=\mathrm{Tr}_\omega(TU)$.

2) If $T\in\mathcal{L}^1(\mathcal{N},\tau)$, then, by (1.21), for all $t>0$,

$$\Phi_t(T)\leqslant\int_0^\infty\mu_s(T)\,ds=\tau(|T|)<\infty.$$

So, by Proposition 2.1.4, $\mathrm{Tr}_\omega(T)=0$. □

This Proposition also implies that the Dixmier trace is not normal.

2.1.2 Measurability of operators

This subsection is based on [LSS].

Definition 2.1.13 *An operator T from $\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ is said to be* measurable *if its Dixmier trace $\tau_\omega(T)$ does not depend on the state ω.*

For an arbitrary subset $A \subseteq \mathcal{N}$, we denote by A_m the set of measurable elements from A.

Lemma 2.1.14 *The set of measurable operators is a linear space.*

The proof is evident.

Definition 2.1.15 *The set of all operators $T \in \mathcal{N}$, which satisfy*

$$\|T\|_{1,\mathrm{w}} := \sup\{t\mu_t(T) \colon t > 0\} < \infty,$$

form a $*$-ideal in $\mathcal{N}$, denoted by $\mathcal{L}^{1,\mathrm{w}}$.

We note that $\|T\|_{1,\mathrm{w}}$ is not a norm. Evidently, $\mathcal{L}^{1,\mathrm{w}} \subset \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$. The ideal $\mathcal{L}^{1,\mathrm{w}}$ is a little bit smaller than $\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$. At the same time, all natural and interesting examples of operators from $\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ belong also to $\mathcal{L}^{1,\mathrm{w}}$.

Lemma 2.1.16 *Let $\mathcal{N}$ be a semifinite factor. If $T \geqslant 0$ in $\mathcal{L}^{1,\mathrm{w}}$, then there exists $S \in \mathcal{L}^{1,\mathrm{w}}_m$ such that $T \leqslant S$, $\mathrm{supp}(S) \leqslant \mathrm{supp}(T)$ and $ST = TS$.*

Proof. If $\mathcal{N}$ is a type II factor, then the assertion follows from [DDP, Theorem 3.5]. The type I case is straightforward. □

Lemma 2.1.17 *If $T \in \mathcal{L}^{1,\infty}_m(\mathcal{N},\tau)$ then $T^*,\ \mathrm{Re}(T),\ \mathrm{Im}(T) \in \mathcal{L}^{1,\infty}_m(\mathcal{N},\tau)$. The same assertion also holds for $\mathcal{L}^{1,\mathrm{w}}$.*

Proof. Let $T = T_1 - T_2 + iT_3 - iT_4$, where $T_1, \ldots, T_4 \geqslant 0$. Then, by the definition of the Dixmier trace τ_ω, we have $\tau_\omega(T) = \tau_\omega(T_1) - \tau_\omega(T_2) + i\tau_\omega(T_3) - i\tau_\omega(T_4)$ and $\tau_\omega(T^*) = \tau_\omega(T_1) - \tau_\omega(T_2) - i\tau_\omega(T_3) + i\tau_\omega(T_4) = \overline{\tau_\omega(T)}$. The latter equality shows that T^* is measurable provided T is measurable. Since $\mathrm{Re}(T)$ and $\mathrm{Im}(T)$ are linear combinations of T and T^*, the assertions follow. □

Remark 3 *The positive and negative parts of a self-adjoint measurable operator are not necessarily measurable. For example, take a positive non-measurable* *diagonal operator* $A = \operatorname{diag}\{a_1, a_2, a_3, \ldots\}$ *from* $\mathcal{L}^{1,\infty}(\mathcal{H})$. *Then define a diagonal operator* B *by* $B = \operatorname{diag}\{a_1, -a_1, a_2, -a_2, \ldots\}$ *Evidently, this latter sequence is measurable, moreover* $\operatorname{Tr}_\omega(b) = 0$ *for all* ω. *However, the positive and negative parts of* b *are not measurable.*

In this section, we will need two auxiliary theorems.

Theorem 2.1.18 (G. H. Hardy, cf. [Ha, section 6.8]) *Let* $b(t)$ *be a positive piecewise differentiable function such that* $tb'(t) > -H$ *for some* $H > 0$ *and all* $t > C$, *where* C *is a constant. Then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t b(s)\,ds = A \quad \textit{for some} \quad A \geqslant 0 \quad \textit{if and only if} \quad \lim_{t\to\infty} b(t) = A.$$

For any $\alpha \in \mathbb{R}$, a **translation operator** T_α on the algebra $L^\infty[0,\infty)$ is defined by formula

$$T_\alpha f(t) = f(t+\alpha), \quad t \geqslant 0,$$

if $\alpha > 0$, and by formula

$$T_\alpha f(t) = \begin{cases} f(t+\alpha), & \text{if} \quad -\alpha \leqslant t, \\ 0, & \text{if} \quad 0 \leqslant t < -\alpha, \end{cases}$$

if $\alpha < 0$. The set $\{T_\alpha \colon \alpha \in \mathbb{R}\}$ is a group up to functions of compact support.

A state L on the algebra $L^\infty[0,\infty)$ is said to be **translation-invariant**, if $L(T_\alpha f) = L(f)$ for all $\alpha \geqslant 0$. Evidently, if for $f, g \in L^\infty[0,\infty)$ there exists $\alpha > 0$, such that $f\big|_{[\alpha,\infty)} = g\big|_{[\alpha,\infty)}$, then $L(f) = L(g)$ for any translation invariant state L. For this reason, we write

$$L(f) = \operatorname*{L-lim}_{t\to\infty} f(t).$$

Let $A\colon L^\infty[0,\infty) \to L^\infty[0,\infty)$ be an operator, defined by formula $Af(x) = f(\log_+ x)$. The inverse A^{-1} is defined up to functions of compact support. So, up to a functions of compact support,

$$A^{-1}D_\alpha A = T_\alpha.$$

So, if ω is a dilation invariant state on $L^\infty[0,\infty)$, then $L = \omega \circ A$ is translation invariant. Indeed,

$$L \circ T_\alpha = \omega \circ A \circ T_\alpha = \omega \circ D_\alpha \circ A = \omega \circ A = L.$$

Similarly, if L is a translation invariant state, then $\omega = L \circ A^{-1}$ is dilation invariant. This can also be expressed by formula

$$\operatorname*{\omega\text{-}lim}_{t\to\infty} f(t) = \operatorname*{L-lim}_{\lambda\to\infty} f(e^\lambda). \tag{2.6}$$

So, the operator A maps bijectively the set of all dilation invariant states to the set of all translation invariant states.

It is known that the set of translation invariant states on $L^\infty[0,\infty)$ is not empty [Gr]. Hence, on $L^\infty[0,\infty)$ there exists a dilation invariant state (though it follows from the same result from [Gr]).

We recall that a positive function $f \in C_b[0,\infty)$ is said to be **almost convergent** to $A \in \mathbb{C}$, if all translation-invariant states take the same value A on this function.

Theorem 2.1.19 (G. Lorentz, cf. [Lo], [LSS, Theorem 3.3]) *If a function $f \in C_b[0,\infty)$ is almost convergent to a number A then the ordinary limit*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)\,ds$$

exists and is equal to A.

Remark 4 *Actually, the theorem of G. Lorentz says that f is almost convergent to A if and only if $\lim\limits_{t\to\infty} \frac{1}{t}\int_a^{a+t} f(s)\,ds$ exists uniformly with respect to a and is equal to A. But we don't need this.*

The aim of this section is to prove the following theorem.

Theorem 2.1.20 [LSS] *A positive operator T from $\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ is measurable if and only if the limit*

$$\lim_{t\to\infty} \frac{1}{\log(2+t)}\int_0^t \mu_s(T)\,ds$$

exists.

Proof. If $\|T\|_{(1,\infty)} = 0$ then the assertion is evident. So, we assume that $\|T\|_{(1,\infty)} > 0$.

Suppose that for all dilation-invariant states ω on $C_b[0,\infty)$ the Dixmier trace

$$\mathrm{Tr}_\omega(T) := \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(2+t)}\int_0^t \mu_s(T)\,ds$$

exists and is equal to A. By (2.6), this implies that for all translation-invariant states L the limit

$$\mathrm{Tr}_L(T) := \mathrm{L}\text{-}\lim_{\lambda\to\infty} \frac{1}{\log(2+e^\lambda)}\int_0^{e^\lambda} \mu_s(T)\,ds \tag{2.7}$$

exists and is equal to A. Now, Lorentz's theorem (Theorem 2.1.19) implies that the limit

$$\lim_{u\to\infty}\frac{1}{u}\int_0^u\left(\frac{1}{\log(2+e^\lambda)}\int_0^{e^\lambda}\mu_s(T)\,ds\right)d\lambda$$

exists and equal to A. So, according to Hardy's theorem (Theorem 2.1.18), the theorem will be proved if we check that the function

$$b(\lambda):=\frac{1}{\log(2+e^\lambda)}\int_0^{e^\lambda}\mu_s(T)\,ds$$

satisfies the inequality $\lambda b'(\lambda)>-\|T\|_{(1,\infty)}$.

We have

$$\begin{aligned}\lambda b'(\lambda)&=\lambda\frac{d}{d\lambda}\left(\frac{1}{\log(2+e^\lambda)}\int_0^{e^\lambda}\mu_s(T)\,ds\right)\\ &\geqslant\lambda\frac{d}{d\lambda}\left(\frac{1}{\log(2+e^\lambda)}\right)\int_0^{e^\lambda}\mu_s(T)\,ds\\ &=-\frac{\lambda e^\lambda}{(2+e^\lambda)\log^2(2+e^\lambda)}\int_0^{e^\lambda}\mu_s(T)\,ds\\ &\geqslant-\frac{\lambda}{\log(2+e^\lambda)}\cdot\frac{1}{\log(2+e^\lambda)}\int_0^{e^\lambda}\mu_s(T)\,ds\geqslant-\|T\|_{(1,\infty)}\end{aligned}$$

for all $\lambda>0$. So, the theorem is proved. □

2.2 Lidskii formula for Dixmier traces

2.2.1 Spectral characterization of sums of commutators

This subsection is based on the work [DK_2]. In this subsection, we assume that $\mathcal{N}$ is a semifinite factor.

Proposition 2.2.1 [DK_2, Proposition 6.5] *If $T\in\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ then there exists a normal operator $S\in\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ with the same Brown measure as that of T.*

The following theorem is a very deep result, due to K. J. Dykema, T. Fack and N. J. Kalton.

Theorem 2.2.2 [DK$_2$, Theorem 6.8] *An operator $T \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ can be represented as finite linear combination of commutators $[A_j, S_j]$ with $A_j \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ and $S_j \in \mathcal{N}$, $j = 1,\dots,N$, if and only if there is a positive operator $V \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ such that for all $r > 0$*

$$\left| \int_{\{z\colon r<|z|\}} z\, d\mu_T(z) \right| \leqslant r\tau(E^V_{(r,\infty)}). \tag{2.8}$$

Theorem 2.2.3 [N. J. Kalton] *If $S \in \mathcal{L}^{1,\mathrm{w}}$ then there exists a normal operator $T \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ such that the Brown spectral measures of S and T coincide and $\tau_\omega(S) = \tau_\omega(T)$.*

Proof. Let T be a normal operator with the same Brown measure as that of S, which exists according to Proposition 2.2.1. Consider the operator

$$A := \begin{pmatrix} T & 0 \\ 0 & -S \end{pmatrix}.$$

From the definition of the Brown measure and Proposition 1.7.20, it follows that the Brown measure of this operator is $\mu_A = \mu_T + \mu_{-S}$. As follows from the definition of the Brown measure, we have $\mu_{-S}(z) = \mu_S(-z)$ (this also follows from Proposition 1.8.28). Hence,

$$\int_{\{z\colon r<|z|\}} z\, d\mu_A(z) = \int_{\{z\colon r<|z|\}} z\, d\mu_T(z) - \int_{\{z\colon r<|z|\}} z\, d\mu_S(z) = 0.$$

Theorem 2.2.2 implies that A can be represented as linear combination of commutators of the form $[A_i, B_i]$ with $A_i \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ and $B_i \in \mathcal{N}$. By Proposition 2.1.12, it follows that $\tau_\omega(A) = 0$. Since $\tau_\omega(A) = \tau_\omega(T) - \tau_\omega(S)$, it follows that $\tau_\omega(S) = \tau_\omega(T)$. $\square$

2.2.2 The Lidskii formula for the Dixmier trace

In this subsection, we assume that $\mathcal{N}$ is a semifinite factor. The aim of this subsection is to prove the Lidskii formula for the Dixmier trace in case of normal operators. The main idea of the proof is that, in the definition of the Dixmier trace, $\Phi_t(T)$ can be replaced by $\Psi_t(T)$, introduced below.

For arbitrary operators the result (Theorem 2.2.11) follows from Dykema-Fack-Kalton theorem (Theorem 2.2.3).

Lemma 2.2.4 *If $T \in \mathcal{L}^{1,\mathrm{w}}$, then for all $t > 0$ $\lambda_{1/t}(T) \leqslant \|T\|_{1,\mathrm{w}}\, t$.*

Proof. Let $M := \|T\|_{1,\mathrm{w}}$. Since for all $t > 0$ $\mu_t(T) \leqslant \frac{M}{t}$, it follows from (1.23) that $t \geqslant \lambda_{M/t}$ for all $t > 0$. Replacing t with Mt, one gets $Mt \geqslant \lambda_{1/t}$ for all $t > 0$. $\square$

Let
$$\Psi_t(T) = \int_0^{\lambda_{1/t}} \mu_s(T)\, ds, \quad t > 0.$$

Proposition 2.2.5 *If* $0 \leqslant T \in \mathcal{L}^{1,\mathrm{w}}$, *then*
$$|\Psi_t(T) - \Phi_t(T)| \leqslant \|T\|_{1,\mathrm{w}} \log \|T\|_{1,\mathrm{w}}.$$
Consequently,
$$\lim_{t\to\infty} \frac{1}{\log(2+t)} |\Psi_t(T) - \Phi_t(T)| = 0.$$

Proof. Again, let $M := \|T\|_{1,\mathrm{w}}$.

If $t \geqslant \lambda_{1/t}(T)$, then, since by (1.23) $s \geqslant \lambda_{1/t}(T)$ if and only if $\mu_s(T) \leqslant \frac{1}{t}$, we have
$$|\Psi_t(T) - \Phi_t(T)| = \int_{\lambda_{1/t}(T)}^{t} \mu_s(T)\, ds \leqslant (t - \lambda_{1/t}(T)) \cdot \frac{1}{t} \leqslant 1.$$
If $t < \lambda_{1/t}(T)$, then, by Lemma 2.2.4, for all $t > 0$,
$$\begin{aligned} |\Psi_t(T) - \Phi_t(T)| &= \int_t^{\lambda_{1/t}(T)} \mu_s(T)\, ds \leqslant M \int_t^{\lambda_{1/t}(T)} \frac{ds}{s} \\ &\leqslant M \int_t^{Mt} \frac{ds}{s} = M \log M. \end{aligned}$$

$\square$

Lemma 2.2.6 *If* $T \geqslant 0$ *in* $\mathcal{L}^{1,\mathrm{w}}$ *then*
$$\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(2+t)} \Psi_t(T). \tag{2.9}$$
If T *is measurable, then the* ω*-limit can be replaced with the true limit.*

Proof. The equality (2.9) follows from Proposition 2.2.5. The second assertion follows Theorem 2.1.20 and Proposition 2.2.5. $\square$

Lemma 2.2.7 *If* $A, B, C \geqslant 0$ *in* $\mathcal{L}^{1,\mathrm{w}}$ *and* $C = A + B$ *then*
$$\lim_{t\to\infty} \frac{1}{\log(2+t)} |\Phi_t(A) + \Phi_t(B) - \Phi_t(C)| = 0 \tag{2.10}$$

and

$$\lim_{t\to\infty} \frac{1}{\log(2+t)} \left|\Psi_t\left(A\right) + \Psi_t\left(B\right) - \Psi_t\left(C\right)\right| = 0. \tag{2.11}$$

Proof. Let $A' \geqslant 0$ and $B' \geqslant 0$ be operators from $\mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ such that $\mu_t(A') = \mu_t(A)$, $\mu_t(B') = \mu_t(B)$ $\forall t > 0$ and $A'B' = 0$. Let $C' = A' + B'$.

Since A' and B' are orthogonal, the formula (1.24) implies

$$\Psi_t\left(A'\right) + \Psi_t\left(B'\right) - \Psi_t\left(C'\right) = 0. \tag{2.12}$$

By Proposition 2.2.5, it follows that

$$\lim_{t\to\infty} \frac{1}{\log(2+t)} \left|\Phi_t\left(A'\right) + \Phi_t\left(B'\right) - \Phi_t\left(C'\right)\right| = 0. \tag{2.13}$$

Lemmas 1.3.27 and 1.3.25 imply that, for all $t > 0$,

$$\Phi_t(C') \leqslant \Phi_t(C) \leqslant \Phi_t(A) + \Phi_t(B) = \Phi_t(A') + \Phi_t(B').$$

This and (2.13) imply (2.10).

The formula (2.11) follows from (2.10) and Proposition 2.2.5. □

Lemma 2.2.8 *Let $T \in \mathcal{L}^{1,\mathrm{w}}$ be normal and let $T = T_1 - T_2 + iT_3 - iT_4$, where $T_1, \ldots, T_4 \geqslant 0$. Then*

$$\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(2+t)} \left(\Psi_t\left(T_1\right) - \Psi_t\left(T_2\right) + i\Psi_t\left(T_3\right) - i\Psi_t\left(T_4\right)\right).$$

If T is measurable, then the ω-limit can be replaced with the true limit.

Proof. The first assertion follows from Lemma 2.2.6 and the linearity of Dixmier traces.

Let T be measurable and self-adjoint. By Lemma 2.1.16, there exists a non-negative measurable operator $S \in \mathcal{L}^{1,\mathrm{w}}$, commuting with T_-, such that $S - T_- \geqslant 0$ and $\mathrm{supp}(S) \leqslant \mathrm{supp}(T_-)$. Since

$$0 \leqslant S,\ T + S \in \mathcal{L}^{1,\mathrm{w}}_m, \tag{2.14}$$

Lemma 2.2.6 implies

$$\begin{aligned}\tau_\omega(T) &= \tau_\omega(S+T) - \tau_\omega(S)\\ &= \lim_{t\to\infty} \frac{1}{\log(2+t)} \Psi_t\left(S+T\right) - \lim_{t\to\infty} \frac{1}{\log(2+t)} \Psi_t\left(S\right).\end{aligned}$$

Since $S+T=(S-T_-)+T_+$ and the operators $S-T_-$ and T_+ are disjoint, by (2.12) we have $\Psi_t(S+T)=\Psi_t(S-T_-)+\Psi_t(T_+)$. This and the formula (2.11) of Lemma 2.2.7 with $A=T_-$, $B=S-T_-$ and $C=S$ imply that

$$\lim_{t\to\infty}\frac{1}{\log(2+t)}\big(\Psi_t(T_+)-\Psi_t(T_-)-\Psi_t(S+T)+\Psi_t(S)\big)=0.$$

Taking the ω-limit, we conclude from Lemma 2.2.6 and (2.14) that

$$\begin{aligned}\tau_\omega(T) &= \omega\text{-}\lim_{t\to\infty}\frac{\Psi_t(T_+)-\Psi_t(T_-)}{\log(2+t)}=\omega\text{-}\lim_{t\to\infty}\frac{\Psi_t(S+T)-\Psi_t(S)}{\log(2+t)}\\ &= \lim_{t\to\infty}\frac{\Psi_t(S+T)-\Psi_t(S)}{\log(2+t)}=\lim_{t\to\infty}\frac{\Psi_t(T_+)-\Psi_t(T_-)}{\log(2+t)}. \qquad (2.15)\end{aligned}$$

If T is normal, the assertion now follows from Lemma 2.1.17. □

Lemma 2.2.9 *If $T\in\mathcal{L}^{1,\mathrm{w}}$ is normal and $a>0$ then*

$$\tau_\omega(T)=\omega\text{-}\lim_{t\to\infty}\frac{1}{\log(2+t)}\int_{\lambda\notin Q_t}\lambda\,d\mu_T(\lambda),$$

where $Q_t=\{x+iy\in\mathbb{C}:\ |tx|\leqslant a,|ty|\leqslant a\}\ \forall\,t>0$. If T is measurable, then the ω-limit can be replaced with the true limit.

Proof. We may take $a=1$, by dilation invariance of ω. Let $T=T_1-T_2+iT_3-iT_4$, where $T_1,\dots,T_4\geqslant 0$. For $T\geqslant 0$ in $\mathcal{L}^{1,\mathrm{w}}$, it follows from (1.24) and Proposition 1.8.29 that

$$\Psi_t(T)=\int_{1/t}^{\infty}\lambda\,d\mu_T(\lambda). \qquad (2.16)$$

Let $\bar{A}$ be the complement of $A\subset\mathbb{C}$, $R_t:=\{\lambda\in\mathbb{C}\colon\ |\mathrm{Re}\,\lambda|\leqslant 1/t\}$ and $I_t:=\{\lambda\in\mathbb{C}\colon\ |\mathrm{Im}\,\lambda|\leqslant 1/t\}$. For any Borel set $B\subseteq\mathbb{R}$, we have

$$\begin{aligned}\int_B\lambda\,d\mu_{\mathrm{Re}(T)}(\lambda)&=\int_{\{\lambda\colon\ \mathrm{Re}(\lambda)\in B\}}\mathrm{Re}(\lambda)\,d\mu_T(\lambda),\\ \int_B\lambda\,d\mu_{\mathrm{Im}(T)}(\lambda)&=\int_{\{\lambda\colon\ \mathrm{Im}(\lambda)\in B\}}\mathrm{Im}(\lambda)\,d\mu_T(\lambda)\end{aligned}$$

and so

$$\begin{aligned}\int_{\bar{Q}_t}\lambda\,d\mu_T(\lambda)&=\int_{\bar{Q}_t}\mathrm{Re}(\lambda)\,d\mu_T(\lambda)+i\int_{\bar{Q}_t}\mathrm{Im}(\lambda)\,d\mu_T(\lambda)\\ &=\int_{\bar{R}_t}\mathrm{Re}(\lambda)\,d\mu_T(\lambda)+\int_{\bar{Q}_t\cap R_t}\mathrm{Re}(\lambda)\,d\mu_T(\lambda)\\ &\qquad+i\int_{\bar{I}_t}\mathrm{Im}(\lambda)\,d\mu_T(\lambda)+i\int_{\bar{Q}_t\cap I_t}\mathrm{Im}(\lambda)\,d\mu_T(\lambda)\\ &=\int_{\{|\xi|>1/t\}}\xi\,d\mu_{\mathrm{Re}(T)}(\xi)+\int_{\bar{Q}_t\cap R_t}\mathrm{Re}(\lambda)\,d\mu_T(\lambda)\\ &\qquad+i\int_{\{|\xi|>1/t\}}\xi\,d\mu_{\mathrm{Im}(T)}(\xi)+i\int_{\bar{Q}_t\cap I_t}\mathrm{Im}(\lambda)\,d\mu_T(\lambda).\end{aligned}$$

By Lemma 2.2.8 and (2.16), the sum of the first and the third terms in the expression above gives $\tau_\omega(T)$ after dividing by $\log(2+t)$ and taking the ω-limit with respect to $t \to \infty$. If T is measurable, then, by Lemma 2.2.8, we may take the ordinary limit.

So, to complete the proof it suffices to show that

$$\lim_{t\to\infty} \frac{1}{\log(2+t)} \int_{\bar{Q}_t \cap R_t} \mathrm{Re}(\lambda)\, d\mu_T(\lambda) = 0$$

and

$$\lim_{t\to\infty} \frac{1}{\log(2+t)} \int_{\bar{Q}_t \cap R_t} \mathrm{Im}(\lambda)\, d\mu_T(\lambda) = 0.$$

It is enough to prove the first equality, the second is proved analogously. In fact, it suffices to prove that

$$\lim_{t\to\infty} \frac{1}{\log(2+t)} \int_{\{\mathrm{Im}(\lambda)>1/t\}\cap R_t} \mathrm{Re}(\lambda)\, d\mu_T(\lambda) = 0.$$

We have

$$\begin{aligned}
\left| \int_{\{\mathrm{Im}(\lambda)>1/t\}\cap R_t} \mathrm{Re}(\lambda)\, d\mu_T(\lambda) \right| &\leqslant \frac{1}{t} \int_{\{\mathrm{Im}(\lambda)>1/t\}\cap R_t} d\mu_T(\lambda) \\
&\leqslant \frac{1}{t} \int_{\{\mathrm{Im}(\lambda)>1/t\}} d\mu_T(\lambda) = \frac{1}{t} \int_{1/t}^{\infty} d\mu_{\mathrm{Im}(T)}(\lambda) \\
&= \frac{1}{t} \tau\left(\chi_{(1/t,\infty)}(T_3)\right) = \frac{1}{t} \lambda_{1/t}(T_3) \leqslant C.
\end{aligned}$$

The last inequality follows from the equivalence of $\mu_{Ct}(T_3) \leqslant 1/t$ and $\lambda_{1/t}(T_3) \leqslant Ct$, see (1.23). $\Box$

Lemma 2.2.10 *Let T be a normal operator from $\mathcal{L}^{1,\mathrm{w}}$ and let G be a bounded Borel neighborhood of $0 \in \mathbb{C}$. Setting $t \in \mathbb{R}$ let $G_t := \{z \in \mathbb{C} : tz \in G\},\ t > 0$, we have*

$$\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(2+t)} \int_{\lambda \notin G_t} \lambda\, d\mu_T(\lambda).$$

If T is measurable, then the ω-limit can be replaced with the true limit.

Proof. For an arbitrary bounded neighborhood G of $0 \in \mathbb{C}$ there exist squares Q_a and Q_b such that $Q_a \subseteq G \subseteq Q_b$. Hence, Lemma 2.2.9 implies that it is sufficient to prove that

$$\lim_{t\to\infty} \frac{1}{\log(2+t)} \int_{Q_{t/b} \setminus Q_{t/a}} |\lambda|\, d\mu_T(\lambda) = 0. \tag{2.17}$$

The set $Q_{t/b}\backslash Q_{t/a}$ consists of four trapeziums and it is suffices to prove the above limit for one of them, $D_t := \left\{z \in Q_{t/b} \setminus Q_{t/a} \colon \ \mathrm{Re}(tz) \in [a,b]\right\}$, for example. We have

$$\begin{aligned}\frac{1}{2}\int_{D_t} |\lambda|\, d\mu_T(\lambda) \leqslant \int_{D_t} \mathrm{Re}(\lambda)\, d\mu_T(\lambda) \leqslant \int_{t\,\mathrm{Re}(\lambda)\in[a,b]} \mathrm{Re}(\lambda)\, d\mu_T(\lambda) \\ = \int_{a/t}^{b/t} \lambda\, d\mu_{\mathrm{Re}(T)}(\lambda) = \int_0^{\lambda_{a/t}} \mu_s\, ds - \int_0^{\lambda_{b/t}} \mu_s\, ds.\end{aligned}$$

By Lemma 2.2.6, we can replace upper limits $\lambda_{a/t}$ and $\lambda_{b/t}$ by t/a and t/b respectively. Then

$$\int_0^{t/a} \mu_s\, ds - \int_0^{t/b} \mu_s\, ds \leqslant \int_{t/b}^{t/a} C/s\, ds \leqslant C \log \frac{b}{a}.$$

□

The following theorem is the main result of this section.

Theorem 2.2.11 *If $S \in \mathcal{L}^{1,\mathrm{w}}$ and G is a bounded Borel neighborhood of $0 \in \mathbb{C}$, then*

$$\tau_\omega(S) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(2+t)} \int_{\lambda\notin G_t} \lambda\, d\mu_S(\lambda).$$

If S is measurable, then the ω-limit can be replaced with the true limit.

Proof. According to Theorem 2.2.3, there exists a normal operator $T \in \mathcal{L}^{1,\infty}(\mathcal{N},\tau)$ with the same Brown measure and Dixmier trace. Hence, the Lemma 2.2.10 implies

$$\begin{aligned}\tau_\omega(S) = \tau_\omega(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(2+t)} \int_{\lambda\notin G_t} \lambda\, d\mu_T(\lambda) \\ = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(2+t)} \int_{\lambda\notin G_t} \lambda\, d\mu_S(\lambda).\end{aligned}$$

If S is measurable then T is also measurable by Theorem 2.2.3. The second assertion now follows from Lemma 2.2.10. □

Corollary 2.2.12 *Let $S, T \in \mathcal{N}$ be such that $ST, TS \in \mathcal{L}^{1,\mathrm{w}}$. Then*

$$\tau_\omega(ST) = \tau_\omega(TS).$$

Proof. It follows directly from Theorems 1.8.31, 2.2.11, noting that $\mathcal{L}^{1,\mathrm{w}} \subset \mathcal{L}^{1+\varepsilon}(\mathcal{N},\tau)$. □

We specialize Theorem 2.2.11 to the case $\mathcal{N} = \mathcal{B}(\mathcal{H})$.

Corollary 2.2.13 *Let T be a compact operator on an infinite-dimensional Hilbert space $\mathcal{H}$ such that $\mu_n(T) \leqslant C/n,\ n \geqslant 1$ for some $C > 0$. If $\lambda_1, \lambda_2, \dots$ is the list of eigenvalues of T counting the multiplicities such that $|\lambda_1| \geqslant |\lambda_2| \geqslant \dots$, then*

$$\begin{aligned}\mathrm{Tr}_\omega(T) &= \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(2+t)} \sum_{\lambda\in\sigma(T),\ \lambda\notin G_t} \lambda\mu_T(\lambda) \\ &= \omega\text{-}\lim_{N\to\infty} \frac{1}{\log N} \sum_{i=1}^{N} \lambda_i,\end{aligned}$$

where $\mu_T(\lambda)$ is the algebraic multiplicity of the eigenvalue λ. If T is measurable, then the ω-limit can be replaced with the true limit.

Proof. The first equality is an immediate consequence of Theorem 2.2.11. By Theorem 2.2.3, it is sufficient to prove the second equality for a normal operator T. Let $G := \{z \in \mathbb{C} : |z| < 1\}$. It is enough to show that $\sum\limits_{k\in A_N\cup B_N} |\lambda_k| <$ const, where $A_N = \{k \in \mathbb{N} : k \leqslant N, |\lambda_k| \leqslant 1/N\}$ and $B_N = \{k \in \mathbb{N} : k > N, |\lambda_k| > 1/N\}$. We have, $\sum\limits_{k\in A_N} |\lambda_k| \leqslant 1$. That $\sum\limits_{k\in B_N} |\lambda_k|$ is bounded follows from the condition $|\lambda_k| < C/k,\ k \in \mathbb{N}$, for some $C > 0$ and estimate (2.17). □

The following corollary follows from the combination of Corollary 2.2.13 and [Co, Prop. IV.2.5]

Corollary 2.2.14 [Fac, Prop 1] *If M is a compact Riemannian n-manifold and T is a pseudo-differential operator of order $-n$ on M, then*

$$\mathrm{Tr}_\omega(T) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{k=1}^{N} \lambda_k.$$

Chapter 3

Spectral shift function in von Neumann algebras

3.1 Spectral shift function for trace class perturbations

As usual, we denote by $\mathcal{N}$ a semifinite von Neumann algebra $\mathcal{N}$ equipped with normal faithful semifinite trace τ, acting in Hilbert space $\mathcal{H}$.

We recall that (Definition 1.7.1)

$$\mathcal{L}^{1,\pi}(\mathcal{N},\tau) = \left\{T \in \mathcal{L}^1(\mathcal{N},\tau) \colon \sigma_T \cap (-\infty,-1] = \varnothing\right\}.$$

We denote by log the single valued branch of the logarithm in $\mathbb{C}\setminus(-\infty,0]$ which takes value 0 at 1. If $T \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$, we define, via the standard Riesz-Dunford functional calculus (see, for example, [Ta, Proposition I.2.7]),

$$\log(1+T) := \frac{1}{2\pi i}\int_\gamma \log(1+\lambda) R_\lambda(T)\, d\lambda \in \mathcal{N}$$

where γ is any positively oriented, simple closed curve in $\mathbb{C}\setminus(-\infty,-1]$ containing σ_T in its interior. On the other hand, following [GK, Chapter IV.1], observe that

$$R_\lambda(T) = 1/\lambda + (1/\lambda) T R_\lambda(T), \quad 0 \neq \lambda \notin \sigma_T. \tag{3.1}$$

Since $T \in \mathcal{L}^1(\mathcal{N},\tau)$, the resolvent equation implies that the function

$$\lambda \in \mathbb{C}\setminus(-\infty,-1] \mapsto \frac{\log(1+\lambda)}{\lambda} T R_\lambda(T) \in \mathcal{L}^1(\mathcal{N},\tau)$$

is $\|\cdot\|_{1,\infty}$ continuous. Now suppose that γ is any positively oriented, simple closed curve in $\mathbb{C} \setminus (-\infty, -1]$ containing $\sigma_T \cup \{0\}$ in its interior. Using (3.1) and Cauchy's theorem, it follows that

$$\begin{aligned}\log(1+T) &= \frac{1}{2\pi i}\int_\gamma \frac{\log(1+\lambda)}{\lambda}\,d\lambda + \frac{1}{2\pi i}\int_\gamma \frac{\log(1+\lambda)}{\lambda} T R_\lambda(T)\,d\lambda \\ &= \frac{1}{2\pi i}\int_\gamma \frac{\log(1+\lambda)}{\lambda} T R_\lambda(T)\,d\lambda\end{aligned}$$

Since the integral on the right exists in the norm $\|\cdot\|_{1,\infty}$, it follows immediately that if $T \in \mathcal{L}^{1,\pi}(\mathcal{N}, \tau)$ then

$$\log(1+T) \in \mathcal{L}^1(\mathcal{N}, \tau). \tag{3.2}$$

We note that if $T \in \mathcal{L}^{1,\pi}(\mathcal{N}, \tau)$ and $\|T\| < 1$, then the usual power series expansion

$$\log(1+T) = \sum_{k=1}^{\infty} (-1)^{k+1} T^k / k$$

is valid with convergence in the norm $\|\cdot\|_{1,\infty}$. This equality in the operator norm is given, for example, in [Ta, Chapter 1.2], while convergence of the series in the norm $\|\cdot\|_1$ follows simply by observing that

$$\left\| \sum_{k=N}^{\infty} (-1)^{k+1} T^k / k \right\|_1 \leqslant \|T\|_1 \, (\sum_{k=N}^{\infty} \|T\|^{k-1} / k)$$

for all $N \in \mathbb{N}$.

We shall need the following representation theorem from complex function theory which is given in [Ya, Theorem 1.2.9 and Corollary 10].

Theorem 3.1.1 *Suppose that F is holomorphic in the open upper half-plane $\mathbb{C}_+$. If $\operatorname{Im} F$ is bounded and nonnegative (or non-positive) and if*

$$\sup_{y \geqslant 1} y \ |F(iy)| < \infty$$

then there exists a nonnegative (respectively, non-positive) real function $\xi \in L^1(\mathbb{R})$ such that

$$F(z) = \int_{-\infty}^{\infty} \frac{\xi(\lambda)\,d\lambda}{\lambda - z}, \quad \operatorname{Im} z > 0.$$

The function ξ is uniquely determined by the inversion formula

$$\xi(\lambda) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \operatorname{Im} F(\lambda + i\epsilon), \quad a.\,e.\ \lambda \in \mathbb{R}.$$

We also need the following simple uniqueness result. We indicate the proof for lack of convenient reference.

Proposition 3.1.2 *If $\xi_1, \xi_2 \in L^1(\mathbb{R})$ are real-valued and if*

$$\int_{-\infty}^{\infty} \frac{\xi_1(\lambda)\, d\lambda}{(\lambda - z)^2} = \int_{-\infty}^{\infty} \frac{\xi_2(t)\, d\lambda}{(\lambda - z)^2}, \quad \operatorname{Im} z > 0, \tag{3.3}$$

then $\xi_1 = \xi_2$.

Proof. We observe that equality (3.3) may be written in the form

$$\frac{d}{dz} \int_{-\infty}^{\infty} \frac{\xi_1(\lambda)\, d\lambda}{\lambda - z} = \frac{d}{dz} \int_{-\infty}^{\infty} \frac{\xi_2(\lambda)\, d\lambda}{\lambda - z}, \quad \operatorname{Im} z > 0. \tag{3.4}$$

If

$$F_i(z) := \int_{-\infty}^{\infty} \frac{\xi_i(\lambda)\, d\lambda}{\lambda - z}, \quad i = 1, 2, \quad \operatorname{Im} z \neq 0,$$

then

$$\sup_{y > 0} y\, |F_i(iy)| \leqslant \|\xi_i\|_1, \quad i = 1, 2.$$

It follows that

$$\lim_{y \to \infty} F_i(iy) = 0, \; i = 1, 2. \tag{3.5}$$

It now follows from (3.4) and (3.5) that $F_1 = F_2$. Using standard properties of the Poisson kernel [Ga] together with the fact that the functions $\xi_i, \; i = 1, 2$ are real-valued, it follows that

$$\xi_1 = \xi_2 = \lim_{\epsilon \to 0^+} \operatorname{Im} F_1(\cdot + i\epsilon) \; (= \lim_{\epsilon \to 0^+} \operatorname{Im} F_2(\cdot + i\epsilon))$$

where the limit taken in the norm of $L^1(\mathbb{R})$. $\square$

3.1.1 Krein's trace formula: resolvent perturbations

Throughout this subsection, we will denote by H a self-adjoint operator affiliated with $\mathcal{N}$, and by V a bounded self-adjoint operator in $\mathcal{L}^1(\mathcal{N}, \tau)$.

Proposition 3.1.3 *Suppose that $z \in \mathbb{C}$, that $\operatorname{Im} z > 0$ and set*

$$X := \left\{ \lambda \in \mathbb{C} \colon \left| \lambda + i \frac{\|V\|}{2 \operatorname{Im} z} \right| \leqslant \frac{\|V\|}{2\, |\operatorname{Im} z|} \right\}.$$

If $V \geqslant 0$, then $\sigma_{R_z(H)V} \subseteq X$ and if $V \leqslant 0$ then $\sigma_{R_z(H)V} \subseteq -X$.

Proof. Suppose first that $V \geqslant 0$. By [Ta, Proposition I.2.1], it follows that

$$\sigma_{R_z(H)V} \cup \{0\} = \sigma_{V^{1/2} R_z(H) V^{1/2}} \cup \{0\},$$

and so it suffices to show that

$$\sigma_{V^{1/2}R_z(H)V^{1/2}} \subseteq X.$$

Now observe that

$$\begin{aligned} W(V^{1/2}R_z(H)V^{1/2}) &= \left\{\langle R_z(H)V^{1/2}\xi, V^{1/2}\xi\rangle : \xi \in \mathcal{H}, \|\xi\| = 1\right\} \\ &\subseteq [0, \|V\|]W(R_z(H)). \end{aligned} \tag{3.6}$$

Since $R_z(H)$ is normal, and using the spectral mapping theorem, it follows from Theorem 1.1.11 that

$$W(R_z(H)) \subseteq \text{conv } \sigma_{R_z(H)} = \text{conv}\left\{(z-\lambda)^{-1} : \lambda \in \sigma_H\right\}. \tag{3.7}$$

Since by Theorem 1.1.10

$$\sigma_{V^{1/2}R_z(H)V^{1/2}} \subseteq \overline{W(V^{1/2}R_z(H)V^{1/2})},$$

the assertion of the Lemma for the case that $V \geqslant 0$ now follows from (3.6) and (3.7). If $V \leqslant 0$, we set $W = -V$ so that $W \geqslant 0$. From what has just been proved, it follows that $\sigma_{R_z W} \subseteq X$ and so $\sigma_{R_z V} = -\sigma_{R_z W} \subseteq -X$. This completes the proof of the Lemma. □

We note that, in particular, it follows that if $V \geqslant 0$ then $1 \pm R_z(H)V$ is invertible. We shall use this fact repeatedly below without further reference.

Remark 3.1.4 *1) Note, that in the proof of the last proposition we didn't use the fact that V is τ-trace class.*

2) In the case of bounded *self-adjoint operator H and (not necessarily positive or negative) bounded self-adjoint operator V, it follows from [W, Theorem 1] that the spectrum of $R_z(H)V$ is a subset of $X_+ \cup X_-$, which is enough to define* $\log(1 + R_z(H)V)$.

3) **Open problem.** *Prove that the spectrum of $R_z(H)V$ is a subset of $X_+ \cup X_-$ for any self-adjoint H and any bounded self-adjoint V.*

A positive solution of this problem would enable one to simplify the following theory of SSF a little bit.

Using (3.2) and the fact that $X \cap (-\infty, -1] = \varnothing$, we obtain the following result.

Corollary 3.1.5 *If $0 \leqslant V \in \mathcal{L}^1(\mathcal{N}, \tau)$, if $H = H^*$ is affiliated with $\mathcal{N}$ and if $z \in \mathbb{C} \setminus \mathbb{R}$ then $\pm R_z(H)V \in \mathcal{L}^{1,\pi}(\mathcal{N}, \tau)$ and $\log(1 \pm R_z(H)V) \in \mathcal{L}^1(\mathcal{N}, \tau)$.*

We now prove the following

Proposition 3.1.6 *If $V \in \mathcal{L}^1(\mathcal{N},\tau)$ and $H = H^*$ is affiliated with $\mathcal{N}$, then*

$$R_z(H+V) - R_z(H) \in \mathcal{L}^1(\mathcal{N},\tau), \quad \operatorname{Im} z \neq 0.$$

Further, if either $V \geqslant 0$ or $V \leqslant 0$ and if

$$F(z) := \tau\left(\log(1 - R_z(H)V)\right), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

then F is holomorphic in $\mathbb{C} \setminus \mathbb{R}$ and

$$\frac{d}{dz}F(z) = \tau\left(R_z(H+V) - R_z(H)\right), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Proof. The first assertion follows from the second resolvent identity (1.2). We now assume that $V \geqslant 0$, since the case that $V \leqslant 0$ is identical. Since the limit

$$\frac{d}{dz}(-R_z(H)V) = R_z(H)^2 V, \quad z \in \mathbb{C} \setminus \mathbb{R}$$

exists in the norm $\|\cdot\|_{1,\infty}$, and since the trace τ is a continuous linear functional on $\mathcal{L}^1(\mathcal{N},\tau)$, precisely the same argument as in [GK, Chapter IV, (1.14)] shows that the function F is holomorphic in $\mathbb{C} \setminus \mathbb{R}$ and that for $z \in \mathbb{C} \setminus \mathbb{R}$

$$\begin{aligned}\frac{d}{dz}F(z) &= \tau\left((1 - R_z(H)V)^{-1}\frac{d}{dz}(-R_z(H)V)\right) \\ &= \tau\left((1 - R_z(H)V)^{-1}R_z(H)^2 V\right).\end{aligned}$$

We now observe that for $z \in \mathbb{C} \setminus \mathbb{R}$

$$\begin{aligned}\tau\left((1 - R_z(H)V)^{-1}R_z(H)^2 V\right) &= \tau\left(R_z(H+V)R_z(H)V\right) \\ &= \tau\left(R_z(H)VR_z(H+V)\right) \\ &= \tau\left(R_z(H+V) - R_z(H)\right),\end{aligned}$$

and this completes the proof. □

Proposition 3.1.7 *If $V \in \mathcal{L}^1(\mathcal{N},\tau)$ and if $H = H^*$ is affiliated with $\mathcal{N}$, then*

$$\tau(V) = \lim_{y \to \pm\infty} iy\tau\big(\log(1 + R_{iy}(H)V)\big).$$

Proof. We note first, via the spectral theorem, that

$$\|R_z(H)\| \leqslant 1/\left|\operatorname{Im} z\right|, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Consequently, by taking $y = \operatorname{Im} z$ sufficiently large, it may be assumed that, for some $M > 0$,

$$\|R_{iy}(H)\| \, \|V\|_{1,\infty} < 1/2, \quad |y| \geqslant M, \tag{3.8}$$

and so

$$\sigma(R_{iy}(H)V) \subseteq \{\lambda \in \mathbb{C}\colon\ |\lambda| < 1/2\}\,.$$

In particular, it follows that $R_{iy}(H)V \in \mathcal{L}^{1,\pi}(\mathcal{N},\tau)$ and $\log(1+R_{iy}(H)V) \in \mathcal{L}^1(\mathcal{N},\tau)$ for $|y| \geqslant M$. It follows that

$$\log(1+R_{iy}(H)V) = R_{iy}(H)V + \sum_{k=2}^{\infty}(-1)^{k+1}(R_{iy}(H)V)^k/k$$

with convergence in the norm $\|\cdot\|_{1,\infty}$ so that

$$\tau\left(\log(1+R_{iy}(H)V)\right) = \tau\left(R_{iy}(H)V\right) + \sum_{k=2}^{\infty}(-1)^{k+1}\tau\left((R_{iy}(H)V)^k/k\right), \quad |y| \geqslant M. \tag{3.9}$$

Now observe that

$$\tau\left(V\right) = \tau\left((iy-H)R_{iy}(H)V\right) = iy\tau\left(R_{iy}(H)V\right) - \tau\left(HR_{iy}(H)V\right). \tag{3.10}$$

Setting $E_n := \chi_{[-n,n]}(H),\ n \in \mathbb{N}$, note that

$$\tau\left(HR_{iy}(H)V\right) = \tau\left(HR_{iy}(H)E_nV\right) + \tau\left(HR_{iy}(H)E_n^{\perp}V\right).$$

Using the spectral theorem, we obtain that, for all $|y| \geqslant M$,

$$\|HR_{iy}(H)E_n\| \leqslant n/\sqrt{n^2+y^2}, \quad \|HR_{iy}(H)\| \leqslant 1,\ n \in \mathbb{N},$$

so that

$$|\tau\left(HR_{iy}(H)E_nV\right)| \leqslant \|HR_{iy}(H)E_n\|\ \|V\|_1 \leqslant n\,\|V\|_1/\sqrt{n^2+y^2} \tag{3.11}$$

for all $n \in \mathbb{N}$ and for all $|y| \geqslant M$. On the other hand, since $VHR_{iy}(H) \in \mathcal{L}^1(\mathcal{N},\tau)$ and E_n converges to 1 in the *so*-topology (by Theorem 1.1.4), it follows from Lemma 1.3.30 that

$$|\tau\left(HR_{iy}(H)E_n^{\perp}V\right)| = |\tau\left(VHR_{iy}(H)E_n^{\perp}\right)| \leqslant \|VHR_{iy}(H)E_n^{\perp}\|_1 \to 0, \tag{3.12}$$

and $n \to \infty$. Consequently, from (3.10), (3.11) and (3.12), it follows readily that

$$\tau\left(V\right) = \lim_{y\to\pm\infty} iy\tau\left(R_{iy}(H)V\right). \tag{3.13}$$

Finally, using (3.8) for $|y| \geqslant M$,

$$\begin{aligned}\left|\sum_{k=2}^{\infty}(-1)^{k+1}\tau\left((R_{iy}(H)V)^k\right)/k\right| &\leqslant \sum_{k=2}^{\infty}\left\|(R_{iy}(H)V)^k\right\|_1 \\ &\leqslant \|R_{iy}(H)\|^2\,\|V\|\,\|V\|_1\sum_{k=2}^{\infty}(\|R_{iy}(H)\|\,\|V\|)^{k-2} \leqslant 2\,\|V\|_1\,\|V\|\,/\,|y|^2.\end{aligned} \tag{3.14}$$

The assertion of the Proposition now follows directly from (3.9), (3.13) and (3.14). □

Lemma 3.1.8 *Suppose that $V \in \mathcal{L}^1(\mathcal{N}, \tau)$ and that $H = H^*$ is affiliated with $\mathcal{N}$.*
If $V \geqslant 0$, then

$$0 \leqslant \operatorname{Im} \tau\left(\log(1 - R_z(H)V)\right) \leqslant \pi\tau\left(\operatorname{supp}(V)\right), \quad \operatorname{Im} z > 0;$$

and

$$-\pi\tau\left(\operatorname{supp}(V)\right) \leqslant \operatorname{Im} \tau\left(\log(1 + R_z(H)V)\right) \leqslant 0, \quad \operatorname{Im} z > 0.$$

Proof. If $V \geqslant 0$, it follows from Proposition 3.1.3 that

$$\sigma_{-R_z(H)V} \subseteq \{\lambda \in \mathbb{C}\colon \operatorname{Im} \lambda > 0\} \cup \{0\},$$

and by Corollary 1.8.19 this implies that the support of the Brown measure $\mu_{-R_z(H)V}$ is contained in the open upper half-plane. Further, by Corollary 1.8.25(ii)

$$\mu_{-R_z(H)V}(\sigma_{-R_z(H)V} \setminus \{0\}) \leqslant \tau\left(\operatorname{supp}_r(-R_z(H)V)\right) \leqslant \tau\left(\operatorname{supp}(V)\right).$$

Since $0 \leqslant \operatorname{Im}(\log(1 + \lambda)) < \pi$ whenever $\operatorname{Im} \lambda > 0$, it now follows from Theorem 1.8.27 that

$$\begin{aligned} \operatorname{Im} \tau\left(\log(1 - R_z(H)V)\right) &= \int_{-\sigma_{R_z(H)V} \setminus \{0\}} \operatorname{Im}(\log(1 + \lambda))\, d\mu_{-R_z(H)V}(\lambda) \\ &\leqslant \pi\tau\left(\operatorname{supp}(V)\right). \end{aligned}$$

and this establishes the first assertion. The second assertion follows similarly. □

We may now state the principal result of this section.

Theorem 3.1.9 *Suppose that $H = H^*$ is affiliated with $\mathcal{N}$ and that $V = V^* \in \mathcal{L}^1(\mathcal{N}, \tau)$ satisfies $\tau\left(\operatorname{supp}(V)\right) < \infty$. Let $V = V_+ - V_-$ be the standard decomposition of V into its positive and negative parts. There exists a unique real-valued function $\xi_{H+V,H} \in L^1(\mathbb{R})$ with $\|\xi_{H+V,H}\|_1 \leqslant \|V\|_1$ such that*

$$\tau\left(R_z(H + V) - R_z(H)\right) = \int_{-\infty}^{\infty} \frac{\xi_{H+V,H}(\lambda)\, d\lambda}{(\lambda - z)^2}, \quad \operatorname{Im} z > 0. \tag{3.15}$$

Further,

$$\int_{-\infty}^{\infty} \xi_{H+V,H}(\lambda)\, d\lambda = \tau\left(V\right) \tag{3.16}$$

and

$$-\tau\left(\operatorname{supp}(V_-)\right) \leqslant \xi_{H+V,H}(\lambda) \leqslant \tau\left(\operatorname{supp}(V_+)\right), \quad a.\ e.\ \lambda \in \mathbb{R}. \tag{3.17}$$

Proof. We define

$$F_+(z) := \tau\left(\log(1 - R_z(H)V_+)\right),$$
$$F_-(z) := -\tau\left(\log(1 + R_z(H+V_+)V_-)\right), \quad \operatorname{Im} z > 0,$$

and set
$$F := F_+ - F_-.$$

By Proposition 3.1.6, each of the functions F_+, F_- (and, consequently, the function F) are holomorphic in $\mathbb{C}_+$ and for $z \in \mathbb{C}_+$

$$\begin{aligned}\frac{d}{dz}F(z) &= \frac{d}{dz}F_+(z) - \frac{d}{dz}F_-(z)\\ &= \tau\left(R_z(H+V_+) - R_z(H)\right) + \tau\left(R_z(H+V_+-V_-) - R_z(H+V_+)\right)\\ &= \tau\left(R_z(H+V) - R_z(H)\right).\end{aligned} \tag{3.18}$$

From Lemma 3.1.8, it follows that

$$0 \leqslant \operatorname{Im} F_\pm(z) \leqslant \pi\tau\left(\operatorname{supp}(V)\right), \quad \operatorname{Im} z > 0. \tag{3.19}$$

Since $\tau\left(\operatorname{supp}(V)\right) < \infty$, it follows that the functions $F_\pm$ are bounded and non-negative in the open upper half-plane. Further, it follows from Proposition 3.1.7 that $\sup_{y\geqslant 1} y\,|F_\pm(iy)| < \infty$. We may therefore apply Theorem 3.1.1 to obtain functions $\xi_+, \xi_- \in L^1(\mathbb{R})$ such that

$$F_\pm(z) = \int_{-\infty}^{\infty} \frac{\xi_\pm(\lambda)\,d\lambda}{\lambda - z}, \quad \operatorname{Im} z > 0,$$

where the functions $\xi_\pm$ are uniquely determined by the formulae

$$\xi_\pm(\lambda) = \frac{1}{\pi}\lim_{\epsilon\to 0^+} \operatorname{Im} F_\pm(\lambda + i\epsilon), \quad \text{a.e. } \lambda \in \mathbb{R}. \tag{3.20}$$

We now set
$$\xi_{H+V,H} := \xi_+ - \xi_-.$$

It follows from (3.19) and (3.20) that

$$0 \leqslant \xi_\pm(\lambda) \leqslant \tau\left(\operatorname{supp}(V_\pm)\right), \quad \text{a.e. } \lambda \in \mathbb{R} \tag{3.21}$$

and that $\xi_{H+V,H} \in L^1(\mathbb{R})$ and is uniquely determined by the formula

$$\xi_{H+V,H}(\lambda) = \frac{1}{\pi}\lim_{\epsilon\to 0^+} \left(\operatorname{Im} F_+(\lambda+i\epsilon) - \operatorname{Im} F_-(\lambda+i\epsilon)\right), \quad \text{a.e. } \lambda \in \mathbb{R}.$$

Further, we obtain that

$$F(z) = \int_{-\infty}^{\infty} \frac{\xi_{H+V,H}(\lambda)\,d\lambda}{\lambda - z}, \quad \operatorname{Im} z > 0. \tag{3.22}$$

The inequalities (3.21) imply that

$$-\tau\left(\operatorname{supp}(V_-)\right) \leqslant \xi(\lambda) \leqslant \tau\left(\operatorname{supp}(V_+)\right), \quad \text{a. e. } \lambda \in \mathbb{R}$$

and this is (3.17). From Proposition 3.1.7, and using the dominated convergence theorem, we obtain that

$$\begin{aligned}\tau\left(V_+\right) &= -\lim_{y\to\infty} iy\tau\left(\log(1 - R_{iy}(H)V_+)\right) \\ &= -\lim_{y\to\infty} iy\int_{-\infty}^{\infty} \frac{\xi_+(\lambda)\,d\lambda}{\lambda - iy} = \int_{-\infty}^{\infty} \xi_+(\lambda)\,d\lambda.\end{aligned}$$

Similarly, we obtain that

$$\tau\left(V_-\right) = \int_{-\infty}^{\infty} \xi_-(\lambda)\,d\lambda.$$

Consequently,

$$\int_{-\infty}^{\infty} \xi_{H+V,H}(\lambda)\,d\lambda = \tau\left(V_+\right) - \tau\left(V_-\right) = \tau\left(V\right),$$

which is (3.16). Further,

$$\begin{aligned}\left\|\xi_{H+V,H}\right\|_1 &= \int_{-\infty}^{\infty} \left|\xi_{H+V,H}\right|(\lambda)\,d\lambda \\ &\leqslant \int_{-\infty}^{\infty} \xi_+(\lambda)\,d\lambda + \int_{-\infty}^{\infty} \xi_-(\lambda)\,d\lambda = \tau\left(V_+\right) + \tau\left(V_-\right) = \tau\left(|V|\right) = \left\|V\right\|_1.\end{aligned}$$

Finally, from (3.18) and (3.22), we obtain that

$$\begin{aligned}\tau\left(R_z(H+V) - R_z(H)\right) &= \frac{d}{dz}\int_{-\infty}^{\infty} \frac{\xi_{H+V,H}(\lambda)\,d\lambda}{\lambda - z} \\ &= \int_{-\infty}^{\infty} \frac{\xi_{H+V,H}(\lambda)\,d\lambda}{(\lambda - z)^2}, \quad \operatorname{Im} z > 0.\end{aligned}$$

This is (3.15) and completes the proof of the Theorem. □

The function $\xi_{H+V,H}$ whose existence is given by Theorem 3.1.9 will be called the Krein **spectral shift function** associated with the self-adjoint operator H for the perturbation V. We remark that the preceding Theorem 3.1.9 specializes to [Kr, Theorem 5.1] in the special case that $\mathcal{N}$ is the von Neumann algebra of all bounded linear operators on some Hilbert space and τ is the canonical trace.

Now we are going to extend Krein's trace formula (3.15) to the case that V does not necessarily have finite support.

The proof of the following lemma, which is [Kr, Theorem 1], is simpler than that of [Kr].

Lemma 3.1.10 *If $H = H^*$ is affiliated with $\mathcal{N}$ and $V = V^*, W = W^* \in \mathcal{L}^1(\mathcal{N}, \tau)$, then $R_z(H+V) - R_z(H+W) \in \mathcal{L}^1(\mathcal{N}, \tau)$ for all $\operatorname{Im} z \neq 0$ and*

$$\|R_z(H+V) - R_z(H+W)\|_1 \leqslant \|V - W\|_1 / |\operatorname{Im} z|^2, \quad \operatorname{Im} z \neq 0.$$

Proof. It follows from (1.2) and (1.4) that

$$\begin{aligned} &\|R_z(H+V) - R_z(H+W)\|_1 = \|R_z(H+V)(V-W)R_z(H+W)\|_1 \\ &\quad \leqslant \|R_z(H+V)\| \, \|(V-W)\|_1 \, \|R_z(H+W)\| \leqslant \|V - W\|_1 / |\operatorname{Im} z|^2 . \end{aligned} \tag{3.23}$$

□

Theorem 3.1.11 *Suppose that $H = H^*$ is affiliated with $\mathcal{N}$ and that $V = V^* \in \mathcal{L}^1(\mathcal{N}, \tau)$. There exists a unique function $\xi_{H+V,H} \in L^1(\mathbb{R})$ with*

$$\|\xi_{H+V,H}\|_1 \leqslant \|V\|_1 \ \text{ and } \tau(V) = \int_{-\infty}^{\infty} \xi_{H+V,H}(\lambda)\, d\lambda$$

and such that

$$\tau\left(R_z(H+V) - R_z(H)\right) = \int_{-\infty}^{\infty} \frac{\xi_{H+V,H}(\lambda)\, d\lambda}{(\lambda - z)^2}, \quad \operatorname{Im} z > 0.$$

Proof. We set

$$V_n := \chi_{[1/n,n)}(|V|)V \in \mathcal{L}^1(\mathcal{N}, \tau), \quad n \in \mathbb{N}$$

and note that

$$\tau\left(\operatorname{supp}(V_n)\right) \leqslant \tau\left(\chi_{[1/n,n)}(|V|)\right) \leqslant n\tau\left(|V|\right) < \infty, \quad n \in \mathbb{N}.$$

Further, by order continuity of the norm $\|\cdot\|_1$, it follows that $\|V - V_n\|_1 \to 0$ as $n \to \infty$. Using Lemma 3.1.10, it follows that

$$\begin{aligned} &|\tau\left(R_z(H+V) - R_z(H)\right) - \tau\left(R_z(H+V_n) - R_z(H)\right)| \\ &\qquad = |\tau\left(R_z(H+V) - R_z(H+V_n)\right)| \leqslant \|V - V_n\|_1 / |\operatorname{Im} z|^2 \to 0 \end{aligned}$$

as $n \to \infty$ for all $\operatorname{Im} z \neq 0$. Using the addition formula given by the first assertion of Corollary 3.1.12 below (note that the proof of this formula in the special case that $\tau\left(\operatorname{supp}(V)\right)$, $\tau\left(\operatorname{supp}(W)\right) < \infty$ depends only on Theorem 3.1.9 and Proposition 3.1.2) and using the norm estimate given in Theorem 3.1.9, we obtain that

$$\|\xi_{H+V_m,H} - \xi_{H+V_n,H}\|_1 = \|\xi_{H+V_m,H+V_n}\|_1 \leqslant \|V_m - V_n\|_1 \to_{n,m} 0.$$

We now set

$$\xi_{H+V,H} := \lim_{n\to\infty} \xi_{H+V_n,H},$$

where the limit is taken in the norm $\|\cdot\|_1$. The assertion of the Theorem now follows readily from the facts that, for all $n \in \mathbb{N}$,

$$\|\xi_{H+V_n,H}\|_1 \leqslant \|V_n\|_1, \quad \tau(V_n) = \int_{-\infty}^{\infty} \xi_{H+V_n,H}(\lambda)\,d\lambda,$$

$$\tau(R_z(H+V_n) - R_z(H)) = \int_{-\infty}^{\infty} \frac{\xi_{H+V_n,H}(\lambda)\,d\lambda}{(\lambda - z)^2}, \quad \operatorname{Im} z > 0,$$

and $\tau(V_n) \to \tau(V)$.

The uniqueness assertion follows immediately from Proposition 3.1.2 and this completes the proof of the theorem. □

We now exhibit several properties of the spectral shift function, given in [Ya, Proposition 8.2.5] for the case that $(\mathcal{N}, \tau)$ is the von Neumann algebra $\mathcal{B}(\mathcal{H})$ equipped with the canonical trace. In this setting, the proof given in [Ya] depends on the theory of perturbation determinants.

Corollary 3.1.12 *If $H = H^*$ is affiliated with $\mathcal{N}$ and if $V, W \in \mathcal{L}^1(\mathcal{N}, \tau)$ are self-adjoint, then*

$$\xi_{H+V+W,H} = \xi_{H+V+W,H+V} + \xi_{H+V,H},$$
$$\xi_{H,H+V} = -\xi_{H+V,H}$$

and

$$\|\xi_{H+W,H} - \xi_{H+V,H}\|_1 \leqslant \|W - V\|_1.$$

Further, if $0 \leqslant V$, W, *then*

$$\xi_{H+V+W,H} \geqslant \xi_{H+V,H}.$$

Proof. It follows from Theorem 3.1.11 that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\xi_{H+V+W,H}(\lambda)\,d\lambda}{(\lambda - z)^2} &= \tau(R_z(H+V+W) - R_z(H)) \\ &= \tau(R_z(H+V+W) - R_z(H+V)) + \tau(R_z(H+V) - R_z(H)) \\ &= \int_{-\infty}^{\infty} \frac{\xi_{H+V+W,H+V}(\lambda)\,d\lambda}{(\lambda - z)^2} + \int_{-\infty}^{\infty} \frac{\xi_{H+V,H}(\lambda)\,d\lambda}{(\lambda - z)^2}, \quad \operatorname{Im} z > 0.\end{aligned}$$

The first assertion of the Corollary now follows from Proposition 3.1.2 and the second by taking $W = -V$ in the first. Replacing W by $W - V$ in the first assertion and using the estimate in Theorem 3.1.11, it follows that

$$\|\xi_{H+W,H} - \xi_{H+V,H}\|_1 = \left\|\xi_{H+V+(W-V),H+V}\right\|_1 \leqslant \|W - V\|_1.$$

The final assertion follows from the first together with the observation that if $V \geqslant 0$, then $\xi_{H+V,H} \geqslant 0$. This observation follows readily from Lemma 3.1.8, and an inspection of the proofs of Theorems 3.1.9, 3.1.11. □

3.1.2 The Krein trace formula: general case

The principal result of this section is the following theorem, due to M. G. Krein [Kr, Kr$_2$] in the special case that $\mathcal{N}$ is the von Neumann algebra $\mathcal{B}(\mathcal{H})$ equipped with the canonical trace.

Theorem 3.1.13 *If $H = H^*$ is affiliated with $\mathcal{N}$ and $V = V^* \in \mathcal{L}^1(\mathcal{N}, \tau)$, then there exists a unique function $\xi_{H+V,H} \in L^1(\mathbb{R})$ such that*

$$\|\xi_{H+V,H}\|_1 \leqslant \|V\|_1 , \qquad \int_{-\infty}^{\infty} \xi_{H+V,H}(\lambda)\, d\lambda = \tau(V),$$

$$-\tau(\operatorname{supp}(V_-)) \leqslant \xi_{H+V,H}(\lambda) \leqslant \tau(\operatorname{supp}(V_+)), \quad a.\,e.\ \lambda \in \mathbb{R},$$

and, for every function $f \in C^1(\mathbb{R})$ whose derivative f' admits the representation

$$f'(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t}\, dm(t), \quad \lambda \in \mathbb{R} \tag{3.24}$$

for some finite (complex) Borel measure m on $\mathbb{R}$, then $f(H+V) - f(H) \in \mathcal{L}^1(\mathcal{N}, \tau)$ and

$$\tau(f(H+V) - f(H)) = \int_{-\infty}^{\infty} \xi_{H+V,H}(\lambda) f'(\lambda)\, d\lambda. \tag{3.25}$$

The proof of the Theorem is based on the corresponding formula for the case of resolvent perturbations given in Theorems 3.1.9 and Theorem 3.1.11. The first two assertions are given in Theorem 3.1.11 and the third assertion is established in (3.17) in the case that $\tau(\operatorname{supp}(V_\pm)) < \infty$. To extend Theorem 3.1.11 to the wider class of functions f specified by Theorem 3.1.13, we follow the same lines as the original proof of Krein [Kr, Kr$_2$] in the type I case. However, some additional technical details are necessary in the type II setting. We proceed as follows.

Lemma 3.1.14 *Let H and V be as in Theorem 3.1.13.*

(i) The operator function

$$t \mapsto e^{it(H+V)} V e^{-itH} \in \mathcal{L}^1(\mathcal{N}, \tau), \quad t \in \mathbb{R},$$

is continuous in the norm $\|\cdot\|_1$.

(ii) For each $t \in \mathbb{R}$,

$$e^{it(H+V)} - e^{itH} \in \mathcal{L}^1(\mathcal{N}, \tau)$$

and

$$\left\| e^{it(H+V)} - e^{itH} \right\|_1 \leqslant |t|\, \|V\|_1 , \quad t \in \mathbb{R}.$$

(iii) There exists a unique function $\xi_{H+V,H} \in L^1(\mathbb{R})$ *such that*

$$\tau\left(e^{it(H+V)} - e^{itH}\right) = it \int_{-\infty}^{\infty} \xi_{H+V,H}(\lambda) e^{it\lambda}\, d\lambda, \quad t \in \mathbb{R}.$$

Proof. (i) By Lemma 1.3.30 and Theorem 1.1.5, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|t - t_0| < \delta$ implies $\left\|Ve^{itH_0} - Ve^{it_0H_0}\right\|_1 < \varepsilon/2$ and $\left\|e^{-itH_1}V - e^{-it_0H_1}V\right\|_1 < \varepsilon/2$. It follows that

$$\begin{aligned}
&\left\|e^{it(H+V)}Ve^{-itH} - e^{it_0(H+V)}Ve^{-it_0H}\right\|_1 \\
&\quad = \left\|e^{it(H+V)}\left(Ve^{itH} - Ve^{it_0H}\right) + \left(e^{it(H+V)}V - e^{it_0(H+V)}V\right)e^{it_0H}\right\|_1 \\
&\quad \leqslant \left\|e^{it(H+V)}\right\| \left\|Ve^{itH} - Ve^{it_0H}\right\|_1 + \left\|e^{it(H+V)}V - e^{it_0(H+V)}V\right\|_1 \left\|e^{it_0H}\right\| \\
&\quad \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

(ii) Since $\mathcal{L}^1(\mathcal{N}, \tau)$ has property (F) (see Lemma 1.3.39), the inclusion follows from Duhamel's formula (Lemma 1.1.8) and Lemma 1.4.8. The inequality follows from Duhamel's formula and Lemma 1.4.14.

(iii) Using (i), (ii) above and Theorem 3.1.11, the proof of (iii) is exactly the same as that in the type I setting given in [Ya, Lemma 8.3.2] and accordingly, the details are omitted. □

Corollary 3.1.15 *If the function* $f \in C^1(\mathbb{R})$ *satisfies* (3.24), *then*

$$f(H+V) - f(H) \in \mathcal{L}^1(\mathcal{N}, \tau)$$

and

$$\|f(H+V) - f(H)\|_1 \leqslant \|V\|_1 |m| (\mathbb{R}).$$

Proof. From (3.24), it follows that

$$f(\lambda) = f(0) + \int_{-\infty}^{\infty} \frac{e^{i\lambda t} - 1}{it}\, dm(t), \quad \lambda \in \mathbb{R}.$$

Using the spectral theorem, we obtain that

$$f(H+V) - f(H) = \int_{-\infty}^{\infty} \frac{e^{it(H+V)} - e^{itH}}{it}\, dm(t). \tag{3.26}$$

It follows from Lemma 3.1.14 that the integral exists in the norm $\|\cdot\|_1$ and this implies that $f(H+V) - f(H) \in \mathcal{L}^1(\mathcal{N}, \tau)$. Finally, the estimate

$$\|f(H+V) - f(H)\|_1 \leqslant \int_{-\infty}^{\infty} \left\|\frac{e^{it(H+V)} - e^{itH}}{it}\right\|_1 d\,|m|\,(t) \leqslant \|V\|_1 |m| (\mathbb{R})$$

follows immediately from Lemma 3.1.14 (ii) and this completes the proof. □

We may now complete the proof of Theorem 3.1.13 as follows. By (3.26), Lemma 3.1.14 (iii) and the fact that $\xi_{H+V,H} \in L^1(\mathbb{R})$, we obtain

$$\begin{aligned} \tau\left(f(H+V)-f(H)\right) &= \int_{-\infty}^{\infty} \frac{\tau\left(e^{it(H+V)}-e^{itH}\right)}{it}\, dm(t) \\ &= \int_{-\infty}^{\infty} dm(t) \int_{-\infty}^{\infty} \xi_{H+V,H}(\lambda) e^{it\lambda}\, d\lambda \\ = \int_{-\infty}^{\infty} \xi_{H+V,H}(\lambda)\, d\lambda \int_{-\infty}^{\infty} & e^{it\lambda}\, dm(t) = \int_{-\infty}^{\infty} \xi_{H+V,H}(\lambda) f'(\lambda)\, d\lambda \end{aligned}$$

and this completes the proof of the theorem.

3.2 Multiple operator integrals in von Neumann algebras

A multiple operator integral is an expression of the form

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(\lambda_0, \ldots, \lambda_n)\, dE_{\lambda_0}^{H_0} V_1\, dE_{\lambda_1}^{H_1} V_2\, dE_{\lambda_2}^{H_2} \ldots V_n\, dE_{\lambda_n}^{H_n},$$

where $H_0, \ldots, H_n$ are self-adjoint operators on the Hilbert space $\mathcal{H}$, and $V_1, \ldots, V_n$ are bounded operators on $\mathcal{H}$. These integrals were first introduced and investigated by Yu. L. Daletskii and S. G. Krein in [DK]. Afterwards a number of works appeared devoted to multiple operator integrals [Pa, SS, St]. These authors defined a multiple operator integral as a repeated integral. We take quite different approach to definition of multiple operator integrals, which seems to make simpler handling them.

We denote by $\mathcal{N}$ a von Neumann algebra acting on Hilbert space $\mathcal{H}$, and by Tr the standard trace on $\mathcal{B}(\mathcal{H})$. In case when $\mathcal{N}$ is semifinite, we denote by τ a fixed faithful normal semifinite trace on $\mathcal{N}$. If S is a measure space we denote by $B(S)$ the set of all bounded measurable complex-valued functions on S.

3.2.1 BS representations

Let n be a non-negative integer. We denote by $B(\mathbb{R}^{n+1})$ the set of all bounded Borel functions on $\mathbb{R}^{n+1}$.

Definition 3.2.1 *Let $\varphi \in B(\mathbb{R}^{n+1})$. A* ***BS representation*** *of the function φ is a representation of it in the form*

$$\varphi(\lambda_0, \lambda_1, \ldots, \lambda_n) = \int_S \alpha_0(\lambda_0, \sigma) \ldots \alpha_n(\lambda_n, \sigma)\, d\nu(\sigma), \tag{3.27}$$

where (S, ν) is a finite measure space and $\alpha_0, \ldots, \alpha_n$ are bounded Borel functions on $\mathbb{R} \times S$.

For any C^1-function $f \colon \mathbb{R} \to \mathbb{C}$, we denote by $f^{[1]}$ the continuous function

$$f^{[1]}(\lambda_0, \lambda_1) = \frac{f(\lambda_1) - f(\lambda_0)}{\lambda_1 - \lambda_0},$$

and for any C^{n+1}-function $f \colon \mathbb{R} \to \mathbb{C}$

$$f^{[n+1]}(\lambda_0, \ldots, \lambda_{n+1}) = \frac{f^{[n]}(\lambda_0, \ldots, \lambda_{n-1}, \lambda_{n+1}) - f^{[n]}(\lambda_0, \ldots, \lambda_{n-1}, \lambda_n)}{\lambda_{n+1} - \lambda_n}.$$

It is well known that $f^{[n]}$ is a symmetric function.

We denote by $C^{n+}(\mathbb{R})$ the set of functions $f \in C^n(\mathbb{R})$, such that the j-th derivative $f^{(j)}$, $j = 0, \ldots, n$, belongs to the space $\mathcal{F}^{-1}(L^1(\mathbb{R}))$, where $\mathcal{F}(f)$ is the Fourier transform of f. Here, the Fourier transform is taken in the sense of tempered distributions. Note, that if $f \in C^{n+}(\mathbb{R})$, then $\mathcal{F}(f^{(n)})(\xi) = (-i\xi)^n \mathcal{F}(f)(\xi)$. See, for example, [Y, Chapter VI.2]. It is not difficult to see that the Schwartz space $S(\mathbb{R}) = \bigcap_{n=0}^{\infty} C^{n+}(\mathbb{R})$.

The next lemma introduces a finite measure space which will be frequently used to construct BS representations.

Lemma 3.2.2 *Let*

$$\Pi^{(n)} = \{(s_0, s_1, \ldots, s_n) \in \mathbb{R}^{n+1} \colon \, |s_n| \leqslant \ldots \leqslant |s_1| \leqslant |s_0| \, , \\ \operatorname{sign}(s_0) = \ldots = \operatorname{sign}(s_n)\},$$

and

$$\nu_f^{(n)}(s_0, \ldots, s_n) = \frac{i^n}{\sqrt{2\pi}} \mathcal{F}(f)(s_0) \, ds_0 \, \ldots \, ds_n.$$

If $f \in C^{n+}(\mathbb{R})$, then $\left(\Pi^{(n)}, \nu_f^{(n)}\right)$ is a finite measure space.

Proof. The total variation of the measure $\nu_f^{(n)}$ on the set $\Pi^{(n)}$ (up to a constant) is equal to

$$\begin{aligned}
\int_{\Pi^{(n)}} |\mathcal{F}(f)(s_0)| \, ds_0 \, \ldots \, ds_n &= \int_{\mathbb{R}} |\mathcal{F}(f)(s_0)| \, \Delta_{s_0} \, ds_0 \\
&= \frac{1}{n!} \int_{\mathbb{R}} |\mathcal{F}(f)(s_0) s_0^n| \, ds_0 \\
&= \frac{1}{n!} \int_{\mathbb{R}} \left| \mathcal{F}(f^{(n)})(s_0) \right| \, ds_0 = \frac{1}{n!} \left\| \mathcal{F}(f^{(n)}) \right\|_1,
\end{aligned}$$

where Δ_{s_0} is the volume of the n-dimensional simplex of size s_0. □

We write for simplicity $\Pi = \Pi^{(1)}$ and $\nu_f = \nu_f^{(1)}$, so that

$$\Pi := \left\{(s_0, s_1) \in \mathbb{R}^2 \colon \ |s_1| \leqslant |s_0|, \operatorname{sign}(s_0) = \operatorname{sign}(s_1)\right\},$$

and

$$d\nu_f(s_0, s_1) := \frac{i}{\sqrt{2\pi}} \mathcal{F}(f)(s_0)\, ds_0\, ds_1. \tag{3.28}$$

The next two lemmas provide concrete BS-representations for divided differences $f^{[n]}$ of functions belonging to the class $C^{n+}(\mathbb{R})$.

Lemma 3.2.3 *If $f \in C^{1+}(\mathbb{R})$, then*

$$f^{[1]}(\lambda_0, \lambda_1) = \iint_\Pi \alpha_0(\lambda_0, \sigma)\alpha_1(\lambda_1, \sigma)\, d\nu_f(\sigma), \tag{3.29}$$

where $\sigma = (s_0, s_1)$, $\alpha_0(\lambda_0, \sigma) = e^{i(s_0 - s_1)\lambda_0}$, $\quad \alpha_1(\lambda_1, \sigma) = e^{is_1\lambda_1}$, $\ s_0, s_1 \in \mathbb{R}$.

Proof. We have

$$\begin{aligned}
&\iint_\Pi \alpha_0(\lambda_0, \sigma)\alpha_1(\lambda_1, \sigma)\, d\nu_f(\sigma) \\
&\qquad = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} ds_0 \mathcal{F}(f)(s_0) \int_0^{s_0} e^{is_0\lambda_0 - is_1\lambda_0 + is_1\lambda_1}\, ds_1 \\
&\qquad = \frac{1}{(\lambda_0 - \lambda_1)\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}(f)(s_0)(e^{is_0\lambda_0} - e^{is_0\lambda_1})\, ds_0 \\
&\qquad = \frac{1}{\lambda_0 - \lambda_1}(f(\lambda_0) - f(\lambda_1)) = f^{[1]}(\lambda_0, \lambda_1),
\end{aligned}$$

where the repeated integral can be replaced by the double integral by Fubini's theorem and Lemma 3.2.2. □

Lemma 3.2.4 *If $f \in C^{n+}(\mathbb{R})$, then, for all $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$,*

$$\begin{aligned}
&f^{[n]}(\lambda_0, \ldots, \lambda_n) \\
&\qquad = \int_{\Pi^{(n)}} e^{i((s_0 - s_1)\lambda_0 + \ldots + (s_{n-1} - s_n)\lambda_{n-1} + s_n\lambda_n)}\, d\nu_f^{(n)}(s_0, \ldots, s_n).
\end{aligned}$$

Proof. By Lemma 3.2.3 and induction, we have

$$\begin{aligned}
&\int_{\Pi^{(n+1)}} e^{i((s_0-s_1)\lambda_0+\ldots+(s_n-s_{n+1})\lambda_n+s_{n+1}\lambda_{n+1})}\, d\nu_f^{(n+1)}(s_0,\ldots,s_{n+1})\\
&\quad= \int_{\Pi^{(n)}} e^{i((s_0-s_1)\lambda_0+\ldots+s_n\lambda_n)} \left(\int_0^{s_n} ie^{is_{n+1}(\lambda_{n+1}-\lambda_n)}\, ds_{n+1}\right)\\
&\qquad\qquad d\nu_f^{(n)}(s_0,\ldots,s_n)\\
&\quad= \frac{1}{\lambda_{n+1}-\lambda_n}\int_{\Pi^{(n)}} e^{i((s_0-s_1)\lambda_0+\ldots+s_n\lambda_n)} \left(e^{is_n(\lambda_{n+1}-\lambda_n)}-1\right)\\
&\qquad\qquad d\nu_f^{(n)}(s_0,\ldots,s_n)\\
&\quad= \frac{1}{\lambda_{n+1}-\lambda_n}\left(f^{[n]}(\lambda_0,\ldots,\lambda_{n-1},\lambda_{n+1}) - f^{[n]}(\lambda_0,\ldots,\lambda_{n-1},\lambda_n)\right)\\
&\quad= f^{[n+1]}(\lambda_0,\ldots,\lambda_{n+1}).
\end{aligned}$$

□

Lemma 3.2.5 *If $f \in C^{n+1+}(\mathbb{R})$, then, for all $\lambda_0,\ldots,\lambda_{n+1}\in\mathbb{R}$,*

$$\begin{aligned}
&f^{[n+1]}(\lambda_0,\ldots,\lambda_{n+1})\\
&\quad= i\int_{\Pi^{(n)}}\int_0^{s_j-s_{j+1}} e^{i((s_0-s_1)\lambda_0+\ldots+u\lambda_{n+1}+(s_j-s_{j+1}-u)\lambda_j+\ldots+s_n\lambda_n)}\\
&\qquad\qquad du\, d\nu_f^{(n)}(s_0,\ldots,s_n).
\end{aligned}$$

Proof. The right hand side is equal to

$$\begin{aligned}
&i\int_{\Pi^{(n)}} e^{i((s_0-s_1)\lambda_0+\ldots+(s_j-s_{j+1})\lambda_j+\ldots+s_n\lambda_n)}\int_0^{s_j-s_{j+1}} e^{iu(\lambda_{n+1}-\lambda_j)}\, du\, d\nu_f^{(n)}(s_0,\ldots,s_n)\\
&\quad= \frac{1}{\lambda_{n+1}-\lambda_j}\int_{\Pi^{(n)}} e^{i((s_0-s_1)\lambda_0+\ldots+(s_j-s_{j+1})\lambda_j+\ldots+s_n\lambda_n)}\\
&\qquad\left(e^{(s_j-s_{j+1})(\lambda_{n+1}-\lambda_j)}-1\right)\, d\nu_f^{(n)}(s_0,\ldots,s_n)\\
&\quad= \frac{1}{\lambda_{n+1}-\lambda_j}\int_{\Pi^{(n)}}\Big(e^{i((s_0-s_1)\lambda_0+\ldots+(s_j-s_{j+1})\lambda_{n+1}+\ldots+s_n\lambda_n)}\\
&\qquad - e^{i((s_0-s_1)\lambda_0+\ldots+(s_j-s_{j+1})\lambda_j+\ldots+s_n\lambda_n)}\Big)\, d\nu_f^{(n)}(s_0,\ldots,s_n)\\
&\quad= \frac{1}{\lambda_{n+1}-\lambda_j}\big(f^{[n]}(\lambda_0,\ldots,\lambda_{j-1},\lambda_{n+1},\lambda_{j+1},\ldots,\lambda_n)\\
&\qquad - f^{[n]}(\lambda_0,\ldots,\lambda_{j-1},\lambda_j,\lambda_{j+1},\ldots,\lambda_n)\big)\\
&\quad= f^{[n+1]}(\lambda_0,\ldots,\lambda_{n+1}).
\end{aligned}$$

□

3.2.2 Multiple operator integrals

In this subsection, we define multiple operator integrals of the form

$$\int_{\mathbb{R}^{n+1}} \varphi(\lambda_0, \ldots, \lambda_n)\, dE^{H_0}_{\lambda_0} V_1\, dE^{H_1}_{\lambda_1} V_2\, dE^{H_2}_{\lambda_2} \ldots V_n\, dE^{H_n}_{\lambda_n}.$$

Definition 3.2.6 *Let $\varphi \in B(\mathbb{R}^{n+1})$ be a function with BS representation (3.27). For arbitrary self-adjoint operators $H_0, \ldots, H_n$ on the Hilbert space $\mathcal{H}$ and bounded operators $V_1, \ldots, V_n \in \mathcal{B}(\mathcal{H})$ the* ***multiple operator integral*** *$T^{H_0,\ldots,H_n}_{\varphi}(V_1, \ldots, V_n)$ is defined as*

$$T^{H_0,\ldots,H_n}_{\varphi}(V_1, \ldots, V_n) := \int_S \alpha_0(H_0, \sigma) V_1 \ldots V_n \alpha_n(H_n, \sigma)\, d\nu(\sigma), \tag{3.30}$$

where the integral is the so*-integral (Definition 1.4.6).

Remark 3.2.7 *By [dPS, Lemma 5.13] and Lemma 1.4.5(i) applied to $\mathcal{E} = \mathcal{B}(\mathcal{H})$, the function $\sigma \mapsto \alpha_0(H_0, \sigma) V_1 \ldots V_n \alpha_n(H_n, \sigma)$ is $*$- measurable and therefore the integral above exists.*

Theorem 3.2.8 *The multiple operator integral is well-defined in the sense that it does not depend on the BS representation of φ.*

Proof. We first prove that if the operators $V_1, \ldots, V_n$ are all one-dimensional, then the right hand side of (3.30) does not depend on the BS representation of φ.

For $\eta, \xi \in \mathcal{H}$, we denote by $\theta_{\eta,\xi}$ the one-dimensional operator defined by formula $\theta_{\eta,\xi}\zeta = \langle \eta, \zeta \rangle\, \xi,\ \zeta \in \mathcal{H}$. It is clear that $\mathrm{Tr}(\theta_{\eta,\xi}) = \langle \eta, \xi \rangle,\ \ A\theta_{\eta,\xi} = \theta_{\eta,A\xi}$ for any $A \in \mathcal{B}(\mathcal{H})$ and that $\theta_{\eta_1,\xi_1} \ldots \theta_{\eta_n,\xi_n} = \langle \eta_1, \xi_2 \rangle \ldots \langle \eta_{n-1}, \xi_n \rangle\, \theta_{\eta_n,\xi_1}$.

Let $V_j = \theta_{\eta_j,\xi_j},\ j = 0, \ldots, n$. Then

$$\begin{aligned}
E &:= \operatorname{Tr}\left(V_0 \int_S \alpha_0(H_0,\sigma)V_1 \ldots V_n\alpha_n(H_n,\sigma)\,d\nu(\sigma)\right) \\
&= \operatorname{Tr}\int_S V_0\alpha_0(H_0,\sigma)V_1 \ldots V_n\alpha_n(H_n,\sigma)\,d\nu(\sigma) \\
&= \operatorname{Tr}\int_S \theta_{\eta_0,\xi_0}\alpha_0(H_0,\sigma)\theta_{\eta_1,\xi_1} \ldots \theta_{\eta_n,\xi_n}\alpha_n(H_n,\sigma)\,d\nu(\sigma) \\
&= \int_S \operatorname{Tr}\left(\theta_{\eta_0,\xi_0}\alpha_0(H_0,\sigma)\theta_{\eta_1,\xi_1} \ldots \theta_{\eta_n,\xi_n}\alpha_n(H_n,\sigma)\right)\,d\nu(\sigma) \\
&= \int_S \operatorname{Tr}\left(\alpha_0(H_0,\sigma)\theta_{\eta_1,\xi_1} \ldots \theta_{\eta_n,\xi_n}\alpha_n(H_n,\sigma)\theta_{\eta_0,\xi_0}\right)\,d\nu(\sigma) \\
&= \int_S \operatorname{Tr}\left(\theta_{\eta_1,\alpha_0(H_0,\sigma)\xi_1} \ldots \theta_{\eta_n,\alpha_{n-1}(H_{n-1},\sigma)\xi_n}\theta_{\eta_0,\alpha_n(H_n,\sigma)\xi_0}\right)\,d\nu(\sigma) \\
&= \int_S \langle\eta_0, \alpha_0(H_0,\sigma)\xi_1\rangle\,\langle\eta_1, \alpha_1(H_1,\sigma)\xi_2\rangle \ldots \langle\eta_n, \alpha_n(H_n,\sigma)\xi_0\rangle\,d\nu(\sigma).
\end{aligned}$$

Now, since $\langle\eta, \alpha(H)\xi\rangle = \int_{\mathbb{R}} \alpha(\lambda)\left\langle\eta, dE_\lambda^H\xi\right\rangle$, we have that

$$E = \int_S \int_{\mathbb{R}} \alpha_0(\lambda_0,\sigma)\left\langle\eta_0, dE_{\lambda_0}^{H_0}\xi_1\right\rangle \ldots \int_{\mathbb{R}} \alpha_n(\lambda_n,\sigma)\left\langle\eta_n, dE_{\lambda_n}^{H_n}\xi_0\right\rangle\,d\nu(\sigma).$$

Since the measure $\langle\eta, dE_\lambda\xi\rangle$ has finite total variation, Fubini's theorem implies

$$\begin{aligned}
E &= \int_S \left(\int_{\mathbb{R}^{n+1}} \alpha_0(\lambda_0,\sigma)\ldots\alpha_n(\lambda_n,\sigma)\left\langle\eta_0, dE_{\lambda_0}^{H_0}\xi_1\right\rangle \ldots \left\langle\eta_n, dE_{\lambda_n}^{H_n}\xi_0\right\rangle\right)\,d\nu(\sigma) \\
&= \int_{\mathbb{R}^{n+1}} \left(\int_S \alpha_0(\lambda_0,\sigma)\ldots\alpha_n(\lambda_n,\sigma)\,d\nu(\sigma)\right)\left\langle\eta_0, dE_{\lambda_0}^{H_0}\xi_1\right\rangle \ldots \left\langle\eta_n, dE_{\lambda_n}^{H_n}\xi_0\right\rangle \\
&= \int_{\mathbb{R}^{n+1}} \varphi(\lambda_0,\ldots,\lambda_n)\left\langle\eta_0, dE_{\lambda_0}^{H_0}\xi_1\right\rangle \ldots \left\langle\eta_n, dE_{\lambda_n}^{H_n}\xi_0\right\rangle.
\end{aligned}$$

We recall that, if A, B are bounded operators, then $A = B$ if and only if the equality $\operatorname{Tr}(VA) = \operatorname{Tr}(VB)$ holds for all one-dimensional operators V. It now follows immediately that the multiple operator integral does not depend on BS representation of φ in the case that the operators $V_1, \ldots, V_n$ are one-dimensional.

By linearity, it follows that the definition of multiple operator integral does not depend on BS representation in the case of finite-dimensional operators $V_1, \ldots, V_n$. Since every bounded operator is an *so*-limit of a sequence of finite-dimensional operators, the claim follows from Proposition 3.2.13. □

Lemma 3.2.9 *If $\mathcal{N}$ is a von Neumann algebra, if $H_0, \ldots, H_n$ are self-adjoint operators affiliated with $\mathcal{N}$ and if $V_1, \ldots, V_n \in \mathcal{N}$, then $T_\varphi^{H_0,\ldots,H_n}(V_1, \ldots, V_n) \in \mathcal{N}$.*

This follows from Lemma 1.4.11.

Lemma 3.2.10 *If $\mathcal{E}$ is an invariant operator ideal with property (F) and if one of the operators $V_1, \dots, V_n$ belongs to $\mathcal{E}$, then*

$$T_\varphi^{H_0,\dots,H_n}(V_1,\dots,V_n) \in \mathcal{E}.$$

In case that $n = 2$, this yields

$$\left\|T_\varphi^{H_1,H_2}\right\|_{\mathcal{E}\to\mathcal{E}} \leqslant \|\varphi\|,$$

where

$$\|\varphi\| = \inf\left\{\int_S \|\alpha(\cdot,\sigma)\|_\infty \|\beta(\cdot,\sigma)\|_\infty \, d\nu(\sigma) \colon \varphi(\lambda,\mu) = \int_S \alpha(\lambda,\sigma)\beta(\mu,\sigma)\, d\nu(\sigma)\right\}.$$

Proof. Follows from Lemmas 1.4.5(i) and 1.4.8. □

Remark 3.2.11 *If $V \in \mathcal{L}^2(\mathcal{N},\tau)$ and if $n = 2$, then the preceding definition coincides with the definition of double operator integral as a spectral integral given in [BS$_2$] and [dPS].*

Corollary 3.2.12 *If $V_1,\dots,V_n \in \mathcal{N}$, $V_j \in \mathcal{L}^1(\mathcal{N},\tau)$ for some $j = 1,\dots,n$, $H_0,\dots,H_n$ are self-adjoint operators affiliated with $\mathcal{N}$, $\varphi \in B(\mathbb{R}^{n+1})$ and $\varphi(\lambda_0,\dots,\lambda_n)$ admits the representation (3.27), then*

$$\tau\left(T_\varphi^{H_0,\dots,H_n}(V_1,\dots,V_n)\right) = \int_S \tau\left(\alpha_0(H_0,\sigma)V_1\alpha_1(H_1,\sigma)\dots V_n\alpha_n(H_n,\sigma)\right) d\nu(\sigma)$$

Proof. It is enough to note that the operator-valued function

$$\sigma \mapsto \alpha_0(H_0,\sigma)V_1\alpha_1(H_1,\sigma)\dots V_n\alpha_n(H_n,\sigma)$$

is $*$- measurable by [dPS, Lemma 5.13] and Lemma 1.4.5(i), so that we can apply Lemma 1.4.13. □

Proposition 3.2.13 *(i) If a sequence of self-adjoint operators $V_j^{(k_j)} \in \mathcal{B}(\mathcal{H})$, $j = 1,\dots,n$, converges to $V_j \in \mathcal{B}(\mathcal{H})$ in the so-topology (respectively, norm topology) as $k_j \to \infty$, then*

$$T_\varphi^{H_0,\dots,H_n}(V_1^{(k_1)},\dots,V_n^{(k_n)}) \to T_\varphi^{H_0,\dots,H_n}(V_1,\dots,V_n)$$

in the so-topology (respectively, norm topology) as $k_1,\dots,k_n \to \infty$.

(ii) If a sequence of self-adjoint operators $H_j^{(k_j)}$, $j = 0,\dots,n$ resolvent strongly converges to H_j as $k_j \to \infty$ and $V_1,\dots,V_n \in \mathcal{B}(\mathcal{H})$, then

$$T_\varphi^{H_0^{(k_0)},\dots,H_n^{(k_n)}}(V_1,\dots,V_n) \to T_\varphi^{H_0,\dots,H_n}(V_1,\dots,V_n)$$

in the so-topology as $k_0,\dots,k_n \to \infty$.

Proof. We prove the part (ii), the proof of part (i) is similar (and simpler). Suppose that

$$\varphi(\lambda_0, \dots, \lambda_n) = \int_S \alpha_0(\lambda_0, \sigma) \dots \alpha_n(\lambda_n, \sigma)\, d\nu(\sigma)$$

is a representation of φ given by (3.27). Since $\alpha(\cdot, \sigma)$ is a bounded function for every $\sigma \in S$, the operators $\alpha(H_j^{(k_j)}, \sigma)$ converge to $\alpha(H_j, \sigma)$ in the *so*-topology by Theorem 1.1.6. Since by Theorem 1.1.1 multiplication of operators is jointly continuous in the *so*-topology on the unit ball of $\mathcal{N}$, the operator $\alpha(H_0^{(k_0)}, \sigma)V_1 \dots V_n\alpha(H_n^{(k_n)}, \sigma)$ converges in the *so*-topology to $\alpha(H_0, \sigma)V_1 \dots V_n\alpha(H_n, \sigma)$, $\sigma \in S$. Now, an application of the Dominated Convergence Theorem for the Bochner integral of $\mathcal{H}$-valued functions (Theorem 1.1.13) completes the proof. $\square$

This new definition of multiple operator integral enables us to give a simple proof of the following

Proposition 3.2.14 *The multiple operator integral has the properties:*
(i) if φ_1 and φ_2 admit a representation of the type given in (3.27), then so does $\varphi_1 + \varphi_2$ and

$$T^{H_1,\dots,H_n}_{\varphi_1+\varphi_2} = T^{H_1,\dots,H_n}_{\varphi_1} + T^{H_1,\dots,H_n}_{\varphi_2}; \tag{3.31}$$

(ii) in the case of double operator integrals, if φ_1 and φ_2 admit a representation of the type given in (3.27), then so does $\varphi_1\varphi_2$ and

$$T^{H_1,H_2}_{\varphi_1\varphi_2} = T^{H_1,H_2}_{\varphi_1} T^{H_1,H_2}_{\varphi_2}.$$

Proof. (i) If we take representations of the form (3.27) with (S_1, ν_1) and (S_2, ν_2) for φ_1 and φ_2 and put $(S, \nu) = (S_1, \nu_1) \sqcup (S_2, \nu_2)$ for $\varphi_1 + \varphi_2$ with evident definition of $\alpha_1, \alpha_2, \dots$, then the equality (3.31) follows from Definition 3.2.6. Here $\sqcup$ denotes the disjoint sum of measure spaces.

(ii) If

$$\varphi_j(\lambda_1, \lambda_2) = \int_{S_1} \alpha_j(\lambda_1, \sigma_1)\beta_j(\lambda_2, \sigma_1)\, d\nu_j(\sigma_1),\ j = 1, 2,$$

set

$$\varphi(\lambda_1, \lambda_2) = \int_S \alpha(\lambda_1, \sigma)\beta(\lambda_2, \sigma)\, d\nu(\sigma),$$

where

$$(S, \nu) = (S_1, \nu_1) \times (S_2, \nu_2)$$

and

$$\alpha(\lambda, \sigma) = \alpha_1(\lambda, \sigma_1)\alpha_2(\lambda, \sigma_2), \quad \beta(\lambda, \sigma) = \beta_1(\lambda, \sigma_1)\beta_2(\lambda, \sigma_2).$$

Consequently,

$$
\begin{aligned}
&T^{H_1,H_2}_{\varphi_1}\left(T^{H_1,H_2}_{\varphi_2}(V)\right)\\
&\qquad=\int_{S_1}\alpha_1(H_1,\sigma_1)T^{H_1,H_2}_{\varphi_2}(V)\beta_1(H_2,\sigma_1)\,d\nu_1(\sigma_1)\\
&\qquad=\int_{S_1}\alpha_1(H_1,\sigma_1)\left(\int_{S_2}\alpha_2(H_1,\sigma_2)V\beta_2(H_2,\sigma_2)\,d\nu_2(\sigma_2)\right)\beta_1(H_2,\sigma_1)\,d\nu_1(\sigma_1).
\end{aligned}
$$

Now, Lemma 1.4.9 and Fubini's theorem (Lemma 1.4.10) imply

$$
\begin{aligned}
&T^{H_1,H_2}_{\varphi_1}\left(T^{H_1,H_2}_{\varphi_2}(V)\right)\\
&\qquad=\int_{S_1\times S_2}\alpha_1(H_1,\sigma_1)\alpha_2(H_1,\sigma_2)V\beta_2(H_2,\sigma_2)\beta_1(H_2,\sigma_1)\,d(\nu_1\times\nu_2)(\sigma_1,\sigma_2)\\
&\qquad=T^{H_1,H_2}_{\varphi_1\varphi_2}(V).
\end{aligned}
$$

□

The following observation is a direct consequence of Lemma 3.2.4 and Definition 3.2.6.

Lemma 3.2.15 *If $f \in C^{n+}(\mathbb{R})$, then*

$$
\begin{aligned}
&T^{H_0,\dots,H_n}_{f^{[n]}}(V_1,\dots,V_n)\\
&\qquad=\int_{\Pi^{(n)}}e^{i(s_0-s_1)H_0}V_1e^{i(s_1-s_2)H_1}V_2\dots V_ne^{is_nB_n}\,d\nu_f^{(n)}(s_0,\dots,s_n).
\end{aligned}
$$

3.3 Higher order Fréchet differentiability

We note that, by Stone's theorem (Theorem 1.1.5) and joint continuity of multiplication of operators (from the unit ball) in the *so*-topology (Theorem 1.1.1) all operator-valued integrals occurring in this and subsequent sections are defined as in subsection 1.4.

Theorem 3.3.1 *Let $\mathcal{N}$ be a von Neumann algebra. Suppose that $H_0 = H_0^*$ is affiliated with $\mathcal{N}$, that $V \in \mathcal{N}$ is self-adjoint and set $H_1 = H_0+V$. If $f \in C^{1+}(\mathbb{R})$, then*

$$f(H_1) - f(H_0) = T^{H_1,H_0}_{f^{[1]}}(V).$$

Proof. It follows from Lemma 1.1.7 that

$$f(H_1) - f(H_0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} ds\mathcal{F}(f)(s)(e^{isH_1} - e^{isH_0}).$$

Hence, by Lemma 1.1.8,

$$f(H_1) - f(H_0) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} ds \mathcal{F}(f)(s) \int_0^s e^{i(s-t)H_1} V e^{itH_0}\, dt. \tag{3.32}$$

Since $f \in C^{1+}(\mathbb{R})$, by Lemma 3.2.2 and Fubini's theorem (Lemma 1.4.10), the repeated integral can be replaced by a double integral, so that

$$\begin{aligned} f(H_1) - f(H_0) &= \frac{i}{\sqrt{2\pi}} \iint_{\Pi} \mathcal{F}(f)(s) e^{i(s-t)H_1} V e^{itH_0}\, ds\, dt \\ &= \iint_{\Pi} e^{i(s-t)H_1} V e^{itH_0}\, d\nu_f(\sigma). \end{aligned} \tag{3.33}$$

It now follows from Lemma 3.2.15 that $f(H_1) - f(H_0) = T^{H_1,H_0}_{f^{[1]}}(V)$. □

Remark 3.3.2 *The preceding result is a generalization of a formula due to Yu. L. Daletskii and S. G. Krein [DK]. It is similar to [dPSW, Corollary 7.2], which applies to a wider class of functions but is restricted to bounded operators in a semifinite von Neumann algebra $\mathcal{N}$. The proof given here is simpler.*

Theorem 3.3.3 *Let $\mathcal{N}$ be a von Neumann algebra, acting in a Hilbert space $\mathcal{H}$. Let $H = H^*$ be affiliated with $\mathcal{N}$ and let $V \in \mathcal{E}_{sa}$, where $\mathcal{E}$ is an invariant operator ideal over $\mathcal{N}$ with property (F). If $f \in C^{2+}(\mathbb{R})$, then the function $f \colon H' \in H + \mathcal{E}_{sa} \mapsto f(H') \in f(H) + \mathcal{E}_{sa}$ is affinely Fréchet differentiable along $\mathcal{E}_{sa}$ and*

$$\mathcal{D}_{\mathcal{E}} f(H) = T^{H,H}_{f^{[1]}}.$$

The function $X \mapsto \mathcal{D}_{\mathcal{E}} f(H + X)$ is continuous in the norm of $\mathcal{E}$ and satisfies the estimate

$$\left\| \mathcal{D}_{\mathcal{E}} f(H+X)(V) - \mathcal{D}_{\mathcal{E}} f(H)(V) \right\|_{\mathcal{E}} \leqslant \left\| \mathcal{F}(f'') \right\|_1 \left\| V \right\|_{\mathcal{E}} \left\| X \right\|_{\mathcal{E}}, \quad X, V \in \mathcal{E}. \tag{3.34}$$

Proof. By (3.33) we have, following [Wi],

$$\begin{aligned} f(H+V) - f(H) &= \iint_{\Pi} e^{i(s-t)(H+V)} V e^{itH}\, d\nu_f(s,t) \\ &= \iint_{\Pi} e^{i(s-t)H} V e^{itH}\, d\nu_f(s,t) \\ &\qquad + \iint_{\Pi} \left(e^{i(s-t)(H+V)} - e^{i(s-t)H} \right) V e^{itH}\, d\nu_f(s,t) \\ &= (I) + (II). \end{aligned} \tag{3.35}$$

(I) is equal to $T^{H,H}_{f^{[1]}}(V)$ and represents a continuous linear operator on $\mathcal{E}$ (see Lemmas 3.2.15 and 3.2.10), so that it will be a Fréchet derivative of $f\colon H+\mathcal{E}\to f(H)+\mathcal{E}$ provided it is shown that the second term is $o(\|V\|_{\mathcal{E}})$. Applying Duhamel's formula (1.6) yields

$$(II) = \iint_{\Pi}\left(\int_0^{s-t} e^{iu(H+V)} iVe^{i(s-t-u)H}\,du\right) Ve^{itH}\,d\nu_f(s,t). \tag{3.36}$$

Since $f\in C^{2+}(\mathbb{R})$, Lemmas 3.2.5, 3.2.15 and Theorem 3.2.8 enable us to rewrite (3.36) as

$$(II) = \iiint_{\Pi^{(2)}} e^{iu(H+V)} Ve^{i(t-u)H} Ve^{i(s-t)H}\,d\nu_f^{(2)}(s,t,u),$$

where $(\Pi^{(2)},\nu_f^{(2)})$ is the finite measure space defined in Lemma 3.2.2. The $\mathcal{E}$-norm of the last expression is estimated by $\left|\nu_f^{(2)}\right| \|V\|\,\|V\|_{\mathcal{E}} \leqslant \left|\nu_f^{(2)}\right| \|V\|_{\mathcal{E}}^2$. So, the function $f\colon H+\mathcal{E}\to f(H)+\mathcal{E}$ is Fréchet differentiable and $\mathcal{D}_{\mathcal{E}}f(H) = T^{H,H}_{f^{[1]}}$.

The norm continuity of this derivative and the estimate (3.34) follow by a similar argument using Duhamel's formula (1.6). □

Remark 3.3.4 *It follows, in particular, from the preceding theorem via Lemma 3.2.10 that the operator* $T^{H,H}_{f^{[1]}}\Big|_{\mathcal{E}}$ *is a bounded linear operator on* $\mathcal{E}$*.*

Remark 3.3.5 [dPS] *Here we show that (a) the function* $\sin\colon f\in L^\infty[0,1]\mapsto \sin(f)\in L^\infty[0,1]$ *is Fréchet differentiable and that (b) the function* $\sin\colon f\in L^1[0,1]\mapsto\sin(f)\in L^1[0,1]$ *is Gateaux differentiable but not Fréchet differentiable.*

Indeed,

$$\begin{aligned}\sin(f+2h)-\sin(f) &= 2\sin(h)\cos(f+h)\\ &= 2\sin(h)\cos(f)+2\sin(h)(\cos(f+h)-\cos(f))\\ &= 2\cos(f)h+2\cos(f)(\sin(h)-h)+4\sin(h)\sin(f+h/2)\sin(h/2)\\ &= 2\cos(f)h+r(f;h).\end{aligned}$$

So, it is clear that $r(f;h)=O(\|h\|_\infty^2)$*. This means that the function* $\sin\colon f\in L^\infty[0,1]\mapsto\sin(f)\in L^\infty[0,1]$ *is Fréchet differentiable.*

It is also clear that the function $\sin\colon f\in L^1[0,1]\mapsto\sin(f)\in L^1[0,1]$ *is Gateaux differentiable. In order to see that it is not Fréchet differentiable it is enough to take* $h_n=n\chi_{[0,1/n]}$*. In this case*

$$r(f;h_n) = 2\chi_{[0,1/n]}\cos 1(\sin 1-1)+4\chi_{[0,1/n]}\sin(1)\sin(f+1/2)\sin(1/2).$$

Clearly, $\|r(f;h_n)\|_1$ *is not* $o(\|h_n\|_1$.

This example shows that, for example, for perturbations of the class $L^1(\mathcal{N},\tau)$ *this theory does not work. One has to take functions from narrower than* $C^{1+}(\mathbb{R})$ *class.*

Theorem 3.3.6 *Let* $\mathcal{N}$ *be a von Neumann algebra on a Hilbert space* $\mathcal{H}$*, let* $H = H^*$ *be affiliated with* $\mathcal{N}$ *and let* $V_1,\ldots,V_n \in \mathcal{E}_{sa}$*. If* $f \in C^{n+1+}(\mathbb{R})$*, then the function* $f\colon H' \in H+\mathcal{E}_{sa} \mapsto f(H') \in f(H)+\mathcal{E}_{sa}$ *is n-times affinely Fréchet differentiable along* $\mathcal{E}_{sa}$ *and*

$$\mathcal{D}^n_{\mathcal{E}} f(H)(V_1,\ldots,V_n) = \sum_{\sigma\in P_n} T^{H,\ldots,H}_{f^{[n]}}(V_{\sigma(1)},\ldots,V_{\sigma(n)}) \in \mathcal{E}, \tag{3.37}$$

where P_n *is the standard permutation group.*

Proof. If $n=1$ then this theorem is exactly Theorem 3.3.3. Set $\tilde H = H + V_{n+1}$. By induction we have

$$\begin{aligned} &D^n f(\tilde H; V_1,\ldots,V_n) - D^n f(H;V_1,\ldots,V_n) \\ &\quad = \sum_{\sigma\in P_n}\left(T^{\tilde H,\tilde H,\ldots,\tilde H}_{f^{[n]}}(V_{\sigma(1)},\ldots,V_{\sigma(n)}) - T^{H,H,\ldots,H}_{f^{[n]}}(V_{\sigma(1)},\ldots,V_{\sigma(n)})\right).\end{aligned}$$

A single term of this sum is

$$\begin{aligned} &T^{\tilde H,\tilde H,\ldots,\tilde H}_{f^{[n]}}(V_{\sigma(1)},\ldots,V_{\sigma(n)}) - T^{H,H,\ldots,H}_{f^{[n]}}(V_{\sigma(1)},\ldots,V_{\sigma(n)}) \\ &= \sum_{j=0}^{n}\left(T^{\tilde H,\ldots,\overset{(j)}{\tilde H},H,\ldots,H}_{f^{[n]}}(V_{\sigma(1)},\ldots,V_{\sigma(n)}) - T^{\tilde H,\ldots,\tilde H,\overset{(j)}{H},\ldots,H}_{f^{[n]}}(V_{\sigma(1)},\ldots,V_{\sigma(n)})\right).\end{aligned}$$

Now, the j-th summand is (Lemma 3.2.15)

$$\begin{aligned} &T^{\tilde H,\ldots,\overset{(j)}{\tilde H},H,\ldots,H}_{f^{[n]}}(V_{\sigma(1)},\ldots,V_{\sigma(n)}) - T^{\tilde H,\ldots,\tilde H,\overset{(j)}{H},\ldots,H}_{f^{[n]}}(V_{\sigma(1)},\ldots,V_{\sigma(n)}) \\ &\quad = \int_{\Pi^{(n)}} e^{i(s_0-s_1)\tilde H} V_{\sigma(1)}\ldots V_{\sigma(j)} e^{i(s_j-s_{j+1})\tilde H} V_{\sigma(j+1)} e^{i(s_{j+1}-s_{j+2})H} \\ &\qquad\qquad V_{\sigma(j+2)}\ldots V_{\sigma(n)} e^{is_nB}\, d\nu^{(n)}_f(s_0,\ldots,s_n) \\ &\qquad - \int_{\Pi^{(n)}} e^{i(s_0-s_1)\tilde H} V_{\sigma(1)}\ldots V_{\sigma(j-1)} e^{i(s_{j-1}-s_j)\tilde H} V_{\sigma(j)} e^{i(s_j-s_{j+1})H} \\ &\qquad\qquad V_{\sigma(j+1)}\ldots V_{\sigma(n)} e^{is_nB}\, d\nu^{(n)}_f(s_0,\ldots,s_n) \\ &\quad = \int_{\Pi^{(n)}} e^{i(s_0-s_1)\tilde H} V_{\sigma(1)}\ldots V_{\sigma(j)}\left(e^{i(s_j-s_{j+1})\tilde H} - e^{i(s_j-s_{j+1})H}\right) \\ &\qquad\qquad V_{\sigma(j+1)}\ldots V_{\sigma(n)} e^{is_nB}\, d\nu^{(n)}_f(s_0,\ldots,s_n).\end{aligned}$$

By Duhamel's formula (Lemma 1.1.8), we have

$$T_{f^{[n]}}^{\tilde{H},\dots,\overset{(j)}{\tilde{H}},H,\dots,H}(V_{\sigma(1)},\dots,V_{\sigma(n)}) - T_{f^{[n]}}^{\tilde{H},\dots,\tilde{H},\overset{(j)}{H},\dots,H}(V_{\sigma(1)},\dots,V_{\sigma(n)})$$
$$= \int_{\Pi^{(n)}} e^{i(s_0-s_1)\tilde{H}} V_{\sigma(1)}\dots V_{\sigma(j)} \left(\int_0^{s_j-s_{j+1}} e^{iu\tilde{H}} iV_{n+1} e^{i(s_j-s_{j+1}-u)H}\,du\right)$$
$$V_{\sigma(j+1)}\dots V_{\sigma(n)} e^{is_nB}\,d\nu_f^{(n)}(s_0,\dots,s_n).$$

Applying Fubini's theorem (Lemma 1.4.10) we get

$$T_{f^{[n]}}^{\tilde{H},\dots,\overset{(j)}{\tilde{H}},H,\dots,H} - T_{f^{[n]}}^{\tilde{H},\dots,\tilde{H},\overset{(j)}{H},\dots,H}$$
$$= i\int_{\Pi^{(n)}} \int_0^{s_j-s_{j+1}} e^{i(s_0-s_1)\tilde{H}} V_{\sigma(1)}\dots V_{\sigma(j)} e^{iu\tilde{H}} V_{n+1}$$
$$e^{i(s_j-s_{j+1}-u)H} V_{\sigma(j+1)}\dots V_{\sigma(n)} e^{is_nB}\,du\,d\nu_f^{(n)}(s_0,\dots,s_n). \quad (3.38)$$

Hence, it follows from formula (3.38), Lemma 3.2.5 and the fact that multiple operator integral is well-defined (Theorem 3.2.8) that

$$T_{f^{[n]}}^{\tilde{H},\dots,\overset{(j)}{\tilde{H}},H,\dots,\overset{(n)}{H}}(V_{\sigma(1)},\dots,V_{\sigma(n)}) - T_{f^{[n]}}^{\tilde{H},\dots,\tilde{H},\overset{(j)}{H},\dots,\overset{(n)}{H}}(V_{\sigma(1)},\dots,V_{\sigma(n)})$$
$$= T_{f^{[n+1]}}^{\tilde{H},\dots,\overset{(j)}{\tilde{H}},H,\dots,\overset{(n+1)}{H}}(V_{\sigma(1)},\dots,V_{\sigma(j)},V_{n+1},V_{\sigma(j+1)},\dots,V_{\sigma(n)}).$$

Since the multiple operator integral on the right hand side minus the same multiple operator integral with the last $\tilde{H}$ replaced by H has the order of $o((\max\|V_j\|)^{n+2})$ by Duhamel's formula, we see that the theorem is proved.

That the value of the derivative (3.37) belongs to $\mathcal{E}$ follows from Lemma 3.2.10. □

The argument of the last proof and Lemma 3.2.15 implies

Corollary 3.3.7 *Let $\mathcal{N}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. If $H = H^*$ is affiliated with $\mathcal{N}$, if $V \in \mathcal{E}_{sa}$ and if $f \in C^{n+1+}(\mathbb{R})$, then*

$$f(H+V) - f(H)$$
$$= T_{f^{[1]}}^{H,H}(V) + T_{f^{[2]}}^{H,H,H}(V,V) + \dots + T_{f^{[n]}}^{H,\dots,H}(V,\dots,V) + O(\|V\|_{\mathcal{E}}^{n+1}).$$

Proof. This corollary is a consequence of Theorem 3.3.6 and Taylor's formula [Sch, Theorem 1.43]. □

3.4 Spectral shift and spectral averaging in semifinite von Neumann algebras

The aim of this subsection is to prove a semifinite extension of a formula for spectral averaging due to Birman-Solomyak [BS].

Lemma 3.4.1 *If $(\mathcal{N}, \tau)$ is a semifinite von Neumann algebra, if $H = H^*$ is affiliated with $\mathcal{N}$ and $V \in \mathcal{L}^1(\mathcal{N}, \tau)$, then the function $\gamma(\lambda, r) = \tau\left(VE_\lambda^{H_r}\right)$ is measurable, where $H_r := H + rV,\ r \in [0,1]$.*

Proof. Let $\varphi_{\lambda,\varepsilon}$ be a smooth approximation of $\chi_{(-\infty,\lambda]}$. We note that $\varphi_{\lambda,\varepsilon}(H) = \varphi_{0,\varepsilon}(H - \lambda)$, and that the unbounded-operator valued function $(\lambda, r) \in \mathbb{R}^2 \mapsto H_r - \lambda$ is resolvent uniformly continuous [RS, VIII.19]. It follows from Theorem 1.1.6 that the function $(\lambda, r) \mapsto \varphi_{\lambda,\varepsilon}(H_r)$ is *so*-continuous, so that Lemma 1.3.30 implies that the function $(\lambda, r) \mapsto \tau\left(V\varphi_{\lambda,\varepsilon}(H_r)\right)$ is continuous. Now, since $\varphi_{\lambda,\varepsilon} \to \chi_{(-\infty,\lambda]}$ pointwise as $\varepsilon \to 0$, the operator $\varphi_{\lambda,\varepsilon}(H_r)$ converges to $\chi_{(-\infty,\lambda]}(H_r)$ in *so*-topology by [RS, Theorem VIII.5(d)]. Hence, again by Lemma 1.3.30, the function $\tau\left(V\chi_{(-\infty,\lambda]}(H_r)\right)$ is measurable as pointwise limit of continuous functions. $\square$

Theorem 3.4.2 *Let $(\mathcal{N}, \tau)$ be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ with a faithful normal semifinite trace τ. Let $H = H^*$ be affiliated with $\mathcal{N}$ and let $V = V^* \in \mathcal{L}^1(\mathcal{N}, \tau)$. If $f \in C^{2+}(\mathbb{R})$, then $f(H+V) - f(H) \in \mathcal{L}^1(\mathcal{N}, \tau)$ and*

$$\tau\left(f(H+V) - f(H)\right) = \int_{\mathbb{R}} f'(\lambda)\, d\Xi(\lambda),$$

where the measure Ξ is given by

$$\Xi(a,b) = \int_0^1 \tau\left(VE_{(a,b)}^{H_r}\right) dr, \quad a, b \in \mathbb{R}.$$

Here $H_r := H + rV,\ r \in [0,1]$ and $dE_\lambda^{H_r}$ is the spectral measure of H_r.

Due to Lemma 3.4.1 the measure Ξ is well-defined.

Proof. If $\varphi(\lambda, \mu) = \alpha(\lambda)\beta(\mu)$, where α, β are continuous bounded functions on $\mathbb{R}$, then by the definition of the multiple operator integral

$$T_\varphi^{H,H}(V) = \alpha(H)V\beta(H).$$

Hence,

$$\tau\left(T_\varphi^{H,H}(V)\right) = \tau\left(\alpha(H)V\beta(H)\right) = \tau\left(\alpha(H)\beta(H)V\right).$$

Since the function $\alpha(\cdot)\beta(\cdot)$ is bounded, the simple spectral approximations to the bounded operator $\alpha(H)\beta(H)$ converge uniformly and so, after multiplying by V, converge in norm of $\mathcal{L}^1(\mathcal{N},\tau)$. This implies that

$$\tau\left(\alpha(H)\beta(H)V\right) = \tau\left(\int_{\mathbb{R}} \alpha(\lambda)\beta(\lambda)\, dE_\lambda^H\, V\right) = \int_{\mathbb{R}} \alpha(\lambda)\beta(\lambda)\tau\left(dE_\lambda^H\, V\right).$$

Hence, for functions of the form $\varphi(\lambda,\mu) = \alpha(\lambda)\beta(\mu)$, it follows that

$$\tau\left(T_\varphi^{H,H}(V)\right) = \int_{\mathbb{R}} \varphi(\lambda,\lambda)\tau\left(dE_\lambda^H\, V\right). \tag{3.39}$$

Let (S,Σ,ν) be a finite (complex) measure space, let $\alpha(\cdot,\cdot),\beta(\cdot,\cdot)$ be bounded continuous functions on $\mathbb{R}\times S$ and suppose that

$$\varphi(\lambda,\mu) = \int_S \alpha(\lambda,\sigma)\beta(\mu,\sigma)\, d\nu(\sigma), \quad \lambda,\mu\in\mathbb{R},$$

is a BS-representation (3.27) of φ. Let $\varphi_\sigma(\lambda,\mu) := \alpha(\lambda,\sigma)\beta(\mu,\sigma)$. It then follows from the definition of the multiple operator integral that $T_\varphi^{H,H}(V) = \int_S T_{\varphi_\sigma}^{H,H}(V)\, d\nu(\sigma)$ and hence by Corollary 3.2.12

$$\tau\left(T_\varphi^{H,H}(V)\right) = \int_S \tau\left(T_{\varphi_\sigma}^{H,H}(V)\right)\, d\nu(\sigma).$$

It follows from (3.39) that

$$\begin{aligned}\tau\left(T_\varphi^{H,H}(V)\right) &= \int_S\int_{\mathbb{R}} \varphi_\sigma(\lambda,\lambda)\tau\left(dE_\lambda^H\, V\right)\, d\nu(\sigma)\\ &= \int_{\mathbb{R}}\int_S \varphi_\sigma(\lambda,\lambda)\, d\nu(\sigma)\tau\left(dE_\lambda^H\, V\right)\\ &= \int_{\mathbb{R}} \varphi(\lambda,\lambda)\tau\left(dE_\lambda^H\, V\right).\end{aligned} \tag{3.40}$$

The interchange of integrals in the second equality is justified by Lemma 1.3.33 and Fubini's theorem. Further, since $f\in C^{2+}(\mathbb{R})$, it follows from Theorem 3.3.3 applied to $\mathcal{E} = \mathcal{L}^1(\mathcal{N},\tau)$ that the Fréchet derivative $\mathcal{D}_{\mathcal{L}^1} f(H_r) = T_{f^{[1]}}^{H_r,H_r}$ exists for all $r\in[0,1]$. By the continuity of the Fréchet derivative given by the estimate (3.34) and the Newton-Leibnitz formula for the Fréchet derivative (Theorem 1.2.2), it follows that

$$\int_0^1 T_{f^{[1]}}^{H_r,H_r}(V)\, dr = \int_0^1 \mathcal{D}_{\mathcal{L}^1} f(H_r)(V)\, dr = f(H+V) - f(H).$$

Taking traces by Lemma 1.4.13 we have

$$\int_0^1 \tau\left(T_{f^{[1]}}^{H_r,H_r}(V)\right)\, dr = \tau\left(f(H+V)-f(H)\right). \tag{3.41}$$

Since $f^{[1]}$ is continuous, $f^{[1]}(\lambda,\lambda) = f'(\lambda)$, so that (3.41) and (3.40) imply

$$\begin{aligned}\tau\left(f(H+V)-f(H)\right) &= \int_0^1 \int_{\mathbb{R}} f^{[1]}(\lambda,\lambda)\tau\left(dE_\lambda^{H_r} V\right)\, dr\\ &= \int_0^1 \int_{\mathbb{R}} f'(\lambda)\tau\left(dE_\lambda^{H_r} V\right)\, dr\\ &= \int_{\mathbb{R}} f'(\lambda) \int_0^1 \tau\left(dE_\lambda^{H_r} V\right)\, dr,\end{aligned}$$

the interchange of the integrals in the last equality being justified by Fubini's theorem due to Lemma 1.3.33 and the fact that f' is a bounded function. □

The next corollary in the case that $\mathcal{N} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{Tr}$ was established in [BS].

Corollary 3.4.3 *The measure Ξ is absolutely continuous and the following equality holds*

$$d\Xi(\lambda) = \xi(\lambda)\, d\lambda,$$

where $\xi(t)$ is the spectral shift function for the pair $(H+V, H)$.

Proof. From Theorems 3.1.13 and 3.4.2 it follows that

$$\int_{\mathbb{R}} f'(\lambda)\, d\Xi(\lambda) = \int_{\mathbb{R}} f'(\lambda)\xi(\lambda)\, d\lambda$$

for all $f \in C_c^\infty(\mathbb{R})$. Consequently, the measures $d\Xi(\lambda)$ and $\xi(\lambda)\, d\lambda$ have the same derivative in the sense of generalized functions. By [GSh, Ch. I.2.6] there exists a constant c such that

$$d\Xi(\lambda) - \xi(\lambda)\, d\lambda = c \cdot d\lambda.$$

Since the measures $d\Xi(\lambda)$ and $\xi(\lambda)\, d\lambda$ are finite, it follows immediately, that $c = 0$. □

Chapter 4

Spectral shift function and spectral flow

4.1 Preliminary results

We denote by $\mathcal{N}$ a semifinite von Neumann algebra acting on Hilbert space $\mathcal{H}$, with a faithful normal semifinite trace τ.

If $D = D^* \eta \mathcal{N}$ and $R_z(D)$ is τ-compact for some (and hence for all) $z \in \mathbb{C} \backslash \mathbb{R}$, then we say that D has τ-compact resolvent.

We constantly use some parameters for specific purposes. The parameter r will always be an operator path parameter, i. e. the letter r is used when we consider paths of operators such as $D_r = D_0 + rV$. Very rarely we need another path parameter which we denote by s. We do not use t as path parameter, since t is used for other purposes later in the paper. The letter λ is always used as a spectral parameter. If we need another spectral parameter we will use μ.

The following elementary fact will be used repeatedly.

Lemma 4.1.1 *If Ω is an open interval in $\mathbb{R}$ and if $f \in C_c^k(\Omega)$, then there exist functions $f_1, f_2 \in C_c^k(\Omega)$ such that f_1, f_2 are non-negative, $f = f_1 - f_2$ and $\sqrt{f_1}, \sqrt{f_2} \in C_c^k(\Omega)$.*

Proof. Let $[a, b]$ be a closed interval, $[a, b] \subseteq \Omega$ and $\operatorname{supp}(f) \subseteq (a, b)$. Take a non-negative C^∞-function $f_1 \geqslant f$ on $[a, b]$ which vanishes at a and b in such a way that $\sqrt{f_1}$ is C^∞-smooth at a and b, and take $f_2 = f_1 - f$. $\Box$

4.1.1 Self-adjoint operators with τ-compact resolvent

In this subsection, we collect some facts about operators with compact resolvent in a semifinite von Neumann algebra.

Lemma 4.1.2 *If $D = D^*\eta\mathcal{N}$ has τ-compact resolvent, then for all compact sets $\Delta \subseteq \mathbb{R}$ the spectral projection E^D_Δ is τ-finite.*

Proof. If D has τ-compact resolvent then the operator $(1 + D^2)^{-1} = (D + i)^{-1}(D - i)^{-1}$ is τ-compact. Since for every finite interval Δ there exists a constant $c > 0$, not depending on D such that $E^D_\Delta \leqslant c(1+D^2)^{-1}$, the projection E^D_Δ is also τ-compact, and hence τ-finite. $\square$

Corollary 4.1.3 *If $D = D^*\eta\mathcal{N}$ has τ-compact resolvent, then for all $f \in B_c(\mathbb{R})$ the operator $f(D)$ is τ-trace class.*

Proof. There exists a finite segment $\Delta \subseteq \mathbb{R}$ such that $|f| \leqslant \mathrm{const}\,\chi_\Delta$, so that $|f(D)| \leqslant \mathrm{const}\,E^D_\Delta$. $\square$

The following lemma and its proof are taken from [CP].

Lemma 4.1.4 [CP, Appendix B, Lemma 6] *If D_0 is an unbounded self-adjoint operator, A is a bounded self-adjoint operator, and $D = D_0 + A$ then*

$$(1 + D^2)^{-1} \leqslant f(\|A\|)(1 + D_0^2)^{-1},$$

where $f(a) = 1 + \frac{1}{2}a^2 + \frac{1}{2}a\sqrt{a^2 + 4}$.

Proof. (A) If $A\,\mathrm{dom}(D_0) \subseteq \mathrm{dom}\,D_0$, so that $\mathrm{dom}\,D^2 = \mathrm{dom}\,D_0^2$, then there exists a positive constant C such that:

$$1 + D_0^2 \leqslant C(1 + D^2) \qquad \text{on} \quad \mathrm{dom}\,D^2 = \mathrm{dom}\,D_0^2.$$

Proof of (A). We need to prove that for some $C > 0$ and for all vectors ξ of norm 1 in $\mathrm{dom}\,D_0^2$:

$$\langle\xi,\xi\rangle + \langle D_0\xi, D_0\xi\rangle \leqslant C[\langle\xi,\xi\rangle + \langle D_0\xi, D_0\xi\rangle + \langle A\xi, A\xi\rangle + \langle D_0\xi, A\xi\rangle + \langle A\xi, D_0\xi\rangle]$$

or, letting $x = \|D_0\xi\|$ and $a = \|A\xi\|$,

$$1 + x^2 \leqslant C\left(1 + x^2 + a^2 + 2xa\right),$$

which would follow from:

$$1 + x^2 \leqslant C\left(1 + x^2 + a^2 - 2xa\right).$$

One easily calculates the maximum value of $\frac{1+x^2}{1+(x-a)^2}$ to be $f(a) = 1 + \frac{1}{2}a^2 + \frac{1}{2}a\sqrt{a^2+4}$. Since $a \leqslant \|A\|$ and f is clearly increasing, we get

$$1 + x^2 \leqslant f(\|A\|)\left(1 + (x-a)^2\right)$$

and so $(1 + D_0^2) \leqslant f(\|A\|)(1 + D^2)$ on $\operatorname{dom} D_0^2$. Lemma 1.1.3 now implies

$$(1 + D^2)^{-1} \leqslant f(\|A\|)(1 + D_0^2)^{-1}.$$

(B) To rid ourselves of the restrictive hypothesis that $A(\operatorname{dom} D_0) \subseteq \operatorname{dom} D_0$, let $E_n = E^{D_0}_{[-n,n]}$ and let $A_n = E_n A E_n$. Then A_n is self-adjoint, $\|A_n\| \leqslant \|A\|$ and $A_n(\operatorname{dom} D_0) \subseteq \operatorname{dom} D_0$. If $D_n = D_0 + A_n$ then by (A)

$$(1 + D_n^2)^{-1} \leqslant f(\|A\|)(1 + D_0^2)^{-1}.$$

Now, since $E_n \to 1$ in *so*-topology by Theorem 1.1.4, we have that $A_n \to A$ in *so*-topology by Theorem 1.1.1, and hence $D_n\xi \to D\xi$ for every $\xi \in \operatorname{dom} D_0$. So, [RS, Theorem VIII.25] implies that $D_n \to D$ in strong resolvent sense, and Theorem 1.1.6 implies that

$$(1 + D_n^2)^{-1} \to (1 + D^2)^{-1}$$

in *so*-topology. This completes the proof. □

Lemma 4.1.5 *Let $D_0 = D_0^* \eta \mathcal{N}$ have τ-compact resolvent, and let $B_R = \{V = V^* \in \mathcal{N} \colon \|V\| \leqslant R\}$. Then for any compact subset $\Delta \subseteq \mathbb{R}$ the function*

$$V \in B_R \mapsto E_\Delta^{D_0+V}$$

is $\mathcal{L}^1(\mathcal{N}, \tau)$-bounded.

Proof. We have $E_\Delta^{D_0+V} \leqslant c_0(1+(D_0+V)^2)^{-1}$ for some constant $c_0 = c_0(\Delta) > 0$ and for every $V = V^* \in \mathcal{N}$. Now, by Lemma 4.1.4 there exists a constant $c_1 = c_1(R) > 0$, such that for all $V \in B_R$

$$(1 + (D_0 + V)^2)^{-1} \leqslant c_1(1 + D_0^2)^{-1}.$$

Hence, since D_0 has τ-compact resolvent, all projections $E_\Delta^{D_0+V}$, $V \in B_R$, are bounded from above by a single τ-compact operator $T = c_0c_1(1 + D_0^2)^{-1}$. This means, that for $t > 0$

$$\mu_t(E_\Delta^{D_0+V}) \leqslant \mu_t(T).$$

Further, by (1.22) $\mu_t(E_\Delta^{D_0+V}) = \chi_{[0,\tau\left(E_\Delta^{D_0+V}\right))}(t)$ and there exists $t_0 > 0$ such that $\mu_{t_0}(T) < 1$. This implies that for all $V \in B_R$, $\tau\left(E_\Delta^{D_0+V}\right) \leqslant t_0$. □

Corollary 4.1.6 *If $D_0 = D_0^* \eta \mathcal{N}$ has τ-compact resolvent, then for any function $f \in B_c(\mathbb{R})$ the function $V \in B_R \mapsto \|f(D_0 + V)\|_{\mathcal{L}^1}$ is bounded.*

Corollary 4.1.7 *Let $D_0 = D_0^* \eta \mathcal{N}$ have τ-compact resolvent, $r = (r_1, \ldots, r_m) \in [0,1]^m$, $V_1, \ldots, V_m \in \mathcal{N}_{sa}$ and set $D_r = D_0 + r_1 V_1 + \ldots + r_m V_m$. Then*
(i) for any compact subset $\Delta \subseteq \mathbb{R}$ the function $r \in [0,1]^m \mapsto \left\| E_\Delta^{D_r} \right\|_1$ is bounded;
(ii) for any function $f \in B_c(\mathbb{R})$ the function $r \in [0,1] \mapsto \|f(D_r)\|_1$ is bounded.

Lemma 4.1.8 *If $D_0 = D_0^* \eta \mathcal{N}$ and if $V = V^* \in \mathcal{N}$, then for any $t \in \mathbb{R}$, $e^{it(D_0+V)}$ converges in $\|\cdot\|$-norm to e^{itD_0} when $\|V\| \to 0$.*

Proof. It follows directly from Duhamel's formula (1.6)

$$e^{it(D_0+V)} - e^{itD_0} = \int_0^t e^{i(t-u)(D_0+V)} iV e^{iuD_0}\, du$$

and Lemma 1.4.14 □

4.1.2 Difference quotients and double operator integrals

Lemma 4.1.11 provides a modification of the BS-representation for $f^{[1]}$ from Lemma 3.2.3 with which we will constantly work.

We include the proof of the following fact for completeness.

Lemma 4.1.9 *(i) If $f \in C_c^1(\mathbb{R})$, then $\hat{f} \in L^1(\mathbb{R})$.*
(ii) the function

$$\phi(x) := \frac{x}{\sqrt{1+x^2}} \tag{4.1}$$

belongs to $C^{2,+}(\mathbb{R})$.

Proof. (i) [BR, Corollary 3.2.33] The Schwartz inequality and the Parseval's identity imply

$$\begin{aligned}
\int_{\mathbb{R}} |\hat{f}|(\xi)\, d\xi &= \int_{\mathbb{R}} |\xi + i|^{-1} |\xi + i| |\hat{f}|(\xi)\, d\xi \\
&\leqslant \left(\int_{\mathbb{R}} |\xi + i|^{-2}\, d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\xi + i|^2 |\hat{f}|^2(\xi)\, d\xi \right)^{\frac{1}{2}} \\
&= \text{const} \left(\int_{\mathbb{R}} |f'(x) + f(x)|^2\, dx \right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

(ii) The derivatives $\phi'(x) = (1+x^2)^{-3/2}$, $\phi''(x) = -3x(1+x^2)^{-5/2}$, $\phi'''(x) = -3(1+x^2)^{-5/2} + \frac{15}{2} x (1+x^2)^{-7/2}$ belong to $L^1(\mathbb{R})$. So, by the argument of (i),

$$\int_{\mathbb{R}} |\xi|\, |\hat{\phi}|(\xi)\, d\xi \leqslant \text{const} \left(\int_{\mathbb{R}} |\phi''(x) + \phi'(x)|^2\, dx \right)^{\frac{1}{2}} < \infty$$

and

$$\int_{\mathbb{R}} |\xi|^2 \, |\hat{\phi}|(\xi) \, d\xi \leqslant \text{const} \left(\int_{\mathbb{R}} |\phi'''(x) + \phi''(x)|^2 \, dx \right)^{\frac{1}{2}} < \infty.$$

□

This lemma means that $C_c^1(\mathbb{R}) \subseteq C^{0,+}(\mathbb{R})$. Applying it to the first n derivatives of a function f from $C_c^{n+1}(\mathbb{R})$, we obtain

Corollary 4.1.10 $C_c^{n+1}(\mathbb{R}) \subseteq C^{n,+}(\mathbb{R})$, $n = 0, 1, 2, \ldots$

The following lemma provides a BS-representation for $f^{[1]}$, $0 \leqslant f \in C_c^2(\mathbb{R})$, which will be used throughout this section.

Lemma 4.1.11 *Let $f \in C_c^2(\mathbb{R})$ be a non-negative function such that $g := \sqrt{f} \in C_c^2(\mathbb{R})$. If $\Omega \supseteq \operatorname{supp}(f)$, then*

$$f^{[1]}(\lambda_0, \lambda_1) = \int_\Pi (\alpha_1(\lambda_0, \sigma)\beta_1(\lambda_1, \sigma) + \alpha_2(\lambda_0, \sigma)\beta_2(\lambda_1, \sigma)) \, d\nu_g(\sigma),$$

where $\sigma = (s_0, s_1)$ and

$$\begin{aligned} \alpha_1(\lambda_0, \sigma) &= e^{i(s_0 - s_1)\lambda_0} g(\lambda_0), \qquad & \beta_1(\lambda_1, \sigma) &= e^{is_1\lambda_1}, \\ \alpha_2(\lambda_0, \sigma) &= e^{i(s_0 - s_1)\lambda_0}, \qquad & \beta_2(\lambda_1, \sigma) &= e^{is_1\lambda_1} g(\lambda_1), \end{aligned} \tag{4.2}$$

so that $\alpha_1(\cdot, \sigma)$, $\beta_2(\cdot, \sigma) \in C_c^2(\Omega)$ for all $\sigma \in \Pi$, and $|\alpha_1(\cdot)|, |\beta_2(\cdot)| \leqslant \|g\|_\infty$, while $\alpha_2(\cdot, \sigma)$, $\beta_1(\cdot, \sigma) \in C^\infty(\mathbb{R})$ for all $\sigma \in \Pi$, and $|\alpha_2(\cdot)|, |\beta_1(\cdot)| \leqslant 1$.

Proof. The assumption $g \in C_c^2(\mathbb{R})$ implies that $g \in C^{1,+}(\mathbb{R})$ (see Corollary 4.1.10). Now,

$$\begin{aligned} f^{[1]}(\lambda_0, \lambda_1) &= \frac{g^2(\lambda_0) - g^2(\lambda_1)}{\lambda_0 - \lambda_1} \\ &= \frac{g(\lambda_0) - g(\lambda_1)}{\lambda_0 - \lambda_1} (g(\lambda_0) + g(\lambda_1)) = g^{[1]}(\lambda_0, \lambda_1) \left(g(\lambda_0) + g(\lambda_1) \right). \end{aligned}$$

Hence, using (3.2.3), we have

$$f^{[1]}(\lambda_0, \lambda_1) = \int_\Pi \Big(\alpha(\lambda_0, \sigma) g(\lambda_0) \beta(\lambda_1, \sigma) + \alpha(\lambda_0, \sigma) g(\lambda_1) \beta(\lambda_1, \sigma) \Big) \, d\nu_g(\sigma).$$

If we set $\alpha_1(\lambda, \sigma) = \alpha(\lambda, \sigma) g(\lambda)$, $\beta_1(\lambda, \sigma) = \beta(\lambda, \sigma)$, $\alpha_2(\lambda, \sigma) = \alpha(\lambda, \sigma)$ and $\beta_2(\lambda, \sigma) = g(\lambda)\beta(\lambda, \sigma)$, then we see that all the conditions of the Lemma are fulfilled. □

The following lemma is a corollary of Lemma 4.1.11 and the definition of the multiple operator integral (Definition 3.2.6).

Lemma 4.1.12 *If $D_0 = D_0^* \eta \mathcal{N}$, if $D_1 = D_0 + V$, $V = V^* \in \mathcal{N}$ and if $f \in C_c^2(\mathbb{R})$ is a non-negative function, such that $g := \sqrt{f} \in C_c^2(\mathbb{R})$, then*

$$T_{f^{[1]}}^{D_1,D_0}(V) = \int_\Pi (\alpha_1(D_1,\sigma)V\beta_1(D_0,\sigma) + \alpha_2(D_1,\sigma)V\beta_2(D_0,\sigma))\, d\nu_g(\sigma),$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$ are given by (4.2).

We need the following weaker version of [ACDS, Theorem 5.3]. See also [BS$_2$].

Proposition 4.1.13 [ACDS, Theorem 5.3] *Let $\mathcal{N}$ be a von Neumann algebra. Suppose that $D_0 = D_0^*$ is affiliated with $\mathcal{N}$, that $V \in \mathcal{N}$ is self-adjoint and set $D_1 = D_0 + V$.*
(i) If $f \in C^{1,+}(\mathbb{R})$, then

$$f(D_1) - f(D_0) = T_{f^{[1]}}^{D_1,D_0}(V).$$

(ii) If $f \in C^{2,+}(\mathbb{R})$, then the function $f\colon D_0 + \mathcal{N}_{sa} \mapsto f(D_0) + \mathcal{N}_{sa}$ is affinely $(\mathcal{N},\mathcal{N})$-Fréchet differentiable, the equality $\mathcal{D}_\mathcal{N} f(D) = T_{f^{[1]}}^{D,D}$ holds and $\mathcal{D}_\mathcal{N} f(D)$ is $\|\cdot\|$-continuous.

4.1.3 Some continuity and differentiability properties of operator functions

We are going to consider spectral flow along ‘continuous’ paths of unbounded Fredholm operators. We will make precise what we mean by continuity in this setting later. However our formulae require more than just continuity. They require us to be able to take derivatives with the respect to the path parameter. For this to be feasible we need the full force of the double operator integral formalism. We present the results we will need as a sequence of lemmas. In the sequel we will constantly need to take functions of a path of operators. We thus need the following continuity result. For the definition of B_R see Lemma 4.1.5.

Proposition 4.1.14 *If $D_0 = D_0^* \eta \mathcal{N}$ has τ-compact resolvent and if $f \in C_c^2(\mathbb{R})$ then the operator-valued function $A\colon V \in B_R \mapsto f(D_0 + V)$ takes values in $\mathcal{L}^1(\mathcal{N},\tau)$ and is $\mathcal{L}^1(\mathcal{N},\tau)$-continuous.*

Proof. That $A(\cdot)$ takes values in $\mathcal{L}^1(\mathcal{N},\tau)$ follows from Lemma 1.6.8 and Corollary 4.1.3. By Lemma 4.1.1 it is enough to prove continuity for a non-negative function f with $g = \sqrt{f} \in C_c^2(\mathbb{R})$. By Proposition 4.1.13(i) and Lemma 4.1.12 we have

$$\begin{aligned} f(D_0+V) - f(D_0) &= T_{f^{[1]}}^{D_0+V,D_0}(V) \\ &= \int_\Pi (\alpha_1(D_0+V,\sigma)V\beta_1(D_0,\sigma) + \alpha_2(D_0+V,\sigma)V\beta_2(D_0,\sigma))\, d\nu_g(\sigma). \end{aligned}$$

Hence, by Lemma 1.4.14, we have

$$\begin{aligned}
\|f(D_0+V)-f(D_0)\|_{\mathcal{L}^1} &\leqslant \int_\Pi \big[\, \|\alpha_1(D_0+V,\sigma)\|_{\mathcal{L}^1} \|V\| \, \|\beta_1(D_0,\sigma)\| \\
&\qquad + \|\alpha_2(D_0+V,\sigma)\| \, \|V\| \, \|\beta_2(D_0,\sigma)\|_{\mathcal{L}^1} \,\big] \, d\,|\nu_g|\,(\sigma) \\
&\leqslant \int_\Pi \big(\, \|g(D_0+V)\|_{\mathcal{L}^1} \|V\| \\
&\qquad + \|V\| \, \|g(D_0)\|_{\mathcal{L}^1} \,\big) \, d\,|\nu_g|\,(\sigma) \\
&\leqslant |\nu_g|\,(\Pi) \, \|V\| \, (\|g(D_0+V)\|_{\mathcal{L}^1} + \|g(D_0)\|_{\mathcal{L}^1}).
\end{aligned}$$

Now, Corollary 4.1.6 applied to g completes the proof. □

Corollary 4.1.15 *If $D_0 = D_0^* \eta \mathcal{N}$ has τ-compact resolvent, $r = (r_1, \dots, r_m) \in [a,b]^m$, if $V_1, \dots, V_m \in \mathcal{N}_{sa}$ and if $D_r = D_0 + rV = D_0 + r_1V_1 + \dots + r_mV_m$, then for any function $f \in C_c^2(\mathbb{R})$ the operator-valued function $A\colon r \in [a,b]^m \mapsto f(D_0+rV)$ takes values in $\mathcal{L}^1(\mathcal{N},\tau)$ and is $\mathcal{L}^1(\mathcal{N},\tau)$-continuous.*

Next we prove the main lemmas of this Section. There are several matters to establish. First we want to be able to differentiate, with respect to the path parameter, certain functions of paths of operators. Then we need to determine formulae for the derivatives and the continuity properties of the derivatives with respect to the path parameter.

Lemma 4.1.16 *If D_1 and D_2 are two self-adjoint operators with τ-compact resolvent affiliated with semifinite von Neumann algebra $\mathcal{N}$, if $X \in \mathcal{N}_{sa}$ and if $f \in C_c^3(\mathbb{R})$ then $T_{f^{[1]}}^{D_1,D_2}(X)$ depends $\mathcal{L}^1$-continuously on $\|\cdot\|$ perturbations of D_1 and D_2.*

Proof. As usual, we can assume that f is non-negative and its square root is C^3-smooth.

Let $Y_1, Y_2 \in \mathcal{N}_{sa}$. Then by Lemma 4.1.12

$$\begin{aligned}
&T_{f^{[1]}}^{D_1+Y_1,D_2+Y_2}(X) - T_{f^{[1]}}^{D_1,D_2}(X) \\
&\qquad = \int_\Pi \big[\alpha_1(D_1+Y_1,\sigma)X\beta_1(D_2+Y_2,\sigma) \\
&\qquad\qquad + \alpha_2(D_1+Y_1,\sigma)X\beta_2(D_2+Y_2,\sigma) \\
&\qquad\qquad - \alpha_1(D_1,\sigma)X\beta_1(D_2,\sigma) - \alpha_2(D_1,\sigma)X\beta_2(D_2,\sigma)\big] \, d\nu_f(\sigma) \\
&\qquad = \int_\Pi \Big(\big[\alpha_1(D_1+Y_1,\sigma) - \alpha_1(D_1,\sigma)\big] X\beta_1(D_2+Y_2,\sigma) \\
&\qquad\qquad + \alpha_1(D_1,\sigma)X\big[\beta_1(D_2+Y_2,\sigma) - \beta_1(D_2,\sigma)\big] \\
&\qquad\qquad + \big[\alpha_2(D_1+Y_1,\sigma) - \alpha_2(D_1,\sigma)\big] X\beta_2(D_2+Y_2,\sigma) \\
&\qquad\qquad + \alpha_2(D_1,\sigma)X\big[\beta_2(D_2+Y_2,\sigma) - \beta_2(D_2,\sigma)\big] \Big) \, d\nu_f(\sigma).
\end{aligned}$$

For every fixed $\sigma \in \Pi$ by Lemma 4.1.8 the norms $\|\beta_1(D_2+Y_2,\sigma)-\beta_1(D_2,\sigma)\|$ and $\|\alpha_2(D_1+Y_1,\sigma)-\alpha_2(D_1,\sigma)\|$ converge to zero when $\|Y_1\|, \ \|Y_2\| \to 0$, and by Corollary 4.1.6 the $\|\cdot\|_{\mathcal{L}^1}$-norms of $\alpha_1(D_1,\sigma)$ and $\beta_2(D_2+Y_2,\sigma)$ are bounded when $\|Y_1\|,\|Y_2\| \to 0$. Hence, for every fixed $\sigma \in \Pi$ the $\|\cdot\|_{\mathcal{L}^1}$-norms of the second and third summands in the last integral converge to zero when $\|Y_1\|,\|Y_2\| \to 0$.

Now we are going to show that the same is true for the first and fourth summands. It is enough to prove that for every fixed $\sigma \in \Pi$, for example, $\|\alpha_1(D_1+Y_1,\sigma)-\alpha_1(D_1,\sigma)\|_{\mathcal{L}^1}$ tends to zero. We have

$$\begin{aligned}
&\|\alpha_1(D_1+Y_1,\sigma)-\alpha_1(D_1,\sigma)\|_{1,\infty} \\
&\qquad = \left\| e^{i(s_0-s_1)(D_1+Y_1)} g(D_1+Y_1) - e^{i(s_0-s_1)D_1} g(D_1) \right\|_{\mathcal{L}^1} \\
&\qquad \leqslant \left\| \left(e^{i(s_0-s_1)(D_1+Y_1)} - e^{i(s_0-s_1)D_1} \right) g(D_1+Y_1) \right\|_{\mathcal{L}^1} \\
&\qquad\quad + \left\| e^{i(s_0-s_1)D_1} \left(g(D_1+Y_1) - g(D_1) \right) \right\|_{\mathcal{L}^1} \\
&\qquad \leqslant \left\| \left(e^{i(s_0-s_1)(D_1+Y_1)} - e^{i(s_0-s_1)D_1} \right) \right\| \|g(D_1+Y_1)\|_{\mathcal{L}^1} \\
&\qquad\quad + \left\| e^{i(s_0-s_1)D_1} \right\| \|g(D_1+Y_1) - g(D_1)\|_{\mathcal{L}^1} .
\end{aligned}$$

It follows from Lemma 4.1.8 that the first summand converges to zero when s_0, s_1 are fixed and $\|Y_1\| \to 0$, and it follows from Proposition 4.1.14 that the second summand also converges to zero.

Since by Corollary 4.1.6 the trace norm of the expression under the last integral is uniformly $\mathcal{L}^1(\mathcal{N},\tau)$-bounded with respect to $\sigma \in \Pi$, it follows from Lemma 1.4.14 that

$$\left\| T^{D_1+Y_1,D_2+Y_2}_{f^{[1]}}(X) - T^{D_1,D_2}_{f^{[1]}}(X) \right\|_{\mathcal{L}^1} \to 0,$$

when $\|Y_1\|,\|Y_2\| \to 0$. □

Theorem 4.1.17 *If the von Neumann algebra $\mathcal{N}$ is semifinite, $D_0 = D_0^* \eta \mathcal{N}$ has τ-compact resolvent and $f \in C_c^3(\mathbb{R})$ then the function $f\colon D \in D_0 + \mathcal{N}_{sa} \mapsto f(D) \in \mathcal{N}_{sa}$ takes values in $\mathcal{L}^1(\mathcal{N},\tau)_{sa}$. Moreover, it is affinely $(\mathcal{N},\mathcal{L}^1)$-Fréchet differentiable, the equality*

$$\mathcal{D}_{\mathcal{N},\mathcal{L}^1} f(D) = T^{D,D}_{f^{[1]}}$$

holds, and $\mathcal{D}_{\mathcal{N},\mathcal{L}^1} f(D)$ is $(\mathcal{N},\mathcal{L}^1)$-continuous, so that

$$f(D_b) - f(D_a) = \int_a^b T^{D_r,D_r}_{f^{[1]}}(V)\, dr, \tag{4.3}$$

where $V \in \mathcal{N}_{sa}$, $D_r = D_0 + rV$ and the integral converges in $\mathcal{L}^1(\mathcal{N},\tau)$-norm.

Proof. We have by Proposition 4.1.13(i) and Lemma 4.1.12

$$\begin{aligned}
f(D_1) - f(D_0) &= T^{D_1,D_0}_{f^{[1]}}(V) \\
&= \int_\Pi (\alpha_1(D_1,\sigma)V\beta_1(D_0,\sigma) + \alpha_2(D_1,\sigma)V\beta_2(D_0,\sigma))\, d\nu_g(\sigma), \\
&= \int_\Pi (\alpha_1(D_0,\sigma)V\beta_1(D_0,\sigma) + \alpha_2(D_0,\sigma)V\beta_2(D_0,\sigma))\, d\nu_g(\sigma) \\
&\qquad + \int_\Pi [\alpha_1(D_1,\sigma) - \alpha_1(D_0,\sigma)]V\beta_1(D_0,\sigma)\, d\nu_g(\sigma) \\
&\qquad + \int_\Pi [\alpha_2(D_1,\sigma) - \alpha_2(D_0,\sigma)]V\beta_2(D_0,\sigma)\, d\nu_g(\sigma) \\
&= T^{D_0,D_0}_{f^{[1]}}(V) + (II) + (III).
\end{aligned}$$

Since α_2 is just an exponent and since $g \in C^{2,+}(\mathbb{R})$ that $\|(III)\|_{\mathcal{L}^1} = O(\|V\|^2)$ can be shown by Duhamel's formula. The argument is as in the proof of [ACDS, Theorem 5.5]. So, it is left to show that $\|(II)\|_{\mathcal{L}^1}$ is $o(\|V\|)$. By Lemma 1.4.14 we have

$$\begin{aligned}
\|(II)\|_{\mathcal{L}^1} &= \left\| \int_\Pi [\alpha_1(D_1,\sigma) - \alpha_1(D_0,\sigma)]V\beta_1(D_0,\sigma)\, d\nu_g(\sigma) \right\|_{\mathcal{L}^1} \\
&\leqslant \int_\Pi \|\alpha_1(D_1,\sigma) - \alpha_1(D_0,\sigma)\|_{\mathcal{L}^1} \|V\| \, \|\beta_1(D_0,\sigma)\| \; d\nu_g(\sigma) \\
&= \|V\| \int_\Pi \|\alpha_1(D_1,\sigma) - \alpha_1(D_0,\sigma)\|_{\mathcal{L}^1} \; d\nu_g(\sigma).
\end{aligned}$$

Now, it follows from $\alpha_1(\cdot,\sigma) \in C^2_c(\mathbb{R})$ (see (4.2)) and Proposition 4.1.14 that $\|\alpha_1(D_1,\sigma) - \alpha_1(D_0,\sigma)\|_{\mathcal{L}^1} \to 0,\ \sigma \in \Pi$, so that by the Lebesgue dominated convergence theorem we conclude that the last integral converges to 0, and hence $\|(II)\|_{\mathcal{L}^1} = o(\|V\|)$.

Finally, that $\mathcal{D}_{\mathcal{N},\mathcal{L}^1} f(D)$ is $(\mathcal{N},\mathcal{L}^1)$-continuous follows from Lemma 4.1.16. □

4.1.4 A class $\mathcal{F}^{a,b}(\mathcal{N},\tau)$ of τ-Fredholm operators

Our technique for handling spectral flow of paths of unbounded operators is to map them into the space of bounded operators using a particular function. We thus need to discuss some continuity properties of paths of bounded τ-Fredholm operators, analogous to those we described in the unbounded case.

Let $a < b$ be two real non-zero numbers. Let $\mathcal{F}^{a,b}(\mathcal{N},\tau)$ be the set of bounded self-adjoint τ-Fredholm operators $F \in \mathcal{N}$ such that $(F-a)(F-b) \in \mathcal{K}(\mathcal{N},\tau)$. For $F_0 \in \mathcal{F}^{a,b}(\mathcal{N},\tau)$ let $\mathcal{A}_{F_0} = F_0 + \mathcal{K}_{sa}(\mathcal{N},\tau)$ be the affine space of τ-compact self-adjoint perturbations of F_0.

Lemma 4.1.18 *If $F_0 \in \mathcal{F}^{a,b}(\mathcal{N},\tau)$ then*

$$\mathcal{A}_{F_0} \subseteq \mathcal{F}^{a,b}(\mathcal{N},\tau).$$

Proof. If $K \in \mathcal{K}_{sa}(\mathcal{N},\tau)$ then $(F_0+K-a)(F_0+K-b) = (F_0-a)(F_0-b) + (F_0-a)K + K(F_0+K-b) \in \mathcal{K}(\mathcal{N},\tau)$. $\square$

Lemma 4.1.19 *If $F \in \mathcal{F}^{a,b}(\mathcal{N},\tau)$ and $h \in B_c(a,b)$ then $h(F) \in \mathcal{L}^1(\mathcal{N},\tau)$.*

Proof. The proof is similar to the proof of Lemma 4.1.2. For any compact subset Δ of (a,b) there exists a constant $c_0 > 0$ such that $\chi_\Delta(x) \leqslant c_0\chi_{[a,b]}(x)(b-x)(x-a)$, so that

$$E^F_\Delta \leqslant c_0 E^F_{[a,b]}(b-F)(F-a). \tag{4.4}$$

Since $(b-F)(F-a) \in \mathcal{K}(\mathcal{N},\tau)$, it follows that $E^F_\Delta \in \mathcal{K}(\mathcal{N},\tau)$ and hence E^F_Δ is τ-finite. Now, for any $h \in B_c(a,b)$ there exists a compact subset Δ of (a,b) and a constant c_1 such that $|h| \leqslant c_1\chi_\Delta$, so that $|h(F)| \leqslant c_1 E^F_\Delta$ and hence $h(F) \in \mathcal{L}^1(\mathcal{N},\tau)$. $\square$

Lemma 4.1.20 *If $F_0 \in \mathcal{F}^{a,b}(\mathcal{N},\tau)$, $K = K^* \in \mathcal{K}(\mathcal{N},\tau)$, and if Δ is a compact subset of (a,b), then*
(i) the function $r \in [0,1] \mapsto E^{F_0+rK}_\Delta$ takes values in $\mathcal{L}^1(\mathcal{N},\tau)$ and is $\mathcal{L}^1(\mathcal{N},\tau)$-bounded;
(ii) there exists $R > 0$ such that the function $K \in B_R \cap \mathcal{K}(\mathcal{N},\tau) \mapsto E^{F_0+K}_\Delta$ takes values in $\mathcal{L}^1(\mathcal{N},\tau)$ and is $\mathcal{L}^1(\mathcal{N},\tau)$-bounded.

Proof. (i) That $E^{F_r}_\Delta = E^{F_0+rK}_\Delta \in \mathcal{L}^1(\mathcal{N},\tau)$ follows from Lemmas 4.1.18 and 4.1.19. By (4.4) we have $E^{F_r}_\Delta \leqslant c_0 E^{F_r}_{[a,b]}(b-F_r)(F_r-a)$ for all $r \in [0,1]$ and hence by Lemmas 1.3.15(ii) and 1.3.19

$$\begin{aligned}\mu_t(E^{F_r}_\Delta) &\leqslant c_0\mu_t\left(E^{F_r}_{[a,b]}(b-F_r)(F_r-a)\right)\\ &\leqslant c_0\mu_t\left((b-F_r)(F_r-a)\right).\end{aligned}$$

Since $(b-F_r)(F_r-a) = (b-F_0)(F_0-a) + rL_1 - r^2L_2$, where $L_1, L_2 \in \mathcal{K}(\mathcal{N},\tau)$, we have by Lemma 1.3.18

$$\begin{aligned}\mu_t(E^{F_r}_\Delta) &\leqslant c_0\left(\mu_{t/3}[(b-F_0)(F_0-a)] + r\mu_{t/3}(L_1) + r^2\mu_{t/3}(L_2)\right)\\ &\leqslant c_0\left(\mu_{t/3}[(b-F_0)(F_0-a)] + \mu_{t/3}(L_1) + \mu_{t/3}(L_2)\right),\end{aligned}$$

so that $\mu_t(E^{F_r}_\Delta) = \chi_{[0,\tau(E^{F_r}_\Delta))}(t)$ is majorized for all $r \in [0,1]$ by a single function decreasing to 0 when $t \to \infty$, since all three operators $(b-F_0)(F_0-a)$, L_1 and

L_2 are τ-compact. The same argument as in Lemma 4.1.5 now completes the proof.

(ii) If $F = F_0 + K$ then $(b - F)(F - a) = (b - F_0)(F_0 - a) + L$, where $L = (b - F_0)K - K(F_0 - a) - K^2 \in \mathcal{K}(\mathcal{N}, \tau)$. Choose the number $R > 0$ such that $\|K\| < R$ implies $\|L\| < 1$. Then by (4.5) the function $t \mapsto \mu_t(E_\Delta^{F+K}) = \chi_{[0,\tau(E_\Delta^{F+K})]}(t)$ will be majorized by a single function decreasing to a number < 1, so that the same argument as in Lemma 4.1.5 again completes the proof. □

Proposition 4.1.21 *Let $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$, $K = K^* \in \mathcal{K}(\mathcal{N}, \tau)$, and let $h \in C_c^2(a, b)$. Then*
(i) the function $r \in \mathbb{R} \mapsto h(F_0 + rK)$ takes values in $\mathcal{L}^1(\mathcal{N}, \tau)$ and is $\mathcal{L}^1(\mathcal{N}, \tau)$-continuous;
(ii) there exists $R > 0$ such that the function $K \in B_R \cap \mathcal{K}(\mathcal{N}, \tau) \mapsto h(F_0 + K)$ takes values in $\mathcal{L}^1(\mathcal{N}, \tau)$ and is $\mathcal{L}^1(\mathcal{N}, \tau)$-continuous;

Proof. The proof of this proposition follows verbatim the proof of Proposition 4.1.14 with references to Lemmas 4.1.18, 4.1.19 and 4.1.20 instead of Lemmas 1.6.8, 4.1.3 and Corollary 4.1.6. □

Lemma 4.1.22 *If $F_1, F_2 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$, $X \in \mathcal{K}_{sa}(\mathcal{N}, \tau)$ and $h \in C_c^3(a, b)$, then the double operator integral*

$$T_{h^{[1]}}^{F_1, F_2}(X)$$

takes values in $\mathcal{L}^1(\mathcal{N}, \tau)$ and is $\mathcal{L}^1(\mathcal{N}, \tau)$-continuous with respect to norm perturbations of F_1 and F_2 by τ-compact operators.

Proof. The proof of this lemma is similar to that of Lemma 4.1.16 with references to Lemma 4.1.20(ii) and Proposition 4.1.21(ii) instead of Corollary 4.1.6 and Proposition 4.1.14.

As usual, we can assume that h is non-negative and its square root $g = \sqrt{h}$ is C^2-smooth. Let $F_{1,s} = F_1 + sK_1$, $F_{2,r} = F_2 + rK_2$. By Lemma 4.1.12

$$T_{h^{[1]}}^{F_{1,s}, F_{2,r}}(X) = \int_\Pi \left[\alpha_1(F_{1,s}, \sigma) X \beta_1(F_{2,r}, \sigma) + \alpha_2(F_{1,s}, \sigma) X \beta_2(F_{2,r}, \sigma)\right] d\nu_g(\sigma),$$

so that

$$
\begin{aligned}
&T_{h^{[1]}}^{F_{1,s},F_{2,r}}(X) - T_{h^{[1]}}^{F_{1,s_0},F_{2,r_0}}(X) \\
&\qquad = \int_\Pi \big[\alpha_1(F_{1,s},\sigma)X\beta_1(F_{2,r},\sigma) + \alpha_2(F_{1,s},\sigma)X\beta_2(F_{2,r},\sigma) \\
&\qquad\qquad - \alpha_1(F_{1,s_0},\sigma)X\beta_1(F_{2,r_0},\sigma) - \alpha_2(F_{1,s_0},\sigma)X\beta_2(F_{2,r_0},\sigma)\big]\, d\nu_g(\sigma) \\
&\qquad = \int_\Pi \Big(\big[\alpha_1(F_{1,s},\sigma) - \alpha_1(F_{1,s_0},\sigma)\big]X\beta_1(F_{2,r},\sigma) \\
&\qquad\qquad + \alpha_1(F_{1,s_0},\sigma)X\big[\beta_1(F_{2,r},\sigma) - \beta_1(F_{2,r_0},\sigma)\big] \\
&\qquad\qquad + \big[\alpha_2(F_{1,s},\sigma) - \alpha_2(F_{1,s_0},\sigma)\big]X\beta_2(F_{2,r},\sigma) \\
&\qquad\qquad + \alpha_2(F_{1,s_0},\sigma)X\big[\beta_2(F_{2,r},\sigma) - \beta_2(F_{2,r_0},\sigma)\big]\Big)\, d\nu_g(\sigma)
\end{aligned}
$$

In the last integral for every fixed $\sigma \in \Pi$, when $(r,s) \to (r_0,s_0)$ the $\mathcal{L}^1(\mathcal{N},\tau)$-norms of the second and third summands converge to zero by Lemmas 4.1.8 and 4.1.19, and the $\mathcal{L}^1(\mathcal{N},\tau)$-norms of the first and fourth summands converge to zero by Proposition 4.1.21. Since the $\mathcal{L}^1(\mathcal{N},\tau)$-norm of the expression under the last integral is uniformly $\mathcal{L}^1(\mathcal{N},\tau)$-bounded with respect to $\sigma \in \Pi$ by Lemma 4.1.20, it follows from the Lebesgue Dominated Convergence Theorem that

$$
\left\| T_{h^{[1]}}^{F_{1,s},F_{2,r}}(X) - T_{h^{[1]}}^{F_{1,s_0},F_{2,r_0}}(X) \right\|_{\mathcal{L}^1} \to 0,
$$

when $(r,s) \to (r_0,s_0)$. □

Theorem 4.1.23 *Let $\mathcal{N}$ be a semifinite von Neumann algebra. If $F_0 \in \mathcal{F}^{a,b}(\mathcal{N},\tau)$, $h \in C_c^3(a,b)$, then the function $h\colon F \in F_0 + \mathcal{K}_{sa}(\mathcal{N},\tau) \mapsto h(F_0) + \mathcal{K}_{sa}(\mathcal{N},\tau)$ takes values in $\mathcal{L}^1(\mathcal{N},\tau)_{sa}$. Moreover, it is affinely $(\mathcal{K},\mathcal{L}^1)$-Fréchet differentiable, the equality*

$$
\mathcal{D}_{\mathcal{K},\mathcal{L}^1} h(F) = T_{h^{[1]}}^{F,F}
$$

holds, and $\mathcal{D}_{\mathcal{K},\mathcal{L}^1} h(F)$ is $(\mathcal{K},\mathcal{L}^1)$ continuous, so that

$$
h(F_{r_1}) - h(F_{r_0}) = \int_{r_1}^{r_0} T_{h^{[1]}}^{F_r,F_r}(K)\, dr, \quad r_0, r_1 \in \mathbb{R}, \tag{4.5}
$$

where $K \in \mathcal{K}_{sa}(\mathcal{N},\tau)$, $F_r = F_0 + rK$ and the integral is in $\mathcal{L}^1(\mathcal{N},\tau)$-norm.

The proof is similar to that of Theorem 4.1.17 with use of Proposition 4.1.21(ii) and Lemma 4.1.22 instead of Proposition 4.1.14 and Lemma 4.1.16, and therefore it is omitted.

4.2 The spectral shift function for operators with compact resolvent

We will take an approach to the notion of spectral shift function suggested by Birman-Solomyak formula (4.6). The key point is that once one appreciates that the spectral shift function of M. G. Krein is related to spectral flow in a specific fashion one can reformulate the whole approach to take advantage of what is known about spectral flow as expounded for example in [BCPRSW]. The theorem in [ACDS] which connects spectral flow and the spectral shift function contains the germ of the idea but one needs the technical machinery of the last Section to exploit this.

We now explain this different way to approach spectral shift theory which is influenced by ideas from noncommutative geometry.

4.2.1 The unbounded case

In order to make our main definition we need to prove a preliminary result which complements [ACDS, Lemma 6.2]. The latter asserts that the function $\gamma(\lambda, r) = \tau\left(VE^{D_r}_\lambda\right)$ is measurable for every $V \in \mathcal{L}^1(\mathcal{N}, \tau)$ and $D = D^*\eta\mathcal{N}$.

Lemma 4.2.1 *Let $(\mathcal{N}, \tau)$ be a semifinite von Neumann algebra and let $D = D^*\eta\mathcal{N}$ have τ-compact resolvent. If $V = V^* \in \mathcal{N}$ then the function $f\colon (a, b, r) \in \mathbb{R}^3 \mapsto \tau\left(VE^{D_r}_{(a,b)}\right)$ is measurable.*

Proof. Without loss of generality, we can assume that $V \geqslant 0$. It is enough to prove that the function f is measurable with respect to the second variable b and with respect to r. Since $\tau\left(VE^{D_r}_{(a,b)}\right) = \tau\left(\sqrt{V}E^{D_r}_{(a,b)}\sqrt{V}\right)$, we know by Lemma 1.4.12 that it is enough to prove that the operator function $(r, b) \mapsto \sqrt{V}E^{D_r}_{(a,b)}\sqrt{V}$ is so^*-measurable. By Proposition 1.4.4 it is enough to prove that for any $\xi, \eta \in \mathcal{H}$ the scalar function $\left\langle \sqrt{V}E^{D_r}_{(a,b)}\sqrt{V}\xi, \eta\right\rangle = \mathrm{Tr}(\theta_{\sqrt{V}\xi,\sqrt{V}\eta}E^{D_r}_{(a,b)})$ is measurable, where $\theta_{\xi,\eta}(\zeta) := \langle\xi, \zeta\rangle\,\eta$. Since the operator $\theta_{\sqrt{V}\xi,\sqrt{V}\eta}$ is trace class, the measurability of this function follows from Lemma 3.4.1. □

Definition 4.2.2 *If $D_0 = D_0^*\eta\mathcal{N}$ has τ-compact resolvent and if $D_1 = D_0 + V$, $V \in \mathcal{N}_{sa}$, then the* spectral shift measure *for the pair (D_0, D_1) is defined to be the following Borel measure on $\mathbb{R}$*

$$\Xi_{D_1,D_0}(\Delta) = \int_0^1 \tau\left(VE^{D_r}_\Delta\right)\,dr, \quad \Delta \in \mathcal{B}(\mathbb{R}). \tag{4.6}$$

The generalized function

$$\xi_{D_1,D_0}(\lambda) = \frac{d}{d\lambda}\Xi_{D_1,D_0}(a,\lambda) \tag{4.7}$$

is called the spectral shift distribution *for the pair* (D_0, D_1).

Evidently, this definition does not depend on a choice of a. By Lemmas 1.6.8, 4.1.2, 4.2.1 and Corollary 4.1.7(i) the measure Ξ exists and is locally finite.

Our task now is to show that the spectral shift distribution is in fact a function of locally bounded variation. The main result we wish to establish next is that the spectral shift measure is absolutely continuous with respect to Lebesgue measure. Moreover its density, which we previously referred to as the spectral shift distribution, is in fact a function of locally bounded variation which we will then refer to as the spectral shift function. It is our extension of M. G. Krein's function to the setting of this paper.

Our method of proof is to first establish some trace formulae.

Lemma 4.2.3 *(i) Let $D = D^*\eta\mathcal{N}$ have τ-compact resolvent. A function $\alpha \in B(\mathbb{R})$ is 1-summable with respect to the measure $\tau\left(E^D_\Delta\right)$ $(\Delta \in \mathcal{B}(\mathbb{R}))$, if and only if $\alpha(D) \in \mathcal{L}^1(\mathcal{N},\tau)$ and in this case*

$$\tau\left(\alpha(D)\right) = \int_{\mathbb{R}} \alpha(\lambda)\, \tau\left(dE^D_\lambda\right).$$

Furthermore, for any $V = V^* \in \mathcal{N}$ the function α is 1-summable with respect to the measure $\tau\left(VE^D_\Delta\right)$, and

$$\tau\left(V\alpha(D)\right) = \int_{\mathbb{R}} \alpha(\lambda)\, \tau\left(VdE^D_\lambda\right).$$

(ii) Let $F \in \mathcal{F}^{a,b}(\mathcal{N},\tau)$. A function $\alpha \in B(a,b)$ is 1-summable with respect to the measure $\tau\left(E^F_\Delta\right)$ $(\Delta \in \mathcal{B}(a,b))$, if and only if $\alpha(F) \in \mathcal{L}^1(\mathcal{N},\tau)$ and in this case

$$\tau\left(\alpha(F)\right) = \int_a^b \alpha(\lambda)\, \tau\left(dE^F_\lambda\right).$$

Furthermore, for any $V = V^* \in \mathcal{N}$ the function α is 1-summable with respect to the measure $\tau\left(VE^F_\Delta\right)$, and

$$\tau\left(V\alpha(F)\right) = \int_a^b \alpha(\lambda)\, \tau\left(VdE^F_\lambda\right).$$

Proof. We give only the proof of (i). Without loss of generality, we can assume that α is a non-negative function. If α is a simple function then the first part of the claim follows from Lemma 4.1.2. Let α_n be an increasing sequence of simple non-negative functions, converging pointwise to α.

Then for each of the functions α_n the first equality is true. The supremum of the increasing sequence of non-negative operators $\alpha_n(D)$ is $\alpha(D)$ and the supremum of the increasing sequence of numbers $\int_{\mathbb{R}} \alpha_n(\lambda)\, \tau\left(dE_\lambda^D\right)$ is $\int_{\mathbb{R}} \alpha(\lambda)\, \tau\left(dE_\lambda^D\right)$. Hence, both non-negative numbers $\int_{\mathbb{R}} \alpha(\lambda)\, \tau\left(dE_\lambda^D\right)$ and $\tau\left(\alpha(D)\right)$ are finite or infinite simultaneously, which proves the first part of the lemma.

For the second part we can assume w.l.o.g. that $V \geqslant 0$. Then again the both parts of the second equality make sense and they are equal for simple functions.

Since the measure $\tau\left(VE_\Delta^D\right) = \tau\left(\sqrt{V}E_\Delta^D\sqrt{V}\right)$, $\Delta \in \mathcal{B}(\mathbb{R})$, is non-negative and the supremum of $\sqrt{V}\alpha_n(D)\sqrt{V} \in \mathcal{L}^1(\mathcal{N}, \tau)$ is $\sqrt{V}\alpha(D)\sqrt{V}$ we have that

$$\begin{aligned}\int_{\mathbb{R}} \alpha(\lambda)\, \tau\left(VdE_\lambda^D\right) &= \lim_{n\to\infty} \int_{\mathbb{R}} \alpha_n(\lambda)\, \tau\left(VdE_\lambda^D\right) \\ &= \lim_{n\to\infty} \tau\left(\sqrt{V}\alpha_n(D)\sqrt{V}\right) = \tau\left(\sqrt{V}\alpha(D)\sqrt{V}\right),\end{aligned}$$

so that $\int_{\mathbb{R}} \alpha(\lambda)\, \tau\left(VdE_\lambda^D\right)$ and $\tau\left(V\alpha(D)\right) = \tau\left(\sqrt{V}\alpha(D)\sqrt{V}\right)$ are finite or infinite simultaneously. $\square$

We need the following version of Fubini's theorem.

Lemma 4.2.4 *(i) For any self-adjoint operator $D\eta\mathcal{N}$ with τ-compact resolvent and $V = V^* \in \mathcal{N}$, let $m_{D,V}(\Delta) = \tau\left(VE_\Delta^D\right)$. Let $D_0 = D_0^*\eta\mathcal{N}$ have τ-compact resolvent and let $D_r = D_0 + rV$. If $g \in B_c(\mathbb{R})$, then*

$$\int_0^1 dr \int_{\mathbb{R}} g(\lambda)\, m_{D_r,V}(d\lambda) = \int_{\mathbb{R}} g(\lambda)\Xi_{D_1,D_0}(d\lambda). \tag{4.8}$$

(ii) For any $F \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ and $V = V^* \in \mathcal{N}$, let $m_{F,V}(\Delta) = \tau\left(VE_\Delta^F\right)$, $\Delta \in \mathcal{B}(a, b)$. Let $F \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ and let $F_r = F_0 + rV$. If $g \in B_c(a, b)$, then

$$\int_0^1 dr \int_a^b g(\lambda)\, m_{F_r,V}(d\lambda) = \int_a^b g(\lambda)\Xi_{F_1,F_0}(d\lambda).$$

Proof. (See also [Ja, VI.2]). We give only the proof of (i). The measurability of the function $r \mapsto \int_{\mathbb{R}} g(\lambda)\, m_{D_r,V}(d\lambda)$ follows from Lemma 4.2.1.

Note, that both integrals are repeated ones. Let $\Omega \supseteq \mathrm{supp}(g)$ be a finite interval. By Corollary 4.1.7 (i) there exists $M > 0$ such that for all $r \in [0, 1]$ we have $|m_{D_r,V}(\Omega)| \leqslant M$.

If $g(\lambda) = \chi_\Delta(\lambda)$, $\Delta \in \mathcal{B}(\Omega)$, then

$$\int_0^1 dr \int_\Omega \chi_\Delta(\lambda)\, m_{D_r,V}(d\lambda) = \int_0^1 m_{D_r,V}(\Delta)\, dr$$
$$= \Xi(\Delta) = \int_\Omega \chi_\Delta(\lambda)\, \Xi(d\lambda).$$

So, (4.8) is true for simple functions. Let now g be an arbitrary function from $B_c(\Omega)$, let $\varepsilon > 0$ and let h be a simple function such that $\|g-h\|_\infty < \varepsilon$. Then the LHS of (4.8) is equal to

$$\int_0^1 dr \int_\Omega (g-h)(\lambda)\, m_{D_r,V}(d\lambda) + \int_0^1 dr \int_\Omega h(\lambda)\, m_{D_r,V}(d\lambda) = (I) + (II),$$

and the RHS of (4.8) is equal to

$$\int_\Omega (g-h)(\lambda)\, \Xi(d\lambda) + \int_\Omega h(\lambda)\, \Xi(d\lambda) = (III) + (IV).$$

We have $(II) = (IV)$. Further, $|(I)| \leqslant M\|g-h\|_\infty \leqslant M\varepsilon$ and $|(III)| \leqslant M\|g-h\|_\infty \leqslant M\varepsilon$. $\Box$

The following theorem complements [ACDS, Theorem 6.3].

Theorem 4.2.5 *If $D = D^* \eta \mathcal{N}$ has τ-compact resolvent, if $V = V^* \in \mathcal{N}$, and if $D_1 = D_0 + V$, then the measure Ξ_{D_1,D_0} is absolutely continuous, its density is equal to*

$$\xi_{D_1,D_0}(\cdot) = \tau\left(E^{D_0}_{(a,\lambda]}\right) - \tau\left(E^{D_1}_{(a,\lambda]}\right) + \text{const} \tag{4.9}$$

for almost all $\lambda \in \mathbb{R}$. Moreover, for all $f \in C_c^3(\mathbb{R})$ $\quad f(D_1) - f(D_0) \in \mathcal{L}^1(\mathcal{N}, \tau)$ and

$$\tau\left(f(D_1) - f(D_0)\right) = \int_{\mathbb{R}} f'(\lambda)\xi_{D_1,D_0}(\lambda)\, d\lambda. \tag{4.10}$$

Proof. By Lemma 1.6.8 and Corollary 4.1.3 $f(D_1) - f(D_0) \in \mathcal{L}^1(\mathcal{N}, \tau)$.

By Lemma 4.1.1 we need only consider the case of a non-negative function f with $g := \sqrt{f} \in C_c^2(\mathbb{R})$.

We have by (4.3)

$$f(D_1) - f(D_0) = \int_0^1 T^{D_r,D_r}_{f^{[1]}}(V)\, dr,$$

where the integral converges in $\mathcal{L}^1(\mathcal{N},\tau)$-norm. Hence, it follows from Lemma 4.1.12 that

$$f(D_1) - f(D_0) = \int_0^1 \int_\Pi \left(\alpha_1(D_r,\sigma)V\beta_1(D_r,\sigma) + \alpha_2(D_r,\sigma)V\beta_2(D_r,\sigma)\right) d\nu_g(\sigma)\, dr. \tag{4.11}$$

Now, for a fixed $\sigma \in \Pi$, we have

$$\begin{aligned}\tau\Big(\alpha_1(D_r,\sigma)V\beta_1(D_r,\sigma) + \alpha_2(D_r,\sigma)V\beta_2(D_r,\sigma)\Big) &= \tau\left(V(\alpha_1(D_r,\sigma)\beta_1(D_r,\sigma) + \alpha_2(D_r,\sigma)\beta_2(D_r,\sigma))\right)\\ &= \int_{\mathbb{R}} \left(\alpha_1(\lambda,\sigma)\beta_1(\lambda,\sigma) + \alpha_2(\lambda,\sigma)\beta_2(\lambda,\sigma)\right) \tau\left(V dE_\lambda^{D_r}\right),\end{aligned}$$

where the last equality uses Lemma 4.2.3. (That $\alpha_1(\lambda)\beta_1(\lambda) + \alpha_2(\lambda)\beta_2(\lambda)$ belongs to $B_c(\mathbb{R})$ follows from Lemma 4.1.11)

Hence using (4.11), and by Lemma 1.4.13 applied to the finite measure space $([0,1]\times\Pi, dr\times\nu_g)$ our previous equality implies that we have:

$$\begin{aligned}A &:= \tau\left(f(D_1) - f(D_0)\right)\\ &= \int_0^1 \int_\Pi \tau\left(\alpha_1(D_r,\sigma)V\beta_1(D_r,\sigma) + \alpha_2(D_r,\sigma)V\beta_2(D_r,\sigma)\right) d\nu_g(\sigma)\, dr\\ &= \int_0^1 \int_\Pi \int_{\mathbb{R}} \left(\alpha_1(\lambda,\sigma)\beta_1(\lambda,\sigma) + \alpha_2(\lambda,\sigma)\beta_2(\lambda,\sigma)\right) \tau\left(V dE_\lambda^{D_r}\right) d\nu_g(\sigma)\, dr.\end{aligned}$$

Now, by Lemma 4.2.3, Fubini's theorem, and Lemma 4.1.11 we have

$$\begin{aligned}A &= \int_0^1 \int_{\mathbb{R}} \int_\Pi \left(\alpha_1(\lambda,\sigma)\beta_1(\lambda,\sigma) + \alpha_2(\lambda,\sigma)\beta_2(\lambda,\sigma)\right) d\nu_g(\sigma)\, \tau\left(V dE_\lambda^{D_r}\right) dr\\ &= \int_0^1 \int_{\mathbb{R}} f'(\lambda)\, \tau\left(V dE_\lambda^{D_r}\right) dr.\end{aligned}$$

Finally, by Lemmas 4.2.4 we have

$$A = \int_{\mathbb{R}} f'(\lambda) \int_0^1 \tau\left(V dE_\lambda^{D_r}\right) dr = \int_{\mathbb{R}} f'(\lambda)\, d\Xi_{D_1,D_0}(\lambda). \tag{4.12}$$

Let $f \in C_c^1(\mathbb{R})$ and take a point a outside of the support of f. Then we have (see [AB, Proposition 8.5.5])

$$\begin{aligned}A &= \tau\left(f(D_1) - f(D_0)\right) = \tau\left(f(D_1)\right) - \tau\left(f(D_0)\right)\\ &= \int_{\mathbb{R}} f(\lambda) d\tau\left(E_{(a,\lambda]}^{D_1}\right) - \int_{\mathbb{R}} f(\lambda) d\tau\left(E_{(a,\lambda]}^{D_0}\right) \qquad \text{(integrating by parts)}\\ &= -\int_{\mathbb{R}} f'(\lambda)\left(\tau\left(E_{(a,\lambda]}^{D_1}\right) - \tau\left(E_{(a,\lambda]}^{D_0}\right)\right) d\lambda.\end{aligned} \tag{4.13}$$

Comparing (4.13) and (4.12) we see that Ξ is absolutely continuous with density equal to

$$\xi_{D_1,D_0}(\lambda) = \tau\left(E^{D_0}_{(a,\lambda]}\right) - \tau\left(E^{D_1}_{(a,\lambda]}\right) + \text{const}\,. \tag{4.14}$$

□

It is worth noting that the formula (4.10) does not determine the function ξ uniquely, but only up to an additive constant.

Remark 5 *This theorem is an analogue of Theorem 3.1.13, in which the existence and absolute continuity of the spectral shift measure were proved for any self-adjoint operator D affiliated with $\mathcal{N}$ and τ-trace class operator $V \in \mathcal{N}$.*

As a result of what we have proved to this point we are now in a position to assert that in fact the spectral shift distribution is an everywhere defined function and hence to change our terminology and refer to ξ as a function. Moreover this last lemma enables one to modify ξ so as to make it a function defined everywhere in a natural way.

Definition 4.2.6 *If the expression (4.14) is continuous at a point $\lambda \in \mathbb{R}$, then we define $\xi_{D_1,D_0}(\lambda)$ via formula (4.14). Otherwise, we define the value of the spectral shift function ξ at a discontinuity point to be half sum of left and right limits.*

Corollary 4.2.7 *The spectral shift function ξ is a function of locally bounded variation.*

Proof. This is immediate because ξ is the difference of two increasing functions by the last formula. □

Lemma 4.2.8 *Let $D_0 \eta \mathcal{N}$ be a self-adjoint operator with τ-compact resolvent, let $V \in \mathcal{N}_{sa}$ and let $D_1 = D_0 + V$. If $f \in B_c(\mathbb{R})$ then*

$$\int_{-\infty}^{\infty} f(\lambda)\xi_{D_1,D_0}(\lambda)\, d\lambda = \int_0^1 \tau\left(V f(D_r)\right)\, dr.$$

Proof. It follows from Lemma 4.2.3 and Lemma 4.2.4 that

$$\begin{aligned}\int_0^1 \tau\left(V f(D_r)\right)\, dr &= \int_0^1 \int_{-\infty}^{\infty} f(\lambda)\tau\left(V dE^{D_r}_\lambda\right)\, dr \\ &= \int_{-\infty}^{\infty} f(\lambda) \int_0^1 \tau\left(V dE^{D_r}_\lambda\right)\, dr.\end{aligned}$$

□

The situation where the operators D and $D+V$, are unitarily equivalent arises naturally in noncommutative geometry in the context of spectral triples. One thinks of the unitary implementing the equivalence as a gauge transformation by analogy with the study of gauge transformations of Dirac type operators. It thus warrants special consideration especially in view of our first result below.

Theorem 4.2.9 *Let D be a self-adjoint operator affiliated with $\mathcal{N}$ having τ-compact resolvent and let $V = V^* \in \mathcal{N}$ be such that the operators $D+V$ and D are unitarily equivalent. Then the spectral shift function $\xi_{D+V,D}$ of the pair $(D+V, D)$ is constant on $\mathbb{R}$.*

Proof. The operators $f(D+V)$ and $f(D)$ are unitarily equivalent and for $f \in C_c^\infty(\mathbb{R})$ they are τ-trace class by Corollary 4.1.3. Hence,

$$\tau\left(f(D+V) - f(D)\right) = 0,$$

so that by Theorem 4.2.5 the equality

$$\int_{\mathbb{R}} f'(\lambda)\xi_{D+V,D}(\lambda)\, d\lambda = 0$$

holds for any $f \in C_c^\infty(\mathbb{R})$. Now, integration by parts shows that $\xi'(\lambda)$ is zero as generalized function on $\mathbb{R}$, which by [GSh, Ch. I.2.6] implies that ξ is equal to a constant function. □

Note, function ξ in this theorem is equal to a constant function everywhere, not just almost everywhere.

Our second major result on the spectral shift function in this special context is the following theorem. We shall show in Section 4.3 below that this theorem extends one of the main results of [CP_2].

Theorem 4.2.10 *Let D_0 be a self-adjoint operator with τ-compact resolvent, affiliated with $\mathcal{N}$. Let $V = V^* \in \mathcal{N}$ be such that the operators $D_1 = D_0 + V$ and D_0 are unitarily equivalent. If $f \in C_c^2(\mathbb{R})$ then*

$$\xi_{D_1,D_0}(\mu) = C^{-1} \int_0^1 \tau\left(Vf(D_r - \mu)\right)\, dr, \quad \forall\ \mu \in \mathbb{R}, \tag{4.15}$$

where $C = \int_{\mathbb{R}} f(\lambda)\, d\lambda$.

Proof. For any fixed μ the operator $D_r - \mu$ has τ-compact resolvent by Lemma 1.6.8 and the function $r \mapsto \tau\left(Vf(D_r - \mu)\right)$ is continuous by Proposition 4.1.14, so that the integral on the RHS of (4.15) exists. By Lemma 4.2.8 and Theorem 4.2.9 we have

$$\int_0^1 \tau\left(Vf(D_r - \mu)\right)\, dr = \int_{\mathbb{R}} f(\lambda - \mu)\xi_{D_1,D_0}(\lambda)\, d\lambda = \xi(0) \int_{\mathbb{R}} f(\lambda)\, d\lambda.$$

□

4.2.2 The bounded case

Our technique in the next Section for discussing spectral flow in the unbounded case is to map into the space of bounded τ-Fredholm operators. We thus need to develop the theory described in the previous subsections *ab initio* for the bounded case. Fortunately, this is not a difficult task as the proofs are much the same. As we will see, because we are considering bounded perturbations of our unbounded operators, it suffices to consider compact perturbations in the bounded case.

Definition 4.2.11 *If $F_0 \in \mathcal{F}^{a,b}(\mathcal{N},\tau)$, $K \in \mathcal{K}_{sa}(\mathcal{N},\tau)$, $F_1 = F_0 + K$ and if $F_r = F_0 + rK$, then the* spectral shift measure *for the pair (F_0, F_1) is defined to be the following Borel measure on (a,b)*

$$\Xi_{F_1,F_0}(\Delta) = \int_0^1 \tau\left(KE_\Delta^{F_r}\right) dr, \quad \Delta \in \mathcal{B}(a,b). \tag{4.16}$$

The generalized function

$$\xi_{F_1,F_0}(\lambda) = \frac{d}{d\lambda}\Xi_{F_1,F_0}(c,\lambda), \quad c \in (a,b), \tag{4.17}$$

is called the spectral shift distribution *for the pair (F_0, F_1).*

Evidently, this definition does not depend on a choice of $c \in (a,b)$. The measurability of the function $r \mapsto \tau\left(KE_\Delta^{F_r}\right)$ may be established following the argument of Lemma 4.2.1, using Lemma 4.1.20. It follows that the measure Ξ exists and is locally-finite on (a,b).

Proposition 4.2.12 *If $F_0 \in \mathcal{F}^{a,b}(\mathcal{N},\tau)$, $K \in \mathcal{K}(\mathcal{N},\tau)$ and if $F_1 = F_0 + K$, then*
(i) the measure Ξ_{F_1,F_0} is absolutely continuous and its density is equal to

$$\xi_{F_1,F_0}(\lambda) = \tau\left(E_{(c,\lambda]}^{F_0} - E_{(c,\lambda]}^{F_1}\right) + \text{const}, \ \lambda \in (c,b),$$

where c is an arbitrary number from (a,b);
(ii) there exists a unique function $\xi_{F_1,F_0}(\cdot)$ of locally bounded variation on (a,b), such that for any $h \in C_c^2(a,b)$ the following equality holds true

$$\tau\left(h(F_1) - h(F_0)\right) = \int_a^b h'(\lambda)\xi_{F_1,F_0}(\lambda)\, d\lambda.$$

The proof is identical to the proof of Theorem 4.2.5, with references to 4.1.18, 4.1.19, (4.5) instead of 1.6.8, 4.1.3, (4.3) and hence we omit it.

Corollary 4.2.13 *In the setting of Proposition 4.2.12, if F_0 and F_1 are unitarily equivalent, then ξ_{F_1,F_0} is constant on (a,b).*

Proof. The proof is similar to the proof of Theorem 4.2.9.

For any $h \in C_c^\infty(a,b)$ the operators $h(F_0)$, $h(F_1)$ are τ-trace class by Lemma 4.1.19, and since they are also unitarily equivalent it follows that $\tau\left(h(F_1) - h(F_0)\right) = 0$. Hence, it follows from Proposition 4.2.12 that

$$\int_a^b h'(\lambda)\xi_{F_1,F_0}(\lambda)\,d\lambda = 0,$$

which implies that the generalized derivative of $\xi_{F_1,F_0}(\cdot)$ is equal to 0 on (a,b), so that $\xi_{F_1,F_0}(\cdot) = \text{const}$ on (a,b). □

Definition 4.2.14 *We redefine the function ξ_{F_1,F_0} at discontinuity points to be half the sum of the left and the right limits of the RHS of the last equality.*

Thus, the function ξ_{F_1,F_0} is defined everywhere on (a,b).

Lemma 4.2.15 *If $F_0 \in \mathcal{F}^{a,b}(\mathcal{N},\tau)$, $K \in \mathcal{K}(\mathcal{N},\tau)$, if $F_r = F_0 + rK$, $r \in [0,1]$ and if $h \in B_c(a,b)$ then*

$$\int_{\mathbb{R}} h(\lambda)\xi_{F_1,F_0}(\lambda)\,d\lambda = \int_0^1 \tau\left(Kh(F_r)\right)\,dr.$$

This Lemma and its proof are bounded variants of Lemma 4.2.8, so we omit the details.

4.3 Spectral flow

4.3.1 The spectral flow function

The next lemma is a strengthening of Carey-Phillips Theorem 1.5.37.

Lemma 4.3.1 *Let P and Q be two projections in the semifinite von Neumann algebra $\mathcal{N}$ and let $a < 0 < b$ be two real numbers. Let κ be a continuous function such that for any $s \in [0, \frac{(b-a)^2}{4}]$ $\kappa(s(P-Q)^2)$ is τ-trace class. Then*

$F_0 = (b-a)P + a$ and $F_1 = (b-a)Q + a$ are self-adjoint τ-Fredholm operators from $\mathcal{F}^{a,b}(\mathcal{N}, \tau)$ as is the path $F_r = F_0 + r(F_1 - F_0)$, and

$$\mathrm{sf}(\{F_r\}) = C_{a,b}^{-1} \int_0^1 \tau\left(\dot{F}_r \kappa[(b-F_r)(F_r-a)]\right) dr,$$

where $C_{a,b} = \int_0^1 (b-a)\kappa\big((b-a)^2(r-r^2)\big)\, dr$ is a constant, and the derivative $\dot{F}_r$ is $\|\cdot\|$-derivative.

Proof. We have
$$\dot{F}_r = F_1 - F_0 = (b-a)(Q-P)$$
and
$$(b-F_r)(F_r-a) = (b-a)^2 r(1-r)(Q-P)^2,$$
so that by assumption $\kappa[(b-F_r)(F_r-a)]$ is τ-trace class for $r \in [0,1]$. For each $r \in (0,1)$ define
$$f_r(x) = (b-a)x\kappa\big((b-a)^2(r-r^2)x^2\big).$$
Then

$$\int_0^1 \tau\left(\dot{F}_r \kappa[(b-F_r)(F_r-a)]\right) dr$$
$$= \int_0^1 \tau\Big((b-a)(Q-P)\kappa[(b-a)^2 r(1-r)(Q-P)^2]\Big)\, dr, \quad (4.18)$$

and by Theorem 1.5.37 we have

$$\begin{aligned}\int_0^1 \tau\left(\dot{F}_r \kappa[(b-F_r)(F_r-a)]\right) dr &= \int_0^1 \tau\left(f_r(Q-P)\right) dr \\ &= \int_0^1 f_r(1)\, \mathrm{ec}(Q,P)\, dr \\ &= C_{a,b}\, \mathrm{ec}(Q,P) = C_{a,b}\, \mathrm{sf}(\{F_r\}),\end{aligned}$$

where the last equality follows from $a < 0 < b$ and Definition 1.6.3 of spectral flow. □

4.3.2 Spectral flow one-forms: unbounded case

The strategy of [CP$_2$] is geometric and follows ideas of [Ge]. The first step in this strategy is summarized in Proposition 4.3.3 in preparation for which we need an explicit formula for the derivative of function of a path of operators. The method by which this is achieved in [CP$_2$] does not apparently generalise sufficiently far to cover the situations considered in this paper. The double operator integral approach of Section 2 overcomes this problem.

Lemma 4.3.2 *Let $D = D^*\eta\mathcal{N}$, let $X, Y \in \mathcal{N}$ and let $f \in C_c^2(\mathbb{R})$ be a non-negative function such that $g := \sqrt{f} \in C_c^2(\mathbb{R})$. If $YT_{f^{[1]}}^{D,D}(X)$ and $XT_{f^{[1]}}^{D,D}(Y)$ are both τ-trace class then*

$$\tau\left(YT_{f^{[1]}}^{D,D}(X)\right) = \tau\left(XT_{f^{[1]}}^{D,D}(Y)\right).$$

Proof. By Lemma 4.1.12 we have

$$\begin{aligned} A &= \tau\left(Y\,T_{f^{[1]}}^{D,D}(X)\right) \\ &= \tau\left(Y\int_{\Pi}\left(\alpha_1(D,\sigma)X\beta_1(D,\sigma) + \alpha_2(D,\sigma)X\beta_2(D,\sigma)\right)\,d\nu_g(\sigma)\right) \\ &= \tau\left(Y\int_{\Pi}\left(e^{i(s_1-s_0)D}g(D)Xe^{is_1D} + e^{i(s_1-s_0)D}Xg(D)e^{is_1D}\right)\,d\nu_g(s_0,s_1)\right). \end{aligned}$$

Making the change of variables $s_1 - s_0 = t_0,\ s_1 = t_1$, and using (3.28) we have

$$\begin{aligned} &\int_{\Pi}\big(\alpha_1(D,\sigma)X\beta_1(D,\sigma) + \alpha_2(D,\sigma)X\beta_2(D,\sigma)\big)\,d\nu_g(\sigma) \\ &\quad= \frac{i}{\sqrt{2\pi}}\int_{\{(t_0,t_1)\in\mathbb{R}^2,\,t_0t_1\geqslant 0\}}\left[e^{it_0D}g(D)Xe^{it_1D} + e^{it_0D}Xg(D)e^{it_1D}\right] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\mathcal{F}(g)(t_0+t_1)\,dt_0dt_1 \\ &\quad= \frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\Sigma_t}\left(e^{it_0D}g(D)Xe^{it_1D} + e^{it_0D}Xg(D)e^{it_1D}\right)\,dl_t\right)\mathcal{F}(g)(t)\,dt, \end{aligned}$$

where $\Sigma_t = \left\{(t_0,t_1)\in\mathbb{R}^2 \colon t_0t_1 \geqslant 0,\ t_0+t_1 = t\right\}$ and dl_t is the Lebesgue measure on Σ_t. Thus, by Lemma 1.4.10

$$\begin{aligned} A &= \frac{i}{\sqrt{2\pi}}\tau\left(Y\int_{\mathbb{R}}\left(\int_{\Sigma_t}\left(e^{it_0D}g(D)Xe^{it_1D} + e^{it_0D}Xg(D)e^{it_1D}\right)\,dl_t\right)\mathcal{F}(g)(t)\,dt\right) \\ &= \frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\Sigma_t}\tau\left(Ye^{it_0D}g(D)Xe^{it_1D} + Ye^{it_0D}Xg(D)e^{it_1D}\right)\,dl_t\right)\mathcal{F}(g)(t)\,dt \\ &= \frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\Sigma_t}\tau\left(Xe^{it_1D}Yg(D)e^{it_0D} + Xe^{it_1D}g(D)Ye^{it_0D}\right)\,dl_t\right)\mathcal{F}(g)(t)\,dt, \end{aligned}$$

where the trace and integral can be interchanged by Lemma 1.4.13. The integral above coincides with $\tau\left(XT_{f^{[1]}}^{D,D}(Y)\right)$. □

The key geometric idea is to regard the analytic formula for spectral flow of [Ge] and [CP_2] as expressing it as an integral of a one form. As we are dealing with an affine space the geometry is easy to invoke as we see in the next result.

Proposition 4.3.3 *Let D be a self-adjoint operator affiliated with $\mathcal{N}$, having τ-compact resolvent and let $f \in C_c^3(\mathbb{R})$. Let $\alpha = \alpha^f$ be a 1-form on the affine* *space $D_0 + \mathcal{N}_{sa}$ defined at the point $D \in D_0 + \mathcal{N}_{sa}$ by the formula*

$$\alpha_D^f(X) = \tau\left(X f(D)\right), \quad X \in \mathcal{N}_{sa}, \quad D \in D_0 + \mathcal{N}_{sa}. \tag{4.19}$$

Then α is a closed 1-form, and, hence, also exact by the Poincaré lemma.

Proof. The proof follows mainly the lines of [CP], with necessary adjustments. As usual, we can assume that $f \geqslant 0$ and $g := \sqrt{f} \in C_c^3(\mathbb{R})$. We note that the operator $Xf(D)$ is τ-trace class, so that the 1-form α is well-defined. Now, by the definition of the exterior differential, for $X, Y \in \mathcal{N}$, we have

$$d\alpha_D(X,Y) = \pounds_X\, \alpha_D(Y) - \pounds_Y\, \alpha_D(X) - \alpha_D([X,Y]),$$

where $\pounds_X$ is the Lie derivative along the constant vector field X. Since the space $D_0 + \mathcal{N}_{sa}$ is flat, we have $[X,Y] = 0$. So, we have to prove that $\pounds_X\, \alpha_D(Y) = \pounds_Y\, \alpha_D(X)$. It follows from Theorem 4.1.17 that

$$\begin{aligned} A := \pounds_X\, \alpha_D(Y) = \left.\frac{d}{ds}\right|_{s=0} \alpha_{D+sX}(Y) = \left.\frac{d}{ds}\right|_{s=0} \tau\left(Y f(D+sX)\right) \\ = \tau\left(Y \mathcal{D}_{\mathcal{N},\mathcal{L}^1} f(D)(X)\right) = \tau\left(Y\, T_{f^{[1]}}^{D,D}(X)\right). \end{aligned}$$

Hence, by Lemma 4.3.2

$$\pounds_X\, \alpha_D(Y) = \tau\left(Y\, T_{f^{[1]}}^{D,D}(X)\right) = \tau\left(X\, T_{f^{[1]}}^{D,D}(Y)\right) = \pounds_Y\, \alpha_D(X),$$

which implies that α_D is a closed 1-form. $\square$

Though closedness of a 1-form already should imply its exactness by the Poincaré lemma and contractibility of the domain we follow [CP$_2$] and give an independent proof of exactness.

Definition 4.3.4 *Let D_0 be a fixed self-adjoint operator with τ-compact resolvent affiliated with $\mathcal{N}$, and let $f \in C_c(\mathbb{R})$. We define the function θ^f on the affine space $D_0 + \mathcal{N}$ by the formula*

$$\theta_D^f = \int_0^1 \tau\left(V f(D_r)\right)\, dr,$$

where $D \in D_0 + \mathcal{N}_{sa}$, $V = D - D_0$ and $D_r = D_0 + rV$. Measurability of the function $r \mapsto \tau\left(V f(D_r)\right)$ follows from Lemma 4.2.1.

Proposition 4.3.5 *Let $f \in C_c^3(\mathbb{R})$ and let $X \in \mathcal{N}$. Then*

$$d\theta_D^f(X) = \alpha_D^f(X).$$

Proof. Without loss of generality, we can assume that X is self-adjoint. By definitions

$$\begin{aligned}
(A) := d\theta_D^f(X) = \frac{d}{ds}\bigg|_{s=0} \theta^f_{D+sX} \\
&= \frac{d}{ds}\bigg|_{s=0} \int_0^1 \tau\left((V+sX)f(D_r+srX)\right)\,dr \\
&= \lim_{s\to 0}\frac{1}{s}\int_0^1 \tau\left((V+sX)f(D_r+srX) - Vf(D_r)\right)\,dr \\
&= \lim_{s\to 0}\int_0^1 \tau\big(Xf(D_r+srX)\big)\,dr \\
&\qquad + \lim_{s\to 0}\frac{1}{s}\int_0^1 \tau\Big(V\big(f(D_r+srX)-f(D_r)\big)\Big)\,dr.
\end{aligned}$$

The first summand of this sum by Proposition 4.1.14 is equal to

$$\int_0^1 \tau\big(Xf(D_r)\big)\,dr.$$

By Proposition 4.1.13(i) the second summand is equal to

$$\begin{aligned}
&\lim_{s\to 0}\frac{1}{s}\int_0^1 \tau\left(VT^{D_r+srX,D_r}_{f^{[1]}}(srX)\right)\,dr \\
&\qquad = \lim_{s\to 0}\int_0^1 \tau\left(VT^{D_r+srX,D_r}_{f^{[1]}}(rX)\right)\,dr \\
&\qquad = \int_0^1 \tau\left(V\,T^{D_r,D_r}_{f^{[1]}}(rX)\right)\,dr \\
&\qquad = \int_0^1 \tau\left(X\,T^{D_r,D_r}_{f^{[1]}}(V)\right)r\,dr,
\end{aligned}$$

where the second equality follows from Lemma 4.1.16 and the last equality follows from Lemma 4.3.2. Hence, by Lemma 1.4.13

$$\begin{aligned}
(A) &= \int_0^1 \tau\left(X\big[f(D_r)+r\,T^{D_r,D_r}_{f^{[1]}}(V)\big]\right)\,dr \\
&= \tau\left(X\int_0^1 \big[f(D_r)+r\,T^{D_r,D_r}_{f^{[1]}}(V)\big]\,dr\right),
\end{aligned}$$

where the integral on the RHS is a so^*-integral. By Proposition 4.1.14 and Lemma 4.1.16 the function $r\in[0,1]\mapsto f(D_r)+r\,T^{D_r,D_r}_{f^{[1]}}(V)\in\mathcal{L}^1(\mathcal{N},\tau)$ is $\mathcal{L}^1(\mathcal{N},\tau)$-continuous, so that the last integral

$$(B) := \int_0^1 \big[f(D_r)+r\,T^{D_r,D_r}_{f^{[1]}}(V)\big]\,dr$$

can be considered as Riemann integral. Let $0 = r_0 < r_1 < \ldots < r_n = 1$ be the partition of $[0,1]$ into n segments of equal length $\frac{1}{n}$. By the argument used in the proof of Lemma 4.1.16, it can be shown that the $\mathcal{L}^1(\mathcal{N},\tau)$-norm of $T_{f^{[1]}}^{D_{r_j},D_{r_j}}(V) - T_{f^{[1]}}^{D_r,D_r}(V)$, $r \in [r_{j-1}, r_j]$, has order $\frac{1}{n}$. Hence

$$\mathcal{L}^1\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \left(\frac{j}{n} T_{f^{[1]}}^{D_{r_j},D_{r_j}}(V) - j\int_{r_{j-1}}^{r_j} T_{f^{[1]}}^{D_r,D_r}(V)\,dr \right)$$
$$= \mathcal{L}^1\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} j\int_{r_{j-1}}^{r_j} \left(T_{f^{[1]}}^{D_{r_j},D_{r_j}}(V) - T_{f^{[1]}}^{D_r,D_r}(V) \right) dr = 0,$$

so that by formula (4.3) applied to the pair $(D_{r_{j-1}}, D_{r_j})$ we have

$$\begin{aligned}
(B) \ &= \mathcal{L}^1\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \left(f(D_{r_{j-1}}) + \frac{j}{n} T_{f^{[1]}}^{D_{r_j},D_{r_j}}(V) \right) \\
&= \mathcal{L}^1\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \left(f(D_{r_{j-1}}) + j\left(f(D_{r_j}) - f(D_{r_{j-1}}) \right) \right) \\
&\qquad + \mathcal{L}^1\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \left(\frac{j}{n} T_{f^{[1]}}^{D_{r_j},D_{r_j}}(V) - j\int_{r_{j-1}}^{r_j} T_{f^{[1]}}^{D_r,D_r}(V)\,dr \right) \\
&= \mathcal{L}^1\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \left(jf(D_{r_j}) - (j-1)f(D_{r_{j-1}}) \right) = f(D_1).
\end{aligned}$$

□

Corollary 4.3.6 *The integral of the 1-form α^f along a piecewise continuously differentiable path Γ in $D_0 + \mathcal{N}$ depends only on the endpoints of the path Γ.*

Proof. The 1-form α_D^f is a derivative, which depends continuously on D due to the equality (Proposition 4.1.13)

$$f(D) - f(D_0) = T_{f^{[1]}}^{D,D_0}(D - D_0).$$

and Lemma 4.1.16. Hence, the integral of α^f depends only on endpoints by Theorem 1.2.3. □

Proposition 4.3.7 *If a self-adjoint operator D_0 affiliated with $\mathcal{N}$ has τ-compact resolvent, $D_1, D_2 \in D_0 + \mathcal{N}_{sa}$, then for all $\lambda \in \mathbb{R}$*

$$\xi_{D_2,D_0}(\lambda) = \xi_{D_2,D_1}(\lambda) + \xi_{D_1,D_0}(\lambda).$$

Remark 6 *We emphasize that this additivity property is not almost everywhere in the spectral variable but in fact holds everywhere.*

Proof. It follows from (4.9) that

$$\xi_{D_2,D_0}(\lambda) = \xi_{D_2,D_1}(\lambda) + \xi_{D_1,D_0}(\lambda) + C,$$

where C is a constant. Multiplying both sides of this equality by a positive C_c^2-function f, and integrating it, by Lemma 4.2.8 we get

$$\int_{\Gamma_{D_2,D_0}} \alpha^f = \int_{\Gamma_{D_2,D_1}} \alpha^f + \int_{\Gamma_{D_1,D_0}} \alpha^f + C \int_{\mathbb{R}} f(\lambda)\, d\lambda,$$

where Γ_{D_i,D_j} is the straight line path connecting operators D_i and D_j. The last equality and Corollary 4.3.6 imply that $C = 0$. □

4.3.3 Spectral flow one-forms: bounded case

Since we obtain our unbounded spectral flow formula from a bounded one we need to study the map $D \mapsto F_D = D(1 + D^2)^{-1/2}$ which takes the space of unbounded self adjoint operators with τ-compact resolvent to the space $\mathcal{F}^{-1,1}(\mathcal{N}, \tau)$ of bounded τ-Fredholm operators F satisfying $1 - F^2 \in \mathcal{K}(\mathcal{N}, \tau)$.

Let $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$, let $h \in C_c^2(a, b)$ and let $K = F - F_0,\ F_r := F_0 + rK$. We define a 0-form θ and a 1-form α^h on the affine space $\mathcal{A}_{F_0}$ by the formulae

$$\theta_F^h = \int_0^1 \tau\left(K\, h(F_r)\right)\, dr,$$

and

$$\alpha_F^h(X) = \tau\left(X\, h(F)\right), \quad X \in \mathcal{K}(\mathcal{N}, \tau).$$

By Lemmas 4.1.18 and 4.1.19, the operators $h(F_r)$ and $h(F)$ are τ-trace class, so that the forms θ^h and α^h are well-defined.

Proposition 4.3.8 *If $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ and if $h \in C_c^2(a, b)$, then*

$$d\theta_F^h(X) = \alpha_F^h(X),$$

where $X \in \mathcal{K}(\mathcal{N}, \tau)$, so that the 1-form α_F^h is exact.

Proof. The proof follows verbatim the proof of Proposition 4.3.5, with references to Proposition 4.1.21 and Lemma 4.1.22 instead of Proposition 4.1.14 and Lemma 4.1.16.

We give the proof for completeness.

Without loss of generality, we can assume that X is self-adjoint. By definition

$$
\begin{aligned}
(A) := d\theta_F^h(X) = \frac{d}{ds}\bigg|_{s=0} \theta_{F+sX}^h &= \frac{d}{ds}\bigg|_{s=0} \int_0^1 \tau\left((K+sX)h(F_r+srX)\right)\,dr \\
&= \lim_{s\to 0}\frac{1}{s}\int_0^1 \tau\left((K+sX)h(F_r+srX) - Kh(F_r)\right)\,dr \\
&= \lim_{s\to 0}\int_0^1 \tau\big(Xh(F_r+srX)\big)\,dr \\
&\qquad + \lim_{s\to 0}\frac{1}{s}\int_0^1 \tau\Big(K\big(h(F_r+srX)-h(F_r)\big)\Big)\,dr.
\end{aligned}
$$

The first summand of this sum, by Proposition 4.1.21, is equal to

$$\int_0^1 \tau\big(Xh(F_r)\big)\,dr.$$

By Proposition 4.1.13(i) the second summand is equal to

$$
\begin{aligned}
\lim_{s\to 0}\frac{1}{s}\int_0^1 \tau\left(KT_{h^{[1]}}^{F_r+srX,F_r}(srX)\right)\,dr &= \lim_{s\to 0}\int_0^1 \tau\left(KT_{h^{[1]}}^{F_r+srX,F_r}(rX)\right)\,dr \\
&= \int_0^1 \tau\left(K\,T_{h^{[1]}}^{F_r,F_r}(rX)\right)\,dr \\
&= \int_0^1 \tau\left(X\,T_{h^{[1]}}^{F_r,F_r}(K)\right)r\,dr,
\end{aligned}
$$

where the second equality follows from Lemma 4.1.22 and the last equality follows from Lemma 4.3.2. Hence, by Lemma 1.4.13

$$
\begin{aligned}
(A) &= \int_0^1 \tau\left(X\big[h(F_r) + r\,T_{h^{[1]}}^{F_r,F_r}(K)\big]\right)\,dr \\
&= \tau\left(X\int_0^1 \big[h(F_r) + r\,T_{h^{[1]}}^{F_r,F_r}(K)\big]\,dr\right),
\end{aligned}
$$

where the last integral is a so^*-integral. By Proposition 4.1.21 and Lemma 4.1.22 the function $r\in[0,1]\mapsto h(F_r)+r\,T_{h^{[1]}}^{F_r,F_r}(K)\in\mathcal{L}^1(\mathcal{N},\tau)$ is $\mathcal{L}^1(\mathcal{N},\tau)$-continuous, so that the last integral

$$(B) := \int_0^1 \big[h(F_r) + r\,T_{h^{[1]}}^{F_r,F_r}(K)\big]\,dr$$

can be considered as a Riemann integral. Let $0 = r_0 < r_1 < \ldots < r_n = 1$ be the partition of $[0,1]$ into n segments of equal length $\frac{1}{n}$. By the argument used in the proof of Theorem 3.3.6, it can be shown that the $\mathcal{L}^1(\mathcal{N},\tau)$-norm of

$T^{F_{r_j},F_{r_j}}_{h^{[1]}}(K) - T^{F_r,F_r}_{h^{[1]}}(K)$ has order $\frac{1}{n}$. Hence

$$\mathcal{L}^1\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}T^{F_{r_j},F_{r_j}}_{h^{[1]}}(K) - j\int_{r_{j-1}}^{r_j} T^{F_r,F_r}_{h^{[1]}}(K)\,dr\right)$$
$$= \mathcal{L}^1\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} j\int_{r_{j-1}}^{r_j}\left(T^{F_{r_j},F_{r_j}}_{h^{[1]}}(K) - T^{F_r,F_r}_{h^{[1]}}(K)\right)dr = 0,$$

so that by (4.5) applied to the pair $(F_{r_{j-1}}, F_{r_j})$ we have

$$\begin{aligned}(B) &= \mathcal{L}^1\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(h(F_{r_{j-1}}) + \frac{j}{n}T^{F_{r_j},F_{r_j}}_{h^{[1]}}(K)\right)\\
&= \mathcal{L}^1\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(h(F_{r_{j-1}}) + j\left(h(F_{r_j}) - h(F_{r_{j-1}})\right)\right)\\
&\qquad + \mathcal{L}^1\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}T^{F_{r_j},F_{r_j}}_{h^{[1]}}(K) - j\int_{r_{j-1}}^{r_j} T^{F_r,F_r}_{h^{[1]}}(K)\,dr\right)\\
&= \mathcal{L}^1\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(jh(F_{r_j}) - (j-1)h(F_{r_{j-1}})\right) = h(F_1).\end{aligned}$$

□

As in the unbounded case we get the following

Corollary 4.3.9 *The integral of the one-form α^h depends only on the end-points.*

Proof. The 1-form α^h_F is a derivative, which depends continuously on F due to the equality (Proposition 4.1.13)

$$h(F) - h(F_0) = T^{F,F_0}_{h^{[1]}}(F - F_0).$$

and Lemma 4.1.22. Hence, the integral of α^h depends only on endpoints by Theorem 1.2.3. □

Corollary 4.3.10 *Let $F_j \in \mathcal{F}^{a,b}(\mathcal{N},\tau)$, $j = 0,1,2$, such that $F_2 - F_1, F_1 - F_0 \in \mathcal{K}(\mathcal{N},\tau)$. Then for any $\lambda \in (a,b)$ the following equality holds true*

$$\xi_{F_2,F_0}(\lambda) = \xi_{F_2,F_1}(\lambda) + \xi_{F_1,F_0}(\lambda).$$

Proof. The proof is similar to that of Proposition 4.3.7.

It follows from Proposition 4.2.12 that for any $h \in C_c^\infty(a,b)$

$$\int_a^b h'(\lambda)\left(\xi_{F_2,F_1}(\lambda)+\xi_{F_1,F_0}(\lambda)\right)d\lambda = \int_a^b h'(\lambda)\xi_{F_2,F_0}(\lambda)\,d\lambda.$$

It follows that for all $\lambda \in (a,b)$

$$\xi_{F_2,F_0}(\lambda) = \xi_{F_2,F_1}(\lambda)+\xi_{F_1,F_0}(\lambda)+\mathrm{const}\,.$$

□

It is easy to see that if $D = D^*\eta\mathcal{N}$ is an operator with τ-compact resolvent, then the operator $F_D := \phi(D)$ belongs to $\mathcal{F}^{-1,1}(\mathcal{N},\tau)$ (see (4.1) for the definition of ϕ).

Proposition 4.3.11 *If $D_0 = D_0^*\eta\mathcal{N}$ is an operator with τ-compact resolvent, and if $V = V^* \in \mathcal{N}$, $D_1 = D_0 + V$, then the following equality holds*

$$\xi_{D_1,D_0}(\lambda) = \xi_{F_{D_1},F_{D_0}}(\phi(\lambda)).$$

Proof. Let $h \in C_c^3(-1,1)$ and $f(\lambda) = h(\phi(\lambda))$. Then by Theorem 4.2.5

$$A := \tau\left(f(D_1)-f(D_0)\right) = \int_{\mathbb{R}} f'(\lambda)\xi_{D_1,D_0}(\lambda)\,d\lambda$$

and since $F_{D_1} - F_{D_0} \in \mathcal{K}(\mathcal{N},\tau)$ by Lemma 1.6.9, we can apply Proposition 4.2.12 to get

$$\begin{aligned} A = \tau\left(h(F_{D_1})-h(F_{D_0})\right) &= \int_{-1}^1 h'(t)\xi_{F_{D_1},F_{D_0}}(t)\,dt \\ &= \int_{-\infty}^{\infty} h'(\phi(\lambda))\phi'(\lambda)\xi_{F_{D_1},F_{D_0}}(\phi(\lambda))\,d\lambda \\ &= \int_{-\infty}^{\infty} f'(\lambda)\xi_{F_{D_1},F_{D_0}}(\phi(\lambda))\,d\lambda. \end{aligned}$$

Since f is an arbitrary C^2-function with compact support, comparing the last two formulas we get the equality

$$\xi_{D_1,D_0}(\lambda) = \xi_{F_{D_1},F_{D_0}}(\phi(\lambda)) + C. \tag{4.20}$$

It is left to show that the constant $C = 0$.

Let h be a non-negative function from $C_c^\infty(-1,1)$. By Lemma 4.2.8 we have

$$\int_{\mathbb{R}} h(\phi(\lambda))\xi_{D_1,D_0}(\lambda)\,d\lambda = \int_0^1 \tau\left(Vh(F_{D_r})\right)\,dr. \tag{4.21}$$

Multiplying the first term of the RHS of (4.20) by $h(\phi(\lambda))$, integrating it and using Lemma 4.2.15, we get

$$\begin{aligned} A := \int_{\mathbb{R}} h(\phi(\lambda))\xi_{F_{D_1},F_{D_0}}(\phi(\lambda))\,d\lambda &= \int_{-1}^{1} h(\mu)\xi_{F_{D_1},F_{D_0}}(\mu)(\phi^{-1})'(\mu)\,d\mu \\ &= \int_0^1 \tau\left(Kh(F_r)(\phi^{-1})'(F_r)\right)\,dr, \end{aligned}$$

where $K = F_{D_1} - F_{D_0}$ and F_r is the straight line path connecting F_{D_1} and F_{D_0}. Let $g(\mu) = h(\mu)(\phi^{-1})'(\mu)$. By Corollary 4.3.9 we have

$$A = \int_0^1 \tau\left(Kg(F_r)\right)\,dr = \int_0^1 \tau\left(\dot{F}_{D_r} g(F_{D_r})\right)\,dr.$$

By Proposition 4.1.13(ii) we have $\dot{F}_{D_r} = T^{D_r,D_r}_{\phi^{[1]}}(V)$. Hence,

$$A = \int_0^1 \tau\left(T^{D_r,D_r}_{\phi^{[1]}}(V)g(F_{D_r})\right)\,dr.$$

Using the BS-representation for $\phi^{[1]}$ given by (3.2.3), it follows from the definition of DOI (3.30), Lemma 4.1.9(ii) and Lemma 1.4.13, that

$$\begin{aligned} A &= \int_0^1 \tau\left(\int_{\Pi} e^{i(s-t)D_r} V e^{itD_r}\,d\nu_\phi(s,t)\cdot g(F_{D_r})\right)\,dr \\ &= \int_0^1 \int_{\Pi} \tau\left(e^{i(s-t)D_r} V e^{itD_r} g(F_{D_r})\right)\,d\nu_\phi(s,t)\,dr \\ &= \frac{1}{\sqrt{2\pi}}\int_0^1 \int_{\mathbb{R}} \tau\left(V e^{isD_r} is\hat{\phi}(s) g(F_{D_r})\right)\,ds\,dr \qquad (4.22) \\ &= \frac{1}{\sqrt{2\pi}}\int_0^1 \tau\left(V g(F_{D_r})\int_{\mathbb{R}} e^{isD_r} is\hat{\phi}(s)\,ds\right)\,dr \\ &= \int_0^1 \tau\left(V g(F_{D_r})\phi'(D_r)\right)\,dr = \int_0^1 \tau\left(V h(F_{D_r})\right)\,dr, \end{aligned}$$

since $g(\phi(\lambda))\phi'(\lambda) = h(\phi(\lambda))$. It follows from (4.20), (4.21) and (4.22) that $C\int_{\mathbb{R}} h(\phi(\lambda))\,d\lambda = 0$ and, hence, $C = 0$. □

4.3.4 The first formula for spectral flow

We establish first a spectral flow formula for bounded τ-Fredholm operators. In this way we avoid a number of difficulties with unbounded operators. Then we make a 'change of variable' to get to the unbounded case.

First we require some additional notation which is important for establishing a convention for how we handle the situation when the endpoints have a kernel.

Let $a < 0$, $b > 0$ and let $\operatorname{sign}_{a,b}$ be the function defined as $\operatorname{sign}_{a,b}(x) = b$ if $x \geqslant 0$, and $\operatorname{sign}_{a,b}(x) = a$ if $x < 0$.

We will write $\widetilde{F} = \operatorname{sign}_{a,b}(F)$, when it is clear from the context what the numbers a and b are.

Definition 4.3.12 *If $F \in \mathcal{F}^{a,b}(\mathcal{N},\tau)$ and κ is a C^2-function on $[0,\infty)$ vanishing in a neighbourhood of 0 then for $h(\lambda) = \kappa\big((b-\lambda)(\lambda-a)\big)$ we define $\gamma_h(F)$ as*

$$\gamma_h(F) = \int_0^1 \alpha^h_{F_r}(\dot{F}_r)\, dr,$$

where α^h is the closed one-form defined before Proposition 4.3.8, and $\{F_r\}_{r\in[0,1]}$ is the straight line connecting F and $\widetilde{F}$.

The following theorem is the analogue of [CP$_2$, Theorem 5.7]. It is the fundamental formula that we need as our starting point. The proof follows ideas of [CP$_2$, Theorem 5.7].

Theorem 4.3.13 *Let $F_0 \in \mathcal{F}^{a,b}(\mathcal{N},\tau)$, let $K \in \mathcal{K}(\mathcal{N},\tau)$ and let $F_1 = F_0 + K$. Let κ be a C^2-function on $[0,\infty)$ vanishing in a neighbourhood of 0, such that the integral of $h(\lambda) = \kappa\big((b-\lambda)(\lambda-a)\big)$ over (a,b) is equal to 1. Then the spectral flow between F_0 and F_1 is equal to*

$$\operatorname{sf}(F_0,F_1) = \int_a^b h(\lambda)\xi_{F_1,F_0}(\lambda)\, d\lambda + \gamma_h(F_1) - \gamma_h(F_0).$$

Proof. By additivity property of spectral flow (Proposition 1.6.6(2)) we have

$$\operatorname{sf}(F_0,F_1) = \operatorname{sf}(F_0,\widetilde{F}_0) + \operatorname{sf}(\widetilde{F}_0,\widetilde{F}_1) + \operatorname{sf}(\widetilde{F}_1,F_1).$$

It directly follows from the definition of spectral flow that $\operatorname{sf}(F,\widetilde{F}) = 0$ for any $F \in \mathcal{F}^{a,b}(\mathcal{N},\tau)$, since all projections $\chi(F + t(\widetilde{F} - F))$ are the same for all t. Hence

$$\operatorname{sf}(F_0,F_1) = \operatorname{sf}(\widetilde{F}_0,\widetilde{F}_1).$$

Now, by Lemma 4.3.1 we have

$$\operatorname{sf}(\widetilde{F}_0,\widetilde{F}_1) = \int_0^1 \alpha^h_{\widetilde{F}_r}(\dot{\widetilde{F}}_r)\, dr,$$

where $\{\widetilde{F}_r\}_{r\in[0,1]}$ is the straight line path, connecting $\widetilde{F}_0$ and $\widetilde{F}_1$. By Corollary 4.3.9 we can replace this path by the (broken) path given on this diagram

$$\begin{tikzcd} F_0 \arrow[r, dashed] & F_1 \arrow[d, dashed, "\gamma_h(F_1)"] \\ \widetilde{F}_0 \arrow[u, dashed, "-\gamma_h(F_0)"] \arrow[r] & \widetilde{F}_1 \end{tikzcd}$$

Then we get

$$\mathrm{sf}(\widetilde{F}_0, \widetilde{F}_1) = -\gamma_h(F_0) + \int_0^1 \alpha^h_{F_r}(\dot{F}_r)\, dr + \gamma_h(F_1),$$

where $\{F_r\}_{r\in[0,1]}$ is the straight line path, connecting F_0 and F_1. But, setting $F_1 - F_0 = K$, we have by Lemma 4.2.15

$$\int_0^1 \alpha^h_{F_r}(\dot{F}_r)\, dr = \int_0^1 \tau\left(Kh(F_r)\right)\, dr = \int_{\mathbb{R}} h(\lambda)\xi_{F_1,F_0}(\lambda)\, d\lambda.$$

□

Theorem 4.3.14 *Let $F_0 \in \mathcal{F}^{-1,1}(\mathcal{N},\tau)$, let $K \in \mathcal{K}(\mathcal{N},\tau)$ and let $F_1 = F_0 + K$. Let κ be a C^2-function on $[0,\infty)$ vanishing in a neighbourhood of 0, such that the integral of $h(\lambda) = \kappa(1-\lambda^2)$ over $(-1,1)$ is equal to 1. Then the spectral flow function for the pair F_0 and F_1 is equal to*

$$\mathrm{sf}(\mu; F_0, F_1) = \int_{-1}^1 h(\lambda)\xi_{F_1,F_0}(\lambda)\, d\lambda + \gamma_{h_{-\mu}}(F_1-\mu) - \gamma_{h_{-\mu}}(F_0-\mu),$$

where $h_{-\mu}(\lambda) = h(\lambda+\mu)$.

Proof. By definition we have

$$\mathrm{sf}(\mu; F_0, F_1) = \mathrm{sf}(F_0-\mu, F_1-\mu).$$

Since $F_j - \mu \in \mathcal{F}^{-1-\mu,1-\mu}(\mathcal{N},\tau)$, by Theorem 4.3.13 we have

$$\mathrm{sf}(F_0-\mu, F_1-\mu) = \int_{-1-\mu}^{1-\mu} h_{-\mu}(\lambda)\xi_{F_1-\mu,F_0-\mu}(\lambda)\, d\lambda + \gamma_{h_{-\mu}}(F_1-\mu) - \gamma_{h_{-\mu}}(F_0-\mu).$$

Since $\xi_{F_1-\mu,F_0-\mu}(\lambda) = \xi_{F_1,F_0}(\lambda+\mu)$, we have

$$\begin{aligned}\mathrm{sf}(F_0-\mu, &F_1-\mu)\\ &= \int_{-1-\mu}^{1-\mu} h(\lambda+\mu)\xi_{F_1,F_0}(\lambda+\mu)\, d\lambda + \gamma_{h_{-\mu}}(F_1-\mu) - \gamma_{h_{-\mu}}(F_0-\mu)\\ &= \int_{-1}^{1} h(\lambda)\xi_{F_1,F_0}(\lambda)\, d\lambda + \gamma_{h_{-\mu}}(F_1-\mu) - \gamma_{h_{-\mu}}(F_0-\mu).\end{aligned}$$

□

Corollary 4.3.15 *If F_0 and F_1 are unitarily equivalent, then*

$$\mathrm{sf}(\mu; F_0, F_1) = \xi_{F_1,F_0}(\mu) = \mathrm{const}\,.$$

Proof. By Corollary 4.2.13 the function $\xi_{F_1,F_0}(\cdot)$ is constant on $(-1,1)$, so that $\int_{\mathbb{R}} h(\lambda)\xi_{F_1,F_0}(\lambda)\,d\lambda = \xi_{F_1,F_0}(0)$.

If F_0 and F_1 are unitarily equivalent, then $\gamma_{h_{-\mu}}(F_1-\mu) = \gamma_{h_{-\mu}}(F_0-\mu)$. Hence, for all $\mu \in (-1,1)$

$$\mathrm{sf}(\mu; F_0, F_1) = \xi_{F_1,F_0}(\mu) = \xi_{F_1,F_0}(0).$$

□

Lemma 4.3.16 *If $F \in \mathcal{F}^{-1,1}(\mathcal{N},\tau)$ and if $\{h_\varepsilon\}_{\varepsilon>0}$ is an approximate δ function (by compactly supported even functions) then for all $\mu \in (-1,1)$ the limit*

$$\gamma_\mu(F) := \lim_{\varepsilon\to 0} \gamma_{h_\varepsilon}(F-\mu)$$

exists and is equal to $\xi_{G,\widetilde{G}}(0)$, where $G = F-\mu$.

Proof. Since h is an even function we have that

$$\int_{-\infty}^{0} h_\varepsilon(\lambda)\xi_{G,\widetilde{G}}(\lambda)\,d\lambda \to \frac{1}{2}\xi_{G,\widetilde{G}}(0-)$$

and

$$\int_{0}^{\infty} h_\varepsilon(\lambda)\xi_{G,\widetilde{G}}(\lambda)\,d\lambda \to \frac{1}{2}\xi_{G,\widetilde{G}}(0+),$$

as $\varepsilon \to 0$. If $\{G_r\}_{r\in[0,1]}$ is the straight line path connecting G and $\widetilde{G}$ then by Lemma 4.2.15 we have

$$\begin{aligned}\gamma_{h_\varepsilon}(G) &= \int_0^1 \alpha_{h_\varepsilon}(\dot{G}_r)\,dr = \int_0^1 \tau\left(\dot{G}_r h_\varepsilon(G_r)\right)\,dr \\ &= \int_{\mathbb{R}} h_\varepsilon(\lambda)\xi_{G,\widetilde{G}}(\lambda)\,d\lambda \;\to\; \frac{1}{2}\left(\xi_{G,\widetilde{G}}(0-) + \xi_{G,\widetilde{G}}(0+)\right) = \xi_{G,\widetilde{G}}(0)\end{aligned}$$

as $\varepsilon \to 0$, by Definition 4.2.6 of ξ at discontinuity points. □

Now we consider the situation when the endpoints are not unitarily equivalent. For this we require some additional facts about the 'end-point correction terms'. The approach used here differs in a fundamental way from the previous point of view in [CP$_2$]. The next few results demonstrate this by showing that the spectral shift function absorbs the contribution to the formula due to the spectral asymmetry of the endpoints leaving only kernel correction terms to be handled.

Lemma 4.3.17 *If $F \in \mathcal{F}^{-1,1}(\mathcal{N},\tau)$ and if $\mu \in (-1,1)$, then the following equality holds true*

$$\gamma_\mu(F) = \frac{1}{2}\tau\left(\mathrm{N}_{F-\mu}\right).$$

Proof. Let $G = F - \mu$. We have

$$\tau\left(\mathrm{N}_G\right) = \tau\left(E^G_{(-\infty,0]} - E^{\widetilde{G}}_{(-\infty,0]}\right).$$

By Proposition 4.2.12(ii) and Definition 4.2.14 the value $\xi_{\widetilde{G},G}(0)$ is the half sum of the last expression and

$$\tau\left(E^G_{(-\infty,0)} - E^{\widetilde{G}}_{(-\infty,0)}\right) = 0.$$

Hence, by Lemma 4.3.16

$$\gamma_\mu(F) = \xi_{G,\widetilde{G}}(0) = \frac{1}{2}\tau\left(\mathrm{N}_G\right).$$

□

Theorem 4.3.18 *If $F_0, F_1 \in \mathcal{F}^{-1,1}(\mathcal{N},\tau)$ such that $F_1 - F_0 \in \mathcal{K}(\mathcal{N},\tau)$, then for all $\mu \in (-1,1)$*

$$\mathrm{sf}(\mu; F_0, F_1) = \xi_{F_1,F_0}(\mu) + \frac{1}{2}\left(\tau\left(\mathrm{N}_{F_1-\mu}\right) - \tau\left(\mathrm{N}_{F_0-\mu}\right)\right). \tag{4.23}$$

Proof. Replace h in Theorem 4.3.14 by $h_{\varepsilon,\mu}$ (thus translate the approximate δ function h_ε by μ) and then let $\varepsilon \to 0$ using Lemmas 4.3.16, 4.3.17. □

We now see that under hypotheses that guarantee both are defined the spectral flow function and the spectral shift function differ only by kernel corrections terms for the endpoints. We should remark that the occurrence of the correction terms $\gamma_\mu(F_j)$, $j = 1, 2$, in the last formula can be explained by the fact that we actually define the spectral flow function and the spectral shift function at discontinuity points in different ways. The spectral shift function is defined as a half-sum of the left and the right limits, while the spectral flow is defined to be left-continuous.

4.3.5 Spectral flow in the unbounded case

The formulae for spectral flow in the bounded case may now be used to establish corresponding results in our original setting of unbounded self adjoint operators with compact resolvent.

By Proposition 4.3.11 $\xi_{D_1,D_0}(0) = \xi_{F_{D_1},F_{D_0}}(0)$ and by definition of spectral flow for unbounded operators [BCPRSW] $\mathrm{sf}(D_0, D_1) = \mathrm{sf}(F_{D_1}, F_{D_0})$. Hence, it follows from (4.23) taken at $\mu = 0$ that

$$\mathrm{sf}(D_0, D_1) = \xi_{D_1,D_0}(0) + \gamma_0(F_1) - \gamma_0(F_0).$$

Since

$$\ker(D) = \ker(F_D)$$

we have the following equality

$$\operatorname{sf}(D_0, D_1) = \xi_{D_1,D_0}(0) + \frac{1}{2}\tau\left(\mathrm{N}_{D_1}\right) - \frac{1}{2}\tau\left(\mathrm{N}_{D_0}\right).$$

If we replace here the operators D_0 and D_1 by the operators $D_0 - \lambda$ and $D_1 - \lambda$ respectively then we get

$$\operatorname{sf}(\lambda; D_0, D_1) = \xi_{D_1-\lambda,D_0-\lambda}(0) + \frac{1}{2}\tau\left(\mathrm{N}_{D_1-\lambda}\right) - \frac{1}{2}\tau\left(\mathrm{N}_{D_0-\lambda}\right).$$

Since $\xi_{D_1-\lambda,D_0-\lambda}(0) = \xi_{D_1,D_0}(\lambda)$ we have proved

Theorem 4.3.19 *If $D_0 = D_0^*\eta\mathcal{N}$ has τ-compact resolvent, $V = V^* \in \mathcal{N}$ and $D_1 = D_0 + V$, then for every $\lambda \in \mathbb{R}$*

$$\operatorname{sf}(\lambda; D_0, D_1) = \xi_{D_1,D_0}(\lambda) + \frac{1}{2}\tau\left(\mathrm{N}_{D_1-\lambda}\right) - \frac{1}{2}\tau\left(\mathrm{N}_{D_0-\lambda}\right). \tag{4.24}$$

The spectral flow formula using infinitesimal spectral flow

The results on the spectral shift function which were established in Section 4.2 now suggest a new direction for spectral flow theory.

Definition 4.3.20 *Let D_0 be a self-adjoint operator affiliated with $\mathcal{N}$ having τ-compact resolvent. The **infinitesimal spectral flow** one-form is a distribution-valued one-form Φ_D on the affine space $D_0 + \mathcal{N}_{sa}$, defined by formula*

$$\langle \Phi_D(X), \varphi \rangle = \tau\left(X\varphi(D)\right), \qquad X \in \mathcal{N}_{sa}, \quad \varphi \in C_c^\infty(\mathbb{R}).$$

Formally,

$$\Phi_D(X) = \tau\left(X\delta(D)\right),$$

where $\delta(D)$ is the δ-function of D.

Theorem 4.3.21 *Let $D_1 \in D_0 + \mathcal{N}_{sa}$. Spectral flow between D_0 and D_1 is equal to the integral of the infinitesimal spectral flow one-form along any piecewise C^1-path $\{D_r\}_{r\in[0,1]}$ in $D_0 + \mathcal{N}$ connecting D_0 and D_1 in the sense that for any $\varphi \in C_c^\infty(\mathbb{R})$ the following equality holds true*

$$\int_{\mathbb{R}} \operatorname{sf}(\lambda; D_0, D_1)\varphi(\lambda)\, d\lambda = \int_0^1 \langle \Phi_{D_r}(\dot{D}_r), \varphi \rangle\, dr.$$

Formally,

$$\operatorname{sf}(D_0, D_1) = \int_0^1 \Phi_{D_r}(\dot{D}_r)\, dr,$$

or

$$\mathrm{sf}(\lambda; D_0, D_1) = \int_0^1 \Phi_{D_r - \lambda}(\dot{D}_r)\, dr.$$

Proof. By Corollary 4.3.6 we can choose the path $\{D_r\}_{r\in[0,1]}$ to be the straight line path $D_r = D_0 + rV$. It follows from Lemmas 1.6.8 and 4.1.2 that the functions $\lambda \mapsto \tau\left(\mathrm{N}_{D_0-\lambda}\right)$ and $\lambda \mapsto \tau\left(\mathrm{N}_{D_1-\lambda}\right)$ can be non-zero only on a countable set. Hence, by (4.24) and Lemma 4.2.8 we have

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{sf}(\lambda; D_0, D_1)\varphi(\lambda)\, d\lambda &= \int_{\mathbb{R}} \xi_{D_1, D_0}(\lambda)\varphi(\lambda)\, d\lambda \\ &= \int_0^1 \tau\left(V\varphi(D_r)\right)\, dr = \int_0^1 \langle \Phi_{D_r}(\dot{D}_r), \varphi\rangle\, dr.\end{aligned}$$

□

We remark that the infinitesimal spectral flow one-form is exact in the sense that its value on every test function is exact.

4.3.6 The spectral flow formulae in the $\mathcal{I}$-summable spectral triple case

The original approach of [CP$_2$] required summability constraints on the operator D_0. We will now see that if indeed D_0 satisfies such conditions then we can weaken conditions on the function f in Theorem 4.2.10.

Lemma 4.3.22 *Let D_0 be a self-adjoint operator with τ-compact resolvent affiliated with $\mathcal{N}$. Let g be an increasing continuous function on $[0, +\infty)$, such that $g(0) \geqslant 0$ and $g\left(c(1+D^2)^{-1}\right) \in \mathcal{L}^1(\mathcal{N}, \tau)$ for all $c > 0$. Let $f(x) = g\left((1+x^2)^{-1}\right)$. Then for any $R > 0$ and for any $V = V^* \in \mathcal{N}$ the operator $f(D+V)$ is trace class and the function*

$$V \in B_R \mapsto \|f(D+V)\|_1$$

is bounded.

Proof. By Lemma 1.3.17 we have for all $t > 0$

$$\mu_t(f(D+V)) = \mu_t\Big(g\left(\left(1+(D+V)^2\right)^{-1}\right)\Big) = g\Big(\mu_t\left(\left(1+(D+V)^2\right)^{-1}\right)\Big).$$

By Lemma 4.1.5 there exists a constant $c = c(R) > 0$ such that for any $V \in B_R$

$$\left(1+(D+V)^2\right)^{-1} \leqslant c\left(1+D)^2\right)^{-1}.$$

Hence, by Lemma 1.3.16 we have

$$\mu_t(f(D+V)) \leqslant g\left(\mu_t[c\left(1+D^2\right)^{-1}]\right) = \mu_t\left(g[c\left(1+D^2\right)^{-1}]\right).$$

Since $g\left(c(1+D^2)^{-1}\right) \in \mathcal{L}^1(\mathcal{N},\tau)$, by Proposition 1.3.21 the last function belongs to $L^1[0,\infty)$, which in its turn again by Proposition 1.3.21 implies that $f(D+V) \in \mathcal{L}^1(\mathcal{N},\tau)$. $\square$

Lemma 4.3.23 *Let D_0, g and f be as in Lemma 4.3.22. An integral of the one-form*

$$\alpha_D^f(X) = \tau\left(Xf(D)\right), \ X \in \mathcal{N}, \ D \in D_0 + \mathcal{N}_{sa},$$

along a piecewise smooth path in $D_0 + \mathcal{N}_{sa}$ depends only on endpoints of that path.

Proof. Let f_n be a increasing sequence of compactly supported smooth functions converging pointwise to f and Γ_1, Γ_2 be two piecewise smooth paths in in $D_0+\mathcal{N}_{sa}$ with the same endpoints. Then by Lemma 4.3.22, Lebesgue dominated convergence theorem and Corollary 4.3.6 we have

$$\begin{aligned}\int_{\Gamma_1} \alpha^f &= \int_{\Gamma_1} \lim_{n\to\infty} \alpha^{f_n} = \lim_{n\to\infty} \int_{\Gamma_1} \alpha^{f_n} \\ &= \lim_{n\to\infty} \int_{\Gamma_2} \alpha^{f_n} = \int_{\Gamma_2} \lim_{n\to\infty} \alpha^{f_n} = \int_{\Gamma_2} \alpha^f.\end{aligned}$$

$\square$

The condition that $g(c(1+D_0^2)^{-1})$ be trace class is a generalized summability constraint. This notion arises naturally for certain ideals $\mathcal{I}$ of compact operators (for example for the Schatten ideals $\mathcal{L}^p(\mathcal{H})$, $p \geqslant 1$, $g(x) = x^{p/2}$ and we have the notion of p-summability).

Now if there is a unitary $u \in \mathcal{N}$ with $V = u^*[D_0, u]$ bounded then we have, for a dense subalgebra $\mathcal{A}$ of the C^*-algebra generated by u, a semifinite 'g-summable' spectral triple $(\mathcal{A}, \mathcal{N}, D_0)$. Moreover $D_0 + V = uD_0u^*$ so we have unitarily equivalent endpoints.

Theorem 4.3.24 *Let f be a non-negative L^1-function such that $f(D_r) \in \mathcal{L}^1(\mathcal{N},\tau)$ for all $r \in [0,1]$, and let $r \mapsto \|f(D_r)\|_1$ be integrable on $[0,1]$. If D_0 and D_1 are unitarily equivalent then*

$$\mathrm{sf}(\lambda; D_0, D_1) = C^{-1} \int_0^1 \tau\left(Vf(D_r - \lambda)\right)\, dr,$$

where $C = \int_{-\infty}^{\infty} f(\lambda)\, d\lambda$.

Proof. Unitary equivalence of D_0 and D_1 implies that two last terms in (4.24) vanish. In case of $f \in B_c(\mathbb{R})$, multiplying (4.24) by $f(\lambda)$ and integrating it we get the required equality by Lemma 4.2.8 and Theorem 4.2.9. For an arbitrary $f \in L^1$ the claim follows from Lebesgue's dominated convergence theorem by approximating f by an increasing sequence of step-functions converging a.e. to f. □

Definition 4.3.25 *An operator $D = D^*\eta\mathcal{N}$ is θ-summable, if for every $\varepsilon > 0$ the operator $e^{-\varepsilon D^2}$ has finite τ-trace. An operator $D = D^*\eta\mathcal{N}$ is p-summable, where $p > 0$, if the operator $(1 + D^2)^{-p/2}$ has finite τ-trace.*

Note that any θ-summable or p-summable operator has τ-compact resolvent.

The following corollary recovers two of the main results of [CP, CP$_2$].

Corollary 4.3.26 *(i) If D_0 is θ-summable with respect to $\mathcal{N}$ and if D_0 and D_1 are unitarily equivalent then*

$$\mathrm{sf}(D_0, D_1) = \sqrt{\frac{\varepsilon}{\pi}} \int_0^1 \tau\left(V e^{-\varepsilon D_r^2}\right) dr.$$

(ii) If D_0 is p-summable (i.e. $(1 + D_0^2)^{-p/2} \in \mathcal{L}^1(\mathcal{N}, \tau)$) with respect to $\mathcal{N}$, where $p > 1$ and if D_0 and D_1 are unitarily equivalent then

$$\mathrm{sf}(D_0, D_1) = C_p^{-1} \int_0^1 \tau\left(V\left(1 + D_r^2\right)^{-\frac{p}{2}}\right) dr,$$

where $C_p = \int_{-\infty}^{\infty} (1 + \lambda^2)^{-\frac{p}{2}}\, d\lambda$.

Proof. Put $f(\lambda) = e^{-\varepsilon\lambda^2}$ and $f(\lambda) = (1 + \lambda^2)^{-\frac{p}{2}}$ for (i) and (ii) respectively in Theorem 4.3.24. The conditions of that theorem are fulfilled by Lemma 4.3.22. □

In particular, one has (Carey-Phillips formulae for spectral flow)

Corollary 4.3.27 *(i) Let $(\mathcal{A}, \mathcal{N}, D)$ be a θ-summable semifinite spectral triple. Then for any unitary $u \in \mathcal{A}$ the following holds*

$$\mathrm{sf}(D, uDu^*) = \pi^{-1/2} \int_0^1 \tau\left(u[D, u^*] e^{-(D + tu[D,u^*])^2}\right) dt.$$

(ii) Let $(\mathcal{A}, \mathcal{N}, D)$ be an n-summable semifinite spectral triple, $p > 1$. Then

$$\mathrm{sf}(D, uDu^*) = C_{p/2}^{-1} \int_0^1 \tau\left(u[D, u^*](1 + (D + tu[D, u^*])^2)^{-p/2}\right) dt,$$

where $C_{p/2} = \int\limits_{-\infty}^{+\infty} (1 + x^2)^{-p/2} dx$.

The last formula is a starting point in [CPRS] for the proof of the Connes-Moscovici Local Index Theorem for spectral flow.

4.3.7 Recovering η-invariants

To demonstrate that we have indeed generalized previous analytic approaches to spectral flow formulae we still need some refinements. What is missing is the relationship of the 'end-point correction terms' to the truncated eta invariants of [Ge].

In fact Theorem 4.3.13 combined with some ideas of [CP$_2$] will now enable us to give a new proof of the original formula (4.25) for spectral flow with unitarily inequivalent endpoints.

Introduce the function

$$\kappa_\varepsilon(\lambda) = \sqrt{\frac{\varepsilon}{\pi}}\lambda^{-3/2}e^{\varepsilon(1-\lambda^{-1})}.$$

and let $h_\varepsilon(\lambda) = \kappa_\varepsilon(1-\lambda^2)$, $f_\varepsilon(\lambda) = \kappa_\varepsilon\left((1+\lambda^2)^{-1}\right)$.

Lemma 4.3.28 *Let $D_0 = D_0^*\eta\mathcal{N}$ be θ-summable, let $V \in \mathcal{N}_{sa}$ and let $D_1 = D_0 + V$. Then*

$$\int_{-1}^{1} h_\varepsilon(\lambda)\xi_{F_{D_1},F_{D_0}}(\lambda)\,d\lambda = \sqrt{\frac{\varepsilon}{\pi}}\int_0^1 \tau\left(Ve^{-\varepsilon D_r^2}\right)\,dr.$$

Proof. Since $h_\varepsilon(\phi(\mu)) = f_\varepsilon(\mu)$, by Proposition 4.3.11 we have

$$\begin{aligned}(A) := \int_{-1}^{1} h_\varepsilon(\lambda)\xi_{F_{D_1},F_{D_0}}(\lambda)\,d\lambda &= \int_{-\infty}^{\infty} h_\varepsilon(\phi(\mu))\xi_{F_{D_1},F_{D_0}}(\phi(\mu))\phi(\mu)'\,d\mu\\ &= \int_{-\infty}^{\infty} f_\varepsilon(\mu)\phi'(\mu)\xi_{D_1,D_0}(\mu)\,d\mu.\end{aligned}$$

Further, by Lemmas 4.2.4 and 4.2.3

$$\begin{aligned}(A) &= \int_{-\infty}^{\infty} f_\varepsilon(\mu)\phi'(\mu)\int_0^1 \tau\left(VdE_\mu^{D_r}\right)\,dr\\ &= \int_0^1 \tau\left(Vf_\varepsilon(D_r)\phi'(D_r)\right)\,dr\\ &= \int_0^1 \tau\left(V\sqrt{\frac{\varepsilon}{\pi}}(1+D_r^2)^{3/2}e^{-\varepsilon D_r^2}(1+D_r^2)^{-3/2}\right)\,dr\\ &= \sqrt{\frac{\varepsilon}{\pi}}\int_0^1 \tau\left(Ve^{-\varepsilon D_r^2}\right)\,dr.\end{aligned}$$

□

As we have emphasized previously, the strategy of our proof follows that of [CP$_2$] in that, we deduce the unbounded version of the spectral flow formula for the theta summable case from a bounded version. To this end introduce $F_s = D(s+D^2)^{-1/2}$.

Lemma 4.3.29 [CP$_2$, Lemma 8.8] *We have*

$$\lim_{\delta\to 0}\int_{\Gamma_\delta}\alpha^{h_\varepsilon}=0,$$

where Γ_δ is the straight line connecting F_0 and F_δ.

Lemma 4.3.30 *If $D = D^*\eta\mathcal{N}$ is θ-summable, then the following equality holds true*

$$\gamma_{h_\varepsilon}(F_D)=\frac{1}{2}\left(\eta_\varepsilon(D)+\tau\left(\mathrm{N}_D\right)\right).$$

Proof. We note that $1-F_s^2 = s(s+D^2)^{-1}$ and that $\dot{F}_s = -\frac{1}{2}D(s+D^2)^{-3/2}$. The path $\Gamma_1 := \{F_s\}_{s\in[0,1]}$ connects $\mathrm{sgn}(F_D)$ with F_D. If we denote by Γ_2 the straight line path connecting $\mathrm{sgn}(F_D)$ with $\widetilde{F} = \mathrm{sign}(F_D)$ then the path $-\Gamma_1+\Gamma_2$ connects F_D with $\widetilde{F}$, so that by Lemma 4.3.23 applied to $f = h_\varepsilon$, and by the argument of [CP$_2$] and Lemma 4.3.29 dealing with discontinuity of the path Γ_1 at zero, it follows that

$$\gamma_{h_\varepsilon}(F_D)=-\int_{\Gamma_1}\alpha^{h_\varepsilon}+\int_{\Gamma_2}\alpha^{h_\varepsilon}.$$

We have for the first summand

$$\begin{aligned}
\int_{\Gamma_1}\alpha^{h_\varepsilon} &= \int_0^1 \alpha^{h_\varepsilon}_{F_s}(\dot{F}_s)\,ds = \int_0^1 \tau\left(\dot{F}_s h_\varepsilon(F_s)\right)\,ds\\
&= -\frac{1}{2}\int_0^1 \tau\left(D(s+D^2)^{-3/2}\kappa_\varepsilon\left(1-F_s^2\right)\right)\,ds\\
&= -\frac{1}{2}\int_0^1 \tau\left(D(s+D^2)^{-3/2}\kappa_\varepsilon\left(s(s+D^2)^{-1}\right)\right)\,ds\\
&= -\frac{1}{2}\sqrt{\frac{\varepsilon}{\pi}}\int_0^1 \tau\left(D(s+D^2)^{-3/2}s^{-3/2}(s+D^2)^{3/2}e^{-\frac{\varepsilon}{s}D^2}\right)\,ds\\
&= -\frac{1}{2}\sqrt{\frac{\varepsilon}{\pi}}\int_0^1 \tau\left(De^{-\frac{\varepsilon}{s}D^2}\right)s^{-3/2}\,ds\\
&= -\frac{1}{2}\sqrt{\frac{\varepsilon}{\pi}}\int_1^\infty \tau\left(De^{-\varepsilon tD^2}\right)\frac{dt}{\sqrt{t}} = -\frac{1}{2}\eta_\varepsilon(D).
\end{aligned}$$

Let $G_r = \operatorname{sgn}(F_D) + r\,\mathrm{N}_D$ be the path Γ_2. Then the second summand is equal to

$$\int_{\Gamma_2} \alpha^{h_\varepsilon} = \int_0^1 \tau\left(\dot{G}_r h_\varepsilon(G_r)\right)\, dr = \int_0^1 \tau\left(\mathrm{N}_D \kappa_\varepsilon(1-G_r^2)\right)\, dr.$$

Since $1 - G_r^2 = (1-r^2)\,\mathrm{N}_D$, it follows that

$$\int_{\Gamma_2} \alpha^{h_\varepsilon} = \int_0^1 \tau\left(\mathrm{N}_D \kappa_\varepsilon(1-r^2)\right)\, dr = \tau\left(\mathrm{N}_D\right) \int_0^1 \kappa_\varepsilon(1-r^2)\, dr = \frac{1}{2}\tau\left(\mathrm{N}_D\right),$$

so that $\gamma_{h_\varepsilon}(F_D) = \frac{1}{2}\left(\eta_\varepsilon(D) + \tau\left(\mathrm{N}_D\right)\right).$ $\Box$

As a direct corollary of these Lemmas and Theorem 4.3.13 we get a new proof of Carey-Phillips formula ([CP$_2$, Corollary 8.10], [Ge, Theorem 2.6]) with η-invariants.

Theorem 4.3.31 *If D_0 is θ-summable then the formula*

$$\operatorname{sf}(D_0, D_1) = \sqrt{\frac{\varepsilon}{\pi}} \int_0^1 \tau\left(V e^{-\varepsilon {D_r}^2}\right)\, dr + \frac{1}{2}\left(\eta_\varepsilon(D_1) - \eta_\varepsilon(D_0)\right) + \frac{1}{2}\tau\left(\mathrm{N}_{D_1} - \mathrm{N}_{D_0}\right) \quad (4.25)$$

holds true, where

$$\eta_\varepsilon(D) := \frac{1}{\sqrt{\pi}} \int_\varepsilon^\infty \tau\left(D e^{-tD^2}\right)\, t^{-1/2}\, dt.$$

is a 'truncated eta invariant'.

Proof. Let h_n be a sequence of smooth non-negative functions, compactly supported on $(-1,1)$, and converging pointwise to h_ε. Recall that $D_r = D_0 + rV$. Then the sequence $\gamma_{h_n}(F_{D_r})$ converges to $\gamma_{h_\varepsilon}(F_{D_r})$ and the sequence $\int_{-1}^1 h_n(\lambda)\xi_{F_{D_1},F_{D_0}}(\lambda)\, d\lambda$ converges to $\int_{-1}^1 h_\varepsilon(\lambda)\xi_{F_{D_1},F_{D_0}}(\lambda)\, d\lambda$ by Lebesgue's DCT, since θ-summability of D_0 implies 1-summability of $h_\varepsilon(F_{D_r})$. Hence the claim follows from Theorem 4.3.13 and Lemmas 4.3.28, 4.3.30. $\Box$

If $p > 1$ then the same argument with the choice

$$\kappa_p(\lambda) = C_p^{-1} \lambda^{p-\frac{3}{2}},$$

where $C_p = \int_{-\infty}^\infty (1+x^2)^{-p/2}\, dx$, leads to a formula with "η_p-invariants" for p-summable case [CP$_2$].

Concluding remarks

In the development of the main ideas of this thesis we will make some concluding remarks.

The Birman-Solomyak formula for the spectral shift function can be written as

$$\xi(\varphi) = \int_0^1 \mathrm{Tr}(V\varphi(H_r)\,dr,$$

where $H_r = H_0 + rV$, and the spectral shift function ξ is considered as distribution. It seems that the Birman-Solomyak spectral averaging formula is a key fundamental formula, which should be taken as definition of the spectral shift function. One of the reasons is that for all pairs of operators H_0, H_1 for which the spectral shift function ξ_{H_1,H_0} exists, the Birman-Solomyak formula holds. Further, the integrand of the Birman-Solomyak formula seems to have special importance. One can interpret the expression

$$\Phi_H(V)(\varphi) = \mathrm{Tr}(V\varphi(H))$$

as *infinitesimal spectral flow*, in a certain sense. Indeed, in the case when H is the operator of multiplication by λ on the Hilbert space $L^2(\mathbb{R}, d\rho(\lambda))$, and V is an integral operator with compactly supported C^1 kernel $v(\lambda,\lambda')$, one has

$$\Phi_H(V)(\varphi) = \mathrm{Tr}(V\varphi(H)) = \int_{\mathbb{R}} v(\lambda,\lambda)\varphi(\lambda)\,d\rho(\lambda).$$

This means that the infinitesimal spectral flow in this case is a measure on the spectrum of H with density $v(\lambda,\lambda)\,d\rho(\lambda)$. This fully agrees with a classical formula from quantum-mechanical perturbation theory [LL, (38.6)]

$$E_n^{(1)} = V_{nn} := \int \psi_n^{(0)} \hat{V} \psi_n^{(0)}\,dq,$$

where $E_n^{(1)}$ is the first correction term for the nth eigenvalue of the perturbed operator $\hat{H} = \hat{H}_0 + \hat{V}$. One can show that $\Phi_H(V)$ is a measure of the spectrum of H. The one-form (7) is the value of infinitesimal spectral flow on the test function $\varphi_\varepsilon(x) = \sqrt{\frac{\varepsilon}{\pi}}e^{-\varepsilon x^2}$, which approximates δ-function as $\varepsilon \to 0$. Similarly, the integrand of the formula (8) is the value of infinitesimal spectral flow on the test function $\varphi_p(x) = C_p^{-1}(1+x^2)^{-p/2}$, which approximates δ-function as $p \to \infty$.

Another idea suggested by Birman-Solomyak formula is that the spectral shift function should be possible to define for pairs of operators H_0, H_1 for which the expression $V\varphi(H_r)$ is trace class for all r. This poses the question of whether the inclusions $V\varphi(H_0)$, $V\varphi(H_1) \in \mathcal{L}^1(\mathcal{H})$ implies that $V\varphi(H_r) \in \mathcal{L}^1(\mathcal{H})$. This problem is still open, but one can observe that in all known cases this implication has been proved. This leads naturally to the notion of *trace compatible operators.* Two operators H_0 and $H_1 = H_0 + V$ are called trace compatible, if $V\varphi(H_r) \in \mathcal{L}^1(\mathcal{H})$ for all $r \in \mathbb{R}$. One can define *trace compatible affine space* as an affine space with some appropriately defined topology, such that any two elements of this space are trace compatible. It was shown in [AS$_2$] that for any pair of operators from a trace compatible affine space the Birman-Solomyak formula holds with $\{H_r\}$ an arbitrary piecewise smooth path, connecting H_0 and H_1, and that the spectral shift distribution is absolutely continuous. Independence from path of integration is a corollary of *exactness* of $\Phi_H(\cdot)$, considered as a one-form on the corresponding trace compatible affine space. For example, one can show that the space $D + C_c(\mathbb{R})$ is trace compatible, where $D = \frac{1}{i}\frac{d}{dx}$. In this case $\Phi_{D+a}(v)$ does not depend on $a \in C_c(\mathbb{R})$ and is constant:

$$\Phi_{D+a}(v) = \frac{1}{2\pi}\int_{\mathbb{R}} v(x)\,dx \cdot \mathrm{mes},$$

where mes is the Lebesgue measure.

The notion of infinitesimal spectral flow was developed further in [AzISF]. In [AzISF] it is shown that the absolutely continuous part (the density of) $\Phi_H^{(a)}(V)(\lambda)$ of infinitesimal spectral flow is actually the trace of an operator valued function $\Pi_H(V)(\lambda)$, which was called *infinitesimal scattering matrix* and which is defined naturally on the absolutely continuous part of the spectrum of H. The reason for this terminology is a formula for the scattering matrix

$$S(\lambda; H_1, H_0) = \mathrm{Texp}\left(-2\pi i \int_0^1 w_+(\lambda; H_0, H_r)\Pi_{H_r}(\dot{H}_r)(\lambda) w_+(\lambda; H_r, H_0)\,dr\right),$$

established in [AzISF], where Texp is the chronological exponential, defined by the formula

$$\mathrm{Texp}\left(\frac{1}{i}\int_a^t A(s)\,ds\right) = 1 + \sum_{k=1}^{\infty} \frac{1}{i^k}\int_a^t dt_1 \int_a^{t_1} dt_2 \ldots \int_a^{t_{k-1}} dt_k A(t_1)\ldots A(t_k).$$

Bibliography

[AB] E. Asplund, L. Bungart, *A First Course in Integration*, New York, Holt, Rinehart and Winston, 1966.

[APS] M. Atiyah, V. Patodi, I. M. Singer, *Spectral Asymmetry and Riemannian Geometry. III*, Math. Proc. Camb. Phil. Soc. **79** (1976), 71–99.

[ASS] J. E. Avron, R. Seiler, B. Simon, *The index of a pair of projections*, J. Functional Analysis **120** (1994), 220–237.

[AzISF] N. A. Azamov, *Infinitesimal spectral flow and scattering matrix*, submitted for publication, arXiv:0705.3282v4.

[ACDS] N. A. Azamov, A. L. Carey, P. G. Dodds, F. A. Sukochev, *Operator integrals, spectral shift and spectral flow*, to appear in Canad. J. Math, arXiv: math/0703442.

[ACS] N. A. Azamov, A. L. Carey, F. A. Sukochev, *The spectral shift function and spectral flow*, Comm. Math. Phys **276** (2007), 51–91.

[ADS] N. A. Azamov, P. G. Dodds, F. A. Sukochev, *The Krein spectral shift function in semifinite von Neumann algebras*, Integral Equations Operator Theory **55** (2006), 347–362.

[AS] N. A. Azamov, F. A. Sukochev, *A Lidskii type formula for Dixmier traces*, C. R. Acad. Sci. Paris Sér. A-B **340** (2005), 107–112.

[AS_2] N. A. Azamov, F. A. Sukochev, *Spectral averaging for trace compatible operators*, to appear in Proc. Amer. Math. Soc., arXiv: 0704.1683v2.

[BCPRSW] M-T. Benameur, A. L. Carey, J. Phillips, A. Rennie, F. A. Sukochev, K. P. Wojciechowski, *An analytic approach to spectral flow in von Neumann algebras*, Analysis, geometry and topology of elliptic operators, 297–352, World Sci. Publ., Hackensack, NJ, 2006.

[BS] M. Sh. Birman, M. Z. Solomyak, *Remarks on the spectral shift function*, J. Soviet math. **3** (1975), 408–419.

[BS_2] M. Sh. Birman, M. Z. Solomyak, *Double Stieltjes operator integrals. III*, Izdat. Leningrad. Univ., Leningrad, (1973), 27–53 (Russian).

[BR] O. Bratteli, D. Robinson, *Operator Algebras and Quantum Statistical Mechanics I*, Springer-Verlag, 1979.

[Br] M. Breuer, *Fredholm theories in von Neumann algebras. I*, Math. Ann. **178** (1968), 243–254.

[Br_2] M. Breuer, *Fredholm theories in von Neumann algebras. II*, Math. Ann. **180** (1969), 313–325.

[Brn] L. G. Brown, *Lidskii's theorem in the type II case*, in *Proc. U.S.-Japan Seminar, Kyoto 1983*, Pitman Research Notes, Math. Ser., **123** (1986), 1–35.

[BK] L. G. Brown, H. Kosaki, *Jensen's inequality in semi-finite von Neumann algebras*, J. Operator Theory **23** (1990), 3–19.

[CFM] A. L. Carey, M. Farber, V. Mathai, *Anomalies, complex determinants and von Neumann algebras*, unpublished notes.

[CP] A. L. Carey, J. Phillips, *Unbounded Fredholm modules and spectral flow*, Canad. J. Math. **50** (1998), 673–718.

[CP_2] A. L. Carey, J. Phillips, *Spectral flow in Fredholm modules, eta invariants and the JLO cocycle*, K-Theory **31** (2004), 135–194.

[CPRS] A. L. Carey, J. Phillips, A. Rennie, F. A. Sukochev, *The local index formula in semifinite von Neumann algebras I: spectral flow*, Adv. in Math. **202** (2006), 451–516.

[$CPRS_2$] A. L. Carey, J. Phillips, A. Rennie, F. A. Sukochev, *The local index formula in semifinite von Neumann algebras II: even case*, Adv. in Math. **202** (2006), 517–554.

[CPS] A. L. Carey, J. Phillips, F. A. Sukochev, *Spectral flow and Dixmier traces*, Adv. in Math. **173** (2003), 68–113.

[CP] R. W. Carey, J. D. Pincus, *Mosaics, principal functions, and mean motion in von Neumann algebras*, Acta Math. **138** (1977), 153–218.

[CDS] V. I. Chilin, P. G. Dodds, F. A. Sukochev, *The Kadec-Klee property in symmetric spaces of measurable operators*, Israel J. Math. **97** (1997), 203–219.

[CMS] L. A. Coburn, R. D. Moyer, I. M. Singer, *C^*- algebras of almost periodic pseudo-differential operators*, Acta Math. **130:3-4** (1973), 279–307.

[Co] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, 1994.

[DK] Yu. L. Daletskiĭ, S. G. Kreĭn, *Integration and differentiation of functions of Hermitian operators and applications to the theory of perturbations*, Voronež. Gos. Univ., Trudy Sem. Funkcional. Anal. **1** (1956), 81–105 (Russian).

[Di] J. Dixmier, *von Neumann Algebras*, North-Holland, Amsterdam, New York, 1981.

[Di_2] J. Dixmier, *Existence de traces non-normales*, C. R. Acad. Sci. Paris Sér. A-B **262** (1966), A1107–A1108.

[DDP] P. G. Dodds, T. K. Dodds, B. de Pagter, *Fully symmetric operator spaces*, Integral Equations Operator Theory **15** (1992), 942–972.

[DDP_2] P. G. Dodds, T. K. Dodds, B. de Pagter, *Noncommutative Köthe duality*, Trans. Amer. Math. Soc. **339** (1993), 717–750.

[DDPS] P. G. Dodds, T. K. Dodds, B. de Pagter, F. A. Sukochev, *Lipschitz continuity of the absolute value and Riesz projections in symmetric operator spaces*, J. Functional Analysis **148** (1997), 28–69.

[DPSS] P. G. Dodds, B. de Pagter, E. M. Semenov, F. A. Sukochev, *Symmetric functionals and singular traces*, Positivity **2** (1998), 47–75.

[DPSSS] P. G. Dodds, B. de Pagter, A. A. Sedaev, E. M. Semenov, F. A. Sukochev, *Singular symmetric functionals*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), Issled. po Linein. Oper. i Teor. Funkts., **290** 30 (2002), 42–71 (Russian).

[DPSSS_2] P. G. Dodds, B. de Pagter, A. A. Sedaev, E. M. Semenov, F. A. Sukochev, *Singular symmetric functionals and Banach limits with additional invariance properties*, Izv. Ross. Akad. Nauk, Ser. Mat., **67** 6 (2003), 111–136 (Russian).

[DS] N. Dunford, J. T. Schwartz, *Linear Operators, Part I: General theory*, Interscience publishers, Inc., New York, 1964.

[DK] K. J. Dykema, N. J. Kalton, *Spectral characterization of sums of commutators II*, J. Reine Angew. Math. **504** (1998), 127–137.

[DK_2] K. J. Dykema, N. J. Kalton, *Sums of commutators in ideals and modules of type II factors*, Ann. Inst. Fourier (Grenoble) **55** (2005), 931–971.

[Fac] T. Fack, *Sums of commutators in non-commutative Banach function spaces*, J. Functional Analysis **207** (2004), 358–398.

[FK] T. Fack, H. Kosaki, *Generalised s- numbers of τ- measurable operators*, Pacific J. Math. **123** (1986), 269–300.

[Far] Yu. B. Farforovskaya, *Example of a Lipschitz function of self-adjoint operators that gives a nonnuclear increment under a nuclear perturbation*, J. Soviet math. **4** (1975), 426–433.

[FKa] B. Fuglede, R. V. Kadison, *Determinant theory in finite factors*, Ann. of Math. **55** 3 (1952), 520–530.

[Ga] J. B. Garnett, *Bounded analytic functions*, Academic Press, 1981.

[GSh] I. M. Gelfand, G. E. Shilov, *Generalized functions, vol. 1*, New York, Academic Press, 1964.

[Ge] E. Getzler, *The odd Chern character in cyclic homology and spectral flow*, Topology **32** (1993), 489–507.

[GK] I. C. Gohberg, M. G. Kreĭn, *Introduction to the Theory of Nonselfadjoint Operators*, Providence, R. I., AMS, Trans. Math. Monographs, AMS, **18**, 1969.

[GMP] Ya. Goldshtein, S. A. Molchanov, L. A. Pastur, *A random onedimensional Schrödinger operator has a pure point spectrum*, Funct. Anal. Appl. **11** (1977), 1–10.

[Gr] F. P. Greenleaf, *Invariant Means on Topological Groups*, Van Nostrand Reinhold, New York, 1969.

[Hal] Halmos, *A Hilbert Space Problem Book, Second edition*, New York, Springer-Verlag, 1982.

[Ha] G. H. Hardy, *Divergent Series*, Clarendon, Oxford, 1949.

[HS] P. de la Harpe, G. Skandalis, *Déterminant associé à une trace sur une algèbre de Banach*, Ann. Inst. Fourier (Grenoble) **34** (1984), no. 1, 241–260.

[HK] W. K. Hayman, P. B. Kennedy, *Subharmonic functions*, London, Academic Press, 1976.

[HPh] E. Hille, R. S. Phillips, *Functional Analysis and Semigroups*, AMS, Providence, R. I., 1957.

[Ja] K. Jacobs, *Measure and integral*, Academic Press, New-York, San Francisco, London, 1978.

[Jav] V. A. Javrjan, *A certain inverse problem for Sturm-Liouville operators*, Izv. Akad. Nauk Armjan. SSR Ser. Mat. **6** (1971), 246–251.

[Kal] N. J. Kalton, *Spectral characterization of sums of commutators I*, J. Reine Angew. Math. **504** (1998), 115–125.

[Kat] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[KM] V. Kostrykin, K. A. Makarov, *On Krein's example*, arXiv: math.SP/0606249v1.

[Ko] S. Kotani, *Lyapunov exponents and spectra for one-dimensional random Schrödinger operators*, Contemporary Math. **50** (1986), 277–286.

[Kr] M. G. Kreĭn, *On the trace formula in perturbation theory*, Mat. Sb., **33** 75 (1953), 597–626.

[Kr_2] M. G. Kreĭn, *Some new studies in the theory of perturbations of self-adjoint operators*, (Russian) First Math. Summer School (Kanev, 1963), Part I, 'Naukova Dumka', Kiev, (1964), 103—187; English transl. in M. G. Kreĭn, Topics in differential and integral equations and operator theory, Birkhäuser, Basel, 1983, pp. 107–172.

[LL] L. D. Landau, E. M. Lifshitz, *Quantum mechanics, 3rd edition*, Pergamon Press.

[Lid] V. B. Lidskii, *Non-selfadjoint operators with a trace*, Dokl. Akad. Nauk SSSR **125** (1959), 485–587 (Russian).

[Lif] I. M. Lifshits, *On a problem in perturbation theory*, Uspekhi Mat. Nauk **7** (1952), 171–180 (Russian).

[LSS] S. Lord, A. A. Sedaev, F. A. Sukochev, *Dixmier traces as singular symmetric functionals and applications to measurable operators*, J. Functional Analysis **224** (2005), 72–106.

[Lo] G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.

[dPS] B. de Pagter, F. A. Sukochev, *Differentiation of operator functions in non-commutative L_p- spaces*, J. Functional Analysis **212** (2004), 28–75.

[dPSW] B. de Pagter, F. A. Sukochev, H. Witvliet, *Double operator integrals*, J. Functional Analysis **192** (2002), 52–111.

[Pa] B. S. Pavlov, *On multiple operator integrals*, "Problems of mathematical analysis", Leningrad University Publ., **2** (1969), 99–122 (Russian).

[Pel] V. V. Peller, *Multiple operator integrals and higher operator derivatives*, J. Functional Analysis **233** (2006), no. 2, 515–544.

[PR] J. Phillips, I. Raeburn, *An index theorem for Toeplitz operators with noncommutative symbol space*, J. Functional Analysis **120** (1994), 239–263.

[Ph] J. Phillips, *Self-adjoint Fredholm operators and spectral flow*, Canad. Math. Bull. **39** (1996), 460–467.

[Ph_2] J. Phillips, *Spectral flow in type I and type II factors – a new approach*, Fields Inst. Comm. **17** (1997), 137–153.

[Pu] A. B. Pushnitski, *Differences of spectral projections and scattering matrix*, arXiv: math/0702253v1.

[RS] M. Reed, B. Simon, *Methods of modern mathematical physics: 1. Functional analysis*, Academic Press, New York, 1972.

[Sch] J. T. Schwartz, *Nonlinear Functional Analysis*, Gordon and Breach Science Publishers, New-York, London, Paris.

[Sh] M. A. Shubin, *Pseudodifferential almost periodic operators and von Neumann algebras*, Trans. Moscow Math. Soc. **1** (1979), 103–166.

[Si] B. Simon, *Trace ideals and their applications*, London Math. Society Lecture Note Series, **35**, Cambridge University Press, Cambridge, London, 1979.

[Si_2] B. Simon, *Spectral averaging and the Krein spectral shift*, Proc. Amer. Math. Soc. **126** (1998), 1409–1413.

[SS] M. Z. Solomyak, V. V. Sten'kin, *On a class of multiple operator Stieltjes integrals*, "Problems of mathematical analysis", Leningrad University Publ., **2** (1969), 122–134 (Russian).

[SW] E. M. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton, New Jersey 1971.

[St] V. V. Sten'kin, *On multiple operator integrals*, Izv. Vyssh. Uch. Zav., Mat. **4** (1977), 102–115 (Russian).

[SZ] S. Stratila, L. Zsido, *Lectures on von Neumann algebras*, Abacus Press, Bucharest, 1979.

[Ta] M. Takesaki, *Theory of Operator Algebras, vol. I*, Springer-Verlag, 2002.

[VTCh] N. N. Vakhaniya, V. I. Tarieladze, S. A. Chobanyan, *Probability distributions in Banach spaces*, Moscow, Nauka, 1985 (Russian).

[Vl] V. S. Vladimirov, *Methods of the Theory of Functions of Many Complex Variables*, Moscow, Nauka, 1964 (Russian).

[Wi] H. Widom, *When are differentiable functions are differentiable?*, "Linear and Complex Analysis Problem Book **3**, Part 1", Editors V. P. Havin et al., Lecture Notes in Mathematics **1573**, Springer-Verlag, Berlin-Heidelberg-New York, 1994, pp. 266–271.

[W] J. P. Williams, *Spectra of products and numerical ranges*, J. Math. Anal. and Appl. **17** (1967), 214–220.

[Ya] D. R. Yafaev, *Mathematical scattering theory: general theory*, Providence, R. I., AMS, 1992.

[Y] K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin, New York, 1980.

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