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LINEAR PROGRAMMING BASED APPROACH TO INFINITE HORIZON OPTIMAL CONTROL PROBLEMS WITH TIME DISCOUNTING CRITERIA

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ABSTRACT

The aim of this thesis is to develop mathematical tools for the analysis and solution of infinite horizon optimal control problems with a time discounting criteria based on the fact that the latter are equivalent to certain infinite dimensional linear programming problems. We establish that near-optimal solutions of these infinite dimensional linear programming problems and their duals can be obtained via approximation with semi-infinite linear programming problems and subsequently with finite-dimensional ("standard") linear programming problems and their respective duals. We show that near-optimal controls of the underlying optimal control problems can be constructed on the basis of solutions of these standard linear programming problems. The thesis consists of two parts. In Part I, theoretical results are presented. These include results about semi-infinite and finite dimensional approximations of the infinite dimensional linear programming problems, results about the construction of near-optimal controls and results establishing the possibility of using solutions of optimal control problems with time discounting criteria for the construction of stabilising controls. In Part II, results of numerical experiments are presented. These results include finding near-optimal controls for the optimisation of a damped mass-spring system, the optimisation of a Neck and Dockner model and the problem of finding stabilising controls for a Lotka-Volterra system.

PUBLICATIONS

Some ideas and figures have appeared previously in the following publications:

† V. Gaitsgory, S. Rossomakhine and N. Thatcher, *Approximate solution of the HJB inequality related to the infinite horizon optimal control problem with discounting.* Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications and Algorithms **19** (2012), no. 1–2b, 65–92.

‡ V. Gaitsgory, L. Grüne and N. Thatcher, *Stabilization with discounted optimal control.* Systems & Control Letters **82** (2015), 91–98.

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Neil Thatcher
30 January 2017

The declaration is not required for final submission

DECLARATION

I, the undersigned, do hereby certify, that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge and belief this thesis does not contain any material previously published or written by any other person except where due reference is made in the text.

Signature

Neil Thatcher (2127408)
30 January 2017

NOTATION

$\lvert\cdot\rvert$	Absolute value
$\lVert\cdot\rVert$	Norm in $\mathbb{R}^n$
B	Closed unit ball
$\bar{co}Q$	Closed convex hull of the set Q
C	Discount rate
$\stackrel{\text{def}}{=}$	Lit. *is defined to be equal to*
C^n	The set of n-times continuously differentiable functions
$\delta_{(y,u)}$	Dirac measure concentrated at the point (y, u)
$G^*(y_0)$	Optimal value of primal problem
$G^N(y_0)$	Optimal value of primal semi-inf. dim. problem
$G^{N,\Delta}(y_0)$	Optimal value of primal finite dim. problem
$\mu^*(y_0)$	Optimal value of dual problem
$\mu^N(y_0)$	Optimal value of dual semi-inf. dim. problem
$\mu^{N,\Delta}(y_0)$	Optimal value of dual finite dim. problem
MPC	Model Predictive Control
$\mathcal{P}(Y \times U)$	A set of probability measures defined on $Y \times U$
$\mathbb{R}^n$	The Euclidean n-dimensional space
$W(y_0)$	Set of γ satisfying an inf. dim. system of lin. constraints
$W^N(y_0)$	Set of γ satisfying an semi-inf. dim. system of lin. constraints
$W^{N,\Delta}(y_0)$	Set of γ satisfying an finite dim. system of lin. constraints
γ	Occupation measure
$(y(\cdot), u(\cdot))$	An admissible pair of functions
$\rho(a, b)$	A metric defined by weak* convergence
$\rho_H(A, B)$	Hausdorff metric between sets
HJB	Hamilton-Jacobi-Bellman (equation)

CONTENTS

INTRODUCTION

The purpose of this research is to further develop linear programming (LP) based techniques for solving optimal control problems with time discounting criteria over the infinite time horizon. We also present results on the use of time discounted optimal control problems for the construction of stabilising controls.

The linear programming approach to control systems is based on the fact that the occupational measures generated by admissible controls and the corresponding solutions of a nonlinear system satisfy certain linear equations representing the system's dynamics in an integral form. Using this fact, one can reformulate the optimal control problem as an infinite dimensional linear programming (IDLP) problem. The linear programming approach to optimal control problems in both deterministic and stochastic settings have been studied by many (see, e.g., [3, 5, 6, 7, 11, 14, 15, 16, 17, 19, 21, 22, 24, 30, 37, 38, 42, 43, 51, 56, 57, 59] and references therein).

In this thesis we continue the line of research started in [23], [25], [26] and [27]. In [23], [26] and [27] the linear programming approach to deterministic long run average problems of optimal control was considered. It was established that these problems are "asymptotically equivalent" to IDLP problems and that these linear programming problems can be approximated by finite dimensional linear programming (FDLP) problems, the solution of which can be used for construction of optimal controls. In [25], the theoretical aspects of the linear programming formulation to infinite horizon optimal control problems with time discounting criteria were dealt with. In this thesis, we use results of [25] and some ideas of [23], [26] and [27] to investigate ways of constructing near-optimal solutions of optimal control problems with time discounting. In particular, we focus on finding smooth approximate solutions of the Hamilton-Jacobi-Bellman (HJB) inequality that corresponds to the latter. We establish that such approximate solutions of the HJB inequality exist (under a simple controllability type condition) and that they can be used for the construction of near-optimal controls. We also show that these approximate solutions of the HJB inequality can be found by solving certain semi-infinite linear programming (SILP) problems [28] and we approximate the latter with "standard" FDLP problems.

It is well known that infinite horizon undiscounted optimal control methods can be used for the design of asymptotically stabilising controls by choosing the objective which penalises states away from the desired equilibrium. These problems are numerically very difficult to solve [9]. In this thesis, we will discuss the construction of a stabilising control based on a linear programming solution of an infinite horizon optimal control problem with time discounting. To this end we utilize a condition involving a bound on the optimal value function which is similar to what can be found in the Model Predictive Control (MPC) literature (see, related results [9, 12, 13, 20, 29, 33, 49, 54] and references therein).

The thesis is in two parts.

In Part I, the theoretical results are presented. In Chapter 1, we introduce the IDLP problem and its dual (which is a maxmin type problem) that are related to the infinite time horizon optimal control problem with time discounting criterion and we review some of the results obtained in [25] that are used later in the text. We then establish that the maxmin dual problem is in a certain sense equivalent to a HJB type inequality. We finish the chapter by stating necessary and sufficient optimality conditions based on the HJB inequality. In Chapter 2, we show that the IDLP problem is approximated by a sequence of semi-infinite linear programming problems. We then introduce a maxmin problem considered on an N-dimensional linear space of smooth functions and we establish that, under certain controllability conditions, an optimal solution of this problem exists and that this solution can serve as an approximate solution of the HJB inequality. We also state and prove a result establishing that, under certain assumptions, the approximate solution of the HJB inequality allows one to construct an admissible pair that can serve as an approximation to the optimal one. In Chapter 3, we further approximate the SILP problem and the corresponding dual problem with an FDLP problem defined on a grid of points and show that by solving the FDLP problem, one can construct a continuously differentiable function which solves the dual IDLP problem approximately and that this function can be used for construction of a near-optimal control. In Chapter 4, we show that stabilising controls can be constructed on the basis of solutions of certain optimal control problems with time discounting criteria.

In Part II, we present numerical experiments illustrating results of Part I. In Chapter 5, a problem of optimal control of a damped mass-spring system is introduced and a general framework for numerical solutions of time discounted optimal control problems is outlined. This problem is then solved numerically. In Chapter 6, an economic model of Neck and Dockner [46, 47] is introduced and the results for various values of the discount rate are presented. In Chapter 7 a series of experiments is presented, which demonstrate the stabilisation of a Lotka-Volterra system using time discounted optimal controls solved using the linear programming method. The Lotka-Volterra system is then used in Chapter 8, where we demonstrate how a series of FDLP problems solved on a grid of initial conditions may be used to derive a control map which is near-optimal for arbitrary initial conditions.

This thesis is submitted to Flinders University in partial fulfilment of the requirements for the Doctor of Philosophy degree in the field of Mathematics.

Part I

THEORETICAL RESULTS

1

THE LINEAR PROGRAMMING APPROACH

In this chapter, we introduce notations and present results which are used throughout this text. Specifically, in Section 1.1 we introduce the dynamical system and define the time discounted optimal control problem. Then in Section 1.2 we introduce the concept of a discounted occupational measure γ. In Section 1.3 we show how the time discounted optimal control problem can be reformulated in terms of these measures and establish that the reformulated problem is in fact, equivalent to an infinite dimensional linear program (IDLP). In Section 1.4 the dual maxmin problem is introduced and we describe the duality relationships with the original IDLP problem. In Section 1.5 we demonstrate the equivalence of the dual maxmin problem and the HJB inequality by showing that the set of solutions of the latter is related to the set of solutions of the former. Finally, in Section 1.6 we state the necessary and sufficient conditions for optimality based on the HJB inequality.

1.1 PRELIMINARIES

We will be considering the control system

$$y'(t) = f(y(t), u(t)),\ t \geqslant 0, \tag{1.1}$$

where the function $f(y, u)\colon \mathbb{R}^m \times U \mapsto \mathbb{R}^m$ is continuous in (y, u) and satisfies the Lipschitz condition in y uniformly with respect to u. The controls are Lebesgue measurable functions $u(\cdot)\colon [0, \infty) \mapsto U$ where U is a compact metric space. The set of these controls is denoted as $\mathcal{U}$.

Definition 1.1. *A pair $(y(\cdot), u(\cdot))$ will be called admissible if equation (1.1) is satisfied for almost all t and if the following inclusions are valid:*

$$y(t) \in Y,\ t \in [0, \infty)$$

and

$$u(t) \in U, \quad \textit{for almost all } t,$$

where Y *is a given compact subset of* $\mathbb{R}^m$.

The cost function of our discounted optimal control problem is defined as:

$$J(y_0, u(\cdot)) \stackrel{\text{def}}{=} \int_0^\infty e^{-Ct} g(y(t), u(t)) dt,$$

where the function $g(y, u)\colon Y \times U \mapsto \mathbb{R}$ is a given continuous function we call the *running cost* and the parameter $C > 0$ shall be referred to as the *discount rate*.

Definition 1.2. *The optimal value function of the discounted optimal control problem is defined as*

$$V(y_0) \stackrel{\text{def}}{=} \inf_{(y(\cdot), u(\cdot))} J(y_0, u(\cdot)), \tag{1.2}$$

where the minimisation is over all admissible pairs that satisfy the initial conditions

$$y(0) = y_0. \tag{1.3}$$

For a given initial value, an admissible control $u^*(\cdot) \in \mathcal{U}$ is called an *optimal control* if $J(y_0, u^*(\cdot)) = V(y_0)$ holds.

1.2 OCCUPATIONAL MEASURE FORMULATION

Now we introduce the concept of discounted occupational measure. Let $\mathcal{P}(Y \times U)$ be the space of probability measures defined on Borel subsets of $Y \times U$ and let $(y(\cdot), u(\cdot))$ be an arbitrary admissible pair. A probability measure $\gamma^{(y(\cdot), u(\cdot))}$ is called the *discounted occupational measure* generated by the pair $(y(\cdot), u(\cdot))$ if, for any Borel set $Q \subset Y \times U$,

$$\gamma^{(y(\cdot), u(\cdot))}(Q) = C \int_0^\infty e^{-Ct} 1_Q(y(t), u(t)) dt, \tag{1.4}$$

where $1_Q(\cdot)$ is the indicator function of the set Q. That is, $1_Q(y, u) = 1, \forall (y, u) \in Q$ and $1_Q(y, u) = 0, \forall (y, u) \notin Q$. The validity of (1.4) for any indicator function leads to the validity of a similar equality for the simple functions (that is, linear combinations of the indicator functions) and, thus, with the help of a standard approximation argument, leads to the validity of the equality

$$\int_{Y\times U} q(y,u)\gamma^{(y(\cdot),u(\cdot))}(dy,du) = C\int_0^\infty e^{-Ct}q(y(t),u(t))dt \tag{1.5}$$

for any $q(\cdot) \in C(Y\times U)$ (the space of continuous functions defined on $Y\times U$).

Thus the integral with respect to the induced occupational measure is proportional to the integral with respect to time over a state-control trajectory. The constant of proportionality being equal to the discount rate C.

Note that the discounted occupational measure generated by a steady state admissible pair $(y(t),u(t)) = (y,u) \in Y\times U$ is just the Dirac measure at the point (y,u).

Let $\Gamma(y_0)$ stand for the set of all discounted occupational measures defined as follows

$$\Gamma(y_0) \stackrel{\text{def}}{=} \bigcup_{(y(\cdot),u(\cdot))} \{\gamma^{(y(\cdot),u(\cdot))}\},$$

where the union is over all admissible controls and the corresponding solutions of (1.1) satisfying the initial condition (1.3).

Note that, due to (1.5), the problem (1.2) can be rewritten as

$$CV^*(y_0) = \inf_{\gamma\in\Gamma(y_0)} \int_{Y\times U} g(y,u)\gamma(dy,du). \tag{1.6}$$

Let us endow the space $\mathcal{P}(Y\times U)$ with a metric ρ,

$$\rho(\gamma',\gamma'') \stackrel{\text{def}}{=} \sum_{j=1}^{\infty}\frac{1}{2^j}\Big|\int_{Y\times U} q_j(y,u)\gamma'(dy,du) - \int_{Y\times U} q_j(y,u)\gamma''(dy,du)\Big|,\ \forall\gamma',\gamma''\in\mathcal{P}(Y\times U), \tag{1.7}$$

where $q_j(y,u)$, $j=1,2,\dots$ is a sequence of Lipschitz continuous functions which is dense in the unit ball of $C(Y\times U)$. Note that this metric is consistent with the weak* topology of $\mathcal{P}(Y\times U)$. Namely, a sequence $\gamma^k \in \mathcal{P}(Y\times U)$ converges to $\gamma\in\mathcal{P}(Y\times U)$ in this metric if and only if

$$\lim_{k\to\infty}\int_{Y\times U} q(y,u)\gamma^k(dy,du) = \int_{Y\times U} q(y,u)\gamma(dy,du)$$

for any continuous $q(\cdot)\in C(Y\times U)$.

Using the metric ρ, one can define the "distance" $\rho(\gamma,\Gamma)$ between $\gamma \in \mathcal{P}(Y \times U)$ and $\Gamma \subset \mathcal{P}(Y \times U)$ and the Hausdorff metric $\rho_H(\Gamma_1,\Gamma_2)$ between $\Gamma_1 \subset \mathcal{P}(Y \times U)$ and $\Gamma_2 \subset \mathcal{P}(Y \times U)$ as follows:

$$\rho(\gamma,\Gamma) \stackrel{\text{def}}{=} \inf_{\gamma' \in \Gamma} \rho(\gamma,\gamma'),$$

$$\rho_H(\Gamma_1,\Gamma_2) \stackrel{\text{def}}{=} \max\Big\{ \sup_{\gamma\in\Gamma_1} \rho(\gamma,\Gamma_2), \sup_{\gamma\in\Gamma_2} \rho(\gamma,\Gamma_1)\Big\}.$$

Note that, although, by some abuse of terminology, we refer to $\rho_H(\cdot,\cdot)$ as a metric on the set of subsets of $\mathcal{P}(Y \times U)$, it is, in fact, a semi-metric on this set (since $\rho_H(\Gamma_1,\Gamma_2) = 0$ implies $\Gamma_1 = \Gamma_2$ only if Γ_1 and Γ_2 are closed).

Note, that, being endowed with such a metric, $\mathcal{P}(Y \times U)$ becomes a compact metric space (see, e.g., [8] or [48]).

1.3 THE LINEAR PROGRAMMING PROBLEM

Define the set $W(y_0) \subset \mathcal{P}(Y \times U)$ by the equation

$$W(y_0) \stackrel{\text{def}}{=} \{\gamma \in \mathcal{P}(Y \times U) \colon \int_{Y\times U} (\phi'(y)^T f(y,u) + C(\phi(y_0) - \phi(y)))\gamma(dy,du) = 0,\ \forall \phi(\cdot) \in C^1\},$$

where C^1 is the space of continuously differentiable functions $\phi(y)\colon \mathbb{R}^n \mapsto \mathbb{R}$ and $\phi'(y)$ is a vector column of partial derivatives of $\phi(y)$.

Assuming that $W(y_0)$ is not empty, let us consider the problem

$$G^*(y_0) \stackrel{\text{def}}{=} \inf_{\gamma \in W(y_0)} \int_{Y\times U} g(y,u)\gamma(dy,du). \tag{1.8}$$

Note that problem (1.8) is an infinite dimensional linear program since its objective function and its constraints are linear in γ (see, e.g., [2]). Note also that an optimal solution of this problem exists if the set $W(y_0)$ is not empty (this follows from the fact that $W(y_0)$ is compact; see Lemma 1.4 below).

We will be interested in studying and exploiting the connections between problem (1.2) and the problem (1.8). Note that the set $W(y_0)$ can be empty and no solution to (1.8) exists. It is easy to see, for example, that $W(y_0)$ is empty if there exists a continuously differentiable function $\phi(\cdot) \in C^1$ such that

$$\max_{(y,u)\in Y\times U} \big\{\phi'(y)^T f(y,u) + C(\phi(y_0) - \phi(y))\big\} < 0.$$

Example 1.3. *For an example of a problem for which the set* $W(y_0)$ *is empty, consider* $y'(t) = y(t)$, $t > 0$ *on* $Y \stackrel{\text{def}}{=} [\frac{1}{2}, 1]$. *In fact, take* $\phi(y) = -y$. *Then*

$$\begin{aligned}\phi'(y)^T f(y,u) + C(\phi(y_0) - \phi(y)) &= -y - Cy_0 + Cy \\ &= -y(1-C) - Cy_0 \\ &< -Cy_0 \\ &< 0,\end{aligned}$$

this being valid for $C < 1$ *and* $y_0 \in Y$. *Hence* $W(y_0) = \emptyset$.

The set $W(y_0)$ is not empty if there exists at least one admissible pair (e.g., a steady state pair), since the discounted occupational measure generated by this pair is completely contained in $W(y_0)$ (see Proposition 1.5 below).

The following simple properties will be useful later on:

Lemma 1.4. *The set* $W(y_0)$ *is convex and compact in the weak* topology of* $\mathcal{P}(Y \times U)$.

Proof of Lemma 1.4. The space $\mathcal{P}(Y \times U)$ is convex if $\mu_1, \mu_2 \in \mathcal{P}(Y \times U)$ and $0 \leqslant \alpha \leqslant 1$, then $\alpha\mu_1 + (1-\alpha)\mu_2 \in \mathcal{P}(Y \times U)$. So, for any γ', $\gamma'' \in W(y_0)$ and for any $\alpha \in [0, 1]$,

$$\begin{aligned}&\int_{Y \times U} (\phi'(y)^T f(y,u) + C(\phi(y_0) - \phi(y)))(\alpha\gamma' + (1-\alpha)\gamma'')(dy, du) \\ &= (\alpha) \int_{Y \times U} (\phi'(y)^T f(y,u) + C(\phi(y_0) - \phi(y)))\gamma'(dy, du) \\ &\qquad + (1-\alpha) \int_{Y \times U} (\phi'(y)^T f(y,u) + C(\phi(y_0) - \phi(y)))\gamma''(dy, du) = 0.\end{aligned}$$

Hence the set $W(y_0)$ is convex.

To show that $W(y_0)$ is closed and compact, consider a sequence of points $\gamma_i \in W(y_0)$ such that

$$\lim_{i \to \infty} \gamma_i = \bar{\gamma},$$

where $\bar{\gamma}$ is a boundary point. For all $\gamma_i \in W(y_0)$,

$$\int_{Y \times U} (\phi'(y)^T f(y,u) + C(\phi(y_0) - \phi(y)))\gamma_i(dy, du) = 0.$$

Therefore

$$\lim_{i\to\infty}\int_{Y\times U}(\phi'(y)^T f(y,u)+C(\phi(y_0)-\phi(y)))\gamma_i(dy,du)$$
$$=\int_{Y\times U}(\phi'(y)^T f(y,u)+C(\phi(y_0)-\phi(y)))\bar{\gamma}(dy,du)=0.$$

Hence $\bar{\gamma}\in W(y_0)$ and the set $W(y_0)$ is closed, hence, compact since $\mathcal{P}(Y\times U)$ is compact. □

Proposition 1.5. *The following relationships are valid*

$$CV^*(y_0)\geqslant G^*(y_0) \tag{1.9}$$

and

$$\bar{co}\Gamma(y_0)\subset W(y_0),\ \forall y_0\in Y, \tag{1.10}$$

where $\bar{co}$ *stands for the closed convex hull of the corresponding set.*

Proof of Proposition 1.5. Take arbitrary $\gamma\in\Gamma(y_0)$. Then, by definition, there exists $u(\cdot)\in U$ and $y(t)=y(t,y_0,u(\cdot))$ such that $\gamma=\gamma^{(y(\cdot),u(\cdot))}$. Using the fact that the equality (1.5) is valid for any continuous function $q(y,u)$, one can obtain through integration by part:

$$\begin{aligned}&\int_{Y\times U}\phi'(y)^T f(y,u)\gamma^{(y(\cdot),u(\cdot))}(dy,du)\\&\quad=C\int_0^\infty e^{-Ct}\phi'(y(t))^T f(y(t),u(t))dt\\&\quad=-C\phi(y_0)+C^2\int_0^\infty e^{-Ct}\phi(y(t))dt\\&\quad=-C\int_{Y\times U}(\phi(y_0)-\phi(y))\gamma^{(y(\cdot),u(\cdot))}(dy,du),\ \forall\phi\in C^1.\end{aligned}$$

This result implies $\gamma=\gamma^{(y(\cdot),u(\cdot))}\in W(y_0)$ and hence $\Gamma(y_0)\subset W(y_0)$. The last inclusion implies (1.9) and also implies (1.10) since $W(y_0)$ is convex and compact. □

The statement of Proposition 1.5 can be strengthened. Namely, under certain assumptions (see Theorem 4.4 in [25]), the following relationships are valid:

$$CV^*(y_0)=G^*(y_0) \tag{1.11}$$

and

$$\bar{co}\Gamma(y_0) = W(y_0), \ \forall y_0 \in Y. \tag{1.12}$$

In what follows it is always assumed that (1.11) and (1.12) are true.

1.4 THE DUAL OF THE LINEAR PROGRAMMING PROBLEM

Next we consider the problem dual with respect to the IDLP problem (1.8). This is the problem defined a follows:

$$\mu^*(y_0) \stackrel{\text{def}}{=} \sup_{\psi(\cdot)\in C^1} \{\mu \colon \mu \leqslant g(y,u) + \psi'(y)^T f(y,u) \\ + C(\psi(y_0) - \psi(y)), \ \forall (y,u) \in Y \times U\}. \tag{1.13}$$

It can be readily seen that, if $W(y_0) \neq \emptyset$ and $\gamma \in W(y_0)$, then for any $\psi(\cdot) \in C^1$,

$$\begin{aligned} \min_{(y,u)\in Y\times U} &\{g(y,u) + \psi'(y)^T f(y,u) + C(\psi(y_0) - \psi(y))\} \\ &\leqslant \int_{Y\times U} (g(y,u) + \psi'(y)^T f(y,u) + C(\psi(y_0) - \psi(y)))\gamma(dy,du) \\ &= \int_{Y\times U} g(y,u)\gamma(dy,du), \end{aligned}$$

which implies

$$\mu^*(y_0) \leqslant G^*(y_0). \tag{1.14}$$

The following statements establish more elaborate connections between the IDLP problem (1.8) and dual problem (1.13) which we shall refer to as the D–IDLP problem.

Proposition 1.6.

(i) The optimal value of the D–IDLP problem (1.13) is bounded (that is, $\mu^*(y_0) < \infty$) if and only if the set $W(y_0)$ is not empty.

(ii) If the optimal value of the D–IDLP problem (1.13) is bounded, the optimal value of the problem (1.13) and the optimal value of the problem (1.2) are related by the equality

$$\mu^*(y_0) = G^*(y_0). \tag{1.15}$$

(iii) *The optimal value of the D–IDLP problem (1.13) is unbounded (that is,* $\mu^*(y_0) = \infty$*) if and only if there exists a function* $\psi(\cdot) \in C^1$ *such that*

$$\max_{(y,u)\in Y\times U} \{\psi'(y)^T f(y,u) + C(\psi(y_0) - \psi(y))\} < 0. \tag{1.16}$$

Proof of Proposition 1.6. The statements of the proposition were proved in [25] (see Theorem 3.1 in [25]). For completeness this proof is reproduced in appendix A. □

Definition 1.7. *A function* $\psi(\cdot) \in C^1$ *will be called a solution of the D–IDLP problem (1.13) if*

$$\min_{y\in Y} \{H(\psi'(y), y) + C(\psi(y_0) - \psi(y))\} = \mu^*(y_0),$$

where $H(p, y)$ *is the Hamiltonian*

$$H(p, y) \stackrel{\text{def}}{=} \min_{u\in U} \{p^T f(y,u) + g(y,u)\}. \tag{1.17}$$

Definition 1.7 is equivalent, of course, to the statement that the function $\psi(\cdot)$ is a solution to problem dual to the IDLP (1.8).

1.5 THE HAMILTON-JACOBI-BELLMAN INEQUALITY

It is well known that, if the optimal value function $V(\cdot)$ is continuously differentiable, then $V(\cdot)$ satisfies the Hamilton-Jacobi-Bellman equation

$$H(V'(y), y) - CV(y) = 0, \tag{1.18}$$

where the Hamiltonian $H(p, y)$ is defined in (1.17).

In this paper we deal with the inequality form of (1.18), and we call a function $\psi(\cdot) \in C^1$ a solution of the HJB inequality on Y if

$$H(\psi'(y), y) - C\psi(y) \geqslant 0, \ \forall y \in Y. \tag{1.19}$$

Note that the concept of a solution of the HJB inequality on Y is essentially the same as that of a smooth viscosity subsolution of the HJB equation (1.18) considered on the interior of Y (see, e.g., [6]), the former being introduced just for convenience of references.

A solution of the HJB inequality on Y that satisfies the additional condition

$$\psi(y_0) = V(y_0) \tag{1.20}$$

(if it exists) can be used for a characterisation of the optimal solution of the problem (1.2) in a similar way as the solution of the HJB equation (1.18) does (see Proposition 1.9 below).

Note that, if $\psi(\cdot) \in C^1$ is a solution of the problem (1.13), then $\tilde{\psi}(\cdot) = \psi(\cdot) + \text{const}$ is a solution of this problem as well.

Proposition 1.8. *If $\psi(\cdot) \in C^1$ is a solution of the HJB inequality (1.19) that satisfies (1.20), then this $\psi(\cdot)$ is also a solution of the problem (1.13). Conversely, if $\psi(\cdot) \in C^1$ is a solution of the problem (1.13), then*

$$\tilde{\psi}(\cdot) \stackrel{\text{def}}{=} \psi(\cdot) - \psi(y_0) + V(y_0) \tag{1.21}$$

is a solution of the HJB inequality (1.19) that satisfies (1.20).

Proof of Proposition 1.8. Let $\psi(\cdot) \in C^1$ be a solution of the HJB inequality (1.19) that satisfies (1.20). Obviously, this $\psi(\cdot)$ will also satisfy the inequality

$$H(\psi'(y), y) + C(\psi(y_0) - \psi(y)) \geqslant CV(y_0), \ \forall y \in Y. \tag{1.22}$$

Hence, by (1.15), $\psi(\cdot)$ is a solution of the problem (1.13). Let now $\psi(\cdot) \in C^1$ be a solution of the problem (1.13). By (1.15), it means that $\psi(\cdot)$ satisfies the inequality (1.22). Since $\tilde{\psi}(\cdot) - \psi(\cdot) = \text{const}$, from (1.22) it follows that

$$H(\tilde{\psi}'(y), y) + C(\tilde{\psi}(y_0) - \tilde{\psi}(y)) \geqslant CV(y_0), \ \forall y \in Y. \tag{1.23}$$

By (1.21), $\tilde{\psi}(y_0) = V(y_0)$. The substitution of the latter into (1.23) gives

$$H(\tilde{\psi}'(y), y) - C\tilde{\psi}(y) \geqslant 0, \ \forall y \in Y.$$

Thus, both (1.19) and (1.20) are satisfied. □

1.6 NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS

The following is a proposition which specifies the necessary and sufficient conditions of optimality based on the HJB inequality.

Proposition 1.9. *Assume that a solution* $\psi(\cdot) \in C^1$ *of the HJB inequality (1.19) that satisfies (1.20) exists. Then an admissible pair* $(y(t), u(t))$ *is optimal in (1.2) if and only if the following relationships are valid:*

$$H(\psi'(y(t)), y(t)) - C\psi(y(t)) = 0, \ \forall t \in [0, \infty) \tag{1.24}$$

and

$$u(t) = \operatorname*{argmin}_{u \in U} \{\psi'(y(t))^T f(y(t), u) + g(y(t), u)\} \ \textit{a.e.}, \forall t \in [0, \infty), \tag{1.25}$$

$$y(t) = \operatorname*{argmin}_{y \in Y} \{H(\psi'(y), y) - C\psi(y)\}, \ \forall t \in [0, \infty). \tag{1.26}$$

In addition, if $(y(t), u(t))$ *is optimal in (1.2), then*

$$\psi(y(t)) = V(y(t)), \ \forall t \in [0, \infty). \tag{1.27}$$

Proof of Proposition 1.9. Note first of all that, from (1.17) and (1.19) it follows that

$$\psi'(y)^T f(y, u) + g(y, u) - C\psi(y) \geqslant 0, \ \forall (y, u) \in Y \times U. \tag{1.28}$$

Let us prove the "only if statement". Assume that $(y(t), u(t))$ is optimal in (1.2). That is,

$$\int_0^\infty e^{-Ct} g(y(t), u(t)) dt = V(y_0). \tag{1.29}$$

We need to show that (1.24), (1.25) and (1.26) are satisfied. By integrating the inequality

$$\frac{d}{dt}(e^{-Ct}\psi(y(t))) = e^{-Ct}(\psi'(y(t))^T f(y(t), u(t)) - C\psi(y(t))), \tag{1.30}$$

We get

$$e^{-Ct}\psi(y(t))\big|_0^\infty = \int_0^\infty e^{-Ct}(\psi'(y(t))^T f(y(t), u(t)) - C\psi(y(t)))dt.$$

$$\int_0^\infty \frac{d}{dt}(e^{-Ct}\psi(y(t)))dt = \int_0^\infty e^{-Ct'}(\psi'(y(t))^T f(y(t), u(t)) - C\psi(y(t)))dt.$$

$$e^{-Ct}\psi(y(t)) - \psi(y_0) = \int_0^\infty e^{-Ct}(\psi'(y(t))^T f(y(t), u(t)) - C\psi(y(t)))dt.$$

And so,

$$-\psi(y_0) = \int_0^\infty e^{-Ct}(\psi'(y(t))^T f(y(t), u(t)) - C\psi(y(t)))dt.$$

Recall that $\int_0^\infty e^{-Ct}dt = 1/C$ and so $\psi(0) = C\int_0^\infty e^{-Ct}\psi(0)dt$. Hence

$$-C\int_0^\infty e^{-Ct}\psi(y_0)dt = \int_0^\infty e^{-Ct}(\psi'(y(t))^T f(y(t),u(t)) - C\psi(y(t)))dt.$$

One can verify that

$$\int_0^\infty e^{-Ct}(\psi'(y(t))^T f(y(t),u(t)) + C(\psi(y_0) - \psi(y(t))))dt = 0. \tag{1.31}$$

Hence, (1.29) can be rewritten as

$$\int_0^\infty e^{-Ct}(g(y(t),u(t)) + \psi'(y(t))^T f(y(t),u(t)) \\ + C(\psi(y_0) - \psi(y(t))))dt = V(y_0), \tag{1.32}$$

which implies

$$\int_0^\infty e^{-Ct}(g(y(t),u(t)) + \psi'(y(t))^T f(y(t),u(t)) \\ + C(-V(y_0) + \psi(y_0) - \psi(y(t))))dt = 0$$

and further implies

$$\int_0^\infty e^{-Ct}(g(y(t),u(t)) + \psi'(y(t))^T f(y(t),u(t)) - C\psi(y(t)))dt = 0, \tag{1.33}$$

where the last equality (1.33) is implied by the previous one due to (1.20). From (1.28) and (1.33) it follows that, for almost all $t \in [0,\infty)$,

$$g(y(t),u(t)) + \psi'(y(t))^T f(y(t),u(t)) - C\psi(y(t)) = 0, \tag{1.34}$$

which implies

$$(y(t),u(t)) = \operatorname*{argmin}_{(y,u)\in Y\times U} \{\psi'(y)^T f(y,u) + g(y,u) - C\psi(y)\}. \tag{1.35}$$

The inclusion (1.35) is equivalent to the inclusions (1.25) and (1.26). Also, the equality (1.34) is equivalent to (1.24).

Let us now prove the "if statement". That is, assume that (1.24), (1.25) and (1.26) are satisfied and show that $(y(t),u(t))$ is optimal in (1.2). From (1.24), (1.25) and (1.26) it follows that (1.34) is valid, which, in turn, implies that (1.33) is valid. Due to (1.20), this leads to the validity of (1.32). The latter along with (1.31) imply (1.29). Hence, $(y(t),u(t))$ is optimal. To complete the proof of the proposition, let us show that (1.27) is true if $(y(t), u(t))$ is optimal in (1.2). From (1.30) and (1.34) it follows that

$$\frac{d}{dt}(e^{-Ct}\psi(y(t))) = -e^{-Ct}g(y(t), u(t))$$
$$\implies e^{-Ct}\psi(y(t)) - \psi(y_0) = -\int_0^t e^{-Ct'}g(y(t'), u(t'))dt'.$$

Using (1.20) and (1.29), one obtains that

$$\begin{aligned} e^{-Ct}\psi(y(t)) &= V(y_0) - \int_0^t e^{-Ct'}g(y(t'), u(t'))dt' \\ &= \int_0^\infty e^{-Ct'}g(y(t', u(t')))dt' - \int_0^t e^{-Ct'}g(y(t'), u(t'))dt' \\ &= \int_t^\infty e^{-Ct'}g(u(t'), y(t'))dt'. \end{aligned} \tag{1.36}$$

Since (by the dynamic programming principle),

$$\begin{aligned} \int_t^\infty e^{-Ct'}g(y(t'), u(t'))dt' &= \int_0^\infty e^{-C(t+\tau)}g(y(t+\tau), u(t+\tau))d\tau \\ &= e^{-Ct}\int_0^\infty e^{-C\tau}g(y(t+\tau), u(t+\tau))d\tau \\ &= e^{-Ct}V(y(t)), \end{aligned}$$

from (1.36) it follows that $e^{-Ct}\psi(y(t)) = e^{-Ct}V(y(t))$. This proves (1.27). □

2

SEMI-INFINITE DIMENSIONAL APPROXIMATIONS

The dual maxmin problem (1.13) is defined upon the space of continuously differentiable functions C^1 for which solutions may not exist. So, we focus on approximating this problem by problems considered on N dimensional subspaces of C^1 (these will be called N-approximating duals).

This chapter is organised as follows; In Section 2.1, we introduce the semi-infinite linear programming (SILP) problem and its dual. In Section 2.2, we proceed with a description of the duality relationships. In Section 2.3, we establish that, under a suitable controllability condition, an optimal solution of the N-approximating dual problem exists. In Section 2.4, we state a result establishing that, under certain assumptions, a solution of the N-approximating dual problem allows one to construct an admissible pair that converges to the optimal admissible pair as $N \to \infty$.

2.1 THE SEMI-INFINITE LP PROBLEM

Let $\{\phi_i(\cdot) \in C^1, i = 1, 2, \ldots\}$ be a sequence of functions having continuous partial derivatives of the second order such that any function $\phi(\cdot) \in C^1$ and its gradient $\phi'(\cdot)$ can be simultaneously approximated on Y by linear combinations of functions from $\phi_i(\cdot)$, $i = 1, 2, \ldots$ and their corresponding gradients. That is, for any $\phi(\cdot) \in C^1$ and any $\delta > 0$, there exist real numbers $\beta_1, \ldots, \beta_k$ such that

$$\max_{y \in Y} \left\{ |\phi(y) - \sum_{i=1}^{k} \beta_i \phi_i(y)| + \|\phi'(y) - \sum_{i=1}^{k} \beta_i {\phi_i}'(y)\| \right\} \leqslant \delta, \tag{2.1}$$

with $\|\cdot\|$ being a norm in $\mathbb{R}^m$. An example of such an approximating sequence is the sequence of monomials $y_1^{i_1} \cdots y_m^{i_m}$, $i_1 \ldots i_m = 0, \ldots,$ where y_j, $j = 1, \ldots, m$ stands for the jth component of y [see 44, p.23].

Note that it always will be assumed that the gradients $\phi_i'(y)$, $i = 1, 2, \dots$ are linearly independent on any open set Q. More specifically, the following assumption is assumed to be valid:

Assumption 2.1. *For any open set* Q *and any* N*, the equality*

$$\sum_{i=1}^{N} \nu_i \phi_i'(y) = 0, \ \forall y \in Q,$$

is valid if and only if $\nu_i = 0$.

Note that this is satisfied automatically if $\phi_i(y)$ are chosen to be monomials. Using these $\phi_i(\cdot)$ functions, one can define a set $\widehat{W}(y_0)$ with a countable system of constraints

$$\widehat{W}(y_0) \stackrel{\text{def}}{=} \{\gamma \in \mathcal{P}(Y \times U) :$$
$$\int_{Y \times U} (\phi_i'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\gamma(dy, du) = 0,$$
$$i = 1, 2, \dots \}. \quad (2.2)$$

Lemma 2.2. *The set* $W(y_0)$ *is equal to the set* $\widehat{W}(y_0)$.

Proof of Lemma 2.2. Obviously $W(y_0) \subseteq \widehat{W}(y_0)$, so we need to show the validity of the converse inclusion. That is, we need to show that, for any $\widehat{\gamma} \in \widehat{W}(y_0)$,

$$\int_{Y \times U} (\phi'(y)^T f(y,u) + C(\phi(y_0) - \phi(y)))\widehat{\gamma}(dy, du) = 0, \ \forall \phi(\cdot) \in C^1. \quad (2.3)$$

Let $M \stackrel{\text{def}}{=} \max_{(y,u) \in Y \times U} \|f(y,u)\|$. For an arbitrary $\phi(\cdot) \in C^1$ and arbitrarily small $\epsilon > 0$ one can choose $\widehat{\phi}(y) \stackrel{\text{def}}{=} \sum_{i=1}^{k} \beta_i \phi_i(y)$ such that

$$\max_{y \in Y} \{\|\phi'(y)^T - \widehat{\phi}'(y)^T\| + \frac{C}{M}|(\phi(y_0) - \widehat{\phi}(y_0)| + \frac{C}{M}|\widehat{\phi}(y) - \phi(y)|\} \leqslant \frac{\epsilon}{M}.$$

Then for any $\widehat{\gamma} \in \widehat{W}(y_0)$ consider

$$
\begin{aligned}
&\|\int_{Y\times U}(\phi'(y)^T f(y,u) + C(\phi(y_0) - \phi(y)))\widehat{\gamma}(dy,du) \\
&\qquad - \int_{Y\times U}(\widehat{\phi}'(y)^T f(y,u) + C(\widehat{\phi}(y_0) - \widehat{\phi}(y)))\widehat{\gamma}(dy,du)\| \\
&\quad \leqslant \int_{Y\times U}(\|\phi'(y)^T f(y,u) - \widehat{\phi}'(y)^T f(y,u)\| \\
&\qquad + C\|\phi(y_0) - \widehat{\phi}(y_0)\| + C\|\widehat{\phi}(y) - \phi(y)\|)\widehat{\gamma}(dy,du) \\
&\quad \leqslant \int_{Y\times U}(\|\phi'(y)^T - \widehat{\phi}'(y)^T\|M \\
&\qquad + C\|\phi(y_0) - \widehat{\phi}(y_0)\| + C\|\widehat{\phi}(y) - \phi(y)\|)\widehat{\gamma}(dy,du) \\
&\quad \leqslant \int_{Y\times U} M(\frac{\epsilon}{M})\widehat{\gamma}(dy,du) = \epsilon.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|\int_{Y\times U}(\phi'(y)^T f(y,u) + C(\phi(y_0) - \phi(y)))\widehat{\gamma}(dy,du)\| \\
&\quad \leqslant \|\int_{Y\times U}(\widehat{\phi}'(y)^T f(y,u) + C(\widehat{\phi}(y_0) - \widehat{\phi}(y)))\widehat{\gamma}(dy,du)\| + \epsilon \\
&\quad \leqslant \epsilon.
\end{aligned}
$$

Since ϵ is arbitrarily small, the equality (2.3) is true. This completes the proof. □

Now let us define the set $W^N(y_0)$ by truncating the system of equations in (2.2):

$$
\begin{aligned}
&W^N(y_0) \stackrel{\text{def}}{=} \{\gamma \in \mathcal{P}(Y \times U): \\
&\int_{Y\times U}(\phi_i{}'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\gamma(dy,du) = 0, \\
&\qquad\qquad i = 1, \dots, N\}, \qquad (2.4)
\end{aligned}
$$

where $\phi_i(\cdot)$ are as in (2.1). Then, consider the semi-infinite linear programming problem

$$
G^N(y_0) \stackrel{\text{def}}{=} \min_{\gamma \in W^N(y_0)} \int_{Y\times U} g(y,u)\gamma(dy,du), \qquad (2.5)
$$

This problem will be referred to as the N-approximating LP problem (or just SILP problem). Note that $W^N(y_0)$ is convex and compact and $W(y_0) \subset W^N(y_0)$, which implies

$$G^*(y_0) \geqslant G^N(y_0).$$

A consequence of the fact that $W(y_0)$ is assumed to be non-empty, it follows that the sets $W^N(y_0)$, $N = 1, 2, \dots$ are not empty. Hence, the set of optimal solutions of (2.5) is not empty for any $N = 1, 2, \dots$. The connection between LP problem (1.8) and SILP problem (2.5) is established by the following proposition [see 26].

Proposition 2.3. *The set $W(y_0)$ is not empty if and only if the set $W^N(y_0)$ is not empty for all $N \geqslant 1$. If $W(y_0)$ is not empty, then the following relationships are valid:*

$$\lim_{N\to\infty} \rho_H(W^N(y_0), W(y_0)) = 0 \tag{2.6}$$

and

$$\lim_{N\to\infty} G^N(y_0) = G^*(y_0). \tag{2.7}$$

Proof of Proposition 2.3. Let $W^N(y_0)$, $N = 1, 2, \dots$ be not empty and let $\gamma^N \in W^N(y_0)$. Then, as can be readily seen, any partial limit (cluster point) of $\{\gamma^N, N = 1, 2, \dots\}$ is contained in $W(y_0)$. Hence, $W(y_0)$ is not empty. As $W(y_0) \subset W^N(y_0)$, to prove (2.6) it is sufficient to show that

$$\lim_{N\to\infty} \sup_{\gamma \in W^N(y_0)} \rho(\gamma, W(y_0)) = 0,$$

where ρ is defined in (1.7). Let us assume this is not the case. Then there exists a positive number δ, a subsequence of positive numbers, $N' \to \infty$, and a sequence of probability measures $\gamma^{N'} \in W_{N'}(y_0)$ such that $\rho(\gamma^{N'}, W(y_0)) \geqslant \delta$. Due to the compactness of $\mathcal{P}(Y \times U)$, one may assume without loss of generality that there exists $\overline{\gamma} \in \mathcal{P}(Y \times U)$ such that

$$\lim_{N'\to\infty} \rho(\gamma^{N'}, \overline{\gamma}) = 0 \implies \rho(\overline{\gamma}, W(y_0)) \geqslant \delta.$$

Due to the fact that $\gamma^{N'} \in W_{N'}(y_0)$ it follows that, for any integer i and $N' \geqslant i$

$$\int_{Y\times U} (\phi_i'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\gamma^{N'}(dy, du) = 0$$

$\Longrightarrow$

$$\int_{Y\times U}(\phi_i'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\overline{\gamma}(dy,du) = 0.$$

Since the latter is valid for all $i = 1, 2, \ldots$ one can conclude that $\overline{\gamma} \in W(y_0)$. This contradicts (2.2), which therefore, means the initial assumption is false and thus proves (2.6). The validity of (2.7) is implied by (2.6). □

Corollary 2.4. *If the optimal solution γ^* of the problem (1.6) is unique, then for any optimal solution γ^N of the problem (2.5) there exists the limit*

$$\lim_{N\to\infty} \gamma^N = \gamma^*. \tag{2.8}$$

Proof of corollary 2.4. The proof follows from (2.7). □

Note that every extreme point of the optimal solutions set of (2.5) is an extreme point of $W^N(y_0)$ and that the latter is presented as a convex combination of (no more than $N+1$) Dirac measures. That is [see 51, Theorem A.5], if γ^N is an extreme point of the optimal solutions set of (2.5) (or, equivalently, it is an extreme point of $W^N(y_0)$, which is an optimal solution of (2.5)), then there exist

$$(y_l^N, u_l^N) \in Y \times U,\ \gamma_l^N > 0,\ l = 1, 2, \ldots,\ K_N \leqslant N+1,\ \sum_{l=1}^{K_N} \gamma_l^N = 1$$

such that

$$\gamma^N = \sum_{l=1}^{K_N} \gamma_l^N \delta_{(y_l^N, u_l^N)}, \tag{2.9}$$

where $\delta_{(y_l^N, u_l^N)}$ is the Dirac measure concentrated at (y_l^N, u_l^N).

2.2 THE SEMI-INFINITE DUAL PROBLEM

Define the finite dimensional space $\mathcal{D}_N(y_0) \subset C^1$ by the equation

$$\mathcal{D}_N(y_0) \stackrel{\text{def}}{=} \{\psi(\cdot) \in C^1 : \psi(y) = \sum_{i=1}^{N} \lambda_i \phi_i(y),\ \lambda = (\lambda_i) \in \mathbb{R}^N\}$$

and consider the following problem dual to (2.5) (which, for convenience, will be referred to as the N-approximating dual problem or D–SILP problem):

$$\mu^N(y_0) \stackrel{\text{def}}{=} \sup_{\psi(\cdot)\in\mathcal{D}_N(y_0)} \{\mu\colon g(y,u) + \psi'(y)^T f(y,u) + C(\psi(y_0) - \psi(y)) \geqslant \mu,\ \forall (y,u) \in Y \times U\}. \quad (2.10)$$

For a fixed N, the relationships between SILP problem (2.5) and D–SILP problem (2.10) are similar to those between (1.8) and (1.13). For example, one can establish that if $W^N(y_0) \neq \emptyset$ and $\gamma \in W^N(y_0)$, then for any $\psi \in \mathcal{D}_N(y_0)$

$$\begin{aligned}\mu^N(y_0) &\leqslant \int_{Y\times U} (g(y,u) \\ &\qquad + \sum_{i=1}^{N} \lambda_i({\psi_i}'(y) f(y,u) + C(\psi_i(y_0) - \psi_i(y)))\gamma(dy, du) \\ &= \int_{Y\times U} g(y,u)\gamma(dy,du),\end{aligned}$$

which implies

$$\mu^N(y_0) \leqslant G^N(y_0). \quad (2.11)$$

The SILP problem (2.5) is related to the N-approximating dual problem (2.10) through the following duality type relationships.

Proposition 2.5.

(i) The optimal value of the D–SILP problem (2.15) is bounded (that is, $\mu^N(y_0) < \infty$) if and only if the set $W^N(y_0)$ is not empty.

(ii) If $W^N(y_0)$ is not empty, then optimal values of (2.5) and (2.15) are equal

$$G^N(y_0) = \mu^N(y_0). \quad (2.12)$$

(iii) The optimal value of D–SILP problem (2.15) is unbounded (that is, $\mu^N(y_0) = \infty$) if and only if there exists $v = (v_1, \dots, v_N)$ such that

$$\max_{(y,u)\in Y\times U} \{{\psi_v}'(y)^T f(y,u) + C(\psi_v(y_0) - \psi_v(y))\} < 0,$$

$$\psi_v(y) \stackrel{\text{def}}{=} \sum_{i=1}^{N} v_i \phi_i(y). \quad (2.13)$$

The proof of Proposition 2.5 is similar to the proof of Proposition 1.6 and it is presented below.

Proof of Proposition 2.5(iii). If the function $\psi_v(\cdot)$ satisfying (2.13) exists, then the inequality

$$\min_{(y,u)\in Y\times U} -\{\psi_v{}'(y)^T f(y,u) + C(\psi_v(y_0) - \psi_v(y))\} > 0$$

holds, and hence,

$$\lim_{\alpha\to\infty} \min_{(y,u)\in Y\times U} \{g(y,u) - \alpha(\psi_v{}'(y)^T f(y,u) \\ + C(\psi_v(y_0) - \psi_v(y)))\} = \infty.$$

This implies that the optimal value of the N-approximating dual problem is unbounded ($\mu^N(y_0) = \infty$). Assume now that the optimal value of the semi-infinite dimensional dual problem is unbounded. That is, there exists a sequence $(\mu^k, \psi_k(\cdot))$ such that

$$\mu^k \leqslant g(y,u) + (\psi_k{}'(y)^T f(y,u) + C(\psi_k(y_0) - \psi_k(y))), \\ \forall (y,u) \in Y \times U, \ \lim_{k\to\infty} \mu^k = \infty,$$

which implies

$$1 \leqslant \frac{1}{\mu^k} g(y,u) + \frac{1}{\mu^k}(\psi_k{}'(y)^T f(y,u) \\ + C(\psi_k(y_0) - \psi_k(y))), \ \forall (y,u) \in Y \times U.$$

For k large enough, $\frac{1}{\mu^k}|g(y,u)| \leqslant \frac{1}{2}$, $\forall (y,u) \in Y \times U$. Hence

$$\frac{1}{2} \leqslant \frac{1}{\mu^k}(\psi_k{}'(y)^T f(y,u) + C(\psi(y_0) - \psi(y))), \ \forall (y,u) \in Y \times U.$$

That is, the function $\psi(y) \stackrel{\text{def}}{=} -\frac{1}{\mu^k}\psi_k(y)$ satisfies (2.13). □

Proof of Proposition 2.5(i). From (2.11) it follows that, if $W^N(y_0)$ is not empty, then the optimal value of problem (2.10) is bounded. Conversely, let us assume that the optimal value $\mu^N(y_0)$ of problem (2.10) is bounded and let us establish that $W^N(y_0)$ is not empty. Assume that it is not true and $W^N(y_0)$ is empty. Define the set $\mathcal{Q}'(y_0)$ by the equation

$$\mathcal{Q}'(y_0) \stackrel{\text{def}}{=} \{x = (x_1, \dots, x_N) : \\ x_i = \int_{Y\times U} (\phi_i'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\gamma(dy, du), \\ \gamma \in \mathcal{P}(Y \times U), \ i = 1, \dots, N\}.$$

It is easy to see that the set $\mathcal{Q}'(y_0)$ is a convex and compact subset of $\mathbb{R}^N$ (the fact that it is closed follows from that $\mathcal{P}(Y \times U)$ is compact in weak* convergence topology). By (2.4), the assumption that $W^N(y_0)$ is empty is equivalent to the assumption that the set $\mathcal{Q}'(y_0)$ does not contain the "zero element" ($0 \notin \mathcal{Q}'(y_0)$). Hence, by a separation theorem [see 52, p.59], there exists $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_N) \in \mathbb{R}^N$ such that

$$\begin{aligned}0 = \bar{\lambda}(0) > & \max_{x \in \mathcal{Q}'(y_0)} \sum_{i=1}^{N} \bar{\lambda}_i x_i \\ &= \max_{\gamma \in \mathcal{P}(Y \times U)} \int_{Y \times U} (\psi_{\bar{\lambda}}'(y)^T f(y,u) + C(\psi_{\bar{\lambda}}(y_0) - \psi_{\bar{\lambda}}(y)))\bar{\gamma}(dy, du) \\ &= \max_{(y,u) \in Y \times U} (\psi_{\bar{\lambda}}'(y)^T f(y,u) + C(\psi_{\bar{\lambda}}(y_0) - \psi_{\bar{\lambda}}(y))),\end{aligned}$$

where $\psi_{\bar{\lambda}}(y) = \sum_{i=1}^{N} \bar{\lambda}_i \phi_i(y)$ (see (A.4)). This implies that the function $\psi(y) \stackrel{\text{def}}{=} \psi_{\bar{\lambda}}(y)$ satisfies (2.13), and, by Proposition 2.5(iii), $\mu^N(y_0)$ is unbounded. Thus, we have obtained a contradiction that proves that $W^N(y_0)$ is not empty. □

Proof of Proposition 2.5(ii). By Proposition 2.5(i), if the optimal value of the dual problem (2.10) is bounded, then $W^N(y_0)$ is not empty and, hence, a solution of the problem (2.5) exists. That is,

$$\exists \gamma \in W^N(y_0)\colon \int_{Y \times U} g(y,u)\gamma(dy, du) = G^N(y_0).$$

Define the set $\hat{\mathcal{Q}}'(y_0) \subset \mathbb{R}^1 \times \mathbb{R}^N$ by the equation

$$\begin{aligned}\hat{\mathcal{Q}}'(y_0) \stackrel{\text{def}}{=} \Big\{(\theta, x)\colon \theta \geqslant \int_{Y \times U} g(y,u)\gamma(dy, du),\ x = (x_1, \dots, x_N), \\ x_i = \int_{Y \times U} ({\phi_i}'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\gamma(dy, du), \\ \gamma \in \mathcal{P}(Y \times U),\ i = 1, \dots, N\Big\}.\end{aligned}$$

The set $\hat{\mathcal{Q}}'(y_0)$ is convex and closed. Also, for any $j = 1, 2, \dots$, the point $(\theta_j, 0) \notin \hat{\mathcal{Q}}'(y_0)$, where $\theta_j \stackrel{\text{def}}{=} G^N(y_0) - \frac{1}{j}$ and 0 is the zero element of $\mathbb{R}^N$. On the basis of a separation theorem [see 52, p.59], one may conclude that there exists a sequence $(\kappa^j, \lambda^j) \in \mathbb{R}^1 \times \mathbb{R}^N$ (with $\lambda^j \stackrel{\text{def}}{=} (\lambda^j_1, \dots, \lambda^j_N)$), where j is an arbitrary natural number $j = 1, 2, \dots$

$$\kappa^j(G^N(y_0) - \frac{1}{j}) + \delta^j \leqslant \inf_{(\theta,x)\in\hat{\Omega}(y_0)} \{\kappa^j\theta + \sum_{i=1}^{N} \lambda^j_i x_i\}$$
$$= \inf_{\gamma\in\mathcal{P}(Y\times U)} \{\kappa^j\theta + \int_{Y\times U} (\psi'_{\lambda^j}(y)^T f(y,u) + C(\psi_{\lambda^j}(y_0) - \psi_{\lambda^j}(y)))\gamma(dy,du)$$
$$\text{s.t. } \theta \geqslant \int_{Y\times U} g(y,u)\gamma(dy,du)\}, \quad (2.14)$$

where $\delta^j > 0$ for all j and $\psi_{\lambda^j}(y) = \sum_{i=1}^N \lambda^j_i \phi_i(y)$. From (2.14) it immediately follows that $\kappa^j \geqslant 0$. Let us show that $\kappa^j > 0$. In fact, if it was not the case, one would obtain that

$$0 < \delta^j \leqslant \min_{\gamma\in\mathcal{P}(Y\times U)} \int_{Y\times U} (\psi'_{\lambda^j}(y)^T f(y,u) + C(\psi_{\lambda^j}(y_0) - \psi_{\lambda^j}(y)))\gamma(dy,du)$$
$$= \min_{(y,u)\in Y\times U} \{\psi'_{\lambda^j}(y)^T f(y,u) + C(\psi_{\lambda^j}(y_0) - \psi_{\lambda^j}(y))\},$$

which implies

$$\max_{(y,u)\in Y\times U} -\{\psi'_{\lambda^j}(y)^T f(y,u) + C(\psi_{\lambda^j}(y_0) - \psi_{\lambda^j}(y))\} \leqslant -\delta^j < 0.$$

The latter would lead to the validity of the inequality (2.13) with $\psi(y) = -\psi_{\lambda^j}(y)$, which, by Proposition 2.5(iii), would imply that the optimal value of the dual problem is unbounded. Thus, $\kappa^j > 0$. Dividing (2.14) by κ^j one can obtain that

$$G^N(y_0) - \frac{1}{j} < (G^N(y_0) - \frac{1}{j}) + \frac{\delta^j}{\kappa_j}$$
$$\leqslant \min_{\gamma\in\mathcal{P}(Y\times U)} \int_{Y\times U} (g(y,u) + \frac{1}{\kappa^j}(\psi'_{\lambda^j}(y)^T f(y,u)$$
$$+ C(\psi_{\lambda^j}(y_0) - \psi_{\lambda^j}(y))))\gamma(dy,du)$$
$$= \min_{(y,u)\in Y\times U} \{g(y,u) + \frac{1}{\kappa^j}(\psi'_{\lambda^j}(y)^T f(y,u) + C(\psi_{\lambda^j}(y_0) - \psi_{\lambda^j}(y)))\}$$
$$\leqslant \mu^N(y_0),$$

which implies

$$G^N(y_0) \leqslant \mu^N(y_0).$$

The latter and (2.11) prove (2.12). □

Let us now establish that the optimal values of the D–SILP problem (2.15) converge to the optimal value of D–IDLP problem (1.13) as $N \to \infty$.

Corollary 2.6. $\mu^N(y_0)$ *converges to* $G^*(y_0)$*, that is,*

$$\lim_{N\to\infty} \mu^N(y_0) = G^*(y_0).$$

Proof of corollary 2.6. It is obvious that, for any $N \geqslant 1$,

$$\mu^1(y_0) \leqslant \mu^2(y_0) \leqslant \cdots \leqslant \mu^N(y_0) \leqslant \mu^*(y_0).$$

Hence, $\lim_{N\to\infty} \mu^N(y_0)$ exists, and it is less or equal than $\mu^*(y_0)$. The fact that it is actually equal to $\mu^*(y_0)$ (and, hence, due to (1.15), equal to $G^*(y_0)$) follows from that, for any function $\psi(\cdot) \in C^1$ and for any $\delta > 0$, there exist N large enough and $\psi_\delta(\cdot) \in \mathcal{D}_N(y_0)$ such that

$$\max_{y\in Y} \left\{|\psi(y) - \psi_\delta(y)| + \|\psi'(y) - {\psi_\delta}'(y)\|\right\} \leqslant \delta.$$

□

2.3 SOLUTIONS TO THE SEMI-INFINITE DUAL PROBLEM

Note that, due to the definition of the Hamiltonian (1.17), problem (2.10) is equivalent to

$$\sup_{\psi(\cdot)\in\mathcal{D}_N(y_0)} \Big\{\mu\colon H(\psi'(y), y) + C(\psi(y_0) - \psi(y)) \geqslant \mu, \quad \forall y \in Y\Big\} = \mu^N(y_0). \tag{2.15}$$

Definition 2.7. *A function* $\psi^N(\cdot) \in C^1$ *will be called a solution of the* N*-approximating dual problem (2.10) if*

$$\min_{y\in Y} \left\{H(\psi^{N\prime}(y), y) + C(\psi^N(y_0) - \psi^N(y))\right\} = \mu^N(y_0).$$

Let $\mathcal{R}_{y_0} \subset Y$ stand for the set of points that are reachable (in finite time) along admissible trajectories of (1.1) satisfying the initial condition (1.3). That is,

$$\mathcal{R}_{y_0} \stackrel{\text{def}}{=} \Big\{y\colon y = y(t),\ (y(\cdot), u(\cdot))\text{-admissible, satisfying (1.3)}, \quad t \in [0, \infty)\Big\} \tag{2.16}$$

Assumption 2.8. *The closure of $\mathcal{R}_{y_0}$ has a nonempty interior. That is,*

$$\text{int}(\text{cl}\,\mathcal{R}_{y_0}) \neq \emptyset. \tag{2.17}$$

Proposition 2.9. *Let Assumption 2.8 be satisfied. Then, for every $N = 1, 2, \dots,$ there exists $\lambda^N = (\lambda_i^N)$ such that*

$$\psi^N(y) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \lambda_i^N \phi_i(y) \tag{2.18}$$

is a solution of the N-approximating dual problem (2.15). That is

$$\min_{y \in Y} \left\{ H(\psi^{N\prime}(y), y) + C(\psi^N(y_0) - \psi^N(y)) \right\} = \mu^N(y_0). \tag{2.19}$$

Proof of Proposition 2.9. The proof follows from Lemmas 2.11 and 2.12 which are presented and proved at the end of this section. □

Corollary 2.10. *Let (2.17) be satisfied. Then, for any $\delta > 0$, there exists N_δ such that, $\forall N \geqslant N_\delta$,*

$$H(\psi^{N\prime}(y), y) + C(\psi^N(y_0) - \psi^N(y)) \geqslant CV(y_0) - \delta, \ \forall y \in Y,$$

where $\psi^N(\cdot)$ is a solution of the N-approximating dual problem (2.15) (the existence of which is established by Proposition 2.9).

Proof of corollary 2.10. The proof follows from Proposition 2.5(ii) and corollary 2.6. □

Lemma 2.11. *Assume that, for*

$$\psi(y) = \sum_{i=1}^{N} \nu_i \phi_i(y). \tag{2.20}$$

the inequality

$$\psi'(y)^T f(y, u) + C(\psi(y_0) - \psi(y)) \geqslant 0, \ \forall (y, u) \in Y \times U \tag{2.21}$$

is valid only if

$$\nu_i = 0, \ \forall i = 1, \dots, N. \tag{2.22}$$

Then a solution (2.18) of the N-approximating ψ-problem (2.15) exists.

Proof of Lemma 2.11. For any $k = 1, 2, \ldots$ let $v^k = (v_i^k) \in \mathbb{R}^N$ be such that the function

$$\psi^k(y) \stackrel{\text{def}}{=} \sum_{i=1}^{N} v_i^k \phi_i(y)$$

satisfies the inequality

$$H(\psi^{k\prime}(y), y) + C(\psi^k(y_0) - \psi^k(y)) \geqslant \mu^N(y_0) - \frac{1}{k}, \ \forall y \in Y.$$

Hence

$$\psi^{k\prime}(y)^T f(y,u) + g(y,u) + C(\psi^k(y_0) - \psi^k(y)) \\ \geqslant \mu^N(y_0) - \frac{1}{k}, \ \forall (y,u) \in Y \times U. \quad (2.23)$$

Show that the sequence v^k, $k = 1, 2, \ldots$ is bounded. That is, there exists $\alpha > 0$ such that

$$\|v^k\| \leqslant \alpha, \ k = 1, 2, \ldots \quad (2.24)$$

Assume that the sequence v^k, $k = 1, 2, \ldots$ is not bounded. Then there exists a subsequence $v^{k'}$ such that

$$\lim_{k' \to \infty} \|v^{k'}\| = \infty, \ \lim_{k' \to \infty} \frac{v^{k'}}{\|v^{k'}\|} \stackrel{\text{def}}{=} \tilde{v}, \ \|\tilde{v}\| = 1. \quad (2.25)$$

Dividing (2.23) by $\|v^k\|$ and passing to the limit along the subsequence $\{k'\}$, one can obtain that

$$\tilde{\psi}'(y)^T f(y,u) + C(\tilde{\psi}(y_0) - \tilde{\psi}(y)) \geqslant 0, \ \forall (y,u) \in Y \times U,$$

where

$$\tilde{\psi}(y) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \tilde{v}_i \phi_i(y).$$

Hence, by the assumption of the lemma, $\tilde{v} = (\tilde{v}_i) = 0$, which is in contradiction with (2.25). Thus, the validity of (2.24) is established. Due to (2.24), there exists a subsequence $\{k'\}$ such that there exists a limit

$$\lim_{k' \to \infty} v^{k'} \stackrel{\text{def}}{=} v^*.$$

Passing to the limit in (2.23) along this subsequence, one obtains

$$\psi^{*\prime}(y)^T f(y,u) + g(y,u) + C(\psi^*(y_0) - \psi^*(y)) \geqslant \mu^N(y_0), \quad \forall (y,u) \in Y \times U, \quad (2.26)$$

where

$$\psi^*(y) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \nu_i{}^* \phi_i(y).$$

From (2.26) it follows that

$$H(\psi^{*\prime}(y), y) + C(\psi^*(y_0) - \psi^*(y)) \geqslant \mu^N(y_0), \ \forall y \in Y.$$

That is, $\psi^*(y)$ is a solution of the N-approximating dual problem (2.15). □

Lemma 2.12. *Let Assumption 2.8 be satisfied. Then, for $\psi(\cdot)$ of the form (2.20), the inequality (2.21) is valid only if (2.22) is valid.*

Proof of Lemma 2.12. Let $\psi(\cdot)$ be presented in the form (2.20) and let it satisfy (2.21). For an arbitrary admissible pair $(y(\cdot), u(\cdot))$ that satisfies the initial condition (1.3), one can obtain that

$$\int_0^\infty e^{-Ct}(\psi'(y(t))^T f(y(t), u(t)) + C(\psi(y_0) - \psi(y(t))))dt = 0.$$

This along with (2.21) imply that

$$\psi'(y(t))^T f(y(t), u(t)) + C(\psi(y_0) - \psi(y(t)) = 0, \ \text{a.e. } t \in [0, \infty). \quad (2.27)$$

From (2.27) it follows that

$$\frac{d}{dt}(\psi(y(t)) - \psi(y_0)) = C(\psi(y(t)) - \psi(y_0)) \implies \psi(y(t)) = \psi(y_0), \ \forall t \in [0, \infty).$$

Consequently, by definition of $\mathcal{R}_{y_0}$ (see (2.16)),

$$\psi(y) = \psi(y_0), \ \forall y \in \mathcal{R}_{y_0} \implies \psi(y) = \psi(y_0), \ \forall y \in \operatorname{cl} \mathcal{R}_{y_0}.$$

The latter implies that

$$\psi'(y) = 0, \ \forall y \in \operatorname{int}(\operatorname{cl} \mathcal{R}_{y_0}),$$

which, in turn, implies that all v_i in (2.20) are equal to zero (due to linear independence of $\phi_i'(\cdot)$, $i = 1, \ldots, N$). □

2.4 THE CONSTRUCTION OF NEAR-OPTIMAL CONTROLS

Let (2.17) be satisfied and let $\psi^N(\cdot)$ be a solution of the N-approximating dual problem (2.15). Define a control $u^N(y)$ by the equation

$$u^N(y) = \operatorname*{argmin}_{u \in U} \{\psi^{N\prime}(y)^T f(y,u) + g(y,u)\} \tag{2.28}$$

and assume that the system

$$y'(t) = f(y(t), u^N(y(t))), \; y(0) = y_0,$$

has a unique solution $y^N(t) \in Y$. Under assumptions that are introduced below, we establish that $u^N(y^N(t))$ converges (almost everywhere) to the optimal control $u^*(t)$, and $y^N(t)$ converges (uniformly on any bounded interval) to the corresponding optimal trajectory $y^*(t)$ as $N \to \infty$.

Assumption 2.13. *Let the optimal solution γ^* of the IDLP problem (1.6) be unique and the optimal pair $(y^*(\cdot), u^*(\cdot))$ (that is, the pair that delivers minimum in (1.2)) exist.*

Remark 2.14. *Note that, due to (1.10), the discounted occupational measure generated by $(y^*(\cdot), u^*(\cdot))$ is an optimal solution of the IDLP problem (1.6). Hence, if γ^* is the unique optimal solution of the latter, it will coincide with the discounted occupational measure generated by $(y^*(\cdot), u^*(\cdot))$.*

Assumption 2.15. *The optimal control $u^*(\cdot)\colon [0,\infty) \mapsto U$ is piecewise continuous and, at every discontinuity point, $u^*(\cdot)$ is either continuous from the left or it is continuous from the right.*

Assumption 2.16.

(i) *For almost all $t \in [0,\infty)$, there exists an open ball $Q_t \subset \mathbb{R}^m$ centred at $y^*(t)$ such that $u^N(\cdot)$ is uniquely defined for $y \in Q_t$ (that is, the problem in the right hand side of (2.28) has a unique optimal solution for $y \in Q_t$), and $u^N(\cdot)$ satisfies Lipschitz conditions on Q_t (with a Lipschitz constant being independent of N and t);*

(ii) *The Lebesgue measure of the set $A_t(N) \stackrel{\text{def}}{=} \{t' \in [0,t], \; y^N(t') \notin Q_{t'}\}$ tends to zero as $N \to \infty$. That is,*

$$\lim_{N \to \infty} \operatorname{meas}\{A_t(N)\} = 0. \tag{2.29}$$

Proposition 2.17. *Let* $f(y,u)$ *and* $g(y,u)$ *be Lipschitz continuous in a neighbourhood of* $Y \times U$, *let (2.17) be valid and let Assumptions 2.13, 2.15 and 2.16 be satisfied. Then*

$$\lim_{N\to\infty} u^N(y^N(t)) = u^*(t) \tag{2.30}$$

for almost all $t \in [0,\infty)$ *and*

$$\lim_{N\to\infty} \max_{t'\in[0,t]} \|y^N(t') - y^*(t')\| = 0, \ \forall t \in [0,\infty). \tag{2.31}$$

Also,

$$\lim_{N\to\infty} V^N(y_0) = V(y_0), \tag{2.32}$$

where

$$V^N(y_0) \stackrel{\text{def}}{=} \int_0^\infty e^{-Ct} g(y^N(t), u^N(y^N(t)))dt.$$

Proof of Proposition 2.17. The proof of the proposition is given on Page 30. □

Lemma 2.18. *Let Assumptions 2.13 and 2.15 be satisfied and let* γ^N *be an optimal solution of (2.5) that is presented in the form (2.9). Then, for any* $t \in [0,\infty)$, $(y^*(t), u^*(t))$ *is presented as the limit*

$$(y^*(t), u^*(t)) = \lim_{N\to\infty} (y^N_{l_N}, u^N_{l_N}), \tag{2.33}$$

where $(y^N_{l_N}, u^N_{l_N}) \in \{(y^N_l, u^N_l), l = 1, \dots, K_N\}$ *(that is,* $(y^N_{l_N}, u^N_{l_N})$ *is an element of the set of the concentration points of the Dirac measures in the expansion (2.9)).*

Proof of Lemma 2.18. The proof of the lemma is given at the end of the section. □

Proposition 2.19. *If* γ^N *is an optimal solution of (2.5) that allows a representation (2.9) and if* $\psi^N(y) = \sum_{i=1}^N \lambda_i^N \phi_i(y)$ *is an optimal solution of (2.15), then the concentration points* (y^N_l, u^N_l) *of the Dirac measures in the expansion (2.9) satisfy the following relationships:*

$$y^N_l = \operatorname*{argmin}_{y\in Y} \{H(\psi^{N\prime}(y), y) + C(\psi^N(y_0) - \psi^N(y))\}, \tag{2.34}$$

$$u^N_l = \operatorname*{argmin}_{u\in U} \{\psi^{N\prime}(y^N_l)^T f(y^N_l, u) + g(y^N_l, u)\}, \ l = 1, \dots, K_N. \tag{2.35}$$

Proof of Proposition 2.19. The proof of the proposition is given on Page 33. □

Proof of Proposition 2.17. Let $t \in [0, \infty)$ be such that $u^N(\cdot)$ is Lipschitz continuous on Q_t and let $(y^N_{l_N}, u^N_{l_N})$ be as in (2.33). By (2.35),

$$u^N_{l_N} = u^N(y^N_{l_N}).$$

Hence,

$$\begin{aligned}
\|u^*(t) - &u^N(y^*(t))\| \\
&\leqslant \|u^*(t) - u^N_{l_N}\| + \|u^N(y^N_{l_N}) - u^N(y^*(t))\| \\
&\leqslant \|u^*(t) - u^N_{l_N}\| + L\|y^*(t) - y^N_{l_N} s\|,
\end{aligned} \tag{2.36}$$

where L is a Lipschitz constant of $u^N(\cdot)$. From (2.33) and (2.36) it follows that

$$\lim_{N \to \infty} u^N(y^*(t)) = u^*(t). \tag{2.37}$$

By Assumption 2.16, the same argument is applicable for almost all $t \in [0, \infty)$. Consequently, the convergence (2.37) is valid for almost all $t \in [0, \infty)$. Taking an arbitrary $t \in [0, \infty)$ and subtracting the equation

$$y^*(t) = y_0 + \int_0^t f(y^*(t'), u^*(t'))dt'$$

from the equation

$$y^N(t) = y_0 + \int_0^t f(y^N(t'), u^N(y^N(t')))dt',$$

one obtains

$$\begin{aligned}
\|y^N(t) - &y^*(t)\| \\
&\leqslant \int_0^t \|f(y^N(t'), u^N(y^N(t'))) - f(y^*(t'), u^*(t'))\|dt' \\
&\leqslant \int_0^t \|f(y^N(t'), u^N(y^N(t'))) - f(y^*(t'), u^N(y^*(t')))\|dt' \\
&\qquad + \int_0^t \|f(y^*(t'), u^N(y^*(t'))) - f(y^*(t'), u^*(t'))\|dt'.
\end{aligned} \tag{2.38}$$

It is easy to see that

$$\begin{aligned}\int_0^t \|f(y^N(t'),u^N(y^N(t'))) - f(y^*(t'),u^N(y^*(t')))\|dt' \\ \leqslant \int_{t'\notin A_t(N)} \|f(y^N(t'),u^N(y^N(t'))) - f(y^*(t'),u^N(y^*(t')))\|dt' \\ + \int_{t'\in A_t(N)} (\|f(y^N(t'),u^N(y^N(t')))\| \\ + \|f(y^*(t'),u^N(y^*(t')))\|)dt' \\ \leqslant L_1 \int_0^t \|y^N(t') - y^*(t')\|dt' + L_2 \mathrm{meas}\{A_t(N)\},\end{aligned}$$

where L_1 is a constant defined (in an obvious way) by Lipschitz constants of $f(\cdot,\cdot)$ and $u^N(\cdot)$, and $L_2 \overset{\text{def}}{=} 2\max_{(y,u)\in Y\times U}\{\|f(y,u)\|\}$. Also, due to (2.37) and the dominated convergence theorem [see 4, p.49]

$$\lim_{N\to\infty} \int_0^t \|f(y^*(t'),u^N(y^*(t'))) - f(y^*(t'),u^*(t'))\|dt' = 0. \tag{2.39}$$

Let us introduce the notation

$$\kappa_t(N) \overset{\text{def}}{=} L_2 \mathrm{meas}\{A_t(N)\} + \int_0^t \|f(y^*(t'),u^N(y^*(t'))) - f(y^*(t'),u^*(t'))\|dt'$$

and rewrite the inequality (2.38) in the form

$$\|y^N(t) - y^*(t)\| \leqslant L_1 \int_0^t \|y^N(t') - y^*(t')\|dt' + \kappa_t(N),$$

which, by Gronwall-Bellman Lemma [see 6, p.218], implies that

$$\max_{t'\in[0,t]} \|y^N(t') - y^*(t')\| \leqslant \kappa_t(N)e^{L_1 t'}. \tag{2.40}$$

Since, by (2.29) and (2.39),

$$\lim_{N\to\infty} \kappa_t(N) = 0, \tag{2.41}$$

(2.40) implies (2.31).

For any $t \in [0,\infty)$ such that $u^N(\cdot)$ is Lipschitz continuous on Q_t, one has

$$\begin{aligned}\|u^N(y^N(t)) - u^*(t)\| \\ \leqslant \|u^N(y^N(t)) - u^N(y^*(t))\| + \|u^N(y^*(t)) - u^*(t)\| \\ \leqslant L\|y^N(t) - y^*(t)\| + \|u^N(y^*(t)) - u^*(t)\|,\end{aligned}$$

the latter implying (2.30) (due to (2.40), (2.41) and due to (2.37)). To prove (2.32), let us recall that

$$V(y_0) = \int_0^{\infty} e^{-Ct} g(y^*(t), u^*(t)) dt.$$

For an arbitrary $\nu > 0$, choose $T_\nu > 0$ in such a way that

$$M \int_{T_\nu}^{\infty} e^{-Ct} dt \leqslant \frac{\nu}{4}, \; M \stackrel{\text{def}}{=} \max_{(y,u) \in Y \times U} \{|g(y,u)|\}.$$

Then

$$\begin{aligned}
&|V^N(y_0) - V(y_0)| \\
&\quad \leqslant \int_0^{T_\nu} e^{-Ct} |g(y^N(t), u^N(y^N(t))) - g(y^*(t), u^*(t))| dt \\
&\qquad + \int_{T_\nu}^{\infty} e^{-Ct} |g(y^N(t), u^N(y^N(t))) - g(y^*(t), u^*(t))| dt \\
&\quad \leqslant \int_0^{T_\nu} e^{-Ct} |g(y^N(t), u^N(y^N(t))) - g(y^*(t), u^*(t))| dt \\
&\qquad + \int_{T_\nu}^{\infty} e^{-Ct} |g(y^N(t), u^N(y^N(t)))| dt \\
&\qquad + \int_{T_\nu}^{\infty} e^{-Ct} |g(y^*(t), u^*(t))| dt.
\end{aligned}$$

$$\begin{aligned}
&|V^N(y_0) - V(y_0)| \\
&\quad \leqslant \int_0^{T_\nu} e^{-Ct} |g(y^N(t), u^N(y^N(t))) - g(y^*(t), u^*(t))| dt \\
&\qquad\qquad + M \int_{T_\nu}^{+\infty} e^{-Ct} dt + M \int_{T_\nu}^{+\infty} e^{-Ct} dt.
\end{aligned}$$

$$\begin{aligned}
&|V^N(y_0) - V(y_0)| \\
&\quad \leqslant \int_0^{T_\nu} e^{-Ct} |g(y^N(t), u^N(y^N(t))) - g(y^*(t), u^*(t))| dt + \frac{\nu}{2} \\
&\quad \leqslant L_3 \int_0^{T_\nu} e^{-Ct} (\|y^N(t) - y^*(t)\| + \|u^N(y^N(t)) - u^*(t)\|) dt + \frac{\nu}{2},
\end{aligned}$$

where L_3 is a Lipschitz constant of $g(\cdot, \cdot)$. Based on the fact that the convergence of $u^N(y^N(t))$ to $u^*(t)$ (for almost all t) and the uniform (on any bounded interval) convergence of $y^N(\cdot)$ to $y^*(\cdot)$ have been already established, one may conclude that there exists N_ν such that, for any $N \geqslant N_\nu$,

$$L_3 \int_0^{T_\nu} e^{-Ct} (\|y^N(t) - y^*(t)\| + \|u^N(y^N(t)) - u^*(t)\|) dt \leqslant \frac{\nu}{2}$$

$$\implies |V^N(y_0) - V(y_0)| \leqslant \nu.$$

Since ν can be arbitrarily small, the latter proves (2.32). □

Proof of Proposition 2.19. Let us prove (2.34) and (2.35). Note that, due to (2.12) and due to the fact that $\psi^N(y)$ is an optimal solution of (2.15) (see (2.19)),

$$\begin{aligned} G^N(y_0) &= \min_{y\in Y}\{H(\psi^{N\prime}(y),y) + C(\psi^N(y_0) - \psi^N(y))\} \\ &= \min_{(y,u)\in Y\times U}\{\psi^{N\prime}(y)^T f(y,u) + g(y,u) \\ &\qquad + C(\psi^N(y_0) - \psi^N(y))\}. \end{aligned} \tag{2.42}$$

Also, for any $\gamma \in W^N(y_0)$,

$$\begin{aligned} \int_{Y\times U} g(y,u)\gamma(dy,du) = \int_{Y\times U} &(g(y,u) + \psi^{N\prime}(y)^T f(y,u) \\ &+ C(\psi^N(y_0) - \psi^N(y)))\gamma(dy,du). \end{aligned}$$

Consequently, for $\gamma = \gamma^N$,

$$\begin{aligned} G^N(y_0) &= \int_{Y\times U} g(y,u)\gamma^N(dy,du) \\ &= \int_{Y\times U} (g(y,u) + \psi^{N\prime}(y)^T f(y,u) \\ &\qquad + C(\psi^N(y_0) - \psi^N(y)))\gamma^N(dy,du). \end{aligned}$$

Hence, by (2.9),

$$\begin{aligned} G^N(y_0) = \sum_{l=1}^{K_N} \gamma_l^N (g(y_l^N,u_l^N) \\ + \psi^{N\prime}(y_l^N)^T f(y_l^N,u_l^N) + C(\psi^N(y_0) - \psi^N(y_l^N))). \end{aligned} \tag{2.43}$$

Since $(y_l^N,u_l^N) \in Y\times U$, from (2.42) and (2.43) it follows that, if $\gamma_l^N > 0$, then

$$\begin{aligned} g(y_l^N,u_l^N) + \psi^{N\prime}(y_l^N)^T f(y_l^N,u_l^N) + C(\psi^N(y_0) - \psi^N(y_l^N)) \\ = \min_{(y,u)\in Y\times U}\{\psi^{N\prime}(y)^T f(y,u) + g(y,u) + C(\psi^N(y_0) - \psi^N(y))\}. \end{aligned}$$

That is,

$$(y_l^N,u_l^N) = \operatorname*{argmin}_{(y,u)\in Y\times U}\{\psi^{N\prime}(y)^T f(y,u) + g(y,u) + C(\psi^N(y_0) - \psi^N(y))\}.$$

The latter is equivalent to (2.34) and (2.35). □

Proof of Lemma 2.18. Let

$$\Theta^* \stackrel{\text{def}}{=} \{(y,u) \colon (y,u) = (y^*(t), u^*(t)) \text{ for some } t \in [0,\infty)\},$$

and let B be the open unit ball in $\mathbb{R}^{n+m}$: $B \stackrel{\text{def}}{=} \{(y,u) \colon \|(y,u)\| < 1\}$. It is easy to see that Assumption 2.15 implies that, for any $(\bar{y},\bar{u}) \in \mathrm{cl}\,\Theta^*$ (the closure of Θ^*) and any $r > 0$, the set $B_r(\bar{y},\bar{u}) \stackrel{\text{def}}{=} ((\bar{y},\bar{u}) + rB) \cap (Y \times U)$ has a nonzero γ^*-measure. That is,

$$\gamma^*(B_r(\bar{y},\bar{u})) > 0. \tag{2.44}$$

In fact, if $(\bar{y},\bar{u}) \in \mathrm{cl}\,\Theta^*$, then there exists a sequence t_i, $i = 1,2,\ldots$ such that $(\bar{y},\bar{u}) = \lim_{i\to\infty}(y^*(t_i), u^*(t_i))$, with $(y^*(t_i), u^*(t_i)) \in B_r(\bar{y},\bar{u})$ for some i large enough. Hence, there exists $\alpha > 0$ such that $(y^*(t'), u^*(t')) \in B_r(\bar{y},\bar{u})$, $\forall t' \in (t_i - \alpha, t_i]$ if $u^*(\cdot)$ is continuous from the left at t_i, and $(y^*(t'), u^*(t')) \in B_r(\bar{y},\bar{u})$, $\forall t' \in [t_i, t_i + \alpha)$ if $u^*(\cdot)$ is continuous from the right at t_i. By the definition of the discounted occupational measure (see (1.4)), this implies (2.44). Assume now the statement of the lemma is not valid. Then there exist a number $r > 0$ and sequences: $(y_i, u_i) \in \Theta^*$, N_i, $i = 1,2,\ldots$ with

$$\lim_{i\to\infty}(y_i,u_i) = (\bar{y},\bar{u}) \in \mathrm{cl}\,\Theta^*, \ \lim_{i\to\infty} N_i = \infty$$

such that

$$d((y_i,u_i),\Theta^{N_i}) \geqslant 2r \implies d((\bar{y},\bar{u}),\Theta^{N_i}) \geqslant r, \ i \geqslant i_0, \tag{2.45}$$

where

$$\Theta^N \stackrel{\text{def}}{=} \{(y_l^N, u_l^N), \ l = 1,\ldots,K_N\}$$

and $d((y,u),Q)$ stands for the distance between a point $(y,u) \in Y \times U$ and a set $Q \subset Y \times U$: $d((y,u),Q) \stackrel{\text{def}}{=} \inf_{(y',u')\in Q}\{\|(y,u) - (y',u')\|\}$. The second inequality in (2.45) implies that

$$(y_l^{N_i}, u_l^{N_i}) \notin B_r(\bar{y},\bar{u}), \ l = 1,\ldots,K_{N_i}, \ i \geqslant i_0.$$

By (2.9), the latter implies that

$$\gamma^{N_i}(B_r(\bar{y},\bar{u})) = 0.$$

From (2.8) it follows that,

$$\lim_{i\to\infty} \rho(\gamma^{N_i},\gamma^*) = 0.$$

Consequently [see 10, thm 2.1],

$$0 = \lim_{i \to \infty} \gamma^{N_i}(B_r(\bar{y}, \bar{u})) \geqslant \gamma^*(B_r(\bar{y}, \bar{u})).$$

The latter contradicts (2.44) and, thus, proves the lemma. □

3

FINITE DIMENSIONAL APPROXIMATIONS

In this chapter, we further approximate the N-dimensional problem (2.5) and the corresponding dual problem (2.10), with a finite dimensional linear programming (FDLP) problem defined on a grid of points in $Y \times U$ and show that, by solving the FDLP problem, one can construct a function $\psi^{N,\Delta}(\cdot) \in C^1$ (where Δ is a parameter of the grid) which solves the dual IDLP problem approximately and that this function can be used for the construction of an approximate control for the problem 1.2.

This chapter is organised as follows; In Section 3.1 we introduce the FDLP problem and show that it approximates the SILP problem studied in Chapter 2. In Section 3.2, we establish relationships between solutions of the problems dual to the finite and semi-finite problems. In Section 3.3, we state a result (Proposition 3.5) about the construction of a near-optimal control for the problem 1.2. In Section 3.4 we outline an algorithm for construction of a near optimal control on the basis of a solution of the FDLP problem.

3.1 THE FINITE DIMENSIONAL LP PROBLEM

Assume first that N is fixed and that, for any $\Delta > 0$, Borel sets $Q^{\Delta}_{l,k} \subset Y \times U$ where $l = 1, \ldots, L^{\Delta}$ and $k = 1, \ldots, K^{\Delta}$ are defined in such a way that they are mutually disjoint. The union of all cells is equal to $Y \times U$ and

$$\sup_{(y,u) \in Q^{\Delta}_{l,k}} \|(y,u) - (y_l, u_k)\| \leqslant a\Delta, \;\; a = \text{const}, \tag{3.1}$$

for some point $(y_l, u_k) \in Q^{\Delta}_{l,k}$. For simplicity of notation, it is assumed (from now on) that U is a compact subset of $\mathbb{R}^n$ and $\|\cdot\|$ stands for a norm in $\mathbb{R}^{n+m}$.

Let us fix the points (y_l, u_k). Then define a polyhedral set $W^{N,\Delta}(y_0) \subset \mathbb{R}^{L^\Delta + K^\Delta}$ by the equation

$$W^{N,\Delta}(y_0) \stackrel{\text{def}}{=} \{\gamma = \{\gamma_{l,k}\} \geqslant 0 \colon \sum_{l,k} \gamma_{l,k} = 1,$$
$$\sum_{l,k} (\phi_i{}'(y_l)^T f(y_l, u_k) + C(\phi_i(y_0) - \phi_i(y_l)))\gamma_{l,k} = 0, \ \forall i = 1, \dots, N\},$$

where $\sum_{l,k} \stackrel{\text{def}}{=} \sum_{l=1}^{L^\Delta} \sum_{k=1}^{K^\Delta}$. Then consider the problem

$$G^{N,\Delta}(y_0) \stackrel{\text{def}}{=} \min_{\gamma \in W^{N,\Delta}(y_0)} \sum_{l,k} g(y_l, u_k)\gamma_{l,k}. \tag{3.2}$$

This is a finite dimensional linear programming problem, which will be referred to as the NΔ-approximating LP problem (or NΔ–LP problem). Note that the polyhedral set $W^{N,\Delta}(y_0)$ is the set of probability measures on $Y \times U$ which assign non-zero probabilities only to the points (y_l, u_k), and as such,

$$W^{N,\Delta}(y_0) \subset W^N(y_0) \implies G^{N,\Delta}(y_0) \geqslant G^N(y_0). \tag{3.3}$$

Note that, by Proposition 3.1, $W^{N,\Delta}(y_0)$ is not empty and, hence, (3.2) and (3.3) are true if Assumption 2.8 is satisfied.

Proposition 3.1. *Let Assumption 2.8 be satisfied and let the set $W^N(y_0)$ be not empty. Then there exists $\Delta_0 > 0$ such that $W^{N,\Delta}(y_0)$ is not empty for $\Delta \leqslant \Delta_0$. Also*

$$\lim_{\Delta \to 0} \rho_H(W^{N,\Delta}(y_0), W^N(y_0)) = 0. \tag{3.4}$$

and

$$\lim_{\Delta \to 0} G^{N,\Delta}(y_0) = G^N(y_0). \tag{3.5}$$

If $\gamma^{N,\Delta}$ is a solution of the problem (3.2) and $\lim_{\Delta' \to 0} \rho(\gamma^{N,\Delta'}, \gamma^N) = 0$ for some sequence of Δ' tending to zero, then γ^N is a solution of (2.5). If the optimal solution γ^N of the problem (2.5) is unique, then for any optimal solution $\gamma^{N,\Delta}$ of the problem (3.2) there exists the limit

$$\lim_{\Delta \to 0} \rho(\gamma^{N,\Delta}, \gamma^N) = 0. \tag{3.6}$$

Proof of Proposition 3.1. The first thing that should be noted that by (3.3), the set $W^N(y_0)$ will not be empty if the set $W^{N,\Delta}(y_0)$ is not empty. Let us assume the set $W^N(y_0)$ is not empty and show that the set $W^{N,\Delta}(y_0)$ is also not empty and that (3.4) is valid. The validity of (3.5) will follow from (3.4).

Due to (3.1) and the fact that the functions $\phi_i'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y))$ are continuous it follows that

$$\sup_{(y,u)\in Q_{l,k}^{\Delta}} |(\phi_i'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y))) \\ - (\phi_i'(y_l)^T f(y_l,u_k) + C(\phi_i(y_0) - \phi_i(y_l)))| \leqslant \kappa(\Delta),\ i = 1,\dots,N \quad (3.7)$$

for some $\kappa(\Delta)$ such that $\lim_{\Delta\to 0}\kappa(\Delta) = 0$.

Now let us define the set $Z^{N,\Delta}(y_0) \subset \mathbb{R}^{L^\Delta + K^\Delta}$ by the equation

$$Z^{N,\Delta}(y_0) \stackrel{\text{def}}{=} \Big\{\gamma = \{\gamma_{l,k}\} \geqslant 0\colon \sum_{l,k}\gamma_{l,k} = 1, \\ |\sum_{l,k}(\phi_i'(y_l)^T f(y_l,u_k) + C(\phi_i(y_0) - \phi_i(y_l)))\gamma_{l,k}| \leqslant \kappa(\Delta), \\ i = 1,\dots,N\Big\}. \quad (3.8)$$

For any Δ, let $\gamma^\Delta \in W^N(y_0)$ such that

$$\rho(\gamma^\Delta, Z^{N,\Delta}(y_0)) = \max_{\gamma\in W^N(y_0)} \rho(\gamma, Z^{N,\Delta}(y_0))$$

(where we know that γ^Δ exists due to the fact that $W^N(y_0)$ is compact) and show that

$$\lim_{\Delta\to 0}\max_{\gamma\in W^N(y_0)} \rho(\gamma, Z^{N,\Delta}(y_0)) = \lim_{\Delta\to 0}\rho(\gamma^\Delta, Z^{N,\Delta}(y_0)) = 0. \quad (3.9)$$

Let $\gamma_{l,k}^\Delta \stackrel{\text{def}}{=} \int_{Q_{l,k}^\Delta}\gamma^\Delta(dy,du)$. By (3.7)

$$|\sum_{l,k}(\phi_i'(y_l)^T f(y_l,u_k) + C(\phi_i(y_0) - \phi_i(y_l)))\gamma_{l,k}^\Delta|$$

$$= |\sum_{l,k}(\phi_i'(y_l)^T f(y_l,u_k) + C(\phi_i(y_0) - \phi_i(y_l)))\gamma_{l,k}^\Delta \\ - \int_{Y\times U}(\phi_i'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\gamma^\Delta(dy,du)|$$

$$= |\sum_{l,k}\int_{Q_{l,k}^\Delta}(\phi_i'(y_l)^T f(y_l,u_k) + C(\phi_i(y_0) - \phi_i(y_l)))\gamma^\Delta(dy,du) \\ - \sum_{l,k}\int_{Q_{l,k}^\Delta}(\phi_i'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\gamma^\Delta(dy,du)|$$

$$= |\sum_{l,k} \int_{Q^\Delta_{l,k}} ((\phi_i'(y_l)^T f(y_l, u_k) + C(\phi_i(y_0) - \phi_i(y_l))) \\ - (\phi_i'(y)^T f(y, u) + C(\phi_i(y_0) - \phi_i(y))))\gamma^\Delta(dy, du)|$$

$$\leqslant \sum_{l,k} \int_{Q^\Delta_{l,k}} |((\phi_i'(y_l)^T f(y_l, u_k) + C(\phi_i(y_0) - \phi_i(y_l))) \\ - (\phi_i'(y)^T f(y, u) + C(\phi_i(y_0) - \phi_i(y))))|\gamma^\Delta(dy, du)$$

$$\leqslant \sum_{l,k} \int_{Q^\Delta_{l,k}} \kappa(\Delta)\gamma^\Delta(dy, du) \text{ by (3.7)}$$

$$= \sum_{l,k} \kappa(\Delta)\gamma^\Delta_{l,k} = \kappa(\Delta).$$

Hence, by denoting $\widetilde{\gamma}^\Delta \stackrel{\text{def}}{=} (\gamma^\Delta_{l,k})$, one can obtain that $\widetilde{\gamma}^\Delta \in Z^{N,\Delta}(y_0)$ and consequently

$$\rho(\widetilde{\gamma}^\Delta, Z^{N,\Delta}(y_0)) = 0. \tag{3.10}$$

Let $q(y, u) \colon Y \times U \to \mathbb{R}^1$ be some arbitrary continuous function and let $\kappa_q(\Delta)$ be such that

$$\sup_{(y,u)\in Q^\Delta_{l,k}} |q(y, u) - q(y_l, u_k)| \leqslant \kappa_q(\Delta), \ \lim_{\Delta\to 0} \kappa(\Delta) = 0.$$

Then

$$|\int_{Y\times U} q(y, u)\gamma^\Delta(dy, du) - \sum_{l,k} q(y_l, u_k)\gamma^\Delta_{l,k}| \\ = |\sum_{l,k} \int_{Q^\Delta_{l,k}} q(y, u)\gamma^\Delta(dy, du) - \sum_{l,k} \int_{Q^\Delta_{l,k}} q(y_l, u_k)\gamma^\Delta(dy, du)| \leqslant \kappa_q(\Delta).$$

The fact that the latter inequality is valid for an arbitrary continuous function $q(y, u)$ implies that $\lim_{\Delta\to 0} \rho(\gamma^\Delta, \widetilde{\gamma}^\Delta) = 0$ which along with (3.10) implies the validity of (3.9). By (3.3)

$$\max_{\gamma\in W^{N,\Delta}(y_0)} \rho(\gamma, W^N(y_0)) = 0.$$

Hence, to prove (3.4) it is enough to establish that

$$\lim_{\Delta\to 0} \max_{\gamma\in W^N(y_0)} \rho(\gamma, W^{N,\Delta}(y_0)) = 0. \tag{3.11}$$

Since (as it can be verified using the triangle inequality),

$$\max_{\gamma\in W^N(y_0)} \rho(\gamma, W^{N,\Delta}(y_0)) \leqslant \max_{\gamma\in W^N(y_0)} \rho(\gamma, Z^{N,\Delta}(y_0)) + \max_{\gamma\in Z^{N,\Delta}(y_0)} \rho(\gamma, W^N(y_0))$$

and since (3.9) has been already verified, equality (3.11) will be established if one shows that

$$\lim_{\Delta\to 0} \max_{\gamma\in Z^{N,\Delta}(y_0)} \rho(\gamma, W^{N,\Delta}(y_0)) = \lim_{\Delta\to 0} \rho(\overline{\gamma}^{\Delta}, W^{N,\Delta}(y_0)) = 0, \tag{3.12}$$

where $\overline{\gamma}^{\Delta} = \{\overline{\gamma}_{l,k}\} \in Z^{N,\Delta}(y_0)$ is such that

$$\rho(\overline{\gamma}^{\Delta}, W^N(y_0)) = \max_{\gamma\in Z^{N,\Delta}(y_0)} \rho(\gamma, W^{N,\Delta}(y_0))$$

for any $\Delta > 0$. Let $q_j(\cdot)$ be a sequence of Lipschitz continuous functions which is dense in the unit ball of $C(Y \times U)$ and consider the following finite dimensional linear program

$$F_J(\Delta) \stackrel{\text{def}}{=} \min_{\gamma=\{\gamma_{l,k}\}\in W^{N,\Delta}(y_0)} \sum_{j=1}^{J} \frac{1}{2^j} |\sum_{l,k} q_j(y_l, u_k)\gamma_{l,k} - \sum_{l,k} q_j(y_l, u_k)\overline{\gamma}^{\Delta}_{l,k}|. \tag{3.13}$$

To prove that (3.12) is valid it is sufficient to show that

$$\lim_{\Delta\to 0} F_J(\Delta) = 0, \quad J = 1, 2, \ldots \tag{3.14}$$

Below it is shown that the optimal value of the problem dual to (3.13) tends to zero as Δ tends to zero. Since the latter coincides with $F_J(\Delta)$, this will prove (3.14). Also from (3.14) it follows that $F_J(\Delta)$ is bounded and, hence, $W^{N,\Delta}(y_0)$ is not empty for Δ small enough. Lets rewrite problem (3.13) in equivalent form

$$F_J(\Delta) = \min_{\gamma=\{\gamma_{l,k}\}\in W^{N,\Delta}(y_0)} \sum_{j=1}^{J} \frac{1}{2^j} \theta_j, \tag{3.15}$$

where

$$|\sum_{l,k} q_j(y_l, u_k)\gamma_{l,k} - \sum_{l,k} q_j(y_l, u_k)\overline{\gamma}^{\Delta}_{l,k}| \leqslant \theta_j$$

which is equivalent to

$$-\theta_j \leqslant \sum_{l,k} q_j(y_l,u_k)\gamma_{l,k} - \sum_{l,k} q_j(y_l,u_k)\overline{\gamma}^{\Delta}_{l,k} \leqslant \theta_j,$$

or

$$-\sum_{l,k} q_j(y_l,u_k)\overline{\gamma}^{\Delta}_{l,k} \leqslant -\sum_{l,k} q_j(y_l,u_k)\gamma_{l,k} + \theta_j,$$

or

$$\sum_{l,k} q_j(y_l,u_k)\overline{\gamma}^{\Delta}_{l,k} \leqslant \sum_{l,k} q_j(y_l,u_k)\gamma_{l,k} + \theta_j. \tag{3.16}$$

The dual problem to (3.15) to (3.16) is

$$F_J(\Delta) = \max_{\lambda_i,\mu_j,\eta_j,\zeta} \sum_{j=1}^{J}(-\mu_j+\eta_j)(\sum_{l,k} q_j(y_l,u_k)\overline{\gamma}^{\Delta}_{l,k}) + \zeta,$$

where $\lambda_i = 1,\dots,N$; μ_j, η_j, $j = 1,\dots,J$ and ζ satisfy the following relationships

$$\sum_{i=1}^{N} \lambda_i(\phi_i{}'(y_l)^T f(y_l,u_k) + C(\phi_i(y_0) - \phi_i(y_l))) + \sum_{j=1}^{J}(-\mu_j+\eta_j)q_j(y_l,u_k) + \zeta \leqslant 0, \tag{3.17}$$

where $l = 1,\dots,L^{\Delta}$, $k = 1,\dots,K^{\Delta}$ and

$$\mu_j + \eta_j = \frac{1}{2^j},\ \mu_j \geqslant 0,\ \eta_j \geqslant 0,\ j = 1,\dots,J. \tag{3.18}$$

Before proving (3.14), lets show that $F_J(\Delta)$ is bounded for Δ small enough (which by (3.13) is equivalent to that $W^{N,\Delta}(y_0)$ is not empty). Assume it is not. Then there exists a sequence Δ^r, $r = 1,2,\dots$ where $\lim_{r\to\infty}\Delta^r = 0$, and sequences λ_i^r, μ_j^r, η_j^r, ζ^r, satisfying (3.17)–(3.18) with $\Delta = \Delta^r$, $r = 1,2,\dots$ such that $\lim_{r\to\infty}(|\zeta^r| + \sum_{i=1}^{N}|\lambda_i^r|) = \infty$ and

$$\lim_{r\to\infty}\frac{\zeta^r}{|\zeta^r| + \sum_{i=1}^{N}|\lambda_i^r|} \stackrel{\text{def}}{=} a \geqslant 0,\ \lim_{r\to\infty}\frac{\lambda_i^r}{|\zeta^r| + \sum_{i=1}^{N}|\lambda_i^r|} \stackrel{\text{def}}{=} \nu_i,$$

where

$$a + \sum_{i=1}^{N}|\nu_i| = 1. \tag{3.19}$$

Dividing (3.17) by $\zeta^r + \sum_{i=1}^N |\lambda_i^r|$ and passing to the limit as $r \to \infty$ one can obtain

$$\sum_{i=1}^{N} \nu_i(\phi_i{}'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y))) + a \leqslant 0, \quad \forall (y,u) \in Y \times U, \quad (3.20)$$

where it is taken account that every point $(y,u) \in Y \times U$ can be represented as the limit of (y_l, u_k) belonging to the sequence of cells $Q_{l,k}^{\Delta_r}$ such that $(y,u) \in Q_{l,k}^{\Delta_r}$. Two cases are possible, $a > 0$ and $a = 0$. If $a > 0$, then the validity of (3.20) implies that the function $\phi(y) = \sum_{i=1}^N \nu_i \phi_i(y)$ satisfies

$$\max_{(y,u)\in Y\times U} \left\{ \phi'(y)^T f(y,u) + C(\phi(y_0) - \phi(y)) \right\} < 0$$

which would lead to $W^N(y_0)$ being empty. The set $W^N(y_0)$ is not empty (by our assumption) and hence the only case to consider is $a = 0$. In this case, (3.20) becomes

$$\sum_{i=1}^{N} \nu_i(\phi_i{}'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y))) \leqslant 0, \ \forall (y,u) \in Y \times U. \quad (3.21)$$

By Lemma 2.11, (3.21) can be valid only with all ν_i being equal to zero. This contradicts (3.19) and thus, proves that $F_J(\Delta)$ is bounded for Δ small enough (and that $W^{N,\Delta}(y_0)$ is not empty).

From that fact $F_J(\Delta)$ is bounded it follows that a solution, λ_i^Δ, $i = 1, \dots, N$; μ_j^Δ, η_j^Δ, $j = 1, \dots, N$; and ζ^Δ of problem (3.17)–(3.18) exists. Using the above solution one can obtain the following estimates

$$\begin{aligned}
0 \leqslant & F_J(\Delta) \\
&= \sum_{j=1}^{J} (-\mu_j^\Delta + \eta_j^\Delta)(\sum_{l,k} q_j(y_l, u_l)\overline{\gamma}_{l,k}^\Delta) + \zeta^\Delta \\
&= \sum_{l,k} \overline{\gamma}_{l,k}^\Delta (\sum_{j=1}^{J} (-\mu_j^\Delta + \eta_j^\Delta) q_j(y_l, u_k)) + \zeta^\Delta \\
&\leqslant \sum_{l,k} \overline{\gamma}_{l,k}^\Delta (-\sum_{i=1}^{N} \lambda_i^\Delta (\phi_i{}'(y_l)^T f(y_l, u_k) + C(\phi_i(y_0) - \phi_i(y_l)))) - \zeta^\Delta + \zeta^\Delta \\
&= -\sum_{i=1}^{N} \lambda_i^\Delta (\sum_{l,k} (\phi_i{}'(y_l)^T f(y_l, u_k) + C(\phi_i(y_0) - \phi_i(y)))\overline{\gamma}_{l,k}^\Delta) \\
&\leqslant \sum_{i=1}^{N} |\lambda_i^\Delta| \kappa(\Delta),
\end{aligned}$$

where the last two relationships are implied by the fact that $\overline{\gamma}^{\Delta} = \{\overline{\gamma}^{\Delta}_{l,k}\} \in Z^{N,\Delta}(y_0)$ (See (3.8)).

It is sufficient to show that $\sum_{i=1}^{N} |\lambda_i^{\Delta}|$ remains bounded as $\Delta \to 0$. Let assume this is not the case. Then there will exist a sequence Δ^r, $r = 1, 2, \ldots$ such that $\lim_{r\to\infty} \Delta^r = 0$, and will exist sequences $\lambda_i^r, \mu_j^r, \eta_j^r, \zeta^r$, that satisfy (3.17) and (3.18) with $\Delta = \Delta^r, r = 1, 2, \ldots$ such that

$$\lim_{r\to\infty} \sum_{i=1}^{N} |\lambda_i^r| = \infty, \quad \lim_{r\to\infty} \frac{\zeta^r}{\sum_{i=1}^{N} |\lambda_i^r|} = 0,$$

$$\lim_{r\to\infty} \frac{\lambda_i^r}{\sum_{i=1}^{N} |\lambda_i^r|} \stackrel{\text{def}}{=} \nu_i, \quad \sum_{i=1}^{N} |\nu_i| = 1.$$

Now if (3.17) is divided by $\sum_{i=1}^{N} |\lambda_i^r|$ and then passed to the limit as $r \to \infty$, one can obtain that the inequality (3.21) is valid. This by Lemma 2.11 implies that $\nu_i = 0$, $i = 1, \ldots, N$. Which then contradicts the last equality of (3.21) and, hence proves (3.14). □

3.2 THE FINITE DIMENSIONAL DUAL PROBLEM

Consider the finite dimensional linear program

$$\mu^{N,\Delta}(y_0) \stackrel{\text{def}}{=} \max_{(\mu,\lambda)\in\mathbb{R}^1\times\mathbb{R}^N} \{\mu \colon \mu \leqslant g(y_l, u_k) + \sum_{i=1}^{N} \lambda_i (\phi'(y_l)^T f(y_l, u_k) + C(\phi(y_0) - \phi(y_l)), \; \forall (y_l, u_k) \in Q^{\Delta}_{l,k}\}, \tag{3.22}$$

which is dual to NΔ–LP and which will be referred to as D–NΔ–LP. From the duality theory of finite dimensional linear programs [see 18] it follows, in particular, that if $W^{N,\Delta}(y_0)$ is not empty, then the optimal value of the NΔ–LP problem (3.2) is equal to the optimal value of the D–NΔ–LP problem (3.22).

$$G^{N,\Delta}(y_0) = \mu^{N,\Delta}(y_0), \tag{3.23}$$

and the solution set of the D–NΔ–LP is not empty:

$$\emptyset \neq \Lambda_{N,\Delta}(y_0) \stackrel{\text{def}}{=} \{\lambda = (\lambda_i) \colon \mu^{N,\Delta}(y_0) = \min_{(y_l,u_k)\in Q^{\Delta}_{l,k}} \{g(y_l, u_k) + \sum_{i=1}^{N} \lambda_i (\phi_i{}'(y_l)^T f(y_l, u_k) + C(\phi_i(y_0) - \phi_i(y_l))\}\}. \tag{3.24}$$

Note that, by Proposition 3.1, $W^{N,\Delta}(y_0)$ is not empty and, hence, (3.23) and (3.24) are true if Assumption 2.8 is satisfied.

Proposition 3.2. *Let Assumption 2.8 be satisfied. Then*

$$\lim_{\Delta\to 0} \max_{\lambda\in\Lambda^{N,\Delta}(y_0)} \operatorname{dist}(\lambda, V^N) = 0, \ \operatorname{dist}(\lambda, V^N) \stackrel{\text{def}}{=} \min_{v\in V^N} \|\lambda - v\|, \tag{3.25}$$

where V^N is the solution set of the D–SILP problem (2.15).

Proof of Proposition 3.2. First, let us show that the set $\Lambda^{N,\Delta}$ is bounded for Δ small enough. That is, show that

$$\sup_{\lambda\in\Lambda^{N,\Delta}} |\lambda| \leqslant c_N = \text{const} \tag{3.26}$$

for $\Delta \leqslant \Delta_N$ ($\Delta_N > 0$). Assume that it is not true and, hence, there exist sequences Δ_s and $\lambda^{N,\Delta_s} \in \Lambda^{N,\Delta_s}$, $s = 1, 2, \ldots$ such that

$$\lim_{s\to\infty} \Delta_s = 0, \ \lim_{s\to\infty} |\lambda^{N,\Delta_s}| = \infty.$$

Without loss of generality one may assume that there exists a limit

$$\lim_{s\to\infty} \frac{\lambda^{N,\Delta_s}}{\|\lambda^{N,\Delta_s}\|} \stackrel{\text{def}}{=} v, \ |v| = 1. \tag{3.27}$$

From the definition of $\Lambda^{N,\Delta}$ (see (3.24)) it follows that the inequality

$$\mu^{N,\Delta}(y_0) \leqslant g(y_l, u_k) + \sum_{i=1}^{N} \lambda_i^{N,\Delta}(\phi_i{}'(y_l)^T f(y_l, u_k) + C(\phi_i(y_0) - \phi_i(y_l))) \tag{3.28}$$

is valid for any grid point $(y_l, u_k) \in Y \times U$. Substituting Δ_s for Δ in (3.28) and then dividing the latter by $|\lambda^{N,\Delta_s}|$ and passing to the limit as $s \to \infty$, one can prove that

$$0 \leqslant \sum_{i=1}^{N} v_i(\phi_i{}'(y)^T f(y, u) + C(\phi_i(y_0) - \phi_i(y))), \ \forall (y, u) \in Y \times U.$$

Note that the fact that the above inequality is valid follows from (2.12), (3.5) and (3.23). Also,

$$\lim_{\Delta\to 0} \mu^{N,\Delta}(y_0) = \mu^N(y_0), \tag{3.29}$$

which, in particular, implies that μ^{N,Δ_s} remains bounded as $s \to \infty$, and also on the fact that any point (y, u) in $Y \times U$ can be presented as a limit of a sequence of grid points. From Lemma 2.11 it now follows that $v = (v_i) = 0$, which contradicts to (3.27). This proves (3.26).

Let us now prove (3.25). Assuming that it is not true, one can come to a conclusion that there exist a positive number α and sequences Δ_s and $\lambda^{N,\Delta_s} \in \Lambda^{N,\Delta_s}$, $s = 1, 2, \ldots$ such that

$$\lim_{s\to\infty} \Delta_s = 0, \ \mathrm{dist}(\lambda^{N,\Delta_s}, V^N) \geqslant \alpha, \ s = 1, 2, \ldots$$

Due to (3.26), one may assume without loss of generality that there exists a limit

$$\lim_{s\to\infty} \lambda^{N,\Delta_s} \stackrel{\text{def}}{=} v_N \implies \mathrm{dist}(v^N, V^N) \geqslant \alpha. \tag{3.30}$$

Substituting Δ_s for Δ in (3.28), taking into account (3.29) and passing to the limit as $s \to \infty$, one can obtain that

$$\mu^N(y_0) \leqslant g(y,u) + \sum_{i=1}^{N} v_i^N(\phi_i'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y))),$$
$$\forall (y,u) \in Y \times U \implies v^N = (v_i^N) \in V^N.$$

The latter contradicts (3.30) and, thus, proves (3.25).

Let us only note here that, by Proposition 3.1, the set $W^{N,\Delta}(y_0)$ is not empty by Lemma 2.11, and hence, (3.23) and (3.24) are valid for Δ small enough. □

3.3 CONVERGENCE TO THE OPTIMAL SOLUTION

Everywhere in this section it is assumed that Assumption 2.8 is satisfied and, hence, the solution set $\Lambda^{N,\Delta}$ of the D–NΔ–LP problem (3.22) is not empty.

Let $\lambda^{N,\Delta} = (\lambda_i^{N,\Delta}) \in \Lambda^{N,\Delta}$ and let

$$\psi^{N,\Delta}(y) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \lambda_i^{N,\Delta} \phi_i(y). \tag{3.31}$$

Proposition 3.3. *For any* $\delta > 0$*, there exist* $N_\delta > 0$ *and* $\Delta_N > 0$ *such that, for* $N \geqslant N_\delta$ *and* $\Delta \leqslant \Delta_N$*, the function* $\psi^{N,\Delta}(\cdot)$ *solves the IDLP problem approximately in the sense that*

$$\mu^*(y_0) - \delta \leqslant g(y,u) + \psi^{N,\Delta\prime}(y)^T f(y,u) \\ + C(\psi^{N,\Delta}(y_0) - \psi^{N,\Delta}(y), \ \forall (y,u) \in Y \times U. \quad (3.32)$$

Proof of Proposition 3.3. Let us choose N_δ in such a way that

$$\mu^*(y_0) - \frac{\delta}{2} \leqslant \mu^N(y_0)$$

for any $N \geqslant N_\delta$ (this is possible due to Proposition 2.3). By Proposition 2.3, the set V^N is not empty and, hence,

$$\mu^*(y_0) - \frac{\delta}{2} \leqslant \min_{(y,u)\in Y\times U} \{g(y,u) \\ + \sum_{i=1}^{N} v_i(\phi_i{}'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\}, \ \forall v = (v_i) \in V^N.$$

From (3.25) it follows that, for any $\Delta \leqslant \Delta_N$ (Δ_N being positive small enough) and any $\lambda \in \Lambda^{N,\Delta}$, there exists $v^{N,\Delta} = (v_i^{N,\Delta}) \in V^N$ such that

$$\min_{(y,u)\in Y\times U} \{g(y,u) + \sum_{i=1}^{N} v_i^{N,\Delta}(\phi_i{}'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\} - \frac{\delta}{2} \\ \leqslant \min_{(y,u)\in Y\times U} \{g(y,u) + \sum_{i=1}^{N} \lambda_i^{N,\Delta}(\phi_i{}'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\} \\ \implies \mu^N(y_0) - \delta \leqslant \min_{(y,u)\in Y\times U} \{g(y,u) \\ + \sum_{i=1}^{N} \lambda_i^{N,\Delta}(\phi_i{}'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\}.$$

The latter proves (3.32). □

Let $u^{N,\Delta}(y)$ be a solution of the problem

$$\min_{u\in U} \{g(y,u) + \psi^{N,\Delta}(y)^T f(y,u)\}. \quad (3.33)$$

That is,

$$u^{N,\Delta}(y) \stackrel{\text{def}}{=} \operatorname*{argmin}_{u\in U} \{g(y,u) + \psi^{N,\Delta}(y)^T f(y,u)\}. \quad (3.34)$$

Assume that the system

$$y'(t) = f(y(t), u^{N,\Delta}(y(t)), \; y(0) = y_0,$$

has a unique solution $y^{N,\Delta}(t) \in Y$. Let, also, Assumptions 2.13 and 2.15 be satisfied. It can be shown that, under an additional assumption similar to Assumption 2.16, the control $u^{N,\Delta}(y)$ converges to the optimal one as $\Delta \to 0$ and $N \to \infty$ and the the results similar to Proposition 2.17 are true. Let us introduce this assumption and give the statement of the theorem that is analogous to Proposition 2.17.

Assumption 3.4.

(i) *For almost all* $t \in [0, \infty)$ *there exists an open ball* $Q_t \in \mathbb{R}^m$ *centred at* $y^*(t)$ *such that* $u^{N,\Delta}(y)$ *is uniquely defined for* $y \in Q_t$ *(that is the problem in the right hand side of* (3.33) *has a unique solution) and* $u^{N,\Delta}(y)$ *satisfies Lipschitz conditions on* Q_t *with a Lipschitz constant being independent of* N, Δ *and* t.

(ii) *The Lebesgue measure of the set* $A_t(N, \Delta) \stackrel{\text{def}}{=} \{t' \in [0, t], \; y^{N,\Delta}(t') \notin Q_{t'}\}$ *tends to zero as* $\Delta \to 0$ *and* $N \to \infty$. *That is,*

$$\lim_{N \to} \overline{\lim_{\Delta \to 0}} \, \text{meas}\{A_t(N, \Delta)\} = 0.$$

Proposition 3.5. *Let* $f(y, u)$ *and* $g(y, u)$ *be Lipschitz continuous in a neighbourhood of* $Y \times U$, *let Assumptions 2.13, 2.15 and 3.4 be satisfied. Then*

$$\lim_{N \to \infty} \overline{\lim_{\Delta \to 0}} \|u^{N,\Delta}(y^{N,\Delta}(t)) - u^*(t)\| = 0$$

for almost all $t \in [0, \infty)$ *and*

$$\lim_{N \to \infty} \overline{\lim_{\Delta \to 0}} \max_{t' \in [0,t]} \|y^{N,\Delta}(t') - y^*(t')\| = 0, \; \forall t \in [0, \infty).$$

Also,

$$\lim_{N \to \infty} \overline{\lim_{\Delta \to 0}} |V^{N,\Delta}(y_0) - V(y_0)| = 0,$$

where

$$V^{N,\Delta}(y_0) \stackrel{\text{def}}{=} \int_0^\infty e^{-Ct} g(y^{N,\Delta}(t), u^{N,\Delta}(y^{N,\Delta}(t))) dt.$$

Proof of Proposition 3.5. The proof the proposition follows exactly the same steps as the proof of Proposition 2.17. □

Remark 3.6. *The assumptions of Proposition 3.5 as well as those of Proposition 2.17 are not easy to verify. However, the control (3.34) can be constructed without the verification of these assumptions, and once it is constructed one can evaluate how close it is to the optimal by comparing the value of the objective function obtained with this control and the optimal value of the corresponding LP problem.*

3.4 ALGORITHMIC SOLUTIONS TO THE OPTIMAL CONTROL PROBLEM

Note that from corollary 2.4 and (3.6) it follows that $\gamma^{N,\Delta} \stackrel{\text{def}}{=} \{\gamma_{l,k}^{N,\Delta}\}$ can be considered as an "approximation" of γ^* for N large and Δ small enough. Due to the fact that γ^* is the discounted occupational measure generated by the optimal pair $(y^*(\cdot), u^*(\cdot))$, an element $\gamma_{l,k}^{N,\Delta}$ of $\gamma^{N,\Delta}$ can be interpreted as an estimate of the discounted "proportion" of time spent by the optimal pair in a "small" vicinity of the point (y_l, u_k) while the fact that $\gamma_{l,k}^{N,\Delta}$ is positive or zero can be interpreted as an indication of whether or not the optimal pair attends this vicinity.

Based on the consideration above, we outline the steps which can be used to construct an approximate solution to the discounted optimisation problem using commercially available linear programming solvers such as IBM CPLEX [39].

(i) Define a $L \times K$ dimensional grid of sufficient span to encompass the dynamics of the problem under study and then choose $\Delta_{l,k}$ small enough that a further reduction of $\Delta_{l,k}$ leads to an insignificant change in the optimal value $G^{N,\Delta}(y_0)$.

(ii) Choose a value for N which is large enough that a further increase of N leads to an insignificant change in the optimal value $G^{N,\Delta}(y_0)$.

In practice the size of N *and* $\Delta_{l,k}$ *will be constrained by the amount of computer memory required for the LP problem generated. Other factors which influence the choice of* N *and* Δ *may be the requirement for a quick numerical result or, if for technical reasons, a specific set of points* (y_l, u_k) *are required to lie on the grid.*

(iii) Find a basic solution $\gamma^{N,\Delta} = \{\gamma_{l,k}\}$ which can be considered an approximation to γ^* for N large enough and $\Delta_{l,k}$ small enough. Then define the set

$$y^{N,\Delta} \stackrel{\text{def}}{=} \{y_l \colon \sum_k \gamma_{l,k}^{N,\Delta} \neq 0\},$$

which is a projection onto the y-plane, of all the grid points associated with significant concentrations of occupational measure. Since γ^* is the discounted occupational measure generated by the optimal pair $(y(\cdot), u(\cdot))$ we can expect the points $\mathcal{Y}^{N,\Delta}$ will attend the vicinity of the curve $\mathcal{Y}^*$ which is the optimal trajectory.

(iv) Then, find a dual solution $\lambda^{N,\Delta}$ and construct the function $\psi^{N,\Delta}(y)$ according to (3.31) and find the control $u^{N,\Delta}(y)$ by solving problem (3.34) for every y in the vicinity of $\mathcal{Y}^{N,\Delta}$. We can expect to obtain a solution of the system (1.1) by integrating from the initial condition y_0 for a integration period T which is defined by a suitable stopping criterion. An example of such a stopping criteria is $T = 10/C$, by which time the ongoing dynamics of the system contribute little to the objective value.

(v) Compute the integral

$$G_{NUM}(T) \stackrel{\text{def}}{=} C \int_0^T e^{-Ct} g(y^{N,\Delta}(t), u^{N,\Delta}(t)) dt$$

and compare it with $G^{N,\Delta}(y_0)$. If the computed value of the integral proves to be close to $G^{N,\Delta}(y_0)$, then the constructed admissible pair is a "good" approximation to the solution of the discounted optimisation problem.

In chapter 5 we consider a numerical example which is to illustrate the above steps.

4 CONSTRUCTION OF STABILISING CONTROLS

It is well known that optimal control methods can be used for the design of asymptotically stabilising controls by choosing the objective in such a way that it penalises states away from the desired equilibrium. For linear systems, the classical (infinite horizon) linear quadratic regulator [41] is one example of this approach (see also, e.g., the textbooks [1, Chapter 3] or [55, Section 8.2]).

Theoretically, the infinite horizon undiscounted optimal control problem can be used to characterise stabilising controls but in practice these problems are numerically very difficult to solve. Direct methods are efficient for solving finite horizon nonlinear optimal control problems [9] fail here since infinite horizon problems are still infinite dimensional after a discretisation in time. Dynamic programming methods apply to infinite horizon problems, however, for non-discounted problems the resulting dynamic programming operator is typically degenerate near the stable equilibrium such that a suitable regularisation is needed before the problem can be discretised numerically [12].

In this chapter, we will discuss the construction of a stabilising control based on a linear programming solution of an infinite horizon optimal control problem with time discounting.

A part of this chapter was earlier published in [29] where it is shown that a condition similar to that found in the model predictive control (MPC) literature can be used to establish that the discounted optimal value function is a Lyapunov function, from which asymptotic stability can be concluded.

The chapter is organised as follows. After defining the problem and the necessary background in Section 4.1, the main stability result is formulated and proved in Section 4.2. To this end we utilize a condition involving a bound on the optimal value function. In Section 4.3 it is shown how different controllability properties can be used in order to establish this bound.

4.1 PROBLEM FORMULATION

For the convenience of the reader, we shall reintroduce some notations and definitions from Chapter 1. In the discussion to follow, we will be considering the control system

$$y'(t) = f(y(t), u(t)), \ t \geqslant 0, \tag{4.1}$$

where the function $f(y,u)\colon \mathbb{R}^m \times U \mapsto \mathbb{R}^m$ is continuous in (y,u) and satisfies the Lipschitz condition in y uniformly with respect to u. The controls are Lebesgue measurable functions $u(\cdot)\colon [0,\infty) \mapsto U$ where U is a compact metric space. The set of these controls is denoted as $\mathcal{U}$.

Definition 4.1. *A pair* $(y(\cdot), u(\cdot))$ *will be called admissible if equation (4.1) is satisfied for almost all* t *and if the following inclusions are valid:*

$$y(t) \in Y, \ t \in [0,\infty)$$

and

$$u(t) \in U, \ \textit{for almost all } t,$$

where Y *is a given compact subset of* $\mathbb{R}^m$.

The cost function of our discounted optimal control problem is defined as

$$J(y_0, u(\cdot)) \overset{\text{def}}{=} \int_0^\infty e^{-Ct} g(y(t), u(t)) dt, \tag{4.2}$$

where the function $g(y,u)\colon Y \times U \mapsto \mathbb{R}$ is a continuous function we call the running cost and the parameter $C > 0$ is referred to as the discount rate.

The optimal value function of the discounted optimal control problem is defined as

$$V(y_0) \overset{\text{def}}{=} \inf_{(y(\cdot),u(\cdot))} J(y_0, u(\cdot)),$$

where the minimisation is over all admissible pairs that satisfy the initial conditions

$$y(0) = y_0.$$

For a given initial value, an admissible control $u^*(\cdot) \in \mathcal{U}$ is called an optimal control if $J(y_0, u^*(\cdot)) = V(y_0)$ holds.

Definition 4.2. *A point* $\bar{y} \in Y$ *is called an equilibrium point of the control system (4.1) if there exists a control* $\bar{u} \in U$ *such that* $f(\bar{y}, \bar{u}) = 0$*. We shall call* $\bar{u}$ *an equilibrium control.*

Definition 4.3. *An equilibrium point* $\bar{y}$ *is said to be stable if any solution* $y(t)$ *of (4.1) with the initial condition* $y(t_0) = y_0$*, with* y_0 *"close" to* $\bar{y}$*, remains in the neighbourhood of* $\bar{y}$ *for all* $t \geqslant t_0$*. That is, for each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$\|y_0 - \bar{y}\| \leqslant \delta \;\Rightarrow\; \|y(t) - \bar{y}\| \leqslant \varepsilon \text{ for all } t \geqslant 0$$

holds. We further remark that, an equilibrium point $\bar{y}$ *of (4.1) is said to be unstable if it is not stable.*

Definition 4.4. *An equilibrium point* $\bar{y}$ *of (4.1) is said to be asymptotically stable if it is stable and if any solution* $y(t)$ *which begins in this neighbourhood of* $\bar{y}$ *converges to* $\bar{y}$*. That is,*

$$\lim_{t \to \infty} y(t) = \bar{y}.$$

Note that the concept of stability requires only that small perturbations of the equilibrium yield a solution that remains close to the equilibrium, whereas asymptotic stability requires that the solution eventually (in infinite time) returns to the equilibrium.

Our goal is to design the running cost g in (4.2) in such a way that a desired equilibrium $\bar{y}$ is asymptotically stable for optimal trajectories.

Remark 4.5. *In the literature, the term asymptotic stability is more commonly used for systems controlled by a feedback control* $u(t) = F(y(t))$*. Here we use it in a more general sense also for time dependent control functions which, of course, may be generated by a feedback law.*

To achieve asymptotic stability, we impose the following structure on g.

Assumption 4.6. *Given an equilibrium point* $\bar{y} \in Y$ *and the equilibrium control* $\bar{u} \in U$*, the running cost* $g \colon Y \times U \to \mathbb{R}$ *satisfies*

(i) $g(y, u) > 0$ *for* $y \neq \bar{y}$ *and*

(ii) $g(\bar{y}, \bar{u}) = 0$*.*

This assumption states that g penalises deviations of the state y from the desired state $\bar{y}$ and the expectation is that this forces the optimal solution which minimises the integral over g to converge to $\bar{y}$. A typical simple choice of g satisfying this assumption is the quadratic penalisation

$$g(y, u) = \|y - \bar{y}\|^2 + \lambda\|u - \bar{u}\|^2 \tag{4.3}$$

with $\lambda \geqslant 0$.

It is well known that undiscounted optimal control can be used in order to enforce asymptotic stability of the optimally controlled system. Prominent approaches using this fact are the linear quadratic optimal controller or model predictive control (MPC). In the latter, the infinite horizon (undiscounted) optimal control problem is replaced by a sequence of finite horizon optimal control problems. Unless stabilising terminal constraints or costs are used, this approach is known to work whenever the optimisation horizon of the finite horizon problems is sufficiently large, cf. e.g., [31, 40, 50] or [33, Chapter 6]. The idea of using discounted optimal control for stabilisation bears some similarities with this finite horizon approach, as in discounted optimal control the far future contributes very weakly to the value of the functional J in (4.2), i.e., the effective optimisation horizon is also finite. It thus comes as no surprise that the conditions we are going to use in order to deduce stability are similar to conditions which can be found in the MPC literature. More precisely, we will use the following assumption on the optimal value function.

Assumption 4.7. *There exists* $K > C$ *such that*

$$KV(y) \leqslant g(y, u). \tag{4.4}$$

holds for all $y \in Y$ *and* $u \in U$.

This assumption in fact involves two conditions. Firstly, the inequality (4.4) expresses that the optimal value function can be bounded from above by the running cost (a similar condition is used in the MPC literature [see 32, 34, 35, 40, 58]). Secondly, the constant K in the left-hand-side of (4.4) should be greater than the discount rate C.

4.2 STABILITY RESULTS

In this section we are going to derive the stability results. Everywhere in what follows it is assumed that the optimal control exists. The optimal control and the corresponding solution of (4.1) are denoted as $u^*(\cdot)$ and $y^*(\cdot)$ respectively. Note that, due to the dynamic programming principle, we have

$$V(y_0) = \int_0^t e^{-Cs} g(y^*(s), u^*(s)) ds + e^{-Ct} V(y^*(t))$$

implying

$$V(y^*(t)) = e^{Ct} V(y_0) - e^{Ct} \int_0^t e^{-Cs} g(y^*(s), u^*(s)) ds.$$

Proposition 4.8. *If Assumption 4.7 is satisfied, then following inequality is valid*

$$V(y^*(t)) \leqslant e^{-(K-C)t} V(y_0), \; \forall t \geqslant 0. \tag{4.5}$$

Proof of Proposition 4.8. Since the map $t \mapsto V(y^*(t))$ is absolutely continuous, we can differentiate $V(y^*(t))$ for almost all t [45, Chap. IX, Section 2, Corollary to Theorem 1] and under Assumption 4.7 we obtain

$$\begin{aligned} \frac{d}{dt} V(y^*(t)) &= Ce^{Ct} V(y_0) - Ce^{Ct} \int_0^t e^{-Cs} g(y^*(s), u^*(s)) ds - g(y^*(t), u^*(t)) \\ &= CV(y^*(t)) - g(y^*(t), u^*(t)) \\ &\leqslant -(K - C) V(y^*(t)). \end{aligned}$$

The Gronwall-Bellman inequality implies (4.5) □

By Proposition 4.8, $V(y^*(t))$ tends to 0 as $t \to \infty$. Below, we show that Assumption 4.7, implies asymptotic stability.

Recall that a function $\alpha \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, unbounded and satisfies $\alpha(0) = 0$. Note that if $\alpha(\cdot) \in \mathcal{K}_\infty$, then the inverse function $\alpha^{-1}(\cdot) \in \mathcal{K}_\infty$ as well. From this point on we shall assume g is presented in the form of the quadratic penalisation (4.3).

Lemma 4.9. *If Assumption 4.7 is satisfied, then there exist, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that the inequality*

$$\alpha_1(\|y - \bar{y}\|) \leqslant V(y) \leqslant \alpha_2(\|y - \bar{y}\|)$$

holds for all $y \in Y$.

Proof of Lemma 4.9. We note that the upper bound in Lemma 4.9 is immediate from Assumption 4.7 as the inequality is satisfied for $y \mapsto \inf_{u \in U} g(y, u)$, where we have

$$KV(y) \leqslant \inf_{u \in U} g(y,u) \leqslant g(y,u).$$

Due to (4.3), this implies

$$V(y) \leqslant \frac{\|y - \bar{y}\|^2}{K}.$$

That is, the right-hand-size inequality in Lemma 4.9 is proved with

$$\alpha_2(r) = r^2/K.$$

Regarding the lower bound, we show existence in the case when f is bounded on $Y \times U$, we first observe that

$$J(y, u(\cdot)) \geqslant \int_0^\infty e^{-Ct}\|y(t) - \bar{y}\|^2 dt.$$

Since the solution satisfies $y(t) = y_0 + \int_0^t f(y(s), u(s))ds$, we obtain

$$\|y(t) - \bar{y}\| \geqslant \|y_0 - \bar{y}\| - \int_0^t \|f(y(s), u(s))\| ds \geqslant \|y_0 - \bar{y}\| - Mt$$

for $M \overset{\text{def}}{=} \sup_{(y,u) \in Y \times U} \|f(y, u)\|$.

Choosing $\tau = \min\{\|y_0 - \bar{y}\|/(2M), 1\}$, we have two cases to consider:

(i) For $\|y_0 - \bar{y}\|/(2M) \geqslant 1$, we could take $\tau = 1$ however the inequality is preserved if we use $\tau = \|y_0 - \bar{y}\|/(2M)$ and obtain

$$\|y(t) - \bar{y}\| \geqslant \|y_0 - \bar{y}\| - M\|y_0 - \bar{y}\|/(2M), \ \forall t \in [0, \tau].$$

Which simplifies to

$$\|y(t) - \bar{y}\| \geqslant \frac{1}{2}\|y_0 - \bar{y}\|, \ \forall t \in [0, \tau].$$

(ii) For $\|y_0 - \bar{y}\|/(2M) < 1$, we take $\tau = \|y_0 - \bar{y}\|/(2M)$ and obtain a result identical to (i) i.e.,

$$\|y(t) - \bar{y}\| \geqslant \frac{1}{2}\|y_0 - \bar{y}\|, \ \forall t \in [0, \tau].$$

Together this yields

$$J(y,u(\cdot)) \geqslant \int_0^\tau e^{-Ct}\|y(t)-\bar{y}\|^2 dt \geqslant e^{-C\tau}\int_0^\tau \frac{1}{4}\|y_0-\bar{y}\|^2 dt.$$

The expression is simplified by taking $\tau = 1$ in the $e^{-C\tau}$ term and using $\tau = \min\{\|y_0-\bar{y}\|/(2M), 1\}$ elsewhere to obtain

$$J(y,u(\cdot)) \geqslant e^{-C}\frac{1}{4}\|y_0-\bar{y}\|^2 \min\{\|y_0-\bar{y}\|/(2M), 1\}.$$

By Assumption 4.7 we have $K > C$ from which we further obtain

$$J(y,u(\cdot)) \geqslant e^{-K}\frac{1}{4}\|y_0-\bar{y}\|^2 \min\{\|y_0-\bar{y}\|/(2M), 1\}.$$

That is, the left-hand-side inequality in Lemma 4.9 is proved with

$$\alpha_1(r) = e^{-K}\min\{r^3/(8M), r^2/4\}.$$

□

Proposition 4.10. *If Assumption 4.7 holds, then the point $\bar{y}$ is asymptotically stable for optimal trajectories $y^*(\cdot)$.*

Proof of Proposition 4.10. Convergence $y^*(t) \to \bar{y}$ follows immediately from the fact that $V(y^*(t)) \to 0$ and $\|y^*(t)-\bar{y}\| \leqslant \alpha_1^{-1}(V(y^*(t)))$, noting that the inverse function of a $\mathcal{K}_\infty$ function is again a $\mathcal{K}_\infty$ function.

In order to prove stability, let $\varepsilon > 0$. For all $t \geqslant 0$ we have

$$\begin{aligned}\|y^*(t)-\bar{y}\| &\leqslant \alpha_1^{-1}(V(y^*(t))) \leqslant \alpha_1^{-1}(V(y_0)) \\ &\leqslant \alpha_1^{-1}(\alpha_2(\|y_0-\bar{y}\|)).\end{aligned}$$

Thus, for $\|y_0-\bar{y}\| \leqslant \delta = \alpha_2^{-1}(\alpha_1(\varepsilon))$ we obtain $\|y^*(t)-\bar{y}\| \leqslant \varepsilon$ and thus the desired stability estimate. □

4.3 CONTROLLABILITY CONDITIONS

In this section we give sufficient controllability conditions under which Assumption 4.4 holds.

4.3.1 *Finite time controllability*

Assumption 4.11. *Let there exist $\beta > 0$ such that for any initial condition $y(0) = y_0 \in Y$ there exists an admissible control $\hat{u}(\cdot) \in \mathcal{U}$ which will drive our system from y_0 to $\bar{y}$ in time $t(y_0) \leqslant \beta \|y_0 - \bar{y}\|^2$.*

Proposition 4.12. *Under Assumption 4.11, the optimal value function for g from (4.3) with any $\lambda \geqslant 0$ satisfies Assumption 4.7 for all $0 < C < \frac{1}{(1+\lambda)M\beta}$, where $M = \max_{(y,u)\in Y\times U}\{\|y-\bar{y}\|^2 + \|u-\bar{u}\|^2\}$.*

Proof of Proposition 4.12. Let $\hat{y}(\cdot)$ denote the solution corresponding to $\hat{u}(t)$ starting in y_0. Since $(\hat{y}(t), \hat{u}(t)) \in Y \times U$, $\|\hat{y}(t) - \bar{y}\|^2 \leqslant M$ and $\|\hat{u}(t) - \bar{u}\|^2 \leqslant M$. We have

$$
\begin{aligned}
V(y) &\leqslant \int_0^{t(y)} e^{-C\tau}(\|\hat{y}(\tau) - \bar{y}\|^2 + \lambda\|\hat{u}(\tau) - \bar{u}\|^2)d\tau \\
&\leqslant (1+\lambda)M \int_0^{t(y)} e^{-C\tau} d\tau.
\end{aligned}
$$

Applying the inequality $1 - e^{-x} \leqslant x$ we obtain

$$
\begin{aligned}
V(y) &\leqslant \frac{(1+\lambda)M}{C}(1 - e^{-Ct(y)}) \leqslant (1+\lambda)Mt(y) \\
&\leqslant (1+\lambda)M\beta\|(y - \bar{y}\|^2 \leqslant (1+\lambda)M\beta g(y,u),
\end{aligned}
$$

which implies (4.4) with $K = \frac{1}{(1+\lambda)M\beta}$. For Assumption 4.7 to be satisfied, we need $K > C$. Hence the assumption holds whenever $C < \frac{1}{(1+\lambda)M\beta}$. □

4.3.2 *Exponential controllability*

Assumption 4.13.

(i) There are constants $\delta > 0$ and $M \geqslant 1$ such that for any initial condition $y(0) = y_0 \in Y$ there exists an admissible control $\hat{u}(\cdot) \in \mathcal{U}$ such that the corresponding solution $\hat{y}(\cdot)$ of (4.1) satisfies

$$
\|\hat{y}(t) - \bar{y}\| \leqslant M e^{-\delta t}\|y_0 - \bar{y}\|,
$$

i.e. the control drives the system from y_0 to $\bar{y}$ exponentially fast.

(ii) The control function from (i) satisfies the inequality

$$\|\hat{u}(t) - \bar{u}\| \leqslant M e^{-\delta t} \|y_0 - \bar{y}\|$$

with $\delta > 0$ *and* $M \geqslant 1$ *from (i).*

Proposition 4.14. *Under Assumption 4.13(i), the optimal value function for* g *with* $\lambda = 0$ *satisfies Assumption 4.7 for all* $0 < C < \frac{2\delta}{(M^2-1)}$. *If, in addition, Assumption 4.13(ii) holds, then Assumption 4.7 also holds for any* $\lambda > 0$ *for all* $0 < C < \frac{2\delta}{((1+\lambda)M^2-1)}$.

Proof of Proposition 4.14. For $\lambda = 0$ we have

$$\begin{aligned} V(y) &\leqslant \int_0^\infty e^{-C\tau} \|\hat{y}(\tau) - \bar{y}\|^2 d\tau \\ &\leqslant \int_0^\infty e^{-C\tau} M^2 e^{-2\delta\tau} \|y_0 - \bar{y}\|^2 d\tau \\ &= \frac{M^2}{C + 2\delta} \|y_0 - \bar{y}\|^2 \\ &\leqslant \frac{M^2}{C + 2\delta} g(y, u). \end{aligned}$$

If Assumption 4.13(ii) holds, for any $\lambda > 0$ we have

$$\begin{aligned} V(y) &\leqslant \int_0^\infty e^{-C\tau} (\|\hat{y}(\tau) - \bar{y}\|^2 + \lambda \|\hat{u}(\tau) - \bar{u}\|^2) d\tau \\ &\leqslant \int_0^\infty e^{-C\tau} M^2 e^{-2\delta\tau} (1 + \lambda) \|y_0 - \hat{y}\|^2 d\tau \\ &= \frac{(1+\lambda)M^2}{C + 2\delta} \|y_0 - \bar{y}\|^2 \\ &\leqslant \frac{(1+\lambda)M^2}{C + 2\delta} g(y, u). \end{aligned}$$

Thus, in both cases we obtain (4.4) with

$$K = \frac{C + 2\delta}{(1+\lambda)M^2}.$$

For Assumption 4.7 to be satisfied, we again need $K > C$ which holds if

$$\frac{C + 2\delta}{(1+\lambda)M^2} > C.$$

Which we rearrange to find

$$C < \frac{2\delta}{(1+\lambda)M^2 - 1}.$$

□

Note in conclusion that an algorithm and code for the solution of two-dimensional linear programs is presented in Chapter 5. The algorithm is then applied to solve a nonlinear discounted optimal control problem. In particular, in Chapters 6 and 7, we apply the linear programming framework to solve problems which stabilise such systems to both a point and a closed curve.

Part II

NUMERICAL EXPERIMENTS

5

DAMPED MASS-SPRING SYSTEM

In this chapter we shall demonstrate how to solve a discounted optimal control problem using the linear programming method outlined in Chapters 1, 2 and 3.

In Section 5.1, a problem of optimal control of a damped mass-spring system is introduced (the results of this chapter can be compared with the results of a similar problem solved in [26] using a long run average optimality criteria). In Section 5.2, we outline a general framework for numerical solutions of discounted optimal control problems with two state variables and one control variable. In Section 5.3, the optimal control problem is solved numerically and the results for two values of the discount factor C are explicitly presented. Finally, in Section 5.4 a working example of the linear programming method is presented. The example is written for MATLAB and CPLEX users who may wish to replicate these results or undertake their own research.

5.1 PROBLEM FORMULATION

Consider a controlled damped mass-spring system

$$\begin{aligned} {y_1}'(t) &= y_2(t), \\ {y_2}'(t) &= -\omega^2 y_1(t) - k y_2(t) + u(t), \\ y(0) &= y_0, \end{aligned} \tag{5.1}$$

with the control restricted by the inequality $|u(t)| \leqslant 1$.

Let $g(y, u) = \beta u^2 - y_1^2$ and thus the objective function takes the form

$$J(y_0, u(\cdot)) = \int_0^\infty e^{-Ct}(\beta u(t)^2 - y_1(t)^2)dt. \tag{5.2}$$

We consider the optimal control problem

$$V(y_0) \stackrel{\text{def}}{=} \inf_{(y(\cdot),u(\cdot))} \int_0^\infty e^{-Ct}(\beta u(t)^2 - y_1(t)^2)dt,$$

where the minimisation of is over the set of admissible pairs $(y(\cdot), u(\cdot))$ satisfying the system of equations 5.1.

For our numerical experiments, we take the constants $k = 0.3$, $\omega = 2$ and $\beta = 1$, then define the following entities,

$$\begin{aligned} y &\stackrel{\text{def}}{=} (y_1, y_2), \\ f(y,u) &\stackrel{\text{def}}{=} \begin{bmatrix} y_2 \\ -4y_1 - 0.3y_2 + u \end{bmatrix}, \\ g(y,u) &\stackrel{\text{def}}{=} u^2 - y_1^2 \text{ and } y_0 = (-4,-4), \end{aligned} \tag{5.3}$$

with

$$\begin{aligned} &u \in U = [-1,1] \subset \mathbb{R}^1, \\ &y = (y_1,y_2) \in Y = \{(y_1,y_2) \colon y_1 \in [-6,6], y_2 \in [-8,8]\} \subset \mathbb{R}^2 \end{aligned}$$

(Note that the set Y is chosen to be large enough to contain all the solutions to the system under consideration).

5.2 THE LINEAR PROGRAMMING FRAMEWORK

It is useful at this point to outline a framework for the construction of an $N\Delta$-approximating problem of two state variables and one control variable which we will use to solve problem (5.1).

The first step is to define the grid of $Y \times U$ by a suitable choice of intervals Δ_{y_1}, Δ_{y_2} and Δ_u with the equations

$$\begin{aligned} y_{1,i}^{\Delta} &\stackrel{\text{def}}{=} y_{1,\min} + i\Delta_{y_1}, \quad \forall i = 0,1,\ldots,(y_{1,\max} - y_{1,\min})/\Delta_{y_1}, \\ y_{2,j}^{\Delta} &\stackrel{\text{def}}{=} y_{2,\min} + j\Delta_{y_2}, \quad \forall j = 0,1,\ldots,(y_{2,\max} - y_{2,\min})/\Delta_{y_2}, \\ u_k^{\Delta} &\stackrel{\text{def}}{=} u_{\min} + k\Delta_u, \quad \forall k = 0,1,\ldots,(u_{\max} - u_{\min})/\Delta_u, \end{aligned} \tag{5.4}$$

where the Δ are chosen in such a way that $1/\Delta$ is an integer. The size of each Δ may differ to match the dynamics of the problem or to include (or exclude) a specific coordinate within the grid system.

Then, using the discretisation described above, the NΔ-approximating LP problem can written in the form

$$G^{N,\Delta}(y_0) \stackrel{\text{def}}{=} \min_{\gamma \in W^{N,\Delta}(y_0)} \sum_{i,j,k} g(y^{\Delta}_{1,i}, y^{\Delta}_{2,j}, u^{\Delta}_{k}) \gamma_{i,j,k}, \tag{5.5}$$

where we let $\gamma^{N,\Delta} = \{\gamma_{i,j,k}\}$ stand for the solution of (5.5). The polyhedral set $W^{N,\Delta}(y_0)$ is defined by the equation

$$\begin{aligned} W^{N,\Delta}(y_0) \stackrel{\text{def}}{=} \Big\{ \gamma = \{\gamma_{i,j,k}\} \geqslant 0 \colon \sum_{i,j,k} \gamma_{i,j,k} = 1, \\ \sum_{i,j,k} (\phi'_{i_1,i_2}(y^{\Delta}_{1,i}, y^{\Delta}_{2,j})^T f(y^{\Delta}_{1,i}, y^{\Delta}_{2,j}, u^{\Delta}_{k}) \\ + C(\phi(y_0) - \phi(y^{\Delta}_{1,i}, y^{\Delta}_{2,j})))\gamma_{i,j,k} = 0,\ (i_1, i_2) \in I_K \Big\}, \end{aligned}$$

where $\sum_{i,j,k} = \sum_{l=1}^{L^\Delta} \sum_{k=1}^{K^\Delta}$ and the indexation of the components of $\gamma \in W^{N,\Delta}(y_0)$ corresponds to the indexation of the grid points. The functions $\phi_{i_1,i_2}(y_1, y_2)$ are the monomials $\phi_{i_1,i_2}(y_1, y_2) \stackrel{\text{def}}{=} y_1^{i_1} y_2^{i_2}$, $i_1, i_2 = 0, 1, \ldots, J$, $i_1 + i_2 > 0$. Note that the number N which characterises the NΔ-approximating problem is equal to $(J+1)^2 - 1$.

The problem dual to the NΔ-approximating problem (5.5) is of the form

$$\begin{aligned} \max_{(\mu, \lambda_{i_1,i_2})} \Big\{ \mu \colon \mu \leqslant g(y^{\Delta}_{1,i}, y^{\Delta}_{2,j}, u^{\Delta}_{k}) \\ + \sum_{(i_1,i_2) \in I_K} \lambda^{N,\Delta}_{i_1,i_2} (\phi'(y^{\Delta}_{1,i}, y^{\Delta}_{2,j})^T f(y^{\Delta}_{1,i}, y^{\Delta}_{2,j}, u^{\Delta}_{k}) \\ + C(\phi(y_0) - \phi(y^{\Delta}_{1,i}, y^{\Delta}_{2,j}))),\ \forall (y^{\Delta}_{1,i}, y^{\Delta}_{2,j}, u^{\Delta}_{k}) \Big\}. \end{aligned} \tag{5.6}$$

Let $\{\mu^{N,\Delta}, \lambda^{N,\Delta}_{i_1,i_2}\}$ stand for the solution of the problem (5.6) and define the equation

$$\psi^{N,\Delta}(y_1, y_2) \stackrel{\text{def}}{=} \sum_{(i_1,i_2) \in I_K} \lambda^{N,\Delta}_{i_1,i_2} \phi_{i_1,i_2}(y_1, y_2). \tag{5.7}$$

From the latter, we can construct the feedback control function $u^{N,\Delta}(y_1, y_2) : Y \mapsto U$ defined as

$$u^{N,\Delta}(y_1, y_2) \stackrel{\text{def}}{=} \operatorname*{argmin}_{u \in U} \big\{ g(y_1, y_2, u) + \psi^{N,\Delta\prime}(y_1, y_2)^T f(y_1, y_2, u) \big\}, \tag{5.8}$$

which is approximately optimal in the sense that the value of the objective function obtained with this control tends to the optimal one as $N \to \infty$ and $\Delta \to 0$.

5.3 THE NUMERICAL SOLUTION

For problem (5.2), the NΔ-approximating LP problem is as follows (see (5.5)),

$$G^{N,\Delta}(y_0) = \min_{\gamma \in W^{N,\Delta}(y_0)} \sum_{i,j,k} (u_k^2 - y_{1,i}^2)\gamma_{i,j,k} \tag{5.9}$$

where

$$\begin{aligned} W^{N,\Delta}(y_0) = \{\gamma = \{\gamma_{i,j,k}\} \geqslant 0 \colon & \sum_{i,j,k} \gamma_{i,j,k} = 1, \\ \sum_{i,j,k} (\frac{\partial(y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{1,i}} y_{2,j} + \frac{\partial(y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{2,j}} & (-4y_{1,i} - 0.3y_{2,j} + u_k) \\ & + C((-4)^{l_1+l_2} - y_{1,i}^{l_1} y_{2,j}^{l_2})))\gamma_{i,j,k} = 0\}. \end{aligned}$$

The problem dual to the FDLP problem is of the form (see (5.6)),

$$\begin{aligned} \max_{(\mu,\lambda_{l_1,l_2})} \{\mu \colon \mu \leqslant u_k^2 - y_{1,i}^2 + \\ \sum_{l_1,l_2} \lambda_{l_1,l_2}^{N,\Delta} (\frac{\partial(y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{1,i}} y_{2,j} + \frac{\partial(y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{2,j}} (-4y_{1,i} - 0.3y_{2,j} + u_k) \\ + C((-4)^{l_1+l_2} - y_{1,i}^{l_1} y_{2,j}^{l_2})), \ \forall (y_{1,i}, y_{2,j}, u_k)\}. \end{aligned} \tag{5.10}$$

It is a characteristic of the CPLEX solver that solutions $\lambda^{N,\Delta}$ for problem (5.10) are generated simultaneously with the primal results $\gamma^{N,\Delta}$. Using the coefficients $\lambda^{N,\Delta}$, we find $\psi^{N,\Delta}(y_1, y_2)$ as the expansion of (5.7) and then construct the control $u^{N,\Delta}(y_1, y_2)$ in accordance with (5.8). For this example we have

$$u^{N,\Delta}(y_1, y_2) = \operatorname*{argmin}_{u \in U} \{u^2 + \frac{\partial \psi^{N,\Delta}(y_1, y_2)}{\partial y_2} u\}.$$

Which is equivalent to

$$u^{N,\Delta}(y_1, y_2) = \begin{cases} a^{N,\Delta}(y_1, y_2), & \text{if} \quad -1 \leqslant a^{N,\Delta}(y_1, y_2) \leqslant 1, \\ -1, & \text{if} \quad a^{N,\Delta}(y_1, y_2) < -1, \\ 1, & \text{if} \quad a^{N,\Delta}(y_1, y_2) > 1, \end{cases}$$

where

$$a^{N,\Delta}(y_1, y_2) = -\frac{1}{2} \frac{\partial \psi^{N,\Delta}(y_1, y_2)}{\partial y_2}.$$

The problem of (5.9) was solved numerically with the CPLEX [39] solver. The discretisation parameters used are $\Delta_{y_1} = 0.01$, $\Delta_{y_2} = 0.01$, $\Delta_u = 0.05$ and $N = 49$. The results presented in this section were computed for the initial condition $y_0 = (-4, -4)$ and a range of discount factors $C = 0.01$ to $C = 1$.

Substituting the control $u^{N,\Delta}(y_1, y_2)$ into the system (5.3) and integrating with the MATLAB ode45 solver allows us to obtain the state trajectory $y^{N,\Delta}(t) = (y_1^{N,\Delta}(t), y_2^{N,\Delta}(t))$, the control $u^{N,\Delta}(y^{N,\Delta}(t))$ and a numerical estimate of the cost function $G_{NUM}(T)$, where

$$G_{NUM}(T) = C \int_0^T e^{-Ct}(u(y_1(t), y_2(t))^2 - y_1(t)^2)dt, \ T > 0.$$

T is chosen large enough that the integral from T to ∞ is sufficiently small. In our numerical experience, we choose $T = 10/C$. The optimal LP values $G^{N,\Delta}(y_0)$ and the numerical estimates of the optimal value function $G_{NUM}(T)$ are shown in Table 5.1.

Table 5.1: Approximately optimal values $G^{N,\Delta}(y_0)$ and $G_{NUM}(T)$ for different discount factors C.

C	T	$G^{N,\Delta}(y_0)$	$G_{NUM}(T)$
+0.0100	1000	−1.8243	−1.7688
+0.0200	500	−2.2270	−2.1723
+0.0500	200	−3.2139	−3.1734
+0.1000	100	−4.4204	−4.3896
+0.2000	50	−6.0063	−5.9878
+0.5000	20	−8.5134	−8.5041
+1.0000	10	−10.8989	−10.8896

To aid our discussion, we define the set

$$\Theta^{N,\Delta} \stackrel{\text{def}}{=} \{(y_l, u_k) \colon \gamma_{l,k}^{N,\Delta} > 0\}.$$

Each element of $\Theta^{N,\Delta}$ being associated with positive components of the LP solution vector $\gamma^{N,\Delta}$. From which, we also define the set

$$\mathcal{Y}^{N,\Delta} \stackrel{\text{def}}{=} \{y_l \colon (y_l, u_k) \in \Theta^{N,\Delta}\} = \{y_l \colon \sum_k \gamma_{l,k}^{N,\Delta} > 0\}, \tag{5.11}$$

which is a projection of the set $\Theta^{N,\Delta}$ onto the (y_1, y_2) plane. The elements of $\mathcal{Y}^{N,\Delta}$ and the state trajectory $y^{N,\Delta}(t)$ are shown in Figures 5.1 and 5.4 for $C = 0.01$ and $C = 1$ respectively.

The controls $u^{N,\Delta}(y^{N,\Delta}(t))$ for $C = 0.01$ and $C = 1$ are shown in Figures 5.2 and 5.5.

In Figure 5.3 the numerical convergence of the optimal value function $G_{NUM}(T)$ to the LP solution $G^{N,\Delta}(y_0)$ is demonstrated. The error $\varepsilon = |G^{N,\Delta}(y_0) - G_{NUM}(1000)|$ in Figure 5.3 is not unexpected and arises from the long integration period when C is small.

The fact that the state trajectory passes near the points $\mathcal{Y}^{N,\Delta}$ and, most importantly, the fact that the value of the objective function obtained via integration is the same (within the given proximity) as the optimal value of the finite dimensional problem indicate that the admissible solution found is a good approximation of the optimal one.

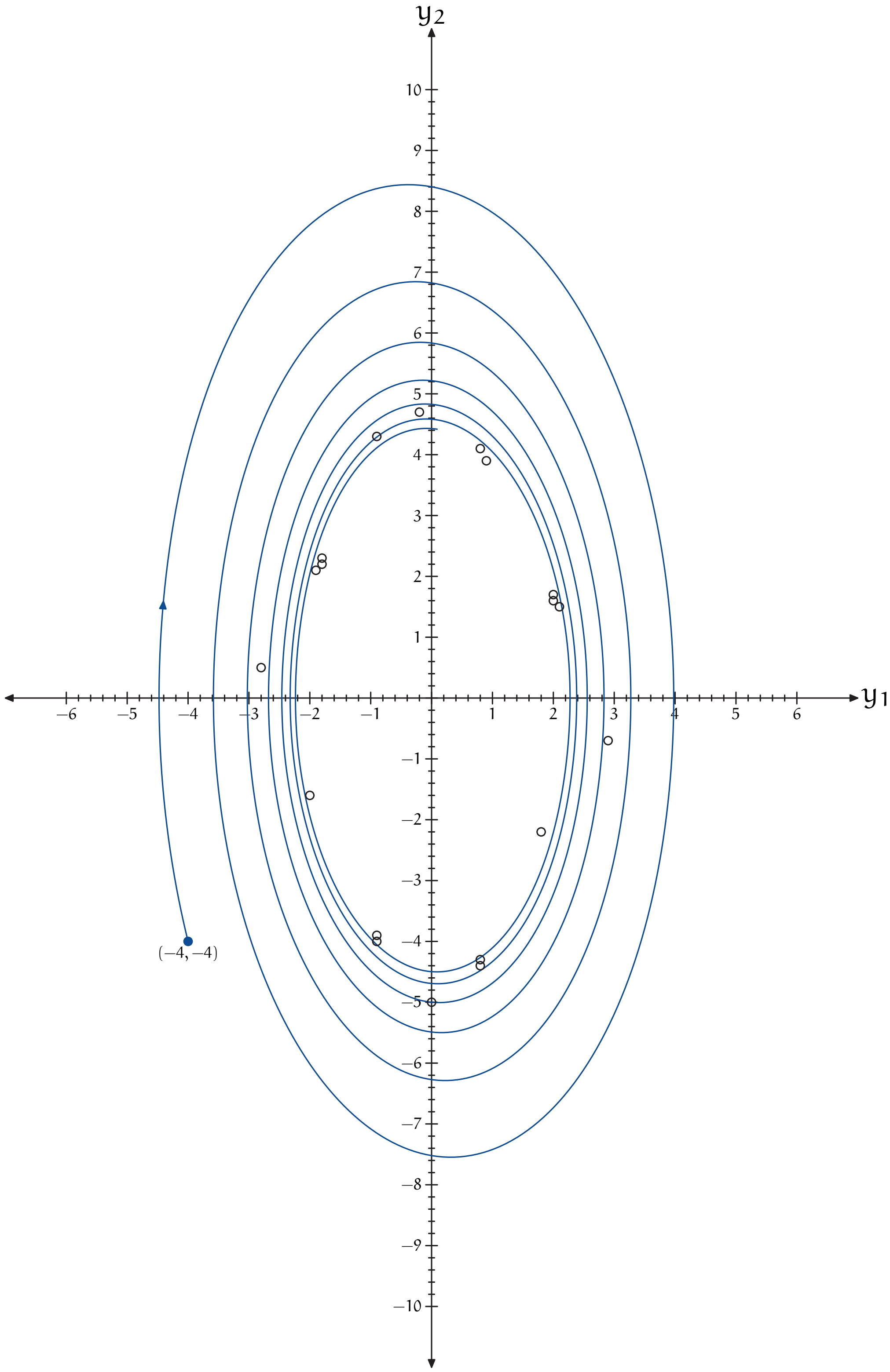


Figure 5.1: The approximate optimal trajectory $y^{N,\Delta}(t)$ of problem 5.2 for the discount factor $C = 0.01$. The points $y^{N,\Delta}$ associated with positive elements of the vector $\gamma^{N,\Delta}$ are indicated by the small black circles on this graph. Integrated for twenty seconds, the solution traces a spiral inwards from the initial condition $y_0 = (-4, -4)$ (shown as a solid blue dot).

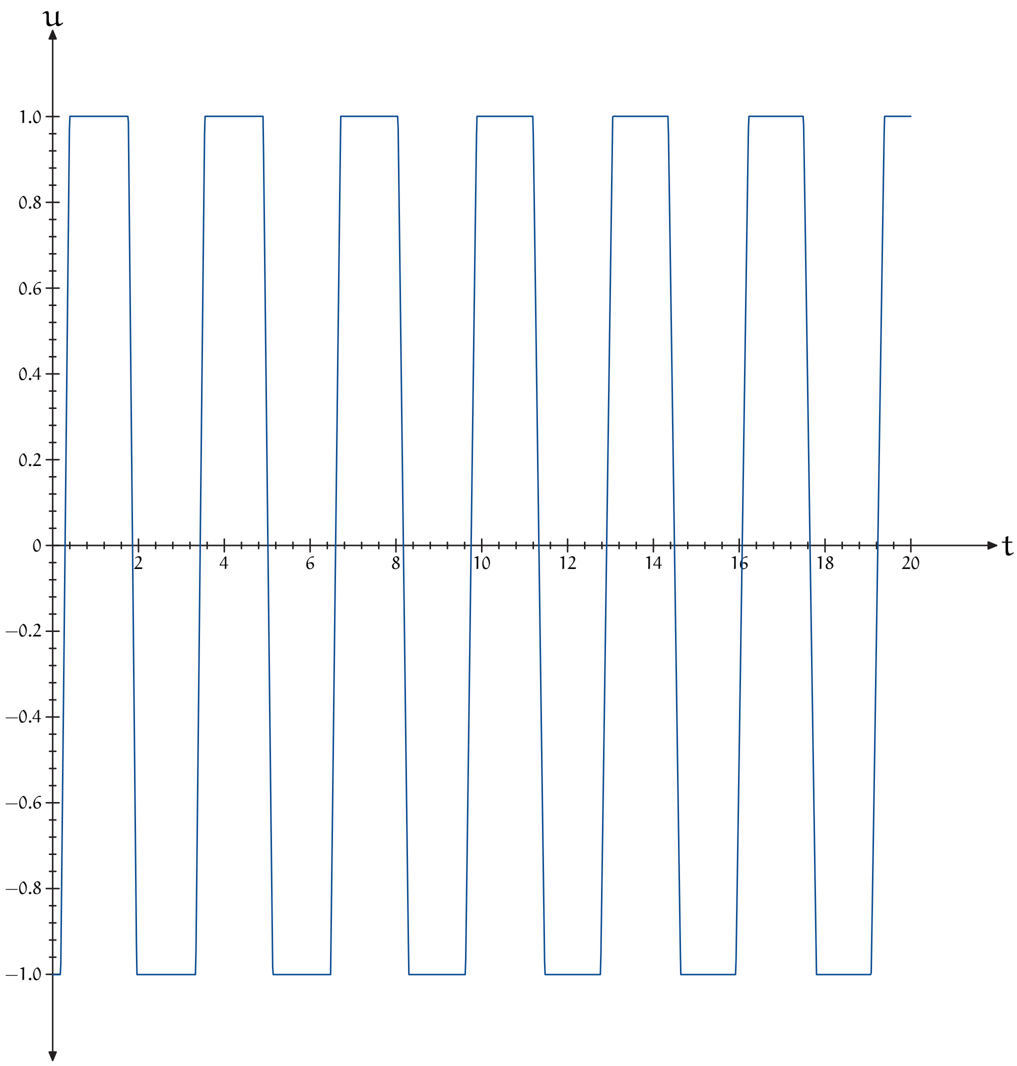


Figure 5.2: The approximate optimal control $u^{N,\Delta}(y^{N,\Delta}(t))$ of problem 5.2 for $C = 0.01$. The control is bound by the restriction $|u(t)| \leqslant 1$.

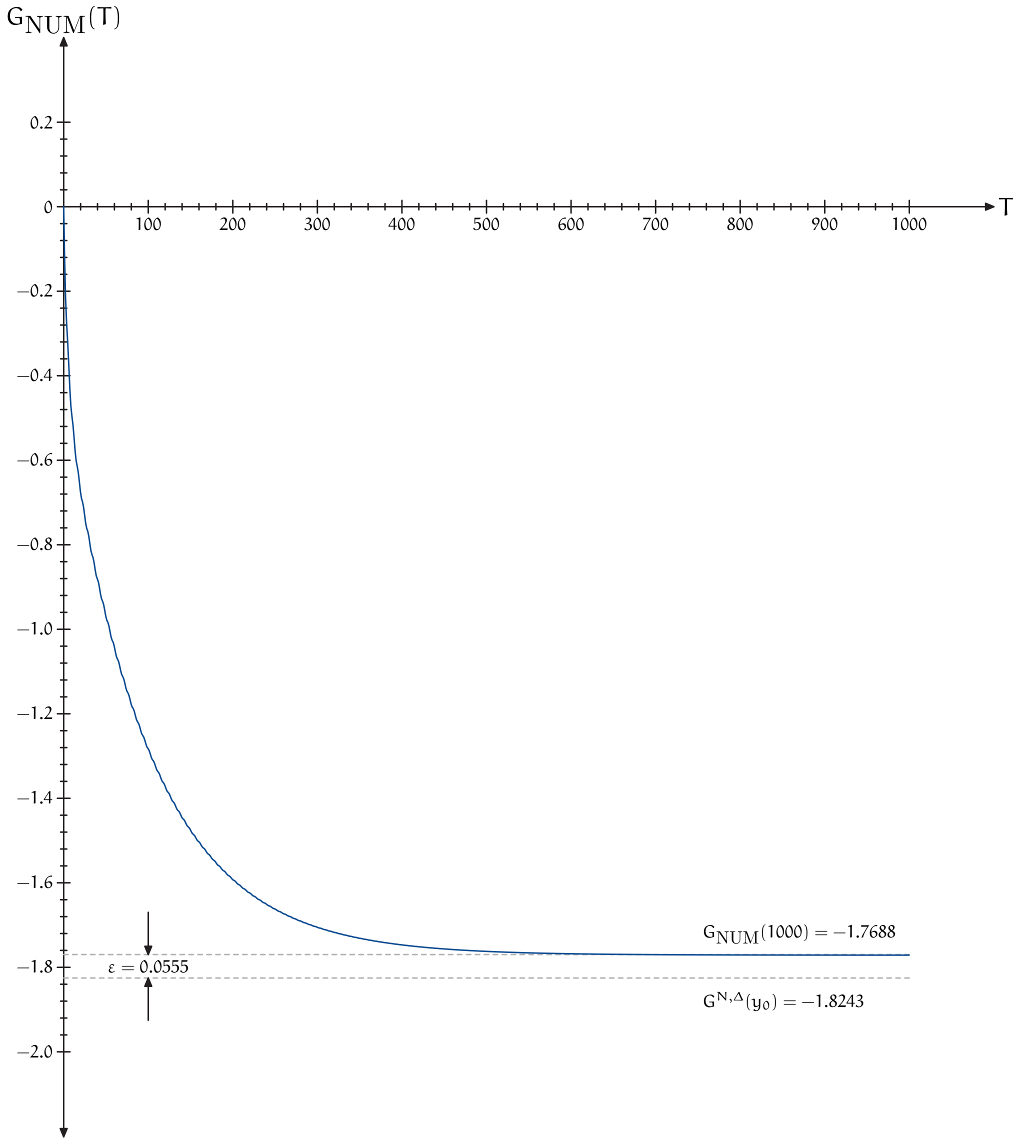


Figure 5.3: Numerical convergence of optimal value function $G_{NUM}(T)$ to the optimal LP value $G^{N,\Delta}(y_0)$ for $C = 0.01$.

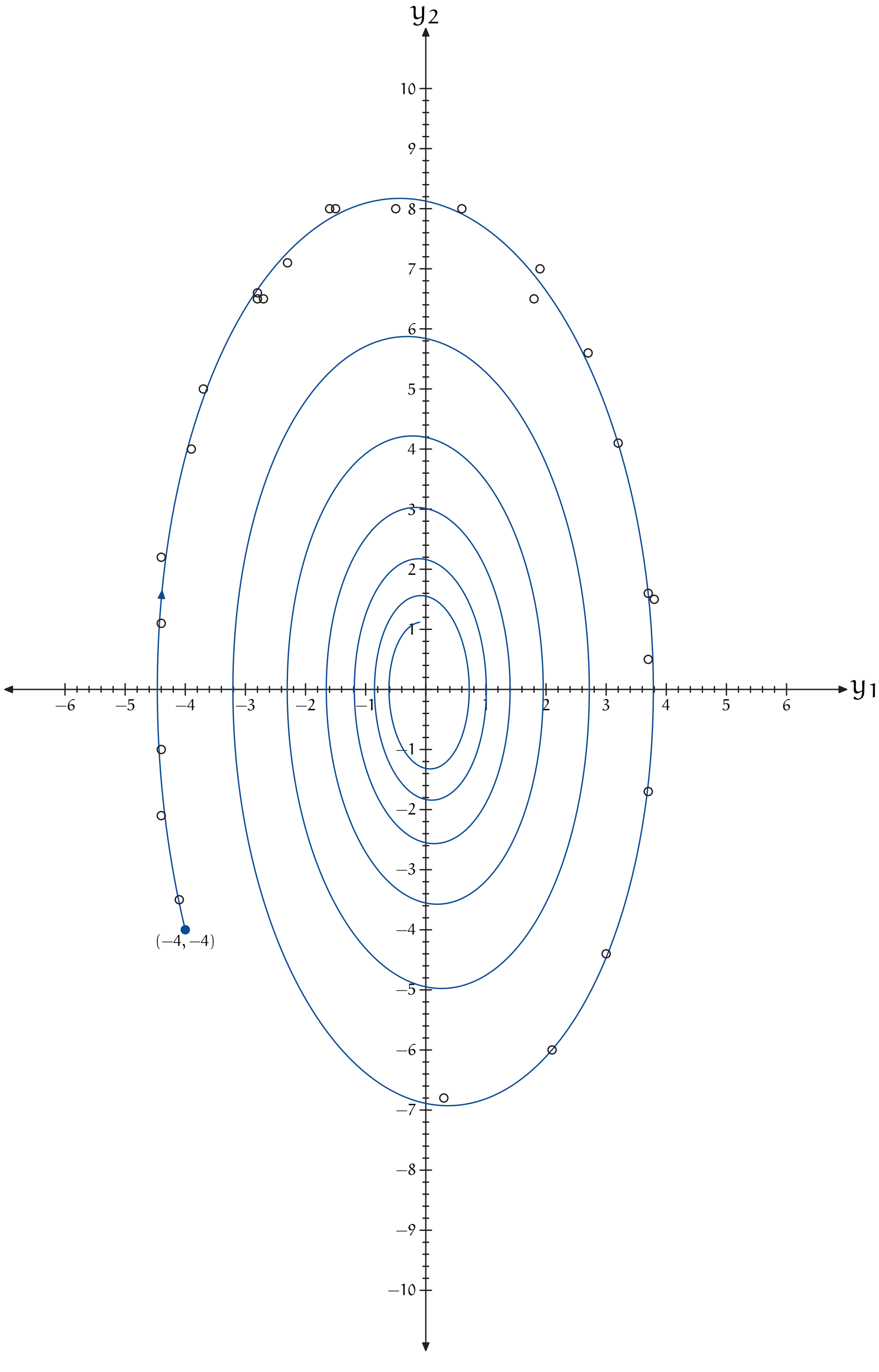


Figure 5.4: The approximate optimal trajectory $y^{N,\Delta}(t)$ of problem 5.2 for the discount factor $C = 1$. The points $y^{N,\Delta}$ associated with positive elements of the vector $\gamma^{N,\Delta}$ are indicated by the small black circles on this graph. Integrated for twenty seconds, the solution traces a spiral inwards from the initial condition $y_0 = (-4, -4)$ (shown as a solid blue dot).

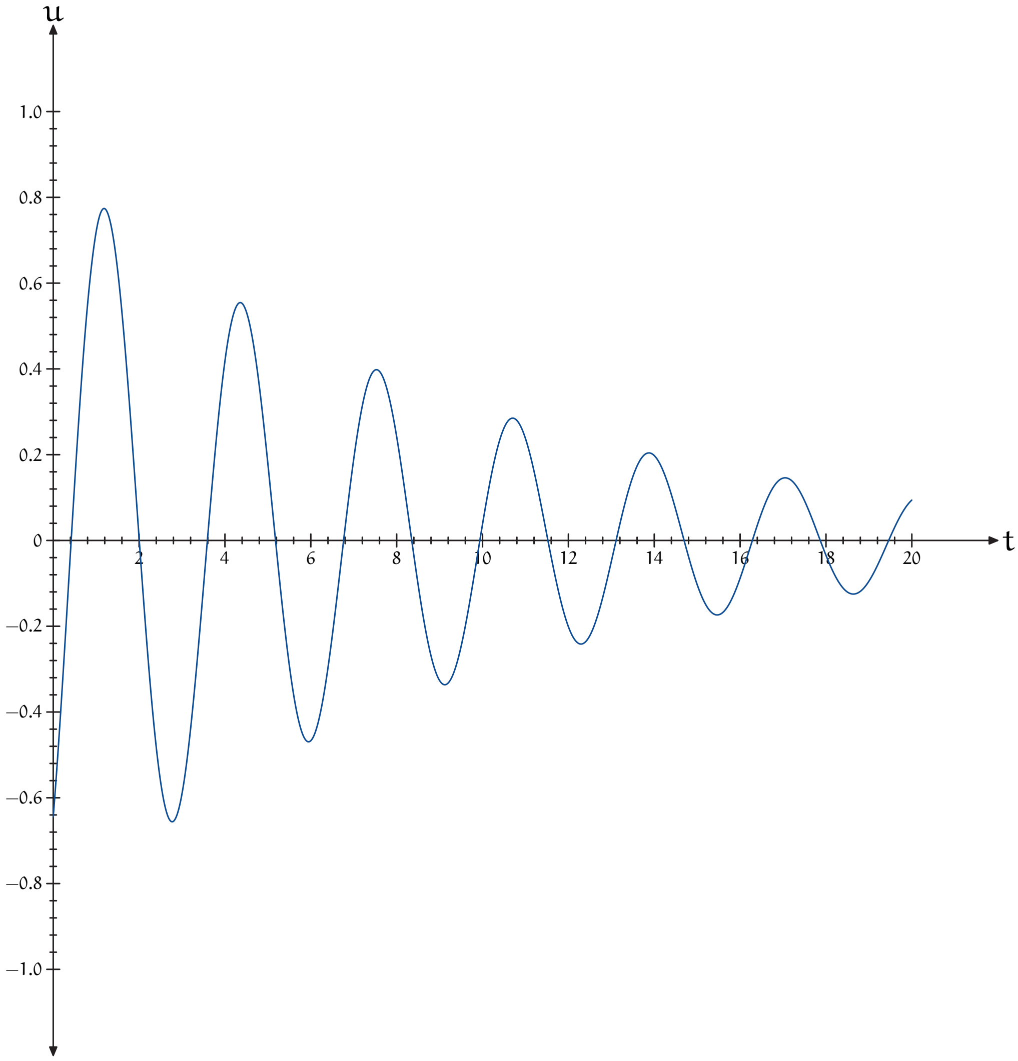


Figure 5.5: The approximate optimal control $u^{N,\Delta}(y^{N,\Delta}(t))$ of problem 5.2 for $C = 1$.

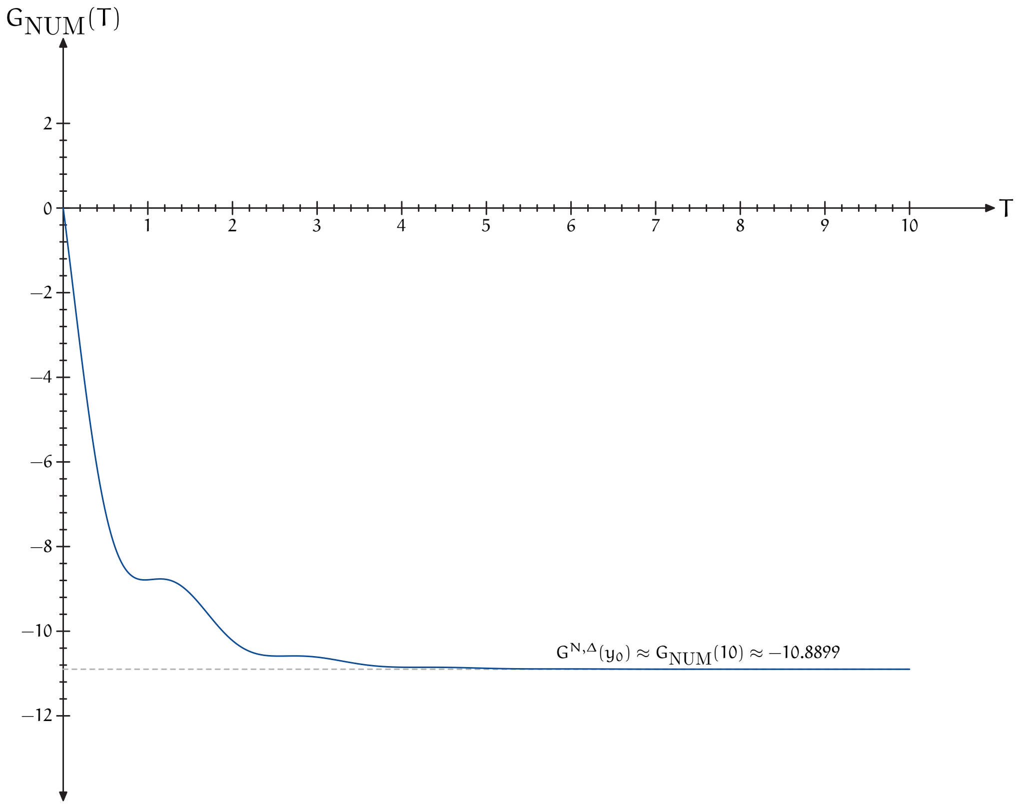


Figure 5.6: Numerical convergence of optimal value function $G_{NUM}(T)$ to optimal LP value $G^{N,\Delta}(y_0)$ for $C = 1$.

There are a number of observations we can make with respect to the examples shown. They are (a) for C small, the control $u^{N,\Delta}(y^{N,\Delta}(t))$ swings between the maximum and minimum values imposed by the restriction $|u(t)| \leqslant 1$ and there is a tendency for the trajectory $y^{N,\Delta}(t)$ to approach a limit cycle; (b) For C large the control $u^{N,\Delta}(y^{N,\Delta}(t))$ operates smoothly within $|u(t)| \leqslant 1$ and there is a tendency for the trajectory $y^{N,\Delta}(t)$ to approach $(0, 0)$.

5.4 A SAMPLE OF MATLAB CODE USING CPLEX

This chapter is concluded with a sample of MATLAB code which can be used to investigate the linear programming method as it is applied to discounted optimal control. The example requires an academic or professional installation of CPLEX. The student versions of CPLEX which are freely available will not support huge linear programs. The author has found that attempts to use the MATLAB `linprog` solver are thwarted by memory issues and other solvers such as GNU GLPK do not cope well with the size of the problems discussed here.

5.4.1 *Solving the* $N\Delta$*-approximating LP*

Listing 5.1 below, is a minimal example of the solver written in MATLAB. The program builds a vector of objective function coefficients (f in CPLEX notation) and a corresponding 3-tuple of y and u coordinates which will be useful for plotting the grid points which are associated with significant concentrations of occupation measure. The program then builds the matrix of constraint coefficients A_{eq} using monomials for the functions $\phi(\cdot)$. There are N rows of constraint coefficients plus an additional row of $1'$s to enforce the constraint $\sum \gamma_{i,j,k} = 1$. The right-hand-side column vector b_{eq} is all $0'$s excepting the last row associated with the $\sum \gamma_{i,j,k} = 1$ constraint which has a coefficient of 1. Finally, lower (zero) and upper bound (unity) vectors constructed.

Listing 5.1: MATLAB example (lpp.m)

```
% A minimal working example to reproduce the results of chapter 5.
% Requires a fully featured version of IBM ILOG CPLEX Optimization
% Studio. The variables containing the results are written to the
% MATLAB file - 'lpp-output.mat'. Tested on MATLAB 7.7.0 (R2008b).

% Parameters
pv = 7; cv = 1; y0 = [-4, -4];

% Intervals
IntU1 = 20; IntY1 = 120; IntY2 = 160;

% Spans
MinU1 = -1; MaxU1 = +1;
MinY1 = -6; MaxY1 = +6;
MinY2 = -8; MaxY2 = +8;

% Vectors for each dimension of the grid
fprintf(1, '\nStarting ..\n');
y1 = linspace(MinY1, MaxY1, IntY1 + 1);
y2 = linspace(MinY2, MaxY2, IntY2 + 1);
u1 = linspace(MinU1, MaxU1, IntU1 + 1);
sz = length(y1) * length(y2) * length(u1);

% Vector of point costs and (y1,y2,u1) tuples
fprintf(1, 'Building f vector\n');
col = 1;
f = zeros(1, sz);
z = zeros(3, sz);
for i = 1 : length(y1)
    for j = 1 : length(y2)
```

```
        for k = 1 : length(u1)
          f(:, col) = u1(k) * u1(k) - y1(i) * y1(i);
          z(:, col) = [y1(i); y2(j); u1(k)];
          col = col + 1;
        end;
    end;
end;

% Populating the Aeq array of coefficients
fprintf(1, 'Populating Aeq[] array\n');
Aeq = zeros((pv + 1) ^ 2, sz);
col = 1;
for i = 1 : length(y1)
  for j = 1 : length(y2)
    for k = 1 : length(u1)
      row = 1;
      for x = 0 : pv
        for y = 0 : pv
          if x + y > 0
            if x == 0
              d1 = 0;
            else
              d1 = x * (y1(i) ^ (x-1)) * y2(j) ^ y;
            end;
            if y == 0
              d2 = 0;
            else
              d2 = y * (y2(j) ^ (y-1)) * y1(i) ^ x;
            end;
            f0 = (y0(1) ^ x) * (y0(2) ^ y);
            fn = (y1(i) ^ x) * (y2(j) ^ y);
            Aeq(row, col) ...
               = d1 * y2(j) ...
               + d2 * (-4 * y1(i) - 0.3 * y2(j) + u1(k)) ...
               + cv * (f0 - fn);
            row = row + 1;
          end;
        end;
      end;
      Aeq(row, col) = 1;
      col = col + 1;
    end;
  end;
end;

% Populating the beq, lb and ub vectors
```

```
fprintf(1, 'Populating beq[], lb[] and ub[] vectors\n');
beq = zeros((pv + 1) ^ 2, 1); beq(end, :) = 1;
lb = zeros(1, sz); ub = zeros(1, sz) + 1;

% Memory usage
fprintf(1, '\nMemory summary\n');
memory

% Invoke the CPLEX solver
fprintf(1, '\nCalculating...\n');
[x, fval, exitflag, output, lambda] = ...
          cplexlp(f, [], [], Aeq, beq, lb, ub);

% Save workspace variables if successful
if exitflag > 0
  fprintf(1, '\nObjValue = %.10E\n', fval);
  save('lpp-output.mat', 'x', 'fval', 'exitflag', ...
       'output', 'lambda', 'pv', 'cv', 'y0', ...
       'MinU1', 'MaxU1', 'z');
else
  fprintf(1, '\nNo solution or error.\n');
end;

% end-of-file
```

5.4.2 *Find the approximate control and integrate*

A successful application of the "lpp.m" example (see Listing 5.1 above) will produce an output file called "lpp-output.m". The output file contains the primal and dual solutions and important properties of the problem such as the values of C and y_0 which we will need to construct an approximate control and numerically integrate the problem. The following MATLAB program "dlp.m" (see Listing 5.2) will load the output file and integrate the damped mass-spring problem from the initial condition for a period of $10/C$ seconds. The control function and the dynamics of the system are contained within the m-file "spring.m" (see Listing 5.3 below). The results are then summarised in a graph.

Listing 5.2: MATLAB example (dlp.m)

```
% This program 'dlp.m' loads the output file from 'lpp.m' and integrate
% the damped mass-spring problem from the initial condition for a period
% of 10/C seconds. The results are then summarised in a graph.
```

```
% Fetch pre-computed variables
s = load('lpp-output.mat');

% Use a 'rule of thumb' for integration times
T = 10/s.cv;

% Integrate from y0 for T seconds. Note that three variables 'y1',
% 'y2' and 'g' are being integrated. This requires us to provide the
% initial condition 'T=0' in addition to 'y0'.
[t, y] = ode45(@(t, y)spring(t, y, ...
         -s.lambda.eqlin(1 : (s.pv + 1)^2 - 1), ...
         s.pv, s.cv, [s.MinU1 s.MaxU1]), [0 T], [s.y0 0]);

% Compute the objective value
G = y(:, 3) * s.cv;
fprintf(1, '\nNumValue = %.10E\n', G(end));

% Plot the trajectory
figure;
plot(y(:, 1),y(:, 2));

% Plot the evolution of the objective function
figure;
plot(t, G);

% end-of-file
```

Listing 5.3: MATLAB example (spring.m)

```
%% This function 'spring.m' computes the control function and system
%% dynamics of the damped mass-spring problem for each time step of
%% the ODE45 solver. The parameters passed are 't' - the time step,
%% 'y' - the state variables, 'dv' - the dual coefficients, 'cv' -
%% the value of C and 'uv' - the restrictions on the control value.

function ret = spring(t, y, dv, pv, cv, uv)

% Compute a value for the derivative of the 'psi' function using
% monomials and dual coefficients.
ix = 1;
kv = zeros((pv + 1)^2 - 1, 2);
for i = 0 : pv
  for j = 0 : pv
    if i + j > 0
      kv(ix, 1) = i * (y(1) ^ (i - 1)) * y(2) ^ j;
      kv(ix, 2) = j * (y(2) ^ (j - 1)) * y(1) ^ i;
      ix = ix + 1;
```

```
    end
  end
end

% Compute the control value and apply the restrictions.
uhat = (1 / 2) * sum(kv(: , 2) .* dv);
u = min(max(uhat, uv(1)), uv(2));

% Advance the dynamics of the system by one time step.
s(1) = y(2);
s(2) = -4 * y(1) - 0.3 * y(2) + u;
s(3) = exp(-cv * t) * (u * u - y(1) * y(1));

ret = s';

% end-of-file
```

6

MODEL OF NECK AND DOCKNER

In this chapter, we apply the linear programming approach to the construction of an optimal control for the Neck and Dockner model proposed in [46, 47] and later studied by Feichtinger and Hartl [36].

In Section 6.1, the discounted optimal control problem is introduced. Then, in Section 6.2 the optimal control problem is solved numerically and the results for various values of the discount rate C are presented. In Section 6.3 we consider a periodic optimisation problem. The optimal solutions of the discounted optimal control problem described in Section 6.2 converge to the optimal solution of this periodic optimisation problem if the discount factor C is small enough.

6.1 THE MODEL OF NECK AND DOCKNER

Following the consideration in [46] and [47], we consider the system

$$y_1{}'(t) = \alpha - \beta y_2(t), \; y_2{}'(t) = u(t), \; y(0) = y_0, \tag{6.1}$$

where $y_1(t)$ is interpreted as the rate of inflation expected by the public at time t, $y_2(t)$ represents the excess of the rate of unemployment over its minimum level, $u(t)$ is the rate of change of unemployment and α, β are known positive constant parameters.

Attached to this system is the objective function

$$J(y_0, u(\cdot)) \stackrel{\text{def}}{=} \int_0^\infty e^{-Ct}(\rho^2(t) + \hat{c}(t) + \frac{1}{2}u^2(t))dt,$$

where $\rho(t) = f_1 - a_1 y_2(t) + y_1(t)$ is the "actual rate of inflation" and $\hat{c}(t) = \ln(b_0 y_2(t) + 1) + b_1 y_2^4(t)$ is the "cost arising from a certain level of unemployment" [see 46, 47].

In the next section, we describe a numerical solution to the discounted optimal control problem

$$V(y_0) \stackrel{\text{def}}{=} \inf_{u(\cdot)\in\mathcal{U}\text{ admissible}} \int_0^\infty e^{-Ct}(f_1^2 + y_1^2(t) - 2a_1 y_1(t) y_2(t) + 2f_1 y_1(t) + \hat{c}(t) + \frac{1}{2}u^2(t))dt, \quad (6.2)$$

where the parameters in the problem are taken to be as follows

$$\alpha = 0.02,\ \beta = 0.5,\ a_1 = 1.5,\ f_1 = 0.02,\ b_0 = 1.5,\ b_1 = 1.0.$$

6.2 THE NUMERICAL SOLUTION

We follow the framework presented in Chapter (5.2).

The $N\Delta$-approximating LP problem can be written in the form (see (5.5))

$$G^{N,\Delta}(y_0) = \min_{\gamma\in W^{N,\Delta}(y_0)} \sum_{i,j,k} (0.0004 + y_{1,i}^2 - 3y_{1,i}y_{2,j} + \ln(1.5y_{2,j}+1) + 0.04y_{2,j}^4 + 0.5u_k^2)\gamma_{i,j,k},$$

where

$$W^{N,\Delta}(y_0) = \{\gamma = \{\gamma_{i,j,k}\} \geqslant 0 \colon \sum_{i,j,k} \gamma_{i,j,k} = 1,$$
$$\sum_{i,j,k} (\frac{\partial(y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{1,i}}(0.02 - 0.5y_{2,j}) + \frac{\partial(y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{2,j}} u_k) + C((0.04)^{l_1+l_2} - y_{1,i}^{l_1} y_{2,j}^{l_2})))\gamma_{i,j,k} = 0\}.$$

The problem dual to the FDLP problem is of the form (see (5.6)),

$$\max_{(\mu,\lambda_{l_1,l_2})} \{\mu \colon \mu \leqslant 0.0004 + y_{1,i}^2 - 3y_{1,i}y_{2,j} + \ln(1.5y_{2,j}+1) + y_{2,j}^4 + 0.04y_{1,i} + 0.5u_k^2 + \sum_{l_1,l_2} \lambda_{l_1,l_2}^{N,\Delta}(\frac{\partial(y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{1,i}}(0.02 - 0.5y_{2,j}) + \frac{\partial(y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{2,j}} u_k + C((0.04)^{l_1+l_2} - y_{1,i}^{l_1} y_{2,j}^{l_2})),\ \forall (y_{1,i}, y_{2,j}, u_k)\}.$$

Using the coefficients $\lambda^{N,\Delta}$, we find $\psi^{N,\Delta}(y_1, y_2)$ as the expansion of (5.7) and then construct the control $u^{N,\Delta}(y_1, y_2)$ in accordance with (5.8). For this example we have

$$u^{N,\Delta}(y_1, y_2) = \operatorname*{argmin}_{u \in U} \{0.5u^2 - \frac{\partial \psi^{N,\Delta}(y_1, y_2)}{\partial y_2} u\}.$$

Which is equivalent to

$$u^{N,\Delta}(y_1, y_2) = \begin{cases} a^{N,\Delta}(y_1, y_2), & \text{if } -0.0275 \leqslant a^{N,\Delta}(y_1, y_2) \leqslant +0.0275, \\ -0.0275, & \text{if } a^{N,\Delta}(y_1, y_2) < -0.0275, \\ +0.0275, & \text{if } a^{N,\Delta}(y_1, y_2) > +0.0275, \end{cases}$$

where

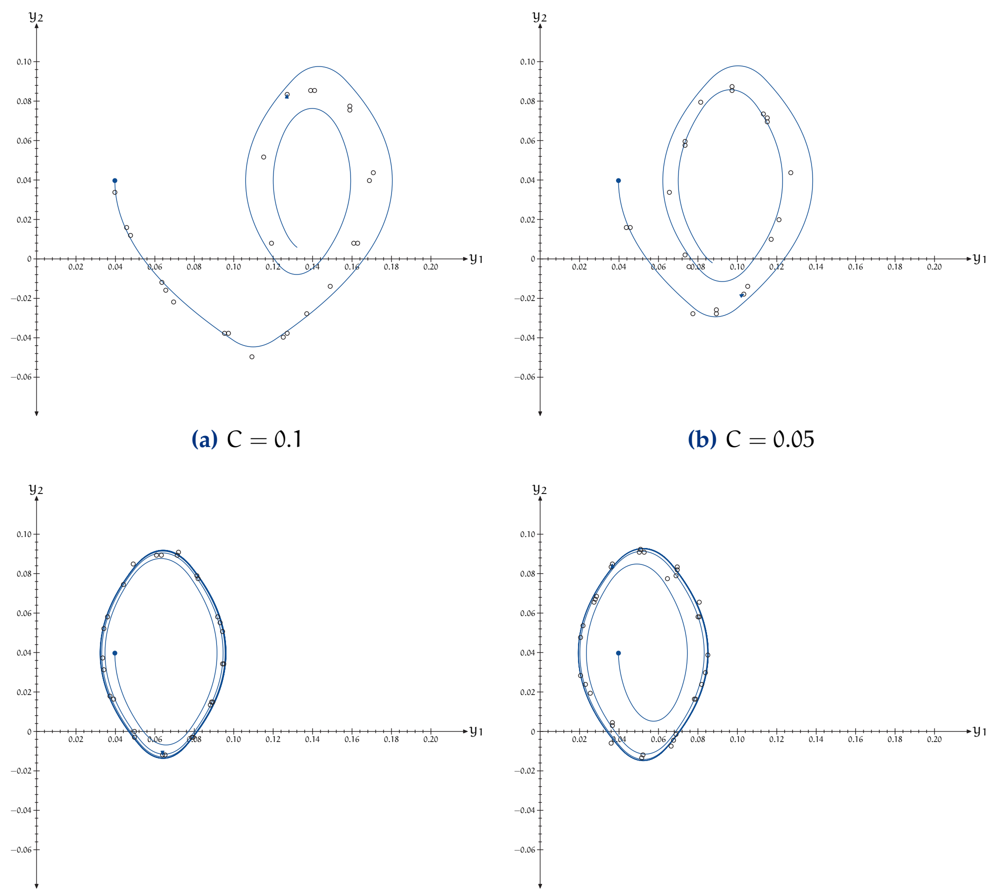
$$a^{N,\Delta}(y_1, y_2) = \frac{\partial \psi^{N,\Delta}(y_1, y_2)}{\partial y_2}.$$

The problem of (6.2) was solved numerically with the CPLEX [39] solver. The discretisation parameters used are $\Delta_{y_1} = 0.0002$, $\Delta_{y_2} = 0.002$, $\Delta_u = 0.001$ and $N = 49$. The results presented in this section were computed for the initial condition $y_0 = (0.04, 0.04)$ and a range of discount rates from $C = 0.001$ to $C = 1$.

Substituting the control $u^{N,\Delta}(y_1, y_2)$ into the system (6.1) and integrating with the MATLAB ode45 solver allows us to obtain the state trajectories $y^{N,\Delta}(t) = (y_1^{N,\Delta}(t), y_2^{N,\Delta}(t))$ as shown in Figure 6.1 and a numerical estimate of the cost function $G_{NUM}(T)$

$$G_{NUM}(T) = C \int_0^T e^{-Ct} (f_1^2 + y_1^2(t) - 2a_1 y_1(t) y_2(t) + 2f_1 y_1(t) + \hat{c}(t) + \frac{1}{2} u^2(t)) dt,$$

where $T = 10/C$.



(a) $C = 0.1$ **(b)** $C = 0.05$

(c) $C = 0.02$ **(d)** $C = 0.01$

Figure 6.1: The approximate optimal trajectory $y^{N,\Delta}(t)$ of problem 6.2 for decreasing values of discount rates $C = 0.1$, 0.05, 0.02 and 0.01. Each problem has the same initial condition $y_0 = (0.04, 0.04)$ (shown as a solid blue dot).

The optimal LP values $G^{N,\Delta}(y_0)$ and the numerical estimates of the cost function $G_{NUM}(T)$ are shown in Table 6.1.

Table 6.1: Approximately optimal values $G^{N,\Delta}(y_0)$ and $G_{NUM}(T)$ for different discount factors C.

C	T	$G^{N,\Delta}(y_0)$	$G_{NUM}(T)$
+0.0010	10000	+0.05680	+0.05681
+0.0050	2000	+0.05677	+0.05677
+0.0100	1000	+0.05666	+0.05666
+0.0200	500	+0.05620	+0.05620
+0.0500	200	+0.05351	+0.05352
+0.1000	100	+0.04549	+0.04554
+0.2000	50	+0.02905	+0.02909
+0.5000	20	+0.01291	+0.01288
+1.0000	10	+0.02482	+0.02480

Finally, the trajectory for $C = 0.001$ is shown in Figure 6.2. The approximately optimal control is shown in Figure 6.3.

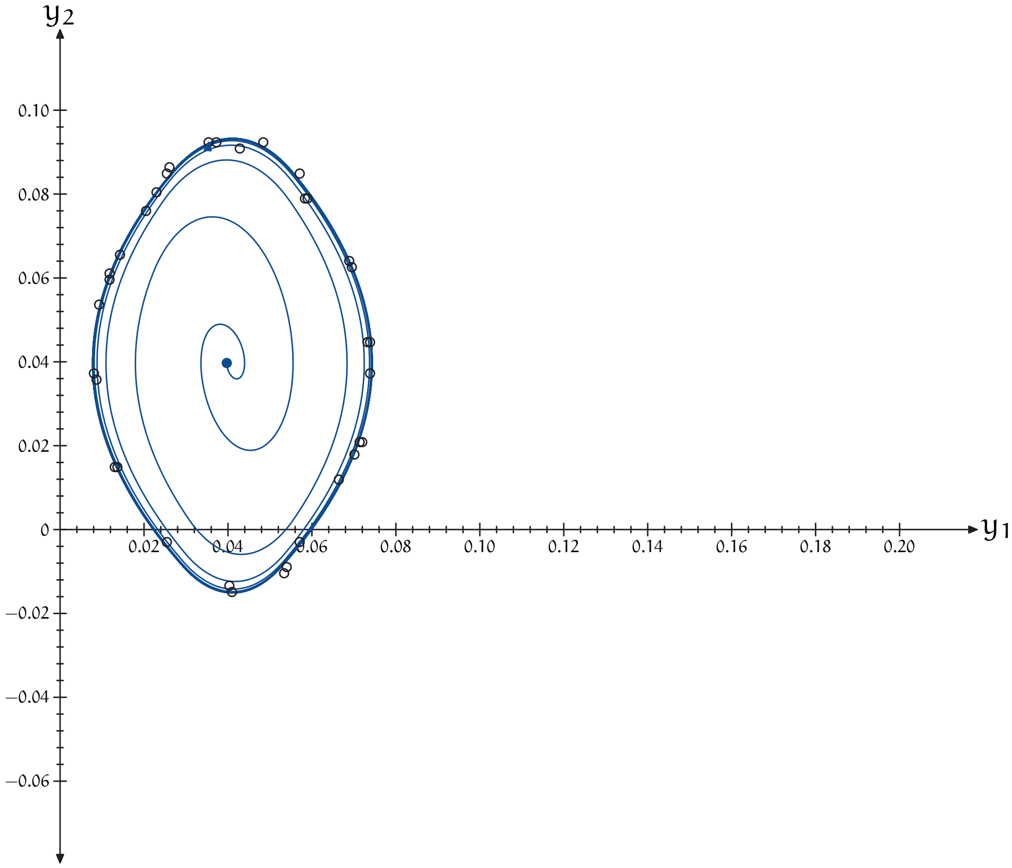


Figure 6.2: The approximate optimal trajectory $y^{N,\Delta}(t)$ of problem 6.2 for the discount factor $C = 0.001$. The points $y^{N,\Delta}$ associated with positive elements of the vector $\gamma^{N,\Delta}$ (see (5.11)) are indicated by the small black circles on this graph. The solution traces a spiral outwards from the initial condition $y_0 = (0.04, 0.04)$ (shown as a solid blue dot).

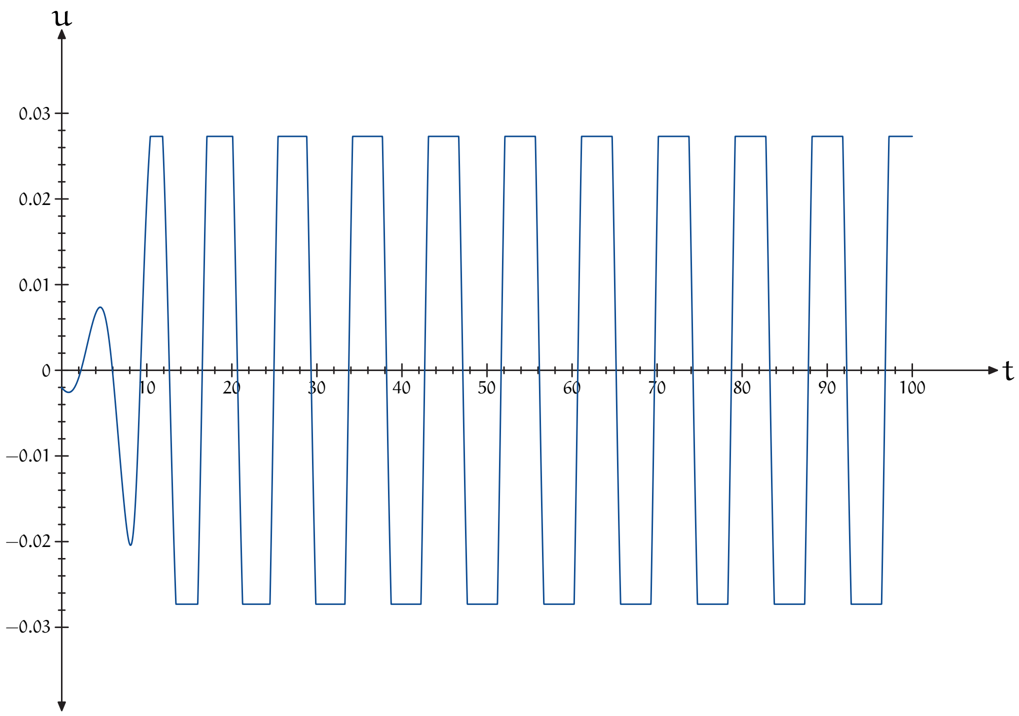


Figure 6.3: The approximate optimal control $u^{N,\Delta}(y^{N,\Delta}(t))$ of problem 6.2 for $C = 0.001$.

6.3 VANISHING DISCOUNT RATE

As can be seen from Figures 6.1c, 6.1d and Figure 6.2, the optimal state trajectories obtained with small values of the discount rate seem to be approaching a certain closed curve. It is natural to conjecture that this closed curve is an optimal solution of the following periodic optimisation problem

$$G_{PER} \stackrel{\text{def}}{=} \inf_{(y(\cdot),u(\cdot),T)} \frac{1}{T}\int_0^T (f_1^2 + y_1^2(t) - 2a_1 y_1(t) y_2(t) \\ + 2f_1 y_1(t) + \hat{c}(t) + \frac{1}{2}u^2(t))dt,$$

where inf is over the length of the time interval T, over admissible controls defined on $[0, T]$ and over the solutions of (6.1) satisfying the periodicity condition $y(0) = y(T)$.

The infinite dimensional LP problem corresponding to this problem is obtained from (6.2) by setting $C = 0$. It has been solved using a technique similar to the one we are using to solve the problem for $C > 0$. The resulting state trajectory is the orbit depicted (blue) in Figure 6.4, this, in fact, being the closed curve towards which the trajectories in Figure 6.1c, 6.1d and Figure 6.2 converge. The orbit depicted (red) in Figure 6.4 is the periodic state trajectory obtained with the control

$$u(t) = \epsilon \sin \omega t, \text{ with } \omega = 2\pi/T, \tag{6.3}$$

where $T = 9$ and $\epsilon = 0.0275$. Note that the use of this control was proposed by Feichtinger and Hartl [36] who indicated that the value of the objective function 0.05684 obtained with this control is an improvement with respect to the value 0.05707 obtained with the steady state solution $\bar{y} = (0.04, 0.04)$, $\bar{u} = 0$. Also note that the value of the objective function achieved with the control obtained through the LP approach 0.05680 is slightly better than the objective value obtained with the periodic control (6.3).

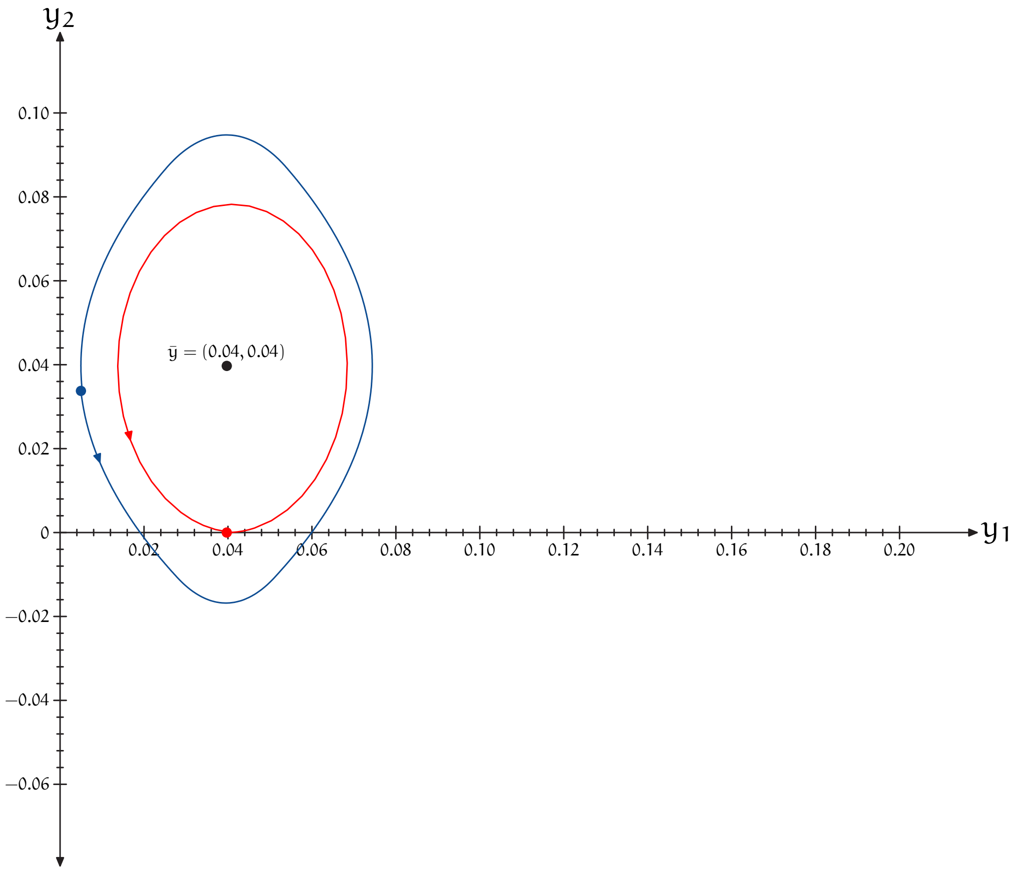


Figure 6.4: The inner trajectory (shown in red) is the trajectory of the system (6.1) obtained with the periodic control (6.3). The outer trajectory (shown in blue) is the trajectory $y^{N,\Delta}(t)$ of the discounted optimal control problem (6.2) for which $C = 0$. The small blue dot is the initial integration condition. The steady state $\bar{y} = (0.04, 0.04)$ which exists under the control $\bar{u} = 0$ is shown as a small black dot.

7 STABILISATION OF A LOTKA-VOLTERRA SYSTEM

The aim of this chapter is to present a series of experiments which demonstrate the stabilisation of a Lotka-Volterra system using discounted optimal controls solved using the linear programming method introduced in earlier chapters.

Section 7.1 introduces the well-known Lotka-Volterra system as the dynamical system to be studied. Sections 7.2 and 7.3 present the results of several numerical experiments which aim to demonstrate the stabilisation of a system to (a) a point in the state space and (b) an orbit.

7.1 THE DYNAMIC MODEL

The Lotka-Volterra [53] system is described by two first-order, nonlinear differential equations

$$\begin{aligned} {y_1}' &= -y_1 + y_1 y_2, \\ {y_2}' &= +y_2 - y_1 y_2, \end{aligned} \tag{7.1}$$

and has a general solution in the form of a closed curve

$$\ln y_2(t) - y_2(t) + \ln y_1(t) - y_1(t) = K,$$

where the constant K is dependent upon the initial conditions. A sample of such curves are shown in Figure 7.1.

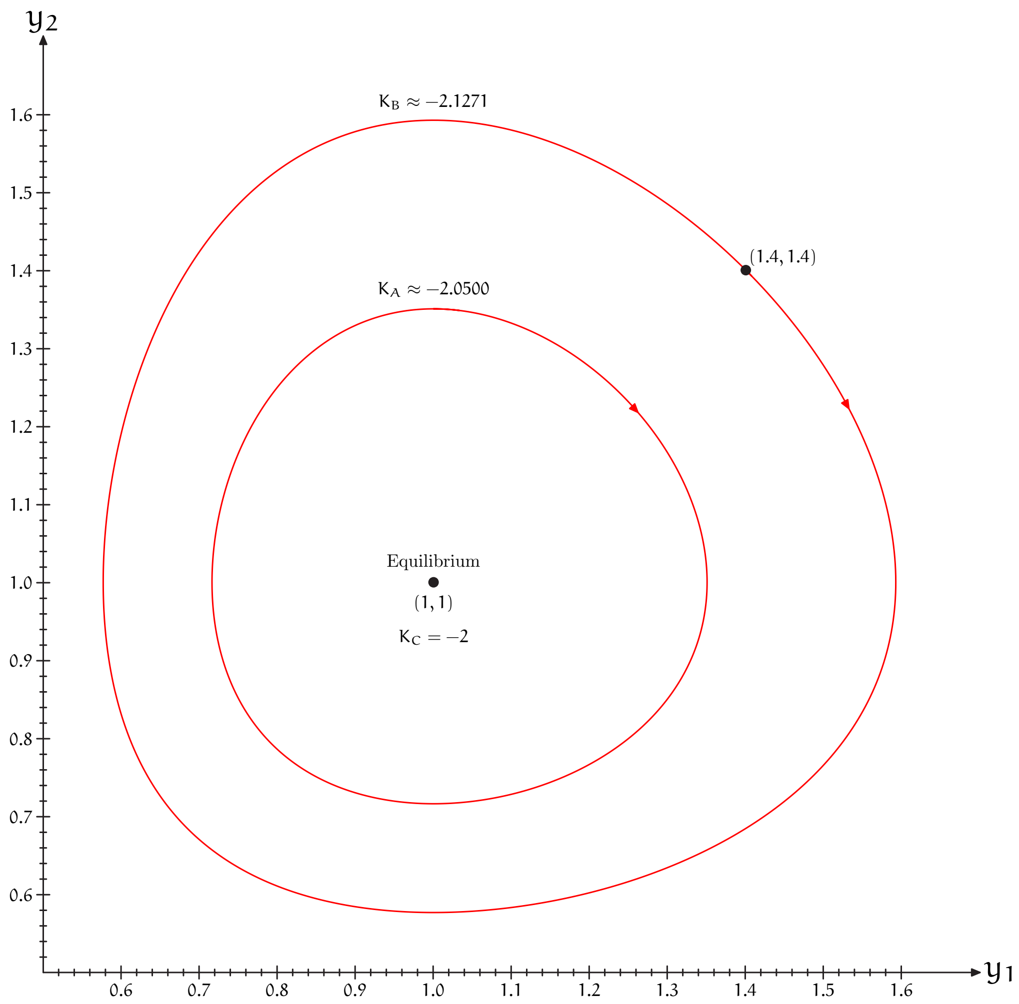


Figure 7.1: Two Lotka-Volterra closed curves characterised by the constants $K_A \approx -2.0500$ and $K_B \approx -2.1271$. The evolution of state is in a clockwise direction about the equilibrium point at $(1, 1)$ which is associated with the constant $K_C = -2$.

In a biological context the Lotka-Volterra system is used to model the interaction between predator and prey species where y_1 represents the number of predators and y_2 represents the number of prey. We shall introduce a positive control $0 \leqslant u \leqslant 1$ which simulates the introduction of a *"poison"* or *"bait"* acting upon the predator species.

The controlled system of equations then becomes:

$$\begin{aligned} y_1{}'(t) &= -y_1(t) + y_1(t)y_2(t) - y_1(t)u(t), \\ y_2{}'(t) &= +y_2(t) - y_1(t)y_2(t), \end{aligned} \tag{7.2}$$

where

$$u \in U = [0,1] \subset \mathbb{R}^1,$$
$$y = (y_1, y_2) \in Y = \{(y_1, y_2) : y_1 \in [0.6, 1.6], y_2 \in [0.6, 1.6]\} \subset \mathbb{R}^2$$

With $u(t) \equiv 0$, the system (7.2) becomes the Lotka-Volterra equations (7.1).

It can readily be seen that the set $\mathcal{S}$ of steady state admissible pairs $(\bar{y}, \bar{u}) \in Y \times U$ such that $f(\bar{y}, \bar{u}) = 0$ is defined by the equation

$$\mathcal{S} = \{(\bar{y}, \bar{u}) : \bar{y} = (1, \bar{u} + 1), \ \forall \bar{u} \in [0, 0.6]\}.$$

Each of the optimal control problems in this chapter were solved using the CPLEX [39] solver for various discount rates between $C = 0.01$ and $C = 1$. The discretisation parameters used are $\Delta_{y_1} = 0.01$, $\Delta_{y_2} = 0.01$, $\Delta_u = 0.05$ and $N = 49$ on the grid defined by (5.4).

7.2 STABILISATION TO A POINT

In this section there are two stabilisation problems. The first problem attempts to stabilise a controlled Lotka-Volterra system (7.2) to an equilibrium point using a cost function which is consistent with the formulation in Chapter 4. The second problem is an earlier study which attempted to control the system to a specified equilibrium point but failed to do so because the cost function was inconsistent. Whilst the result trajectories did converge to a point. The location of this point was dependent on the value of the discount rate C.

7.2.1 *A properly constructed cost function*

Consider the problem of stabilising the system to a point $\bar{y} = (1, 1.26)$ from the initial condition $(1.4, 1.4)$. In accordance with the results obtained in Chapter 4, the stabilising control can be found by solving the optimal control problem

$$V(y_0) = \inf_{u(\cdot) \in \mathcal{U} \text{ admissible}} \int_0^\infty e^{-Ct}((y_1(t) - 1)^2 + (y_2(t) - 1.26)^2 + (u(t) - 0.26)^2) dt. \quad (7.3)$$

From the results of Chapters 1, 2 and 3 it follows that an approximately optimal solution of the problem 7.3 can be constructed on the basis of solution of the finite dimensional (FD) linear programming (LP) problem

$$G^{N,\Delta}(y_0) = \min_{\gamma \in W^{N,\Delta}(y_0)} \sum_{i,j,k} ((y_{1,i}-1)^2 + (y_{2,j}-1.26)^2 + (u_k - 0.26)^2)\gamma_{i,j,k}$$

where

$$\begin{aligned} W^{N,\Delta}(y_0) = \{\gamma = \{\gamma_{i,j,k}\} \geqslant 0 \colon \sum_{i,j,k} \gamma_{i,j,k} = 1, \\ \sum_{i,j,k} (\frac{\partial (y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{1,i}} (-y_{1,i} + y_{1,i}y_{2,j} - y_{1,i}u_k) + \frac{\partial (y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{2,j}} (y_{2,j} - y_{1,i}y_{2,j}) \\ + C(1.4^{l_1+l_2} - y_{1,i}^{l_1} y_{2,j}^{l_2})))\gamma_{i,j,k} = 0\}. \end{aligned} \quad (7.4)$$

The problem dual to the FDLP problem is of the form

$$\begin{aligned} \max_{(\mu, \lambda_{l_1,l_2})} \{\mu \colon \mu \leqslant (y_{1,i}-1)^2 + (y_{2,j}-1.26)^2 + (u_k - 0.26)^2 \\ + \sum_{l_1,l_2} \lambda_{l_1,l_2}^{N,\Delta} (\frac{\partial (y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{1,i}} (-y_{1,i} + y_{1,i}y_{2,j} - y_{1,i}u_k) \\ + \frac{\partial (y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{2,j}} (y_{2,j} - y_{1,i}y_{2,j}) + C(1.4^{l_1+l_2} - y_{1,i}^{l_1} y_{2,j}^{l_2})), \\ \forall (y_{1,i}, y_{2,j}, u_k)\}. \end{aligned} \quad (7.5)$$

Denote by (μ, λ) an optimal solution of the problem (7.5) and declare $\psi^{N,\Delta}$ as the function

$$\psi^{N,\Delta}(y_1, y_2) = \sum_{l_1,l_2} \lambda^{N,\Delta}{}_{l_1,l_2} y_{1,i}^{l_1} y_{2,j}^{l_2}.$$

Denote also

$$a^{N,\Delta}(y_1, y_2) = \frac{1}{2} \frac{\partial \psi^{N,\Delta}(y_1, y_2)}{\partial y_1} y_1 + 0.26.$$

In Section 5.2 it has been shown that the control

$$u^{N,\Delta}(\cdot) = \begin{cases} a^{N,\Delta}(y_1,y_2), & \text{if } \ 0 \leqslant a^{N,\Delta}(y_1,y_2) \leqslant 1, \\ 0, & \text{if } \ a^{N,\Delta}(y_1,y_2) < 0, \\ 1, & \text{if } \ a^{N,\Delta}(y_1,y_2) > 1. \end{cases} \quad (7.6)$$

that minimises the expression

$$\min_{u \in U} \left\{ (u - 0.26)^2 + \frac{\partial \psi^{N,\Delta}(y_1, y_2)}{\partial y_1}(-y_1 u) \right\},$$

can serve as an approximation for the optimal control for N large enough and Δ small enough.

The control system (7.2) was integrated using the control rule (7.6) and terminated at $T = 10/C$. The final values for the state and control variables lie close to the anticipated equilibrium value $\bar{y} = (1, 1.26)$ and $\bar{u} = 0.26$. The numerical value of the cost function $G_{NUM}(T)$ and LP objective value $G^{N,\Delta}(y_0)$ are close. The terminal characteristics of this problem are summarised in the Table 7.1.

Table 7.1: Terminal characteristics of the optimal control problem 7.3.

C	y_1	y_2	u	$G^{N,\Delta}(y_0)$	$G_{NUM}(T)$
+0.0100	+1.0001	+1.2600	+0.2599	+0.0017	+0.0016
+0.0200	+1.0000	+1.2588	+0.2603	+0.0033	+0.0033
+0.0500	+1.0000	+1.2598	+0.2599	+0.0081	+0.0081
+0.1000	+1.0011	+1.2606	+0.2607	+0.0159	+0.0158
+0.2000	+0.9995	+1.2586	+0.2592	+0.0301	+0.0300
+0.5000	+1.0000	+1.2599	+0.2599	+0.0644	+0.0643
+1.0000	+1.0016	+1.2689	+0.2600	+0.1011	+0.1010

The resulting state trajectories are shown in Figure 7.2 and the approximately optimal controls are shown in Figure 7.3.

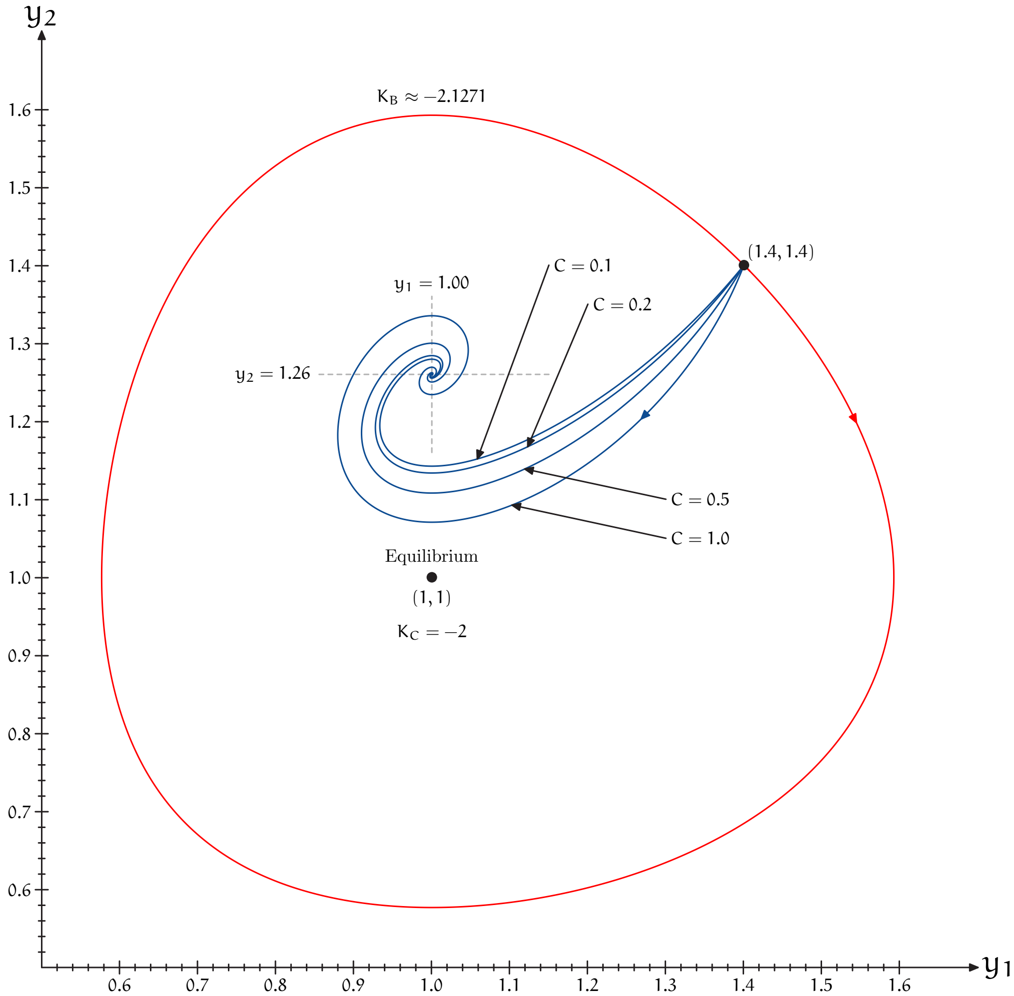


Figure 7.2: A sample of approximately optimal state trajectories for the optimal control problem 7.3 where each trajectory shows convergence to the point $\bar{y} = (1, 1.26)$.

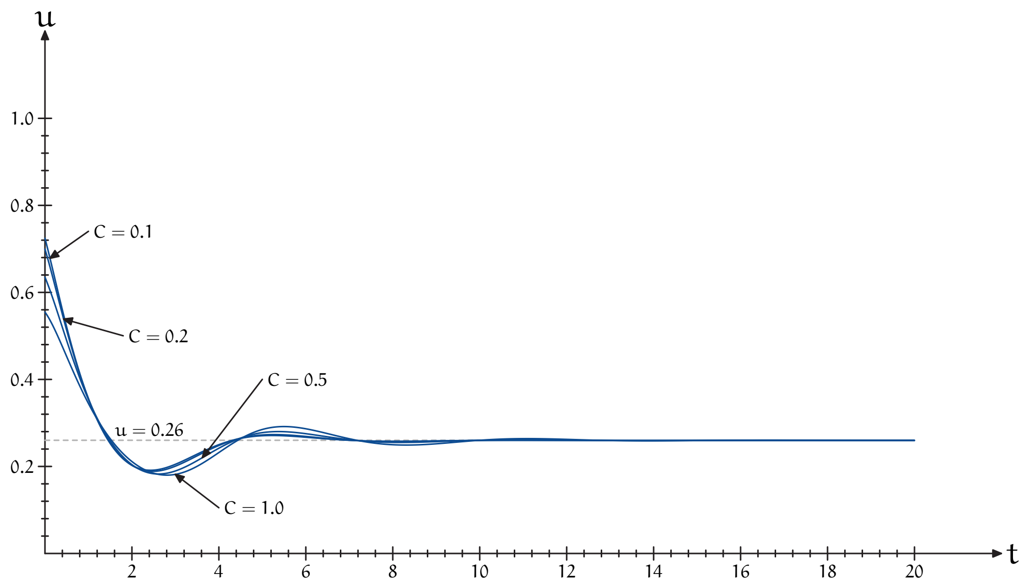


Figure 7.3: A sample of approximately optimal control rules for the optimal control problem 7.3.

As one can see, in all cases, the state trajectories converge to the selected steady state point $(1, 1.26)$.

7.2.2 *A poorly constructed cost function*

Consider the problem of stabilising the system to a point $\bar{y} = (1, 1.5)$ from the initial condition $(1.4, 1.4)$. In this example the cost function is inconsistent with the findings of Chapter 4 in that $f(\bar{y}, \bar{u}) \neq 0$. We find that the trajectory converges to a point $(\hat{y}, \hat{u}) \neq (\bar{y}, \bar{u})$ which is dependent on C.

$$V(y_0) = \inf_{u(\cdot)\in\mathcal{U} \text{ admissible}} \int_0^\infty e^{-Ct}((y_1(t)-1)^2 + (y_2(t)-1.5)^2 + u^2(t))dt. \quad (7.7)$$

From the results of Chapters 1, 2 and 3 it follows that an approximately optimal solution of the problem 7.7 can be constructed on the basis of solution of the finite dimensional linear programming problem

$$G^{N,\Delta}(y_0) = \min_{\gamma\in W^{N,\Delta}(y_0)} \sum_{i,j,k} ((y_{1,i}-1)^2 + (y_{2,j}-1.5)^2 + u_k^2)\gamma_{i,j,k}$$

where $W^{N,\Delta}(y_0)$ is defined in 7.4.

The problem dual to the FDLP problem is of the form

$$\max_{(\mu,\lambda_{l_1,l_2})} \{\mu\colon \mu \leqslant (y_{1,i}-1)^2 + (y_{2,j}-1.5)^2 + u_k^2$$
$$+ \sum_{l_1,l_2} \lambda^{N,\Delta}_{l_1,l_2} \Big(\frac{\partial (y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{1,i}} (-y_{1,i} + y_{1,i}y_{2,j} - y_{1,i}u_k)$$
$$+ \frac{\partial (y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{2,j}} (y_{2,j} - y_{1,i}y_{2,j}) + C(1.4^{l_1+l_2} - y_{1,i}^{l_1} y_{2,j}^{l_2})\Big),$$
$$\forall (y_{1,i}, y_{2,j}, u_k)\}. \quad (7.8)$$

Denote by (μ, λ) an optimal solution of the problem (7.8) and declare $\psi^{N,\Delta}$ as the function

$$\psi^{N,\Delta}(y_1, y_2) = \sum_{l_1,l_2} \lambda^{N,\Delta}{}_{l_1,l_2} y_{1,i}^{l_1} y_{2,j}^{l_2}.$$

Denote also

$$a^{N,\Delta}(y_1, y_2) = \frac{1}{2} \frac{\partial \psi^{N,\Delta}(y_1, y_2)}{\partial y_1} y_1,$$

In Section 5.2 it has been shown that the control

$$u^{N,\Delta}(\cdot) = \begin{cases} a^{N,\Delta}(y_1,y_2), & \text{if } \; 0 \leqslant a^{N,\Delta}(y_1,y_2) \leqslant 1, \\ 0, & \text{if } \; a^{N,\Delta}(y_1,y_2) < 0, \\ 1, & \text{if } \; a^{N,\Delta}(y_1,y_2) > 1. \end{cases} \quad (7.9)$$

that minimises the expression

$$\min_{u \in U} \{u^2 + \frac{\partial \psi^{N,\Delta}(y_1, y_2)}{\partial y_1} (-y_1 u)\},$$

can serve as an approximation for the optimal control for N large enough and Δ small enough.

The control system (7.2) was integrated using the control rule (7.9) and terminated at $T = 10/C$. The final values for the state and control variables are dependent on the value of the discount rate C. The numerical value of the cost function $G_{NUM}(T)$ and LP objective value $G^{N,\Delta}(y_0)$ are close. The terminal characteristics of this problem are summarised in the Table 7.2.

Table 7.2: Terminal characteristics of the optimal control problem 7.7.

C	y_1	y_2	u	$G^{N,\Delta}(y_0)$	$G_{NUM}(T)$
+0.0100	+1.0000	+1.2499	+0.2499	+0.1284	+0.1283
+0.0200	+0.9999	+1.2498	+0.2499	+0.1317	+0.1317
+0.0500	+0.9989	+1.2489	+0.2482	+0.1415	+0.1414
+0.1000	+1.0001	+1.2487	+0.2489	+0.1573	+0.1572
+0.2000	+1.0000	+1.2459	+0.2460	+0.1868	+0.1867
+0.5000	+0.9998	+1.2269	+0.2265	+0.2571	+0.2569
+1.0000	+0.9999	+1.1748	+0.1749	+0.3177	+0.3176

The resulting state trajectories are shown in Figure 7.4 and the approximately optimal controls are shown in Figure 7.5.

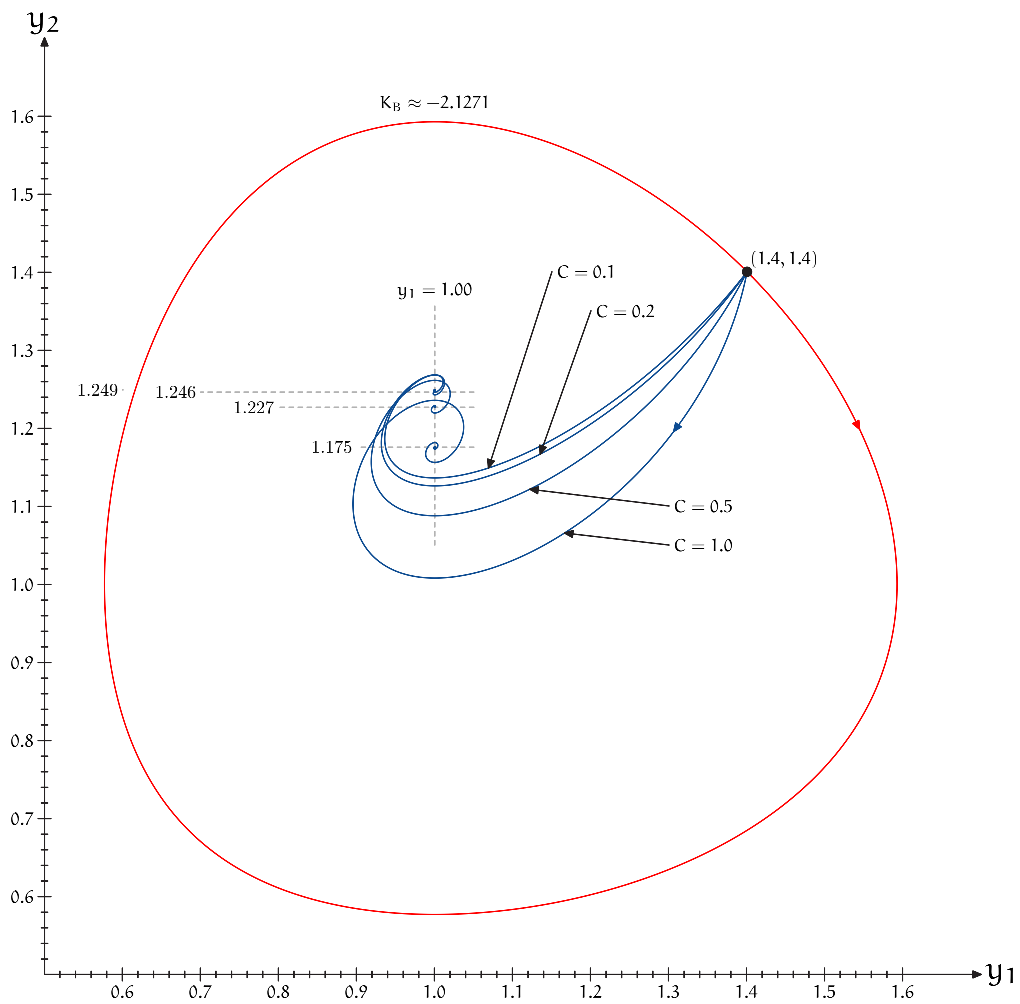


Figure 7.4: A sample of approximately optimal state trajectories for the optimal control problem 7.7.

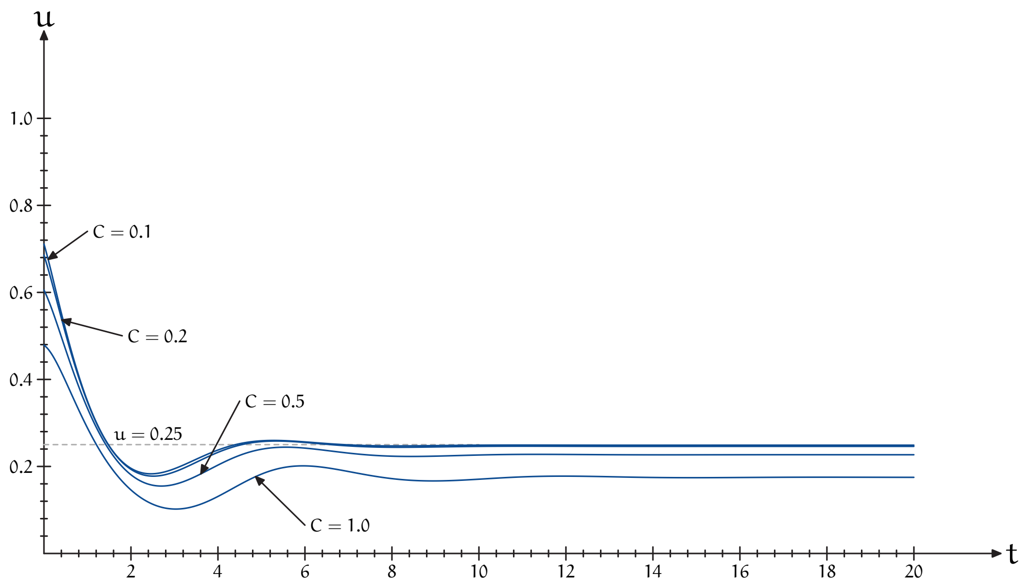


Figure 7.5: A sample the approximately optimal control rules for the optimal control problem 7.7.

As one can see, in all cases, the state trajectories converge to a steady state point dependent upon the value of C. Note that the dependence of the optimal solution on the discount rate is consistent with results described in Chapter 6 of [15].

7.3 STABILISATION TO A CURVE

The problems of the previous section are concerned with problems of optimal control related to asymptotic stabilisation to the point $(\bar{y}, \bar{u})$. In this section, we consider an optimal control problem which stabilises a Lotka-Volterra system to an orbit characterised by a constant K.

The system of equations (7.2) is to be controlled from an initial condition $y_0 = (1.4, 1.4)$ lying on the characteristic curve $K_B \approx -2.127$ to converge to a closed curve characterised by the constant $K_A \approx -2.0500$. Consider the problem

$$V(y_0) = \inf_{u(\cdot)\in\mathcal{U} \text{ admissible}}$$
$$\int_0^\infty e^{-Ct}((\ln(y_1(t)) + \ln(y_2(t)) - y_1(t) - y_2(t) + 2.05)^2 + u^2(t))dt. \quad (7.10)$$

From the results of Chapters 1, 2 and 3 it follows that an approximately optimal solution of the problem 7.3 can be constructed on the basis of solution of the finite dimensional linear programming problem

$$G^{N,\Delta}(y_0) =$$
$$\min_{\gamma\in W^{N,\Delta}(y_0)} \sum_{i,j,k} ((\ln(y_{1,i}) + \ln(y_{2,j}) - y_{1,i} - y_{2,j} + 2.05)^2 + u_k^2)\gamma_{i,j,k}$$

where $W^{N,\Delta}(y_0)$ is defined in 7.4.

The problem dual to the FDLP problem is of the form

$$\max_{(\mu,\lambda_{l_1,l_2})} \{\mu\colon \mu \leqslant (\ln(y_{1,i}) + \ln(y_{2,j}) - y_{1,i} - y_{2,j} + 2.05)^2 + u_k^2$$
$$+ \sum_{l_1,l_2} \lambda^{N,\Delta}{}_{l_1,l_2}(\frac{\partial(y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{1,i}}(-y_{1,i} + y_{1,i}y_{2,j} - y_{1,i}u_k)$$
$$+ \frac{\partial(y_{1,i}^{l_1} y_{2,j}^{l_2})}{\partial y_{2,j}}(y_{2,j} - y_{1,i}y_{2,j}) + C(1.4^{l_1+l_2} - y_{1,i}^{l_1} y_{2,j}^{l_2})), \ \forall(y_{1,i}, y_{2,j}, u_k)\}.$$

Denote by (μ, λ) an optimal solution of the problem (7.10) and declare $\psi^{N,\Delta}$ as the function

$$\psi^{N,\Delta}(y_1, y_2) = \sum_{l_1,l_2} \lambda^{N,\Delta}{}_{l_1,l_2} y_{1,i}^{l_1} y_{2,j}^{l_2}.$$

Denote also

$$a^{N,\Delta}(y_1, y_2) = \frac{1}{2} \frac{\partial \psi^{N,\Delta}(y_1, y_2)}{\partial y_1} y_1.$$

In Section 5.2 it has been shown that the control

$$u^{N,\Delta}(\cdot) = \begin{cases} a^{N,\Delta}(y_1, y_2), & \text{if} \quad 0 \leqslant a^{N,\Delta}(y_1, y_2) \leqslant 1, \\ 0, & \text{if} \quad a^{N,\Delta}(y_1, y_2) < 0, \\ 1, & \text{if} \quad a^{N,\Delta}(y_1, y_2) > 1. \end{cases} \tag{7.11}$$

that minimises the expression

$$\min_{u \in U} \{u^2 + \frac{\partial \psi^{N,\Delta}(y_1, y_2)}{\partial y_1}(-y_1 u)\},$$

can serve as an approximation for the optimal control for N large enough and Δ small enough.

The control system (7.2) was integrated using the control rule (7.11) and terminated using the following stopping rule:

Remark 7.1. *Unlike the problems in Section 7.2 where numerical integration proceeded for a fixed period of time a stopping rule was applied to problem 7.10 of the form*

Stop if $t \geqslant T$ *or* $|\ln(y_1) + \ln(y_2) - y_1 - y_2 - K_A| < \delta$,

where $T = 40$ *seconds,* $K_A = -2.05$ *and* $\delta = 0.0001$.

The state trajectories for discount rates $C = 0.1$ and $C = 1$ are shown in Figures 7.6 and 7.7 respectively, with the associated control rules in Figure 7.8. For each integration the trajectory leaves the initial condition y_0 and takes a different path which ultimately converges to the same curve $K_A = -2.05$. The same remarks apply for the control variable u which converges to zero. In each case the integrated objective value $G_{NUM}(T)$ agrees closely with the objective value obtained from the LP solution $G^{N,\Delta}(y_0)$. From this result we can conclude that the trajectories generated by the control rule (7.11) are approximately optimal. The terminal characteristics of this problem are summarised in Table 7.3.

Table 7.3: Terminal characteristics of the problem 7.10. Numerical values for the objective function $G_{NUM}(T)$ agree closely with the LP objective values $G^{N,\Delta}(y_0)$.

C	K	u	$G^{N,\Delta}(y_0)$	$G_{NUM}(T)$
+0.0100	−2.0443	+0.0017	+0.0001	+0.0002
+0.0200	−2.0386	+0.0017	+0.0002	+0.0004
+0.0500	−2.0352	+0.0018	+0.0007	+0.0008
+0.1000	−2.0463	+0.0000	+0.0014	+0.0014
+0.2000	−2.0324	+0.0024	+0.0024	+0.0024
+0.5000	−2.0500	+0.0026	+0.0041	+0.0040
+1.0000	−2.0375	+0.0043	+0.0051	+0.0050

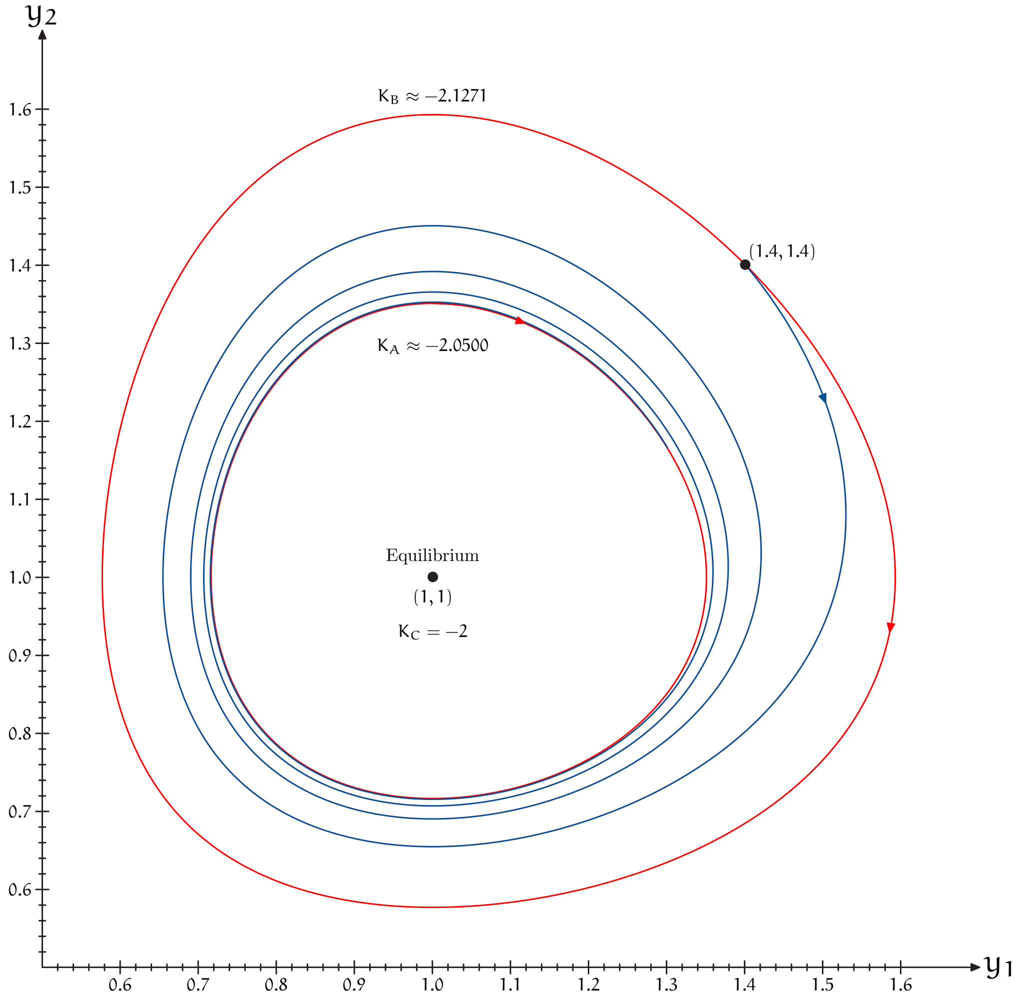


Figure 7.6: Approximately optimal state trajectory for problem 7.10 with a discount rate $C = 0.1$. The trajectory converges to a curve characterised by the constant $K_A \approx -2.0500$.

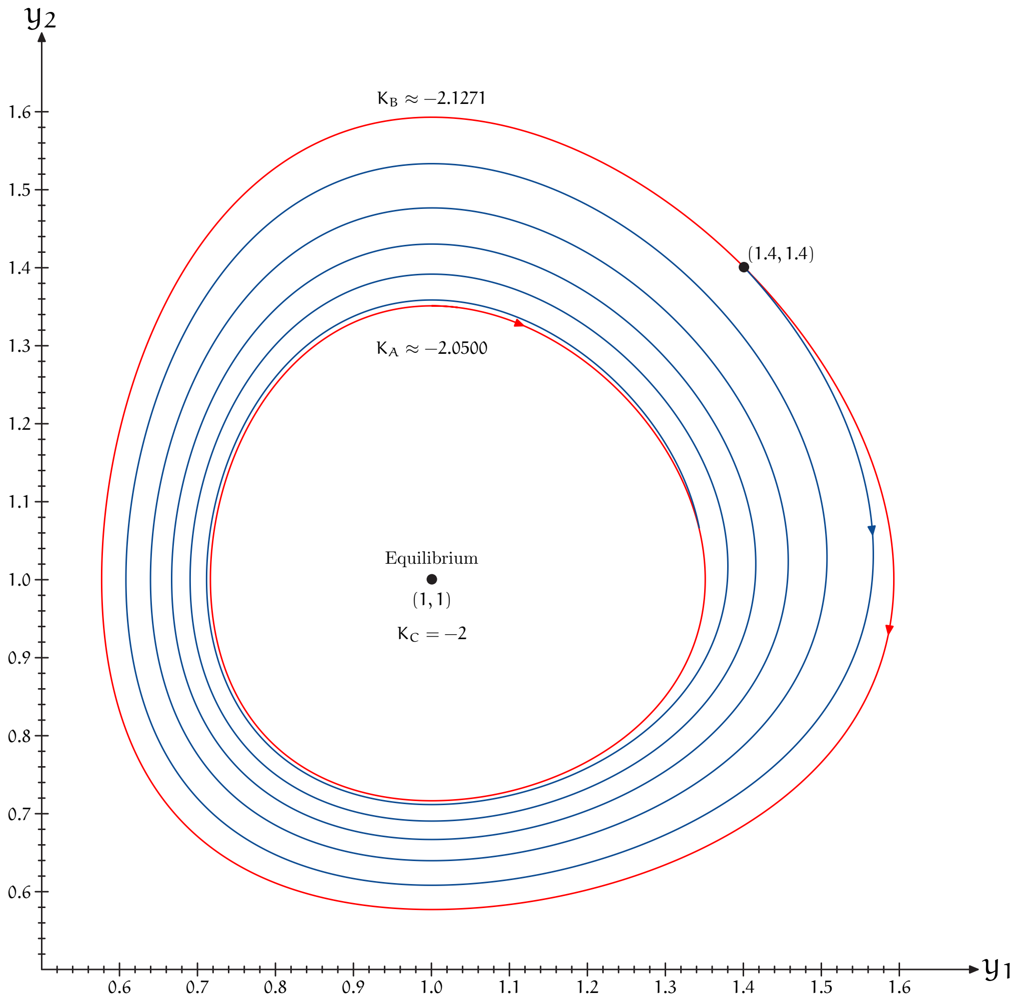


Figure 7.7: Approximately optimal state trajectory for problem 7.10 with a discount rate $C = 1.0$. The trajectory converges to a curve characterised by the constant $K_A \approx -2.0500$.

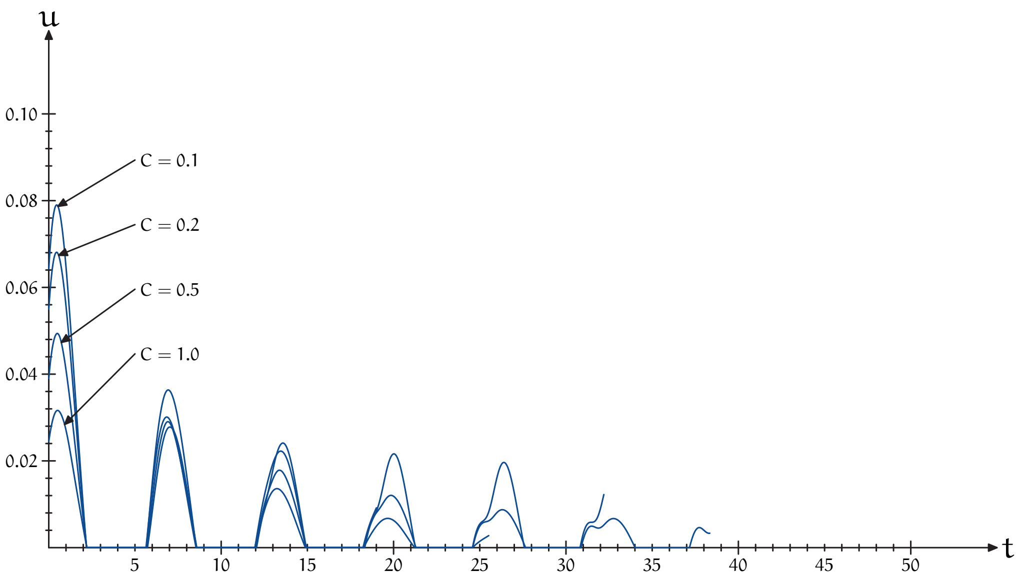


Figure 7.8: A sample of control rules for Problem 7.10. Each control rule appears to be converging to zero.

7.4 CONCLUSION

The problems studied in this chapter demonstrate the applicability of the linear programming method to the stabilisation of a dynamical system. We have demonstrated that the controls constructed on the basis of a solution of the corresponding LP problems steer the Lotka-Volterra system to a desired equilibrium or orbit.

8

SYNTHESIS OF NEAR-OPTIMAL CONTROL MAPS

In the numerical examples of Chapters 5, 6 and 7 we generated approximately optimal controls for time discounted optimal control problems using the linear programming method. We note that these controls are constructed for a specific initial condition $y(0) = y_0$ and are near-optimal only on a trajectory which satisfies such a condition. Using a Lotka-Volterra example from Chapter 7.3, we discuss a way to construct a feedback control that is near-optimal from an arbitrary initial condition y_0'.

In Section 8.1, the Lotka-Volterra problem is solved by the linear programming method outlined in Chapter 5.2. Using two different initial conditions, located inside and outside the characteristic closed curve K, we show that the solution to each problem exhibits stabilising behaviour.

In Section 8.2 we show that the controls which were stabilising in Section 8.1, lead to non-stabilising behaviour when the system is integrated from a starting point y_0' which does not coincide with the initial condition y_0.

In Section 8.3, we describe how a control map can be synthesised by solving time discounted optimal control problems on a regular grid of initial conditions.

8.1 STABILISATION TO A CURVE

We consider the controlled Lotka-Volterra system first described in Chapter 7.3 which is reproduced below

$$\begin{aligned} {y_1}'(t) &= -y_1(t) + y_1(t)y_2(t) - y_1(t)u(t), \\ {y_2}'(t) &= +y_2(t) - y_1(t)y_2(t), \end{aligned} \tag{8.1}$$

where

$$u \in U = [0,1] \subset \mathbb{R}^1,$$
$$y = (y_1, y_2) \in Y = \{(y_1, y_2) : y_1 \in [0.5, 1.7], y_2 \in [0.5, 1.7]\} \subset \mathbb{R}^2.$$

(The set Y has been expanded to accommodate the dynamics of the optimal control problems needed for the construction of the optimal control map described later in this chapter.)

Then consider the problem of stabilisation to an orbit characterised by the constant $K = -2.05$,

$$V(y_0) = \inf_{u(\cdot) \in \mathcal{U} \text{ admissible}} \int_0^\infty e^{-Ct}((\ln(y_1(t)) + \ln(y_2(t)) - y_1(t) - y_2(t) + 2.05)^2 + u^2(t))dt. \quad (8.2)$$

In Chapter 7.3 it has been shown that the control

$$u^{N,\Delta}(\cdot) = \begin{cases} a^{N,\Delta}(y_1, y_2), & \text{if } \quad 0 \leqslant a^{N,\Delta}(y_1, y_2) \leqslant 1, \\ 0, & \text{if } \quad a^{N,\Delta}(y_1, y_2) < 0, \\ 1, & \text{if } \quad a^{N,\Delta}(y_1, y_2) > 1. \end{cases} \quad (8.3)$$

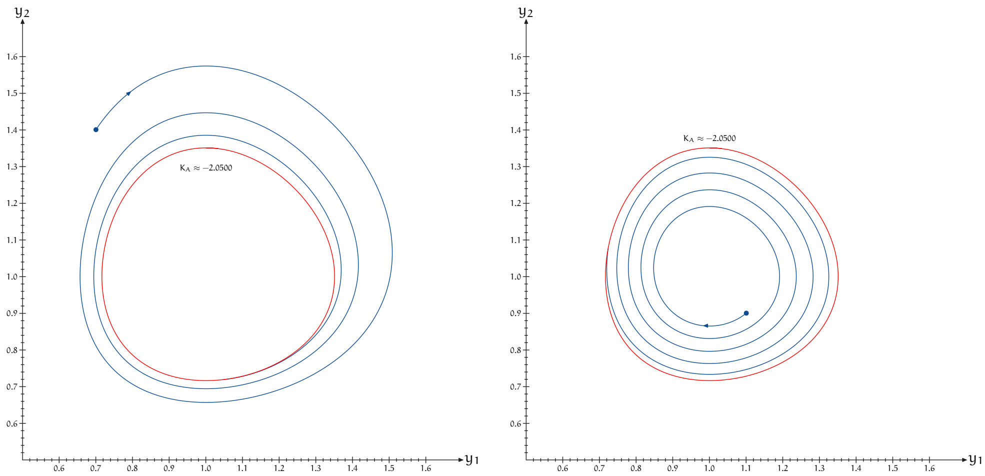
that minimises the expression

$$\min_{u \in U} \{u^2 + \frac{\partial \psi^{N,\Delta}(y_1, y_2)}{\partial y_1}(-y_1 u)\},$$

steers the solution $y^{N,\Delta}(t)$ to the orbit characterised by the constant K.

Two time discounted optimal control problems were solved using the CPLEX [39] solver for the discount rate $C = 0.05$. The first problem used an initial condition $y_0 = (0.7, 1.4)$ which is external to the closed curve K (see, Figure 8.1a) and the second problem used an initial condition $y_0 = (1.1, 0.9)$ which is internal to the closed curve K (see, Figure 8.1b). The discretisation parameters used are $\Delta_{y_1} = 0.01$, $\Delta_{y_2} = 0.01$, $\Delta_u = 0.05$ and $N = 49$ on the grid defined by (5.4).

The near-optimal trajectories $y^{N,\Delta}(t)$ obtained by the numerical integration of the system (8.1) from the respective initial conditions, under the influence of the control (8.3), spiral inwards or outwards (as the case may be) from the initial condition to stabilise on the desired orbit.



(a) Trajectory $y^{N,\Delta}(t)$ from $y_0 = (0.7, 1.4)$. **(b)** Trajectory $y^{N,\Delta}(t)$ from $y_0 = (1.1, 0.9)$.

Figure 8.1: The trajectory $y^{N,\Delta}(t)$ of problem 8.2. Each trajectory is near-optimal with respect to the initial condition y_0 for which the control was constructed. (y_0 is shown as a solid blue dot).

8.2 INITIAL CONDITION DEPENDENCY

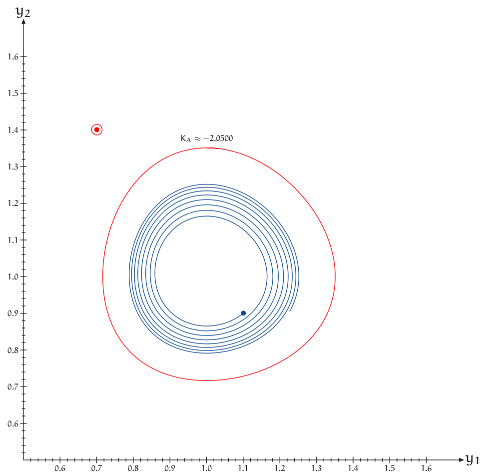
The controls in Section 8.1 are constructed for a specific initial condition and optimal only on a trajectory which includes the initial condition y_0 on it's path. To illustrate this point, Figure 8.2a shows a trajectory $y^{N,\Delta}(t)$ originating from a starting point $y_0' = (1.1, 0.9)$ which differs from the initial condition $y_0 = (0.7, 1.4)$ under which the integrating control was constructed. The trajectory fails to stabilise to the orbit K (c.f. Figure 8.1b). Figure 8.2b shows a trajectory $y^{N,\Delta}(t)$ originating from a starting point $y_0' = (0.7, 1.4)$ which differs from the initial condition $y_0 = (1.1, 0.9)$ under which the integrating control was constructed. This trajectory also fails to stabilise to the orbit K (c.f. Figure 8.1a).

8.3 SYNTHESISED OPTIMAL CONTROL MAP

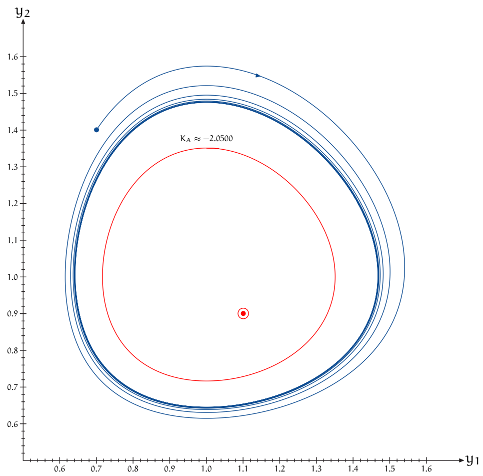
Below, we indicate a way to construct a control map $\hat{u}^{N,\Delta}(y)$, the use of which leads to the stabilising behavior of solutions for any initial conditions. Construction is based on the solution of LP problems related to (8.2) for a set (regular grid) of initial conditions Y_0:

$$Y_0 \stackrel{\text{def}}{=} \{(y_1, y_2) \colon y_1 \in \{0.6, 0.7, \ldots, 1.6\}, y_2 \in \{0.6, 0.7, \ldots, 1.6\}\}.$$

Denote by $u^{N,\Delta}_{y_{0,1}, y_{0,2}}(\cdot)$ the control (8.3) constructed on the basis of the solution of the corresponding LP problem with $(y_1(0), y_2(0)) = (y_{0,1}, y_{0,2}) \in Y_0$.



(a) $y_0 = (1.1, 0.9) \neq y_0' = (0.7, 1.4)$.



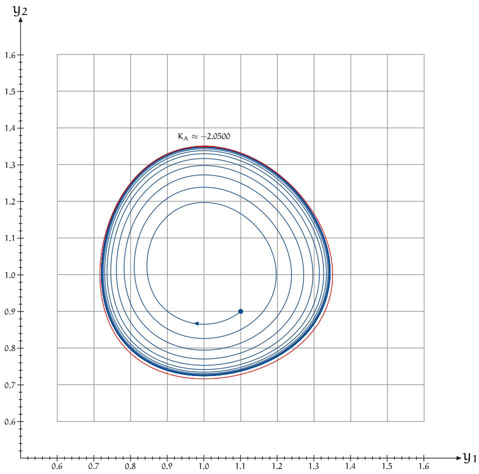
(b) $y_0 = (0.7, 1.4) \neq y_0' = (1.1, 0.9)$.

Figure 8.2: The trajectory $y^{N,\Delta}(t)$ of problem 8.2. The trajectories are integrated from a starting point y_0' (shown as a solid blue dot) using a control constructed from a different initial condition $y_0 \neq y_0'$ (shown as red circle and dot). Neither trajectory stabilises to the desired orbit.

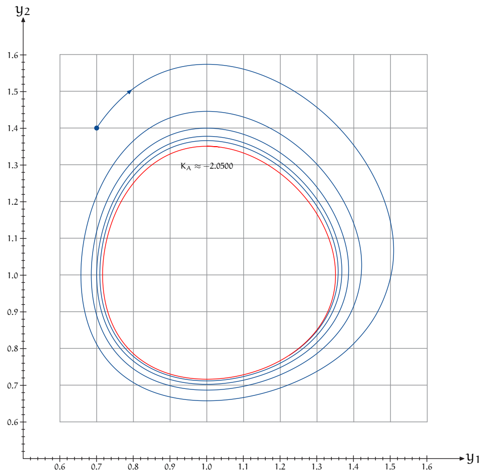
For $(y_1, y_2) \in Y_0$, take $\hat{u}^{N,\Delta}(y_1, y_2) \stackrel{\text{def}}{=} u^{N,\Delta}_{y_1,y_2}(y_1, y_2)$. For $(y_1, y_2) \notin Y_0$, define $\hat{u}^{N,\Delta}(y_1, y_2)$ via interpolation. The trajectory obtained under the interpolated control $\hat{u}^{N,\Delta}(y_1, y_2)$ we denote as $\hat{y}^{N,\Delta}$. The two dimensional interpolation used in these examples is the MATLAB bilinear interpolation function `interp2()` with the "linear" option selected.

Each of the 121 optimal control problems associated with Y_0 were solved using the CPLEX [39] solver for the discount rate $C = 0.05$. The discretisation parameters used are $\Delta_{y_1} = 0.01$, $\Delta_{y_2} = 0.01$, $\Delta_u = 0.05$ and $N = 49$ on the grid defined by (5.4).

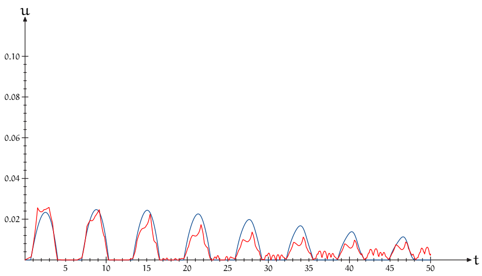
The two examples in Section 8.2 have been integrated again from the same starting points using the synthesised optimal control $\hat{u}^{N,\Delta}(y)$. The trajectories in Figure 8.3a and 8.3b now stabilise to the desired orbit K.



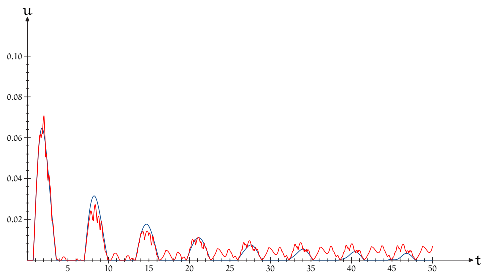
(a) Trajectory $\hat{y}^{N,\Delta}(t)$ from $y_0' = (1.1, 0.9)$. Stabilisation to the desired orbit is restored. c.f. Fig. 8.2a.



(b) Trajectory $\hat{y}^{N,\Delta}(t)$ from $y_0' = (0.7, 1.4)$. Stabilisation to the desired orbit is restored. c.f. Fig. 8.2b.



(c) $u^{N,\Delta}(y^{N,\Delta}(t))$ (blue) and $\hat{u}^{N,\Delta}(y^{N,\Delta}(t))$ (red) for Fig 8.3a.



(d) $u^{N,\Delta}(y^{N,\Delta}(t))$ (blue) and $\hat{u}^{N,\Delta}(y^{N,\Delta}(t))$ (red) for Fig 8.3b.

Figure 8.3: In (a) and (b) above, the near-optimal trajectory $\hat{y}^{N,\Delta}(t)$ steered by the synthesised optimal control is shown in blue. The starting point y_0' is shown as a solid blue dot. In (c) and (d), the near-optimal controls (blue) obtained in Section 8.1 and the synthesised optimal controls (red) are shown as functions of time.

A

APPENDIX

Proof of Proposition 1.6

The following is a proof for Proposition 1.6 which was originally published in [25]. The proof is reproduced here for convenience.

Due to the approximating property of the sequence of the functions $\phi_i(\cdot)$, $i = 1, 2, \ldots$ (see (2.1) and Lemma 2.2), the set $W(y_0)$ can be presented in the form

$$W(y_0) = \{\gamma \in \mathcal{P}(Y \times U) : \\ \int_{Y \times U} (\phi_i'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\gamma(dy, du) = 0, \\ i = 1, 2, \ldots\}, \quad \text{(A.1)}$$

where, without loss of generality, one may assume that the functions $\phi_i(\cdot)$ satisfy the following normalisation conditions:

$$\max_{y \in \hat{D}} \{|\phi_i(y)|, \|\phi_i'(y)\|\} \leqslant \frac{1}{2^i}, \ i = 1, 2, \ldots \quad \text{(A.2)}$$

where $\|\phi_i'(y)\|$ is a norm of $\phi_i'(y)$ in $\mathbb{R}^m$, and $\hat{D}$ is a closed ball in $\mathbb{R}^m$ that contains Y in its interior.

Let l_1 and l_∞ stand for the Banach spaces of infinite sequences such that, for any $x = (x_1, x_2, \ldots) \in l_1$, $\|x\|_{l_1} \overset{\text{def}}{=} \sum_i |x_i| < \infty$ and, for any $\lambda = (\lambda_1, \lambda_2, \ldots) \in l_\infty$, $\|\lambda\|_{l_\infty} \overset{\text{def}}{=} \sup_i |\lambda_i| < \infty$. It is easy to see that, given an element $\lambda \in l_\infty$, one can define a linear continuous functional $\lambda(\cdot) : l_1 \to \mathbb{R}^1$ by the equation

$$\lambda(x) = \sum_i \lambda_i x_i, \ \forall x \in l_1, \ \|\lambda(\cdot)\| = \|\lambda\|_{l_\infty}. \quad \text{(A.3)}$$

It is also known (see, e.g., [52], p.86) that any continuous linear functional $\lambda(\cdot) : l_1 \to \mathbb{R}^1$ can be presented in the form (A.3) with some $\lambda \in l_\infty$.

By (A.2), $(\phi_1(y), \phi_2(y), \dots) \in l_1$ and $(\frac{\partial \phi_1}{\partial y_j}, \frac{\partial \phi_2}{\partial y_j}, \dots) \in l_1$ for any $y \in Y$. Hence, the function $\psi_\lambda(y)$,

$$\psi_\lambda(y) \stackrel{\text{def}}{=} \sum_i \lambda_i \phi_i(y), \ \lambda = (\lambda_1, \lambda_2, \dots) \in l_\infty, \tag{A.4}$$

is continuously differentiable, with $\psi'_\lambda(y) = \sum_i \lambda_i \phi'_i(y)$.

Proof of Proposition 1.6(iii). If the function $\psi(\cdot)$ satisfying (1.16) exists, then

$$\min_{(y,u)\in Y\times U} \{ -\psi'(y)^T f(y,u) - C(\psi(y_0) - \psi(y))\} > 0$$

and, hence,

$$\lim_{\alpha\to\infty} \min_{(y,u)\in Y\times U} \{g(y,u) + \alpha(-\psi'(y)^T f(y,u) - C(\psi(y_0) - \psi(y)))\} = \infty.$$

This implies that the optimal value of problem 1.13 is unbounded ($\mu(y_0) = \infty$).

Assume now that the optimal value of problem 1.13 is unbounded. That is, there exists a sequence $(\mu_k, \psi_k(\cdot))$ such that $\lim_{k\to\infty} \mu_k = \infty$,

$$\mu_k \leqslant g(y,u) + (\psi'_k(y)^T f(y,u) + C(\psi_k(y_0) - \psi_k(y))), \ \forall (y,u) \in Y \times U,$$

$$\Longrightarrow \ 1 \leqslant \frac{1}{\mu_k} g(y,u) + \frac{1}{\mu_k}(\psi'_k(y)^T f(y,u) \\ + C(\psi_k(y_0) - \psi_k(y))), \ \forall (y,u) \in Y \times U.$$

For k large enough, $\frac{1}{\mu_k}|g(y,u)| \leqslant \frac{1}{2}$, for all $(y,u) \in Y \times U$. Hence

$$\frac{1}{2} \leqslant \frac{1}{\mu_k}(\psi'_k(y)^T f(y,u) + C(\psi_k(y_0) - \psi_k(y))), \ \forall (y,u) \in Y \times U.$$

That is, the function $\psi(y) \stackrel{\text{def}}{=} -\frac{1}{\mu_k}\psi_k(y)$ satisfies (1.16). □

Proof of Proposition 1.6(i). From (1.14) it follows that, if $W(y_0)$ is not empty, then the optimal value of problem (1.13) is bounded.

Conversely, let us assume that the optimal value $\mu^*(y_0)$ of problem (1.13) is bounded and let us establish that $W(y_0)$ is not empty. Assume that it is not true and $W(y_0)$ is empty. Define the set $\mathcal{Q}$ by the equation

$$\mathcal{Q} \stackrel{\text{def}}{=} \{x = (x_1, x_2, \dots): \\ x_i = \int_{Y\times U} (\phi_i'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\gamma(dy,du),\ \gamma \in \mathcal{P}(Y \times U)\}.$$

It is easy to see that the set $\mathcal{Q}$ is a convex and compact subset of l_1 (the fact that $\mathcal{Q}(y_0)$ is relatively compact in l_1 is implied by (A.2); the fact that it is closed follows from that $\mathcal{P}(Y \times U)$ is compact in weak* convergence topology).

By (A.1), the assumption that $W(y_0)$ is empty is equivalent to the assumption that the set $\mathcal{Q}$ does not contain the "zero element" ($0 \notin \mathcal{Q}$). Hence, by a separation theorem (see, e.g., [52], p.59), there exists $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots) \in l_\infty$ such that

$$\begin{aligned} 0 = \bar{\lambda}(0) &> \max_{x\in\mathcal{Q})} \sum_i \bar{\lambda}_i x_i \\ &= \max_{\gamma\in\mathcal{P}(Y\times U)} \int_{Y\times U} (\psi_{\bar{\lambda}}'(y)^T f(y,u) + C(\psi_{\bar{\lambda}}(y_0) - \psi_{\bar{\lambda}}(y)))\gamma(dy,du) \\ &= \max_{(y,u)\in Y\times U} \{\psi_{\bar{\lambda}}'(y)^T f(y,u) + C(\psi_{\bar{\lambda}}(y_0) - \psi_{\bar{\lambda}}(y))\}, \end{aligned}$$

where $\psi_{\bar{\lambda}}(y) = \sum_i \bar{\lambda}_i \phi_i(y)$ (see (A.4)). This implies that the function $\psi(y) \stackrel{\text{def}}{=} \psi_{\bar{\lambda}}(y)$ satisfies (1.16), and, by Proposition 1.6(iii), $\mu^*(y_0)$ is unbounded. Thus, we have obtained a contradiction that proves that $W(y_0)$ is not empty. □

Proof of Proposition 1.6(ii). By Proposition 1.6(i), if the optimal value of problem (1.13) is bounded, then $W(y_0)$ is not empty and, hence, a solution to problem (1.8) exists.

Define the set $\hat{\mathcal{Q}} \subset \mathbb{R}^1 \times l_1$ by the equation

$$\hat{\mathcal{Q}} \stackrel{\text{def}}{=} \{(\theta, x): \theta \geqslant \int_{Y\times U} g(y,u)\gamma(dy,du),\ x = (x_1, x_2, \dots), \\ x_i = \int_{Y\times U} (\phi_i'(y)^T f(y,u) + C(\phi_i(y_0) - \phi_i(y)))\gamma(dy,du),\ \gamma \in \mathcal{P}(Y \times U)\}.$$

The set $\hat{\mathcal{Q}}$ is convex and closed. Also, for any $j = 1, 2, \dots$, the point $(\theta_j, 0) \notin \hat{\mathcal{Q}}$, where $\theta_j \stackrel{\text{def}}{=} G^*(y_0) - \frac{1}{j}$ and 0 is the zero element of l_1. On the basis of a separation theorem (see [52], p.59), one may conclude that there exists a sequence $(\kappa^j, \lambda^j) \in \mathbb{R}^1 \times l_\infty$, $j = 1, 2, \ldots$ (with $\lambda^j \stackrel{\text{def}}{=} (\lambda_1^j, \lambda_2^j, \ldots)$) such that

$$\kappa^j(G^*(y_0) - \frac{1}{j}) + \delta^j \leqslant \inf_{(\theta,x)\in\hat{Q}} \{\kappa^j\theta + \sum_i \lambda_i^j x_i\}$$
$$= \inf_{\gamma\in\mathcal{P}(Y\times U)} \{\kappa^j\theta + \int_{Y\times U} (\psi'_{\lambda^j}(y)^T f(y,u) + C(\psi_{\lambda^j}(y_0) - \psi_{\lambda^j}(y)))\gamma(dy,du)$$
$$\text{s.t. } \theta \geqslant \int_{Y\times U} g(y,u)\gamma(dy,du)\}, \quad (A.5)$$

where $\delta^j > 0$ for all j and $\psi_{\lambda^j}(y) = \sum_i \lambda_i^j \phi_i(y)$. From (A.5) it immediately follows that $\kappa^j \geqslant 0$. Let us show that $\kappa^j > 0$. In fact, if it was not the case, one would obtain that

$$0 < \delta^j \leqslant \min_{\gamma\in\mathcal{P}(Y\times U)} \int_{Y\times U} (\psi'_{\lambda^j}(y)^T f(y,u) + C(\psi_{\lambda^j}(y_0) - \psi_{\lambda^j}(y)))\gamma(dy,du)$$
$$= \min_{(y,u)\in Y\times U} \{\psi'_{\lambda^j}(y)^T f(y,u) + C(\psi_{\lambda^j}(y_0) - \psi_{\lambda^j}(y))\}$$

which implies

$$\max_{(y,u)\in Y\times U} \{-\psi'_{\lambda^j}(y)^T f(y,u) - C(\psi_{\lambda^j}(y_0) - \psi_{\lambda^j}(y))\} \leqslant -\delta^j < 0.$$

The latter would lead to the validity of the inequality (1.16) with $\psi(y) = -\psi_{\lambda^j}(y)$, which, by Proposition 1.6(iii), would imply that the optimal value of the dual problem (1.13) is unbounded. Thus, $\kappa^j > 0$.

Dividing (A.5) by κ^j one can obtain that

$$G^*(y_0) - \frac{1}{j}$$
$$< (G^*(y_0) - \frac{1}{j}) + \frac{\delta^j}{\kappa_j}$$
$$\leqslant \min_{\gamma\in\mathcal{P}(Y\times U)} \{ \int_{Y\times U} (g(y,u) + \frac{1}{\kappa^j}[\psi'_{\lambda^j}(y)^T f(y,u))$$
$$+ C(\psi_{\lambda^j}(y_0) - \psi_{\lambda^j}(y)))\gamma(dy,du)\}$$
$$= \min_{(y,u)\in Y\times U} \{g(y,u) + \frac{1}{\kappa^j}(\psi'_{\lambda^j}(y)^T f(y,u) + C(\psi_{\lambda^j}(y_0) - \psi_{\lambda^j}(y)))\}$$
$$\leqslant \mu^*(y_0)$$

$$\Longrightarrow \; G^*(y_0) \leqslant \mu^*(y_0).$$

The latter and (1.14) prove (1.15). □

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