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Identifying Mathematics Teaching Knowledge for Saudi Female Mathematics Teachers in Middle School

by

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In the name of

Allah

The merciful the compassionate

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List of Abbreviations

COACTIV	Cognitive Activation in the classroom (project)
CCK	Common Content Knowledge
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
LMT	Learning Mathematics for Teaching (research project at University of Michigan)
MKT	Mathematical Knowledge for Teaching
MCK	Mathematical Content Knowledge
MOE	Ministry of Education
PCK	Pedagogical Content Knowledge
PISA	Programme for International Student Assessment
PUFM	Profound understanding of fundamental mathematics
SCK	Specialised Content Knowledge
TEDS-M	The Teacher Education and Development Study in Mathematics

Dedication and Acknowledgements

In the name of Allah, the Most Gracious and the Most Merciful I thank Allah (Scheiner, Montes, Godino, Carrillo, & Pino-Fan) for giving me the opportunity, ability, and power to carry out and complete my doctoral study.

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Abstract

Teacher's mathematical content knowledge is an essential aspect of the work of teaching. To teach effectively and apply the required curriculum, other aspects like pedagogical content knowledge and beliefs are rather more important. Student learning is highly connected to teachers and the work of teaching. Over the last few decades, different (theoretical and empirical) frameworks addressing the mathematics knowledge needed for teaching have been proposed, including The Michigan Project (US), COACTIV (Germany), TEDS-M (International), and Rowland's Knowledge Quartet (UK). According to Neubrand (2018), the work of teaching is complex and many scholars argue that describing a multifaceted process like teaching requires more than just examining teachers' knowledge; it must also include related contextual factors. This research argues that a holistic view of mathematics teachers' mathematics knowledge for teaching (MKT) is essential in order to determine how aspects of MKT (Subject Matter Knowledge (SMT) and Pedagogical Content Knowledge (PCK)) and beliefs impact pedagogical decisions. Indeed, relational understanding of the complex work of teaching and the role of teachers' mathematical knowledge for teaching makes new contributions to theory and practice.

The new approach of this research sheds light both on what teachers do and why, by investigating three areas: 1) The gap between knowing and acting; 2) The importance of the cultural context; 3) The affective component. This has been applied through examining three major aspects: 1) knowledge (Types of cognitive content knowledge); 2) beliefs about (nature of mathematics-mathematics teaching- and mathematics learning); and 3) culture**.** It provides additional information about the relationships between them and understanding of how these aspects may impact on teachers' classroom practices. The research evaluates the teaching process through a cultural lens within the Saudi Arabian context. A qualitative study employed three central data collection methods, written test, interviews and observations, to identify Saudi Female Mathematics Teachers' (SFMTs') mathematics knowledge for teaching in middle school in Saudi Arabia. As a result of this investigation, the current study suggests that there are factors other than teachers' knowledge that guide their actual teaching. Finding shows that In the first area, 1) The gap between knowing and acting: teachers' content knowledge focussed primarily on knowledge of facts and rules (Type 1) are significantly linked with the rule-based approach to teaching in Saudi Arabia. In the second area, the importance of the cultural context: teaching is a cultural activity (Stigler & Hiebert, 1999), and Saudi's culture is highly influenced by Confucian epistemological beliefs that shape the processes of teaching in the classroom and the practice of teaching within the country. In the third area, the affective component: Platonist beliefs about the nature of mathematics held by SFMTs are not consistent with a new Saudi curriculum reform where the problem-solving approach in the new curriculum is underpinned by constructivist learning. This thesis provides evidence and insight to guide future teacher education professional development program designers and policy makers in Saudi Arabia and elsewhere.

Declaration

I certify that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge and belief it does not contain any material previously published or written by another person except where due reference is made in the text.

Hdil Alatallah

1/07/2020

CHAPTER 1: INTRODUCTION

This chapter introduces the rationale and problem context to be addressed within this thesis as well as giving a background to important contextual information for the study and an overview of the research topic. General information about the research as a whole is clearly stated. The significance of the study and structure of the thesis are also presented.

1.1 The rationale and problem context of this research

Globally, a basic problem for policy makers is to strengthen the education system in order to improve students' learning outcomes. In Saudi Arabia, government and policy leaders have commenced numerous development programs to promote and enhance its educational system, in particular, implementing a new curriculum with a strong emphasis on constructivist approaches that stress the quality of teachers as being student-centred rather than teacher-centred (Wilson, Cooney, & Stinson, 2005). Mathematicians and educators in Saudi Arabia, and from all over the world, have a rising concern about underachieving students and ask that changes and improvements in teaching mathematics are addressed (Atweh & Brady, 2009). The main aim for the policy maker in Saudi Arabia is to improve student learning, especially after the low level of students' performance noted in the report of the Trends in International Mathematics and Science Study (TIMSS) 2007 (Wiseman, Sadaawi, & Alromi, 2008). However, the Saudi education system appears to not produce and prepare students who have appropriate problem solving strategies and critical thinking skills (Al-Essa, 2009; Allamnakhrah, 2013; Elyas, 2008; Kafa, 2009). Rather, it produces passive students who are thought to receive the information without flexible and creative thinking during their study (Alsaleh, 2019). Research in the literature has indicated that this situation may be due to the priority given to developing mathematics education outcomes over focusing on students' learning and by neglecting the important role of the teachers and their quality of teaching (Alsharif, 2011; Knight et al., 2015). Yet, what students learn is highly connected to the work of teaching and what and how teachers teach. Teachers should draw on a depth of mathematical content knowledge, pedagogical knowledge, curriculum knowledge, and students' knowledge (Callingham, Oates, & Hay, 2019). Shulman (1986) noted the following:

> . . . What we miss are questions about content of the lessons taught, the questions asked, and the explanations offered. . . . Our work does not intend to denigrate the importance of pedagogical understanding or skill in the development of a teacher or in enhancing the effectiveness of instruction. Mere content knowledge is likely to be as useless pedagogically as [a] content-free skill. But to blend properly the two aspects of a teacher's capacities requires

> that we pay as much attention to the content aspects of teaching as we have recently devoted to the elements of teaching process (p. 8).

Scholars from the ‘reform’ context in the USA highlight the crucial role of teachers’ knowledge and beliefs and the relationship between them and practice in implementing a new curriculum (Buzeika, 1996; Cooney, Sanchez, & Ice, 2000).

Alrwathi, Almazroa, Alahmed, Scantbly, and Alshaye (2014) said that teachers’ classroom competency has a great impact when implementing the new mathematics and science curricula in Saudi Arabia. Many international researchers have focused on the importance of teacher knowledge as it is a key principle for high-quality teaching (Blömeke et al., 2015; Bruckmaier, Krauss, Blum, & Leiss, 2016; Davis & Simmt, 2006; Hill, Schilling, & Ball, 2004; Ma, 1999a; Rowland, Huckstep, & Thwaites, 2005; Stigler & Hiebert, 1999). In addition, the essential role of teachers’ beliefs and conception play out in their instructional practices (Calderhead, 1996; Kloosterman, Raymond, & Emenaker, 1996). Buzeika (1996) stated that,

> . . . if pre-service and in-service approaches are to be effective in promoting practice which supports the new curriculum it is critical to clarify the relationship between beliefs and practice. If one element of this relationship is found to be of greater influence than the other, then professional development work could be focused in that direction (p. 99).

Therefore, understanding the different features that influence teacher quality has encouraged researchers to examine teacher knowledge for teaching (Ball, Thames, & Phelps, 2008; Rowland et al., 2005), teachers’ professional skills (Day & Sachs, 2004), their understanding that student achievement parallels teacher knowledge (Tchoshanov, Lesser, & Salazar, 2008), and teachers’ beliefs about mathematics and how this is impacting their teaching methods (Beswick, 2012; Skemp, 1978; Sullivan & Mousley, 2001).

Additionally, it is surprising that there is a shortage of research in Saudi Arabia that addresses the importance of teachers’ knowledge, beliefs and teaching practice, as much research has been done in the United States and European countries. According to Linde (2003), “teacher education has to be analysed and understood in the context of where it takes place” (p. 110). Thus, teachers’ knowledge as well as beliefs in the context of Saudi Arabia were taken into account for this research in order to understanding the process of teaching and what teachers do in the classroom to assist students’ learning.

1.2 Aims of current study

This research seeks to identify teachers' mathematics content knowledge (MCK), pedagogical content knowledge (PCK), and beliefs through the work of teaching in Saudi Arabia. According to Neubrand (2018) the work of teaching is complex, which makes many scholars argue that describing such a complex process like teaching requires more than just examining teachers' knowledge, it also requires examining the related contextual factors, especially when they are connected to the field. Thus, three areas are investigated:

1- The gap between knowing and acting;
2- The importance of the cultural context;
3- The affective component

This research attempts to fill the gap in the literature noted by Neubrand (2018) by examining all of these aspects and how they interact with each other in the teaching process and influence the quality of teaching in mathematics in middle classrooms in the context of Saudi Arabia, with an emphasis on the role of Saudi culture in their practice.

1.3 Background information related to the context of Saudi Arabia

1.3.1 The education system in Saudi Arabia

The education system in Saudi Arabia is formed and maintained by the Ministry of Education, in the kingdom which was first established in 1954 (Alanazi, 2016). The Ministry is mainly responsible for creating and assessing the curriculum, teachers' training and development, and educational school programmes which have been designed based on long and short terms goals. Moreover, it is responsible for school buildings and distribution around the country. Public and private schools in Saudi Arabia are all regulated by MOE, however private schools have the right to choose different curriculum as long as they get approval from the MOE (Saudi Ministry of Education, 2019).

The educational system in Saudi Arabia is strongly linked with and designed to suit the country's economic and social needs. In fact, this system was heavily influenced by the dominant religion in the country (Al-Essa, 2009). As with any other education system in the world, the prime aim of education in Saudi Arabia is to equip its people with the knowledge and skills needed to boost the country's economy, science and many other sectors of the recently announced Saudi Arabia Vision 2030 which focusses on moving the country toward independency and self-sufficiency. Many

Saudi citizens would argue that this vision has pushed the country into a new phase of openness and diversity that is unprecedented in Saudi Arabia. However, the Saudi government realises that the ultimate goals of Vision 2030 necessitate investment in education. In fact, education was heavily influenced by the introduction of Vision 2030 (Saudi Minstry of Education, 2020).

Saudi Arabia’s government gives a massive consideration to the education system in the country. It has dedicated around 20% of its budget to education (Saudi Ministry of Finance, 2020). This budget includes building schools and universities. Schools in Saudi Arabia consist of three levels: primary, secondary and high schools. Since the foundation of the country, schools in Saudi Arabia have been designed to be single sex schools. Starting from year 1 up to high educational, females do not mix with males in the educational setting although recently the MOE started a new model of schooling in the kingdom which allows selected primary schools to be mixed. Students have to spend six years, three years and three years respectively in schools, which is in total a minimum of twelve years for each student during school life. In the last year of high school, students are required to perform tests in all subjects in the school and then, in order to apply for any domestic university, students have to undertake another test which is set by the MOE and not the school (Saudi Ministry of Education, 2019).

1.3.2 Saudi educational reform

As mentioned above, Saudi Arabia has made a huge commitment to education in the country. Therefore, the motive of education development has led the government to seek a continuous assessment for their educational programmes and students’ level of knowledge and achievements. Trends in International Mathematics and Science Study (TIMSS) is one assessment tool of many that has been used in Saudi Arabia to evaluate students’ achievements in Science and Mathematics (Alanazi, 2016). In fact, the report of the 2016 TIMSS indicated poor performance by Saudi students, being below the average among their peers all over the world (Alanazi, 2016; Tayan, 2017).

In response to students’ poor achievements in mathematics, the Ministry of Education in Saudi Arabia has investigated the reasons for this issue. Results of investigations pointed out many reasons that caused such low achievements, with teaching quality being one of them. This is because teaching strategies that were used in classrooms were mainly focusing on memorizing and were not presented in ways that challenged the students or encouraged them to be self-sufficient in their studying (Alanazi, 2016).

As a result, for such investigations and the strong need to reform the whole education system, the Ministry of Education in Saudi Arabia set plans to introduce certain programs to help improve students' achievements (Alanazi, 2016; Tayan, 2017) (see Figure 1 below).

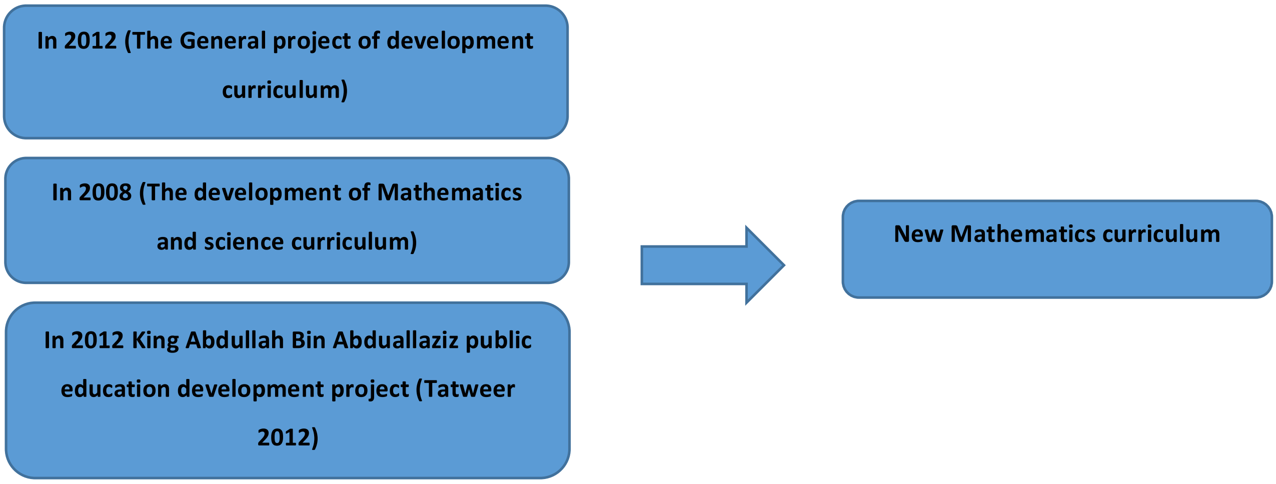


Figure 1: The educational development projects

In 2003 the Ministry of Education implemented and monitored what is called the general projects for curriculum development. In 2007 students had to perform a mathematics test. TIMSS results and achievement on the test were not satisfactory for officials in the education sector (Alanazi, 2016). Therefore, in 2008 MOE introduced another project called the Development of the Mathematics and Science Curricula Project and sought an overseas expert to assist in designing this project. In fact, the developers of the new program thought that changes this time had to be wider, and more factors like structure of curriculum had to be examined (Alanazi, 2016).

1.3.3 New Mathematics curriculum

Since students' knowledge of mathematics was a main focus of educational policy makers in Saudi Arabia, the Ministry of Education introduced in (2008) a new math curriculum based on constructive approaches. The new textbooks for this curriculum were designed to be challenging for the students and help them to be more independent in math learning. It encouraged them to be free thinkers and build more comprehensive knowledge, rather than learning through repeating or memorizing. Those textbooks were intended to motivate students to gain and apply problem solving techniques and establish a dialogue between teachers and students (Alghamdi & Al-Salouli, 2013).

In addition, all efforts were made to improve education in the kingdom. MOE continued to seek improvement and development in the education sector. There was an intention for the MOE to seek a further development for curriculum as mentioned above, and also training for teachers. Since teachers are in the front line in dealing with students, in 2012 MOE introduced a new project called the King Abdullah Bin Abdulaziz Public Education Development Project (Tatweer) (Alanazi, 2016; Tayan, 2017). This project focused on improving the learning environment in schools as well as teacher's development and training (Alanazi, 2016). Tatweer is an Arabic word meaning development and this project is currently conducted in many schools and is planned to be applied in all the schools around the kingdom of Saudi Arabia (Alyami, 2014).

1.3.4 Mathematics teachers' professional development

Within the context of improving students' knowledge and overall performance, the Saudi Government has realized the need for substantial changes in the structure of education in order to meet the ambitious plans and desires of the country and for the policy makers to open up our society to the world and integrate with it even more effectively (Alanazi, 2016). The Tatweer project was then introduced to boost the learning environment through continuous development in curriculum, enhancing teaching methodologies and teacher training (Alanazi, 2016; Alyami, 2014; Elyas & Picard, 2013). Ultimately, in this project, schools had to shift their teaching and learning environment from being highly structured to giving more freedom to students and creating environments that stimulate collaborative working between teachers and students and between students themselves (Tayan, 2017).

On top of that, teachers at schools were classified into different categories and they were given different titles based on their knowledge, qualifications and experience. More experienced teachers were leading certain programmes that related to their experiences and less experienced teachers were required to help their experienced peers to implement the new programmes within the schools (Alanazi, 2016). However, it was believed that this structural reforming within the schools would not have a positive outcome without providing educational professionals with training and opportunities for self-development (Alanazi, 2016).

The Tatweer project emphasises the importance of training for all staff within the school, from leaders to less experienced teachers. The fundamental goals for those training programmes are to improve teaching skills and knowledge for teachers; to assist teachers to create new learning environments within the classrooms; to promote diverse teaching and learning process in the classrooms; to equip staff in the school with the required management skills to maintain such new learning environments; and to improve overall student achievement (Alanazi, 2016; Tayan, 2017). In parallel with curriculum development and teachers' training, the Tatweer project has emphasised the importance of ICT in the educational settings. WIFI has to be available to all students and staff, classes have to be equipped with smart boards, and the Internet is considered a source of learning (Tayan, 2017).

Policy makers in MOE believe that in order to succeed with the ultimate goals of improving students' achievements, other factors have to be taken into consideration to ensure the successful implementation of Tatweer, such as:

1- In the new project families were strongly advised to be involved in the educational plan for each student.

2- Professionals were given the required authority and assisted to apply all the changes required for school transformation policy and practice into the new project.

3- To ensure effectiveness of the new project, governing rules for schools and professionals were modified. Teachers were given different titles based on their experience and performance; a new reward system was introduced for professionals, and leaders were given more freedom and authority to apply and maintain the new project at schools.

4- Students' health was to be closely monitored and staff and parents had to establish communication channels for the benefits of students. (Alanazi, 2016)

To sum up, MOE in Saudi Arabia pays a great deal of attention to education. They have been investing for a long time in improving education, with a large budget that has been set up to serve the purpose. However, Almannie (2015) argued that in spite of increasing investments in the education sector and the improvement of new mathematics and science curricula, the educational productivity did not reflect this investment as was expected. Further, students' achievements were not satisfactory, as illustrated by the poor performance on standardized mathematics tests. As a result, MOE proposed and applied many educational developmental programmes to tackle this issue, such as new mathematics curriculum reform. Yet, Badri et al. (2016) illustrated that education is a complex system because the enhancement and development of the curriculum is not the only factor that could improve education outcomes and lead to success. There are other important factors that might have contributed to the students' poor performance and contribute to quality mathematics classes having been neglected. For example, teachers' MCK, PCK and beliefs have not been taken into consideration, even though numerous research papers have given evidence of the importance of these factors on the quality of teaching and the quality of knowledge being delivered to students. Therefore, this research has been designed to take these factors into considerations.

1.4 Significance of the study

This research has the potential to make several contributions to the mathematics education research community and teachers' education development and policy in Saudi Arabia.

Firstly, this research adds to the literature by offering a new approach in examining mathematics teaching which aims to address the three limitations in the literature which had been suggested by Neubrand (2018). Instead of treating the work of teaching by describing teachers' knowledge and/or beliefs, this research addresses both of these essential aspects and how they interact with each other, as well as the influence of culture on the work of teaching.

Secondly, the findings of this research could give an insight to Saudi teacher education programs, as they "may provide important insights into validity of the measures and the subtleties of the impacts of teacher education programs. Whether or not this proves to be the case, the current policy environment makes evidence-based teacher education a priority" (Beswick & Goos, 2012, p. 87). Thus, it will help teachers and policy makers lead their decisions and efforts to focus on the essential aspects that can achieve effectiveness. Also, this research has the potential to impact on future teacher preparation programs in Saudi Arabia as it provides information about the factors that could impact their training practice as well as how teachers implement new teaching practices required for teaching the new curricula.

Thirdly, the mathematics education community could benefit as this research adds to the literature by increasing opportunities for researchers to study female teacher knowledge in Saudi Arabia through using the Mathematical Knowledge for Teaching (MKT) items (Ball & Hill, 2008) in Saudi Arabia. This is because there is not much research available on mathematics teachers who are from the culture of the Middle East; and research on female teachers is rare as well. Also, this research adds to the international work of the Learning Mathematics for Teaching (LMT) research group (the authors of MKT items).

Further, the mathematics education community could gain new insights as this research adds to the literature on how Saudi culture influences the way teachers are teaching; particularly, it broadens the opportunities for researchers to study the culture in Saudi Arabia and compare it with cultures across other countries. This could have the potential to impact teacher education policy in Saudi Arabia by taking the culture into account when analysing teachers' teaching for future development as it could help enhance and improve teacher effectiveness.

1.5 Structure of the thesis

This thesis is organized into six chapters:

Chapter two is the main literature review for this research and is divided into three objectives. The first objective is a review of research that offers a foundational understanding as well as empirical research on mathematical knowledge for teaching and how culture could impact mathematical knowledge for teaching. The section contains two parts: the first part discusses different approaches to understand effective teaching by focusing on the cognitive lens and the affective lens; the second part discusses the relationship between the affective lens and culture and how culture impacts beliefs about education. The second objective is a critical analysis of the literature by addressing three limitations: the gap between knowing and acting, the narrow understanding of the role of cultural context, and the still-developing work of the affective aspect of mathematical knowledge for teaching. The third objective is to express the theoretical framework which this thesis follows to address the new approach in order to answer the research questions concerning the factors that impact Saudi middle school mathematics teachers' knowledge for teaching.

Chapter three provides illustrations and justification of the methods that were carried out in order to address the research questions, and the underlying theoretical conjectures of the researcher are taken into consideration as they might have influenced the research design. The three central data collection methods used for this qualitative methods research (written test, interviews and observations) are then considered in turn.

Chapter four details the finding from the three data collections methods and analyses them based on the theoretical framework of this research which comprises three frameworks (Cognitive type of content knowledge; Categories of teacher beliefs; the Knowledge quartet framework).

Chapter five comprises three main discussion points from this research (examining Saudi teachers' cognitive type of content knowledge, examining SFMTs' beliefs system, and examining SFMTs' practice) in a cultural context. Each discussion section takes into consideration the related existing empirical research and how the implication of findings is related to this literature.

Chapter six presents the overall conclusions of this research, limitations of the research sample, data collection methods and the analyses of the data. Finally, recommendations are suggested for further research.

CHAPTER 2: LITERATURE REVIEW

2.1 Introduction

> We cannot make major headway in raising student performance and closing the achievement gap until we make progress in closing the teaching gap (Darling-Hammond, 2015, p. 4).

A major goal for educational researchers around the world is to find ways to improve students' learning. There are various factors that influence students' learning outcomes: a seminal meta-analysis of 800 previous meta-analyses of studies of student learning outcomes showed that these factors include the student's characteristics, the school curriculum, the environment of the school, and the student's family (Hattie, 2008). Although Hattie's meta-analysis showed that demographic factors describing the student were the strongest predictors of students' learning outcomes, this meta-analysis also indicated that the factors within educators' and policymakers' control, factors related to teachers and teaching approaches, were strong predictors of learning outcomes. Therefore, results of empirical studies indicate that, if researchers want to contribute most to understanding how to improve student learning outcomes, the most fruitful approach is to better understand why teachers choose their teaching approaches (Ball & Forzani, 2009).

Of course, the literature on the role of teachers in structuring effective learning environments, and creating effective educational discourse, remains rife with productive controversy and debate (Chen, Kalyuga, & Sweller, 2016; Hattie, 2009; Kirschner, Sweller, & Clark, 2006; Sweller, 2016). Still, a general consensus exists regarding teachers of mathematics: mathematics teachers who have deep understandings of mathematical content are able to scaffold learners' learning process more flexibly and more purposively (Ball et al., 2008; Beswick & Goos, 2012; Burghes, 2011; Chapman, 2015; Jacobson & Kilpatrick, 2015; Lai & Murray, 2012; Tatto et al., 2008; Zhang & Stephens, 2013). In short, there is ample empirical evidence that the more extensive and deep a mathematics teacher's knowledge of the mathematics content (s)he has mastered is, the more agency(ies) he/she has in selecting teaching approaches.

This matters in relation to understanding student outcomes because there is also ample empirical evidence that the quality of classroom teaching is an essential component of the quality of students' mathematics learning (Boaler, 2001; Cohen, Raudenbush, & Ball, 2003; Sanders, 1998). Indeed, some research indicates that students are more able to enhance their mathematics knowledge, thinking skills, and abilities in a classroom setting than in any other setting (Cai, 2007). For this reason, understanding the factors that contribute to effective mathematics teaching is absolutely crucial for scholars of mathematics education, as effective mathematics teaching is essential to enhancing students' learning of mathematics (National Council for Teachers of Mathematics, 2000; Stigler & Hiebert, 2009).

There is already an extensive range of research dedicated to understanding how to better enhance teaching in mathematics (Ball, 2000; Fennema & Franke, 1992; Rowland, 2013). To understand how to improve mathematics teaching, it is essential to have a foundational notion of what constitutes effective teaching, and yet, effective teaching remains an elusive notion (Stronge, 2007). Indeed, according to Hemmi and Ryve (2015) there is no clear definition of what is effective classroom teaching and what constitutes quality teaching. Effective teaching varies across different cultures and is influenced by the educational systems which establish demands for mathematics teaching and the expected learning results of the students (Hiebert & Grouws, 2007). Therefore, notions of effective teaching can differ greatly based on cultural context and geographic location.

Further, it is very challenging to judge and assess the quality of classroom teaching (Li, 2011). For one thing, what constitutes effective teaching can change across cultures, depending on each culture's conceptions, imperatives, and systems surrounding learning (Li, 2011). Further, the question of what makes mathematics teaching effective has a fundamental, unavoidable normative component, because it involves decisions about what people think is important to know (Krainer, 2005). Nevertheless, a clearer definition of effective teaching is important as it has a high impact on teachers' decisions and planning for teaching. The way that teachers conceive of effective teaching, or the standard of effective teaching that they hold, will influence the way they plan their teaching as well as the decisions they make in the classroom (Krainer, 2005). Therefore, this study uses the following definition of effective teaching, borrowing from Krainer (2005): "interventions mounted by a mathematics teacher that make a relevant difference to a learner's knowledge, understanding, or skill in mathematics” (p. 78).

Previous literature has established that mathematics teachers’ mathematical knowledge for teaching is related to their ability to structure and execute their teaching within the classroom (Ball et al., 2008; Beswick & Goos, 2012; Burghes, 2011; Chapman, 2015; Jacobson & Kilpatrick, 2015; Lai & Murray, 2012; Tatto et al., 2008; Zhang & Stephens, 2013). Therefore, in order to analyse how effectively mathematics teachers teach within the classroom, a clear understanding of what is meant by ‘mathematical knowledge for teaching’ is crucial. As Petrou and Goulding (2011) argued, "…without some common understanding of what subject knowledge means and what it looks like in practice, there can be no coherent approach to ... answering research questions about the role of teachers' mathematical knowledge in teaching" (Petrou & Goulding, 2011, p. 9).

In other words, just as teachers need mathematical knowledge for teaching in order to effectively execute classroom mathematics teaching strategies, researchers interested in understanding what constitutes effective mathematics teaching need a clear concept of mathematical knowledge for teaching in order to understand how mathematical knowledge for teaching influences mathematics teaching. As Ball stated:

> As a field, we wanted to understand how teachers’ mathematical knowledge matters for teaching and learning. We wanted to know this with more practical relevance and more theoretical clarity. We assumed that something about mathematical knowledge would affect the quality of teaching and learning. But what we need to be talking more clearly about is mathematical knowing and doing inside the mathematical work of teaching. This change from nouns— “knowledge” and “teachers”—to verbs— “knowing and doing” and “teaching”(2017, p. 14).

Indeed, the way that mathematics teachers choose to approach their work within the classroom is in part driven by their conceptions of “knowing and doing” and “teaching” (Ball, 2017).

However, as with the concept of effective teaching, how to best define mathematical knowledge for teaching is still highly contested within the literature (Scheiner et al., 2019). Still, some consensus does exist within the literature. First, mathematical knowledge for teaching is distinct from overall knowledge of mathematics as a discipline because mathematical knowledge for teaching specifically concerns knowledge of mathematics for the purpose of teaching (Shulman, 1986). This is distinct from knowledge of mathematics as a discipline because it also includes knowledge of the most useful ways of externally representing or explaining the mathematical concepts that the teacher is responsible for teaching (Ball, 1988; Shulman, 1986).

For the purposes of this thesis, the researcher defines mathematical knowledge for teaching as the knowledge of mathematics content, mathematics pedagogy, and broader pedagogy that the mathematics teacher uses to teach mathematics. This definition draws upon the Scheiner et al. (2019) observation that various approaches to understanding mathematics knowledge have alternately emphasized three interrelated but distinct aspects of mathematical knowledge for teaching: additional knowledge, qualitatively different knowledge, and teaching-oriented action (p. 160).

Researchers who emphasize additional knowledge explore how mathematics teachers need a knowledge of mathematics that expands beyond the mathematics concepts they are directly tasked with teaching in the classroom (Scheiner et al., 2019). In particular, this approach emphasises the requirement of different dimensions of mathematical knowledge as well as teacher knowledge in general (Scheiner et al., 2019). It reflects a type of knowledge introduced by Shulman (1986) as that "which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (p. 9) and it was termed pedagogical content knowledge (PCK). He explained further that teacher knowledge consists of three types of knowledge: pedagogical content knowledge, knowledge of curriculum, and knowledge of subject matter. Moreover, Shulman (1986) defined PCK as encompassing:

> for the most regularly taught topics in one's subject area, the most useful forms of [external] representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others [...] [and] an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons (p. 9).

So according to Shulman's (1986) previous definition, PCK comprises two features: 'knowledge of representations of subject matter' and 'knowledge of specific learning difficulties and students' conceptions'.

Moreover, PCK was also described by (Shulman, 1986; Shulman, 1987) as a knowledge that combined pedagogy knowledge, content knowledge, and knowledge of learners, in order to understand and find out the best method of teaching and explaining specific concepts to students that would suit their ability, personality, skills, and interest, as well as in order to understand how to simplify students' learning of specific concepts. Moreover, PCK includes teachers' understanding of students' ability as well as obstacles and limitations to their learning, and also teachers' understanding of different methods and forms of representation to deliver knowledge to students to make it easier and more interesting for them. Thus, when thinking about PCK, these two elements (i.e., understanding students' abilities and understanding different methods) are often considered to be reference points (Scheiner et al., 2019).

Some scholars in mathematics education have explained and extended the original considerations of PCK. For example, Marks (1990) offered four dimensions of PCK in the context of teaching mathematics which are: knowledge of subject matter for instructional purposes, knowledge of instructional processes, knowledge of students' understanding, and knowledge of media for instruction. In addition, the PCK notion was further investigated and extended by Ball and her colleagues (2008). They suggested that PCK is a combination of knowledge that assists teachers to select the type of tasks that they will teach in the classroom. It also helps teachers in their explanation and representation of mathematical concepts in the classroom. Moreover, PCK would give teachers the ability to interpret and organize classrooms dialogues as well as understand the reasons for students' failure and difficulties.

Similarly, the work of Schwab (1978), Ball (1990), and Ball and Bass (2009) introduced the notion of knowledge of the practice in mathematics (KPM). Those researchers had investigated mathematical knowledge and syntactic knowledge and they emphasised that teachers should be able to utilise different type of reasoning when teaching mathematics and they should be able to modify the types of reasoning to another type based on the circumstances in the classroom. Moreover, KPM can be explained further as the knowledge of teachers in how to communicate and discuss mathematical knowledge with students by exploring, proceeding, and creating knowledge in mathematics, for instance, including ways to use heuristic methods for solving problems, the role of generalization in mathematics, and knowledge about selecting and constructing related components.

Researchers who emphasize qualitatively different knowledge (i.e., the second type of teaching knowledge proposed by Scheiner et al. (2019)) examine how mathematics teachers need to hold a different conceptual and pedagogical knowledge than do teachers of different disciplines, such as writing, art, or biology, and other practitioners of mathematics who do not teach mathematics, such as engineers (Scheiner et al., 2019). Mathematics subject matter knowledge is special for mathematics teachers and the teaching of mathematics, and different from subject matter knowledge required by those who teach other subjects (Scheiner et al., 2019). In fact, Baumert et al. (2010) argued that although PCK has proved its effectiveness in quality of instructions and students' progress, "PCK is inconceivable without a substantial level of CK [content/subject knowledge]" (p. 163). This is because subject knowledge is crucial to PCK (Cai, Mok, Reddy, & Stacey, 2016; Krainer, Hsieh, Peck, & Tatto, 2015).

Also, it has been argued by Ball et al. (2008) that teaching may require "a specialized form of pure subject matter knowledge" (p. 396):

> Pure because it is not mixed with knowledge of students or pedagogy and is thus distinct from the pedagogical content knowledge identified by Shulman and his colleagues and specialized because it is not needed or used in settings other than mathematics teaching. (Ball et al., 2008, p. 396).

Special Content Knowledge (SCK) is a particular form of mathematical skills and knowledge specific for teaching mathematics. It involves conceptual and procedural understanding of mathematics as well as identifying and understanding students' errors and misconceptions in mathematics (Ball et al., 2009; Ball et al., 2008). Hill, Rowan, and Ball (2005), Ball, Hill, and Bass (2005) gave more illustrations of the concept of SCK. They stated that SCK is a type of knowledge that requires skills and comprehensive understanding for teachers to be able to understand and analyse students' behaviours and responses, provide an appropriate explanation for mathematical content, and use different means when teaching mathematics, for example applying visual aids by using a suitable image to deliver certain concepts. As noted earlier, SCK is a special type of knowledge that only mathematics teachers need to have; it is not necessary for other practitioners who utilise mathematics in their work (Silverman & Thompson, 2008). For example, an accountant and a doctor do not have to provide a mathematical reason for finding a common denominator when adding fractions, whereas reasoning and explanation is essential in mathematics teachers' work of teaching in the classroom (Lai & Clark, 2018).

Researchers who emphasize teaching-oriented action (i.e., the third type of teaching knowledge proposed by Scheiner et al. (2019)) study how mathematics teachers organize content knowledge to present it to students in a way that is accessible to them (Scheiner et al., 2019). This third approach, extending and underpinning the second orientation (qualitatively different knowledge), focuses on teachers' actions and how they deal with the subject matter knowledge, through their transforming, demonstrating, and delivering the subject matter (Scheiner et al., 2019). It is the notion of transforming the subject matter knowledge of a specific discipline which, as Shulman (1987) said, is an exclusive feature of teachers' professional work: a teacher must "transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students" (Shulman, 1987, p. 15). The key intention of transformation is to shape, organise, construct, and represent the subject matter knowledge for the subject in a way "that is appropriate for students and peculiar to the task of teaching" (Grossman, Wilson, & Shulman, 1989, p. 32).

In recent times, professionals working in the Knowledge Quartet research program have been widely elaborating on the concept of transformation (Rowland, 2009; Rowland et al., 2005) as a part of their understanding of different situations where the mathematical knowledge is important in teaching. Tim Rowland and his colleagues considered transformation as concerning "knowledge in action as demonstrated both in planning to teach and in the act of teaching itself. A central focus is the representation of ideas to learners in the form of analogies, examples, explanations, and demonstrations" (Rowland, 2009, p. 373). The Knowledge Quartet framework focuses on teachers' knowledge in action and teachers' activities in the transformation of the content knowledge. The definition of mathematical knowledge for teaching articulated previous attempts to combine these three elements of mathematical knowledge for teaching.

Therefore, to understand the role that mathematical knowledge for teaching plays in effective pedagogy, we need to examine it as a whole concept, and equally consider teachers' knowledge of mathematics content, knowledge of mathematics pedagogy, and knowledge of broader pedagogy. Only by examining teachers' mathematical knowledge for teaching in all its constituent parts can we clarify how this knowledge impacts mathematics teachers' teaching practices and, in turn, their teaching efficacy (Scheiner et al., 2019). Since one of the bedrock assumptions of this research, built on previous empirical observations, is that mathematics teachers' knowledge for teaching lies at the heart of effective teaching, unpacking the nature of mathematics teachers' knowledge for teaching promises to yield important insights on ways to improve teaching efficacy (Ball et al., 2008; Beswick & Goos, 2012; Burghes, 2011; Chapman, 2015; Jacobson & Kilpatrick, 2015; Lai & Murray, 2012; Tatto et al., 2008; Zhang & Stephens, 2013).

The next section of this chapter has three objectives. The first objective is to review the literature that provides foundational understandings and empirical demonstrations of mathematical knowledge for teaching and how mathematical knowledge for teaching can be conditioned by culture. The second objective is to critically analyse this literature and offer a discussion of three short-comings of the field: the underexplored gap between knowing and acting, the limited understanding of the role of cultural context, and the still-developing notion of the affective component of mathematical knowledge for teaching. The third objective is to articulate the theoretical framework that this thesis uses to undergird the empirical approach to answering the central research question about the factors that influence Saudi middle school female mathematics teachers' knowledge for teaching.

To accomplish these three objectives, this chapter continues in three broad parts. The first part reviews the empirical and theoretical literature that sheds light on the central research question. It begins with a section that contains a more substantial discussion of the various approaches to understanding what constitutes effective teaching (Chapter 1, section: 2.1), with a special emphasis on the affective lens (Chapter 1, section: 2.1.1) and the cognitive lens (Chapter 1, section: 2.1.2). It then moves to explain how the affective lens relates to questions of culture, and narrows down to a discussion of broad findings of how culture influences beliefs about education, an explication of the categories of mathematics teachers' beliefs about mathematics, and explaining each category of beliefs about mathematics and how the category interacts with culture to condition mathematical knowledge for teaching.

The second major section of the review of the literature on effective teaching details research that uses the cognitive lens to understand mathematical knowledge for teaching, with a special emphasis on research that explores how culture influences mathematics teachers' cognition about mathematics. To supplement this, this section also discusses several major research projects that have used the cognitive lens to better understand mathematical knowledge for teaching, including Shulman (1986) theoretical framework; the Michigan Project (Ball et al., 2005; Ball et al., 2008; Hill et al., 2005; Petrou & Goulding, 2011); the COACTIV Project (Baumert et al., 2010; Cole, 2011; Kunter et al., 2013; Neubrand, 2018; Warburton, 2015); TEDS-M (Cole, 2011; Neubrand, 2018; Senk, Peck, Bankov, & Tatto, 2008; Tatto et al., 2008); and the Knowledge Quartet (Clarke & Clarke, 2002; Neubrand, 2018; Rowland, Huckstep, & Thwaites, 2003, 2004; Rowland & Turner, 2007; Weston, 2013).

This chapter's second objective is to provide a substantive, productive critique of the current literature on mathematical knowledge for teaching. To accomplish this, the third major section of the chapter discusses three research gaps in the current literature: the underexplored gap between knowing and acting, the limited understanding of the role of cultural context, and the still-developing notion of the affective component of mathematical knowledge for teaching. These discussions point the way to the final section of the chapter, which offers a theoretical framework that attempts to fill the research gaps in current approaches to understanding mathematical knowledge for teaching.

2.2 Effective Teaching: The Elements of Effective Teaching

Teachers' beliefs about what constitutes effective mathematics teaching are conditioned by many aspects, some of which include: desirable goals of the mathematics teacher education programs, teachers' roles in teaching, the students' roles, relevant and applicable classroom activities, desirable instructional methodologies and teachers' mathematical techniques (Thompson, 1992). Culture plays a key role in conditioning beliefs about many of these aspects (An, Kulm, Wu, Ma, & Wang, 2006; Cai, 2005, 2007; Cai, Kaiser, Perry, & Wong, 2009; Cai & Wang, 2006; Ma, 1999a; Perry, Tracey, & Howard, 1999; Stigler & Hiebert, 1999).

Indeed, understanding teachers' conceptions of what constitutes good or effective mathematics teaching is a complex pursuit. It is culturally and contextually contingent, as culture and context affect the different philosophies, beliefs, approaches and practices available to and accepted by different teachers, as well as in their society at large (Pratt, Kelly, & Wong, 1999). In particular, mathematics teachers develop philosophies about mathematics, and these philosophies constitute epistemologies of mathematics and mathematics education or, to put it differently, they are individuals' beliefs and ideas about mathematics and its teaching and learning.

Cross-national and comparative research has showed that, around the world, mathematics education is highly influenced by cultural and social factors that shape and guide teaching goals, beliefs and teaching methods (An et al., 2006; Cai, 2005; Cai et al., 2009; Cai, Perry, & Wong, 2007; Cai & Wang, 2006; Ma, 1999a; Perry et al., 1999; Stigler & Hiebert, 1999).

Cultures and societies can vary widely regarding their philosophies and beliefs about teaching and learning mathematics (An et al., 2006).Current understandings of how culture conditions notions of effective teaching tend to centre around an implicit comparison of Western and East Asian cultures (Leung, 2001; Tweed & Lehman, 2002; Watkins, 2000). Indeed, much research on cultural beliefs about pedagogy have emphasized the power of Socratic and Confucian philosophies in the educational traditions of Western and East Asian cultures respectively (Leung 2001; Tweed & Lehman 2002; Watkins 2000).

Indeed, reviews of empirical studies of the role of culture in perceptions of mathematics teaching efficacy indicate that there are great differences of opinion about what constitutes good mathematics teaching between educators and students in East Asian and Western societies (e.g., Cai 2005; Cai and Wang 2006; Cai et al. 2007; Cai et al. 2009; Ma 1999; Stigler and Hiebert 1999; Perry et al. 1999). For this reason, there is a consensus among researchers that the goal of education researchers should not be to find universal criteria meant to be applied across all cultures (Kaiser, 2005). Rather, research on conceptions of teaching efficacy in mathematics should focus on better understanding the extent to which normative factors condition expectations of teaching efficacy, and move toward better understanding of factors, especially teaching approaches, that influence whether classroom teaching meets culturally conditioned expectations of mathematics teaching efficacy (Kaiser, 2005).

However, this should not be taken to mean that research that focuses on mathematics teaching efficacy should only focus on teaching practices and strategies. Indeed, there is also a broad consensus that understanding whether classroom teaching is effective or ineffective, and explaining the reason for its efficacy or lack thereof, is a much broader question than what happens in the classroom (Roberts et al., 2010). As Hersh (1986) argued:

> One's conceptions of what mathematics is affects one's conception of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it. … The issue, then, is not, What is the best way to teach? But, What is mathematics really about? (p.13)

In other words, no matter how a given culture or society conceives of effective mathematics teaching, in order for researchers to understand not only what effective teaching is in a given context, but also why it happens when it does, it is crucial to understand teachers' views and understandings of the nature of mathematics, because teachers' views on the nature of mathematics are key to teachers' beliefs about optimal teaching approaches (Hersh, 1979). For this reason, investigating how teachers understand the nature of mathematics, and how people learn mathematics, is crucial to providing a clearer picture of how teachers view effective teaching, and therefore of their teaching practices (Cai, 2007).

2.2.1 Culture and Affective Lens

A crucial factor that influences mathematics teachers' approach to teaching, and therefore learning outcomes, is mathematics teachers' affective lens. This affective lens can best be understood as the beliefs that mathematics teachers hold about the nature of mathematics itself, as well as how mathematics teaching and learning happens (Ernest, 1989b). This leads to the important question of what constitutes a belief. While it is beyond the scope of this thesis to fully explore this question, the researcher has employed Richardson (1996) definition of beliefs: "psychologically held understandings, premises, or propositions about the world that are felt to be true" (p. 103). Understanding teachers' beliefs is crucial to constructively recognizing the way that mathematics teachers conceptualize mathematics, mathematics teaching, and mathematics learning. Indeed, some researchers have suggested that beliefs are one of the essential aspects that impact teachers' practices (Pajares, 1992; Richardson, 1996; Thompson, 1992).

In this research, the researcher understands that beliefs in this area involve teachers' perceptions, ideas and personal philosophies, but due to the focus of the research questions, this thesis specifically focuses on those beliefs that shape teaching practice (Lepik, Pipere, & Hannula, 2013). This research focus on beliefs makes a central assumption that an individual's behaviour reflects his/her own beliefs, an assumption that is customary in the literature (Ajzen & Fishbein, 1980). Again, since the research focus of this thesis concerns mathematics teachers' pedagogical approaches, this thesis also assumes, in keeping with much of the literature, that a teacher's methods of teaching mathematics indicate their priorities for teaching mathematics, which are in turn indicative of the teacher's beliefs about the nature of mathematics, mathematics teaching, and mathematics learning (Thompson, 1992). Previous research has shown that this, in turn, will influence how students understand mathematics (Cai, 2004). For this reason, understanding mathematics teachers' affective lens, or their beliefs about the nature of mathematics, mathematics teaching, and mathematics learning, is essential to understanding teachers' pedagogical behaviour and learners' outcomes.

Beyond this, there is extensive empirical work that supports the notion that belief is a factor that shapes teachers' approaches to their teaching and learning, and therefore teachers' pedagogical decisions (Ball, 1988; Leikin & Levav-Waynberg, 2007; Liljedahl, 2008; Schoenfeld, 2010) . In a seminal theoretical work, Ball (1988) argued that, from childhood, mathematics teachers begin developing "a web of interconnected ideas about mathematics," which includes ideas about teaching and learning mathematics, and about schools" (p. 1). This notion has been substantially expanded upon in subsequent empirical work. In a qualitative study based on data collected from interviews with twelve mathematics teachers, Leikin and Levav-Waynberg (2007) found that teachers' opinions and decisions about how best to teach mathematics were influenced by socially constructed webs of beliefs that affected their mathematical and pedagogical understandings. According to Liljedahl (2008), researchers argue that it is essential to understand that this affective lens consists of much more than just feelings about mathematics; rather, it is a system of beliefs that forms and shapes teachers' thinking about mathematics itself and mathematics teaching and learning.

There has been considerable theoretical and empirical research dedicated to understanding how culture can influence mathematics teachers' webs of beliefs. For the purposes of this thesis, the researcher understands culture to be a group of people's ways of thinking, believing, and valuing that offers tools, norms, values, and habits that impact on all people's behaviours (Bruner, 1996; Cole, 1996; Heath, 1983). Teaching and learning are two such behaviours, meaning that culture is a crucial aspect in understanding learning outcomes (Bruner, 1996; Cole, 1996). Therefore, before more fully describing and analysing the literature that disaggregates mathematics teachers' webs of beliefs, this section briefly discusses the different cultural conceptions that have been found in the literature as influencing the beliefs mathematics teachers have, and how these beliefs impact on their teaching. The following section disaggregates the concept of mathematics teachers' web of beliefs and explains the categories of this belief structure.

East Asian culture is characterized by collectivism, that is, a relative emphasis on the needs and desires of the community over the needs and desires of each individual, and an emphasis on interconnectedness rather than the individuality of each group member. Thus, individuals are considered in terms of how they fit into the group, which is linked to a relative emphasis on authority and hierarchy (Haller, Fisher, & Gapp, 2007). For this reason, the academic culture of many East Asian countries stresses the hierarchical relationship between older figures of authority and younger people, because this emphasis is also present within the broader culture (Haller et al., 2007). In other words, the collective consciousness of groups in East Asian countries, including academic groups, is oriented around hierarchical relationships (Haller et al., 2007).

In contrast, Western cultures tend to be highly individualistic, meaning that individuals tend to act more in their capacity as autonomous individuals rather than as members of a wider group (Hofstede, 1980). Therefore, in their approach to thinking and education, Western cultures tend to emphasize individualistic perspectives, where people are encouraged to criticize, express doubts, and assess and evaluate others' thinking with the goal of creating new ideas. The goal of education and thinking is not the cohesion or benefit of the group, but the creation of new ideas. However, it is critical to note that this priority is also present in East Asian cultures' intellectual heritage. Indeed, in his dialogues, Confucius frequently stressed the importance of individuality in learning, and said that learning should happen partially for the sake of the self (Lee, 1996). In fact, Confucius held that education would only be meaningful if it led to the perfection of the self, and that "the purpose of learning is to cultivate oneself as an intelligent, creative, independent, autonomous being" (Lee, 1996,

p. 34).

Another important way of understanding how culture conditions priorities for and systems of education is to consider classical cultural exemplars for education and the life of the mind. In Eastern Asian cultures, the most dominant exemplar is Confucius (Cai & Wang, 2010). In Western cultures, it is Socrates (Cai & Wang, 2010). Confucius emphasized effortful learning, pragmatic achievement of crucial knowledge, behavioural development and respectful learning (Tweed & Lehman, 2002). On the other hand, Socrates (e.g., Plato as a disciple) exemplified the private and public questioning of common knowledge and criticism of authority figures and established ideas. The Socratic tradition encouraged students to evaluate others' beliefs and to create their own philosophies (Tweed & Lehman, 2002). The subsequent sections of this literature review show how there are clear parallels between these cultural exemplars and mathematics teachers' beliefs about and approaches to teaching mathematics. Therefore, understanding these cultural contexts is crucial to understanding teachers' conceptions about mathematics, which is in turn crucial to understanding what makes mathematics teaching effective and to promoting effective mathematics teaching. Before exploring how these cultural scripts manifest in beliefs about mathematics teaching, the following section re-engages in further depth with the literature on the web of beliefs.

2.1.1.1 The categories of belief structure

The first section of this literature review (section 2.1.1) discussed the web of beliefs that mathematics teachers have and that characterize the affective component of their mathematical knowledge for teaching. The literature that employs the affective lens has evolved to disaggregate this web of beliefs into three kinds of beliefs: beliefs about the nature of mathematics, beliefs about mathematics teaching, and beliefs about mathematics learning (Ball, 1988; Ernest, 1989a). Of course, these three beliefs are interrelated and they condition each other. The objective of this section is to trace the evolution of this literature, which is summarized in the following table (Table 1), to describe the categories of belief structure, and to begin to survey the literature on how culture influences the relationships between the various categories of belief structure.

Table 1: Relationships between beliefs

Categories of teacher beliefs	*Components of teachers' beliefs*		
Beliefs about the nature of mathematics (Ernest, 1989a)	**Instrumentalist**	**Platonist**	**Problem-solving**
Beliefs about mathematics teaching (Van Zoest et al., 1994)	**Content-focused with an emphasis on performance**	**Content-focused with an emphasis on understanding**	**Learner-focused**
Beliefs about mathematics learning (Ernest, 1989a)	**Skill mastery, passive reception of knowledge**	**Active construction of understanding**	**Autonomous exploration of own interests**

The foundation of the categories of belief structure comes from Ernest's seminal (1989a) article; even though it is 30 years old it is important to be included here. According to Ernest (1989a), teachers' beliefs about the nature of mathematics shape the main philosophy of mathematics, even though some of the teachers' views might not be noticed in that philosophy. Such views are better to be consciously held, rather than being implicitly held philosophies (Ernest, 1989b). In addition, Ernest (1989a) noted that teachers' beliefs about the nature of mathematics have had a great impact on teachers' view of teaching and learning mathematics. Indeed, teacher perceptions of mathematics influence their perceptions of how it should be taught (Hersh, 1979). This is not only an important conceptual observation, it also has methodological implications, as it means that teachers' ways of teaching reflect their essential teaching beliefs, an insight that is foundational to this thesis's empirical approach (Hersh, 1979).

Ernest (1989a) typologized three different conceptions of the nature of mathematics as psychological systems of belief that can be surmised to shape a hierarchy: the instrumentalist conception is at the lowest level; at the next level is the Platonist conception; and the problem-solving conception lies at the highest level. The instrumentalist conception holds that "mathematics is an accumulation of facts, rules and skills to be used in the pursuance of some external end…a set of unrelated but utilitarian rules and facts" (Ernest, 1989a, p. 250). The Platonist conception of mathematics is that mathematics is "a static but unified body of certain knowledge" that is meant to be "discovered, not created" (Ernest, 1989a, p. 250) Then, the problem-solving conception of mathematics is that mathematics is “a dynamic, continually expanding field of human creation and invention, a cultural product…[it] is a process of enquiry and coming to know, not a finished product, for its results remain open to revision” (Ernest, 1989a, p. 250).

Ernest also argued that beliefs about the nature of mathematics were directly related to beliefs about mathematics learning (1989a). In brief, teachers who believe the instrumentalist view will see themselves as instructors, teachers who believe the Platonist view will see themselves as explainers, and teachers who believe the problem-solving view will see themselves as facilitators (Ernest, 1989a). For instrumentalist instructors, teaching is effective when learners achieve mastery of skills and perform tasks correctly, and learning is conceptualized as the mastery of skills (Ernest, 1989a). For Platonist explainers, teaching is effective when students achieve conceptual understandings with unified knowledge, and learning is conceptualized as receiving knowledge (Ernest, 1989). For problem-solving facilitators, teaching is effective when students achieve confidence in posing and solving problems, and learning is conceptualized as actively constructing understandings (Ernest 1989a).

Different philosophies about the nature of mathematics have practical outcomes in the classroom Ernest (1989a). For instance, the instrumentalist view of mathematics emphasises the style of teaching that concentrates on teachers explaining and clarifying concepts and encouraging students to follow procedures rather than constructing new knowledge (Lindblom-Ylänne, Trigwell, Nevgi, & Ashwin, 2006). Teaching from a Platonist view has a dual emphasis on both the content and student understanding, While the main emphasis is on the content, attention is also paid to understanding the facts and procedures that underlie the content (Cross, 2009). The problem-solving view focuses on a learner-focused model of teaching and supports making a learner-focused classroom environment which promotes individual sense-making (Cobb & Steffe, 1983).

Different parallels could be drawn between the views of the nature of mathematics and some of the prime curricular reforms in mathematics education. The first parallel is drawn between the Platonist view and the 'modern maths' curricular reform of the early 1960s, with “its stress on structure, the laws of number, and central and unifying concepts of mathematics, such as sets and functions” (Ernest, 1989b, p. 21). Secondly, the view underlying the 'back-to-basics' movement links to the instrumentalist view, as this movement focused on basic skills and knowledge of facts, rules and procedures without any deep connections in this knowledge(Ernest, 1989b). The final parallel is drawn between the problem-solving view of mathematics and the recent curriculum movements (Cockcroft, 1982; DES, 1987; HMI, 1985; National Council of Teachers of Mathematics, 1980), since they emphasize mathematical activities and their central goal of mathematics teaching is to develop children's ability to be creative and confident problems solvers (Ernest, 1989b, p. 21). The view of problem-solving is also reflected in the recommendations of new mathematics curriculum reform in Saudi Arabia. Teachers' views of mathematics evidently affect the extent to which such curriculum innovations or movements take hold, through the way mathematics is taught (Cooney, 1988; Thom, 1973).

The research literature then evolved to explore how mathematics teachers' beliefs about the nature of mathematics and about mathematics learning led to beliefs about mathematics teaching. The step here was the publication of Van Zoest, Jones, and Thornton's seminal (1994) article. In this article, the researchers presented a theoretical model to categorize pre-service teachers' beliefs about mathematics instructions (Van Zoest et al., 1994). They theorized that beliefs about mathematics teaching would be learner-focused with an emphasis on social interactions, content-focused with an emphasis on conceptual understanding, or content-focused with an emphasis on performance (Van Zoest et al., 1994, p. 42).

Van Zoest et al. (1994) interviewed and observed six elementary pre-service mathematics teachers involved in an intervention program and compared them with the 32 teachers not involved in such an intervention program. The authors utilised a variation of the Kuhs and Ball (1986) model for their study, which specified four central views of how mathematics must be taught: a) learner-focused; b) content-focused with an emphasis on conceptual understanding; c) content-focused with an emphasis on performance; and d) classroom focus. Because there was a significant overlap between the two content-focused views and the classroom-focused view (Kuhs & Ball, 1986), Van Zoest et al.'s framework (1994) was constructed on three views which were learner-focused, content-focused with an emphasis on conceptual understanding, and content-focused with an emphasis on performance. These three views provide a continuum to evaluate and estimate pre-service teacher beliefs about mathematics teaching; at one end (view 1 - learner-focused) can be assessed as a socio-constructivist orientation and at the other extreme (view 3 - content-focused with an emphasis on performance) as a performance driven orientation.

Van Zoest et al. (1994) found that the participating pre-service teachers gained socio-constructivist beliefs of learning and teaching mathematics, yet in the early stages of the program they were not able apply their views in their practice. This inconsistency could be due to different reasons such as: teachers' lack of pedagogical skill that directs students' problem-solving activities; the requirement times to complete the activity; and teachers' attention to students' level and skills to solve the problem.

Although Van Zoest et al. (1994) did not argue this in their article, these three views of belief about mathematics teaching clearly align with the three belief systems about the nature of mathematics that Ernest (1989a) articulated. Indeed, Beswick identified these connections in her 2005 article and stated that content-focused with an emphasis on performance closely aligns with the instrumentalist view of mathematics and it conforms to the idea that learning mathematics is about mastering skills and passively receiving knowledge (Beswick, 2005). Content-focused with an emphasis on understanding corresponds with the Platonic view of mathematics and it conforms to the belief that learning mathematics primarily concerns the active construction of understanding (Beswick, 2005). Finally, learner-focused with an emphasis on social interaction fits with the problem-solving view of mathematics, which holds that learning mathematics is about the learner's autonomous exploration of their own interests (Beswick 2005). This is known as the Categories of Teacher Beliefs (CTB) framework (Beswick, 2005).

In addition to synthesizing existing theoretical frameworks on mathematics teachers' beliefs about the nature of mathematics, mathematics teaching, and mathematics learning, Beswick also articulated important theoretical expectations for how this web of beliefs influences mathematics teaching practices (2005). She held context as a crucial mediating variable because "context can influence both the development and enactment of teachers' beliefs" (Beswick, 2005, p. 41). She argued that this conforms to previous arguments that the relationship between beliefs and teaching practices is dialectical, and expanded upon this understanding by arguing that context mediates how teachers come by their beliefs about their mathematics, how teachers enact their beliefs in practice, and the processes by which beliefs and practice affect each other (Beswick, 2005). Further, Beswick argued that all apparent contradictions between mathematics teachers' beliefs and mathematics teachers' teaching practices can be explained by contextual factors (2005).

Since Beswick's framework, known as the CTB framework, incorporates three distinct, but interrelated beliefs—beliefs about the nature of mathematics, mathematics teaching, and mathematics learning—and because it provides cogent expectations for how these beliefs affect teaching practice, it is very useful for providing theoretical structure to empirical explorations of the affective components of mathematics teachers' mathematical knowledge for teaching. For this reason, to examine mathematics teachers' beliefs about the nature of mathematics, mathematics teaching and mathematics learning in Saudi Arabia, as well as how these beliefs influence teaching practice, this research adopted the CTB framework.

Most of the research regarding teachers' beliefs focuses more on how teachers view the nature of mathematics and its teaching and learning. However, it has been recognised that people's beliefs are not constructed in isolation but rather are contingent upon the context in which teachers developed, both as people and as educators, and upon the context in which they work as educators (Beswick, 2005, 2012). Cultural factors can be an essential part of this context. Indeed, as mentioned in the last section, both cultural factors and teachers' beliefs about what constitutes effective teaching influence the affective lens that applies to teaching mathematics (Cai & Wang, 2010). Therefore, an understanding of the epistemological beliefs of female mathematics teachers in Saudi Arabia provides an interesting window through which to study mathematics teaching in that country. In addition, knowledge of teacher beliefs may have the potential to inform pre-service and in-service teacher education or curricular developments.

Examining this question in the Saudi Arabian context is an important approach because previous work has shown that different cultural contexts are associated with different prevalent webs of beliefs about mathematics. Scholars have argued that Asian teachers most commonly evidence and enact the Platonist view of mathematics in their teaching while Western teachers tend to believe most in the problem-solving conceptualization

(Bryan, Wang, Perry, Wong, & Cai, 2007; Stigler & Hiebert, 2009). According to Stigler and Hiebert (2009), Asian teachers hold the Platonist view because they teach mathematics in a logical way, and they believe mathematics expresses fixed relationships between ideas, concepts, facts, rules and procedures. On the other hand, Bryan et al. (2007) found that Australian and American mathematics teachers see mathematics as a tool to be used every day for solving real life problems. The following subsections survey the respective literatures on how each subtype of beliefs influences mathematics teaching practices and explains how culture interacts with each of these kinds of beliefs to influence mathematics teaching practices. The following section discusses the relationship between culture and the prevalent categories of belief structures.

2.1.1.1.1 Beliefs about the nature of mathematics.

For better or for worse, the vast majority of theoretical and empirical works on how culture conditions the affective lenses of mathematics teachers focus on the contrasts between Western and East Asian cultures (Bryan et al., 2007). However, it is important to be cautious about placing too much emphasis on the contrasts between these two cultures for two reasons. First, both cultures hold certain beliefs in common; most fundamentally, both East Asian and Western cultures believe that mathematics is highly related to real life problems and it is a very essential skill for living (Bryan et al., 2007). Second, it is hopefully extremely obvious that there are more than two cultures in the world, and that contrasts between two of the many cultures in the world paint a limited picture of the role that culture can play in conditioning beliefs about mathematics. Indeed, this is an important potential contribution of this thesis: Saudi culture is neither Western nor East Asian and, therefore, a systematic understanding of how Saudi culture influences Saudi middle school mathematics teachers' webs of beliefs about mathematics promises to broaden the academic perspective on the relationship between culture and the affective aspect of mathematics knowledge for teaching.

Although there has not yet been any study of how Saudi culture conditions mathematics teachers' web of beliefs, the literature on the affective lens and how culture influences belief structures about mathematics provides helpful direction. Teachers' beliefs about the nature of mathematics have a great impact on teachers' view of teaching and learning mathematics (Ernest, 1989a). Indeed, teacher perceptions of mathematics influence their perceptions of how it should be taught, and the ways of teaching reflect teachers' essential beliefs in teaching (Hersh, 1979).

2.1.1.1.2 Beliefs about mathematics teaching

Beliefs are constant and often individuals are unaware of them, or how to have immediate control of them. Beliefs are strongly considered to be predictors of an individual's behaviour as they determine how individuals frame activities and structure tasks (Thompson, 1992; Torff & Warburton, 2005). Many researchers demonstrate in the literature that mathematics teachers' beliefs influence their practice in the classroom, even though the nature of the relationship is complex and dialectically related (Pajares, 1992; Thompson, 1985). Mathematics teachers' beliefs are fundamental to the relationship between beliefs and practice (Raymond, 1997). They affect teachers' practice, their recognition of teaching, learning, and assessment in the classroom and also influence how they see students' skills and capabilities (Barkatsas & Malone, 2005). Beliefs develop from experiences over many years and are impacted by societal mores. They are resistant to change despite educational achievement or teaching experience (Torff & Warburton, 2005). For example, in a study of K-6 teachers' beliefs and practices done by Anderson, White, and Sullivan (2005), they interviewed two teachers who were categorised as having a 'contemporary' view of teaching, which is consistent with an inquiry-based approach to teaching, and found that the views of these teachers had been affected by their previous negative experiences in learning mathematics and they were not willing to teach in the same way they were taught.

In terms of the relationship between teachers' beliefs and their practice, Cobb, Wood, and Yackel (1990) stated that the direction of the relationship between beliefs and practice was not linearly either way; they affected one another and enhanced each other. Some studies have described various degrees of consistency (Frykholm, 1995; Thompson, 1992) between teacher beliefs and their practices (Speer, 2005; Stipek, Givvin, Salmon, & MacGyvers, 2001; Thompson, 1985; Wilkins, 2008). For example, Thompson (1985) illustrated that the relationship is consistent after observing that a mathematics teacher who held constructivist beliefs followed a constructivist approach of teaching, as during her instructions she encouraged the students to construct their knowledge through making an idea explicit as well as explaining their reasoning. Stipek et al. (2001) specified that teachers who held traditional beliefs applied traditional practices as they did not give their students much freedom to participate in the learning process.

However, the relationship between practice and beliefs was found to be a “subject of controversy and is acknowledged to be both subtle and complex” (Beswick, 2007, p. 96) as it has not always been found to be consistent (Barkatsas & Malone, 2005; Cooney, 1985; Raymond, 1997; Thompson, 1992). For instance, the research of Raymond (1997) revealed that while beginning mathematics teachers expressed belief in constructivist principles of teaching, they mostly applied traditional methods in their classroom practice. Similarly, the study of Barkatsas and Malone (2005) found that although mathematics teachers' beliefs were more constructivist, their real practice was more traditional due to the classroom environment. According to Beswick (2005), the inconsistencies could be linked to the different contexts where beliefs are expressed and then observed. In practice, there is evidence from empirical research that indicates teachers’ beliefs about mathematics teaching vary across different mathematical content and contexts (Beswick, Callingham, & Watson, 2012). In other words, a single teacher could have different beliefs about mathematics teaching depending on the content she is tasked with teaching and/or depending on the context in which she is teaching (Beswick et al., 2012). A further complication is that the findings of several studies have revealed that there can be a gap between teachers’ espoused beliefs and their actual teaching practices (Ernest, 1989a; Hoyles, 1992; Webb & Webb, 2004).Teachers at the beginning of their teaching careers might be especially likely to teach in a way that does not conform to their expressed beliefs about teaching mathematics (Beswick, 2012). Such differences can be easily clarified in terms of the various contexts where beliefs are expressed and then observed (Beswick, 2003; Beswick, 2012).

Hoyles (1992) stated that the contextual nature of beliefs is very essential in resolving manifest discrepancies between teachers’ beliefs and practices. As a result of Hoyles (1992) meta-analysis of case-studies on the relationship between teachers' beliefs and practice, she defined all beliefs as situated, which means all teachers' beliefs are structured from previous experiences that take place in different contexts. Likewise, Pajares (1992) emphasized the importance of the contextual nature of beliefs and the conclusions that they are parts of belief systems, which have in fact been described by Green (1971) as being found in isolated clusters.

Green (1971) described the relationship between different beliefs and between beliefs and behaviours through his framework of beliefs system which has three dimensions. The first dimension is quasi-logical as it is not based on the content of the belief, yet it describes the organization of beliefs and how they are held. The second dimension is about the psychological strength of a belief. It is similar to the first dimension as it is not based on the content of the beliefs, however it is linked to how beliefs are held. There are two types of beliefs: beliefs held with great psychological strength which are core beliefs and other ones described as peripheral beliefs. The third dimension of the belief system is the method of clustering beliefs. This method of grouping offers both support and protection for any contradictions and inconsistencies, which makes it possible for the individuals to hold incompatible and opposite core beliefs (Cross, 2009). According to Green (1971), beliefs that are shaped in different contexts, such as place or time, are likely to be separated into different clusters because these separated beliefs clusters did not occur during normal events. Beswick (2006) linked Green's (1971) notion of clustering in the third dimension to how contradictory teachers' beliefs may arise. Beswick (2006) explained this complex connection when she wrote, "[A] person may hold beliefs that contradict one another without being aware of the contradiction" because "beliefs within a system can be held in groups that are isolated from other beliefs" (p.17).

Schoenfeld (1998) explained that beliefs have a practical as well as context-bound nature, and this could be understood because social and culture mental constructs are likely to have been obtained in educational settings in different countries that significantly vary in their historical traditions. A key finding of the Third International Mathematics and Science Study (TIMSS) was teaching is basically a cultural activity (Stigler & Hiebert, 1999). Indeed, teachers' cultural beliefs help them form a normative framework of their objectives, norms and values, all of which influence their way of teaching (Cai, 2004). Therefore, it is unsurprising that many studies have indicated that there are distinct differences in ideas about what constitutes effective classroom teaching across different cultural contexts (Cai 2005; Cai & Wang 2006; Cai et al. 2007; Cai et al. 2009; Li, 2010; Ma 1999; Stigler and Hiebert 1999; Perry et al. 1999). These differences in beliefs about effective teaching hearken back to deeper, culturally conditioned beliefs about the nature of mathematics, mathematics teaching, and mathematics learning (Cai, 2005; Cai et al., 2009; Cai et al., 2007; Cai & Wang, 2006; Ma, 1999a; Perry et al., 1999; Stigler & Hiebert, 1999).

Teachers' beliefs about what constitutes effective mathematics teaching are conditioned by many aspects, some of which include: desirable goals of the mathematics teaching program, their own role in teaching, the students' role, applicable classroom activities, desirable instructional methodologies and emphases, and personal mathematical techniques (Thompson, 1992). As has been previously detailed in this literature review, many researchers have developed theories about teachers' beliefs systems, and about how these belief systems affect teaching practices (e.g., Ernest, 1989b; Thompson, 1992; Van Zoest, Jones & Thornton, 1994). The objective of this subsection is to review the literature that explores how culture interacts with beliefs about mathematics teaching to influence teaching efficacy and learning outcomes.

General goals of teaching

Education is goal-directed learning, and curriculum describes learning goals (Cai & Howson, 2013) . A country's mathematics curriculum can be seen as a cornerstone of the entire mathematics education picture; it is affected by political and cultural traditions of the country which, in the end, have an effect on students' learning and performance (Wu, Kyungmee, & Fredericd, 2006). Curriculum is an aspect of education that can be involved at different levels, and there are different notions about curriculum (Jackson, 1992). It is applied through different institutional levels of society (Kuiper, Folmer, & Ottevanger, 2013) firstly the intended curriculum, which is made at government level (macro-level), this includes formally written documents which reflect political and administrative decisions about the curriculum. Secondly the implemented curriculum, is undertaken at school and classroom level (meso-level), involving the teachers' interpretation and execution of the curriculum. Thirdly the attained curriculum is undertaken at the learner level (micro-level) and refers to the output of students' learning and achievements (Cai & Howson, 2013; National Research Council, 2004; Thijs & van den Akker, 2009). Determination of the educational goals is made at each level and various actors are involved (Van den Akker, 2003). In mathematics education, there is a mismatch between these three levels (intended, implemented, attained curriculum) (Cuban, 1993), as well as at primary and secondary school level (Levitt, 2001; Smith & Southerland, 2007). The intended curriculum is usually planned by the educational organisational system (macro level) in different countries in the world (Van den Akker, 2003, 2010). It includes goals and expectations that are set by the curriculum policy makers and developers, along with official syllabi, textbooks or curriculum standards made by a specific institution or organisation (Kuiper et al., 2013; NRC, 2004; van den Akker, 2003; Cai & Howson, 2013). Wong (2004) analysed curriculum documents in different countries and regions such as Western countries (Australia, France, Germany, the UK, and the USA), and Eastern countries (Mainland China, Hong Kong, Taiwan, Singapore, and Japan), and compared their goals for school mathematics. As a result of his analysis, "higher-order thinking skills" was the most significant goal across all compared countries. Indeed, nowadays, having a large amount of knowledge is not adequate. Students' abilities to think and work critically and independently, to be creative, and to learn how to learn are the most fundamental characteristics that teachers could help students to develop (Cai & Howson, 2013).

For example, in the USA, the National Council of Teachers of Mathematics (1989) published a standards document that listed five goals for students to learn: 1) to value mathematics; 2) to reason mathematically; 3) to communicate mathematically; 4) to be confident of their mathematical skills and abilities: 5) to be problem solvers in mathematics. In addition, in order to achieve these goals in teaching mathematics, the National Council of Teachers of Mathematics (1991, p. 3) indicated key movement shifts were required:

1) toward classrooms as mathematical communities—away from classrooms as simply collections of individuals;
2) toward logic and mathematical evidence as verification—away from the teacher as the sole authority for right answers;
3) toward mathematical reasoning—away from merely memorizing procedures;
4) toward conjecturing, inventing, and problem solving—away from an emphasis on mechanistic answer-finding;
5) toward connecting mathematics, its ideas, and its applications—away from treating it as a body of isolated concepts and procedures.

Further, in China, Cai and Howson (2013) described the most recent curriculum reform which began in 2001. This reform emphasised developing new curriculum standards, teaching procedures, textbooks, and assessment methods. The Chinese Ministry of Education in 2001 published Curriculum Reform Guidelines for the Nine-Year Compulsory Education (Basic Education Curriculum Material Development Center, 2001), where the main goals for this new reform included:

> 1) Shifting from emphasising transmission of knowledge and information toward more focus on students’ active learning roles as well as developing students' skills and abilities such as working and attaining new knowledge independently, becoming problems solvers, and cooperating and interacting with others;
>
> 2) Shifting the design of the curriculum from focusing on separate school subjects toward greater stress on incorporation of school mathematics; and
>
> 3) Shifting from old complex curriculum content toward curriculum that reflects students' real life with the new contemporary science and technology.

The new Chinese curriculum development system is followed and the new Compulsory Education Mathematics Curriculum Standard is used as a guide for all teaching and learning activities in all different levels (Li, Zhang, & Ma, 2014). These new mathematics curriculum standards involve three goals, "knowledge and skills, processes and methods, affective demeanor and value" (Ni, Li, Li, & Zhang, 2011, p. 102). The purpose of these goals is to help students: 1) to gain important mathematical knowledge and develop problem-solving skills that are essential in their life-long learning; 2) to apply knowledge of mathematics and other connected skills as well as to observe, analyse, and solve problems in other subjects and daily life through using mathematical methods; and 3) to value the close relationship between mathematics, nature, and society. Further, these goals assist students to acquire more than basic knowledge and skills and give students more opportunities to reason about evidence and explanation, analyse, and employ learned knowledge and skills to solve real life issues, to make them more confident, independent, and interested in learning (Ni et al., 2011).

In mathematics education, curriculum plays a fundamental part as it identifies what students learn, when they learn it, and the best way of learning. Nevertheless, regardless of how well the curriculum is designed, the way it is implemented in the classroom determines its value (Cai & Howson, 2013). Lloyd (1999) and Collopy (2003) stated that teachers' ways of reading, interpreting, analysing, and using the teachers' guides varies tremendously and argued that the characteristics of the curriculum and teachers' beliefs impact the process of teaching. The teacher is one of the essential factors in implementing the mathematics curriculum, thus studying mathematics education in different cultures requires more focus on the teachers (Leung, Graf, & Lopez-Real, 2006). Indeed, the implementation of the new curriculum reform requires teachers to be equipped with congruent beliefs about mathematics and its teaching and learning (Lloyd, 1999). This is because the degree of the implementation of the innovative concepts will be affected if there is a mismatch between teachers' beliefs and the underpinning beliefs of the new curriculum reform (Handal & Herrington, 2003).

It is assumed that the effectiveness of teaching and curriculum is linked to students' high performance and achievement (National Academy of Education, 1999). Identification of learning goals requires teachers to know mathematics, pedagogical methods, curriculum focus, and students. Further, it is related to teachers' beliefs and views about mathematics as well as notions of teaching mathematics (Cai & Howson, 2013). Teachers' belief systems about teaching and learning mathematics reflect their views on the nature of knowledge and knowing and their favoured approach to teaching mathematics and how to learn it, for example, their view of ideal teaching activities, what intellectual activity and performance are involved in mathematics learning, and what mathematics learning activities are appropriate (Chan & Elliott, 2004; Ernest, 1989b; Thompson, 1992). According to Thompson (1984), teachers' beliefs "seemed to be manifestations of unconsciously held views of expressions of verbal commitments to abstract ideas that may be thought of as part of a general ideology of teaching" (p. 112).

Researchers have used different groupings to categorize teachers' instructional practice, for example, traditional instruction (or teacher-centred) and constructivist instruction (or student-centred) (Hogan et al., 2013). Two approaches to mathematics teaching and learning (in German culture) were identified in the TEDS-M framework of teachers' beliefs (Tatto et al., 2008). The first approach was a knowledge and information transmission or traditional approach to mathematics teaching; this approach emphasized a process where students are passive receivers of knowledge from their teachers. The second approach was a constructivist approach, where the process and focus of teaching was on facilitating and helping students to construct their knowledge (Blömeke & Kaiser, 2014; Tatto et al., 2008). Further, some studies indicated consistency between teachers' beliefs and their teaching. For example, Thompson (1985) stated that teachers who were observed using a constructivist approach usually were explicit and precise in describing their concepts and clarifying their reasons in teaching, and they also encouraged students to be independent in constructing their knowledge, which was consistent with constructivist beliefs. In addition, Stipek et al. (2001) explained that teachers with traditional approaches held traditional beliefs; they did not encourage their students and give them much autonomy to be independent in their learning processes.

Another example, culturally, comes from Eastern mathematics teachers such as teachers from Mainland China and Hong Kong who hold a Platonic view (as explained previously), and for whom the structure of mathematical knowledge is greatly stressed in teaching (Bryan et al., 2007), and who aim to let the students understand the "truth" (Cai, 2004). According to Bryan et al. (2007), the most important task underpinning Eastern teachers' classroom teaching is transmission of knowledge, which lets them make more effort to get themselves well prepared as well as to have a well-structured lesson, whereas in Western countries such as the U.S. and Australia, mathematics teachers emphasise students' construction of their learning and make mathematics teaching and learning appropriate to real life. They see that the ''functional'' understanding of mathematics, understanding of mathematical knowledge, and their knowledge of the students and what they need, are more essential than reliance on planning, knowledge of the syllabus and textbooks. Therefore, they prefer less structured more flexible lessons to reflect their students' needs (Bryan et al., 2007).

The introduction to this chapter noted that the question of teaching efficacy has a normative component, because ideas about what makes teaching effective are linked with beliefs about what concepts people should learn, and how well they should learn (Kaiser, 2005). In other words, the general goals of teaching are a key affective component of teaching efficacy and of mathematical knowledge for teaching. Teaching is an inherently goal-related activity: before teaching, the teacher will set a goal structure as a starting point, and transform that structure into reality during actual classroom teaching (Steffe, 1991).

Thus, the goals that teachers themselves set for teaching, which are conditioned by their affective beliefs about mathematics teaching, are a key determinant of their actions and decisions during classroom teaching.

According to Stigler and Hiebert (1999, p. 87), the teaching gap (i.e., "The teaching gap we describe refers to the differences between the kinds of teaching needed to achieve the educational dreams of the American people and the kind of teaching found in most American schools"(p. xviii)), highlights the importance of teacher beliefs as well as the cultures by which they are surrounded:

> Cultural activities, such as teaching, are not invented full-blown but rather evolve over long periods of time in ways that are consistent with the stable web of beliefs and assumptions that are part of the culture. The scripts for teaching in each country appear to rest on a relatively small and tacit set of core beliefs about the nature of the subject, about how students learn, and about the role that a teacher should play in the classroom. These beliefs, often implicit, serve to maintain the stability of cultural systems over time. ... these systems of teaching, because they are cultural, must be understood in relation to the cultural beliefs and assumptions that surround them.

Culture can be an extremely important factor that conditions the goals that teachers will set for their teaching. As discussed previously in this chapter, mathematics instructors, as well as society at large, tend to have different goals for mathematics across East Asian and Western countries. These differences in goals and values in teaching mathematics can be traced back to broader, deeper cultural differences (Cai, 2007). These broad, deep cultural differences can be conceptualized in terms of the virtues that various cultures emphasize in their goals and methods of teaching. The current consensus is that the educational and social systems of East Asian countries tend to emphasize the promotion of the virtues of responsibility, adherence to authority, and perseverance among students (Smith & Hu, 2013). On the other hand, the educational and social systems of Western countries tend to emphasize individual development, strengthening and supporting justice, and democratic values (Smith & Hu, 2013). These differences in priorities translate to culturally conditioned differences in how East Asian and Western students are encouraged to learn. East Asian systems tend to be concerned about the moral development of students, while Western systems tend to encourage critical thinking and creativity (Smith & Hu, 2013). This is related to another important, culturally conditioned, difference in learning goals. In general, the East Asian perspective embraces the idea that learning can be oriented by extrinsic goals, because of a cultural belief that striving for extrinsic goals can also encourage the achievement of intrinsic goals (Tweed & Lehman, 2002). On the other hand, the Western perspective tends to hold that learning's benefits will be compromised if they are oriented towards an extrinsic goal, and therefore that learning should be an end in and of itself (Dewey, 1916).

Indeed, previous research conducted across East Asian and Western contexts has empirically demonstrated that the way that teachers view mathematics affects the way that they present mathematics concepts in the classroom, which in turn affects the way that students learn mathematics (Cai, 2004; Thompson, 1992; Stigler & Hiebert, 1999). Indeed, culturally conditioned differences in teaching methods have been posited as a key explanation of the teaching gap that has led to systematic differences in student outcomes across geographic and cultural contexts (Stigler & Hiebert, 1999). Then, the juxtaposition of two studies of how Chinese and American students selected problem-solving methods for algebra problems and how Chinese and American teachers wrote comments to student responses to algebraic problems indicated that both students and teachers had different, and likely culturally conditioned, approaches to solving mathematics problems (Cai, 2004). The most important difference that Cai (2004) identified was that Chinese teachers expected students to use generalized strategies to solve algebraic problems, indicating that Chinese culture values an understanding of generalized problem solving methods rather than just a simple demonstration that one can solve the problem at hand (Cai, 2004). Therefore, a key difference in mathematics teaching goals that can be culturally conditioned is the extent to which students are expected to show understanding of general mathematical principles.

The roles of teachers in classroom instruction

The question of the roles that teachers play in classroom instruction touches on a more fundamental question of the relationship between learning and teaching. The interaction between teaching and learning is extremely complex; according to Schwartz (1980):

> How students learn and how teachers teach are complicated processes, difficult to understand and even harder to master. It is not surprising that professors of many years' experience feel they have never quite got it right, and are amazed and gratified when the will to learn and the desire to teach come together in a few moments of excitement,

pleasure, and joyful discovery (p. 235).

In other words, having a thorough understanding of teaching, or many years of experience of teaching, does not cleanly translate to having a perfect understanding of how learners learn.

Accepting that the role of teachers in learning is perhaps too broad a question to ever completely understand, significant studies have explored the interactions between teachers and students, which is a fundamental, and often culturally conditioned, aspect of the roles that teachers play in classroom instruction. One of the most well-known approaches for analysing and understanding the ways that cultures condition differences in teaching and learning was established in a seminal study by Hofstede (1980). As mentioned in a previous section, Hofstede (1986) distinguished different, distinct dimensions of teaching culture. These dimensions can be used to predict cultural differences in teaching and learning environments (Hofstede, 1980). Further, empirical research has affirmed that the interactions between teachers and students are affected by cultural norms, beliefs and values (Salili, 2001). Consequently, it is fundamental to understand the role of the teachers in the classroom across cultures.

Several of the dimensions that Hofstede (1980) articulated concern cultural norms about the social positions of teachers and students, which in turn conditioned teachers' roles in the classroom environment. These dimensions were power distance, uncertainty avoidance, masculinity, individualism/collectivism, while long-term orientation and indulgence dimensions were added later (Hofstede, 1980).

One key, culturally conditioned factor that affects the role of teachers in classroom instruction is power distance. Power distance is linked to the notion of power (Habermas & McCarthy, 1977; Lahlali, 2003); it is a concept that describes the extent to which people are empowered or disempowered by the inequality of power distribution, and the leadership and privilege of the people that the culture empowers (Hofstede, 1980). According to Hofstede (1980), cultures that endorse the low power distance relationship in institutions, such as schools, are more democratic and representative and consultative. Also, a low Power distance score shows a more equal society, whereas a high Power distance relationship shows different level of power and authority in social structures. Moreover, in education, institutions with high Power distance scores show unequal roles between teachers and students, which point to a greater teacher-centred education, whereas institutions with low Power distance scores show a possibility of student-centred education. In high Power distance classrooms, whole class routine teaching is common, and students are rarely proactive or challenging. Also, students respect and treat their teachers, especially older teachers, like parents despite what students feel, while in low Power distance classrooms, it is common to see small group work, and students can express their feelings and challenges and doubts to their teachers.

Overall, countries with high Power distance scores are in Asia, the Arab region, Africa, and Latin America, while countries where the power distance scores are low are Germanic and Anglo regions (Hofstede, 1980).

In addition, Hofstede (1980) and Triandis (2001) proposed different models of culture and offered seven and eleven dimensions respectively. Both models help in understanding the role of culture in shaping people's decisions, and particularly the processes of education (Andrews, 2016). For example, Hofstede examined elementary teachers' education-related values in more than 40 countries. He found that a conservative culture which emphasises the "maintenance of the status quo, propriety, and restraint of actions or inclinations that might disrupt the solidary group or the traditional order" will have a major difference in educational opportunities from an autonomous culture where an individual discovers "meaning in his or her own uniqueness, who seeks to express his or her own internal attributes (preferences, traits, feelings, motives) and is encouraged to do so" (Schwartz, 1999, p. 24) . He thought teachers "play an explicit role in value socialisation", are likely to be "key carriers of culture, and... reflect the mid-range of prevailing value priorities in most societies" (Schwartz, 1999, p. 34).

Many researchers have highlighted the effect of teachers on students, as power and authority are fundamental characteristics of teachers' work (Seddon & Palmieri, 2009). Yet, the degree of power distance within a culture can impact the role of the teacher in classroom instruction because it conditions which approach to teaching the students will best respond to (Chan & Drover, 1997). For example, Western cultures tend to be less accepting of power distance, and therefore the Western educational system tends to emphasize horizontal relationships between students and teachers (Li, 2011) and expects teachers to respect the autonomy of their students, and to approach their role more as an advisor than as an authoritarian (Chan & Drover, 1997). Chen, Zee, Koomen, and Roorda (2019) conducted a comparison research between Dutch and Chinese primary school teachers and students; the research was about the quality of their mutual relationships. The sample involved 789 primary school students and 35 teachers from the Netherlands, and 587 primary school students and 14 teachers from China (Zhejiang). As a result of the power distance differences between teachers and students in the Netherlands and China, less conflict in their mutual relationships appeared between Chinese teachers and students compared to Dutch teachers and students. Chen et al. (2019) further explained that one of the essential features of the traditional culture of China is stressing children's compliance and obedience to their parents as well as teachers, which results in a big power distance between teachers and students (Bear et al., 2014; Yang et al., 2013). Consequently, students should behave properly and respect their teachers and stay away from deviant manners (Bear et al., 2014; Yang et al., 2013), while the Dutch culture places a great emphasis on an equal opinion and viewpoints, with a lower power distance between teachers and students (Hofstede, Hofstede, & Minkov, 2010). Consequently, students have more freedom in their behaviours and actions and teachers allow them to freely give their opinions about curriculum and teaching procedures (Boekaerts, 2003). In the context of power distance, Saudi Arabia could be categorized with high power distance countries (Alshahrani, 2017).These dynamics are expanded further in the next subsection, which explains how culture conditions the student–teacher relationship.

The teacher-student relationship

The previous subsection discussed how culture can condition the roles that teachers play in the classroom. A distinct, but related, subtopic is how culture can condition the relationship between teachers and students. Again, research on this topic is dominated by the conception of two contrasting cultures of education: Western and East Asian (Cortazzi, 1990). One cultural perspective is that the relationship between teachers and students should be hierarchical: the teacher is perceived as a main source of knowledge and, therefore, the role of the teacher is to transmit this knowledge directly to learners (Wang, 2006). In China, a teacher is the authority figure in the classroom, “a respected elder transmitting to a subordinate junior" (Ginsberg, 1992, p. 6). East Asian cultures tend to adopt this perspective because it corresponds well with East Asian cultures' emphasis on "continuity, stability, and group identity" (Cortazzi, 1990, p. 58).

Further, Confucianism emphasizes respect for hierarchical relationships between society members; therefore, it logically follows that the role of teachers would be to transmit knowledge to students, who occupy a space below them on the social hierarchy (Haller et al., 2007). In this system, students are expected to learn by showing persistence, maintaining a respectful attitude and thinking silently about the subject (van Egmond, 2011). One of the powerful heritages of Chinese educators is that Chinese students view the teacher as a person who has a deep knowledge and is able to respond to all of their questions and inquiries; also they see teachers as good moral models (Cortazzi & Jin, 2001). Thus, teachers should be role models and connect students' mental growth with their moral and personal development (Gao & Watkins, 2002). In addition to teachers' broad knowledge and information, there is a saying in Chinese regarding teachers' and students' relationships: "If someone taught you as a teacher for one day, you should respect him as your father for the rest of your life" (Wan, 2001, p. 41). So, teachers should be considered by their students as their parents, because teachers take care of their students and look after them with care and love (Wang, 2006). Thus, the traditional relationship of parent–child is shown in in the teacher–student relationship (Guo, 1996).

Societies with a Confucian philosophy, such as East Asian societies, are based on collectivism as well as obedience to family, which makes the relationship between teacher and students more personal. So, when teachers give more attention and care to students even for small amounts of time (e.g., saying "do it; I know you can do it") this will motivate them and lead to more positive changes in their learning achievement (Reza, 2017).

In Hofstede's (1980) terms, as explained in the previous section, in these cultural systems, the power distance between teachers and students is high. According to Hofstede (1997), this deeply impacts the nature of the relationship between teacher and student:

> in the large power distance situation . . . the educational process is highly personalized . . . what is transferred is not seen as an impersonal 'truth', but as the personal wisdom of the teacher . . . In such a system the quality of one's learning is virtually exclusively dependent on the excellence of one's teachers (p. 34).

According to Reza (2017) , the hierarchical learning environments in non-western countries such as Eastern countries provides a non-reciprocal dependency (e.g., pseudo-emotional dependency) relationship between teachers and students. This is shown clearly in students' behaviours when they wait for feedback from their teachers to allow improvement in their studies. Further, in a system where the quality of one's learning depends on one's teachers, the relationship between student and teacher becomes paramount.

Another cultural conception is that the relationships between teachers and students should be democratic, and not dominated by notions of hierarchy (Wang, 2006). Western societies tend to adopt this conception, as it corresponds with Western culture's emphasis on individual development, creativity and equality (Wang, 2006). The Western educational system emphasizes "horizontal" relationships between individuals—that is, relationships that are not characterized by conceptions of hierarchy (Li, 2011). Therefore, Western teachers tend to have a student-centred approach in which the teacher respects the student's autonomy and tends to act as an advisor to the student (Chan & Drover, 1997). Further, teachers encourage students to be independent in actively promoting and emphasising self-study.

Societies that emphasise independent learning approaches, such as Western societies, are based on individualism which permits much "space" in the relationship between teachers and students. Importantly, these societies have a less competitive structure than Eastern societies, which makes higher education relate more to their interest and not to their social status.

Therefore, to hearken back to Hofstede's (1980) theory, as explained in the previous section, in Western cultures power distance tends to be low, so students and teachers are less tolerant of strictly hierarchical relationships. These cultural differences in teacher-student relationships between Eastern and Western societies were studied by Jin and Cortazzi (1998) in research conducted with British and Chinese secondary school students. The British students preferred their teachers to be sympathetic, understandable and understanding, and patient with students who had difficulty in following the lessons. They also described a good teacher as the one who was explaining the topic very clearly by using effective teaching methods as well as organising various activities to increase students' interest and understanding, which is typical of Western teachers' ‘teaching skills’. The Chinese students believed the relationship between students and a good teacher should be warm-hearted and friendly beyond the classroom. They characterised a good teacher as one who had a deep knowledge and was a good moral model.

How culture conditions instructional focus

Just as culture can condition the roles that teachers play in classroom teaching, as well as how teachers and students relate to each other, culture can also condition teachers’ instructional focus. Many researchers have highlighted the impact of Confucian and Socratic philosophies on the educational traditions of Western and East Asian cultures (Leung, 2001; Tweed & Lehman, 2002; Watkins, 2000). For example, with regard to Confucian versus Socratic debate, Leung (2001) offered six dichotomies between East Asian and Western mathematics classrooms: (1) product (content) versus process; (2) rote learning versus meaningful learning; (3) studying hard versus pleasurable learning; (4) extrinsic versus intrinsic motivations; (5) whole class teaching versus individualised learning; (6) competence of teachers: subject matter versus pedagogy. Nevertheless, according to Andrews (2010), these proposed differences were possibly sometimes inaccurate and certainly crude. Moreover, Mason (2007) stated that these differences were inappropriate and clumsy and sometimes showed a racist stereotype.

Over the past century, there have been various terminological frameworks proposed and used in the mathematics teaching and learning literature in order to define knowledge outcomes (Star & Stylianides, 2013), such as meaning theory (Brownell, 1945) , relational/instrumental understanding (Skemp, 1976), and routine and adaptive expertise (Hatano & Inagaki, 1986). For instance, Skemp (1976) explained ‘instrumental understanding’ as a constant and fixed plan when students learn to solve a particular type of activity. This type of understanding was criticised by Skemp because students do not get a comprehensive understanding of the relationship between the overall goal of the activity and its individual steps. Further, he stressed ‘relational understanding’, when students are able to produce and show various plans and solutions with conceptual structure for getting from any starting point to any finishing point (Skemp, 1976). Thus, this type of understanding would be the main goal itself with no longer fixed plans for a specific activity and task in the classroom, which is the case with instrumental understanding (Skemp, 1976).

Nevertheless, since the mid-1980s, the most common framework is the one that involves two essential types of knowledge, conceptual knowledge and procedural knowledge, which became widely recognised after Hiebert published his book in 1986 (Star & Stylianides, 2013).

Teachers are representative of their educational system's principles and values. Certainly, there is much evidence which proves that teachers consciously or unconsciously operate in a way that reflects those educational objectives and principles, and they will seek specific outcomes by following particular methods and strategies that are unique to their country and which make them different from other countries (Andrews, 2010). Further, teachers' approaches to teaching mathematics are influenced by their personal "conceptual knowledge" and "procedural knowledge" (Star & Stylianides, 2013).

Indeed, mathematics education in different cultures tends to emphasize either conceptual knowledge or procedural knowledge (Hiebert & Carpenter, 1992). Although these two types of knowledge cannot be separated, different cultures tend to emphasise one type over the other (Rittle-Johnson & Alibali, 1999). Rittle-Johnson and Alibali (1999) defined conceptual knowledge as "explicit or implicit understanding of the principles that govern a domain and of the interrelations between pieces of knowledge in a domain" (p. 175). On the other hand, they defined procedural knowledge as "action sequences for solving problems" (Rittle-Johnson & Alibali, 1999, p. 175). Likewise, conceptual instruction emphasises the main principles and ideas of the field, whereas procedural instruction emphasises step-by-step procedures (Rittle-Johnson, Fyfe, & Loehr, 2016).

Many scholars have recommended focusing on both concepts and procedures for direct instruction (Kirschner et al., 2006), while other scholars have argued that teachers' instruction should emphasis concepts alone (Hiebert et al., 1996; Von Glasersfeld, 1995). Several experimental studies (Matthews & Rittle-Johnson, 2009; Perry, 1991; Rittle-Johnson & Alibali, 1999) provide evidence that lessons where teachers focused on conceptual knowledge led to greater learning than lessons where teachers focused on procedural knowledge (Rittle-Johnson et al., 2016). For instance, according to Perry (1991), students who attended lessons that emphasised conceptual knowledge were able to create solution steps that could transfer and be adapted to different situations to solve problems, while other lessons that emphasized procedural knowledge were less effective in enhancing strategies for procedural transfer. Further, emphasising conceptual knowledge during instruction was essential for retention of the knowledge, whereas emphasising procedural knowledge could lead to difficulties in knowledge recall after a while (Baroody, Feil, & Johnson, 2007; J. Hiebert & H. Lefevre, 1986). Nowadays, indeed, there is wide agreement that both types, conceptual and procedural knowledge, are fundamental, since the relationship between both is normally bidirectional (Rittle-Johnson, Schneider, & Star, 2015).

However, both theoretical and empirical research has indicated that teachers from East Asian cultures tend to emphasize procedural knowledge over conceptual knowledge, while teachers from Western cultures tend to emphasize conceptual knowledge over procedural knowledge (Li, 2006; Tweed & Lehman, 2002). For Western teachers, this affects instructional focus because it means they tend to prioritize fostering their students' conceptual understanding of mathematical ideas and rules before practising the rules with their students (Li, 2006). On the other hand, numerous studies have reported that East Asian teachers tend to follow a more traditional style of teaching, which is mainly oriented around lectures delivered by the teacher, with the goal of preparing for examinations (Fan, Wong, Cai, & Li, 2004; Leung, 2006; Watkins & Biggs, 2001). This accounts for an emphasis on procedural rather than conceptual knowledge, and the practice of teaching tends to be more directed toward memorization (Marton, Watkins, & Tang, 1997).

Western educators have criticized this emphasis on procedural over conceptual knowledge, and especially the instructional focus on memorization and practice, because when viewed through the Western cultural lens, these practices constitute "passive transmission" and "rote drilling" of knowledge (Gu, Hunag, & Marton, 2004). Indeed, some Western researchers have even argued that East Asian students tend to take a shallow approach to learning (Tweed & Lehman, 2002). This perception is partially founded in Western educators' emphasis on conceptual over procedural knowledge, which, in terms of instructional focus, leads to the belief that students should be encouraged to understand what they learn and not simply memorize procedures (Purdie, Hattie, & Douglas, 1996).

Another explanation for this critique is the difference in how East Asian and Western cultures tend to perceive memorization (Marton, Dall'alba, & Tse, 1996; Pratt et al., 1999). Indeed, Chinese learners tend to conceive of learning as a four-stage process which includes memorizing, understanding, applying and, finally, questioning or modifying (Pratt et al., 1999). Therefore, while the Western culture's emphasis on conceptual knowledge encourages learners to question and modify from the very beginning, at times even before practising procedural knowledge, the East Asian culture's emphasis on procedural knowledge is more likely to hold questioning and modifying knowledge as the final step of learning. Yet, the East Asian old-fashioned teaching style has produced Chinese students with high mathematical achievements who outperform the Western students in many international comparative studies such as TIMSS (Beaton et al., 1997; Mullis, Martin, & Foy, 2008) and PISA (OECD, 2004; Organisation for Economic Co-operation Development, 2010).

Many researchers (Marton et al., 1997) have supported the hypothesis that the outstanding academic performance of East Asian learners is because of the composition of memorizing and understanding which is an uncommon practice among Western learners. Indeed, according to Marton (2008), after his cross-cultural research of more than 25 years, this Chinese paradoxical phenomenon is related to Chinese variation practice pedagogy (Sun, 2011). As Marton (1997) stated in a public lecture, incessant practice with increasing variations could help deepen students' understanding; which reflects Confucius' saying, "Learn the new when revising the old" (Analects, 2:l I), as well as the first stance of the Analects of Confucius on learning, "Learn and practice frequently" (Analects, l:l). This shows that Confucius did not support rote learning and over drilling (Wong, 2006).

As stated by Needham and Wang (1959, p. 171),

> "the East has its own mathematics and developed different ideological methods from the western traditional deduction system,… that is to produce new methods from practical problems, promote them up to the level of general method, generalize them into "shu" (術) and deploy these "shu" to solve various similar problems which are more complicated and more abstruse".

Indeed, variation approach was the core of teaching and learning mathematics in ancient China (Sun, 2011). The idea of teaching and learning mathematics by using variation approach is very widespread in China which is reflected in the old Chinese maxim, "Only by comparing can one distinguish" (Gu, Huang, & Gu, 2017). Variation theory is posited on the view that "when certain aspects of a phenomenon vary while its other aspects are kept constant, those aspects that vary are discerned" (Ling, Chik, & Pang, 2006, p. 3). In other words, the main idea of using variation theory in teaching mathematics is focusing on the major features of the concepts by changing the non-major features (Gu et al., 2004). Thus, discernment appears when features of the concept are focused and marked from the whole tentatively (Leung, 2003). Therefore, for students to grasp and discern a concept or a group of concepts, they should encounter different forms of variation where the essential features of the concept will be sometimes be "varied" or "not varied" (Lai & Murray, 2012).

Based on a series of longitudinal mathematics teaching experiments, Gu (1994) identified two primary types of variation after systematically synthesizing and analysing the concepts of teaching with certain dimensions of variation. The two primary types of variations in teaching mathematics were conceptual variation and procedural variation. Conceptual variation intends to provide students with different experiences and viewpoints of mathematical concepts (Gu et al., 2004). Procedural variation intends to provide a process of developing a mathematical concept step by step as well as providing varying problems and transferring strategies which enrich students' experience in solving problems (Gu et al., 2004). Some researchers have pointed out some features of teaching mathematical concepts by using variation approach in China. For example, Huang and Leung (2004) found that students can gradually develop problem solving experiences, gain knowledge stepwise, and shape a well-structured knowledge through teachers using variation approaches in teaching. Likewise, teaching with variation helps to present a series of interconnected problems which assist students to understand the concepts and grasp problem-solving methods which enhance students' mathematical knowledge.

Different studies have been conducted about mathematics classroom features in East Asia. For instance, Zhang, Li, and Tang (2004) found that in China the most coherent and apparent norms that were explicitly put forward in teaching mathematics were through focusing on foundation and the "basic knowledge and basic skills" principles in mathematics instruction. As the Chinese education system highly emphasises drilling and procedural skills (Lai & Murray, 2012), procedural variation gives students a considerable opportunity for systematic practice; this type of variation has been used in China for a long time either consciously or naturally (Gu et al., 2004). Also, variation approach can be used to assess and test students' understanding of mathematical concepts at different levels as well as whether they have acquired the required skills or not. Gu et al. (2004) pointed out that procedural variation could come from three forms of problem solving: varying a problem, multiple methods of solving a problem, and multiple applications of a method. Varying a problem refers to varying and extending the original problem through changing the conditions, result, and/or generalization. Multiple methods of solving a problem refers to varying the processes of solving a problem and connecting to different methods of solving a problem. Multiple applications of a method refer to applying the same method to a set of similar but different problems.

Bowden and Marton (1998) reinterpreted previous findings using phenomenography and inferred that, since discernment was a fundamental feature of learning and variation was very essential in leading to discernment, systematic repetition with variation could promote learning and understanding. As a Chinese teacher explained, “in the process of repetition, it is not a simple repetition because each time I repeat, I would have some new idea of understanding, that is to say I can understand better” (Marton et al., 1996, p. 81). This perspective is echoed in mathematics education research conducted by researchers in East Asia, with one study suggesting that teaching with variation could assist teachers to bridge between the basic knowledge and the higher-order thinking abilities (Zhang & Dai, 2004).

In addition, there are several western theories that support the appropriate implementation of variation theories as they could be effective for students’ mathematical learning in a large classroom, particularly Marton’s theory of variation which provides an epistemological and conceptual foundation for teaching practice in China (Marton & Booth, 1997). According to Zheng (2006a) the main ideas of Marton’s theory of variation are: (1) learning is to learn to discern [critical features of a learning object], and (2) discernment relies on comparison (differences) (as cited in Zhang, Wang, Huang, & Kimmins, 2017, p. 216). Therefore, it is really essential to give students chances to examine and explore dimensions of variations in order to widen their learning space (Zhang et al., 2017). Biggs (1996) said,

> In the West, we believe in exploring first, then in the development of skill; the Chinese believe in skill development first, ... after which there is something to be creative with (p.55).

According to Sun (2011), this practice is a typical natural approach which helps deepen Chinese students' understanding in native curriculum as a daily routine and which makes it an "indigenous" unique practice; it rarely appears in the west. This approach is uniquely rooted in deep philosophical backgrounds in China, as it can be applied to any single teaching material (e.g., textbooks or teaching plans) as well as any single learning material (e.g., student exercises or worksheets). For instance, Nie (2004) surveyed 102 Chinese teachers; they were asked to specify how often they applied variation approach in problem activities in their classroom. The result showed that all of the participating teachers used these approaches in different degrees.

Therefore, teaching with variation echoes the main idea of constructivism (i.e., viewing learners as constructors of meaning) as it supports students actively learning and trying things out, and then encourages them to build mathematical concepts and ideas that meet a specified constraint, with related components richly interconnected (Watson & Mason, 2005). Stigler and Stevenson (1991) conducted research in China, Taiwan and Japan, in which they observed teachers' classroom teaching. Stigler and Stevenson used the term 'constructivist' to describe the most common teaching approach that teachers applied, as they saw the teachers posing provocative questions when they taught mathematical tasks, and then they waited for a while allowing time for reflection. Teachers also used different techniques that suited individual students, which reflects Confucius' view of learning.

2.1.1.1.3 Beliefs about mathematics learning

To survey literature on beliefs about mathematics learning, it is important to begin with a somewhat unified conception of what exactly learning is. For the purposes of this thesis, learning is people's perception and interpretation of meaning and its process in order to understand reality (Säljö, 1982). Although culture does not condition whether or not people can learn, the psychological phenomena of learning varies across cultures (van Egmond, Kühnen, & Li, 2013). The literature has particularly emphasized how the psychological phenomena of learning varies between East Asian and Western countries (van Egmond, Kühnen, & Li, 2013).

Researchers have offered numerous explanations for why learning, even as psychological phenomena, differs across East Asian and Western countries. Li (2005) introduced two categories that can guide researchers in understanding and interpreting the differences between how East Asian and Western cultures conceive of learning. These two categories are 'mind orientation' and 'virtue orientation' beliefs (Li, 2005).

The essence of mind orientation is that people learn in order to improve their thinking and develop their cognitive skills and, in the process of learning, they will doubt established knowledge and evaluate their own and others' established beliefs (van Egmond, 2011). This orientation reflects Socrates' epistemological beliefs, which emphasized the importance of evaluating and questioning pre-existing knowledge by deeper, analytical questions, and by valuing self-generated knowledge (Tweed & Lehman, 2002). The essence of the virtue orientation is that people learn in order to develop the moral and social aspects of who they are (van Egmond, 2011). This conforms with Confucius' epistemological beliefs, as Confucius taught that behavioural reform and virtuous behaviour should be the main objective and measure of educational success because it will ratify individual achievement and create societal harmony (Tweed & Lehman, 2002).

Both the mind orientation and the virtue orientation were theorized to occur at the abstract, collective, cultural level, and were not theorized to explain the beliefs of individuals (Li, 2003). This is because individuals' beliefs proceed partially from the individual's environment, and are not solely a function of static, eternal Socratic or Confucian concepts (van Egmond, 2011). Indeed, the categories of the learning concepts can also be systematically different across cultures. The categories of learning concepts comprise four categories: the purpose of learning, which refers to the higher order level of meaning to the concept of learning; the processes of learning, which refers to learning strategies learners use; the affect and motivation of learning, which refers to whether learners enjoy their approach to learning; and the social perception of learning, which refers to how people view successful learners, unsuccessful learners, and teachers (van Egmond & Kühnen, 2014). The concepts of the mind orientation and virtue orientation can be combined with the categories of learning concepts to allow researchers to holistically and meaningfully integrate the differences in the conception of learning between both cultures (van Egmond, 2011; van Egmond, Kühnen, & Li, 2013).

The purpose of learning

The concepts of the nature of learning and the goals of learning vary based on cultural context. It is essential to understand how the cultural context conditions conceptions of the nature of learning and the purpose of learning, because different concepts on the nature and purpose of learning and, in particular, different conceptualizations of the relationship between intellectual and moral development, play an essential role in students' motivations to learn, as well as in the way that teachers guide students' learning processes (van Egmond, Kühnen, & Li, 2013). Again, research on the way that culture conditions beliefs about the nature and purpose of learning centres around a dichotomously conceived contrast between East Asian and Western cultures.

Abstractly, learning goals can be characterized into two essential developmental aspects: the enhancement of mental skills and knowledge, and development of personal aspects, such as moral and social sensibilities (van Egmond, 2011). However, different cultures emphasise different aspects (van Egmond, Kühnen, & Li, 2013). For example, the prime purpose of a Confucian education is for developing the moral aspects of a person, and therefore the objective of intellectual achievement is the cultivation of self-ethics (Leys, 1997). The ultimate objective of a Confucian education is not to have education, but to be wise, and knowledge is not an end in and of itself, but a tool that is used to cultivate a desire for wisdom (Leys, 1997). Correspondingly, in an empirical study of Chinese students, Li (2003) found that knowledge is related to crucial aspects of students' lives, and did not just have cognitive meaning, but also had relevance for the moral and social aspects of life.

Therefore, East Asian students' views of learning are conceptualized as 'virtue orientation' (van Egmond, Kühnen, & Li, 2013). In the virtue orientation, the ultimate goal of learning is to cultivate virtue. According to Li (2010), Chinese beliefs of learning focus on several purposes which are interrelated and part of Confucian beliefs about learning and actively supported by Asian families, communities, schools, and societies. They are moral and social self-perfection, gaining skills and knowledge for oneself, building oneself economically, attaining social position and honour, and participating in society. Consequently, in order to achieve these purposes of learning, one needs to enhance the following set of learning virtues: 1) the notion of resolve, which " refers to the determination of a learner to come to a course of action and the degree to which he or she is prepared to follow through on his or her commitment" (Li, 2010, p. 52); 2) diligence, which refers to recurrent study behaviour and how much time someone spends on learning, and it also ensures familiarity which gives more chances for mastery (Li, 2001); 3) endurance of hardship, which means overcoming the difficulties, barriers, and obstacles that anyone could face in their learning; 4) perseverance, which refers to someone's tendencies, general attitudes and behaviour toward learning which is important as there are no shortcuts to learning and acquiring knowledge; and 5) concentration, which describes a general learning disposition not related exclusively to specific tasks, and which focuses on consistency and dedication during studying. Thus, Chinese learners believe that these learning virtues are more important than actual learning activities such as reading, as they believe that when someone has developed these learning virtues, they will be able to use them to facilitate all their learning activities and processes (Roberts et al., 2010).

Several researchers (Fuligni, 1997; Stevenson, Hofer, & Randel, 2000; Stevenson & Stigler, 1992) provided similar findings that Chinese learners learn and achieve better than Western learners especially in math and science, which is due to different factors. One of these factors is that Chinese learners regard effort as a stable cause for learning and achievement (Hau & Salili, 1991; Salili & Mak, 1988), while Western learners believe that effort is an changeable cause (Weiner, 1986).

In contrast to East Asian culture, Western culture believes that the primary goal of education should be learning for its own sake, without any extrinsic, motivating goal such as the cultivation of good moral and social behaviour (Tweed & Lehman, 2002). This involves active learning, particular reasoning processes, analysis and scientific inquiry, task management skills, and communication such as discussion (Roberts et al., 2010). Western culture views the best learners as those who actively use their minds, and improve their minds through questioning established concepts and inquiring about the world (Li, 2005). Li (2003) argued that the American perspective of the purpose of learning placed more emphasis on the cognitive aspects of learning and neglected the other kinds of development, and thus stated that the American conception of learning did not connect cognitive development to moral development. Therefore, the Western conception of the purpose of learning is categorized as “mind oriented” (van Egmond, Kühnen, & Li, 2013). For example, Gao and Watkins (2001) found, as a result of their mixed method research, that Chinese teachers stressed the cultivation of adaptive attitudes towards learning and moral guidance among students, whereas Western teachers rarely encouraged and stressed this belief, which aligned with the findings of Western researchers about moral development and education (Damon, 2003). Moreover, according to Li (2010) the purpose of American conceptions of learning is centred on enhancing the mind and understanding the world. For instance, Li (2010) noted that when Chinese and American college students were asked to define knowledge, 79% of Chinese college students (but only 15% of European-American students) described it like the "need to perfect oneself" and "spiritual wealth/power"; where as 96% of American students (but 32% of Chinese students) described it as information, facts, skills, and understanding of the world. In fact, the mindsets of American students contain different type of thinking such as inquiry, analysis, deductive and inductive reasoning, and scientific discovery, which reflect the Socratic goal of learning.

The processes of learning

This literature review has established that the process of learning varies across cultures, and that the concepts of 'mind orientation' and 'virtue orientation' can be used to understand how learning processes vary across cultures. Western students, who tend to have a mind orientation, tend to follow learning processes that involve questioning and evaluating the ideas they are taught, both in school and outside of school (Tweed & Lehman, 2002). Further, in a reflection of Western culture's emphasis on individualism, learning processes tend to emphasize practices that help individuals develop their minds in order to help the learner gain knowledge, cognitive skills, and critical thinking skills (Merriam & Kim, 2008; Van Egmond, Kühnen, Haberstroh, & Hansen, 2013). Indeed, Western societies tend to view good learning as synonymous with critical thinking (Doddington, 2007). Thus, even though Western societies view learning as an essential part of students' lives, they emphasize students' cognitive development over their moral and emotional development (Tweed & Lehman, 2002). This ties to Western cultures' beliefs that morality is independent of cognition (Tweed & Lehman, 2002). For these reasons, Li (2003) conceptualized Western learners as being "mind oriented" because their learning processes reflect the characteristics of mind orientation.

By contrast, within the virtue oriented viewpoint, morality and cognition are related, and both are essential for the learning process (van Egmond, 2011). The Confucian philosophy views classrooms from a collectivist perspective and emphasizes the importance of discipline and authority figures (Smith & Hu, 2013). Therefore, in settings that emphasize virtue oriented learning processes, students are expected to accept teachers' instructions without evaluating and questioning the materials (Smith & Hu, 2013).

Rillero (2016) has found that deep learning and surface learning are two styles of learning that are completely different from one another. Deep learners are more likely to think and analyse, through understanding concepts and applying the concepts "to real life situations, or question conclusions", whereas surface learning is "marked by memorization, rote learning, and unquestioning acceptance of information" (Rillero, 2016, p. 16). Moreover, Marton and Säljö (1976, p. 14) have pointed out these qualitative disparities in learning levels "in terms of whether the learner is engaged in surface-level or deep-level processing." Thus, a surface approach to learning requires limited interaction with, for example, a task, usually concentrating on memorization or implementing procedures that do not require reflection or comprehension. In comparison, a deep learning approach includes, for example, an emphasis on relationships between different aspects of the content with the goal of understanding and enforcing meaning. Given that education is considered important for its intrinsic value in the Confucian tradition, it is by definition connected to the deep approach rather than the surface approach to learning. In addition, the importance of reflective thinking in the learning process in the Confucian tradition is strongly emphasised (Wang, 2006).

In mathematics education, the dialogue about procedural understanding and conceptual understanding has wide and deep roots (J. Hiebert & P. Lefevre, 1986; Star, 2005), and words such as 'rote learning' and 'real understanding' have dominated the discussion on learning for many decades (Schoenfeld, 2007). In this sense, 'rote learning' is often used in connection with memorization of facts or procedures for surface learning. 'Rote learning' is also used alongside terms such as procedural knowledge or procedural understanding (Hiebert & Carpenter, 1992; J. Hiebert & P. Lefevre, 1986), and instrumental understanding (Skemp, 1976). 'Real understanding' is frequently used in conjunction with deep learning and concepts like relational understanding (Skemp, 1976) and conceptual understanding (Hiebert & Carpenter, 1992; J. Hiebert & P. Lefevre, 1986). More detailed distinctions, however, exist (De Jong & Ferguson-Hessler, 1996). Such researchers believe that the knowledge base of an individual is made up of different forms of knowledge such as conceptual and procedural knowledge, and that this knowledge base can be distinguished by qualities and attributes such as deep or surface knowledge levels.

Confucian philosophy of learning and teaching not only stresses deep understanding more than surface knowledge (Wang, 2006) but also individuality in learning (Lee, 1996). In recent times, Kennedy (2002) stated there were some existing “Confucian confusions”. While the “Confucian values” of collectivism and conformity are frequently emphasised in the research literature on “the Chinese learner”, it should be noted that Confucius often emphasised individuality in learning, “learning for the sake of the self”. Education is important only if it contributes to the self-perfection; “the purpose of learning is therefore to cultivate oneself as an intelligent, creative, independent, autonomous, and an authentic being”. Confucius also “promoted reflection and inquiry” in the learning process (Lee, 1996, pp. 25-41). Likewise, Biggs (1991) shared a similar opinion and pointed out that, primarily, the Confucian tradition stresses a deep learning approach. Perhaps for this reason, although Western researchers have characterized East Asian students as passive learners, they show high and deep levels of conceptual understanding (Biggs, 1991; Watkins & Biggs, 2001). This deep conceptual understanding might not occur in spite of East Asian cultural norms about education but rather because of them:

> Confucius himself saw learning as deep: “seeing knowledge without thinking is labour lost; thinking without seeking knowledge is perilous [Analects II. 15]”, his methods were individual and Socratic, not expository; his aim was to shape social and familial values in order to conserve a particular political structure. These do not appear particularly conducive to surface learning. However, Confucius did inspire several themes and variations (Biggs, 1991, p. 30).

This observation is supported by empirical work, especially in comparative studies of Chinese and Western students. For example, one set of researchers stated that Chinese students believe that understanding is a long process that requires high mental effort, while Western students typically perceive understanding as a process of sudden insight (Dahlin & Watkins, 2000). This ties to the Confucian perspective that learning is contingent on effort and hard work (Tweed & Lehman, 2002). For this reason, Chinese students believe that to develop understanding, both repetition and 'attentive effort' are required because, by repeating things, students can check their understanding as well as develop it (Dahlin & Watkins, 2000).

Contrary to Western conceptions, Confucius did not encourage accepting whatever the teacher teaches, but simply believed that questioning of established ideas should occur at the end of the learning process, after learners had achieved understanding of established ideas (Cheng, 2000; Haller et al., 2007). For this reason, memorization is a vital aspect of the learning process in the Confucian philosophy because it is a crucial first step towards deeper learning (Biggs, 1996; Kennedy, 2002). It is worth noting that memorization in the Confucian tradition is seen as an integral part of learning but should not be equated with rote learning. Memorization precedes comprehension and is intended for deeper insight not as an end in itself (Wang, 2006).

In situations like the planning for an exam or results, "memorizing lines or already understood facts may be required to ensure success and is considered to be a deep approach" (Ho, Salili, Biggs, & Kit-Tai, 1999, p. 48). Research reveals that many teachers and the best students see memorising and learning not as separate, but instead as interlocking systems, and high quality learning typically involves both processes (Biggs, 1996; Marton et al., 1996; Marton et al., 1997; Watkins & Biggs, 1996). The main components of learning are memorising, remembering, thinking, and questioning. They are interrelated and interconnected, and should be replicated in order to know more in the future and for deep learning (Lee, 1996).

To conclude, the Western view of learning concentrates more on cognitive processes, as captured by the characterization of mind oriented learning systems (Li, 2003). The goal of learning processes in Western systems is to develop individuals' cognitive ability and encourage students to think critically (van Egmond, Kühnen, & Li, 2013). Learning processes in East Asian countries emphasize the development of moral and social attributes, such as diligence, concentration and humility, as well as questioning authority, because all of these qualities can prepare students to contribute to society (van Egmond, Kühnen, & Li, 2013).

Approaches to learning: the role of affect and motivation

In the process of learning, students use different learning strategies consistent with their motives. In the literature, these combinations of motive and strategies are called approaches (Wong, Lin, & Watkins, 1996).

As noted earlier, Marton & Säljö (1976, 1984) have identified two approaches to learning, which they term “surface” and “deep” approaches. A surface approach is characterized by a focus on the signal, or the materials used to convey content knowledge, in which learning reflects a passive approach and relies on memory (Marton & Säljö, 1976, 1984). On the other hand, a deep approach is characterized by a focus on what the signals more deeply signify, and the process of learning stresses the meanings underlying the learning materials (Marton & Säljö, 1976, 1984). Correspondingly, learners who have a deep approach are metacognitive in planning and observing their progress (Biggs, 1994).

The most important aspect of the distinction between the surface approach and deep approach lies in the learner’s motivation, or the absence of motivation. According to Biggs (1996) learners who take a surface approach tend to be extrinsically motivated or are motivated by a fear of failure. Learners who take a deep approach are more motivated by the subject matter of the task itself (Wong et al., 1996). Therefore, to understand a learner’s approach to learning, it is crucial to understand what motivates the learner.

However, this is also complicated by cultural factors because, while in Western cultural contents, extrinsic and intrinsic motivations are construed as mutually exclusive, in East Asian cultural contexts, extrinsic and intrinsic motivations are considered to be interrelated (Tweed & Lehman, 2002). Specially, in Chinese culture, research has indicated that external goals such as obtaining a job, pleasing parents, or achieving high grades are very connected to intrinsic motivations for learning (Tweed & Lehman, 2002). Thus, if Chinese students tend to use different strategies such as memorization and repetition to achieve and internalize their goals of learning that are connected to their lives, these students reflect a deep approach to learning. These learning strategies, such as repetition and memorization, that East Asian students tend to apply are understood in East Asian cultural contexts to lead to deep understanding (van Egmond, Kühnen, & Li, 2013). Therefore, there can be a complementary relationship between memorization and understanding, which contradicts the Western conception that memorization is only associated with rote learning and a surface level approach to learning (Purdie & Hattie, 2002).

Western research into student academic motivation has long been controlled by the theory of achievement goals (e.g., Ames (1992); Dweck (1986)). For example, such theorists specified two prime goals of achievement: mastery and performance goals. Mastery goal refers to the assumption that effort drives to success, and the learning's intrinsic value is salient. In contrast, a performance goal refers to an emphasis on one's self-esteem. Further, talent is shown by outperforming others, overstepping expectations, or making no effort to achieve success (Watkins, 2010). Western learners have appeared to value their performance and ability, whereas their Asian peers prefer their learning efforts (Hau & Salili, 1991).

These result align with the motive / strategy model of learning approaches (Biggs, 1987), where external, intrinsic, and achievement motivation drives a student to follow, respectively, surface, deep, and achievement-oriented learning strategies (Watkins, 2010). For the quality of learning product, mastery goals tend to be aligned with deep-level learning strategies and performance goals with surface learning strategies (Covington, 2000). Although most pedagogical philosophies in the West stress the value of intrinsic motivation (Deci & Ryan, 1985), extrinsic motivation is highly regarded in East Asia where exams are very competitive and teaching is aimed at testing to the degree that many Western teachers find inappropriate and restricted (Leung, 2001). Therefore, while Asian cultures encourage interdependence, school contexts tend to place even greater emphasis on individual competition than in the Western world.

In mathematics education, some Western scholars and scholars who are working in The University of Hong Kong such as Prof Leung (2000) indicated some features of China's mathematics education: East Asians think their Western peer have gone too far into the extreme process. They re-stress the importance of mathematics content in the learning process of mathematics. Western educators value intrinsic motivation in studying mathematics, and regard extrinsic motivation as detrimental to studying, such as that produced from the burden of evaluation and examination. Nonetheless, in East Asian countries an acceptable level of pressure is thought to be safe. East Asia claims that intrinsic as well as extrinsic encouragement can be used to encourage the learning of mathematics for students.

In addition, attitudes are affective responses which accompany a motivational behaviour (Guthrie & Knowles, 2001). Therefore, attitudes can be directly related to motivation and provide key information for a better understanding of attitudinal and motivational processes. In the field of mathematics education, there has been little research done to study the connection between motivation and attitudes (Mata, Monteiro, & Peixoto, 2012). Reynolds and Walberg (1992) utilised structural equation modelling to evaluate different elements influencing math performance and attitudes with students in 11th grade, identifying mathematics attitudes as a major impact on motivation. Moreover, a study with 10th grade students conducted by Hemmings and Kay (2010) also confirmed that effort was linked positively and significantly to math attitudes.

Social perception of learning

Culture does not only condition overt behaviours, but also social rules, values and principles that influence how people behave and how they view themselves (Kennedy, 2002). Cultural influences on teaching and learning practices have been labelled “academic cultures,” which refers to the expectations, beliefs, norms, and practice about how to preform academically (Cortazzi & Jin, 1997). Many researchers have pointed out that perception of learning should be examined with an awareness of social and cultural contexts (Marton & Booth, 1997). Therefore, learners from different cultures are socialized to different expectations, learning approaches, perceptions, norms, and beliefs about success and failure (Ho et al., 1999).

The Confucian philosophy characterizes “a set of principles for appropriate behaviour, so as to retain harmonious interaction among people” (Whitcomb, Erdener, & Li, 1998, p. 847). For example, in the Chinese context, to be considered an educated person, it is very essential to also be considered a good person and a good family member(Fryberg & Markus, 2007). Further, in a reflection of Confucian epistemological views where knowledge and truth must be attained from teachers as authority figures and not created by students themselves, Chinese teachers are likely to have a teacher-led view of classrooms (Cai & Wang, 2010) . In this context, students are taught to respect, obey, and follow teachers' instruction (Haller et al., 2007). In contrast, American teachers tend to have a more student-centred view of teaching, which reflects Socratic epistemological views that knowledge and truth must be created by individual learners through searching and questioning authority figures such as teachers (Cai & Wang, 2010).

In general, East Asian teachers view effective teaching as "a process of teachers' transmitting existing coherent knowledge precisely" (Cai & Wang, 2010, p. 284). Further, Haller et al. (2007) pointed out that in cultures with Confucian heritage, students have a highly structured relationship with their teacher, follow a collective approach to learning, and are concerned with reflective practices which are very important for deep learning. In contrast Western culture stresses individual development (Wang, 2006), and Western societies expect learners to cultivate themselves as individuals with their own ideas, independent minds, and creativity (Haller et al., 2007). Therefore, Western teachers see effective teaching to be a process of facilitating and encouraging students' exploration, use, and creation of knowledge for themselves (Cai & Wang, 2010). To conclude, these different views between East Asian culture and Western culture reflect "virtue orientation" and "mind orientation" respectively.

Wang and Leichtman (2000) offered an empirical support to this theoretical hypothesis in their comparative analysis of narratives of children from China and North America. Such authors consider that Chinese children in their narratives are already making more didactic statements about social standards and moral rules than Western children. Therefore, learning and schooling are not only perceived as cognitive functions, but are viewed by Chinese children as an indicator of morality prior to comprehensive education. For example, Chang and Wong (2008) assessed different components of Chinese student achievement motivation in Singapore, which involved socially oriented goals of achievement (for example: "I study so my peers would respect me"; "I study so my community will be proud of me"; p. 885). The authors hypothesised that this social aspect of academic achievement is an integral part of the Chinese students' learning target orientation. Results showed a positive relationship between this social orientation towards academic achievement and the success goal and performance, mastery goal, and competitive motivation of the students. Such empirical findings underlie the virtue-oriented belief that academic achievement involves a strong social dimension and that good scholarship is highly regarded.

Ignacio, Nieto, and Barona (2006) utilised the expression mathematics self-concept to link to personal beliefs related to the world of mathematics, to the set of ideas, judgments, beliefs, and attributions that the person has consistently built up in the school environment during his or her learning process. Personal beliefs impact the interest of the person in mathematics, efficiency in the performance of mathematical tasks, motivation and enjoyment in mathematics, attribution of causes to academic success or failure and self-conception as integrated to a certain social group. Hannula (2007) noted that a learner of mathematics likes or dislikes mathematics because of his / her structure of belief.

Educational research found that Asian students have greater motivation for achievement than Western students because Asians believe that all performance is related to internal and controllable source-effort, while Western students believe more in fixed capacity (Tweed & Lehman, 2003). This belief impacts Asian parents to put higher academic expectations on their children (Stevenson et al., 1990).

Reglin and Adams (1990) found that, even with Asian students raised in the American culture, these children are more affected by the desire for achievement of their parents than their non-Asian peers are. In this way, the desire of Asian students to meet the academic expectations of their parents, combined with their belief in learning through effort rather than it being a fixed ability, can be interpreted as them showing a higher level of self-control efforts to achieve academic success.

2.2.2 Teachers' Cognitive Lens: How Teachers Know

While research that focuses on the affective lens of teachers' mathematical knowledge for teaching explores the web of beliefs that teachers have about mathematics, researchers need to understand more than a teacher's beliefs to understand her mathematical knowledge for teaching. For this reason, another strain of research uses the concept of cognitive lens to understand teachers' mathematical knowledge for teaching. The cognitive lens focuses not on understanding teachers' beliefs about mathematics, but on identifying and understanding their knowledge (Petrou & Goulding, 2011; Rowland, 2014b). Understanding the cognitive lens is crucial for understanding mathematical knowledge for teaching, as there is widespread agreement that mathematical knowledge for teaching is crucial for teachers as it underpins the way they teach (Rowland, Turner, Thwaites, & Huckstep, 2009). Mathematical knowledge for teaching has been empirically demonstrated to affect the way that teachers choose appropriate examples, make good representations, and help students make effective connections during classroom teaching (Rowland et al., 2009). Further, teachers' mathematical knowledge for teaching has been empirically demonstrated to affect how teachers respond to students' ideas (Rowland et al., 2009). Indeed, identifying and understanding mathematics teachers' knowledge is a complex area of research in the mathematics education field. The aim of identifying teachers' mathematical knowledge goes back several decades and is complicated by the fact that there are currently multiple conceptualizations of mathematics teacher knowledge (Petrou & Goulding, 2011; Rowland, 2014). The reasons that these multiple conceptualizations exist is because researchers use different frameworks to describe and identify various kinds of mathematical knowledge for teaching that are central for teaching mathematics (Ball et al., 2008; Baumert et al., 2010; Blömeke, Hsieh, Kaiser, & Schmidt, 2014). The existence of numerous frameworks for describing, identifying, and understanding teachers' mathematical knowledge for teaching is evidence of the complexity of the mathematics education field.

The aim of the next section of the literature review is to provide a survey of the literature that uses the cognitive lens to understand mathematical knowledge for teaching. To accomplish this aim, it sketches out a few of the primary theoretical frameworks that are used to categorise and measure mathematics teachers' knowledge for teaching. It presents these frameworks chronologically, in subsequent sections, with the goal of tracing how conceptualizations and categorizations of mathematical knowledge for teaching have evolved over the past thirty years. The first framework to be surveyed is a seminal paper by Shulman (1986), and the section then proceeds to review the work of Ball and her colleagues, the COACTIV framework (Krauss, Baumert, & Blum, 2008) , the TEDS-M (Senk et al., 2008), and the Knowledge Quartet (Rowland & Turner, 2007). In contrast to other studies, the Knowledge Quartet considers knowledge in practice rather than categorizing and measuring knowledge, and therefore is considered last, and out of chronological order (Rowland & Turner, 2007). As each framework is surveyed, its meaning, importance, and limitations are considered.

2.1.2.1 Shulman's theoretical framework

A seminal attempt to create a theoretical framework to identify, describe, and categorize mathematical knowledge for teaching was contained in Shulman's 1986 paper. In this work, Schulman created seven categories to typologize teachers' professional knowledge that is essential for teaching, a similar concept to mathematical knowledge for teaching (1986). These categories were: subject content knowledge (CK), pedagogical content knowledge (PCK), curriculum knowledge, general pedagogical knowledge, knowledge of learners and their characteristics, knowledge of educational contexts, and knowledge of educational goals and values (Shulman, 1986).

The first three categories encompassed the content-specific dimensions of knowledge that teachers need in order to effectively teach (Shulman, 1986). In typologizing these, Shulman argued that he had articulated the 'missing paradigm' in research on teaching (1986). On the other hand, the last four categories referred to general dimensions of knowledge that teachers need in order to teach effectively, and were not the main focus of Shulman's conceptualization (Ball et al., 2008). Therefore, the discussion that follows treats Schulman's content-specific dimensions of knowledge in much more detail than it treats the general dimensions of knowledge.

Shulman disaggregated the content knowledge that teachers need to effectively teach into three categories: subject matter content knowledge (CK), pedagogical content knowledge (PCK) and curriculum knowledge (1986). Content knowledge "refers to the amount and organization of knowledge per se in the mind of the teacher" because it requires that teachers go "beyond knowledge of the facts or the concepts of a domain" (Shulman, 1986, p. 9). The concept of content knowledge comprises what Schwab (1978) termed the substantive and syntactic structures of a discipline. Substantive knowledge focuses on the key facts, principles, concepts, models, and frameworks, while syntactic knowledge focuses on the process and procedures for gaining knowledge and includes valid theories and principles (Schwab, 1978). Therefore, in terms of mathematical knowledge, substantive knowledge includes concepts such as the orders of operations, while syntactic knowledge includes the ability to do mathematical proofs (Schwab, 1978). Then, the concept of curriculum knowledge refers to the way of understanding the design and the structure of school subjects, as well as understanding how to use related materials, such as the curriculum and textbooks (Shulman, 1986).

The most influential of Schulman's three categories of content knowledge is pedagogical content knowledge (PKC) (Ball et al., 2008). Indeed, the concept most closely relates to the concept of mathematical knowledge for teaching, because it concerns not just knowledge of the discipline in and of itself, but knowledge of how best to externally represent certain aspects of the discipline so that other people can learn them. For Schulman (1986), pedagogical content knowledge merited special consideration among the seven categories in his typology because it distinguished between content knowledge and pedagogical knowledge and struck to the core of what he termed the "missing paradigm" in education research. Schulman (1986) defined pedagogical content knowledge as "that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (p. 8). Schulman argued that pedagogical content knowledge "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (1986, p.9).

Schulman's (1986) framework, and especially his typology of content knowledge and his emphasis on the importance of pedagogical content knowledge for teachers' professional efficacy, provoked a new focus on the role of content knowledge in teaching efficacy (Ball et al., 2008; Petrou & Goulding, 2011). Indeed, the deep influence of Shulman's (1986) conceptualization of pedagogical content knowledge is evidenced by the succession of researchers who have used and refined the concept in their theoretical and empirical work (Ball et al., 2008; Grossman, 1990; Hill et al., 2004; Marks, 1990).

However, although Shulman's (1986) typology of content knowledge was extremely impactful, subsequent research has argued that the 1986 typology left the differences between the concepts of subject content knowledge and pedagogical content knowledge unclear (Ball et al., 2008). The source of the confusion was the fact that sometimes Shulman's (1986) definition of pedagogical content knowledge encompassed content knowledge, and then, in some instances, Shulman's (1986) definition of content knowledge encompassed pedagogical content knowledge only (Ball et al., 2008). Further, subsequent theorizing has pointed out that Shulman (1986) did not address the interaction between different types of teachers' knowledge for teaching (Hashweh, 2005). It is likely that the seven different aspects of teachers' knowledge for teaching interact to condition each other, but Shulman's (1986) framework treats them as distinct, artificially independent categories (Hashweh, 2005).

Another set of scholars argued that Shulman's (1986) framework was overly cognitive in emphasis, and in particular that the concept of pedagogical content for teaching needed to be broadened to account for beliefs and emotions in order to be effectively descriptive (Friedrichsen, Driel, & Abell, 2011; Zembylas, 2007). In other words, they argued that Shulman's (1986) cognitive framework needed to do more to account for affective factors. They contended that erasing the role of beliefs and emotions from pedagogical content knowledge ignored the dynamic nature of knowledge, and as a result neglected to focus on the interactions between teachers and students in the classroom (Fennema & Franke, 1992).

2.1.2.2 The Michigan project: "mathematical knowledge for teaching"

Subsequent work on the cognitive aspect of mathematical knowledge for teaching recognized the magnitude of Shulman's (1986) contributions, but also sought to correct the shortcomings of his theoretical framework (Ball et al., 2008). One of these shortcomings was the blurry distinction between content knowledge and pedagogical content knowledge (Ball et al., 2008). Therefore, Ball et al. (2008) sought to build on Shulman's (1986) notion of pedagogical content knowledge by articulating types of domains of mathematics knowledge for teaching that covered both content knowledge and pedagogical content knowledge. These domains are illustrated in Figure 2 (Ball et al., 2008).

These domains of mathematics knowledge are perhaps the most influential reconceptualization of pedagogical content knowledge within the field of mathematics education research (Depaepe, Verschaffel, & Kelchtermans, 2013). Ball and her co-authors developed this reconceptualization of pedagogical content knowledge using their empirical research; they termed their conceptualization of the domains of mathematics knowledge "practice-based theory of the mathematical resources entailed by the work of teaching" (Ball & Bass, 2009, p. 1). In this way, Ball and her co-authors articulated a framework to describe mathematical knowledge for teaching, which is the kind of knowledge that is required in the work of teaching, and which places an emphasis on the use of knowledge for teaching, rather than simply content knowledge for its own sake (Ball et al., 2008). It is crucial to note that the concept of mathematical knowledge for teaching was predicated on a broad conception of what constitutes teaching:

> everything that teachers do to support the instruction of their students. . .the interactive work of teaching lessons in classrooms, and all the tasks that arise in the course of that...Each of these tasks involves knowledge of mathematical ideas, skills of mathematical reasoning. . .fluency with examples, and thoughtfulness about the nature of mathematical proficiency (Ball et al., 2005, p. 17).

Therefore, mathematical knowledge for teaching encompasses all the knowledge that mathematics teachers use in order to instruct their students across a variety of educational contexts and settings.

This has been removed due to copyright restriction. Available online from [https://www.researchgate.net/publication/255647628_Content_Knowledge_for_Teaching_What_Makes_It_Special / Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes It Special? Journal of Teacher Education, 59(5), 389-407.].

Figure 2: Mathematical knowledge for teaching

As Figure 2 shows, the Ball et al. (2008, p. 403) framework of mathematical knowledge for teaching divided subject matter knowledge into three categories: common content knowledge, specialized content knowledge, and horizon knowledge. Common content knowledge is the kind of knowledge that can be used in various settings, can be understood by everyone, not only mathematics teachers, and is not unique to mathematics teaching (Ball et al., 2008). Specialized content knowledge is knowledge that non mathematics teachers do not need: it is unique for mathematics teachers, as it is the mathematical knowledge and skills distinctive for mathematics teaching that are not required for purposes beyond teaching mathematics (Ball et al., 2008). Horizon knowledge is an understanding of the relationship between mathematical topics in the curriculum, which matters because it helps teachers to know what related knowledge students will study in the future, which can help them ensure that students build a foundation of mathematical knowledge for upcoming mathematical topics (Ball et al., 2008).

Then, as Figure 2 also indicates, Ball et al. (2008) divided the concept of pedagogical content knowledge into three domains: knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. Knowledge of content and students refers to the kind of knowledge that teachers must know about mathematics and know about students' conceptions and misconceptions of mathematics to teach effectively (Ball et al., 2008). Knowledge of content and teaching refers to the kind of knowledge that teachers must have about mathematics itself, as well the knowledge about teaching instruction to enable teachers to effectively choose examples, teaching resources and mathematical language that meets students' learning needs at different stages (Ball et al., 2008).

Ball et al. (2008) pointed out that specialized content knowledge is distinct from the knowledge that could be learned in university mathematics courses. They argued that numerous mathematical tasks that teachers do in the classroom require knowledge of planning and instructional design (Ball et al., 2008). For instance, teachers need to decide and choose which example to start with, what type of manipulatives should be used and what mathematical terms are most amenable to building students' understanding (Ball et al., 2008). Some teachers may have strong content knowledge in mathematics but still lack sufficient knowledge to effectively teach or, in Ball et al.'s (2008) terms, knowledge of content and teaching. Thus, Ball and her colleagues "began to hypothesize that there were aspects of subject matter knowledge—in addition to pedagogical content knowledge—that need to be uncovered, mapped, organized, and included in mathematics courses for teachers" (Ball et al., 2008, p. 389).

The concept of mathematical knowledge for teaching not only represented a conceptual advance, but it also constituted an advance in measurement and therefore methodology. It provided a path as well as domains for educators to investigate teachers' subject matter knowledge in mathematics through examining teachers' practical teaching by evaluating mathematical difficulties related to their teaching (Ball et al., 2008). Further, subsequent researchers developed a measurement framework of Mathematical Knowledge for Teaching (MKT-test) that operationalized Shulman's concept by developing a valid measurement of mathematical knowledge (Depaepe et al., 2013).

Still other researchers pointed out several important shortcomings in the framework of mathematical knowledge for teaching (Petrou & Goulding, 2011). Some of these critiques, again, focused on a lack of incorporation of affective aspects (Petrou & Goulding, 2011). Critics argued that this was an essential aspect of any framework dedicated to explaining mathematical knowledge for teaching, because the inevitable objective of knowledge for teaching is effective teaching. Indeed, the aim of Ball et al.'s (2008) project was not just to describe the mathematical knowledge necessary "to carry out the work of teaching mathematics", but also to create a measure using multiple-choice items that represented "teaching-specific mathematical skills [that] can both reliably discriminate among teachers and meet basic validity requirements for measuring teachers' mathematical knowledge for teaching" (Hill et al., 2005, p. 373). In other words, the point of measures of mathematical knowledge for teaching would be to guide researchers and teachers towards more effective teaching (Hill et al., 2005). Indeed, the central contribution of Ball et al.'s project was construed to be a method for better discovering the relationship between teacher knowledge and students' success in mathematics (Petrou & Goulding, 2011).

In spite of these ambitions, Ball et al.'s (2008) framework did not address the essential role of teachers' beliefs in their actual teaching (Petrou & Goulding, 2011). In addtion, the mathematical knowledge for teaching (MKT) framework did not allow for practical evaluation of teachers in practice, and instead relied on multiple-choice, self-report items. However, understanding whether teaching is effective is a question that depends on context, and self-report multiple-choice items cannot adequately account for the role of context (Watson, 2008). Further, this measure assessed teachers' knowledge but, as Watson and Barton (2011) argued, effective teaching is "not just a question of what teachers know, but how they know it, how they are aware of it, how they use it and how they exemplify it" (p. 67). As previous sections of this literature review have established, how teachers know, use, and exemplify what they know fundamentally concerns their beliefs about the nature of their subject, as well as their beliefs about pedagogy. In other words, it is impossible to understand the extent and effect of a teacher's mathematical teaching using only a cognitive framework such as the MKT test, because affective factors can condition the relevance of this framework.

2.1.2.3 The COACTIV project

At roughly the same time that Ball and her colleagues (2008) developed the framework of mathematical knowledge for teaching, Baumert and his colleagues (2010) in Germany conducted an empirical study of the mathematical knowledge of secondary school mathematics teachers, called the COACTIV project (Baumert et al., 2010). According to Neubrand (2018), the COACTIV project was based on the previous work of Shulman (1986) and Ball and Bass (2009); however, it built on these previous projects because Shulman (1986) emphasized subject knowledge but not mathematics-specific knowledge, and Ball and Bass (2003) focused on primary teaching.

Baumert et al. (2010) conception of knowledge was that the knowledge domains of content knowledge and pedagogical content knowledge were theoretically distinct (Warburton, 2015). They defined content knowledge as "deep understanding of the domain itself" and pedagogical content knowledge as an understanding of "how to make the subject comprehensible to others" (Baumert et al., 2010, p. 1). This definition of content knowledge was based on Shulman's conception (1986), which held that mathematics teachers' content knowledge was distinct from the mathematical knowledge that all adults should have, and that was not necessarily covered by school curricula (Baumert et al., 2010, p. 1). Baumert et al.'s (2010) conceptualization is also similar to Ma's (1999a) conceptualization of content knowledge as the profound mathematical understanding of school mathematics (Cole, 2011). In contrast, pedagogical content knowledge includes three sub dimensions: awareness of student misconceptions and difficulties, selecting tasks, and using representations, analogies, illustrations, and examples (Baumert et al., 2010, p. 1).

The COACTIV project aimed to develop a reliable test to assess secondary mathematics teachers' knowledge, and thus expanded beyond the self-report measure that Ball et al. (2008) had previously developed to assess mathematical knowledge for teaching (Baumert et al., 2010). The COACTIV assessment included 13 items designed to test content knowledge and 35 open-ended items designed to test pedagogical content knowledge (Warburton, 2015). In addition, some of these items could also be used to assess Ball et al.'s (2008) concept of specialized content knowledge (Cole, 2011). Baumert and his colleagues (2010) used the COACTIV assessment to examine mathematics teachers in Germany whose classes played a part in the 2003/4 wave of the Programme for International Student Assessment (PISA) (Warburton, 2015). Since its first use, the COACTIV assessment has influenced many studies, and has broadened their focus to include teachers' enthusiasm, motivation, and beliefs (Kunter et al., 2013). One such study that the COACTIV assessment influenced was the TEDS-M study (Kunter et al., 2013) , which is discussed in more detail in the following subsection.

2.1.2.4 TEDS-M

The Teacher Education and Development Study in Mathematics (TEDS-M) was an international study conducted in 18 countries in order to classify the nature and extent of primary and lower secondary school mathematics trainee teachers' knowledge for teaching (Senk et al., 2008). The motivation of the TEDS-M was to investigate teacher preparation and training (Tatto et al., 2008). To do this, it needed to measure mathematical knowledge for teaching, which comprises both 'mathematical content knowledge' and 'mathematics pedagogical content knowledge' (Tatto et al., 2008.) This conceptual distinction was similar to the distinctions specified in both Ball et al. (2008) and Shulman (1986), and closely matched Ball et al.'s (2008) conception of specialized content knowledge (Cole, 2011).

However, there are some essential differences between Ball et al.'s (2008) conception of specialized content knowledge and the way that the TEDS-M identified and assessed mathematics teachers' content knowledge. For example, the TEDS-M instrument identified teacher knowledge based on teacher trainees' understanding of the content of mathematics taught in school, not by assessing teacher trainees' understanding of the mathematical difficulties of teaching (Senk et al., 2008). Finally, the TEDS-M study did not assess mathematics teachers' quality of instruction to insure the validity of their measures (Cole, 2011). Nonetheless, a notable strength of the TEDS-M measurement is that it included items to identify teachers' beliefs about the nature of mathematics as a discipline, mathematics achievement, and learning mathematics (Neubrand, 2018).

2.1.2.5 The Knowledge Quartet

The Knowledge Quartet was a distinct conceptualization of mathematical knowledge for teaching that partially evolved from Ball et al.'s (2008) work (Neubrand, 2018). However, the Knowledge Quartet was distinct from that framework of mathematical knowledge for teaching because the Knowledge Quartet was developed for a different reason: the need for an empirical framework to observe and investigate teachers' subject matter knowledge, as well as how their subject matter knowledge related to their beliefs and actual teaching practices (Rowland & Turner, 2007). The Knowledge Quartet approach differed considerably from Shulman's (1986) conceptualizations of subject matter knowledge and pedagogy content knowledge because the Knowledge Quartet concentrated on mathematics content of the lessons that teachers teach, rather than focusing more broadly on general features of teaching which include additional aspects, such as behaviour management (Rowland & Turner, 2007). For this reason, the Knowledge Quartet sought to identify and classify situations within classrooms in which mathematical knowledge is employed in actual teaching (Petrou & Goulding, 2011). Therefore, based on analysis of video recordings of trainee teachers teaching in the classroom over one academic year, researchers created four categories of mathematical knowledge as it is used in the classroom: foundation knowledge, transformation, connection and contingency (Petrou & Goulding, 2011). Figure 3 illustrates the relationships between these four categories, which together are known as the Knowledge Quartet.

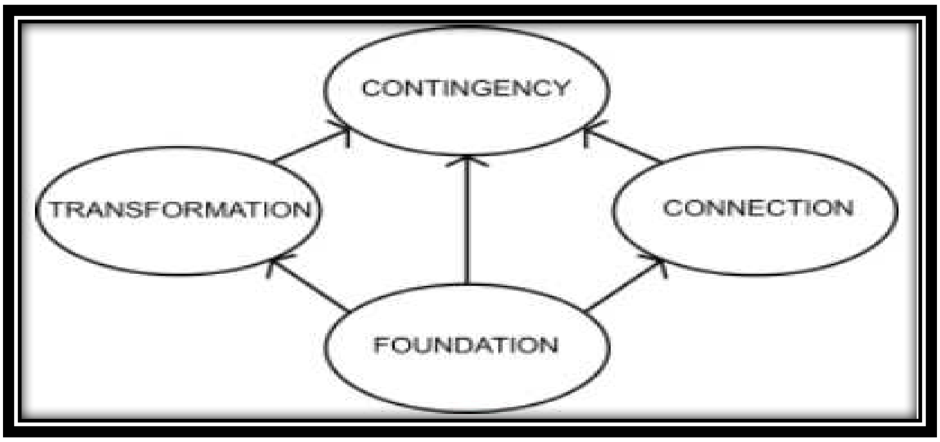


Figure 3: The relationships between the four dimensions that comprise the Knowledge Quartet

Foundation, one of the four categories in the Knowledge Quartet, is a component of beliefs that was not discussed in Shulman's (1986) or Ball et al.'s (2008) work. Foundation is the dimension that "consists of trainees' knowledge, beliefs and understanding acquired in the academy, in preparation for their role in the classroom" (Rowland et al., 2003, p. 97). The key components of the dimension of foundation are:

> knowledge and understanding of mathematics per se; knowledge of significant tracts of the literature and thinking which has resulted from systematic enquiry into the teaching and learning of mathematics; and espoused beliefs about mathematics, including beliefs about why and how it is learnt (Rowland & Turner, 2007, p. 112).

This dimension matters, because the knowledge and beliefs it contains fundamentally inform teachers' pedagogical decisions and strategies (Rowland et al., 2003, p. 97).

Transformation, another of the four categories in the Knowledge Quartet, refers to how teachers present knowledge "in the form of analogies, illustrations, examples, explanations and demonstrations" in order to make the content more accessible to students (Rowland et al., 2004, p. 123). Transformation "concerns knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself" (Rowland & Turner, 2007, p. 113).

Connection, another category in the Knowledge Quartet, refers to a dimension that "binds together certain choices and decisions that are made for the more or less discrete parts of mathematical content" (Rowland et al., 2004, p. 123). The concept of connection draws upon the facts that learners often need their experiences to be sequenced in ways that are optimal for their comprehension, that an essential aspect of the teacher's task is to predict what learners will find difficult or simple to comprehend, and to anticipate which errors learners will typically make in trying to learn a new mathematics concept (Weston, 2013, p. 287).

Contingency, the last category in the Knowledge Quartet, refers to how teachers react to unexpected classroom situations for which they could not plan. Rowland and Turner (2007) explained the contingency dimension as "the ability to 'think on one's feet'" (p. 114). This category in the Knowledge Quartet was not included in other frameworks that have been described in this literature review.

While The Knowledge Quartet was proposed to analyse primary school teachers' classroom teaching, it was also found to be effective in analysing secondary school teachers' classroom teaching (Weston, Kleve, & Rowland, 2012). The Knowledge Quartet has been used to support the development of early career teachers in England (Turner, 2006) and as a framework for lesson observation and mathematics learning development at Cambridge University (Rowland & Turner, 2007).

2.3 Limitations of Previous Approaches

Within this section, three limitations of previous approaches will be addressed. This is because teaching requires more than just examining teachers' knowledge, it also requires examining the related contextual factors, especially when they are connected to the field (Neubrand, 2018).

2.3.1 Limitation one: the gap between knowing and acting

Although many researchers have posited that mathematical knowledge for teaching, both in its cognitive and affective components, influences the decisions that teachers make in their classroom instruction, there are still many questions about how the affective and cognitive factors that constitute mathematical knowledge for teaching interact to produce pedagogical outcomes (Ball et al., 2008; Senk et al., 2008).

2.3.2 Limitation two: the importance of the cultural context

Further, previous research has argued that context plays a crucial role in this process, but again, no one has yet offered a structured theory of how context interacts with affective and cognitive elements of mathematical knowledge for teaching to produce teaching outcomes (Hiebert & Grouws, 2007; Li, 2011). This thesis provides empirical insights that might help form the foundation for subsequent theories that sketch out an explanation of how context, culture, affective elements of mathematical knowledge for teaching, and cognitive elements of mathematical knowledge for teaching, interact to produce the enactment of teaching.

2.3.3 Limitation three: the affective component

Since Shulman (1986) published his foundational typology of the kinds of professional knowledge that teachers need in order to successfully educate students, researchers have argued that those who take the cognitive approach to understanding mathematical knowledge for teaching need to better account for how affective factors might condition cognitive knowledge (Ball et al., 2008; Friedrichsen et al., 2011; Zembylas, 2007). They have argued that knowledge is dynamically constructive, and they draw on empirical findings from psychology that show that processes of cognition are conditioned by affective factors (Friedrichsen et al., 2011; Zembylas, 2007).

Subsequent iterations of cognitive frameworks for understanding mathematics knowledge for teaching also received the same criticism (Petrou & Goulding, 2011). Indeed, when Ball and her co-authors (2008) developed a conceptualization and empirical measure of mathematics knowledge for teaching, this measure was criticized by other researchers because, in spite of its important contribution, it consisted of self-report measures of mathematics teachers' knowledge, and did not account for an assessment of mathematics teachers' beliefs, nor an empirical assessment of how mathematics teachers' knowledge for teaching actually manifested in the classroom (Watson & Barton, 2011). The contention was that, to fully understand how mathematics teachers' cognitive understandings of the knowledge they need to teach mathematics actually translates to teaching efficacy, researchers also needed to account for the affective components that condition how cognitive knowledge translates to teaching practice.

Importantly, the creation of the TEDS-M assessment represented progress in this area (Neubrand, 2018). The TEDS-M measurement included items to identify teachers' beliefs about the nature of mathematics as a discipline, mathematics achievement, and learning mathematics, and thus included affective elements in an assessment of teachers' cognitive frameworks (Neubrand, 2018). In addition, the Knowledge Quartet is another cognitive framework that also better accounts for the affective lens, especially through its category of Foundation, which is one of the four categories of knowledge that constitutes the Knowledge Quartet.

2.4 Theory: A Proposed Approach to Mathematical Knowledge for Teaching

A few Saudi researchers examined Saudi's mathematics teachers' knowledge and beliefs to determine their impact on their teaching. For example, Alsaleh (2019) investigated 105 Saudi female pre-service teachers (PSTs) to find how well prepared they felt to teach mathematics at secondary or middle schools. She found that (PSTs) felt inadequately prepared in some respects for their teaching roles, and needed more support and guidance.

Also, Al Dalan (2015) observed and interviewed five Saudi mathematics trainee primary teachers to outline the possible impact of teachers' knowledge, both subject matter knowledge (SMK) and pedagogical content knowledge (PCK), on their handling of students' contributions in the classroom. He found that, teachers' SMK and PCK have influenced their response to the students' answers. Their response approach was primarily formed by the teachers' beliefs of how mathematics is best learnt, while PCK influenced the quality of their responses.

The motivation of this study is to understand the mathematics teaching knowledge of female mathematics teachers in middle schools in Saudi Arabia. More descriptive and explanatory insight on Saudi female teachers' mathematical teaching knowledge is an important research contribution. It stands to expand our understanding of how culture affects mathematical knowledge for teaching, and of how culture conditions the way mathematical knowledge for teaching affects pedagogical practices and educational outcomes, because very little (e.g., Al Dalan, 2015; Alanazi, 2016; Alqahtani, Kanasa, Garrick, & Grootenboer, 2016; Alzaghibi & Salamah, 2011; Haroun, Ng, Abdelfattah, & AlSalouli, 2016; Madani & Forawi, 2019) of the extensive literature that addresses how culture interacts with knowledge for teaching explores the Saudi Arabian context. Since so much of the literature on the role of culture in knowledge for teaching focuses on the contrast between Asian and American or European cultural contexts, exploration of cultural contexts beyond these cultures holds the potential to broaden our understanding of how culture affects pedagogy.

The literature on mathematics knowledge for teaching indicates that this research focus necessitates four research questions, which are as follows:

RQ1: What cognitive type of content knowledge do Saudi Female Mathematics Teachers in middle school have?

RQ2: What beliefs do Saudi Female Mathematics Teachers hold regarding the nature of mathematics, mathematics teaching, and mathematics learning?

RQ3: How do Saudi Female Mathematics Teachers' cognitive types of content knowledge and their beliefs impact their pedagogical decisions and quality of teaching?

RQ4: How do the culture beliefs of teachers influence their cognitive type of content knowledge, beliefs and pedagogical decisions?

The following section articulates a theoretical framework that can be used to answer these research questions, which will itself constitute a new proposed approach to understanding mathematical knowledge for teaching in the Saudi middle school context. The theoretical framework can be holistically summarized by the following figure (Figure 4), which is then explained in the subsequent paragraphs.

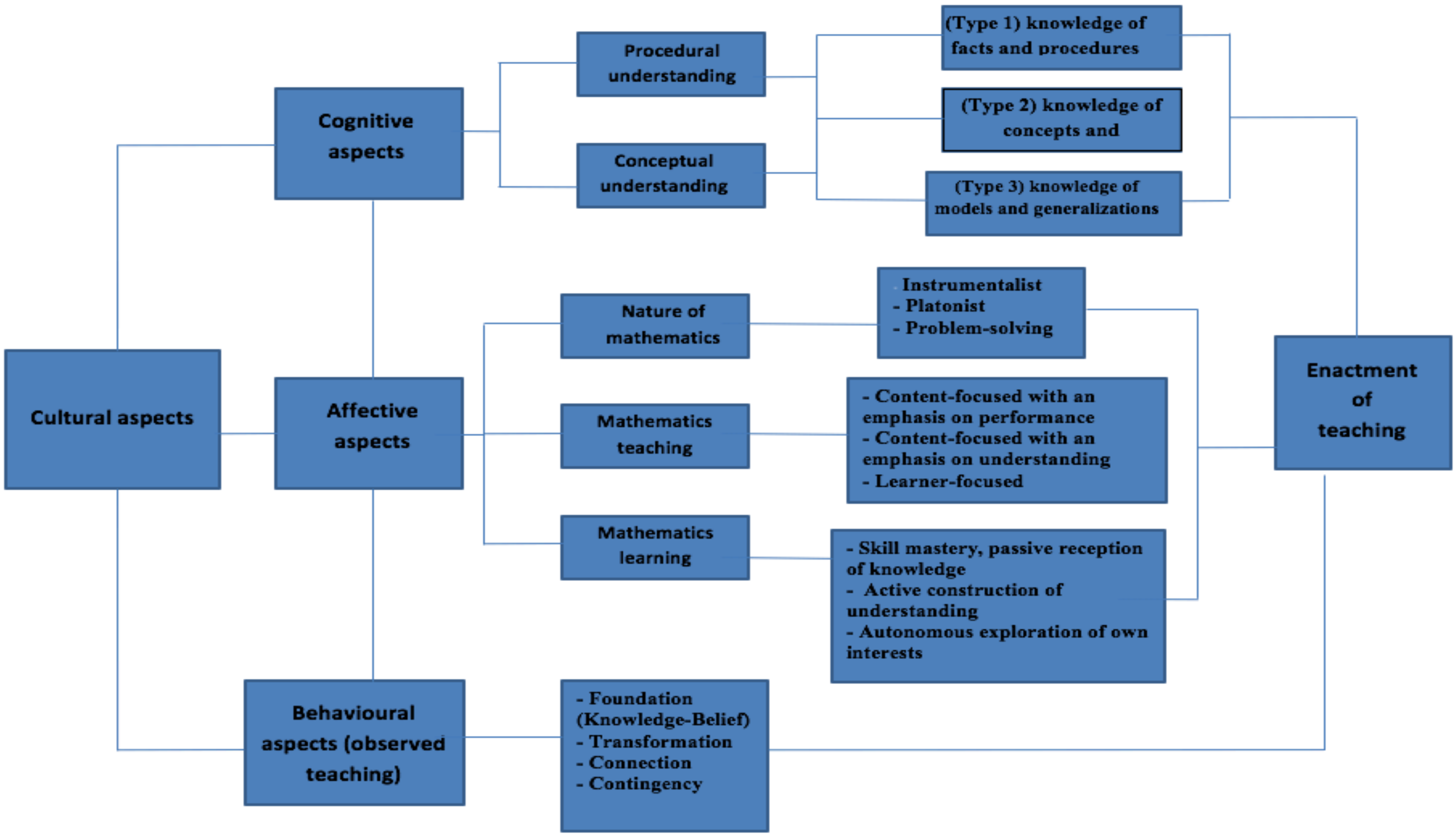


Figure 4: Cognitive, Affective, and Behavioural aspects of enactment of teaching mathematics

Figure 4 shows that there are three broad categories of aspects that affect the enactment of teaching: cognitive aspects, affective aspects, and behavioural aspects. Cognitive aspects refer to conceptual and procedural understandings about what 'mathematics' is and how it work, affective aspects refer to beliefs about the nature of 'mathematics' and how they should be taught, and behavioural aspects refer to the actions that teachers take when instructing learners. Each of these aspects has several constituent sub-categories that bear further explanation.

The cognitive aspects that contribute to the enactment of teaching refer to conceptual understandings and procedural understandings of how mathematics is done. According to Tchoshanov (2011) these are disaggregated into three types. Type 1 conceives of mathematics as knowledge of facts and procedures. Type 2 conceives of mathematics as concepts and connections. Type 3 conceives of mathematics as knowledge and generalizations. The first research question in this thesis is a descriptive question about which type of content knowledge about mathematics Saudi female teachers tend to have. The objective of this research question was to establish whether a Saudi middle school female mathematics teacher is most likely to think of mathematics as the study of facts and procedures, concepts and connections, or knowledge and generalizations.

Moving to the third research question, this focuses on how Saudi female teachers' type of content knowledge impacts on their pedagogical decisions and their quality of teaching. The question of content knowledge best relates to the cognitive aspect of mathematical knowledge for teaching, and previous studies of mathematical knowledge for teaching do not offer clear explanations about how teachers' content knowledge affects the choices that they make as they are teaching in the classroom. This research question touches on the Knowledge Quartet's conceptions of contingency, or the decisions that teachers need to make to address unexpected classroom situations, and transformation, which concerns how teachers choose to communicate their content knowledge to students (Senk et al., 2008). The Knowledge Quartet lends an expectation that teachers' content knowledge should affect the decisions they make as they teach in the classroom.

Indeed, the fourth research question is specifically designed to assess how Saudi middle school female mathematics teachers' cognitive types of mathematical knowledge, beliefs and pedagogical decisions interact with their cultural beliefs. Previous literature on how culture affects mathematical knowledge for teaching has emphasized how culture conditions affective components of mathematical understandings much more than how culture might condition cognitive components of mathematical knowledge. Therefore, this is an important potential contribution of this thesis. Further, none of the previous studies have focused on the Saudi context, and so this research question in particular stands to expand our understandings of how culture conditions the cognitive aspect of mathematics for teaching.

Then, returning to the second research question, it focuses on the affective aspects of mathematics teaching, which are summarized in the central cells of Figure 4 above. The affective aspects of the theory comprise three categories: an educator's beliefs about the nature of mathematics, about mathematics learning, and about mathematics teaching. All these beliefs can be culturally conditioned.

Previous scholars have demonstrated that beliefs about the nature of mathematics in turn condition beliefs about mathematics learning (Beswick, 2005). Those who hold instrumentalist beliefs about the nature of mathematics tend to take an approach to teaching mathematics that focuses on content and emphasizes performance (Beswick, 2005). Those who hold Platonist beliefs tend to approach teaching mathematics by focusing on content, but emphasizing understanding rather than performance (Beswick, 2005). Finally, those who hold problem-solving beliefs about the nature of mathematics tend to approach teaching mathematics by focusing on the learner (Beswick, 2005).

Finally, previous scholars have also demonstrated not only that beliefs about the nature of mathematics condition beliefs about learning mathematics, but that these beliefs also condition the processes by which teachers expect learners to learn mathematics (Cai, 2007). Those who hold instrumentalist beliefs, and who believe that teaching mathematics concerns content knowledge and performance, tend to emphasize skill mastery and the passive reception of knowledge in their pedagogical approach (Beswick, 2005). Those who hold Platonist beliefs, and who believe that teaching mathematics concerns content knowledge and understanding, tend to emphasize learners' active construction of understanding (Beswick, 2005). Those who hold problem-solving beliefs, and who believe that teaching mathematics is about focusing on the learner, tend to emphasize learners' autonomous exploration of their own interests (Beswick, 2005).

However, it is possible that the relationship between beliefs about the nature of mathematics, mathematics learning, and mathematics teaching is not so straightforward. Indeed, culture might condition how a mathematics teacher's beliefs about the nature of knowledge affect their beliefs about mathematics teaching and learning. This is why the third research question matters—if culture can condition how beliefs about the nature of mathematics translate to beliefs about and approaches to mathematics teaching and learning, it is important to inquire about how Saudi middle school female mathematics teachers' culture interacts with their beliefs about the nature of mathematics to influence their beliefs about and approaches to mathematics teaching and learning.

2.5 Conclusion

This chapter began with three objectives. The first objective was to critically survey the literature on how culture conditions the affective and cognitive aspects of mathematical knowledge for teaching, and on how culture conditions the impact of mathematical knowledge for teaching on teachers' actual teaching practices. The second objective was to identify and detail some of the limitations of this literature. The third objective was to explain the theoretical framework that undergirds this study's empirical examination of Saudi middle school female mathematics teachers' knowledge for teaching and explain how the theoretical framework can guide the process of answering each of the four research questions. This section has set the stage for the following chapter, which explains and justifies the empirical approach taken in this thesis.

CHAPTER 3: METHODOLOGY AND METHODS

This chapter describes and justifies the methods used in this research to address the research questions. It commences with a discussion of theoretical perspectives and the epistemological concepts that may impact the research design. The three main data collection methods (written test, interviews and observations) are then discussed in turn.

3.1 Theoretical Positions, Ontological and Epistemological

> We believe that epistemological awareness is an important and informative part of the transparent research process that needs to be addressed and communicated to readers... Moreover, when authors make their epistemological awareness and desired knowledge(s) within a particular research project unambiguous and explicit, this process of self-reflection can assist authors in selecting methods that instantiate and support their knowledge building... as well as choosing a theoretical perspective that is suited to the purposes of their research (Koro-Ljungberg, Yendol-Hoppey, Smith, & Hayes, 2009, p. 689).

This current research seeks to identify mathematics teaching knowledge (knowledge – beliefs) and how it influences the work of teaching in the Saudi context. In line with the views of Warburton (2015), a consideration of mathematical knowledge for teaching requires consideration of the ontology and the epistemology of mathematics. According to Askew (2008),

> If mathematical constructs are shaped through "analogies, metaphors, image, and logical constructs" then are these part of pedagogical content knowledge or part of subject knowledge? And does it matter? The answer comes down to a philosophical position on the epistemology of mathematics. If one believes that there are idealised mathematical forms that exist independently of representations, illustrations, examples and so forth, that there is a signifier/signified distinction (Walkerdine, 1988), then such 'unpacking' (or repackaging) is going to be seen as a pedagogic skill. If, on the other hand, one views mathematics as a 'language game' (Wittgenstein, 1953) only brought into being through representations, illustrations, examples and not existing outside these, then this is an aspect of subject knowledge as much as pedagogic (pp. 28-29).

Teachers have educational beliefs that may show in their instructional practices. Teachers' beliefs about the nature of mathematics is one of them (Thompson, 1992). "A teacher's conception of the nature of mathematics may be viewed as that teacher's conscious or subconscious beliefs, *concepts, meanings, rules, mental images, and preferences* concerning the discipline of mathematics" (Thompson, 1992, p.132).

It is the researcher's belief that mathematical concepts are fixed and exist independently because of their numeric nature. However, the process of teaching mathematics is subject to the interpretation of teachers. Therefore, the phenomenon considered in the study is a static but unified body of knowledge that reflects critical realities, acknowledges independent realities, and defines an objective reality as “one that exists independently of individual perception but also recognizes the role that individual subjective interpretation plays in defining reality” (Edwards, O'Mahoney, & Vincent, 2014). Also, this knowledge is constructed by humans through their individual processes and interactions with one another, which reflects the assumption of social constructivism (Ernest, 2010). As Frazer and Lacey (1993) put it, “even if one is a realist at the ontological level, one could be an epistemological interpretivist. . . our knowledge of the real world is inevitably interpretive and provisional rather than straightforwardly representational” (p. 182). Therefore, this research was carried out from the ontological viewpoint of critical realism and the epistemological viewpoint of social constructivism.

Practically speaking, the critical realism approach to ontology adopted in this study was reflected in the objectivity of mathematical knowledge and, at the same time, the subjectivity of the teachers’ understandings of that knowledge. By examining the extent to which teachers’ subject matter knowledge aligns with the objective knowledge of mathematical content that it is meant to reflect, this study critically examined the interface between objective and experienced reality. This contrast was illuminated using a knowledge test portion of the study and, more broadly, by the use of a multifaceted case study research design. However, as the critical reality perspective also acknowledges the importance of subjective, experienced realities, an epistemological viewpoint which reflects that subjective experience was required.

According to Ernest (1998), mathematical knowledge is expressed in different activities and symbolic conversations through a range of various social contexts; thus, the best approach to use in studying mathematics education is social constructivism. Social constructivists interpret learning within social and cultural settings from a situated perspective (Smith, 1999). In addition, the individual is “integral with cultural, political and historical evolution, in specific times and places, and so resituates psychological processes cross-culturally, in social and temporal contexts” (Owen, 1995, p. 161). This perspective is closely associated with many contemporary theories, particularly the developmental theories of Vygotsky and

Bruner and the social cognitive theory of Bandura (Schunk, 2012). Thus, without historical and sociocultural dimensions a deep construction (of the work of teaching) would be impossible (Crotty, 2010).

Practically speaking, the epistemological viewpoint of social constructivism was reflected most strongly in the interviews, but also in the recordings and observations. Both sources of data, as discussed below, reflect an engagement with the subjective reality experienced by the participants and the social determinants thereof. The case study research design which focuses on contextualization also represents a key manifestation of the study's social constructivist epistemological viewpoint. By looking deeply at the context surrounding the phenomenon, it was possible to develop a more nuanced view of how the social context in which the participants exist may have shaped their experience as teachers of mathematics and their content knowledge.

Although many researchers seek to investigate what teachers do with students in classrooms and view teaching from sociocultural standpoints, their research assessment and measurement tools have been based on individualistic and cognitivist standpoints, which means that the main and fundamental goal of research was changed from examining mathematics in teaching to assessing individual teachers' cognitive abilities (Ball, 2017). Consequently, the guiding philosophy of this research is that "teaching is co-constructed in classrooms through a dynamic interplay of relationships, situated in broad socio-political, historical, economic, cultural, community, and family environments" (Ball, 2017, p. 15). Hence, any approach toward identifying mathematics knowledge for teaching must deal with the multifaceted whole of the work of teaching rather than just with some facets of teacher knowledge or types of teacher beliefs; this research therefore undertook an interpretive qualitative research approach, as described in the next section.

3.2 Research Approach

To achieve the purposes of this study, this research utilised an interpretive qualitative approach. The interpretive qualitative approach allows for an interpretation of data collected in the field. Qualitative research involves studying a phenomenon in its natural setting through interpreting the experiences, meanings, and beliefs of individuals (Denzin & Lincoln, 2000). The underpinning approach in qualitative research is the interpretive approach to social reality, as it describes the experiences of human beings (Denzin & Lincoln, 2000). The interpretivist viewpoint focuses on in-depth analysis of personal and shared meaning regarding a phenomenon wherein more leeway is given for how data are interpreted and presented, as opposed to quantitative approaches (Guest, Namey, & Mitchell, 2013). The key tenet of the interpretive research approach is characterized by understanding through direct experiences of people, rather than objectively observing the action from outside (Guest et al., 2013).

Bryman (2008) clarified several criticisms that have been directed at qualitative research. First, qualitative research is too subjective, as it relies too much on the researcher's opinions about what is important; also, it could develop a personal relationship between the researcher and the participant. The second criticism was that there is no standard process to be followed, which makes it hard to imitate (Bryman, 2008). Thirdly, the small size of the sample is one of the criticisms of qualitative research; this is due to the difficulty in generalising the findings generated from the sample, as it might not be a good representation of the population being studied (Bryman, 2008).

Nevertheless, Bryman (2008) argued that the main concern of qualitative research is to generate theory through understanding behaviour in the context being studied and not to make the sample representative of the population. Therefore, an interpretive qualitative approach provides opportunities for researchers to address the processes of interaction among humans in specific contexts as well as helps the researcher to have deep interpretation of participants' situations from the perspective of their personal and cultural experiences in a natural setting (i.e., a school or classroom).

3.3 Design of the Study

This research was carried out using a case study design, one of the key approaches in qualitative research. According to Punch and Oancea (2014), examining a case provides a deep insight into a wider issue. Woodside (2010) defined a case study as "an inquiry that focuses on describing, understanding, predicting, and or controlling the individual (i.e., process, animal, person, household, organisation, group, industry, culture, or nationality)" (p. 1), which offers a holistic understanding of a phenomenon and issue in its social context (Hesse-Biber & Leavy, 2011). A case study research designs offers two key benefits over other qualitative research designs. First, though all qualitative research is contextual in nature, a case study design is especially so, offering a deep, multifaceted look at the phenomenon under study (Woodside, 2010). A second advantage of case study research is derived somewhat from the first. In order to develop a richly contextualized understanding of the case, a case study research design is conducted using multiple sources of data, which are then triangulated in order to draw stronger conclusions (Hesse-Biber & Leavy, 2011). This triangulating aspect of a case means that, although the case study is a qualitative approach examining issues through a subjective lens, the convergence of multiple data sources paints a more complete picture of the underlying reality upon which the subjective perceptions are based.

In this regard, the case study design served as a bridge between the study's ontological and methodological foundations, linking together the existence of an underlying reality in mathematical terms with the need to explore the subjective realities experienced by teachers. The orienting question of this study was, "What are the aspects of mathematics teaching knowledge for Saudi female mathematics teachers that influence their teaching in middle school?" This question was asked of a group of female mathematics teachers, who were examined according to their MCK and PCK and their beliefs. To examine these factors, this research entailed four data collection methods through four different phases. The phases included: adopting the MKT measure test for female mathematics teachers, comprised of items designed to evaluate their SCK and KCT; a semi-structured pre-lesson interview session; and observation of teachers' actual teaching inside the classroom. Data gathered were analysed using a thematic framework to generate common themes in the data that would address the research question posed in the study.

3.4 Population and Sample

The broad population under study in this research was Saudi mathematics teachers. In particular, the population was delimited to female mathematics teachers of middle school mathematics. Since cultural norms prohibiting a female researcher from interviewing or observing male teachers, the sample group was limited to female teachers. In addition, given the highly technical nature of mathematics, focusing on a particular subject matter, such as that taught in middle school, was a necessary population delimitation in order to focus the study on a feasible set of content. In addition, all participants had been teachers for at least one academic year, ensuring the relevancy of their experiences. By studying this population, it was possible to examine the mathematics teaching experiences of an increasingly important part of the Saudi educational workforce, female teachers, with respect to subject matter that has traditionally been a male domain.

To recruit school mathematics female teachers for the test, the educational guidance section of the Department of Education in Aljouf City was contacted, seeking their permission to conduct research in the schools. School principals in Aljouf City were then informed by a letter detailing the aim and process of the study and requesting the principals to invite teachers to participate. After that, the researcher met with each principal in each school to explain the goals of the research and invited school mathematics teachers in that school to participate. Teachers who opted to be part of the research were provided with an informed consent form. Teachers were asked to read and sign the informed consent form, after which they were included in the study. Fifteen female mathematics teachers who were teaching mathematics in middle school in Aljouf City were selected to participate in phase 1 (written test) and phase 2 (interview). In phase 3 (lesson observation), a sub-sample of six participants from the original sample was invited to participate in lesson observations, which aimed to observe their teaching in "natural" situations, to obtain information through "naturally unfolding behaviour" (Ginsburg, 2009) that could not have been obtained through the other data collection methods. A non-probability sampling method was used based on the criteria of: the individual is working as a mathematics teacher in the middle school in Aljouf City (Bryman, 2015), and (2) the individual is willing to participate in this study. All the schools were located in Aljouf City.

3.5 Data Sources

This research consisted of three main sources of data to identify the mathematics teaching knowledge of Saudi female mathematics teachers in middle school in the context of Saudi Arabia. The first source was the responses of 15 mathematics teachers to a U.S.-developed MKT measures test (Ball et al., 2008). The test was adapted for use in Saudi Arabia female schools for the first time; this has the potential to extend knowledge and add to the work of the Learning Mathematics for Teaching (LMT) research group mentioned earlier. The second source of the data was the audio-recorded interviews of the same teachers who completed the test. The audio-recorded interviews were transcribed for analysis. The third source of the data was the audio recording of observed lessons of the six teachers who were interviewed. The data collection and analysis are described below:

Table 2: An overview of data collection and analysis

Identifying mathematics teaching knowledge for Saudi female mathematics teachers in middle school

Guiding question	**Data**	**Analysis/ Technique**	**Goals**
1- What cognitive type of content knowledge do Saudi Female Mathematics Teachers in middle school have?	**U.S.-developed MKT measures**	**- Adaptation of measures to suit Saudi school cultural context.** **- Qualitative analysis of teachers' answers to examine their type of understanding (Skemp, 1976) as well as their cognitive type of content knowledge (Tchoshanov, 2011)**	**Find out Saudi middle school female teachers' cognitive type of content knowledge** **Are teachers' answering about the items in expected ways?**
2- What beliefs do Saudi Female Mathematics Teachers hold regarding the nature of mathematics, mathematics teaching, and mathematics learning?	**Responses to semi-structured interviews administered to 15 teachers** **Audio recording lessons taught by 6 selected Saudi teachers.**	**- Coding of teacher responses for consistency with the categories of beliefs structure framework (Beswick, 2005).** **- Observing lessons for analysing mathematical teaching quality by using the Knowledge Quartet checklist**	**Are teachers reasoning about the question in expected ways? For instance, are their answers to the questions reflecting their way of teaching?**
3- How do Saudi Female Mathematics Teachers' cognitive types of content knowledge and their beliefs impact their pedagogical decisions and quality of teaching?	**U.S.-developed MKT measures** **- Audio recording lessons taught by the 6 selected Saudi teachers.**	**Investigation of teacher responses to test items and observing mathematical lessons** **Type of understanding (Skemp, 1976)** **Cognitive Type (Tchoshanov, 2011)** **- the Knowledge Quartet checklist**	**Investigating mathematical knowledge demands of answering the questions and choosing and using representations in observed lessons.**

4- How do the culture beliefs of teachers influence their cognitive type of content knowledge, beliefs and pedagogical decisions?	**Responses to U.S.-developed MKT measures and semi- structured interviews, and observing selected mathematics lessons**	**Audio recording of lessons for mathematical teaching quality, explanations, mathematical representations**	**- Examining the quality of teachers' teaching in the lessons** **- How teachers' cognitive types of mathematical content knowledge link with their cultural beliefs**

3.5.1 Written Test

There are hundreds of LMT items available for use. To measure teachers' MCK and PCK, 14 multiple-choice items only were selected, due to practical constraints. Five items were related to number concepts and operations (NCOP); three items were related to patterns, functions, and algebra; three items were related to geometry; two items were related to rational numbers; and one item was related to proportional reasoning. These items focused more on the concept of numbers because: first, the knowledge structure in other mathematical concepts, such as geometry, is built on numbers; and secondly, the number strand outweighs the other topics in the Saudi middle school curriculum because it is fundamental across all strands within mathematics. More importantly, due to the time limitation of this research, it would have been impractical to investigate deeply all types of mathematical content knowledge.

3.5.2 Adaptation of Measures

Different researchers in different countries have adopted MKT, such as Delaney (2008) in Ireland, Kwon (2009) in South Korea, Ng (2009) in Indonesia, Mosvold and Fauskanger (2009) in Norway, and Cole (2011) in Ghana. Delaney, Ball, Hill, Schilling, and Zopf (2008) were the first scholars to adapt the LMT items for use in Ireland.

Delaney (2008) stated that the theory of MKT "describes a generalized approach to and belief about knowledge that can be related to all countries" (Delaney, 2008, p. 22), as it is "etic" in nature (Pike, 1954). Delaney et al. (2008) developed categories of changes to serve as a guide for adapting LMT items to different countries. Other researchers such as Cole (2011) used Delaney et al. (2008) categories of changes. In this research, the researcher adapted the LMT items for Saudi Arabia by considering two factors, accessibility and validity.

Regarding accessibility, the written test items were in English, but the primary language in Saudi Arabia is Arabic. Thus, to improve the accessibility of the measures, these items were translated into the Arabic language and trialled with a group of mathematics teachers to check and improve the clarity of language. That group consisted of one experienced teacher with eight years' teaching experience in middle and secondary school and an experienced mathematics teacher with more than 10 years' teaching experience in middle school. They both spoke two languages, Arabic and English, which facilitated the translation from English to Arabic. Also, the author of this research is a lecturer in Saudi Arabia. This group translated and examined all questions in the test by considering the accessibility and validity of the questions. They tried to keep the cognitive aspect of each question clear. Some minor changes were made to relate more to teachers' culture, such as names, cognisant that Hambleton (1994) had stated that people who take a test could be distracted by unfamiliar names and contexts:

> Second, in term of test items validity, Ball et al. (2008), describe LMT item, On the one hand, the generality of our results may be limited because our data are limited to only a few classrooms all situated in the U.S. context. On the other hand, our results are likely to be broadly applicable because our conception of the work of teaching is based, not on a particular approach to teaching, but on identifying fundamental tasks entailed in teaching (p. 396).

However, the group decided to remove many of the test items that were related to Knowledge of Content and Students (KCS) because of the focus of the research which was on Specialised Content Knowledge (SCK) and Knowledge of Content and Teaching (KCT). Also, even though there are similarities between the U.S. and Saudi Arabia in the nature of mathematical knowledge that teachers are expected to know, the group suggested the exclusion of some items that included terms whose meaning they were not sure of, such as 'tessellations'. Moreover, teachers who took the test agreed on the test's validity, as it was understandable, and the test content was familiar to them.

The final knowledge test that was used in the study, therefore, consisted of 14 questions (refer to Appendix 1). Question 1 pertained to the combination of two negative numbers. Question 2 focused on teachers' knowledge of the concepts of ratio and proportion. Question 3 measured knowledge of mathematical equivalence, or the notion that two sides of an equation represent the same value. Question 4 pertained to knowledge of algebra, particularly the use of patterns and functions to improve conceptual understanding of algebra. Question 5 addressed performing division or multiplication with inequality, like dividing or multiplying parts of equations. Question 6 was intended to investigate teachers' knowledge of rational numbers, and especially their understanding of decimal operations and their reasoning about drawn representations of decimals. Questions 7, 8, 9, and 10 assessed teachers' knowledge of content and teaching as that pertained to number concepts and operations. Questions 11 and 12 were aimed at measuring teachers' knowledge of teaching and content pertaining to geometry. Question 13 addressed mathematical proofs. The final question, Question 14, was developed to assess teachers' knowledge of content and teaching about rational numbers.

3.5.3 Interviews

The pre-lesson interview consisted of 10 semi-structured questions (refer to Appendix 2), in order to "rely on a certain set of questions and try to guide the conversation to remain, more closely, on those questions" (Hesse-Biber & Leavy, 2011, p. 102). The questions were designed to elicit teachers' beliefs and feelings toward the nature of mathematics and its teaching and learning, what factors impacted their teaching of mathematics, and their philosophy of teaching mathematics in general. These semi-structured interviews followed a predetermined interview protocol but allowed for follow-up questions, depending on interviewees' responses (Galletta, 2013). Additional questions were asked depending on the teachers' responses in the interview, which was "deliberately non-standardized" to allow space for the researcher to unfold interviewees' thinking (Ginsburg, 2009). Further, through listening to the interviewees, the researcher was enabled to elicit and evaluate their beliefs regarding their knowledge of mathematics and mathematics teaching. Each interview was conducted one-on-one and took approximately one hour. One-on-one interviews allowed the researcher to clarify any ambiguity and offered flexibility within conversations. A transcription was obtained for data analysis and reflection. A member check procedure was used (Creswell, 2012). Participants used a member checking form to verify the interview notes and transcription to ensure accuracy of understanding of their responses and to clarify any misunderstanding.

More importantly, during the interview the researcher might have faced some challenges to obtaining the goal of the interview. For example, the researcher had to allow the participant's perceptions to emerge, facilitate the meaning of the interview questions, and interpret the idea being discussed (Tomlinson, 1989). Also, the researcher needed to remain neutral and use a positive tone of questioning (Creswell, 2012), with no differences in emphasis, in order to overcome any possible researcher bias (Hunting, 1997). However, according to Hunting (1997), nonverbal signs such as body language could be more difficult to hide and could accidentally bias the interview. To avoid these issues, Tomlinson (1989) developed a systematic approach named "hierarchical focusing", which was taken in this research. That approach involves investigating participants' responses using 'utterance linking' (Tomlinson, 1989, p.170), which involved the following proposed strategy:

> (1) Carry out and explicitly portray an analysis of the content and hierarchical structure of the domain in question as you, the researcher, construe it.
>
> (2) Decide on your research focus: identify those aspects and elements of your topic domain whose construal you wish to elicit from interviewees.
>
> (3) Visually portray a hierarchical agenda of questions to tap these aspects and elements in a way that allows gradual progression from open to closed framing, combining this as appropriate with contextual focussing. Include with this question hierarchy a skeleton of the same structure for use as a guide and record.
>
> (4) Carry out the interview as open-endedly as possible, using the above strategies within a non-directive style of interaction to minimise researcher framing and influence. Tape-record the proceedings.
>
> (5) Make a verbatim transcript and analyse the protocols, with use of the audiotape record where appropriate. (Tomlinson, 1989, p. 162)

This approach required looking for participants' clarification about points that were introduced by them and then using their language to explain those points (Warburton, 2015). By following this approach, the interviewer aimed to elicit "construals with a minimum of framing and [using] a hierarchical interview agenda to raise topics only as necessary" (Tomlinson, 1989, p. 165). One of the strengths of this approach is that it elicits teachers' underlying beliefs while trying to minimise the impact of the researcher on them. The interviews had one fundamental purpose in this research, which was to elicit mathematics teachers' beliefs about the nature of mathematics and its teaching and learning.

3.5.4 Observations

After gathering information theoretically about teachers' MCK and PCK and beliefs, the researcher observed teachers' actual teaching inside the classroom. This was done in order to suggest potential reasoning as to how teachers use their knowledge in their actual teaching, what influences their decisions inside the classroom, and where their explanations come from. Rowland and Turner (2009, p. 31) stated that, "Contingent moments arise one after the other when teachers interact with a class… and their responses to these opportunities draw in various ways on their mathematical content knowledge". Thus, observing teachers in their classroom was anticipated to provide data that could not be obtained by other methods; by identifying actual behaviour instead of just recording perceptions (Creswell, 2012). According to Jersild and Meigs (1939), there are some occasions where observations can be used, including:

> …a desire to probe aspects of behavior not accessible to the conventional paper and pencil, interview, or laboratory technics; a desire to obviate some of the subjective errors likely to enter into the customary rating procedures; an emphasis on the need for studying children in "natural" situations, and for studying the functioning child, including his [her] social and emotional behaviour rather than to rely exclusively on static measurements of mental and physical growth. (p. 472)

In this research, lesson observation was used as a data collection method because the aim of the current research was to investigate mathematics teachers in their natural teaching setting in order to examine the non-static process of how teachers' knowledge comes into being in the actual classroom. Moreover, "Teachers might know things in a theoretical context but be unable to activate and apply that knowledge in a real teaching situation" (Kersting, Givvin, Sotelo, & Stigler, 2010, p. 178).

The researcher used an observational protocol (refer to Appendix 3) that was informed by Rowland and his colleagues' seminal work (Rowland et al., 2003). The Knowledge Quartet (KQ) framework was used to guide the researcher to observe and focus on different features in mathematics lessons for investigating teachers' mathematical knowledge (e.g., MCK and PCK) and beliefs. The KQ Foundation dimension helped the researcher to gain deep awareness of the teachers' MCK from the plan and the structure of the lesson. Also, the other dimensions (transformation, connection and contingency) assisted the researcher to clarify the types of mathematical knowledge that teachers have.

3.6 Data Collection

After gathering participants who signed the informed consent form, data were collected. This research entailed four data collection methods through three different phases. The first phase of the study was an exploration of adopting the MKT measure test for female mathematics teachers, comprised of items designed to evaluate their cognitive type of content knowledge. One additional question was added to each item: teachers were asked about their confidence when answering these mathematical questions. The second phase was a semi-structured pre-lesson interview, aimed to elicit teachers' beliefs regarding the nature of mathematics and its teaching and learning.

After gathering that information, the third phase was to observe teachers' actual teaching inside the classroom to suggest potential reasoning as to how teachers use their knowledge in their actual teaching, what influences their decisions inside the classroom, and where teachers' explanations come from. Audio-recorded interview responses and observation data were transcribed to prepare for data analysis. All data collected were stored in a password-protected computer only accessible to the researcher.

The first portion of the data collection was the knowledge test. The LMT items in this study were conducted as a 'test' in which teachers worked independently on their own. The LMT test is unique because the test items are grounded in the work of teaching (Cole, 2011). No overall time limit was set; teachers were told to "Calculate about 2 minutes per problem stem included on the assessment, or alternatively, 1 minute per item" (Hill, Ball, Schilling, & Bass, 2007). Teachers' responses to the test were recorded in Microsoft Word. Responses were scored using the answer key developed by the LMT project. Teachers' responses were scored as 'incorrect' if they selected a wrong answer, and 'not sure' if they selected two options instead of one. However, there were some missing data when teachers did not choose one of the options, as it was not compulsory for them to answer all questions. Interpreting missing data is essential because it affects the way of describing teachers' knowledge (Cole, 2011). Very intelligent teachers might choose to not respond to a question because they know enough about the content to know that they cannot answer the question correctly (De Ayala, Plake, & Impara, 2001).

Following the knowledge tests, fourteen interviews were audio-recorded; in one interview, the teacher did not agree to record her voice because of personal reasons. To avoid any possible biases in selection or interpretation, Ary, Jacobs, Sorensen, and Razavieh (2010) recommended a direct transcription of words. In the current study, translating and transcribing were done simultaneously. While transcribing, any information that could have been connected to the interviewees was removed from the transcripts to maintain confidentiality.

The third stage of the data collection comprised the recording and observations. All lessons were audio recorded only, due to cultural mores that do not allow video recording. In addition to the recording, the researcher took notes. Glesne (1999) stated that, as a researcher using note writing, “You develop your thoughts, by getting your thoughts down as they occur, no matter how preliminary or in what form, you begin the analysis process” (p. 131). Each individual teacher, according to their availability, was observed for two lessons; the content of the lessons and the classes taught were not considered when arranging visits. However, the day and the time of the lesson were considered in order to have sufficient time to observe arrival at the school and different teachers in the same day when possible. However, the researcher could not choose the observed lesson topic, as the teachers have to follow the curriculum as prescribed for that time of year. The purpose of the observation was to examine teachers’ MCK, PCK, and beliefs in the actual classroom. The researcher was a passive participant (McMillan, 2004). In other words, the researcher did not participate in the lesson and did not interact with the teachers and students.

3.7 Data Analysis

As with the data collection, the data analysis was carried out in separate stages, one of which aligned with each data collection approach. These three steps of the data analysis served to comprehensively analyse the meaning inherent in the copious qualitative data that were collected throughout the study.

To analyse the knowledge test data, a qualitative holistic approach was followed. This approach was used to analyse teachers’ answers in order to examine their type of understanding (instrumental or relational; Skemp, 1976) as well as their cognitive type of content knowledge (Tchoshanov, 2011). Teachers’ mathematical content knowledge (MCK) and pedagogical content knowledge (PCK) are essential to them effectively teaching their students (Ball et al., 2008; Ernest, 1989b). The quantity of teachers' content knowledge is not the only key aspect to assess; instead, per Tchoshanov (2011), the cognitive type of content knowledge they hold is also important. MKT is not used as an analytical framework in this study ; instead cognitive types of teachers' content knowledge (Tchoshanov, 2011) were used as an analytical framework of the types of knowledge measured by the teachers' responses because the Type categories more closely align to the purpose of this research. The cognitive type refers to the kind of teacher content knowledge and thinking processes required to achieve a task successfully, in terms of knowledge of facts and procedures (Type 1), knowledge of concepts and connections (Type 2), and/or knowledge of models and generalizations (Type 3). Although the connection between components of MKT and the types of teacher knowledge proposed by Tchoshanov (2011) is not straightforward, information could be acquired about teachers' understanding of the questions and cognitive type of content knowledge through analysing teachers' answers to these items. Thus, items were analysed to check a number of features: teachers' understanding of the question (relational understanding—instrumental understanding) (Skemp, 1976), and teachers' cognitive type of content knowledge (Tchoshanov, 2011): Type 1: knowledge of facts and procedures; Type 2: knowledge of concepts and connections, which is related to "knowing why"; and Type 3: knowledge of models and generalizations

The second analytical step pertained to the interviews. As a prelude to analysis, the interview audio recordings were completely transcribed. The main aim of qualitative data analysis is to make meaning from the data (Willis & Artino Jr, 2013). From the interview transcripts, in this study it was effective to analyse the interview data by organizing the teachers' beliefs into three categories, with respect to the nature of mathematics, mathematics teaching and mathematics learning (Beswick, 2005).

Table 3: The categories of beliefs structure (Beswick, 2005, p.40)

Relationships Between Beliefs

Categories of	**Components of teachers' beliefs**

teacher beliefs			
Beliefs about the nature of mathematics (Ernest, 1989a)	**Instrumentalist**	**Platonist**	**Problem-solving**
Beliefs about mathematics teaching (Van Zoest et al., 1994)	**Content-focused with an emphasis on performance**	**Content-focused with an emphasis on understanding**	**Learner-focused**
Beliefs about mathematics learning (Ernest, 1989a)	**Skill mastery, passive reception of knowledge**	**Active construction of understanding**	**Autonomous exploration of own interests**

Using these various categories was very effective in analysing the data (Anderson & Piazza, 1996; Perry et al., 1999; Van Zoest et al., 1994) due to their logical interrelationships and the theory foundation (Skemp, 1978). The interview data from 15 teachers were subject to thematic analysis as described by Braun & Clarke (2006). A thematic framework is defined as a qualitative analytic method for classifying, investigating, analysing, and recording themes within data, which helps to describe the data in detail (Braun & Clarke, 2006).

The thematic framework was chosen for the following reasons (Roberts et al., 2010): it provides coherence and structure to otherwise cumbersome, qualitative data (i.e., interview transcripts); it facilitates systematic analysis, thus allowing the research process to be explicit and replicable; and despite the inherent structure, the process of abstraction and conceptualization allows the researcher to be creative with the data. By using this framework, the data were analysed in phases. After the data transcription, the researcher generated initial codes about the nature of mathematics, mathematics teaching, and mathematics learning then collected information related to these codes. The researcher organized codes into possible themes and gathered relevant information regarding teachers' various beliefs. The researcher checked and reviewed the generated themes and their relation to the gathered data from all utilised methods. Following this, the researcher started to define and review the essence of each theme and check its relevance. Finally, the researcher linked the data to the research questions to start writing the final analysis statements (Braun & Clarke, 2006).

A final step in the analysis was the lesson recordings and observations. This analysis served to connect the interview results with the practical ways in which the teachers were found to teach. According to Ernest (1989a), mathematics teachers' beliefs with respect to the nature of mathematics, mathematics teaching, and mathematics learning are very relevant to their practice. Therefore, these statements were then compared with teachers' practice. For instance, the interview asked teachers, "What is your approach to mathematics teaching?", so any statement regarding their teaching approach was linked and compared to their actual teaching.

The Knowledge Quartet is a theoretical tool that can be used for observing, analysing and reflecting on the work of actual mathematics teaching (Breen, Meehan, O'Shea, & Rowland, 2018). The Knowledge Quartet was used to analyse audio recording of all observed mathematics lessons (Rowland & Turner, 2007). The Knowledge Quartet is the only observation schedule recognized that "provides a framework for analysis of the mathematical content knowledge that informs teacher insights when they are brought together in practice" (Turner & Rowland, 2011, p. 196). The online Knowledge Quartet 'coding manual' (Weston et al., 2012) helps the researcher in coding the audio data. In addition, the following method (as used by Anne Thwaites) was outlined as one way to analyse observations using the Knowledge Quartet (as cited in Warburton, 2015, p. 142):

> 1. Write a detailed account of the lesson soon after the observation took place complete with times for each main 'episode' within the lesson.
>
> 2. Add to this detailed account any codes from the Knowledge Quartet which relate to the episodes. Not all episodes will have a code.
>
> 3. Select the more useful/ interesting episodes from the lesson description and write a more fluid description of what happened complete with some background/ context for the reader.
>
> 4. Add to this fluid account screen shots from the lesson and/ or transcript sections from the interviews where relevant.
>
> 5. Finally, cut down on the information and add references from other authors where applicable which may help to account for the behaviour observed.

According to Petrou (2010), the Knowledge Quartet proved to be comprehensive in describing most of the lessons and it is a valid tool for analysing the observed lessons. Nevertheless, the context and knowledge used by Saudi and English teachers are not the same. Therefore, it is essential to consider any differences when adopting the framework for use in different contexts from where it was originally developed.

3.8 Trustworthiness

Lincoln and Guba (1985) say that to secure trustworthiness of the research as a multi-dimensional concept involves building on; credibility, transferability, dependability and confirmability of the research. Issues of trustworthiness in this study were addressed through the principles of credibility, confirmability, and generalisation.

Credibility:

This concept links to the trust of 'truth' of the findings and is building on the believable or acceptable results of qualitative research throughout the process of credible data collection and data analysis. The credibility of this research was established through triangulation process of data collection. I employed several methods to collect the data (written test-interview- observation) to explore the issues from various angles, Yin (2009) argues that using many sources of evidence is important to establish validity. These methods were then analysed carefully as described in the data analysis chapters.

Dependability:

This applies to the degree to which the results of the research are tested to be reliable or can be replicated over time. Shenton (2004) recommends that the researcher disclose in detail the processes inside the analysis so that the review can be replicated by future researchers. The investigator recorded all processes in detail for this analysis and then shared them with advisors to help assess the processes and validate reliability.

Confirmability:

This applies to the degree to which the researcher may be impartial or non-judgemental when analysing and reporting the data collected (Lincoln & Guba, 1985). This means that the ideas presented in terms of results arise from the participants ' perceptions and perspectives rather than being influenced by the researchers (Shenton, 2004). To establish the confirmability, I ensured that triangulation was used as a technique to decrease the impact of research bias. In addition, field notes and written reflection after every field experience reduced research bias.

3.9 Ethical Issues

All the participants were mature professionals and were asked to join the research voluntarily. There were a few initiatives the researcher implemented before collecting the data. Firstly, the researcher discussed and informed the educational administrator of the aim and the process of collecting the data to gain permission to conduct research in girls' middle schools. After gaining ethical approval (see Appendix 4), the educational administrator informed the school in an official way. There was no requirement for ethics approval within the Saudi school system. Participants were asked to read and sign an informed consent form to ensure that they agreed to the activities involved in the study. Moreover, the interviews were to be face-to-face and held in a private room within the school at an appropriate time for the interviewees. Furthermore, the researcher informed the participants that all the data would be stored safely and described anonymously. All data would be stored in a password-protected computer only accessible to the researcher.

3.10 Conclusion

This chapter has defined and justified the methodology of this study in terms of theoretical position, research approach, design of the study, data collection methods, data analysis, trustworthiness, and ethical issues. The current study used qualitative approaches and in the form of case studies. This research applied the concept of triangulation, by utilising data from multiple data sources (written test, interviews and lesson observation). Participants were female middle school mathematics teachers in Aljouf city in Saudi Arabia. In order to identify mathematics teaching knowledge of mathematics teachers three current frameworks for mathematics and teaching practice defined in the literature review were used (i.e., cognitive type of content knowledge, the Knowledge Quartet, and the categories of beliefs structure).

CHAPTER 4: FINDINGS

4.1 Introduction

This chapter presents the data collection findings. Findings from each data collection method are presented in turn: written test, interviews and observations. This provides the basis for the next chapter, where the results are connected to demonstrate how the research questions are answered.

4.2 Written test finding

4.2.1 Written test questions

4.2.1.1 Question one

Students sometimes remember only part of a rule. They might say, for instance, "two negatives make a positive." For each operation listed, decide whether the statement "two negatives make a positive" sometimes works, always works, or never works. (Mark SOMETIMES, ALWAYS, NEVER, or I'M NOT SURE)

	Sometimes works	Always works	Never works	I am not sure
a) Addition	1	2	3	4
b) Subtraction	1	2	3	4
c) Multiplication	1	2	3	4
d) Division	1	2	3	4

The teachers in this item were asked to decide whether the statement "two negatives make a positive" works always, sometimes, or never for all the mathematical operations (addition-subtraction-multiplication-division). This question investigated teachers' knowledge of number concepts and operations (NCOP), and particularly their understanding of principles for operations with negative numbers. There were 6 teachers who provided correct answers overall for this question; these answers were further categorized into questions 1 (a), 1 (b), 1 (c) and 1(d), in which 12, 7, 15 and 15 teachers obtained correct answers respectively. All SFMTs in this research answered correctly for the multiplication and division operations. Among those who did not give correct answers for addition and subtraction operations, 7 answered that this statement never worked for subtraction and 1 teacher answered that this statement always worked for addition operations. In the following paragraph, the researcher shows how to mathematically work out the truthfulness of the statement “two negatives make a positive” for the four operations in order to unfold the type of knowledge that the SFMTs had mastered for this mathematics topic.

“Two negatives make a positive” is a rule taught to students when they learn about multiplication and division of integers; it helps students to quickly choose the sign of the result. Yet, this rule does not always apply for addition and subtraction of integers (Karp, Bush, & Dougherty, 2015). We can all think of many, many times where a positive is not rendered by two negatives. For instance, when carrying out an addition operation, the statement "two negatives make a positive" never works because $-1 + -3$ is not equal to 4 rather it is equal to -4. Using the number line model is extremely useful for adding and subtracting operations, by visualising the direction of movement in the number line as the second negative simply changes the outcome direction. Subtracting a positive puts you in the negative direction, but subtracting a negative changes the direction towards the positive end. Another example, using a debt model: a loan acquisition of $3 from a bank and then borrowing $2 more from a friend implies that one has $-3 + -2 \text{ which}$ means that an individual has five amounts less money which means -5 dollars. In subtraction, the statement sometimes works. In the operation $-5 - (-3)$ is equal to-2, the statement fails to work since the resultant of two negatives is a negative number instead of a positive. On the other hand, the statement “two negatives make a positive” works where one subtracts a negative digit from a positive one (Aubrey, 2013). As such, $5 - (-2)$ is equal to 7 which supports the statement. Again, when you subtract a negative integer from another bigger negative one, the statement is true since with -5-(-6) the answer is 1 which is positive.

From a mathematics didactic point of view, negative numbers is an interesting mathematical topic that is taught in classrooms (Thomaidis, 1993), and it is not easy for many people to understand. Even though some people such as the SFMTs in this study may be able to competently operate with negative numbers, they may not have the conceptual knowledge underlying those operations. For instance, integer subtraction is a very challenging topic to teach and understand conceptually (Gregg & Gregg, 2007; Kinach, 2002). When you ask educated people how to compute $+5 - (-7)$, you will probably get the correct answer very quickly, but if you ask them how to explain to students the underlying concept for why the answer is positive 12, most of them will find it difficult to answer (Charalambous, Hill, & Mitchell, 2012). It is because individuals can be skilful in performing subtractions when they internalise the negative numbers (Sfard, 1991). The SFMTs displayed this feature in this test item and it explained why all SFMTs could give correct answers for multiplication and division but not addition and subtraction. Indeed, many studies have shown that people can have procedural knowledge without the corresponding conceptual knowledge (e.g. (Hiebert & Wearne, 1996; Rittle-Johnson & Alibali, 1999). The findings of this test item echoed the results of these previous studies. The SFMTs had gained an operational (instrumental) understanding simply through memorising the rules of arithmetic operations on integers without thorough conceptual and relational understanding of negative numbers (Skemp, 1971). Both signs "+" and "-" do not just represent an operation, they also represent aspects of the number which can change the nature of the operation (Bellamy, 2015), which is conceptually very confusing to many students and perhaps the SFMTs as well. Consequently, the SFMTs could not make an abstraction of operation properties because they did not have sufficient conceptual knowledge to make this abstraction (Klein, Loftus, Trafton, & Fuhrman, 1992). The SFMTs needed to distinguish between the attribute of the number and the operation, which entails a higher level of abstraction in understanding both the operation and the attribute of the number (Bellamy, 2015). Overall, the SFMTs had mastered the Type 1 knowledge in this test item.

4.2.1.2 Question Two

Mr. Garrison's students were comparing different rectangles and decided to find the ratio of height to width. They wondered, though, if it would matter whether they measured the rectangles using inches or measured the rectangles using centimetres.

As the class discussed the issue, Mr. Garrison decided to give them other examples to consider.

For each situation below, decide whether it is an example for which different ways of measuring produce the same ratio or a different ratio. (Circle PRODUCES SAME RATIO, PRODUCES DIFFERENT RATIO, or I'M NOT SURE for each.

	Produce same ratio	Produce different ratio	I'm not sure
a) The ratio of two people's heights, measured in (1) feet, or (2) meters.	1	2	3
b) The noontime temperatures yesterday and today, measured in (1) Fahrenheit, or (2) Centigrade.	1	2	3
c) The speeds of two airplanes, measured in (1) feet per second, or (2) miles per hour.	1	2	3
d) The growths of two bank accounts, measured in (1) annual percentage increase, or (2) end-of-year balance minus beginning-of-year balance.	1	2	3

Question 2 focused on teachers' knowledge of the concepts of ratio and proportion. Specifically, this question asked teachers to decide whether each example could produce an equivalent ratio or not, by choosing one of these answers: Produce same ratio, produce different ratio, or I'm not sure. The main purpose of this question was to investigate teachers' proportional reasoning by solving problems with ratio and proportion. The concept of ratio and proportion is an essential topic for mathematics and is applied in many fields. Conceptual understanding of this concept is central for mathematical awareness and more importantly is crucial to developing analytical mathematical reasoning as it promotes relational reasoning or proportional reasoning (Ben-Chaim, Keret, & Ilany, 2012).Teachers' ability to solve proportional problems shows that they possess proportional reasoning as well as abstract thinking (Ben-Chaim et al, 2012). Lamon (2007) shared the opinion of various researchers by stating that "proportional reasoning is a long-term developmental process in which the understanding at one level forms a foundation for higher levels of understanding" (p. 637). Overall, in this test there was no one teacher who provided correct answers overall for this question, which indicated that the SFMTs lacked the ability to solve ratio and proportion problems with proportional reasoning. The SFMTs had only gained an instrumental understanding of ratio and proportion.

SFMTs' answers were further categorized into questions: a (The ratio of two people's heights, measured in (1) feet, or (2) meters.), b (The noontime temperatures yesterday and today, measured in (1) Fahrenheit, or (2) Centigrade.), c (The speeds of two airplanes, measured in (1) feet per second, or (2) miles per hour.) and d (The growths of two bank accounts, measured in (1) annual percentage increase, or (2) end-of-year balance minus beginning-of-year balance.), in which 5, 5, 6 and 7 of SFMTs obtained correct answers respectively. In the following paragraph, the researcher explains mathematically the answers for each sub-question in order to unfold the type of knowledge that the SFMTs had mastered for this mathematics topic.

For instance, in the first sub-question (a), the teachers were asked if 'the ratio of two people's heights, measured in (1) feet, or (2) meters produced the same ratio or not'; comparing the heights of two individuals with the use of ratio measured in feet or meters would result in the same ratio. Given that one measures two objects using the same measurement, for instance meters, and then (s)he measures the same objects using a different dimension, in this case feet, then the ratios of two objects in meters and feet will be the same (Kajander & Holm, 2013). For example, if I measure the height of an individual as 2 meters, which is 2 times a 1-meter unit, and I want to change the measurement unit to a foot, the ratio of 2 meters to 1 foot is 6.56168 : 2 which means that 2 meters is 6.56168 times greater than 1 foot. Consequently, because the height of an individual is 2 times greater than 1 meter, and 2 meters is 6.56168 greater than 1 foot; then, the height of an individual is 13.12336 feet (i.e., 2 times 6.56168). According to (Lobato, Ellis, & Zbiek, 2010), individuals' reasoning in this way about measurement is illustrative of proportional understanding/approach.

In the second sub-question (b), teachers were asked if 'the noontime temperatures yesterday and today, measured in (1) Fahrenheit, or (2) Centigrade' also produced the same ratio or not. The temperatures of today and yesterday for noontime, when measured in Fahrenheit or Centigrade, would result in a different ratio. Additionally, each unit of heat energy added in either of the scales results in a different additional value, hence it is impossible to conclude that doubling the °F or °C value would double the heat energy. Thus, it is difficult to determine how much energy in °C or °F is. Fahrenheit and Centigrade scales start at dissimilar arbitrary points which implies that there cannot be a ratio between the two (Karp et al., 2015).

In example (c), two bank accounts' growth when measured in annual percentage increase in relation to subtraction of the beginning of year amount, the end of year balance will yield a different ratio. For example, bank A at the beginning of the year had $10,000 which yielded to $12,000 at the end of that year. Bank B invested $8,000 and yielded $14,000 at the end of the year. Calculating percentage ratio bank A to bank B is 4:15 while when carrying out a subtraction ratio it will amount to 1:3. Therefore, the growth of two banks will produce different ratios. Overall, the SFMTs had mastered the Type 1 knowledge in this test item.

4.2.1.3 Question Three

Students in Mr. Carson's class were learning to verify the equivalence of expressions. He asked his class to explain why the expressions a – (b + c) and a – b – c are equivalent. Some of the answers given by students are listed below. Which of the following statements comes closest to explaining why a – (b + c) and a – b – c are equivalent? (Mark ONE answer.)

a) They're the same because we know that a – (b + c) doesn't equal a – b + c, so it must equal a – b – c.
b) They're equivalent because if you substitute in numbers, like a=10, b=2, and c=5, then you get 3 for both expressions.
c) They're equal because of the associative property. We know that a – (b + c) equals (a – b) – c which equals a – b – c.
d) They're equivalent because what you do to one side you must always do to the other.
e) They're the same because of the distributive property. Multiplying (b + c) by –1 produces –b – c.

This question asked teachers about which of the given statements would be closest to explain why $a - (b + c)$ and $a - b - c$ are equivalent. This question looks to measure the knowledge of mathematical equivalence, the notion that two sides of an equation represent the same value. This concept is a foundational concept in early algebraic thinking, and this knowledge is developed in elementary and middle school (Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011).However, distributive property is the key concept for explaining the equivalence between the expression $a - (b + c)$ and the expression $a - b - c$. It is a vital concept that promotes flexible computations, solves algebraic equations and is very important for generalizations and proofs (Carpenter, Franke, & Levi, 2003). In order to clarify the type of knowledge that the SFMTs had mastered for this mathematics topic, a mathematical explanation of the distributive property and the given statements is justified below.

The distributive property in mathematics is symbolically stated by the two expressions (Chan & Elliott, 2004), symbolically represented as $(a + b)c = ac + bc$. It is a very fundamental notion of the partial products $[e.g., 38\times27 = (30 + 8) \times (20 + 7) = 30 \times (20 + 7) + 8 \times (20 + 7) = 30\times20 + 30\times7 + 8\times20 + 8\times7]$ which contributes in understanding of arithmetic algorithms and algebraic learning (Carpenter et al., 2003).The monomial factor -1 in the expression $a - (b + c)$ is distributed to each term of the binomial factor $b + c$ resulting in the equivalent expression $a - b - c$.

The operation $a - (b + c) = a - b - c$ has a distributive property which implies that every digit that is inside the bracket multiplied by integers that are outside the parenthesis makes the result to be added together (Kinzer & Stanford, 2014) .The -1 in operation $a - (b + c)$ is allocated in every term in the parenthesis $(b + c)$ which results in an expression $a - b - c$. The two operations are the same because they have a distributive property implying that multiplying $(b + c)$ by –1 produces $a - b - c$. Hence, it cannot be associative property because one cannot add or multiply the expression regardless of how the integers are grouped for it would result in a wrong answer which is not the same as $a - b - c$.

Eight participants chose the correct answer which is (e) – They're the same because of the distributive property because multiplying $(b + c)$ by –1 produces $-b - c$.
Two and five participants gave (a) and (b) (refer to next paragraph for the

statements of options (a) and (b)) as their answers respectively. The SFMTs who answered correctly had mastered conceptual and relational understanding of the equivalence of expressions as they were able to apply transformational rules from the basic arithmetical laws (e.g., distributivity). Thus, SFMTs who answered incorrectly did not understand algebraic expression as “a system of meaningless signs (being transformed according to arbitrary rules), but as pattern generalizers of arithmetical or geometrical pattern” (Kieran, 2004, p. 23).

Half of the SFMTs might have had difficulties in understanding the main concept of this question or they interpreted the question differently. SFMTs who chose statement (b) – They are equivalent because if you substitute in numbers, like $a = 10$, $b = 2$ and $c = 5$, then you get 3 for both expressions, paid more attention to the procedural and instrumental reasoning than structural and relational reasoning. Thus, they preferred evaluating input – output to understand the equivalence of algebraic expressions as they focused more on the rules in manipulating symbolic expressions than the conceptual notions underpinning these rules (Kieran, 2004). So, they emphasized the transformational aspects of algebraic activity. In addition, the 2 SFMTs who chose statement (a) – They are the same because we know that $a - (b + c)$ doesn’t equal $a - b + c$, so it must equal a-b-c, had no knowledge or understanding of the distributive property.

Globally, over the last few years, the teaching and learning of algebra has become an interesting topic for many researchers (Artigue, Assude, Grugeon, & Lenfant, 2001; Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005). However, very little investigation has been done on teachers’ knowledge of algebra for teaching (Huang, 2014). According to Lee and Wheeler (1987), “algebra emerges as an activity, something you do, an area of action...” (p. 187). These activities can be seen in school algebra in three different types: generational (i.e., an activity of algebra includes the creating of the expressions that are the objects of algebra), transformational (i.e., rule-based activities), and global/meta-level (i.e., algebra is a tool which is not exclusive to algebra (e.g., problem solving)) (Kieran, 1996). Therefore, SFMTs’ approach in explaining algebra activities to students depends on their understanding of the mathematical content underpinning the activities. Consequently, overall, according to the Tchoshanov (2011) framework of teachers’ cognitive types of content knowledge, half of the SFMTs had mastered the Type 1 knowledge in this test item as they had instrumental understanding of the knowledge of mathematical equivalence, while the other half had gained the Type 2 cognitive content knowledge as they showed conceptual and relational understanding of the equivalence of expressions.

4.2.1.4 Question Four

Ms. Whitley was surprised when her students wrote many different expressions to represent the area of the figure below. She wanted to make sure that she did not mark as incorrect any that were actually right. For each of the following expressions, decide whether the expression correctly represents or does not correctly represent the area of the figure.
(Mark REPRESENTS, DOES NOT REPRESENT, or I'M NOT SURE for each.)

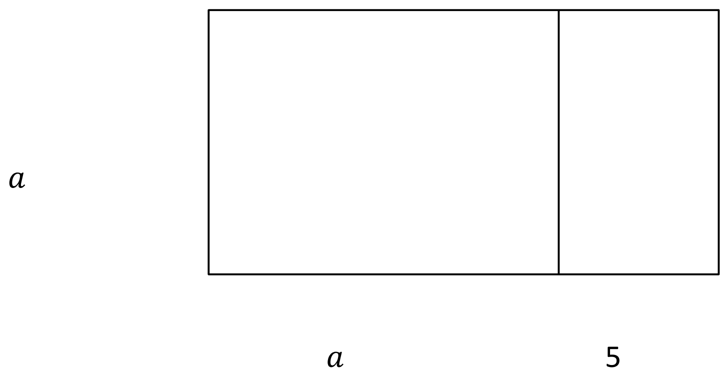


Figure 5: Area of rectangle problem

	Correctly represents	Does not correctly represent	I'm not sure
a) $a^2 + 5$	1	2	3
b) $(a + 5)^2$	1	2	3
c) $a^2 + 5a$	1	2	3

d)	$(a + 5)$a	1	2	3
e)	$2a + 5$	1	2	3
f)	$4\ a + 10$	1	2	3

This question focused on teachers' knowledge of algebra, and the use of patterns and functions to improve conceptual understanding of algebra. The National Council of Teachers of Mathematics (National Council for Teachers of Mathematics, 2000) emphasized the significance of this approach while describing the algebra strand as: "…systematic experience with patterns can build up to an understanding of the idea of function, and experience with numbers and their properties lays a foundation for later work with symbols and algebraic expressions" (p. 37). Teachers were asked to decide which of the following expressions ((a) $a^2 + 5$, (b) $(a + 5)2$, (c) $a^2 + 5\ a$, (d) $(a + 5)\ a$, (e) $2\ a + 5$, (f) $4\ a + 10$) represented the area of rectangle (Figure 5) correctly. Algebra is a way of thinking and a set of concepts and skills that enable us to analyse mathematical situations.

Overall, 11 SFMTs gave (a) and (b) as their answers, 14 SFMTs gave (c) 8 SFMTs gave (d) and 13 SFMTs gave (e) and (f). In the following paragraph, the researcher explains mathematically how SFMTs analysed each expression to get the correct answer. By explaining that, SFMTs type of knowledge is identified.

When we write A = LW, the area formula for a rectangle, we are describing a relationship among three quantities. The area of the figure is calculated by multiplying the length by width (Reddy, 2014). Length is "a" and width is "$a + 5$ ", which is equal to $a^2 + 5\ a$. A 2-digit multiplication relies strongly on the distributive property where the distributive property is represented by dividing a rectangle (Cooper & Arcavi, 2018). Analysis of this question required SFMTs' understanding of the operation of multiplication that displays the distributive property (Russell, Schifter, & Bastable, 2011) and connection between distributive property and the area of the figure (Erbas, 2004) .There are two correct expressions which reflect two forms of distributive property expressions: factored form and expanded form, $(a + 5)\ a$ and $a^2 + 5\ a$ respectively. Fourteen SFMTs chose (c) $a^2 + 5a$ as their answer, which is the expanded form, whereas 7 SFMTs, which was almost half of them, counted the factored form (d) $(a + 5)$a as the incorrect expression. This indicates that some SFMTs were having some difficulty with understanding the distributive property so they could not see that $(a + 5)\ a$ is equivalent to $a^2 + 5a$. Consequently, SFMTs' thinking and answers relied on rote and memorization procedures of the properties of addition and multiplication. This shows that SFMTs had mastered the Type 1 knowledge in this test item.

4.2.1.5 Question Five

Ms. Hurlburt was teaching a lesson on solving problems with an inequality in them. She assigned the following problem.

$$– x < 9$$

Marcie solved this problem by reversing the inequality sign when dividing by
– 1, so that x > – 9. Another student asked why one reverses the inequality when dividing by a negative number; Ms. Hurlburt asked the other students to explain. Which student gave the best explanation of why this method works? (Mark ONE answer.)

a) Because the opposite of x is less than 9.

b) Because to solve this, you add a positive x to both sides of the inequality.

c) Because –x < 9 cannot be graphed on a number line, we divide by the negative sign and reverse the inequality.

d) Because this method is a shortcut for moving both the x and 9 across the inequality. This gives the same answer as Marcie's, but in different form: –9 < x.

This question focused on teachers' knowledge of patterns, functions, and algebra. When performing division or multiplication with inequality, it is similar to dividing or multiplying parts of equations. A rule that applies in both division and multiplication of inequality is that whenever one divides or multiplies an inequality by a negative integer, one must flip the inequality sign so that the variable is left with a negative sign (Rump, 2018). Therefore, the correct answer for $-\mathrm{x} < 9$ is $\mathrm{x} > -9$. The aim of solving inequality is always to leave it on its own to ease calculations so that one may pay more attention to the direction which the inequality favours. Based on the inequality, multiplying or dividing both sides with a positive does not apply in this case; a negative sign cannot be cancelled out using a positive. Therefore, using a negative sign is more appropriate for it eliminates the negative allocated on the variable side which results in changing direction of the inequality. Most adults can remember to reverse the inequality sign when dividing by a negative, yet there are others who do not know the reason behind this (Hill, Sleep, Lewis, & Ball, 2007). This is an algorithm that is a shortcut for moving both the x and 9 across the inequality. Furthermore, " $-x < 9$, is equivalent to $0 < 9 + x$, also equal to $-9 < x$, which can also be expressed as $x > -9$ " (Hill, Sleep, et al., 2007, p. 99).

All the SFMTs' answers were categorized as incorrect except for one teacher. Thirteen SFMTs' chose the choice (c) which is $-x < 9$ cannot be graphed on a number line, we divide by the negative sign and reverse the inequality, which is not true (Hill, Sleep, et al., 2007). In addition, one of the SFMTs' chose the choice (a) which explained the question in different form.

It might be inferred that the reason behind SFMTs' answers is that they had mastered instrumental understanding of this question since they emphasized the transformational aspects of algebraic activity (e.g., understanding algebraic expressions as a system of meaningless signs which is transformed as systematic rules) (Zwetzschler & Prediger, 2013). SFMTs saw algebra as rule-based like other mathematical rules (e.g., multiplication of two negative numbers equals a positive number) (Yılmaz & Erbaş, 2017). Just one of the SFMTs emphasised the generational aspect of algebraic activities (e.g., understanding algebraic expressions as pattern generalisers of arithmetical or geometrical pattern (Mason, Graham, Pimm, & Gower, 1985)), as she chose the choice (d) which gave the same answer but in different form: $-9 < x$). Overall, the SFMTs had mastered the Type 1 knowledge in this test item.

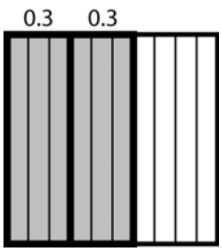
4.2.1.6 Question six

Ms. Austen was planning a lesson on decimal multiplication. She wanted to connect multiplication of decimals to her students' understanding of multiplication as repeated addition. She planned on reviewing the following definition with her class:

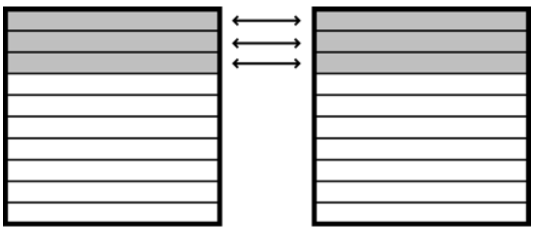
The repeated addition interpretation of multiplication defines a x b as b added together a times, or a groups of b.

After reviewing this definition of repeated addition, she planned to ask her students to represent the problem 0.3 x 2 using the repeated addition interpretation of multiplication. Which of the following representations best illustrates the repeated addition definition of 0.3 x 2? (Circle ONE answer.)

a)



b)



c)

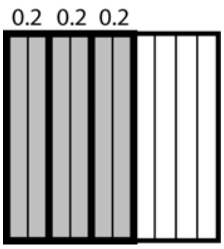


Figure 6: The best representation of the problem 0.3 x 2

d) These representations illustrate the repeated addition definition of 0.3 x 2 equally well.

e) Multiplication of decimals cannot be represented using a repeated addition interpretation of multiplication.

In this question the teachers were asked to choose the best representation of the problem 0.3 x 2 using the repeated addition interpretation of multiplication. This question aimed to investigate teachers' knowledge of rational numbers, and especially their understanding of decimal operations and their reasoning about drawn representations of decimals. There were just 2 SFMTs out of 15 who provided the best answer, which was choice (b) (Figure 6), whereas 7 SFMTs chose answer (a) (Figure 6); 1 SFMT chose (c) (Figure 6); 1 SFMT's choice was (d) (i.e., these representations illustrate the repeated addition definition of $0.3 \text{ x } 2$ equally well), and 4 SFMTs chose answer (e) (i.e., multiplication of decimals cannot be represented using a repeated addition interpretation). In the following paragraph, the researcher mathematically explains the best representation of the problem in order to unfold the type of knowledge that the SFMTs had mastered for this mathematics topic.

It takes only basic mathematical knowledge for SFMTs and other mathematics teachers to know that answer choices (e) and (d) are incorrect, because one would need to know that the model of multiplication as repeated addition could work by multiplying a decimal with a whole number but it did not work by multiplying two decimal numbers (i.e., $0.3 \times 0.5 = 0.15$ is not the same as $0.5 + 0.5 + 0.5 = 1.5$. However, five SFMTs chose options (e) and (d) which determined that those teachers lacked the basic mathematical knowledge presented in picture. This knowledge is not only an essential knowledge of SFMTs, but it is also crucial that the mathematics teachers know the meaning of the operations more deeply than another educated person (Hill, Sleep, et al., 2007).

Each grid array of this item has a different interpretation (additive Vs multiplicative interpretation) of the problem $0.3 \text{ x } 2$. Devlin (2008) indicated the difference between the two interpretations: "Adding numbers tells you how many things (or parts of things) you have when you combine collections. Multiplication is useful if you want to know the result of scaling some quantity" (p. 1). Although choice (a), which indicates multiplicative interpretation, is a more powerful way of thinking (Hurst, 2015), the best answer for this item is (b), which indicates additive interpretation. This is because the question asked the teachers to choose a grid array that represented the problem $0.3 \text{ x } 2$ using the repeated addition interpretation of multiplication. Choice (a) was the most popular answer that the SFMTs chose as it shows the correct quantity of the solution regardless of the operation being asked. This leads to the conclusion that SFMTs were more concerned with the solution quantity than with the required parts of the question, and this in turn suggests what Hurst (2015) found, which was that teachers' choice of drawn representations could be relying on answer-oriented understandings not on conceptual knowledge of operations.

The concept of decimal has been known as one of the most difficult areas to learn and teach, and it is a vital element for understanding (Stacey et al., 2001). According to Tirosh (2000), teachers' understanding of rational number arithmetic is very essential to being able to connect the subject in a meaningful way. In addition, teachers' knowledge of fractions and decimals arithmetic has a great impact on their knowledge of teaching the subject to students (Depaepe et al., 2013). Issues that include decimals below 1 and any whole number are noted very often (Lortie-Forgues & Siegler, 2017). One area of noted difficulty for teachers, especially pre-service teachers (and perhaps the SFMTs as well), is that they try to understand the relationship between decimal fraction multiplication and the operation on whole numbers (Izsák, 2008; Thipkong & Davis, 1991). Some researchers found that in-service teachers (perhaps the SFMTs as well) were struggling to explain the procedure for multiplying a decimal with ten (Chick, 2003; Tirosh & Graeber, 1989). Furthermore, existing research (e.g., Izsák, 2008) indicated that teachers found difficulties in using clear representations for classroom instruction. Overall, this indicates that teachers could lack their own understanding of these grid arrays or they could lack the pedagogical knowledge (Hurst, 2015). Therefore, there is a gap in teachers' (perhaps the SFMTs as well) mathematical knowledge for teaching (Ball et al., 2008). This demonstrates that SFMTs had mastered the Type 1 knowledge in understanding test item.

4.2.1.7 Question seven

To introduce the idea of grouping by tens and ones with young learners, which of the following materials or tools would be most appropriate? (Circle ONE answer.)

a) A number line
b) Plastic counting chips
c) Pennies and dimes
d) Straws and rubber bands
e) Any of these would be equally appropriate for introducing the idea of grouping by tens and ones.

Teachers were asked which of five options would be the most appropriate for introducing the concept of grouping by tens and ones with young students. This question was aimed at assessing teachers' knowledge of content and teaching as it pertained to number concepts and operations. The correct answer was “d”; while three SFMTs answered correctly, five SFMTs selected “e”, three selected “a”, and four chose “b”. In the next paragraph, the researcher mathematically explains the importance of the concept of grouping for learners and the most appropriate option in the question in order to unfold the type of knowledge that the SFTs had mastered for this mathematics topic.

The concept of grouping was the focus of the problem. Grouping in mathematics involves classifying numbers into clusters (Reys, Lindquist, Lambdin, & Smith, 2014). The notion of grouping as an early mathematics concept is used to prepare young learners for more advanced concepts such as understanding place value and decimals. Because the notion of grouping can be more complex for learners than counting or similar skills, activities which give learners the opportunity to visualize grouping can be beneficial (Bruno & Noda, 2019). All the options in this test item could facilitate the visualization and/or categorization of numbers. The option of number line (a) could be used to display where different numbers are in relation to one another. The option of plastic counting chips (b), as well as option (c) pennies and dimes, could be placed into groups. However, the option of straw and rubber band (d) is the most appropriate option because it gives students the opportunity to physically count out ten straws and then bundle them together with a band to represent groups of ten; groups of ten that are physically bound can then easily be added together, subtracted, or otherwise used to represent math operations involving multiples of ten.

After children learn to count consecutively (i.e. 1, 2, 3, 4...) the notion of counting and grouping by tens is one that is often learned relatively easily in comparison to multiples of other numbers, such as three (Chan, Au, & Tang, 2014). There is limited existing research on which methods are most appropriate for teaching the concept of grouping by tens, or grouping in general (Bruno & Noda, 2019). Recent research conducted by Young-Loveridge and Bicknell (2016) centred on analysing the early development of place-value comprehension among 84 five-seven year old children. Participants were asked to complete various math activities. To introduce the notions of place value and/or multiplication, the researchers suggested word problems or visualizations involving groupings of physical objects which could be contained in another object; examples that were provided included egg cartons full of eggs or boxes of chocolates. Though an egg carton would not be an effective visual tool for demonstrating groupings of ten, it is similar to the straws and rubber bands example in that, once the eggs are grouped into a carton, the groupings can then be used to represent mathematical operations (Young-Loveridge & Bicknell, 2016). Chan et al. (2014) also recently developed the strategic counting task and found it was a valid and useful means of determining how children conceptualized two and three-digit numbers. Children who participated were asked to count the total number of small squares in a given picture; each picture contained small squares, bars (comprised of ten small squares), and large squares (comprised of ten bars). Thus, the children were able to see that while counting each individual square would still provide them with the answer to the question, using the grouping method which was demonstrated by the bars and large squares allowed them to more efficiently add up the total number of small squares by groups of ten. While the activities presented by Young-Loveridge and Bicknell (2016) and Chan et al. (2014) differ, they are similar to the straws and rubber bands option presented in this question because they all involve the visual representation of groupings of numbers as a solid unit, as opposed to merely placing objects in piles or otherwise dividing them. Therefore, SFMTs' choices indicated that they did not understand the conceptual idea underpinning the options, which reflected their instrumental understanding of the topic. Overall, SFMTs had mastered the Type 1 knowledge in understanding this test item.

4.2.1.8 Question eight

Mr. Foster's class is learning to compare and order fractions. While his students know how to compare fractions using common denominators, Mr. Foster also wants them to develop a variety of other intuitive methods.

Which of the following lists of fractions would be best for helping students learn to develop several different strategies for comparing fractions? (Circle ONE answer.)

a. 1/4 1/20 1/19 1/2 1/10

b. 4/13 3/11 6/20 1/3 2/5

c. 5/6 3/8 2/3 3/7 1/12

d. Any of these would work equally well for this purpose.

Question eight entailed selecting which of four sets of fractions would be the most useful for students in developing tactics for comparing fractions. Like question seven, question eight was intended to measure teachers' knowledge of teaching and content on the topic of number concepts and operations. The best answer was "c", which was chosen by six SFMTs. Two SFMTs chose "d", six chose "a", and one opted to explain their logic rather than selecting an answer. While many of the fourteen questions require nuanced understandings of each topic in order to choose the best answer, selecting an answer to question eight was a bit more straightforward. Answer "a", selected by six SFMTs, was not a good answer simply because all fractions have the numerator 1, thus students could easily figure out which fraction is bigger than the other by comparing the denominators. In addition, answers "a" and "b"(which no SFMTs selected), each contain a couple of fractions which would be effective for the purpose described in the question. However, denominators such as 19 and 13, which are present in the first two answers, make it difficult for young learners to make comparisons; this is because multiples of 19 and 13 are not taught as common mathematics skills as numbers twelve and below are. For example, most middle-school students could readily grasp that $\frac{1}{6}$ is equivalent to $\frac{2}{12}$ because it is quickly apparent that multiplying the numerator and denominator of $\frac{1}{6}$ by two will give you $\frac{2}{12}$. However, fractions such as $\frac{2}{38}$ or $\frac{3}{39}$ are more difficult to reduce and compare because most learners do not readily know that 38 is a multiple of 19 and 39 is a multiple of 13. Moreover, having two SFMTs selecting "d", the "all of the above" option, is understandable if SFMTs could not discern a significant difference between the other possible answers.

There is a significant body of existing research which has centred on how best mathematics teachers such as SFMTs can teach the concept of comparing fractions. Eisenreich and Mainali (2016) suggested activities for comparing fractions, which asked students to draw out the ideas represented by fractions. For example, drawing a pizza with seven of eight slices missing can easily represent the idea of $\frac{1}{8}$; students can then compare a drawing of a pizza with $\frac{2}{4}$ slices missing to the $\frac{1}{8}$ pizza as a way to compare and visualize the difference between different fractions. Other strategies, such as benchmarking and residual thinking, have also been explored in recent literature (Thanheiser et al., 2016). Benchmarking involves comparing different fractions to a simple and easily understood value, such as $\frac{1}{2}$; however, it is important to note that benchmarking is only effective when the compared fractions represent the same whole value. Another technique, residual thinking, entails comparing the fractions that are one away from the whole value; for example, when comparing $\frac{2}{3}$ and $\frac{4}{5}$, it can be gathered that $\frac{4}{5}$ is the larger value because $\frac{1}{5}$ is smaller than $\frac{1}{3}$ and thus, $\frac{4}{5}$ is closer to 1 (Thanheiser et al., 2016). Further, Gould (2005) found that one of the first strategies students use when comparing fractions is based on their knowledge of the individual whole numbers which make up a fraction; for example, students just learning to compare fractions would likely conceptualize $\frac{1}{3}$ by comparing the size of the number "3" to the number "1". Thus, students with general knowledge of multiples up to twelve could more readily grasp and compare the differences between the fractions listed in option c than those listed in options a or b, as they have the most distinct knowledge of the multiplication and division of numbers up to twelve and the multiples of these numbers. Overall, from the above mathematical explanation of the best way to teach the concept of comparing fractions and the results of SFMTs answers to this item, it could be indicated that SFMTs had mastered the Type 1 knowledge in their understanding of this test item.

4.2.1.9 Question nine

Ms. Brockton assigned the following problem to her students:

How many 4s are there in 3?

When her students struggled to find a solution, she decided to use a sequence of examples to help them understand how to solve this problem. Which of the following sequences of examples would be best to use to help her students understand how to solve the original problem? (Circle ONE answer.)

A) How many:
4s in 6?
4s in 5?
4s in 4?
4s in 3?

B) How many:
4s in 8?
4s in 6?
4s in 1?
4s in 3?

C) How many:
4s in 1?
4s in 2?

D) How many:
4s in 12?
4s in 8?

4s in 4?	4s in 4?
4s in 3?	4s in 3?

The focus of question nine was the same as the two preceding questions: knowledge of teaching and content related to number concepts and operations. The teachers were asked to select the best example of four which could be used to help students solve a division problem. Surprisingly, no SFMTs answered this item correctly; the correct answer was "b". Nine SFMTs selected "c", five selected "d", and one chose "a". In the following paragraph, the researcher mathematically demonstrates which option is best to assist students' understanding to solve this division problem in order to unfold the type of knowledge that the SFMTs had mastered for this mathematics topic.

This was one of the more nuanced questions included in this activity, as may be gathered from the lack of correct responses. This test item was about how to help students understand a division such as "how many 4s are there in 3". While all the options included division problems with four as the numerator, only "b" included a progression which would aid in solving the problem. While all of the examples included numbers which most students would readily know multiples of, option "b" progresses in a way that facilitates understanding, from a simple division problem, to two easily simplified fractions (i.e., $\frac{4}{6}$ and $\frac{1}{4}$, then to the original problem. The example of "How many 4s in 1?" is best to precede the original problem because the answer is the common fraction $\frac{1}{4}$, or 0.25; this is a relatively easy fraction for students to grasp because there are four quarters in a dollar. From there, students can conceptualize how to determine how many 4s are in 3. While "How many 4s in 1?" is included as a part of "c" as well, it is the first example in the sequence and would not give students much of an opportunity to understand easily-divisible multiples or fractions which can be reduced prior to attempting the original problem. In addition, several researchers (Ball, 1988; Ball & Wilson, 1990; Flores, 2002; Ma, 1999b) have identified that teaching the division of fractions requires mathematics teachers such as the SFMTs to have a strong understanding of both the concepts and processes. Therefore, it is insufficient for the teachers such as the SFMTs to just know how to divide a fraction, it requires much more than that (Hill, Sleep, et al., 2007). Learning fractions needs students to be aware of differentiating between natural numbers properties and rational numbers properties (Siegler & Lortie-Forgues, 2015). So, teachers (including the SFMTs) should have a solid understanding of the partitive and the quotitive/measurement interpretation of division (Greer, 1992).This topic is conceptually rich and complex, as it could be explained through its connections with other mathematical knowledge, different representations, and real-world problems (Li, 2008). As a result, in Ma's (1999a) comparisons between U.S. and Chinese elementary teachers, she defined "profound understanding of fundamental mathematics" as a combination of connectedness, fundamental ideas, and various perceptions that occurred through their teaching. To have a profound understanding of fraction division, teachers such as the SFMTs should understand deeply all related ideas that develop this concept (Clarke, Roche, & Mitchell, 2011). Scholars verified that three fundamental concepts are very essential to develop a proper understanding of fraction (units and unitizing, partitioning and iterating, and equivalence) (Barnett-Clark, Fisher, Marks, & Ross, 2010). The unit is counted as the most essential element of rational number understanding, as it is used in all the fundamental concepts mentioned above. In addition, previous research indicates that knowledge of whole number is vital to start understanding fraction (Appleton, 2012).For example, according to Mitchelmore and White (2000) students' knowledge of whole number (e.g., understanding of decomposition of number) could help them divide a whole into fair shares and unit fractions. Moreover, students can use the same counting strategies for whole number to count unit fractions (e.g., $\frac{1}{4}$ can be counted three times to get $\frac{3}{4}$. This explains the concept underpinning the choice (b) and why it is the best answer. Thus, the development of fractional sense is essential for teachers such as the SFMTs to construct conceptual understanding of fractions and the related mathematical concepts. Yet, this test item is very demanding in terms of SFMTs' mathematical and pedagogical knowledge so they are able to teach the concept effectively. Further, it is insufficient for the learner to recognise the essential concepts by simply giving an answer for a division; there should be a variety of connected examples that promote understanding of the main concept (Watson & Mason, 2005).

Mathematics operations such as division and multiplication which include decimals or fractions are a significant point of difficulty for many students; large-scale survey data of eighth grade students from across the country in the US collected in two separate studies showed that only 24% and 21% of participants chose the correct answers for simple mathematics problems involving fractions and decimals, respectively (Lortie-Forgues, Tian, & Siegler, 2015). Some reasons why all SFMTs selected the wrong answer for question nine may coincide with middle-school students' difficulty in performing operations which include decimals and/or fractions; everyone thinks about fractions and decimals differently in relation to where and when they have interacted with them in everyday life, such as when handling money or dividing up a pizza among friends. Thus, the way division is taught when it involves fractional or decimal answers can have a significant impact on how effectively, and how soon, the concept is grasped (Lortie-Forgues et al., 2015).

The Common Core State Standards Initiative (2010) is the official educational policy of most (>80%) U.S. states which determines which mathematics competencies should be taught at each grade level. Recommendations provided in the CCSI were that, prior to learning about fractional and decimal division, students should be taught to add and subtract fractions with common denominators. Subsequently, they should learn about the addition and subtraction of fractions with different denominators, then multiplication of fractions, followed by division of fractions. Further, it was recommended that students in the seventh and eighth grades be given problems which involve using the concepts of fractional and decimal division in conjunction with ratios and rates to solidify the concepts in practical scenarios (Lortie-Forgues et al., 2015). In line with the ideas of progression recommended by the CCSI (2010), option "b" progresses from simple division ($\frac{8}{4} = 2$), to an answer that is easily-simplified ($\frac{6}{4} = \frac{3}{2}$), to a simple fraction ($\frac{1}{4}$), and then to the original problem; the original problem is very similar to the example which precedes it because it also results in an answer which is a common fraction. Thus, if students use fractions to conceptualize each answer, determining how many "4s are in 1" directly before determining how many "4s are in 3", that process sets learners up to use the same thought process to solve both problems. Therefore, as a result of not one of the SFMTs' choosing the correct answer, SFMTs showed lack of understanding of the conceptual idea underpinning the options which reflects their instrumental understanding of the topic. Overall, SFMTs had mastered the Type 1 knowledge in understanding test item.

4.2.1.10 Question ten

Ms. Williams plans to give the following problem to her class:

Baker Joe is making apple tarts. If he uses 3/4 of an apple for each tart, how many tarts can he make with 15 apples?

Because it has been a while since the class has worked with fractions, she decides to prepare her students by first giving them a simpler version of this same type of problem. Which of the following would be most useful for preparing the class to work on this problem? (Circle ONE answer.)

I. Baker Ted is making pumpkin pies. He has 8 pumpkins in his basket. If he uses 1/4 of his pumpkins per pie, how many pumpkins does he use in each pie?

II. Baker Ted is making pumpkin pies. If he uses 1/4 of a pumpkin for each pie, how many pies can he make with 9 pumpkins?

III. Baker Ted is making pumpkin pies. If he uses ¾ of a pumpkin for each pie, how many pies can he make with 10 pumpkins?

a) I only

b) II only

c) III only

d) II and III only

e) I, II, and III

Question ten also assessed the teachers' knowledge of content and teaching number concepts and operations. The test item was about how to prepare students to learn a division of fractions: "if Joe uses $\frac{3}{4}$ of an apple for each tart, how many tarts can he make with 15 apples?". The SFMTs were to select which of the five options would be the best to give students in preparation for a similar, but more complex, problem. Two SFMTs selected the correct answer, which was "b". Seven SFMTs chose "e", three SFMTs chose "a", and three chose "d". In the next paragraph, the researcher justifies mathematically each option and why option "b" is the best answer, in order to unfold the type of knowledge that the SFMTs had mastered for this mathematics topic.

While two SFMTs answered this question correctly, the large number of wrong answers indicates some confusion. Option "a" (scenario I) would not be the most appropriate because it would require multiplication instead of division as required in the original problem. Options “c” and “d” include the similar problem which could be potentially helpful, as it involves the same operation as the original problem; however, multiplying the reciprocal of $\frac{3}{4}$ by ten (in scenario III) is not much simpler than the original problem. Thus, option “b” (scenario II) is the correct answer. From Option “b”, students can understand that there are four $\frac{1}{4}$ s in one whole pumpkin thus there will be $(9 \times 4)\,\frac{1}{4}$ s in 9 pumpkins. Also, it provides a scenario with dividing a number by a fraction which, when converted to its reciprocal value prior to multiplication, results in $\frac{9}{1} \times \frac{4}{1}$. Because the denominators are equal without any further action, it is a simple problem which results in the answer of 36. Once students grasp the simpler problem presented in option “b”, it will then be easier for them to move on to learning how to divide and multiply fractions when the denominators are not equal, as would be the case for the original problem $\frac{15}{\frac{3}{4}} = \frac{15}{1} \times \frac{4}{3}$.

Word problems involving fractions are sometimes easier for young learners to comprehend than equations alone because they provide the opportunity for the fractions and numbers being discussed to be visualized (Lortie-Forgues et al., 2015); rather than having to conceptualize how many times a number that is less than one can be divided, students can seek to understand how many parts or sections of an object are used in a way that could easily be drawn (Fuchs et al., 2016). This problem, particularly, builds on the familiar idea of baking. Though young students may not personally have baked a pie before, they have likely seen someone else bake and, thus, could visualize the process of combining portions of different ingredients to make whole pies.

Overall fraction knowledge is largely dependent on interpretation of fractions and how they are divided (Poon & Lewis, 2015). Because fractions can be represented in many forms, understanding all representations of fractions is essential to mastering mathematics operations using fractions. Fractions can be represented as: 1) comparisons of part-to-whole, 2) ratios, 3) decimals, 4) quotients, 5) means of measuring discrete or continuous quantities, and 6) operators (Poon & Lewis, 2015).

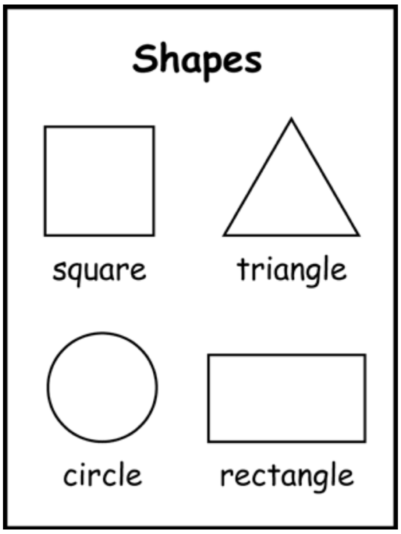
The measurement interpretation of fractions is a mechanism by which students can link the idea of measuring using tools (i.e. a measuring cup) with fractions presented in mathematics class (Fuchs et al., 2016). If they understand this concept then they can better answer the problem which involves fractions with the same denominator or a denominator of one; the correct option “b”, is an effective example of a problem which is easily conceptualized with the measurement interpretation of fractions (Fuchs et al., 2016). Without even writing out a proper equation for the problem presented by choice II and formally converting it to a multiplication problem, some students may be able to interpret that if four pies can be made with one pumpkin, 4 × 9 would result in the number of pies which could be made with nine pumpkins. While the measurement interpretation of fractions mechanism might assist young learners with understanding or visualizing what scenario III was asking, it is very unlikely that students could easily come up with the answer without fully writing out the equation, changing $\frac{3}{4}$ to $\frac{4}{3}$, converting $\frac{10}{1}$ to $\frac{30}{3}$, and then completing the required operation. Thus, though all options may be potentially useful, option “b”, or scenario II, would be the most useful for preparatory purposes. Overall, this indicates that the SFMTs had mastered the Type 1 knowledge as just two of them displayed features of understanding the underlying conceptual idea of this test item.

4.2.1.11 Question eleven

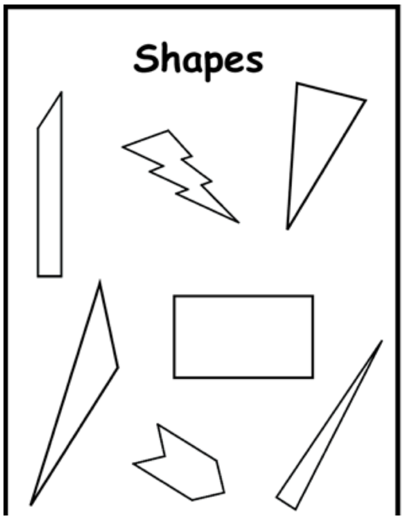
Ms. Miller wants her students to write or find a definition for triangle, and then improve their definition by testing it on different shapes. To help them, she wants to give them some shapes they can use to test their definition.

She goes to the store to look for a visual aid to help with this lesson. Which of the following is most likely to help students improve their definitions? (Circle ONE answer.)

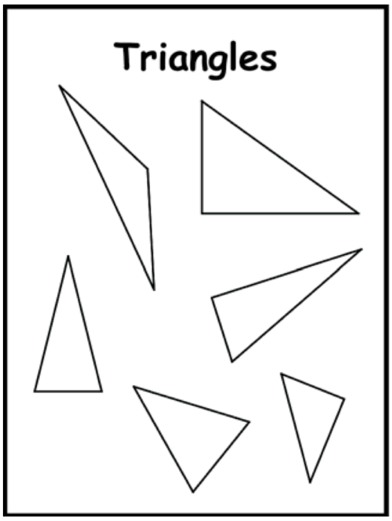
a)



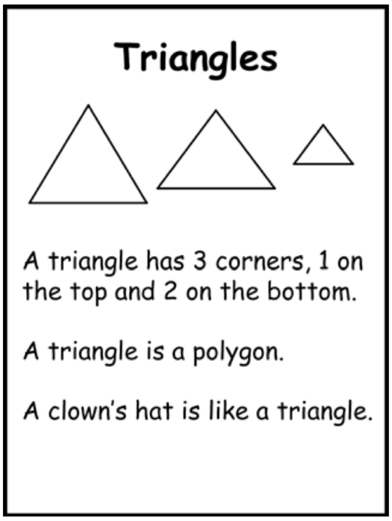
b)



c)



d)



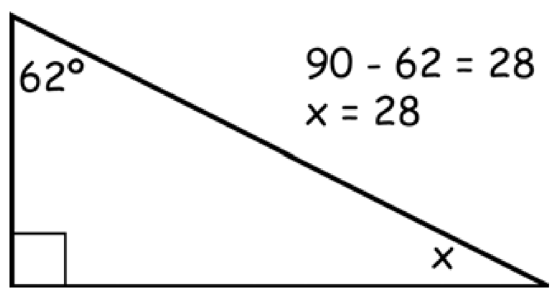
For question eleven, teachers were asked to choose which of four sets of visual aids could most effectively assist students to learn the definition of a triangle (see Appendix 1). Question eleven was aimed at measuring teachers' knowledge of teaching and content pertaining to geometry. One of the SFMTs selected the correct answer, which was "b", five SFMTs selected "a", two selected "c", six selected "d", and one chose both "c" and "d". In the paragraph below, the researcher shows the mathematical idea behind each option in order to unpack the type of knowledge that the SFMTs had mastered for this mathematics topic.

A triangle is characterized by having three angles and three sides (Seah, 2015). Teaching students the definitions of different shapes is primarily done in order to distinguish the shapes from one another. Thus, a visual aid can be used to assist in teaching the definition of a triangle primarily to help distinguish triangles from other shapes based on ascribed characteristics (Seah, 2015). Based on this assertion, answer “b” is the most appropriate choice. While option “a” still shows other shapes and would give students the opportunity to distinguish the characteristics of a triangle, there is a limited number of shapes presented and the other “shapes” such as square and circle are familiar to many elementary students. Further, including the names of each shape allows learners to rely more on associating the name with the shape, rather than focusing on how the characteristics of a triangle are different and distinct from other shapes. Option “c” only shows triangles; many examples are provided, but there is no opportunity for students to distinguish triangles from other shapes. Lastly, option “d” provides examples of triangles but does not demonstrate examples which differ along any characteristic besides size. Further, the text included below the pictures notes characteristics including, “A triangle has three corners, 1 on the top and 2 on the bottom”; such guidelines may confuse learners if a triangle is not oriented in that way specifically, such as some triangles included in visual aid “c”.

There is little existing research on the topic of visual aids used when teaching geometry, though there is considerable literature on the topic of teaching geometric thinking and the definitions of different shapes (Srinivas, Khanna, Rahaman, & Kumar, 2016). Recent research by Srinivas et al. (2016) was aimed at assessing the geometric thinking capabilities of secondary school students; the authors found that questions students received the lowest scores for on the assessment involved differentiating whether a square was still considered a square when tilted, or identifying a triangle within a set of other shapes. Srinivas et al.’s (2016) results suggest that two of the most difficult facets of understanding geometry for students may be understanding the distinct characteristics and definitions associated with shapes. In the case of the tilted square, many respondents believed it was a rhombus or diamond because the square was tilted; however, those shapes have distinctly different characteristics when compared to a square, and tilting a square does not change that a square is defined as a shape with four right angles and four sides equal in length (Seah, 2015). In the case of the triangle task, students who did not correctly choose the triangle explained themselves by saying it did not look like a triangle, or it seemed too thin to be a triangle (Srinivas et al., 2016). Again, when students are too reliant on what they believe a shape usually looks like as opposed to judging a shape's distinct characteristics (i.e. number of angles, number of sides), errors are more likely to occur. Thus, option "b" is the most appropriate visual aid for helping students to learn the definition of a triangle based on its distinct characteristics, as well as learning to differentiate what defines a triangle from the characteristics of other shapes they may or may not recognize. If SFMTs teach the definition of a triangle in this way, as opposed to teaching simple recognition based on appearance, it can contribute to a foundation of logical mathematical thinking which can later make learning more complex operations easier (Herbst, Fujita, Halverscheid, & Weiss, 2017). However, SFMTs did not recognize the conceptual mathematical idea behind option "b" as they understood the question instrumentally which indicates their mastery of Type 1 knowledge of this item.

4.2.1.12 Question twelve

Mr. Donaldson's class was working on an assignment where they had to find the measures of unknown angles in triangles. One student consistently found the measures of unknown angles in right triangles by subtracting the known angle from 90. For example:



Ms. Donaldson was concerned that this student might run into difficulty when trying to find the measures of unknown angles in more general triangles. Which of the following questions would be best to ask the student in order to help clarify this issue? (Circle ONE answer.)

a) "What do you get when you add 90 + 62 + 28?"
b) "Why does subtracting 62 from 90 give you the measure of the unknown angle?"
c) "How could you find the missing angle in an isosceles triangle?"

d) “How did you know that this was a right triangle?”

e) “What if this angle measured 17 ° instead of 62 °?”

Like question eleven, question twelve measured teachers’ knowledge of content and teaching geometry. Teachers were to select the best of five follow-up questions intended to help students who encountered a problem finding the measure of a missing angle. The correct answer, “b”, was chosen by three SFMTs, nine SFMTs chose “a”, one selected “$\mathrm{c} - \mathrm{d}$”, one chose “$\mathrm{a} - \mathrm{b}$”, and one chose “$\mathrm{a} - \mathrm{b} - \mathrm{c}$”. In the next paragraph, the researcher explains the mathematical idea behind each option and what is the best option that requires teachers to have a comprehensive understanding of why certain actions help to develop students’ mathematical reasoning in geometry. This unfolds the type of knowledge that the SFMTs had mastered for this mathematics topic.

The logic for the correct answer is comparable to the previous question; by subtracting the known angle measure from 90, the student has found a quick trick for solving some problems like the example. However, the issue potentially created by this method is similar to the issue encountered by the students who participated in Srinivas et al.’s (2016) study. Students were able to identify a shape based on similarity of appearance but could not consistently identify some because they did not understand the concept of the unchanging characteristics of shapes. Similarly, the issue which could arise with the method of finding the measure of a missing angle presented in question twelve is related to students only learning and using the quickest method of finding the right answer to specific problems, as opposed to learning strategies which consistently result in the right answer to a larger range of problems. Thus, “b” is correct because it could help the student understand why they answered the problem correctly using subtraction from 90 method; then, the student could more readily devise or learn a new strategy for approaching similar problems that don’t involve a right angle. Answer “a” would merely help the student realize that there are 180° in a triangle if they did not know. Answer “c” reflects a similar approach as answer “b” but may be difficult for the student to articulate if they are relying on their ‘subtract from 90’ method. Lastly, answers “d” and “e” would not address the potential issue identified by the teacher, as they would not lead the student in the direction of developing a method of finding the measure of a missing angle that would work for all triangles.

As mentioned in the previous question's explanation, learning geometric concepts and methods of approaching geometry problems can be indicative of a student's capacity for developing the logic and reasoning needed to be successful in more advanced forms of mathematics (Seah, 2015). While some concepts taught to young learners, such as learning multiplication tables, are based more on memorization than anything else, more advanced mathematical operations often require a comprehensive understanding of why certain actions lead to the right answer while others do not. This process, known as deductive reasoning, involves gathering information from statements or a problem that is posed and coming to a logical conclusion as to what the correct answer is and why it is correct (Srinivas et al., 2016). Thus, when learning geometric concepts such as finding missing angles, it is imperative that young learners understand why certain equations and operations lead them to the right answer so that they develop a basis of mathematical reasoning, rather than a collection of memorized facts and processes (Srinivas et al., 2016). Bokosmaty, Sweller, and Kalyuga (2015) highlighted the benefits and consequences of teaching geometry in a way which is reliant on examples versus problem solving strategies. Teaching with a focus on practical examples helps inexperienced students pick up geometric concepts more readily than a problem-solving approach; conversely, more experienced students may experience diminished performance when an emphasis is placed on examples rather than problem-solving strategies. Thus, in classrooms with diverse learner capabilities, it is important not to rely solely on one strategy or the other to ensure students' performance does not falter (Bokosmaty et al., 2015). Incorporating examples and quick approaches to solving some problems, such as the scenario presented in question twelve, is not necessarily a bad thing as long as the student utilizing the 'subtract from 90' rule is introduced to a problem solving method which could help them find angle measures in non-right-angled triangles. Overall, the SFMTs had mastered the Type 1 knowledge in this test item.

4.2.1.13 Question thirteen

As an early introduction to mathematical proof, Ms. Cobb wants to engage her students in deductive reasoning. She wants to use an activity about the sum of the angles of a triangle, but her students have not yet learned the alternate interior angle theorem. They do, however, know that a right angle is 90 degrees and that a point is surrounded by 360 degrees. Which of the following activities would best fit her purpose? (Circle ONE answer.)

a) Have students draw a triangle and a line parallel to its base through the opposite vertex. From there, have them reason about the angles of the triangle and the angles the triangle makes with the parallel line.
b) Have the students use rectangles with diagonals to reason about the sum of the acute angles in a right triangle.
c) Have students use protractors to measure the angles in several different triangles and from there reason about the sum of the angles of a triangle.
d) Have students cut out a triangle then tear off the three corners and assemble them, and from there reason about the sum of the angles of a triangle.

Question thirteen entailed teachers choosing which of four activities would be the most effective for introducing the concept of mathematical proofs. This question was also intended to measure teachers' knowledge of geometry teaching and content. The test item required the SFMTs to choose one activity that could best fit the purpose of introducing the angles sum of a triangle. Four SFMTs selected the correct answer, "b", five SFMTs chose "c", two teachers chose "a", and four selected "d". In the following paragraph, the researcher explains mathematically the four activities, and which one is most effective to prepare students think deductively, in order to unfold the type of knowledge that the SFMTs had mastered for this mathematics topic.

Four SFMTs chose the correct answer for this question, making it one of the more agreed upon items by the participating SFMTs. A mathematical proof is defined as an inferential argument in regards to a mathematics statement; a proof is expressed in a series of steps which result in the original statement and, if done properly, prove it to be true (Erickson, Boileau, Huisinga, & Herbst, 2017). Thus, geometric proofs involve providing an inferential argument in order to prove a provided geometric statement (Erickson et al., 2017).

The correct answer to this question is the answer which would most effectively prepare students to think in the deductive manner necessary to eventually solve geometric proofs. As was mentioned above, deductive reasoning entails using information that is provided to come to a logical conclusion (Srinivas et al., 2016). In the case of proofs, deductive reasoning must be used to consider the provided statement and logically construct the steps that are required to come to that conclusion. Option “a” would require students to understand that a line is a straight angle and has 180° before the proper information could be deduced. Option “c” would likely lead to students deducing that there are 180° in a triangle, but they may be unable to explain why; this is the same issue which was mentioned in the explanation of the previous two questions. When students rely on shortcut “rules” which are not universally applicable, or use arbitrary characteristics such as appearance to solve geometry problems, errors are likely to occur and it is less likely that learners will develop the basis of reasoning necessary to move on to more complex topics (Srinivas et al., 2016). With a similar issue, “d” would not be the most appropriate choice because that activity would help students visualize the angles in a triangle, but reassembling the triangle without additional information would do little to help students reason about the sum of the angles. Thus, answer “b” would be the most effective choice; placing a point in the middle of the described rectangle with diagonals, students would know that the angles surrounding the central point add up to 360°. Using that information, students could then find the missing angle values within each triangle by subtracting the central angle measures from the 90° angles at each of the four corners. This example affords students the opportunity to deduce the ideas presented in the alternate interior angle theorem based on the limited information they already know to be true (a right angle is 90 degrees and that a point is surrounded by 360 degrees).

If learning general geometric ideas and reasoning serves as the first step on the journey to understanding the notion of mathematical reasoning (Srinivas et al., 2016), learning how to solve proofs is a considerable benchmark on the road to sound mathematical logic. The deductive reasoning needed to solve proofs can be incredibly beneficial for understanding more complex forms of mathematics later. However, the notion of proofs and introducing deductive mathematical reasoning can be just as difficult for some teachers as it is for some students (Dimmel & Herbst, 2017). One such reason is that while some proofs have one, or a small number, of potential solutions, others have many. Proofs are the mathematical equivalent of an inferential argument and thus, different people often use different arguments, or steps in their proof, to arrive at the same conclusion. As such, it is important for SFMTs to make significant attempts to understand students’ reasoning or arguments presented in proofs before marking them as incorrect, as it is possible that a valid solution or approach to a mathematical argument may just be incompatible with an individual teacher's approach to a given proof (Dimmel & Herbst, 2017). Overall, the SFMTs had mastered the Type 1 knowledge in this test item.

4.2.1.14 Question fourteen

Mr. Shephard is using his textbook to plan a lesson on converting fractions to decimals by finding an equivalent fraction. The textbook provides the following two examples:

Convert 2/5 to a decimal: 2/5 = 4/10 = 0.4

Convert 23/50 to a decimal: 23/50 = 46/100 = 0.46

Mr. Shephard wants to have some other examples ready in case his students need additional practice in using this method. Which of the following lists of examples would be best to use for this purpose? (Circle ONE answer.)

a) 1/4 8/16 8/20 4/5 ½

b) 1/20 7/8 12/15 3/40 5/16

c) 3/4 2/3 7/20 2/7 11/30

d) All of the lists would work equally well.

The final question, question fourteen, was developed to assess teachers' knowledge of content and teaching about rational numbers. Teachers were asked to choose which of the four options would be the best list of examples to assist students in learning to convert fractions to decimals by finding an equivalent fraction. No SFMTs selected the correct answer, which was "b". Six SFMTs chose "d", seven SFMTs selected "a", one SFMT chose "c", and one SFMT was unsure of which answer to choose. In the below paragraph, the researcher explains the mathematical idea behind the four options in order to unpack the type of knowledge that the SFMTs had mastered for this mathematics topic.

Question fourteen was one of two questions that no SFMTs answered correctly. SFMTs appeared to experience a similar issue with both problems whereby they found it difficult to determine the differences between each respective option. In the case of the choices provided for question fourteen, option "b" was the only choice which did not include a fraction which many learners would already likely know the decimal equivalent of; option "a" included the common fractions $\frac{1}{4}$ and $\frac{1}{2}$ which most students could quickly identify as 0.25 and 0.5, while option "c" included $\frac{3}{4}$ which most students would know to be 0.75. Though answer choices "a" and "c" may have still been useful and would have likely been solved sooner than option "b", all numbers included in the set would not have required conversion as was the point of the lesson described in question fourteen.

Representational flexibility is key for grasping concepts related to fractions and decimals, such as understanding how a decimal may represent many different equivalent fractions (Deliyianni, Gagatsis, Elia, & Panaoura, 2016). Among other previously-explained reasons why fractions and decimal operations can be difficult to grasp, young learners' understanding of fractions and decimals is largely influenced by how such numbers are represented; for example, learners may readily know the answer to $\frac{1}{4} + \frac{3}{4}$ but may become stuck or frustrated when $\frac{1}{4} + \frac{15}{20}$ is presented. While $\frac{15}{20}$ is the fractional equivalent of 3/4, the visual representation of a fraction before it is simplified or converted can be regarded in a completely different way by learners in comparison to the converted or simplified fraction. Thus, it is important that examples contribute to students' understanding of uncommon equivalent fractions and do not allow students to bypass the process of learning to convert fractions in lieu of memorizing simple fractions and their equivalents, such as $\frac{2}{3}$ and $\frac{4}{6}$ (Tian & Siegler, 2018). Examples provided in option "b" would facilitate the conversion of numbers which are not commonly used and increase students' understanding of the process of conversion. Because any decimal can be converted to a fraction and every fraction can be converted to a decimal, it is imperative that students do not merely learn to solve problems involving fractions or decimals because they typically include common fractions and decimals. Thus, including uncommon fractions such as $\frac{3}{40}$ which requires a conversion to multiples of 1,000 helps students to develop problem solving flexibility (Deliyianni et al., 2016). As a result, from the above discussion SFMTs had difficulty in recognizing the conceptual mathematical differences behind each option as they understood the question instrumently, which indicates their mastering Type 1 knowledge of this item.

4.2.2 Summary of written test findings

4.2.2.1 The case of Number concepts and operations (NCOP) items

Written test items 1, 7, 8, 9, and 10 are related to NCOP. Most of the teachers did not identify the correct answer in all of these items as being six, three, six, zero, and two SFMTs identified the key answer for items 1 (i.e., understanding of principles for operations with negative numbers), 7 (i.e., introducing the concept of grouping by tens and ones with young students), 8 (i.e., developing tactics for comparing fractions), 9 (i.e., division problem), and 10 (i.e., fraction) respectively. This result shows that most of the SFMTs demonstrated Type 1 cognitive content knowledge as they had gained an instrumental understanding by simply memorizing the rules without any conceptual and relational understanding of the NCOP domain. Findings indicated that they had a narrow understanding of the ideas underlying the conceptual knowledge of this domain.

4.2.2.2 The case of Patterns, functions, and algebra (PFA) items

Items 3, 4, and 5 are related to PFA. Eight teachers identified the key answer for item 3 (i.e., equivalence of expressions); seven teachers gave the correct answer for item 4 (i.e., knowledge of algebra); whereas with item 5 (i.e., solving problems with an inequality in them) just one teacher answered it correctly. According to the SFMTs' responses, their cognitive type of content knowledge varied between Type 1 and Type 2 across these three questions. This explains that there are some teachers who have relational understanding and deep and conceptual knowledge of this domain as they identified the key answer.

4.2.2.3 The case of Geometry (GEO) items

Three questions in the domain of GEO, items 11, 12, and 13, were provided in the written test. The majority of the SFMTs did not give the correct answer for these items. In item 11 only one teacher answered it correctly; the key answer for item 12 was identified by three teachers, and four teachers answered correctly for item 13.

This result shows that SFMTs possess more knowledge of fact and procedures than the other two cognitive types. It means that the teachers had more procedural knowledge and instrumental understanding.

4.2.2.4 The case of Rational numbers (RAT) items

Written test items 6, and 14 are related to the RAT domain. Almost all of the SFMTs did not find the correct answer for these two questions. Interestingly, for item 14 not one teacher out of 15 SFMTs found the correct answer, whereas only two SFMTs answered item 6 correctly. This result demonstrated Type 1 content knowledge as most of SFMTs possessed procedural knowledge and instrumental understanding of the concept of RAT.

4.2.2.5 The case of Proportional reasoning (PR) items

One question out of 14 related to the PR domain, which is item 2 (i.e., concepts of ratio and proportion). Overall, not one SFMT identified the key answer for all the question's sections. Yet, 5, 5, 6 and 7 of SFMTs obtained correct answers for 2a, 2b, 2c, and 2d respectively. Consequently, SFTs possessed instrumental understanding, the cognitive Type 1 knowledge of facts and procedures.

4.3 Interview findings

4.3.1 Introduction

The interviews were conducted with the same teachers who did the written test. The interview format consisted of asking teachers some questions related to three areas of teachers' beliefs about the nature of mathematics, beliefs about mathematics teaching, and beliefs about mathematics learning; in fact, teachers' beliefs do not exist in isolation (Beswick, 2006). Following analysis, summary tables of the Saudi female mathematics teachers' beliefs in each of the areas were developed. In this section, results of the analysis for all 15 teachers are discussed; direct quotes have been contextually translated.

4.3.2 Saudi female teachers' beliefs about the nature of mathematics

Teachers' view of the nature of mathematics is a basis of teachers' belief systems regarding the nature of mathematics as a whole (Ernest, 1989a). In this research, all the SFMTs' responses to the interview suggested that they held beliefs consistent with a Platonist orientation to mathematics and there was no apparent contradiction among their responses. Table 4 provides a summary showing how teachers viewed the nature of mathematics. Their interview responses can be summarized from commonly expressed themes: (1) mathematical knowledge and real life, and (2) the interconnected and abstract nature of mathematics.

Table 4: Summary of SFMTs' view about the nature of mathematics

Saudi female teachers' views about the nature of mathematics

Description of teachers' views	**Number of Saudi female teachers believed**
Knowledge from daily life	**15**
Abstract and interconnected	**15**

4.3.2.1 Mathematical knowledge and real life

All 15 SFMTs believed that the nature of mathematical content is interrelated to human everyday life, so students can construct their understanding by linking mathematical content with real world activities. Making such connections provides better understanding of both life and mathematics. One of the SFMTs, Maha, stated: "Mathematics is applicable to real life problems and that it is an essential skill for living." Another SFMT, Hind, made a similar statement: "If the students understand mathematics well, it will help them in their daily life problems." In addition, they believed that real life questions are a stepping-stone to having a good understanding of mathematical concepts.

Furthermore, all the teachers believed that mathematical knowledge is concrete as well as abstract. Its application to real life problems is sometimes nontrivial. However, once we attain this knowledge, we can apply it in real life. The SFMTs' idea is depicted in Malak's statement: "Some mathematical topics are hard to connect to the real world because they are abstract, once the students have mastered the knowledge, they can directly apply it in their life."

Another SFMT made a slightly different but related comment. Amira said: "Mathematics is rules and facts students have to remember and they can apply in their daily life."

Interestingly, almost all of the SFMTs (14 teachers) believed that mathematical facts and rules are not discovered, and the mathematical rules are connected to each other. For example, Yara reported: "In order to make the students understand the new topic or rules, we must link it with their previous related knowledge."

4.3.2.2 The nature of mathematics – its abstractness and interconnectedness

All 15 SFMTs expressed the same belief about the nature of mathematics when they were asked about the nature of mathematics and their teaching. They believed that mathematics is an interconnected pattern and teachers must link mathematical rules with related ones as well as abstract concepts, facilitating students' ability to acquire foundational knowledge in order to develop their abstract thinking. For instance, Hana said: "I try to connect any new skill or topic with previous related skills, because Mathematics is connected topics and we have to teach it in a structured way." Consistent with Hana, Maha held a similar belief as she stated that "mathematics topics are connected and when I explain any topic, I really focus on linking related topics together."

In addition, they also indicated that mathematics is abstract in nature and one of the curriculum objectives is to teach students abstract thinking. They talked about the process of developing abstract, logical thinking in the classroom. Hind indicated: "Teaching abstract rules to students requires linking the new rules to students' prior mathematics rules and foundational knowledge." However, many of the SFMTs said students' background knowledge and abilities prevented them from teaching abstract thinking. They complained that some of the students did not have the basic knowledge that even primary students should have, making it difficult to teach them abstract knowledge.

4.3.3 Beliefs about mathematics learning

During the interviews, the SFMTs expressed their beliefs about the purpose of learning mathematics, processes of learning, the role of affect and motivation, and social perception of learning. Table 5 provides a summary showing how the SFMTs viewed mathematics learning.

Table 5: Summary of SFMTs' view of mathematics learning

Saudi female mathematics teachers' views about mathematics learning

Description of teachers' views		Number of Saudi female teachers believed
The purpose of learning	**Understanding is the ultimate goal of learning mathematics**	**15**
	Memorization plays an important role in understanding	**15**

The processes of learning	**Learning through practice**	**Fluency and accuracy of students answers**	**5**
		Useful for students understanding	**10**
Approaches to learning	**Students' understanding**		**2**
	Students' exams		**13**

4.3.3.1 The purpose of learning: (memorization and understanding)

During the interviews, all the SFMTs expressed that understanding is the ultimate goal of learning mathematics and they possessed a similar perspective on the nature of mathematics understanding. They believed that memorization, using real-life activities, and concrete examples could assist students' mathematical understanding and emphasised the role that memorization could play in mathematical understanding, but memorization without understanding does not lead to deep understanding.

The SFMTs highlighted the important role of two types of memorization: memorization before understanding and after understanding. They stated that memorization is an intermediate or transitional step towards understanding mathematics and indicated that they tried to encourage their students to combine both memorizing and understanding. Maha stressed the importance of memorization during the lesson and said: "memorization is very essential after the students have understood the main concept of the lesson." She further explained that: "memorizing previous facts and rules is extremely important because it helps me focus more on explaining the main concept without wasting time on previous facts and rules."

Further, all the SFMTs believed that memorisation helps students to do well in exams and achieve good results. According to Yara: "some students need rote memorization of facts and rules to do well in the exams." Similarly, Nora stated: "memorization is an important component that influences students' understanding as they must remember facts in order to apply them flexibly to different questions in exams." They also thought that students would benefit from memorizing previous topics as it would help them to get the answers quickly. Hind said: "students really need some memorization when I ask them to solve a question quickly and in automation." Similarly, Hana expressed: "through memorizing the rules and formulas students will have more confidence to answer the questions quickly.".

In addition, all the SFMTs agreed that there were two ways that helped them to determine the depth of students' understanding. The first way was through examination. Sara reported: "examination is one way to see students' understanding". She further stressed the importance of questioning them during the lessons, as she explained: "giving them different types of questions for the same concept and making them explain their reasons behind the methods for getting the correct answer is another way." Mariam also said: "giving the students a variety of questions for the same concepts will show me if they have any difficulty in their understanding". Thus, all the SFMTs tended to agree that understanding and memorization are highly connected and they are hard to separate.

4.3.3.2 The processes of learning (the role of practice)

All the SFMTs considered practice as an essential way to enhance students' knowledge and understanding. They all agreed about the necessity of practice, believing that students learn mostly through practising the same skill. For instance, Huda indicated: "practice helps students to have more understanding of the topic by repeating similar exercises, it also helps them to be more familiar with the topic and memorise it easily". Soad had a similar thought: "after explaining the main concept of the lesson and discussing the main rules, students need to do some practice… I let them solve few examples individually and in groups to make sure that they understand the main concept."

Despite all the SFMTs stressing the importance of practice by giving the students different activities, they were divided into two groups. Some preferred to give students the same challenging level of activities (repetition); the second group opted to provide students with a variety of activities, with different challenging levels.

The SFMTs of the first group focused on the fluency and accuracy of students' answers. For instance, Mariam said: "by repeating similar activities, students can solve the question more accurately and smoothly". Huda had a similar view and expressed that: "similar activities will help them focus on the way of finding the correct answer and it will help them to clear up any confusion". Five SFMTs (Hind, Maha, Mariam, Malak, and Hana) expressed the same point: "giving students similar level of activities provides an opportunity for students to know their mistakes, correct them, and learn from their mistakes".

The SFMTs of the second group thought that increasing the challenge level of the activities is very useful for students' understanding. Yara reported that: "in my lesson, I prefer to start with a real-life activity and then gradually increase the challenging level of the activities, and at the end of the lesson, I will give them a high challenging level, which is targeting the smart students."

However, a few SFMTs believed in the importance of practice with variation but, at the same time, they had difficulty applying that with their students due to their students' low level of mathematics. For example, Amira said:

"I have problem with my students, they come from primary school without knowing the basic mathematical concepts (e.g., division) …so, in my lesson I waste time just to explain some rules that they should have already known and this makes it difficult for me to give them questions with different challenging levels."

Furthermore, Asma reported similarly: "I try to ask different type of questions with different challenging levels, but the problem is that just few of the students will answer these my questions while the rest will not answer because of their low mathematical level."

4.3.3.3 Approaches to learning: the role of affect and motivation

All the SFMTs agreed that their main approach to learning was to improve students' understanding. Amira said: "I really focus on my students' understanding more than anything else". Maha had similar thoughts and stated: "students' understanding is my main concern, which makes me pay more effort during the lesson". However, most of the SFMTs stressed the importance of the exams and the exam results. There were two main reasons for the emphasis given to exams. First, the teachers believed that good exam results reflected good understanding. For instance, Huda believed that: "Students' good exams result is a sign of their understanding and this is what I aim to and influence my teaching". Sara also noted that: "I really feel happy when I see my students understand the topic, but at the same time I focus more on their exams results".

The second reason is that the Department of Education assesses teachers' levels with regard to their students' exam results. Hana expressed that: "I focus on students' understanding of the concept, however, the education department assess me by the students' results so I am more concerned about their marks." Mariam also said: "I really want the students to understand the underlying concept in their answer, but at the same time I am really concerned more about their solutions' steps to get to this answer. This is because it will impact their exam results if they didn't show correct steps in their solutions, which at the end will impact my assessment."

4.3.4 Beliefs about mathematics teaching

All the SFMTs' responses suggested that they held a Platonist view toward the nature of mathematics as they believed that mathematics is a fixed knowledge—it is discovered, not invented by individuals. However, their beliefs about mathematics teaching were not consistent with their views of the nature of mathematics. Teachers' responses varied between content-focused with an emphasis on performance and content-focused with an emphasis on understanding. Table 6 provides a summary showing how the teachers viewed mathematics teaching.

Table 6: Summary of SFMTs' view of mathematics teaching

Saudi female mathematics teachers' views about mathematics teaching

Description of teachers' views			Number of Saudi female teachers believed
General goals of effective teaching	Curriculum objective	Procedural understanding	7
		Conceptual understanding	4
The roles of teachers in classroom instruction	Communication and class group discussion		15
	Student–peer interaction;		3
	Concrete examples and real-life		15

The teacher-student relationship	**Teachers is the main source of information**	**15**

4.3.4.1 General goals of effective teaching

The general goals of teaching are a key indicator of teachers' beliefs about mathematics teaching. All the SFMTs' responses emphasised the curriculum objective as a main goal of their teaching. For example, Huda noted that: "when I teach I have to focus on achieving the main concept of the lesson". Similarly, Malak said: "Students should accomplish the main objective of the lesson through practising different exercises". However, their focus varied between procedural and conceptual understanding of the content. Seven SFMTs focused on procedural understanding. Yara, for instance, stated that: "when I teach the students how to add fractions, I am just concerned about explaining the procedural steps and make sure that they know it". Likewise, Mariam said: "I try to give the students similar questions, so they have a chance to practise the solution methods".

Four SFMTs who focused on conceptual understanding did not prioritise the procedural steps. For example, Nora expressed that: "I don't mind if student skip any step if I see that she knows what she is doing and understanding it". In addition, Huda focused more on the type of questions that she used in the lesson and said: "I try to choose different levels of questions to suit all students' levels as well as to develop their understanding of the concept".

4.3.4.2 The roles of teachers in classroom instruction

All the SFMTs agreed that the main role of the teachers is to try their best to teach and facilitate mathematical topics to students. Different SFMTs emphasized different instructional skills that helped them to facilitate their teaching. Communication and class group discussion is one of the instructional skills that they stressed most. All the SFMTs believed that communication and class group discussion is a very effective instructional skill. Yara thought that: "communication and class group discussion make an interesting atmosphere for the students, and at the same time it shows me if the students have any misconception". Also, Hana believed that: "through communication students can provide different methods which can be discussed during the lesson". Three SFMTs (Maha, Soad, and Hind) mentioned the importance of student–peer interaction. For example, Soad said: "student–peer interaction is very enjoyable for the students, and some students learn more from their friends, so it is really a good way to facilitate students' learning". Further, Maha mentioned a similar point that: "some students find it difficult to solve a question individually but with other students' help, they will know how to solve it, so in my lesson I usually ask the students to work with their peers in the first question I give to them, and in the second question I let them work individually."

In addition, most of the SFMTs saw that the language for demonstrating the information to the students plays a key role in facilitating their learning. Hana expressed that: "When I teach the students, I try to use simple language to make easier for them; I also try to make it fun for them". Likewise, Asma explained that: "the language of mathematics is abstract and sometime is boring for the students so when I explain the topics I try to make it more fun and I use easy language and it works for my students, they love my demonstration".

All the SFMTs saw that the use of concrete examples and real-life questions is very important to facilitate students' learning. Amira noted that: "the use of concrete examples makes the lesson more active and it facilitates delivering information to the students". Also, Malak said: "it helps for conceptual understanding". However, some SFMTs found two obstacles in using concrete examples. The first obstacle was the availability of the resources in the school. Nora complained that: "some topics are very abstract, and it would be useful to use concrete materials, but they are not available in the school". Another SFMT stated: "We have mathematics resources room in our school; however, it is not always available as there are other teachers who use it".

Fast teaching pace is the second obstacle that sometimes prevents the SFMTs from using concrete examples. Most of the teachers mentioned that they must cover all the content specified in the curriculum which is required by the Education Department. As a result, Huda argued that: "we don't have time to use concrete examples in our demonstration so students just follow our operation and verbal explanation". Soad also claimed that the: "Education Department requires us to cover all of the content in a specific period of time and therefore using concrete examples impacts my teaching pace".

4.3.4.3 The teacher-student relationship

Even though all the teachers thought that communication and class group discussion is a very effective way to teach, they preferred to be the main source of information in the class for different reasons. Firstly, three SFMTs believed that they are the main figure in the class, and that students learn best by following their demonstration. For example, Hana clarified that: “I prefer being independent in my teaching and being the main figure in the class. I start teaching every new topic with giving and explaining an example of related topic and then I start explaining the new topic and asking students to focus and follow my verbal demonstration.” In addition, Malak mentioned that: “teachers are the expert in the class, so the main source of information is the teachers”. Lesson time is another reason, and seven SFMTs mentioned that. For instance, Soad said:

“The Education Department is always asking me to be more student-centred and I try but the problem is the constraint of time; if I let the students discover everything by themselves I will waste the lesson time and I will need more lessons to finish the topic. This will affect my teaching pace later.”

4.3.5 Summary of interview finding

The findings of the interviews showed that SFMTs held a Platonist view of the nature of mathematics. Further, mathematics was seen by the SFMTs as a set of relationships between ideas, data, and procedures. They believed that mathematical knowledge was abstract and interconnected, and they realised that in real life, mathematical knowledge had large applications. Moreover, the SFMTs concentrated on the substance content of mathematics and stressed their students’ performance of mathematics.

4.4 Observation findings

4.4.1 Introduction

This section presents observation results from lessons taught by six SFMTs as part of their teaching classes in the schools. Two observations were made from each teacher. Results are presented for each teacher in turn. A lesson overview is presented for each observation, followed by a consideration of the teacher’s mathematical knowledge as demonstrated in the lesson using the Knowledge Quartet (Rowland & Turner, 2007). With reference to the ‘Foundation’, ‘Connection’ and the ‘Contingency’ dimensions, the codes proposed in each dimension (see table 5) could be used to describe all the situations that were significant in understanding SFMTs teaching. Ann Thwaite's methods were used to analyse the lessons (as cited in Warburton, 2015). As indicated in step three (Section: 3.7 Data Analysis), the more notable episodes from the lesson are discussed with Knowledge Quartet codes; they are indicated by single quotation marks and italicised (e.g. '*concentration on procedures*').

Table 7: the KQ coding elements

Dimensions' codes	
FOUNDATION	**identifying errors**
	concentration on procedures
	use of terminology
	overt subject knowledge
	Theoretical underpinning of pedagogy
	Awareness of purpose
	Adherence of textbook
TRANSFORMATION	**Teacher demonstration**
	Use of instructional materials
	Choice of representation
	Choice of examples
CONNECTION	**making connections between procedures**
	making connections between concepts
	anticipation of complexity
	decisions about sequencing
	recognition of conceptual appropriateness
CONTINGENCY	**responding to children's ideas (use of opportunities)**
	deviation from lesson agenda
	teacher insight
	responding to the (un)availability of tools and resources

4.4.2 Case one (Mariam)

4.4.2.1 Observation one

Lesson overview:

The objective of a Year 9 lesson was to multiply a monomial by polynomial. Multiplying monomials is a foundational skill for being able to multiply binomials and polynomials more generally, and the teacher began the lesson by writing an example of a previous concept on the board about multiplication of monomials (e.g. $2x^2 \times 5x^{10} = 10x^{12}$). One of the students answered the question and the teacher repeated what the student said and explained further to the whole class. Then the teacher continued giving examples to the students to review some previous related concepts (dividing monomials($15x^2y^3 \div 3xy = 5\,x^{(2-1)}y^{(3-1)} = 5x^1y^2$), monomials to the zero power($y^0 = 1$), dividing monomials with negative exponents ($x^5 \div y^{(-7)} = y^7\,x^5$). Students were then asked to explain the meaning of polynomial and to give an example of it. Following this, the teacher asked the students to explain how they knew the degree of a polynomial and to describe the two methods to add and subtract polynomials (the vertical method and horizontal method).

The teacher reminded the students how to multiply the polynomial and asked them to provide an example of multiplying polynomial. Following this, she asked students to multiply a monomial by a polynomial. While discussing the answer, Mariam focused on using the distributive property as a key to multiply a monomial by a polynomial. The rest of the lesson followed this format, with other examples provided to the students to answer in groups.

Knowledge Quartet Analysis:

Throughout this lesson, there was evidence of Mariam's '*concentration on procedure*' as identified by the Knowledge Quartet framework. Mariam began the lesson by reviewing some procedures of previous concepts:

Mariam : Can you solve this example of multiplying monomial $2x^2 \times 5x^{10}$?

Student: $2\times5 = 10$ and $x^2 \times x^{10} = x^{12}$, so the output becomes $10x^{12}$.

Mariam : So, we multiply the coefficients and add the exponents. And how can we divide monomial?

Students: We subtract the exponents for the division.

Mariam : Ok, who can divide this monomial ($15x^2y^3 \div 3xy$)?

Student: We divide the coefficient $\frac{15}{3} = 5$

(15/3) = 5

Then we write the variable with subtracting the exponents.

Mariam : $x^{(2-1)}$and $y^{(3-1)} = 5xy^2$.

Here, Mariam reminded students about the procedures and the rules of multiplying monomials and how to apply them in order to start explaining the procedure of multiplying a monomial by a polynomial. After reviewing some procedures of previous concepts, Mariam demonstrated further evidence of '*concentrating on procedures*' as Mariam asked "Do you remember how to solve an equation? Today's lesson we will follow the same procedures". Mariam built on the known procedure for solving equations to teach the procedure for multiplying monomials by polynomials.

Furthermore, during this lesson, there was an '*overt display of Mariam's subject knowledge*' about real-life contextual examples of multiplying monomials by polynomials. This was clear through Mariam '*choice(s) of examples*'; she chose to start the new topic with a challenging example that related to a real life context. The question was: "A sports club wants to build a special sports hall the width to be more than three times its length by three metres. To know its area for covering with a special carpet for exercise, we multiply the width of the hall by its length $\left(z(3z+3)\right)$." Then Mariam started discussing the question with the students:

Mariam: What is the shape of the sports hall?

Students: A rectangle.

Mariam: How do you know?

Students: Because its length is longer than its width.

Mariam: Alright, imagine a hall in a rectangular shape with a length and width (z) and the length of this hall is three times bigger than the width and three meters more. Assuming the width is five m, what is the length of this hall.

Student: 18

Mariam: If the width is four m, the length becomes?

Student: The length becomes 15.

Mariam: Ok the hall is rectangular in shape and I want to know its area, so how can I know that?

Student: I find the area of the rectangle which is multiply the length by the width.

Mariam: Who can apply this formula in this question?

Student: $(3z + 3)z$.

Mariam: is it right if I write it in this way $z(3z + 3)$?

Student: Yes, because multiplication is commutative.

Mariam: Does (z) represent a monomial or a polynomial? And what do (3z +3) represent?

Students: (z) is monomial and $(3z + 3)$

(3z +3) is polynomial.

Mariam: So that means we multiply a monomial by a polynomial which is what today's lesson is all about.

Mariam: If we assume that the width is 6 how we can solve the question?

Students: $(3\times 6 + 3)6$; first we multiply $3\times 6 = 18$, it will be $(18 + 3)6$, and then we multiply 6 by each number $(6\times 18 + 6\times 3)$, it will become $(108 + 18) = 126\ m^2$.

Mariam: So we found the answer by using the distributive property.

Through the discussion, Mariam reminded the students of an important mathematics principle (the distributive property) when multiplying a monomial by a polynomial. This rule governs the expansion of parentheses and is important to understand in order to simplify expressions. She began the lesson with this example to encourage students to use their problem-solving skills with an example of multiplying monomials and polynomials that they could relate to. The underlying thinking in Mariam's decision was directly related to her own subject knowledge. Also, she demonstrated her knowledge of *'connection between [the]* *concept[s]'* when she asked the students "what is the area of the sports hall?".

In addition, Mariam demonstrated *'choice of representation'* and '*decisions about sequencing*' during the main body of the lesson as she followed the textbook structure by starting the lesson with a real life activity followed by other activities of multiplying a monomial by a polynomial and finally, a high-level cognitive activity which is listed in the textbook in order to highlight important and fundamental ideas to understand and to master the multiplication procedure. Mariam understood the underlying concept of multiplying a monomial by a polynomial and how to simplify the final expressions. This demonstrated a sound knowledge of procedure as she fully explained the procedure to the students. Moreover, it revealed that her teacher's beliefs about teaching mathematics and 'Theoretical Underpinning' of her teaching were emphasising the procedural knowledge and understanding.

4.4.2.2 Observation two

Lesson overview:

The lesson was a completion of a previous Year 9 lesson which was on multiplying polynomials. The lesson focused on multiplying polynomials with a various number of terms and a quadratic statement. The teacher began the lesson in the same way as the previous lesson, by reviewing all previous related topics with the students. She wrote on the board the main words in the lesson (Distributive property - Quadratic statement). For the whole lesson, students worked on two different questions about multiplying polynomials; the first question was completed with the teacher and the second question in small groups.

Knowledge Quartet Analysis:

Mariam '*concentrates on procedures*' when teaching the students how to multiply Polynomials. She focused on reminding the students of the properties, skills and ideas that they use when multiplying polynomials. This is demonstrated by the following exchange:

Mariam : What type of equations have we studied last week?

Students: Equations involving polynomials.

Mariam : How many ways do we have of multiplying polynomials?

Students: Horizontal and vertical.

Mariam : We said when we have binomials what is the best way to do it?

Students: Distributive property.

Mariam : How?

Students: Multiplying the first two terms and the sides.

Mariam then went on to explain to the students that after multiplying the polynomials, we then simplify the product by combining the like terms and evaluating (adding or subtracting) them (‘*concentrating on procedures’*). Mariam’s understanding of the concepts involved in the lesson is demonstrated through ‘*choice of examples*’. One aspect of mathematical knowledge for teaching is a teacher’s ability to select or create examples with which his/her students will interact (Ball et al., 2008; Rowland et al., 2009). She discussed with the students an activity that allowed students to see the multiplication of two polynomials using an area of rectangle with authentic context as well as matching the purpose of the lesson; it was a useful example since it relates to a ‘real-life’ object:

A rectangular swimming pool is 7 m long and 5 m wide. The pool is surrounded by path that is x m wide. Write a phrase representing the surface area of the pool and the path together.

Mariam : What is the area of the pool?

Students: The area of the rectangle is multiplying the length by the width, which means $5\times 7 = 35$.

Mariam : What about the pool with the path? Since the path is regular on all sides of the pool, the length of the rectangle representing the pool and the path is greater than the length of the pool $2x$, as well as the width, so the length can be represented $2x + 7$ and the width can be represented $2x + 5$.Then, the area of a rectangle will be (length × width) = $2x + 7)\times (2\mathrm{x} + 5)$. So, what do we have now?

Students: A polynomial multiplies a polynomial.

Mariam : How can we solve it?

Students: By using distributive property.

Through discussing this question, Mariam focused on high cognitive demand as she encouraged her students to focus on the use of procedures in order to dig deeper into their understanding of mathematical concepts and ideas. Although these activities can be solved by utilizing procedures, these procedures cannot be followed or applied mindlessly (Stein, Smith, Henningsen, & Silver, 2009). Also, in this lesson Mariam demonstrated '*use of terminology*' as she concentrated on the students' responses to the question:

Mariam : What do we call this expression (4^2)?

Students: A quadratic expression.

Mariam: If it is to the power of 3?

Students: A cubic expression.

Mariam showed her concerns about clarity and used more precise terminology when discussing different expressions and how to write them. To sum up, to understand multiplying polynomials, one must first understand multiplying monomials and binomials, and know the rules of index law. Mariam reviewed in sequence these concepts before introducing the new concept for the students, which demonstrated her good foundation knowledge of mathematics as a discipline ('*overt display of subject knowledge'*).

4.4.2.3 A summary of Mariam's mathematical knowledge for teaching

The analysis using the Knowledge Quartet framework indicates that Mariam was a teacher who showed a confidence in her subject matter knowledge and pedagogical content knowledge. Her main resource for her activities was the textbook. She gave the impression that her 'Theoretical Underpinning' belief reflected the concept of problem solving beliefs. This was clear from her emphasis on conceptual understanding of the concept through her choices and sequences of challenging levels of examples and activities. However, her own type (Type 1) of cognitive content knowledge (as a result of the written test) influenced the way she explained these activities to her students which made her explanation more procedural.

4.4.3 Case two (Malak)

4.4.3.1 Observation one

Lesson overview:

The lesson was on multiplying a monomial by a polynomial in Year 9. The teacher started the lesson by verbally reviewing previous concepts and their rules (multiplying and dividing monomials, monomials to the zero power, adding and subtracting polynomials). Malak then explained the new concept by writing on the board an example of multiplying a monomial by a polynomial: $x^2\ (5x+3)=5x^3+3x^2$. This was followed by a further example to clarify the distributive property. After that, she showed students how to combine like terms. Students then worked individually and in groups on problems in their textbooks and the teacher marked their answers. The same procedure was repeated until the students were confident with the methods they had learned.

Knowledge Quartet Analysis:

In this lesson, Malak made explicit '*connections between procedures*' for multiplying a monomial by a polynomial. She started the lesson by revising and repeating all previous rules and procedures. Malak began by asking: "how we multiply like terms, and what do we do with the exponents… what about division, what do we do?" She further demonstrated focusing on '*connecting between procedures*' through the first example of the lesson $3(5x^2+2x-4)-x(7x^2+2x+3)$. Within this example, Malak made connections between previous rules and the new rule. She encouraged her students to make connections between related rules of monomials and polynomials. She said to the students: "I deliberately chose this example because it combines the three operations (addition – subtraction – multiplication)".

Malak helped the students to understand the procedure of multiplying a monomial by a polynomial through '*concentrat[ing] on procedures*'. She gave instructions such as: "today's lesson is about multiplying a monomial by a polynomial, and the way of doing that is by multiplying a monomial by what is inside the brackets either in vertical way or horizontal way… and we apply distributive property." Another example of '*concentrat[ion] on* *procedures*' was after she showed the students a few examples of multiplying a monomial by a polynomial, she said: "after we multiply the monomial by the polynomial by using distributive property, we come to the step of collecting similar terms, which we studied yesterday."

This instruction was procedural rather than conceptual as Malak focused on the mechanical rules and the steps of doing the multiplication to reach an answer. Malak '*choice(s) of examples*' within this lesson was very basic. Four similar examples were used to demonstrate the procedures for multiplying a monomial by a polynomial. The way in which she deliberately presented the new concept in this indicated that she lacked '*theoretical underpinning of pedagogy'* within her planning and subsequent teaching. Her choice of the examples and the strategies that she implemented were underpinned by her beliefs that the teacher is the authority figure and source of knowledge in the classroom.

4.4.3.2 Observation two

Lesson overview:

This Year 9 lesson focused on multiplying polynomials. Malak started the lesson by asking one of the students to write the answer of the previous lesson's homework, which was about multiplying a monomial by a polynomial. Then the teacher read to the students the main vocabulary of the lesson from the textbook. After that she asked the students to answer one of the examples from the textbook with her assistance and by focusing on the distributive property and combining like terms. Following this, Malak provided additional problems to solve individually or in groups. By the end of the lesson, Malak used the Frayer model strategy (Brunn, 2002)(i.e., Malak mentioned that to her students) as she started asking the students about the polynomial definition and the main properties of it, also asking the students to provide examples of polynomials and counter examples. Finally, she asked the students to answer one of the high-level demand examples of the topic provided in the textbook.

Knowledge Quartet Analysis:

Throughout this lesson, Malak '*concentrated mainly on procedures*' when teaching the students, the method for multiplying polynomials. She explained:

$(3m + 4)(m + 5)$.

The solution for this would be to distribute each term to a polynomial:

$$(3m \times m) + (3m \times 5) + (4m) + (5 \times 4) = 3m^2 + 15m + 4m + 20$$

Then we add the like terms, making $3m^2 + 19m + 20$.

Here, Malak described the procedure and explained the steps for multiplying polynomials. She then instructed the students to follow the same procedure for the next example. Although this concept can be represented using an area model and connected to real world situations, the lesson focus remained mainly procedural. Her intention was to demonstrate the steps of the multiplication procedure and she instructed students to mimic her using an analogous procedure for similar problems. So, this lesson had a good evidence of '*Teacher demonstration of a procedure*'.

Malak later tried to remind the students of the definition and the properties of polynomial thereby evidencing the Frayer strategy (Brunn, 2002)(i.e., Malak mentioned that to her students) of asking different questions of the students verbally . For example:

How do we define polynomial?

If the variable has a negative exponent would it be a polynomial?

Can you give me an example of a polynomial?

Can you give me a counter example of polynomial?

By using Frayer strategies (Brunn, 2002), the information should be placed on a chart that is divided into four sections to provide a visual representation for students. The strategy prompts students to understand words within the larger context; it also helps to activate prior knowledge of a topic and builds connections. However, Malak did not use the strategy properly to accomplish its purpose, which meant that the teacher lacked '*awareness of purpose*' of what the students needed to know or practice in order to understand the concept of multiplying polynomials.

Malak's '*choices of the examples*' throughout the lesson illustrated her focus on the procedural knowledge. Even when she asked the students about one higher-order demand example that was suggested in the book, she chose a procedural example: "write binary and trinomial terms each containing one variable, then write the product of their multiplication."

Malak's interpretation of the textbook and choice of the examples showed that the '*theoretical underpinning*' of her teaching was more focused on procedural understanding than conceptual understanding.

4.4.3.3 A summary of Malak's mathematical knowledge for teaching

The analysis of Malak's teaching with reference to the KQ framework indicates that Malak was a teacher who showed evidence of weakness in her subject matter knowledge regarding the concepts she taught. This was clear from her interpretation of the textbook and choice of the examples, as she chose activities that focused on simply applying rules and facts of the concept. She repeated similar activities that focused on applying the same rules and facts. This aligned with the result of the written test as her own type of cognitive content knowledge was Type 1. Further, her way of teaching was an indication of her instrumentalist '*theoretical underpinning*' belief of teaching as she was focusing on transforming the procedural knowledge to her students and controlling the class.

4.4.4 Case three (Yara)

4.4.4.1 Observation one

Lesson overview:

This Year 8 lesson objective was to find the volume of a pyramid. Yara started the lesson by revising students' prior knowledge through some questions. She then asked each student to write everything they knew about a pyramid. At the same time, Yara continued asking the students questions about a pyramid. After that, she addressed the intended objective of the lesson, to find the volume of a pyramid. She asked the students to work in groups to answer the given activities from the textbook to discover the formula for calculating the volume of a pyramid. However, she did not let students finish these activities. Then, she started explaining the formula and how to calculate the volume of a pyramid. Yara continued to give the students different examples to work on to make sure that they knew how to apply the formula and find the volume of the pyramid.

Knowledge Quartet Analysis:

Throughout this lesson, there was clear evidence that Yara was '*concentrating on procedures*'. She began the lesson by asking about the previous formulas of some shapes and how to find them. Then she started explaining the new concept by telling the students the formula of the volume of a pyramid straight away as she said: "the volume of a pyramid is the same as the volume of a prism but we add 1/3 to the formula".

Here also, Yara showed lack of '*subject knowledge*' as she didn't explain to the students the relationship between the volume of a pyramid and a prism and the reason behind adding 1/3 to the formula. She just focused on telling the students what the procedure and formula were. Furthermore, this lesson showed some confusion in students' understanding, which is clearly related to the bad '*choice of representations*'.

After Yara had told the students the formula, she asked them to solve a problem before making sure that they knew how to apply it and find the volume. The students had not been shown appropriate examples for finding the volume of a pyramid and this could have confused the learners. She started the lesson by asking the students to answer the chosen example and just focused on how the students got the right answers as well as used the correct formula. She said when the students answered the question:

Yara: Well done, girls... but you must write the correct formula. It is necessary that the first thing you do is to write the formula for any question you solve.

This lesson demonstrated Yara's '*theoretical underpinning*' of her teaching that she had a strong belief about using the traditional teaching style. She concentrated more on memorising the rules and formulae than on applying them.

4.4.4.2 Observation two

Lesson overview:

This Year 8 lesson focussed on finding the volumes of cones. Yara started the lessons by reviewing some previous rules and formulas. She began the new lesson by showing the students a video demonstrating the relationship between a cone and a cylinder to find the volume of the cone. Following this, Yara gave the students a few activities to work on in groups, and then she asked one of the groups who had a correct answer to write the steps of their answer on the board, to allow all of the other groups check their answers. By the end of the lesson she asked the students to work on a higher-level demand activity individually.

Knowledge Quartet Analysis:

In this lesson, Yara made explicit '*connections between procedures*' for finding the volume of a cone. Although she showed a video representing the relationship between a cone and a cylinder and demonstrated her understanding of the underlying reasons why the methods for finding volumes of cone and cylinders are related, the teacher just concentrated on explaining what the formula of a cone was. Yara said: "the volume of a cone is the same as the volume of a cylinder but we add 1/3, so it will be (V= 1/3 π r2 h)", and she wrote it on the board so the students used the formula while they were working on other examples. Yara's '*choice(s) of examples*' within this lesson focused on applying the steps to find the volume of a cone. Three examples were chosen for the students to find the volumes of cones with different sizes; this required the students to simply apply the formula and follow the steps to complete the calculations.

At the end of the lesson, Yara asked the students a higher-level demand task selected from the textbook to answer individually. The task was about which has a greater influence on the volume of the cone: double its radius or double its height. Justify your answer. One of the students answered: "its radius has more impact on the volume of a cone because it is the base of the cone". She didn't respond to the student's ideas but instead ignored her answer and continued explaining to the students that: "the most influence is the radius, because when the radius is doubled, the volume is doubled to four times, but when the height is doubled, the volume doubles to two times". Her unwillingness to *'respond to students' ideas'* or answer could be attributed to her narrow understanding or lack of confidence. In addition, when the students read the higher-level demand question and discussed it with the teacher, her lack of response to the students reflected a bad example of '*teacher insight*' which could have had a great impact on students' confidence in answering challenging questions. This is evidenced in the following classroom interaction:

Students: There are no numbers to use in order to find the answer.

Yara: This question is just for clever students.

4.4.4.3 A summary of Yara's mathematical knowledge for teaching

The results of applying the KQ on Yara's lessons showed her weakness in the subject matter knowledge that she taught. This reflected her own Type 1 cognitive content knowledge (as a result of the written test). This was evident from her explanation during the lessons, as her main focus was on memorising the steps and procedures of answering the activities without making connection between the underlying concepts. Also, her unwillingness to respond to students' questions showed her weakness regarding content. In addition, from the above examples, it is clear that Yara's 'theoretical underpinning of pedagogy' was more procedural, as she concentrated more on procedural tasks than conceptual activities. There was also evidence of her belief that the teacher is the source of knowledge and that students are empty vessels to be filled up by the teacher.

4.4.5 Case four (Maha)

4.4.5.1 Observation one

Lesson overview:

The focus of this Year 8 lesson was to find the volume of the pyramid and the cone. Maha began the lesson by writing the important vocabulary of the lesson on the board, and then started reviewing the last topics that they studied in the same unit. Afterwards, she moved to the main section of the lesson and asked the students about the definition of pyramid and cone while reminding the students of related topics that they studied in previous years. Next, Maha asked the students to do an activity from the textbook. She gave them some papers to make two different shapes (pyramid - a quadratic prism) then asked them to find their surface areas. After they found the surface areas, she further explained that they both had same surface areas as they had congruent bases. Then, Maha did an activity by herself to show the students how to discover the formula of the volume of a pyramid.

Her activity was to fill the pyramid with sand and wipe the top with a ruler to flatten the surface, then pour the sand into the cube, and repeat the process until the cube was full. After that she asked the students, how many times did we need to fill the prism. Then, she asked, what is the ratio of the pyramid to the prism? After the students had answered that the ratio is 1/3, she then asked, what was the formula of the volume of a pyramid? Then Maha applied the formula with an activity from the textbook and asked the students to work in groups to solve additional problems.

Knowledge Quartet Analysis:

In this lesson, Maha made explicit '*connections between procedures*' for finding the volume and surface areas of pyramids and prisms. These procedures were written on the white board as a method to follow. She demonstrated her deep understanding of the underlying reasons why the methods for finding volumes of pyramid and prism are related. Yet, she just procedurally explained: "The volume of pyramid formula is 1/3 the volume of prism and that is 1/3 the area of its base times its height; just one thing could change, which is the area especially quadripartite area."

Maha used an appropriate '*choice(s) of examples*' and made appropriate '*decisions about sequencing*' in this lesson. Firstly, she started the new topic by asking the students to make a paper model of two shapes (pyramid - a quad prism). So, they could simply see the relationship in terms of volume and area between prisms and pyramids as a prism and pyramid have the same base and height. Then, Maha did a second discovery activity for the whole class to discover the relationship between the volumes of prisms and pyramids.

Through this activity, the students derived their own formula and deepened their understanding of the volume of a pyramid. After the students discovered the volume of a pyramid, Maha gave them another example to apply the formula to confirm their understanding.

Maha's '*choice of representation*', the paper model of a pyramid and sand model activity, was highly appropriate for physically demonstrating the relationship, especially when she explained that when the bases' areas and heights are congruent, then the volume of the pyramid is 1/3 the volume of the prism; the students could actually visualise it.

With regard to Maha's '*theoretical underpinning of pedagogy*', her theoretical underpinning beliefs were demonstrated in the planning and delivery of the lesson (e.g., her choice of discovery examples); and through highlighting important fundamental ideas that cut across the topic. Maha reviewed with the students the important topics and formulas that are related to the new lesson. For example, she stated: "you studied the same unit (area and volume of shapes) in the Grade 6, but they were simplified as you just learnt about the volume of prism, after that in Grade 7 you studied it more broadly as you learnt about the volume of cylinder."

Here, Maha reminded the students of the previous related topics and then she asked them about a definition of a pyramid; her definition was descriptive as she just described the shape: "the pyramid has only one base and its sides are triangles". To conclude, Maha's teaching in this lesson was an example of in-depth theoretical subject knowledge underpinning her planning and subsequent teaching. She focused on her content and procedural knowledge as was clear from her choice of examples and way of representations. This showed her approach to how students best learn mathematics, especially when she tried to enhance students' conceptual understanding of the topic but at the same time she also focused on performing the procedures.

4.4.5.2 Observation two

Lesson overview:

The lesson was about statistics and analysis in Year 7; the main objective of the lesson was understanding the concept of statistics and the concept of point representation. Maha began the lesson by reviewing all previous related topics with the students. She wrote on the board the main vocabulary in the lesson (statistics - cluster – range). Then, she gave the students an activity and drew some pictures (i.e., plants with different heights) on the board to explain the main points of the lesson. After that, Maha asked the students to work in groups and answer one of the activities.

Knowledge Quartet Analysis:

Within this lesson, Maha '*concentrated on procedures*' when teaching the students how to analyse and represent the data. This is demonstrated by the following exchange:

Maha : Which is the smallest plant on the board?

Student: 3 cm

Maha : Ok, the largest plant?

Student: 15 cm.

Maha : Ok, what are the varied and frequent numbers?

Student: 4 and 5.

Maha : Do you remember how to find the range?

Students: No answers.

Maha : We subtract the largest value from the smallest value. So, how can we find the range of the plants?

Student: 15-3 = 12 cm

Maha : What is the farthest number from them?

Students: 15.

Maha : 15 is the farthest number which we call it extreme value.

Here, the procedure of finding the range and the extreme value was the focus and Maha offered no explanation about the meaning of this. Further evidence of Maha '*concentrating on procedures*' was observed in the next problem where she explained another example to show the students how to do the point representation of the data. She said: "there are a group of numbers in a table and it is asked to write them on a number line and then represent the points". She started to explain step by step. First, she drew the number line; then she asked, "what is the smallest number and the biggest number". When she got the answers from the students, she demonstrated the procedures for representing the points.

4.4.5.3 A summary of Maha's mathematical knowledge for teaching

The analysis using the Knowledge Quartet framework indicates that Maha was a teacher who showed a confidence in her subject matter knowledge and pedagogical content knowledge. This was clear from her attempt to develop students' conceptual understanding of the topic through her choices and sequencing of examples as well as her instructional material. However, her main focus was on students performing the procedures and following the steps in answering the activities. Thus, this illustrated that although Maha showed a confidence in her knowledge, she also showed her lack of understanding of the foundational knowledge as she followed a procedural approach without a conceptual approach. This indicates her beliefs and approach toward teaching and learning mathematics as she focused on enhancing students' understanding of the content, however, her way of teaching was more procedural which could be related to her type of content knowledge in this topic.

4.4.6 Case five (Huda)

4.4.6.1 Observation one

Lesson overview:

The lesson was about statistics and analysis in Year 7. The focus of the lesson was on display and analysis of data using point representation. At the start of this lesson, students had to recall and write some information about the statistics and probability (e.g., statistics definition, the importance of statistics in real life and some examples of it). Analysing data using point representation was then introduced through an example of some data of different shoes prices. Data analysis was then demonstrated by Huda writing on the board and discussing with the students. After that, for the rest of the class, Huda asked the students to work in groups to analyse another example and then discussed it with them.

Knowledge Quartet Analysis:

This lesson demonstrated Huda's foundation knowledge of the use, collection, display, and interpretation of data in real life situations. This knowledge was shown through her discussion with the students about the importance of statistics. She asked students: "I want you to think why these data are important; why would I want to use this data? What would this data be good for? So I've just got some numbers, how's it going to be useful to me?" Here, Huda tried to make the students think about the importance of doing statistics and the reason behind using them and how statistics can be used in real life before they started calculating anything (e.g., mean, median and mode). Following this, Huda explained to the students how to analyse the data by using point representation and calculation (mean, median and mode). Then, she spent the rest of the lesson concentrating on procedures.

To sum up, Huda understood the importance of data analysis in real life and did have a good understanding of these statistical methods. The overall lesson approach of teaching was both procedural and conceptual, demonstrating her '*theoretical underpinning of pedagogy*'. She focused on the content with an emphasis on understanding the procedures of this content.

4.4.6.2 Observation two

Lesson overview:

This Year 7 lesson was on data handling and the lesson objective was to describe data by using mean, median, mode. Huda started the lesson by reviewing all previous related topics. She then gave the students an activity that was suggested in the textbook to start the lesson with and she used a concrete material to demonstrate the activity for the students. She provided five cups and each cup had several pieces. She explained that the number of cups represented the number of Muhammad's tests in mathematics and the number of pieces in each cup represented the result of Muhammad in each test. She also focused on performing the procedures. She transferred the pieces between the cups so that each cup had the same number of pieces and asked: "what is the average score for the five tests? If Muhammad gets a score of 14 on a sixth test, how many pieces will be in each cup?"

So, students were asked to work in groups to solve this problem. After they finished, she explained how they got the result. Then, she further explained that the number 8 (i.e., the answer for the average score) was the centre of a set of data which we could call the central tendency measure.

After that Huda asked the students to use an iPad to find the definition of "central tendency". After discussing the definition with the students, she showed the students a video which explained the measures of central tendency (mean, median and mode) and asked them to write notes about it. After the video, she spent the rest of the lesson explaining some examples of how the students could apply the procedures to find these different central tendencies measures.

Knowledge Quartet Analysis:

Huda demonstrated *‘choice of representation’* and *‘use of instructional materials’* during the main body of the lesson. She used a sequence of instruction, moving from concrete-to-representational-to-abstract instruction. At the start of the lesson she used concrete material to demonstrate the first example of the lesson and assist the students to explore and understand the measures of central tendency. Huda instructed students:

Huda: From the presented data, can you think how can we find the average (mean)?

Students: The average is 8.

Huda: So, the procedure to find the average (mean) is by finding the sum of the data and dividing it by the number of data.

The purpose of the activity was very much to help students understand the meaning behind the values, However, Huda simply focused on calculating averages. She lacked an ‘*awareness of purpose*’ because she did not address the purpose of the task (think about the reasons behind calculating statistics). It could also be an example of a teacher who ‘*concentrates on procedures*’ as she concentrated on the procedure of finding an average (mean). This showed Huda’s emphasis on using different representations of the topic to attract students’ attention to the lesson.

This lesson demonstrated that Huda understood how to find the mean, median, mode but had not understood what circumstances or in what context the different central tendencies were more appropriate. Huda’s instructional materials may help students to just focus on the mathematical aspect (i.e., calculation) but she ignored the variability of data in statistical reasoning. Huda demonstrated a lack of her ‘*subject knowledge’* as she focused on her procedural knowledge, which affected her teaching approach in this lesson in which she focused more on students’ performance of mathematical calculations for finding mean, median and mode.

4.4.6.3 A summary of Huda's mathematical knowledge for teaching

The analysis of Huda's teaching with reference to the KQ framework indicates that Huda was a teacher who showed an evidence of weakness of her subject matter knowledge regarding the concepts she taught. This was evident from her interpretation of the textbook and choice of the examples as well as her way of teaching and explaining the main purpose of the lesson. She repeated similar activities that focused on procedurally applying the same rules and facts. This aligned with the result of the written test as her own type of cognitive content knowledge was Type 1. In addition, her style of teaching was an indication of her instrumentalist '*theoretical underpinning*' belief of teaching and learning as she was focusing on transforming the procedural knowledge to her students and controlling the class.

4.4.7 Case six (Hind)

4.4.7.1 Observation one

Lesson overview:

This Year 9 lesson objective was to add and subtract polynomials and demonstrate understanding of the rules for adding and subtracting polynomials. Hind began the lesson by reviewing all the previous lessons that the students had studied in the same unit. After that she started the new topic by explaining to the students an activity on the board to identify characteristics of polynomials, including terms, coefficients and degree. Students then worked on four different activities individually and in groups. During their work, Hind assisted them and discussed the correct answer with them on the board.

Knowledge Quartet Analysis:

Throughout this lesson there was evidence of Hind '*concentrating on procedures*'. When explaining the method for adding polynomials, she stated:

"When you add polynomials, we are simply going to add the like terms. For example, ((3apples+4oranges+5bananas) + (7apples+3bananas)); so, we simply add like types together and we can't add different types. There are two methods that you can use to add polynomials: the vertical method or horizontal method."

Here, Hind told the students what the procedure was for adding polynomials: “adding polynomials is just as simple as assigning regular numbers with no variables attached. The only difference is that now you must add based on the "like terms" in the polynomial”. Further evidence of Hind ‘*concentrating on procedures*’ was when she explained the second example of the lesson “$(2x^2 + 5x - 7) + (3 - 4x^2 + 3x)$”: “in this expression we have x^2, x and constants. We can use the vertical method or horizontal method. The first step we do is adding a variable with the same power. And then add the constants. So, the answer will be $-2x^2 + 8x - 4$.”

Hind understood the mathematical convention of adding the various coefficients of the different variables, but she did not communicate this to the students. For example, she said: “you look at the power (quantity of the exponent) of each variable in the two polynomials, if the polynomials share a variable with the power, you may combine them by adding or subtracting the coefficient in front of it. If there are single powers, they simply are included in the final polynomial as they are”. The effects of ‘*concentrating on procedures*’ is demonstrated by the confusion some pupils had when attempting to solve questions involving various powers and different signs. This was clear when Hind asked the students to work in groups to solve this expression: $(x^4 - 3x + 7) + (2x^3 + x^2 - 2x^4 - 11)$.

Group 1 answer: $(-x^4 - x + 2x^3 + 4)$.

Group 2 answer: $(-x^4 - x - 4)$.

Hind did not focus on students’ errors although it was evident that the students misunderstood the procedures. She simply wrote the correct answer on the board without any explanation of the students’ mistakes. Her intention was to demonstrate to the class how to carry out the procedure of adding polynomials, and that they, in turn, would mimic her demonstration. Thus, Hind’s reasoning for the ‘*choice of examples*’ was clear, as she emphasized procedural understanding with no apparent concern for conceptual understanding. This indicates her belief about teaching and learning mathematics *‘theoretical underpinning of pedagogy’* as her instructional decisions were more procedural.

4.4.7.2 Observation two

Lesson overview:

This Year 7 lesson objective was to teach the students how to calculate percentages in a given problem and how to find the increment or deduction in different models of percentage applications in daily life. After reviewing all the previous lessons in the same unit, Hind focussed the lesson on increment. She explained and discussed with the students how to find the increment by using two different methods. Then, she asked the students to choose one of these methods and work in groups to solve other problems. Following this, she sought explanations from the students to demonstrate their methods for solving the problems.

Knowledge Quartet Analysis:

In this lesson, Hind made explicit '*concentration on procedures*' when she described to the students how to calculate the increment by using two different methods without any explanation of the relationships between percentages, decimals, and fractions. When talking through the method for finding the increment, Hind illustrated two methods:

"The first step is we take the percentage which is 5.75 and divide it by 100, so when we divide by 100, we move the comma two places with the result 0.0575, after that I multiply the result by 400. The result becomes 23, which is the amount of the increase. Finally, we add 23 with 400 to find the current price which is 423… the second method is we add the percentage 5.75% with 100%, which means (100% + 5.75%) = 105.75%, and the next step is (105.75% / 100) then we move the comma two places, at the end multiply the result by 400".

Here, Hind described both methods to tell the students what the procedures were. Then she asked the students to describe the two methods again to make sure they understood the procedures, which showed that she was concerned more about procedures. Hind later tried to '[*make] connections between procedures*' and she described: "the first method to calculate the increment is similar to the method of finding the percentage of a number". The above statements showed that she was familiar with the procedure for finding the increment by using two different methods and chose to follow a procedural approach because of insecurity about using more conceptually oriented approaches, suggesting some weakness in her Foundation knowledge. This was also indicative of her beliefs about mathematics teaching and learning: that is mathematics is a series of rules to be learned through focusing on students' performance.

4.4.7.3 A summary of Hind's mathematical knowledge for teaching

The results of applying the KQ on Hind's lessons indicated her lack of subject matter knowledge regarding the concepts she taught. There were many evidences of that; for example, Hind taught the lessons in a procedural way as she followed a traditional style of teaching. She transformed the information to her students without explaining the purposes and the responses underling the main concept. She controlled the class and asked the students to repeat the solutions' steps without knowing why they should do it that way. This aligned with the result of the written test as her own type of cognitive content knowledge was Type 1. In addition, as mentioned previously, Hind's '*theoretical underpinning*' belief of teaching and learning was that mathematics is a sequence of rules that should be learned by practising and repeating them instrumentally.

4.4.8 Summary of observation findings

The lessons observed posed a few problems with the cognitive knowledge and beliefs of SFMTs. For example, how the SFMTs made decisions about their teaching approach, sequencing the content and work examples, using materials, how to react to unexpected answers, and what to follow up depended on a wide variety of factors. These factors were discussed in chapter 5.

4.5 Conclusion

This chapter has covered the finding of the qualitative data gathered from (written test-interviews- lessons observation). According to (Schilling & Hill, 2007) the domain of mathematical knowledge for teaching in the written test can be "distinguished by both subject matter area (e.g., number and operations, algebra) and the types of knowledge deployed by teachers" (p.79). Results of the current study indicate that most of the female Saudi teachers involved did not have type 2 (from their answers) but whether they have had the type 1 could not be answered solely by their choice of incorrect answers. However, it is generally accepted that they must have at least the minimum type 1 for teaching in middle school. All the teachers have a specialised mathematics education degree and therefore it is appropriate to assume they have a minimum of Type 1 knowledge This assumption is also supported by the lesson observations. In addition, the semi-structured interviews showed that the Platonist view of the essence of mathematics was held by all the participating female Saudi teachers. However, two categories of beliefs of mathematics teaching and learning were identified in the interviews. Finally, even though there were two categories of beliefs of mathematics teaching and learning all SFMTs stressed procedural knowledge in their teaching which was linked to their superficial procedural knowledge (Type 1) cognitive content knowledge.

CHAPTER 5: DISCUSSION

The objective of this research was to identify Mathematics Teaching Knowledge for Saudi Female Mathematics Teachers (SFMTs) in Middle School. The term Mathematics Teaching Knowledge is used inclusively to cover two aspects, namely knowledge (mathematical content knowledge – pedagogical content knowledge) and beliefs (about mathematics teaching). This chapter, therefore, reflects on key aspects of the findings in relation to the main aim of the study and the research questions guiding this study, and discusses these with reference to the literature. The four research questions that guided this study are as follows:

RQ1: What cognitive type of content knowledge do Saudi Female Mathematics Teachers in middle school have?

RQ2: What beliefs do Saudi Female Mathematics Teachers hold regarding the nature of mathematics, mathematics teaching, and mathematics learning?

RQ3: How do Saudi Female Mathematics Teachers' cognitive types of content knowledge and their beliefs impact their pedagogical decisions and quality of teaching?

RQ4: How do the culture beliefs of teachers influence their cognitive type of content knowledge, beliefs and pedagogical decisions?

Three discussion points tie together results from the three data collection methods of knowledge written tests, interviews, and lesson recordings/observations in order to address the research questions with detailed discussion. Each point of discussion contributes to knowledge or practice based on the current study. How the results relate to existing literature is also considered. The first discussion point addresses the first research question by examining Saudi middle school female mathematics teachers' cognitive type of content knowledge in a cultural context. The second discussion point addresses the second question by examining Saudi middle school female mathematics teachers' beliefs about effective teaching in a cultural context. The third discussion point addresses the third and fourth questions by examining how Saudi middle school female mathematics teachers' cognitive type of content knowledge and their beliefs impact their pedagogical decisions and quality of teaching in a cultural context. The terminology of MKT is used to organise the discussion in each section because the concept of MKT is well understood in the field of mathematics education.

5.1 Examining SFMTs' cognitive type of content knowledge in a cultural context

When considering the Saudi female mathematics teachers' cognitive type of mathematical content knowledge in the Saudi context: What cognitive type of content knowledge do Saudi middle school female mathematics teachers have?

5.1.1 Introduction

This section discusses what the results from this research suggest about the cognitive type of mathematical content knowledge the SFMTs have. Specifically, the relationship between their results on the Mathematical Knowledge for Teaching (MKT) items and their cultural approach to teaching and learning are discussed to provide initial ideas about general constructs which may be associated with the mathematics-related knowledge required for teaching. Woven into the discussion are suggestions of how this research fits with other studies presented in the literature review.

5.1.2 Discussion of the Findings

As part of the research study, 15 SFMTs answered 14 MCK questions (see Appendix 1) in the domains of number sense, operation and patterns, functions and algebra, geometry, rational numbers, and proportional reasoning. The aim was to investigate their understanding of mathematics concepts and to ascertain the most common type of cognitive content knowledge that SFMTs have in their teaching. Table 7 shows that Type 1 is the most common type in all domains.

5.1.2.1 Teachers' responses

When studying teachers' responses to the multiple-choice items, no question was answered correctly by all teachers. Interestingly, there were two questions that no teacher gave a correct answer to: they were items number 9 and 14 which were in the domain of number concepts and operations (NCOP), and rational numbers (RAT) respectively, and they are related to Knowledge of content and teaching Items.

The results from the analysis of the teachers' response to the multiple-choice items (Table 7) indicate that they primarily used Type 1 content knowledge; few teachers drew upon Type 2. The analyses can be interpreted by using cognitive types of content knowledge (Tchoshanov, 2011) linked with type of understanding (Skemp, 1976). In the following sections, different examples are used to illustrate types of teachers' cognitive content knowledge in different domains.

Table 8: Type of cognitive content knowledge that SFMTs have in their teaching

MKT item *(Ball et al., 2008)*	***Type of understanding*** *(Skemp, 1976)*	***Cognitive Type*** *(Tchoshanov, 2011)*
Items related to Number concepts and operations (items 1, 7, 8, 9, and 10)	Most of the teachers have **instrumental understanding**	**Type 1 -** knowledge of facts and procedures
Items related to Patterns, functions, and algebra (items 3, 4, and 5)	Half of the teachers have **relational understanding** and the other halve have **instrumental understanding**	**Type 1 -** knowledge of facts and procedures) and **Type 2-** knowledge of concepts and connections which is related to "Knowing why"
Items related to Geometry (items 11, 12 and 13)	Most of the teachers have **instrumental understanding**	**Type 1 -** knowledge of facts and procedures
Items related to Rational numbers (items 6 and 14)	Most of the teachers have **instrumental understanding**	**Type 1 -** knowledge of facts and procedures
Item related to Proportional reasoning (item 2)	Most of the teachers have **instrumental understanding**	**Type 1 -** knowledge of facts and procedures

5.1.2.2 Cognitive type of content knowledge

Content knowledge is one of the core components of teacher knowledge in all of the frameworks mentioned previously in Chapter 2 (see section: 2.1.2), and mainly concerns the subject matter and the organization of its structure (Shulman, 1986). In addition, teachers' understanding is part of their content knowledge, as teachers need to understand both "that something is so" and "why it is so" (Shulman, 1986, p. 9).

In the MKT framework (Ball et al., 2008), teacher knowledge is mainly described as subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Teachers' SMK responses to the test items showed that, even though written test items require specialized content knowledge (SCK) for the work of teaching, SFMTs mostly relied on their common content knowledge (CCK) to answer them, which showed their weakness on their specialized content knowledge. Yet, teachers have to draw on mathematical knowledge, comprising both making sense of each step indicated in each example, and then evaluating whether the steps may be meaningful and work for all numbers, which is not a common way of doing things for adults who do not teach (Hill, Blunk, et al., 2008). The first seven written items were entirely mathematical, not pedagogical; teachers must be able to determine and evaluate the mathematics of these options — often quickly, on the spot — to make sound educational decisions (Hill, Blunk, et al., 2008). SFMTs' difficulties in answering the items might have been due to their incompetence in understanding unconventional algorithms, and their struggle to use proper terms and definitions. This reflected their poor performance on the specialized content knowledge and their weakness in the depth of their mathematics.

The findings discussed in the previous chapter indicated a limited understanding of the underpinning conceptual knowledge in the written test, since they were much weaker in relational understanding (Skemp, 1976) and content and conceptual knowledge (Type 2). This finding aligned with the findings in the Haroun et al. (2016) study, which examined 197 Saudi teachers' mathematics teaching knowledge using Mathematical Knowledge for Teaching (MKT) measures. They found that Saudi teachers' content knowledge was weak when compared with U.S. benchmarks. Furthermore, when looking across the cases in this study of Saudi female mathematics teachers, the SFMTs' responses indicated their understanding of the underlying principles of the questions was mainly instrumental knowledge (Skemp, 1976): the related practical and procedural knowledge Type 1 (Tchoshanov, 2011). Most of the SFMTs did not get correct answers in the written test; some of them chose two answers and several of them left some questions without answers. The quantity and quality of the links between mathematical procedures and ideas forms part of the conceptual mathematical understanding which is the base of cognitive Type 2 knowledge (Tchoshanov et al., 2017). Thus, in order to find the key solution, teachers need to have conceptual understanding of the meaning of each item. The results reflected that the mathematics knowledge that the SFMTs possessed and understood was superficial and "not rich in connections" (Star, 2005, p. 207), which showed their weakness in SCK. Therefore, in terms of the quality of Type 1 cognitive knowledge, the SFMTs in this study had only mastered superficial procedural knowledge (e.g., memorization and application of essential mathematical rules, and facts). This is because "someone with only superficial knowledge of procedures likely has no recourse but to use a standard technique, which may lead to less efficient solutions or even an inability to solve unfamiliar problems" (Star, 2005, p. 409). The findings in this research echo the two previous studies. Al Nazeer (2004) and Khashan (2014) reported that a lot of Saudi male and female pre-service and in-service mathematics teachers have only mastered the basic (i.e., superficial) understanding of mathematical concepts and procedures.

In terms of the pedagogical content knowledge (PCK), SFMTs had difficulties in choosing the key answer to items that were related to teachers Knowledge of content and teaching (KCT) which could be affected by their cognitive type of content knowledge (Type 1), as well as their reflection on and understanding of these items. It is known that Type 2 teacher cognitive content knowledge "crosses boundaries" with other teacher knowledge categories, including but not limited to knowledge of the pedagogical content (e.g., KCT) (Tchoshanov, 2010). It is evident from the SFMTs' responses that, not only is the awareness of teaching content essential, but the capability to teach it effectively also matters (Tchoshanov, 2010).

In this study, it was found that the Saudi Arabia culture is similar to the Chinese culture in terms of beliefs about teaching and learning mathematics, as both of them reflect similar epistemological beliefs. Broadly, teachers' teaching approach should be shaped by the balance of their conceptual and procedural knowledge (Star & Stylianides, 2013), however, in this study, the SFMTs emphasized an operational approach to teaching mathematics which usually produced instrumental understanding (Type 1). Instrumental understanding was defined as mastery of rules without thoroughly mastering the underlying concepts (i.e., superficial procedural knowledge). Such an approach though allowed teachers to impart practical mathematics knowledge, but it did not link that knowledge to abstract concepts. The SFMTs in this study demonstrated weak and relatively shallow understanding of algorithmic procedures, typically superficial and manifested as rote learning without rich connection. Indeed, good teachers should be able to encourage and support deep learning approaches. However, the SFMTs in this study lacked this depth of knowledge in a way that likely impaired their teaching; this can be seen in Ma's analogy that "teachers' knowledge of mathematics for teaching must be like an experienced taxi driver's knowledge of a city, whereby one can get to significant places in a wide variety of ways, flexibly and adaptively" (Ma, 1999a, p. 123). The written test results, in which no problem was answered correctly by all SFMTs, reflected this shallow understanding. This shows a specific mathematical area in which the teachers could improve. It is as Ball et al. (2005) stated, "knowing mathematics for teaching demands a kind of depth and detail that goes well beyond what is needed to carry out the algorithm reliably" (p. 22).

5.2 Examining Saudi teachers' beliefs system cultural context

When considering teachers' mathematics teaching knowledge in the Saudi context, an important question considered in this research is: What beliefs do female mathematics middle school teachers hold regarding the nature of mathematics, mathematics teaching, and mathematics learning? Cultural beliefs do not exclusively determine teaching practices,however, teachers use their culture normative framework of values, principles, and goals as a guide for their teaching (Bruner, 1996). This section discusses SFMTs' beliefs regarding effective teaching, including their views of the nature of mathematics and its teaching and learning.

5.2.1 Discussion of Saudi female mathematics teachers' belief system

In the current study, all SFMSTs were experienced mathematics teachers and specialised in mathematics teaching. The findings of the interviews revealed that SFMTs held a Platonist view of the nature of mathematics, which reflected the Confucian culture. It also showed some similarities with East Asian teachers such as those from mainland China and Hong Kong SAR. The teachers from mainland China and Hong Kong SAR saw mathematics as an abstract body of knowledge as they focused on the inner, coherent structure of mathematics (Bryan et al., 2007). In this study, the SFMTs saw mathematics as a set of relationships among concepts, facts, and procedures. They believed mathematical knowledge was abstract and interconnected and they acknowledged mathematical knowledge had wide applications in real life. The findings in this study were inconsistent with those of the Ernest (1989a) study, in which teachers' views of the nature of mathematics were found to indicate teachers' mental models of teaching and learning of mathematics. In addition, the SFMTs focused on the content of mathematics in their teaching as well as emphasizing students' mathematics performance.

5.2.1.1 Themes in Saudi female mathematics teachers' belief system

Analysing SFMTs' interview findings led to the identification of two themes that cover and represent all of the SFMTs' beliefs about mathematics teaching and learning. Those themes are discussed below and demonstrate SFMTs' view of PCK including Knowledge of content and students (KCS), Knowledge of content and teaching (KCT), and Knowledge of curriculum Ball et al. (2008)

5.2.1.1.1 Approach to learning: Understanding, memorization, and rote learning

To discuss SFMTs approach towards mathematics learning, two areas are discussed, and they are (i) the nature of understanding and (ii) memorization and rote learning.

During the interviews, when expressing views about mathematics understanding, all the SFMTs agreed that understanding was of utmost important in mathematics learning. The SFMTs saw mathematics understanding as students' ability to know the mathematical concepts, gain understanding of the materials, connect abstract knowledge pieces and transfer them to a new context. As mentioned before, the SFMTs perceived mathematics as a subject with concepts, rules, facts and procedures but in their conception of 'understanding', all of these had to be memorized. They had strong belief that memorization played a crucial role in students' understanding. To them, memorization using real-life questions and concrete experiences should be emphasized as it played an essential role in facilitating mathematical understanding. The teachers from mainland China and Hong Kong shared similar belief to the SFMTs but from different perspectives and interpretations.

There are two types of memorization and they are memorization before understanding and memorization after understanding (Bryan et al., 2007). Mainland Chinese and Hong Kong teachers believe that memorization may occur before or after understanding. At this point, the SFMTs shared the same belief and stressed the importance of memorization before and/or after understanding. However, they had different views on the role of memorization in learning. All SFMTs emphasized that students should start learning with rote memorization to be able to learn a new concept, but the Chinese teachers saw memorization before understanding as a transitional step toward deep learning, especially for some low-level students and it served for final conceptual understanding (Cai & Wang, 2010).

Furthermore, it seemed that both SFMTs and Chinese teachers believed that memorization worked as a tool for understanding but they defined understanding differently. Chinese teachers saw memorization as a major step for approaching deep understanding and thus strongly disagreed with mere mechanical and rote memorization (Cai & Wang, 2010). They emphasized the need for memorizing factual knowledge but did not recognise it as "dead knowledge" even though this kind of knowledge may be hardly applied to our daily life and other mathematics topics (Bryan et al., 2007). To the Chinese teachers, this type of memorization could be classified as a 'surface-learning strategy' but it should not be equated to rote learning (Biggs, 1996). Researchers such as Marton et al. (1997) and Watkins and Biggs (1996) saw the synthesis of memorization and understanding as the reason for the Asian students' excellent academic performance (such as Chinese students) in the international mathematics comparison studies (e.g., TIMSS and PISA).

Likewise, Bryan et al., (2007) stated that teachers in Mainland China and Hong Kong SAR emphasized both ''knowing how'' and ''knowing why''. In other words, in their teaching, they aimed to facilitate students to understand the "how" and "why" in addition to "what". However, in this study, the SFMTs' responses to the interview questions showed that they mainly focused on students' performance and procedural understanding only. The SFMTs possessed limited perspectives of mathematics learning and understanding. They expressed that the best way to assess students' understanding was through examination and presentation of solution steps as they believed memorizing rules and facts was the main indicator of bringing about students' knowing "how" and "why", and their ability to solve mathematics problems. It reflected that the SFMTs have equated memorizing rules, facts and procedures with understanding; their definition of understanding was very different from the Chinese teachers.

As Biggs (1991) stated, examination was the main assessment strategy used in Confucius culture classes (Biggs, 1996) however it emphasized a low-level cognitive process. Yet, the performance of Saudi students was lower than the international average in all TIMSS results (Alanazi, 2016). The Saudi Ministry of Education stated that the low performance was due to Saudi teachers' teaching methods that relied heavily on memorization without focusing on students' conceptual understanding (Alanazi, 2016), which was also evident in the lessons observed in this research (described further in the next section). The memorization-driven teaching method in Saudi could be explained by two major factors. First, many of the Saudi teachers still hold a very traditional view of teaching. They view themselves as the only authority in classrooms and thus are reluctant to put in the effort to change their teaching method (Alanazi, 2016). Second, the historical heritage of religious education in which rote learning for memorizing the Qur'an is essential has adversely impacted teachers' conception of learning (Alanazi, 2016). This phenomenon was also evident in the lesson observation in this study (described further in section: 5.3.1).

As mentioned earlier, the SFMTs have equated memorizing rules, facts and procedures with understanding; they have unconsciously embraced rote learning in their teaching. The aim of rote learning is to spend minimal time and effort to meet the requirement of desired learning goals (Biggs, 1994). However, rote learning can be understood from two different perspectives. Rote learning can be "a mechanical way without thought of meaning" (Biggs, 1994, p. 26), and it can also be understood as a "repetitive learning which uses repetition as a means of ensuring accurate recall" (Biggs, 1994, p. 26). The difference between these two perspectives lies in the learner's intentions with respect to learning (Biggs, 1994). The SFMTs' responses in this study stressed the importance of both and they saw repetitive learning as not only important but indispensable. Similarly, the Chinese teachers emphasized the importance of repetition and believed that "repetition creates new insight" and "practice makes perfect" (Zheng, 2006b, p. 387); they saw repetitive practice not only as means for getting correct answers and high-speed arithmetic computation but also as essential to deep understanding. But the SFMTs equated repetitive practice as exercise drilling. They expressed that getting correct answers and using high-speed arithmetic computation were important means for assessing students' mathematics understanding regardless of how much deep or conceptual understanding students had already mastered. It reflected again their emphasis on rote memorization of mathematical rules and facts and equated these to understanding.

5.2.1.1.2 Approach to teaching: Goals, Characteristics of an effective teacher and teaching methods

To discuss SFMTs approach towards mathematics teaching, two areas are discussed, and they are (i) teaching goals and (ii) characteristics of an effective teaching

With respect to goals, all of the SFMTs in this study agreed that the main goal of their teaching was developing and fostering students' mathematical understanding. So, they claimed that they focused on students' understanding of the content. They believed that the teacher was the authority figure in the classroom as they had the responsibility to explain mathematical content clearly to the students. Hence, their belief was a mixture of authoritarianism and student-centeredness. This ''teacher-led, yet student-centred'' (or "learning centred") is also seen as one of the features of the Confucian culture classroom (Wong, 2006). All of the SFMTs agreed on the importance of encouraging students to answer questions using students' own methods under teachers' guidance, but at the same time SFMTs thought that students would be wasting their lesson time just on the discussion of the question. So, they preferred a didactic teaching style in which the teacher who is the authority figure transmits the requisite mathematical knowledge to the students (Scherer & Steinbring, 2006). Also, they emphasized group work activities and they tried to use this strategy throughout the lesson as they believed that students would learn more from working with their peers. This was like the Hong Kong SAR students (who share some of the same culture beliefs) as they found peer tutoring worked well in their schools (Winter, 1996).

In addition, SFMTs expressed that the new mathematics curriculum reform in Saudi Arabia endeavoured to make meaningful links between students' lives and their educational experiences. This was because the curriculum adopted a constructivist approach and emphasized critical thinking and problem-solving methods instead of relying on memorizing texts. All SFMTs felt that they needed the textbook as a guide for teaching the content in sequence and meeting the goals and objectives as suggested in the curriculum. The textbook was also a guide for SFMTs in applying the student-centred teaching methods proposed in the curriculum. In addition, the textbook is an essential regulator of the implemented curriculum and also could be considered as part of the intended curriculum (Cao, Seah, & Bishop, 2006). However, teachers themselves have major roles in successfully executing the curriculum reform (Stallings, 1998). Thus, their teaching style and their choices of examples from the textbook and how they implement them are very crucial.

However, SFMTs expressed that they have experienced some barriers that have influenced their teaching style and restrained them from innovation. The most common barrier was curricular coverage and time constraints, as these were the criteria that the Education Department used to assess their teaching. Heavy teaching load was another barrier. Thus, SFMTs found using a knowledge transmission approach enabled them to cover the curriculum content more quickly. This observation aligned with the findings of Madani and Forawi (2019) that Saudi teachers were very concerned about content coverage in specific sequence and in a specified period of time; thus it would be difficult to apply the problem solving approach that needs deep understanding of subjects. In addition, they found that teachers usually did not give enough time and effort to search and learn about new integration practices (Madani & Forawi, 2019). In contrast, the Japanese and especially Chinese teachers, who shared similar cultural beliefs with Saudi teachers, have much lighter teaching loads, allowing time to carefully prepare and source teaching materials, as well as providing time to interact with students outside the classroom (Stevenson & Stigler, 1992). Furthermore, Asian teachers do not spend much time on lecturing but instead pose interesting questions to students to work on and encourage them to create different approaches and methods to answer the questions (Stevenson & Stigler, 1992). Overall, one of SFMTs' main goals of teaching and learning mathematics was to cover the curriculum content quickly but at a superficial level of understanding that required accuracy more than deep understanding.

Another barrier is that the Education Department uses students' test scores as one of the criteria to assess teacher effectiveness. SFMTs also expressed that their only effective way to assess their students' understanding of the content was through an exam. Indeed, when teachers prioritise working through the curriculum content on time, and at the same time focusing on examination results, they spend less time on teaching the conceptual knowledge of mathematics.

In terms of characteristics of effective teaching, in this study SFMTs emphasized the importance of connecting mathematical ideas by using different strategies. Similarly, Alqahtani et al. (2016) also found that Saudi secondary mathematics and science teachers stressed, in their definition of effective teaching, the ability to use a variety of teaching strategies. In terms of students' understanding, findings of this current study revealed that all the SFMTs agreed that connecting students' real-life experiences with mathematics activities and involving concrete examples are effective ways of teaching that can facilitate mathematical understanding. However, the problem here is the interpretation of the word "understanding" as it can mean different things in different cultures and it varies from one person to another (Skemp, 1976). As a matter of fact, the findings of this research reveal that SFMTs believed that instrumental understanding is essential for effective teaching, since it was clear that they focused on developing procedural knowledge for connecting between mathematics topics. This echoes the findings from the lesson observations in the actual classroom that SFMTs emphasized procedural knowledge more than conceptual knowledge in their teaching strategies as they tended to use similar examples and taught procedures only, as well as focused on memorizing the previous knowledge (described further in next section). This finding was also in line with Al Dalan (2015) that, in an examination of 15 Saudi trainee primary mathematics teachers, he found that the teachers tended to teach procedures more than the underlying concepts.

Moreover, SFMTs believed that repeated practice could shape and develop students' understanding. They supposed that their students' skill could be improved by applying mathematics in different contexts. Similarly, "teaching through variation and repetition" is the Chinese approach of teaching and the teachers teach the same concepts while varying the related backgrounds or contexts and using different ways of presentation (Zheng, 2006b) in order to develop deeper understanding. Chinese teachers focused on three aspects of understanding: knowledge, skills, and reasons, as these are essential characteristics of mathematical understanding (Hannula, Maijala, & Pehkonen, 2004). In fact, they emphasized not just the quantity of repetitive practice but also the quality of variation (Wong, 2006), whereas SFMTs focused on two aspects, procedural knowledge and skills, and did not give attention to the underlying reason for the concepts.

Therefore, the reason behind SFMTs' approach to teaching could be related to their cognitive type of content knowledge which indirectly influenced their style and focus of teaching. Consequently, although SFMTs stressed the importance of developing students' mathematics understanding, their level of cognitive knowledge of mathematics had a great impact on their teaching: ''beliefs may be dependent on the existence or, perhaps, the absence of knowledge'' (Cooney & Wilson, 1993, p. 150). In other words, the essence of one's mathematical understanding will give rise to a particular belief as to how best to teach mathematics (Wilkins, 2008). In addition, teachers' value that teachers placed on specific content affects their teaching method (Nespor, 1987). Therefore, these ideas support the thought that beliefs can be a mediator between content knowledge and instruction (Wilkins, 2008).

5.2.2 Summary

Given that all of the SFMTs appeared to hold beliefs about the nature of mathematics that were consistent with a Platonist view (Ernest, 1989a), all of them held a belief that was a mixture of Authoritarianism and student-centredness. The results indicated that although the new Saudi curriculum reform encouraged a constructivist approach to teaching, the SFMTs preferred to teach content-focused with an emphasis on instrumental understanding (Van Zoest et al., 1994). Skemp (1976) gave three reasons for teaching instrumental knowledge and they were:

> 1-Within its own context instrumental knowledge is usually easier to understand; sometimes much easier.
>
> 2-So the rewards are more immediate and more apparent.
>
> 3-Just because less knowledge is involved, one can often get the right answer more quickly and reliably by instrumental thinking than relational (p. 5).

Even though all of these reasons could influence teachers' preferred teaching style, Skemp (1976) did not mention teachers' cognitive type of content knowledge which is a major reason that the result of this research about SFMTs has revealed. SFMTs mastered Type 1 cognitive content knowledge which had a great influence on their way of understanding the mathematical concepts they were teaching. Moreover, their perspective of the meaning of relational understanding was more about procedural knowledge, since they focused on knowing what solution methods they followed and how, but not on knowing why. Instrumental understanding requires memorizing which activity a method works for and which it does not (Skemp, 1976). This was evident in their interview responses, which also could be related to their cultural beliefs.

5.3 Examining Saudi female mathematics teachers' practice

Explaining the complexity of mathematics teaching to others can be challenging. The Knowledge Quartet framework provides a fresh lens through which to reflect on mathematics classroom practice. When considering Saudi female mathematics teachers' practice: How do Saudi middle school teachers' cognitive types of content knowledge and their beliefs impact their pedagogical decisions and quality of teaching? What is the role of SFMTs' cognitive type of content knowledge and their belief in teaching mathematics? Understanding the work of teaching, how it works, and how teachers' cognitive types of content knowledge and their beliefs impact their pedagogical decisions and quality of teaching are the main focus in this section. Examining teachers' work in this way was not about making judgements about the teachers but rather trying to explain the actions and processes that they used and how these were influenced by their cognitive content knowledge and beliefs system. This section addresses the relationship of SFMTs' cognitive content knowledge, belief system on their practice by using Knowledge quartet dimensions.

5.3.1 Discussion: the knowledge quartet and SFMTs' practice

From the perspective of the KQ, the knowledge and beliefs evidenced in mathematics teaching can be seen in four dimensions: foundation, transformation, connection and contingency. In this study, the observed lessons raised a few issues about SFMTs' cognitive content knowledge and beliefs, and some of these issues were raised in the written test and interviews as well. In fact, how the SFMTs made decisions about their teaching approach, sequencing the content and work examples, using materials, how to react to unexpected answers, and what to follow up depended on a wide variety of factors. These factors were discussed by looking at contributory codes for each dimension in the order of foundation, transformation, connection, and contingency as well as the MKT framework.

5.3.1.1 Foundation

Lesson observations raised some issues regarding SFMTs' mathematics teaching knowledge (SMK, PCK, beliefs). SFMTs' Knowledge of curriculum was demonstrated through their adherence to the textbook and awareness of its purpose as it influenced how they conducted their lessons. As mentioned in the previous section, teachers in Saudi Arabia are required by the government to use mathematics textbooks as a guide for them to cover all the mathematical topics in a specific order and a specific time. All SFMTs considered their textbooks the main resource for their lesson planning and teaching. Sometimes, they combined the textbooks with other resources from the Internet if necessary.

The new Saudi curriculum was intended to meaningfully connect students' daily lives and their learning experiences through implementing new teaching strategies that focused on students' problems solving and investigation learning strategies (Madani & Forawi, 2019). However, the findings of this research showed that it was challenging for SFMTs to follow both the new curriculum suggestions and the sequence of instruction and procedures suggested in the textbooks. A typical lesson in these textbooks had many sections. It started with the main idea box which summarised the aim of the lesson, which led to the first section 'Get Ready' where the students were required to do some thinking as a warm-up exercise. The 'Real World Example' section came next, in which two work examples were given to introduce the idea of the lesson. That was followed by the 'Check What You Know' section where the students did some exercises under teachers' supervision and then moved to the section 'Practice and Problem Solving' where the students solved some problems in a real life context either individually or in groups. The lesson ended with the 'High Thinking Skills' section where some high-level problems challenged the students (Al Dalan, 2015).

Despite this common presentation of teaching sequence suggested in the textbook, the ways that the SFMTs adapted the text in their classes differed significantly, which indicated their lack of knowledge about curriculum. The design of mathematics textbooks in Saudi Arabia is based on constructivist views of learning and teaching and the activities are designed to assist students in constructing their own understanding of different mathematical concepts. Hence, such textbooks offer multiple applicable and appropriate activities for different mathematical concepts. Some of the SFMTs in this study followed the textbook's suggestion and used more than one teaching resource/materials in their teaching of mathematical concepts to make the lesson more interesting. However, the focus of SFMTs in their teaching was mostly on procedural knowledge, which was observed in their explanations of the activities.

SFMTs understood differently the aims of the mathematics activities and the purpose behind them, and this influenced their ways of using the textbook in their teaching (Ball & Cohen, 1996). Evidence showed that in many cases SFMTs could not understand the underlying concepts of mathematics activities in the textbook, and so they could not make good use of the activities or they did not choose them. Looking at SFMTs' results on the written test, in the cognitive type of their content knowledge Type 1 appeared more than Type 2 and Type 3. Even though there were some situations where the SFMTs tried to apply Type 2 and teach conceptual knowledge, their efforts were more likely to regress into Type 1. The SFMTs' procedural knowledge featured significantly in the Transformation, Connection and Contingency dimensions of the KQ. All the SFMTs appeared to use different aspects of their transformation knowledge by using different choices of examples and different ways of representations. However, their focus was on procedural understanding and this focus was reflected in their ways of teaching. All SFMTs believed in the importance of making connections between mathematics and real-life application, and between mathematics concepts and ideas for effective teaching, and this was clear throughout observed lessons. However, SFMTs' connections focus was mainly on mathematics procedures. The ability of SFMTs to act contingently by responding to students' ideas and answers was limited to two types of actions: confirmation actions and questioning actions, which were affected by SFMTs' type of cognitive knowledge as well as their beliefs.

5.3.1.2 Transformation

All the SFMTs followed the same lesson structure. They followed traditional approaches (i.e., teacher-centred approach) in their teaching, even though some of them tried to make their lessons more constructivist. Every mathematics lesson has a definite aim of learning mathematical concepts and skills. SFMTs designed the mathematics lesson around this aim and segmented it into three distinctive parts. They started with orally recalling previous topics and related facts then proceeded to the main body of lesson and activity. The body of lesson comprised an introduction, followed by the activities. There is great variation in the amount and type of the activities in the textbook that teachers could use in their teaching. SFMTs' choice of activities is a crucial code for the Transformation dimension. Choosing good examples is key but is complicated in practice (Zaslavsky & Zodik, 2007). Different factors such as the teaching goals (SFMTs' knowledge of content and teaching) and teachers' awareness of their students' previous knowledge (SFMTs' knowledge of content and teaching) could influence their choice of activities, examples and classroom exercises (Liz et al., 2006). This may align with SFMTs' cognitive type of content knowledge as teachers' decisions about instructional activities are influenced by their knowledge of mathematics (SMK and PCK) (Walshaw, 2012). Likewise, with SFMTs' respective beliefs, which are an even stronger influential factor (Yang, Kaiser, König, & Blömeke, 2020). It is likely that while two teachers might have similar mathematical knowledge, they may teach mathematics differently because of their different beliefs about the nature of mathematics and its teaching and learning (Ernest, 1989a).

SFMTs focused on using activities in their lesson as they believed that repeated exercises and activities are an essential to "route to understanding" (Hess & Azuma, 1991), and subsequently practice is important in developing students' learning (Wong, 1998). Procedural teaching does not necessarily indicate rote learning or learning without understanding, as understanding is a continuum process (Leung, 2001). However, one of the major issues that has been revealed from the findings of this study was the quality of variation that SFMTs demonstrated in choosing activities and tasks in their teaching. As a matter of fact, SFMTs broadly seemed to share the beliefs held in East Asian cultures, that through repeated practice students gain understanding (Leung, 2006), whereas the Chinese mathematics teachers tried to deepen students' understanding of mathematical knowledge by using different examples and tasks. Yet, Chinese teachers closely linked the idea of "repetition creates new insight" with the method "teaching through variation" (Leung et al., 2006).

Indeed, practising through a systematic variation is more important than the quantity of practice (Wong, 2006). Though all SFMTs used textbook activities in their lessons, their choices of the specific activities differed. Two different approaches of choosing activities were identified among the SFMTs. The first group of SFMTs followed the same structure of activities as was suggested in the textbook with some modification to suit their class context and they made the lesson more meaningful and interesting for their students. This group showed a desire to develop students' conceptual understanding by using a variety of activities including low and high cognitive demand activities and different instructional materials that supported their teaching and made their lesson more enjoyable and attractive for their students. However, when they attempted this, it often appeared that they did not understand the purpose and intention of these activities or they just understood them superficially. The second group of SFMTs either chose the activities that matched their particular belief and teaching approach, as this group favoured memorization and procedural teaching, or they did not adapt the textbook activities appropriately and thus altered the activities in ways that meant the focus of teaching was changed (Petrou, 2010). The SFMTs in this group did not take into consideration what was suggested in the textbook and chose the activities that focused on developing students' procedural understanding only.

In general, there are four types of cognitive tasks demands. Two types are low cognitive demand tasks and they rely on memorization and procedures without connections. Two types are high cognitive demand tasks and they require procedures with connections and rich mathematics (Stein & Smith, 1998). To provide meaningful variation, textbooks suggest that teachers should sequence the questions and activities from low cognitive demand tasks (i.e., memorization and procedures without connections levels) to high cognitive demand tasks (i.e., procedures with connections, and rich mathematics levels). SFMTs had a clear teaching purpose in mind for their lessons as they had similar learning goals and presentation structures because all teachings were based on the Saudi mathematics textbook, yet they were unaware of the limitations they made with their choices of activities and this impacted their instruction greatly.

Both groups of SFMTs, discussed above, sought to include both high- and low- cognitive demand activities. Both low and high cognitive demand tasks require procedures, but high cognitive demand tasks require students to also think about the connections and relationships between concepts and to construct the underlying meaning (Stein et al., 2009). However, in this study, even the SFMTs who used different cognitive demand tasks did not achieve the desired target of learning. This was because, even when SFMTs taught high cognitive demand tasks, they functionally worked them as low cognitive demand tasks. This was due to their implementation and representation of the task as well as their communication with the students. Implementing mathematical knowledge in the classroom involves three factors: 1) choosing a challenging task and activity, 2) making a good and an interesting learning environment to assist mathematical reasoning and offering good opportunities for students to show their understanding, and 3) providing mathematical discourse that supports and encourages reasoning, justification, conjecturing, and communication in constructive ways (National Council of Teachers of Mathematics, 2007). Both groups of SFMTs tried to choose a variety of challenging activities (high and low), still, they had issues in providing a good learning environment that encouraged communication, problem solving, and reasoning. This is because making a constructive learning environment requires teachers to be equipped with a special knowledge in order to encourage evaluation, justification, and guided questions (Lampert, 2001), as well as student collaboration (Hiebert et al., 1997; National Council of Teachers of Mathematics, 2007). Thus, without focusing on these ideas, a high cognitive demand activity will convert into a low cognitive demand activity (Stein et al., 2009).

Hence, SFMTs concentrated on procedures rather than on encouraging the students' discussion. SFMTs' beliefs, perception, and knowledge regarding mathematics have a great impact on their beliefs of how it should be presented (Thompson, 1992). Even when they let their students to work collaboratively in groups or individually, the teachers did not give students much time to think because of time limits.

Choice of representation of mathematical concepts is another important code for the Transformation dimension. As previously noted, those teachers who held a Platonist view believed in the importance of linking mathematics activities with students' real lives, and in using different types of representations (concrete, visual, verbal) and technology in their lessons. The Platonist view sees that different representations could help students to understand abstract mathematical concepts. SFMTs' choices of representation play a key role in delivering mathematical concepts to the students. Representation of the content is a fundamental part of PCK (Shulman, 1986) and it supports mathematics teachers' teaching in order to make mathematics more understandable for the students (Ball, 1993; Principles, 2000).

More importantly, teachers should make clear the relationships between different types (concrete, visual, verbal) of representations in order to achieve the desired understanding (Moru, 2006). Nevertheless, some of the SFMTs had difficulty in making connections among different representations in the lesson and they relied heavily on verbal presentation. Some of the SFMTs did not pay attention to the importance of choice of representation. Instead, they emphasized verbal language with concentration on abstract symbols (symbolic representation) and without images (iconic) or action (enactive) representation (Bruner, 1974). This showed SFMTs' weakness in connecting different representations of certain mathematical concepts as well as in knowing what type of representation would support and develop students understanding. However, a large number of the SFMTs used concrete examples and technology to mediate students' understanding of mathematical concepts. Despite this, their demonstration of the lessons was mostly procedural. Their explanations and questionings of students were too superficial to facilitate the learning of conceptual knowledge.

In general, this shows SFMTs did not have sufficient knowledge: they lacked mathematical specialized content knowledge, knowledge of curriculum, and knowledge of students to vary the complexity of activities, examples, tasks, and representations to develop students' understanding and learning. The completion of the activities mostly relied on retrieving memorized information and rules or on giving the students a hint to make the task easier. The SFMTs usually explained the mathematics concepts without giving the students much time to think critically before leading them towards the answers. Next, the SFMTs engaged students to work on the same/similar types of questions in groups or in pairs, followed by a concluding plenary. Consequently, the students who practised just the procedures found it difficult to use different ways or to apply the knowledge to different contexts because they understood the knowledge (instrumentally) in a very fixed fashion (Li, 2006).

5.3.1.3 Connection

One of the codes in the Connection dimension that was notable in the SFMTs' lessons was in their decisions about an activities sequence. They showed an explicit connection between lessons as they followed the textbook lesson sequence without modification. All SFMTs started their lessons in the same way—by revisiting previous related topics and rules, as they believed the importance of memorizing before understanding. They tried to establish connections between older and newer concepts and procedures within their teaching, which reflected Confucius' saying, "Learn the new when revisiting the old" (Wong, 2006). Similar to Japanese teachers, most of the SFMTs in this study placed great emphasis on preparation for introducing new mathematical concepts and skills to make the lessons more enjoyable through connecting the content to real life examples (Kaiser, Hino, & Knipping, 2006). They had a good awareness of connections between topics; however, their awareness of the connection between lesson components was different. In this regard, the issue was the same as that discussed under the heading of Transformation: that the SFMTs tended to undermine the difference between low and high cognitive demand activities by equalizing the difficulty through hints or other actions.

Teachers should know how to guide students' thinking through the use of appropriate questioning (Hiebert et al., 1997; Lampert, 2001; National Council of Teachers of Mathematics, 2007). Indeed, the way that teachers encourage and support students in their communication and discussion is very critical (Hiebert et al, 1997, Stein et al, 2009). This relates to the key notion of scaffolding as an instructional approach, in which a teacher helps a student to understand content by progressively introducing new concepts and helping students move toward independence, and where the progression of difficulty and cognitive demand in the texts used is likely intended to promote scaffolding of student thinking. However, perhaps due to the SFMTs' own superficial understanding of the content knowledge, teachers struggled to help students effectively move through this progression or achieve the ultimate goal of greater intellectual independence. Some SFMTs tried to provide rich activities but they were unable to encourage rich mathematical discussions because their main teaching method was memorization-driven which focused on students' performance without a conceptual mathematical connection. Very little conversation took place between SFMTs and their students as it mostly consisted of one-word answers. There were insufficient connections made across the concepts and few connections to real life. They infrequently extended students' thinking by giving them critical questions and exchanging ideas with them. By contrast, Chinese teachers, who share similar cultural beliefs with SFMTs, help students to construct connections between and within mathematical concepts by using different teaching strategies and representations; this is due to Chinese teachers having more breadth, depth and flexibility in mathematics understanding (Ma, 1999a). The SFMTs in the current study did not have a breadth of knowledge as they lacked knowledge of content, curriculum, and students, which made them less knowledgeable in terms of effective scaffolding strategies.

Consequently, this showed that SFMTs' shallow understanding of mathematical content knowledge (Type 1), as evident in the written tests, may have further hindered them from choosing appropriate challenging activities for students. Also, it showed a lack of sequence instruction which encourage and support students' understanding as well as making connection between mathematical concepts.

5.3.1.4 Contingency

The fourth dimension of interest is contingency. One constituent component of this category that arose from the data and was notable across all of the lessons was teachers' responses to students' ideas, which links to SFMTs' knowledge of content and students as well as of knowledge of content and teaching. Even though a constructivist view of learning is the base for the new mathematics curriculum reform in Saudi Arabia and it emphasises students' roles and contributions inside classrooms, these SFMTs did not practice a constructivist view of learning. All SFMTs' teaching style was very mechanical and orderly, and their main focus was to cover as much content as they could in the lesson. Hence, when they posed a challenging question, they did not give students sufficient time for reflection but immediately sought a correct answer. Then, SFMTs did not use students' errors constructively, or redirect the students to think and evaluate the original question in a different way. They paid less attention to classroom discussion and dialogue between them and the students. This aligned with their responses in the interview, where they prioritized the curriculum and lesson plan.

Although Saudi Arabia and China share similar teaching cultures in some regards, their responses to students' answers and ideas differ. Chinese teachers emphasise students' reasoning, thereby getting at the reason for their errors (Schleppenbach, Flevares, Sims, & Perry, 2007). Findings in this current study showed that the SFMTs' responses to students' ideas and answers to their questions were like the Saudi mathematics teachers' responses revealed in the Al Dalan (2015) research. The responses included two types of actions. The first was confirmation actions, where the teachers confirmed the correctness of answers and then different actions could be applied after their acceptance or rejection of the answers (e.g., writing the correct answer on the board, providing explicit rejections and correcting the answer). Second, there were questioning actions in which the teachers usually asked more questions before deciding any action about the answer, but they usually posed some questions before accepting or rejecting the answer in order to examine students' understanding, to question other students, or by repeating the answers to challenge students (Al Dalan, 2015).

The SFMTs' main focus in questioning in this study was on the procedures rather than on developing conceptual understanding. This is because their responses to students' answers and ideas tended to rely on procedural aspects, and most of the questions posed by them required short answers. This finding aligned with the Alzaghibi and Salamah (2011) description of Saudi classrooms. These 'responding' moves were the lynch pins of a lesson and played an essential part in the lesson structure and sequence (Brown & Wragg, 1993). Undoubtedly, the quality of SFMTs' responses to the students, at least in part, was determined by teachers' knowledge (Rowland et al., 2005). SFMTs' confirmation actions have strong connections to pedagogical knowledge and teachers' beliefs, whereas questioning actions are generally linked to PCK (Al Dalan, 2015). Thus, both instrumental understanding and relational understanding can be activated in contingent teaching situations. However, it was clear how the SFMTs' mathematical understanding guided their responses to students' participation. The findings of this research showed that SFMTs have mainly Type 1 cognitive content knowledge and this is linked to an instrumental understanding.

Overall, this study has found that SFMTs' actions in responding to students' answers and their ways of handling their mathematical errors were clearly related to both their knowledge and their beliefs.

D 4.5.4 Synthesis of discussion

Discussing the findings from Research Question (RQ) 1 with respect to cognitive content knowledge, with respect to beliefs per RQ2, and examining SFMTs' practice in RQ3, provides evidence of the impact that these components have on Saudi female mathematics teachers' actual teaching in their classrooms. Accordingly, to bring together the key ideas, this final section synthesises the discussed findings from all RQs to show what this research contributes to "Identifying Mathematics Teaching Knowledge for Saudi Female Mathematics Teachers in Middle School".

This research took a holistic view of Saudi female mathematics teachers' practice and determined what aspects of MKT appeared in their actual teaching. Broadly, based on an in-depth examination of experienced teachers in Saudi Arabia, the study's results indicate that the teaching of middle school mathematics in Saudi Arabia is in a transitional period. Teachers are seeking to implement a new constructivist curriculum but their teaching practice still contains significant influences from traditional teaching styles. They are not well-enough prepared to use the new curriculum approaches that emphasise both student-centred learning and understanding concepts; instead they rely on rote learning. Although the new mathematics curriculum approach was introduced in 2008 (Alsaleh, 2019), Saudi teachers may require more training and support to be able to appropriately apply a constructivist approach to learning. One significant reason for this is that the teachers' own knowledge is essentially insufficient, meaning that they struggle to impart knowledge of subject matter that they do not comprehensively understand.

In this study, two groups of SFMTs were identified: one group tried to apply the new curriculum approach by implementing high cognitive demand tasks and using different teaching strategies and the other group did not pay attention to or apply the new approach in their teaching but kept to their traditional teaching styles. Interestingly, both groups followed memorization-driven teaching methods for different reasons. The first reason was teachers' knowledge: there is a strong relationship between teachers' knowing (knowledge), how they know this knowledge, and how they can actually apply and do it by transforming it in their teaching (Hill et al., 2008). Findings from the written test showed that SFMTs have Type1 (knowledge of facts and procedures) cognitive content knowledge in which they lack in-depth understanding. Clearly, this reason did influence the first group as it limited their teaching methods. The second reason was teachers' beliefs: teachers' beliefs about teaching knowledge could impact their teaching practices (Sinatra & Kardash, 2004) as well as their behaviour (Ajzen & Fishbein, 1980; Cooney, 2001; Pajares, 1992). The third reason is teachers' cultural habits, as Saudi Arabia's culture is influenced by Confucian culture. In teaching and learning, the Confucian tradition stresses the authority of teachers and students' hard work (Xie & Cai, 2018). This aligns with the Saudi teachers' expressed beliefs about their role and students' roles in the classroom. These unseen cultural beliefs have a great influence on teachers' habits and their norms of mathematics teaching (Hodgen, 2011) and what they see as teachers' main goal to be achieved in the practice.

To sum up, improved curriculum resources are not a key solution for SFMTs. Rather, SFMTs' knowledge (SMK and PCK) and beliefs are significant, and changing the curriculum without addressing SFMTs' beliefs or improving their knowledge will fail to achieve effective educational reform (Petrou & Goulding, 2011). SFMTs' knowledge and teacher beliefs act together to form teachers' choices, decisions, and actions in classrooms (Charalambous, 2015).

Overall, the findings of this study suggest that teachers' knowledge of facts and procedures has the potential to be a good predictor of teachers' memorization-driven teaching methods, which might limit students' mathematics achievement. This is also consistent with teachers' beliefs toward mathematical teaching and learning, which were content focused with an emphasis on performance and procedural understanding. Generally, although there were some key cultural parallels between Saudi teachers and Chinese teachers, the differences were significant in that Chinese teachers could leverage similar methods to achieve a much higher degree of success through subtle but extremely important differences.

CHAPTER 6 CONCLUSION

This chapter provides an overview of this current study before addressing possible limitations of the study. Finally, some recommendations for further research are suggested.

6.1 Overview of the thesis

The poor mathematics results of Saudi students in all TIMSS has driven the Ministry of Education in Saudi Arabia to raise the quality of teaching by focusing on all factors related to improving education; one of these factors is teacher professional development. Generally, there is no comprehensive agreement on teacher quality and what is effective teaching. However, there is a widespread consensus that the quality of primary and secondary school mathematics teaching is fundamentally dependent on teachers' subject-related knowledge (Rowland & Ruthven, 2011). Researchers have concentrated on teacher knowledge as a fundamental element in improving students learning (Hill, Ball, & Schilling, 2008), as well as teacher cultural beliefs and views of effective teaching which influence students' understanding of mathematics (Cai, 2007; Thompson, 1992)

The literature review for the current study has discussed the conceptions of effective teaching through two different lenses: 1- a cultural and affective lens, and 2- teachers' cognitive lens. Within the literature, several approaches to knowledge for teaching mathematics (MCK, PCK, beliefs) were proposed (see chapter two, section: 2.1.2). However, the literature review has revealed that these approaches have focused on teachers' knowledge without emphasizing the contextual factors of teaching.

Given the problems and limitations of existing approaches to Mathematics Knowledge for Teaching, this thesis has followed a new approach. Rather than seeking to describe the process as teaching based on aspects of knowledge, this research has sought to identify and explain the factors (Knowledge- culture- belief) that might impact on teaching approach.

This research has taken the ontological view of critical realism in which mathematical concepts are fixed and exist independently because of their numeric nature, whereas if taking the epistemological view of social constructivism, the process of teaching mathematics would be subject to the interpretation of teachers, and teacher knowledge would be constructed by people through their distinct processes and communications with one another. Drawing on these underlying ontological and epistemological perspectives, this thesis has addressed the limitations of previous approaches when examining Saudi teachers' Mathematics Knowledge for Teaching through considering the following: 1- the gap between knowing and acting, 2- the importance of the cultural context, and 3- the affective component (Neubrand, 2018) (see chapter two, section: 2.2).

By considering these limitations, a close insight to the relationship of teachers' cognitive type of content knowledge, belief, and culture as well as teachers' practice identifies how they influence teachers' effective teaching. Consequently, this reflects the main aim of this research: Identifying Mathematics Teaching Knowledge for Saudi Female Mathematics Teachers in Middle School. In particular, the research questions for this study were as follows:

RQ1: What cognitive type of content knowledge do Saudi Female Mathematics Teachers in middle school have?

RQ2: What beliefs do Saudi Female Mathematics Teachers hold regarding the nature of mathematics, mathematics teaching, and mathematics learning?

RQ3: How do Saudi Female Mathematics Teachers' cognitive types of content knowledge and their beliefs impact their pedagogical decisions and quality of teaching?

RQ4: How do the culture beliefs of teachers influence their cognitive type of content knowledge, beliefs and pedagogical decisions?

In order to address these research questions, and while being consistent with underlying theoretical viewpoints, the current research used qualitative methods and applied the idea of triangulation, which is using data from different data sources (i.e., written test, interviews and lesson observation) (Depaepe et al., 2013).

In order to identify mathematics teaching knowledge of mathematics teachers, three existing mathematics and teaching practice frameworks (i.e., cognitive type of content knowledge, the Knowledge Quartet, and the categories of beliefs structure) identified in the literature review were used within the context of middle school female mathematics teachers in Aljouf city in Saudi Arabia. Data collection involved three main different data sources to explore these three approaches, which in this study were written test, interviews and observations.

The items of the Mathematical Knowledge for Teaching (MKT) measures (established by Ball and colleagues in the USA) were used in the written test to investigate the in-service mathematics teachers' MKT as a first step of data collection, marking the first known instance of these measures being used for in-service female mathematics teachers in Saudi Arabia. The MKT items used to measure the cognitive type of middle school teachers' content knowledge were adapted and translated to Arabic language in order to suit the research sample and covered a range of main mathematics topics in the Saudi middle school mathematics National Curriculum.

Fifteen female in-service mathematics teachers were invited voluntarily to respond to the written test as well as participate in semi-structured interviews following completion of the written test in this study. In the interviews SFMTs were asked about their beliefs concerning the nature of mathematics and mathematics teaching and learning. The categories of beliefs structure (Beswick, 2005) was used to analyse the interviews. Six of the 15 teachers were also invited to participate in lesson observation. These provided case studies of their teaching practice in their classrooms. The Knowledge Quartet was employed to analyse their lessons.

Results of the current study show that participating female Saudi teachers have superficial procedural knowledge classified as Type1 cognitive content knowledge (Tchoshanov, 2011). This suggests that the MKT test items are culture free as they were able to capture the kind of knowledge that the Saudi teachers possessed and used in their teaching.

The semi-structured interviews indicated that all the participating female Saudi teachers held the Platonist view of the nature of mathematics. However, two categories of beliefs of mathematics teaching and learning were identified in the interviews: one group was content-focused with an emphasis on performance, and the other group was content-focused with an emphasis on understanding. Finally, one main difference was found when comparing the lesson observation results with the written test and the interview data and when two groups of teachers were identified: the first group of teachers used a Platonist view/ constructivist teaching approach, while the other group used a Platonist view/traditional teaching approach. However, both groups emphasized procedural knowledge in their teaching which was linked to their superficial procedural knowledge (Type 1) cognitive content knowledge.

Furthermore, when exploring and comparing the cultural context of Saudi mathematics teachers with the cultural context of Chinese mathematics teachers, it was found that Confucius culture was highly reflected in Saudi teachers' beliefs system and their style of teaching. The analysis of the lesson observations suggested that the Confucian traditional beliefs had a great impact on teachers' teaching methods and were also found to be linked to their beliefs toward the nature of mathematics and mathematical teaching and learning. Nevertheless, the findings of this research revealed that the Saudi teachers' Type 1 cognitive content knowledge (Tchoshanov, 2011) influenced greatly what and how the mathematics content knowledge was delivered in their teaching, which provided evidence that teachers whose knowledge appeared to be more factual and procedural focused more on facts and rules in their teaching. Thus, teachers' Type 1 cognitive content knowledge has the potential to be a good predictor of teachers' teaching method (memorization-driven), which in turn might influence students' mathematics achievement.

In addressing the research questions, this research revealed important points that deserve further investigation and discussion, and the current study has contributed to widening the literature review in the examined area.

In mathematics, when we look at the complex work of teaching, there are different factors other than teachers' knowledge that guide teaching (Neubrand, 2018). The findings of this study have revealed Saudi female teachers' type of cognitive content knowledge as well as their belief systems toward mathematics and its teaching and learning. This has enriched our understanding of mathematics teaching knowledge from a different culture, in this case Saudi Arabia, a major country in the Middle East of the world. Furthermore, the topic that this research investigated is not known well in our existing literature and knowledge system.

This research took a holistic view of Saudi female teachers' practice and determined what aspects of MKT appeared in their actual teaching. This research used different frameworks, theoretical and empirical (The cognitive type of content knowledge framework – The Categories of Teacher Beliefs (CTB) framework – The Knowledge Quartet (KQ) framework – the Mathematics Knowledge for Teaching (MKT) framework), to accomplish the aim of this research. The type of cognitive content knowledge framework was used for the first time in Saudi Arabia as an analytical lens to examine female mathematics teachers' knowledge type. Also, this research provides the first example of the categories of beliefs structure framework employed to identify teachers' beliefs in Saudi Arabia. Further, this research demonstrated the first instance of the MKT items (established in the USA) being tested on female in-service mathematics teachers in Saudi Arabia. Likewise, with the Knowledge Quartet framework which was used to analyse the observed lessons conducted by female in-service mathematics teachers in Saudi Arabia.

The study's results indicate that the teaching of middle school mathematics in Saudi Arabia is in a transitional period. Teachers are trying to apply a new curriculum with a constructivist perspective, but their teaching practice is still influenced by their type of content knowledge as well as their beliefs. Therefore, these four instances offer a basis for other researchers wishing to use these frameworks within Saudi Arabia, or within other cultural contexts.

6.2 Limitations

This research was based on the participation of volunteers and aimed to examine and understand participants' mathematics knowledge and beliefs, and this section highlights and discusses several limitations.

6.2.1 Limitations of sample

Generally, since Saudi society and culture does not allow a female researcher to conduct interviews with and observation of male teachers, the sample group was limited to female teachers. Also, this research was undertaken in one Saudi city and, therefore, there is no evidence as to whether the findings could be applied to male teachers in boys' schools, nor in other settings. The research findings cannot be generalized beyond the context given for this study. However, data were triangulated from three different instruments (written test, interview, and observation) to provide rich data about teachers' mathematics knowledge for teaching.

6.2.2 Limitations of data collection

6.2.2.1 Limitations of data collection: MKT items

Even though the MKT items interpret the work of teaching comprehensively and can be utilised widely (Hill, Sleep, et al., 2007), the format of the MKT multiple-choice items has been criticized for its coverage of some of the knowledge under investigation (Schoenfeld, 2007). It has been argued that multiple-choice items could limit the focus to memorizing facts and procedures and thus the depth and breadth of teachers' knowledge may be ignored (Beswick et al., 2012). Schoenfeld (2007) recommended more investigation into the use of open-ended items with MKT multiple-choice items. However, in this research the data were triangulated from three different instruments to overcome this problem.

The language of the MKT items could be considered universal and applicable in terms of the symbolic expressions (Fauskanger & Mosvold, 2012). However, mathematics terminologies may vary in different cultures, as teaching practices differ significantly across cultures (Stigler & Hiebert, 1999). For instance, there are some technical terms such as "tessellation" that exist in the U.S. context but are not used in Saudi Arabia, thus, some of the MKT items were omitted because of the variations in cultural contexts. While the MKT items are useful, they do need to be screened before being used with particular cultural groups because of such variations in context. This could impact for wider cultural comparative studies.

6.2.2.2 Limitations of data collection: interviews

Time constraints, existing workloads and fear of scrutiny were some reasons that constrained teachers' willingness to participate in the interviews. The researcher attempted to tackle this issue by clearly expressing the voluntary nature of participation that confined the process to teachers who were willing to participate, and assured the anonymity of the participant, before starting the process of collecting the data as well as stressing the flexibility of the time and the special measures undertaken to protect the confidentiality and privacy of the obtained information and the participants.

6.2.2.3 Limitations of data collection: observations

A limitation of the observations method in this study could be that teachers' observed teaching practice may be different from their usual practice in normal lessons. It was acknowledged that the attendance of the researcher may make the teachers more conscious of their behaviours and teaching practice during the lesson observation. Without a doubt "the presence of an observer might be expected to produce self-consciousness or other reactions that would distort the behaviour which is being studied" (Jersild & Meigs, 1939, p. 480). The researcher could not choose the observed lesson topic, as the teachers have to follow the curriculum as prescribed for that time of year. Therefore, learning intentions, activities and the act of teaching might be address different types of knowledge.

6.2.3 Limitations of data analysis

6.2.3.1 Limitations of data analysis: cognitive type of content knowledge

The cognitive type of content knowledge framework was used as an analytical lens to describe what kind of content knowledge and understanding teachers owned and so it only referred to what was known. Findings showed that most of the Saudi female participant teachers had superficial Type 1 (i.e., Knowledge of Facts and Procedures) cognitive content knowledge. However, one challenge was highlighted when using this framework to analyse SFMTs' responses to the written test questions: this framework did not link to the quality of teachers' understanding of the type of knowledge, which is important in determining the influence of teachers' knowledge on their teaching. Thus, viewing teachers' type of cognitive content knowledge as focusing on the qualities of knowledge is needed in order to deepen our understanding of what superficial and deep types of cognitive content knowledge might mean in teachers' actual teaching.

6.2.3.2 Limitations of interview data analysis: categories of teacher beliefs

This research examined the ways in which Saudi female teachers' beliefs influenced their classroom teaching by using the Beswick (2012) categories of teacher beliefs which focused on teachers' beliefs about the nature of mathematics, mathematics teaching, and mathematics learning. The research attempted to link the categories of teacher beliefs with their actual teaching. The validity of this linkage may be problematic as teachers' beliefs about the knowledge they require as a teacher and Beswick's category of teacher beliefs are considered as two separate systems (Fives & Buehl, 2010). Therefore, as this research aimed to identify the relationship between teachers' belief and practice, it would perhaps have been beneficial if this study had also examined teachers' belief about mathematical knowledge required for teaching by adding a new category linked to teachers' belief toward MKT (Mosvold & Fauskanger, 2013) (see section: 6.3).

6.2.3.3 Limitations of observation data analysis: The Knowledge Quartet

The Knowledge Quartet – an existing observation checklist (see Appendix 3) – was used to analyse the data collected from lesson observation. This research endeavoured to make comparisons in accordance with analyses of other collected data (written test and interview) of Saudi female mathematics teachers and to focus attention on aspects useful in identifying the mathematics knowledge for teaching that impacts Saudi female teachers' teaching. This research found the Knowledge Quartet to be a useful tool for analysing the observed lessons in the Saudi classroom.

In other KQ studies, lessons were usually videotaped and the analysis was based on both the teachers' actions and voices. However, the observed lessons in this research were only able to be audio recorded because of cultural reasons, which limited the overview of the lesson and meant the researcher might have missed recording some relevant and critical aspects of the lessons such as students' responses and reactions.

The online Knowledge Quartet 'coding manual' (Weston et al., 2012) was used to aid the analysis of the data. However, teachers' interpretation of the textbook was not addressed by the framework and the importance of this was a significant finding in this study. This suggests a careful adaption of the Knowledge Quartet for different contexts other than the U.K. where it was originally developed (Petrou, 2010).

6.3 Recommendations

Four recommendations are proposed to extend this present study for further research:

1- This research found that all the Saudi participating female mathematics teachers' focus was on procedural knowledge throughout their teaching. Thus, this research could be extended to include a larger sample of female teachers from other cities and rural areas in

Saudi Arabia. This could help to determine whether other teaching patterns might exist, for example, whether there are different types of cognitive content knowledge or beliefs system that Saudi female teachers have which in turn influence their teaching practice. A similar study could also be extended to include male teachers in order to find similarities and differences between genders.

2- This research could be extended by adopting Schoenfeld (2007) suggestion that the researcher could associate some related open-ended questions with the multiple-choice MKT items. This recommendation arises from the negative criticism of other researchers about the effectiveness of the multiple-choice format of the MKT items as the questions could not examine the depth of teacher knowledge (see section: 6.2.2.1). A combination of open-ended questions and multiple choice questions could enrich and widen the research areas of teachers' type of cognitive content knowledge as well as the possible relationships in teachers' knowledge revealed in the two item formats (Fauskanger, 2015).

3- This research argues that in thinking about teachers' responses to the written test multiple choice questions, it is important to investigate both the type of teachers' cognitive content knowledge and the quality of teachers' mathematical content knowledge (Deep and Superficial) (Star & Stylianides, 2013). Examining the quality of teachers' knowledge will inform us about the level of information known by someone (e.g., teacher) and how deeply it is understood (Star & Stylianides, 2013). Deep-level knowledge is connected with someone's understanding, estimation, flexibility, evaluation, and critical decision making (De Jong & Ferguson-Hessler, 1996). In contrast, superficial or surface-level knowledge is related to rote learning, memorizing, and inflexibility (Glaser, 1991). Therefore, it is suggested that the type of cognitive content knowledge developed by Tchoshanov (2010) should be extended to encompass the quality of knowledge (Deep and Superficial) (Table 8). This would enrich and widen educators' understanding of the quality of teachers' cognitive content knowledge, which in turn would support the development of teaching education programs.

Table 9: Type and quality of teacher cognitive content knowledge

Quality of knowledge	Type of cognitive content knowledge

	Type 1	Type 2	Type 3
Deep level			
Superficial level			

4- This researcher agrees with Mosvold and Fauskanger (2013) that it is important to extend the categories of teachers' beliefs (developed by Beswick) to include a category that incorporates teachers' beliefs about the set of mathematics knowledge that teachers require for teaching. The proposed extension of Beswick (2012) categories of teacher beliefs is presented in Table 9. This proposal is expected to provide a tool to deepen educators' understanding and interpreting of teachers' beliefs regarding specific aspects of MKT and their related decisions, choice of examples and representations in the classroom (Mosvold & Fauskanger, 2013).

Table 10: Proposed extension of Beswick's (2012) categories of teacher beliefs

Beliefs mathematics	Beliefs about mathematics teaching	Beliefs about mathematics learning	Beliefs about MKT
Instrumentalist	**Content for performance**	**Mastery of skills**	**Remembering content**
Platonist	**Content with understanding**	**Construction of understanding**	**Understanding content**
Problem solving	**Learner focused**	**Autonomous exploration**	**Adjusting and differentiating**

5- The research could be extended to include both female and male pre-service mathematics teachers in Saudi Arabia. As the research would be designed to investigate teachers' type of cognitive knowledge, their beliefs and how it affects their teaching approach, this proposed extension would be expected to inform the educators about the improvement/enrichment of the learning content in the teacher education programs and how to prepare pre-service teachers for their practicum in Saudi Arabia.

6- This research provides a very sound basis for the development of a focused professional education program designed specifically to develop the quality of teaching and improve student achievement especially in Saudi Arabia. It highlights how the cultural context conditions teachers' knowledge and beliefs, thus giving a more nuanced perspective on the relationship between knowledge, beliefs, and classroom practice than one finds in much of the existing research conducted in Western contexts.

6.4 Concluding remarks

A new approach and implications for professional development is required.

A parallel can be made between teachers' teaching and an iceberg. The tip of the iceberg represents teachers' actual teaching, representation, and decision making in the classroom, while the majority of the iceberg that sits under the water is represented by a combination of teachers' unseen characteristics, such as teachers' knowledge and beliefs as well as teachers' culture, which shapes their actual teaching. Consequently, in order to identify the role of teachers' mathematical knowledge for teaching, this research has considered the previous approaches (see section: 2.1.2) then developed and followed a new approach by taking a holistic overview of teachers' mathematical knowledge for teaching and focusing on the major part of the iceberg, which is the unseen teachers' characteristics including type of cognitive content knowledge and belief, which is where those characteristics come from, with culture as one of many factors.

Rational understanding of the complex work of teaching and the role of teachers' mathematical knowledge for teaching makes new contributions to both theory and practice. In fact, the process of teaching in the classroom arises from the patterns of behaviour that result from the interaction between teachers' mathematical content knowledge and beliefs towards the nature of mathematics and mathematics teaching and learning, as well as teachers' culture. The new approach taken in this research sheds light on both what teachers do and why they do it by investigating three major factors (knowledge – beliefs - culture); it helps provide more details about the relationship between them and to understand how these factors may have an impact on teachers' classroom practices, and it reflects on and evaluates cultural contexts as a base for the teaching process.

According to Clarke and Hollingsworth (2002), “the optimization of the outcomes of a process is facilitated by the understanding of that process. If we are to facilitate the professional development of teachers, we must understand the process by which teachers grow professionally and the conditions that support and promote that growth” (p. 947).

As a result of this research there are different factors other than teachers’ knowledge that guide their actual teaching, and this result should be involved in the professional knowledge of teachers. A high-quality professional development is a fundamental part of improving education (Guskey, 2002). The goal of professional development programs is to improve the quality of teachers’ practices. In particular, the fundamental goal of the new curriculum reform in Saudi Arabia is to improve students’ achievement which showed as being at a low level in the international tests. The new curriculum focuses on the use of problem-solving strategies in mathematics classrooms by encouraging teachers to shift from traditional teacher-centred approaches to student-centred practices.

In terms of teachers’ content knowledge, findings showed that teacher knowledge of facts and rules is significantly linked to the quality of teaching in middle school mathematics. This provides similar evidence to Tchoshanov (2010) who argued that teachers whose knowledge did not appear to be connected and conceptual were found to be more rule-based in their teaching. Interestingly, regarding teachers’ beliefs, all SFMTs appear to hold a Platonist view about the nature of mathematics (Ernest, 1989a) which is not consistent with a new Saudi curriculum reform which emphasizes a problem-solving view (Ernest, 1989a) and a constructivist learning environment.

Recognition of the complexity of the relationship between teachers’ knowledge, beliefs and practices helps in applying the new curriculum reform through influencing their teaching approach. Therefore, an important recommendation from this research is to consider developing an adequate professional education program designed specifically to develop the quality of teaching and improve student achievement. In particular, more emphasis should be placed on developing teachers’ knowledge of other types of cognitive content knowledge (e.g., Type 2 concepts and connections). Also, there should be more focus on deconstructing teachers' traditional beliefs and memory-driven approaches by enhancing teachers' abilities in using problem solving strategies in their classroom practice. In these ways, any challenges for Saudi teachers that could appear in applying the new curriculum approach as a result of their type of knowledge or beliefs could be addressed.

The practical implications of this research provide useful evidence for professional development program designers and policy makers in other cultures. Since teaching is a cultural activity (Stigler & Hiebert, 1999) and every culture is unique, culture shapes the processes of teaching in the classroom and the practice of teaching within countries. As Stigler and Perry (1988) stated:

> Cross-cultural comparison also leads researchers and educators to a more explicit understanding of their own implicit theories about how children learn mathematics. Without comparison, teachers tend not to question their own traditional teaching practices and are not aware of the better choices in constructing the teaching process (p. 199).

Indeed, when cultural traits are emphasized and compared with other cultures, it leads to deeper understanding of the features of one's own culture (Knipping, 2003). It "significantly shape[s] ways of thinking about how teachers develop as professionals" (Rowland, 2014a, p. 238). According to Beswick and Goos (2012) it is very important to link any newly developed approach in teachers' professional knowledge to teacher education programs; this is because the resulting outcome "may provide important insights into validity of the measures and the subtleties of the impacts of teacher education programs. Whether or not this proves to be the case, the current policy environment makes evidence-based teacher education a priority" (p. 87).

This research provides a very solid basis for the creation of a centred professional education programme specifically designed to improve the standard of teaching in Saudi Arabia and improve student success.

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APPENDICES

Appendix 1

This has been removed due to copyright restriction. Available online from [http://www.umich.edu/~lmtweb/files/lmt_sample_items.pdf / Ball, D., & Hill, H. (2008). Mathematical knowledge for teaching (MKT) measures. Mathematics released items 2008. Ann Arbor: University of Michigan.].

Appendix 2

Interview protocol

What is your approach to mathematics teaching?

١-ماهو الأسلوب المتبع لديكِ في تدريس الرياضيات؟

Conceptual: Teaching Goal: explaining “why” or explaining “how”.

When explaining why a maths procedure works, how much can students understand?

أ) أسلوب مفاهيمي: الهدف منه شرح "لماذا" أو "كيف".

-عند استخدامك لهذا الأسلوب في شرح المسائل الحسابية إلى أي مدى يمكن للطالبات فهم المسألة؟

Procedural: Algorithms: memorize steps or discover steps.

How does a student come to understand the steps of an algorithm?

ب) أسلوب إجرائي: الخوارزمية والهدف من هذا الأسلوب حفظ خطوات حل المسألة أو اكتشافها

-كيف يمكن للطالبة فهم الخطوات في حل المسائل الحسابية؟

Connections between concepts: Numbers of skills: teaching sequential isolated skills or mixing math concept.

Should skills be taught in isolation or mixed?

ج) أسلوب الربط بين المفاهيم: ويشمل عدة مهارات بحيث يتم تدريس عدة مهارات بشكل تسلسلي أو باستخدم الخلط/التنويع بين مفاهيم الرياضيات.

-هل يجب أن يتم تدريس تلك المهارات بشكل منفصل أو منوع؟

Why do you have this teaching approach?

٢-ماهو سبب استخدامك لهذا الأسلوب في التدريس؟

What teaching methods in terms of helping students to learn are effective? Why?

٣- ماهي طريق التدريس الأكثر تأثيراً في مساعدة الطالبات على الفهم؟ ولماذا؟

Role of teachers: teachers demonstrate or teachers facilitate

What is the role of the teachers in your math class?

أ) دور المعلمة: تقوم بدور الشرح أم بدور المساعدة والتسهيل

-ما هو دور المعلمة في حصة الرياضيات؟

Calculators: for problem solving or for computations.

What is the role of calculators in your classroom?

ب) استخدام الآلة الحاسبة هل هو لحل المسائل الحسابية أو للمساعدة على الحساب؟

-ما هو دور الآلات الحاسبة في حصة الرياضيات؟

Wrong answers: should be corrected or should lead to discussions

What do you do when a student give a wrong answer?

ج) طريقة تعاملك مع إجابات الطالبات الخاطئة، هل يجب تصحيحها أم أنها يجب أن تقود للمناقشات

- ماذا تفعلين حين تعطي إحدى الطالبات إجابة خاطئة؟

Goal for students: Understanding or speed and accuracy

How important is developing students speed and accuracy of getting answers?

د) دور الطالبات في الفصل، هل هو الفهم/الاستيعاب أم السرعة والدقة في الحل

- إلى أي مدى يعد مهماً تطوير السرعة والدقة في إجابة الطالبات؟

Focus: concept development or skill drill

What is the most important thing for students to learn in mathematics?

هـ) التركيز لتطوير المفاهيم أو التدريب على المهارات.

- ما هو الشيء الأكثر أهمية التي يجب للطالبات أن يتعلمنه خلال دراسة الرياضيات؟

Questioning: justify reasoning or recite facts

What types of questions are important to ask in your mathematics class?

و) الإستفسار/ السؤال: تبرير الأسباب أو تعداد الحقائق

- ماهي الأسئلة المفضل استخدامها في حصة الرياضيات؟

Manipulatives: to explore or to model

How and when do your students use manipulatives?

ز) استخدام الوسائل الحسيّة للاستكشاف الحل أو لتوضيح الفكرة

- متى وكيف يستخدمن الطالبات الوسائل الحسيّة؟

What makes a good mathematics teaching?

٤-ماهو الشيء الذي يجعل تدريس الرياضيات ناجح أو جيد؟

a) Solution process: one right way or many right ways

How important is it to learn a solution process for particular type of problems?

أ) خطوات الحل اتباع طريقة حل صحيحة واحدة أو عدة طرق

- إلى أي مدى يعد مهماً تعلم طريقة الحل لنوع محدد من المسائل؟

b) problems solving or solving a word problem

What type of problem solving do students experience in your class?

ب) أسلوب حل المشكلات أو حل المسائل الكلامية؟

- ما هو نوع حل المشكلات الذي تستخدمه الطالبات في حصة الرياضيات؟

What aspect of mathematics teaching concerns/worries you? Why?

٥-ما هو المفهوم الرياضي الذي يقلقك في تدريس الرياضيات؟ ولماذا؟

How would you describe mathematics teaching in Saudi Arabia?

٦) كيف تصفين تدريس الرياضيات في المملكة العربية السعودية؟

How do your colleagues, school principals, parents and the government effect your classroom practices? Why?

٧- كيف يمكن لزميلاتك، إدارة المدرسة، إدارة التعليم أو الوالدين التأثير على ممارساتك في حصة الرياضيات؟ ولماذا؟

Generally, how confident are you when teaching mathematics? Explain.

[1 2 3 4 5 6 7 8 9 10]

٨-بشكل عام من ١-١٠، ما هو تقييمك لثقتك في تدريس الرياضيات؟ اشرحي.

What are the most and least confident about in term of mathematics topic?

٩- ما الموضوع الذي تجدين نفسك أكثر/أقل ثقة في شرحه في تدريس الرياضيات؟

Describe a lesson that you think are more effective and why?

١٠-أعطِ مثالاً لأحد الدروس التي ترين نفسك أكثر فاعلية في تدريسه، ولما

Appendix 3

Observation Checklist

Teacher's name:

Class: Topic:

Date: Observation: 1st 2nd 3rd

Dimensions' codes		*Field notes*
FOUNDATION	identifying errors	
	concentration on procedures	
	use of terminology	
	overt subject knowledge	
	Theoretical underpinning of pedagogy	
	Awareness of purpose	
	Adherence of textbook	
TRANSFORMA TION	Teacher demonstration	
	Use of instructional materials	
	Choice of representation	
	Choice of examples	
CONNECTION	making connections between procedures	
	making connections between concepts	
	anticipation of complexity	
	decisions about sequencing	
	recognition of conceptual appropriateness	
CONTINGENCY	responding to children's ideas (use of opportunities);	
	deviation from lesson agenda	
	teacher insight	

responding to the (un)availability of tools and resources

Appendix 4

FINAL APPROVAL NOTICE

Project No.:

7736

Project Title:

Identifying mathematics teaching knowledge for Saudi mathematics teachers in middle school

Principal Researcher:

Mrs Hdil Alatallah

Email:

alat0018@flinders.edu.au

Approval Date:

22 September 2017

Ethics Approval Expiry Date:

1 October 2020

The above proposed project has been approved on the basis of the information contained in the application, its attachments and the information subsequently provided with the addition of the following comment(s):

Additional information required following commencement of research:

1. Permissions

Please ensure that copies of the correspondence granting permission to conduct the research from a) the educational administrator in Aljouf city; and b) the principals of all schools to be involved are submitted to the Committee on receipt. Please ensure that the SBREC project number is included in the subject line of any permission emails forwarded to the Committee. Please note that data collection should not commence until the researcher has received the relevant permissions (item D8 and Conditional approval response – number 11).

RESPONSIBILITIES OF RESEARCHERS AND SUPERVISORS

1. Participant Documentation

Please note that it is the responsibility of researchers and supervisors, in the case of student projects, to ensure that:

- all participant documents are checked for spelling, grammatical, numbering and formatting errors. The Committee does not accept any responsibility for the above mentioned errors.
- the Flinders University logo is included on all participant documentation (e.g., letters of Introduction, information Sheets, consent forms, debriefing information and questionnaires – with the exception of purchased research tools) and the current Flinders University letterhead is included in the header of all letters of introduction. The Flinders University international logo/letterhead should be used and documentation should contain international dialling codes for all telephone and fax numbers listed for all research to be conducted overseas.
- the SBREC contact details, listed below, are included in the footer of all letters of introduction and information sheets.

This research project has been approved by the Flinders University Social and Behavioural Research Ethics Committee (Project Number 'INSERT PROJECT No. here following approval'). For more information regarding ethical approval of the project the Executive Officer of the Committee can be contacted by telephone on 8201 3116, by fax on 8201 2035 or by email human.researchethics@flinders.edu.au.

2. Annual Progress / Final Reports

In order to comply with the monitoring requirements of the National Statement on Ethical Conduct in Human Research (March 2007) an annual progress report must be submitted each year on the 22 September (approval anniversary date) for the duration of the ethics approval using the report template available from the Managing Your Ethics Approval SBREC web page. Please retain this notice for reference when completing annual progress or final reports.

If the project is completed before ethics approval has expired please ensure a final report is submitted immediately. If ethics approval for your project expires please submit either (1) a final report; or (2) an extension of time request and an annual report.

Student Projects

The SBREC recommends that current ethics approval is maintained until a student's thesis has been submitted, reviewed and approved. This is to protect the student in the event that reviewers recommend some changes that may include the collection of additional participant data.

Your first report is due on 22 September 2018 or on completion of the project, whichever is the earliest.

3. Modifications to Project

Modifications to the project must not proceed until approval has been obtained from the Ethics Committee. Such proposed changes / modifications include:

- change of project title;
- change to research team (e.g., additions, removals, principal researcher or supervisor change);
- changes to research objectives;
- changes to research protocol;
- changes to participant recruitment methods;
- changes / additions to source(s) of participants;
- changes of procedures used to seek informed consent;
- changes to reimbursements provided to participants;
- changes / additions to information and/or documentation to be provided to potential participants;
- changes to research tools (e.g., questionnaire, interview questions, focus group questions);
- extensions of time.

To notify the Committee of any proposed modifications to the project please complete and submit the Modification Request Form which is available from the Managing Your Ethics Approval SBREC web page. Download the form from the website every time a new modification request is submitted to ensure that the most recent form is used. Please note that extension of time requests should be submitted prior to the Ethics Approval Expiry Date listed on this notice.

Change of Contact Details

Please ensure that you notify the Committee if either your mailing or email address changes to ensure that correspondence relating to this project can be sent to you. A modification request is not required to change your contact details.

4. Adverse Events and/or Complaints

Researchers should advise the Executive Officer of the Ethics Committee on 08 8201-3116 or human.researchethics@flinders.edu.au immediately if:

- any complaints regarding the research are received;
- a serious or unexpected adverse event occurs that effects participants;
- an unforeseen event occurs that may affect the eth