Wavelets and C^*-algebras

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Abstract

A wavelet is a function which is used to construct a specific type of orthonormal basis. We are interested in using C^*-algebras and Hilbert C^*-modules to study wavelets. A Hilbert C^*-module is a generalisation of a Hilbert space for which the inner product takes its values in a C^*-algebra instead of the complex numbers. We study wavelets in an arbitrary Hilbert space and construct some Hilbert C^*-modules over a group C^*-algebra which will be used to study the properties of wavelets.

We study wavelets by constructing Hilbert C^*-modules over C^*-algebras generated by groups of translations. We shall examine how this construction works in both the Fourier and non-Fourier domains. We also make use of Hilbert C^*-modules over the space of essentially bounded functions on tori. We shall use the Hilbert C^*-modules mentioned above to study wavelet and scaling filters, the fast wavelet transform, and the cascade algorithm. We shall furthermore use Hilbert C^*-modules over matrix C^*-algebras to study multiwavelets.

Key Words and Phrases. Wavelet, filter, C^*-algebra, Hilbert C^*-module, cascade algorithm.

Declarations

I certify that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge and belief it does not contain any material previously published or written by another person except where due reference is made in the text.

Peter Wood, Candidate

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Contents

List of Notation

$\mathbf{N}$	Natural numbers	
$\mathbf{Z}$	Integers	
$\mathbf{Z}_p$	Integers mod p	
$\mathbf{R}$	Real numbers	
$\mathbf{C}$	Complex numbers	
$\mathbf{T}$	Unit circle	
$\mathcal{H}$	Hilbert space	
$\mathcal{F}, \wedge$	Fourier transform	1
$\mathcal{A}$	Involutive algebra	4
$\mathbf{1}$	Unit in a unital involutive algebra	4
$B(\mathcal{H})$	Algebra of bounded operators on a Hilbert space $\mathcal{H}$	4
$U(\mathcal{H})$	Unitary operators on a Hilbert space $\mathcal{H}$	
G	Locally compact group	1
$C(X)$	Continuous functions on X	
$C_c(X)$	Compactly supported continuous functions on X	
$C_0(X)$	Continuous functions on X which vanish at infinity	
$C^*(G)$	Group C^*-algebra of G	9
$\langle\ ,\ \rangle$	Inner product in Hilbert space	
$[\ ,\ ]_E$	C^*-algebra valued inner product in right Hilbert module E	11
${}_E[\ ,\ ]$	C^*-algebra valued inner product in left Hilbert module E	
$\mathcal{H}_\mathcal{A}$	Standard Hilbert $\mathcal{A}$-module	12
$\mathcal{L}(E)$	Adjointable operators on Hilbert module E	14
$\mathcal{K}(E)$	Generalised compact operators on Hilbert module E	14
Γ	Group of translations for a multiresolution structure	20
$\mathcal{D}$	Dilation for a multiresolution structure	20
m	Index of a multiresolution structure	20
ψ	Orthonormal wavelet	20
$\{\psi, \ldots, \psi^M\}$	Orthonormal multiwavelet	20
Δ	$\Delta(\gamma) = \mathcal{D}^{-1}\gamma\mathcal{D}$	22
Γ^n	$\Gamma^n = \mathcal{D}^n\Gamma\mathcal{D}^{-n}$	22
$\{V_n\}_{n\in\mathbf{Z}}$	Generalised multiresolution analysis	22
φ	Set of scaling functions for a multiresolution analysis of degree 1	22
$\{\varphi^1, \ldots, \varphi^r\}$	Set of scaling functions for a multiresolution analysis of degree r	22
$\tilde{\mathcal{D}}$	Group homomorphism used to generate dilation for a harmonic multiresolution structure	27
π^n	Representation of Γ on $\mathcal{H}$ which corresponds to action of Γ^n on $\mathcal{H}$ for a harmonic multiresolution structure	27
X_θ	Hilbert $C^*(\mathbf{Z}^d)$-module which corresponds to embedding $\theta : \mathbf{Z}^d \to \mathbf{R}^d$	57
$[\ ,\ ]_\theta$	$C^*(\mathbf{Z}^d)$-valued inner product for X_θ	57

Introduction

In this thesis we shall use some constructions employing C^*-algebras to prove results about wavelet theory. The main way that we shall do this is by constructing a Hilbert C^*-module using the C^*-algebra which is generated by a set of translations associated with a multiresolution analysis.

Wavelets are a tool that can be used to analyse an arbitrary function in terms of resolution and frequency. They do this by decomposing spaces of functions into an orthonormal basis, or more generally a Riesz basis or a frame. An orthonormal basis is a basis where each element is orthogonal to the others and has norm equal to one. A Riesz basis is the image of an orthonormal basis under an invertible operator. A frame is a set which spans the space but need not be linearly independent. Both orthonormal bases and Riesz bases are also frames.

Wavelets have numerous applications including image compression, artificial vision, telecommunications, denoising, seismic signal processing, and medical signal processing including tomography, computer aided mammography, and analysis of both ECG and EEG signals, to mention a few. More applications are described in [Me2], [Da1], and [KL]. Wavelet theory is relatively new, beginning in the early 1980's. Since then there have been literally thousands of papers published on the subject. Although modern wavelet theory began quite recently, there are deep connections between wavelet theory and earlier research, such as Littlewood-Paley theory [EG, LP1, LP2], Calderon-Zygmund operators, pyramid algorithms, and subband coding schemes.

C^*-algebras are normed Banach algebras which have an involution. They also have the property that they can be realised as bounded operators on a separable Hilbert space. Any commutative C^*-algebra can also be realised as an algebra of continuous functions on a compact Hausdorff space. Associated with any group there is a group C^*-algebra, and most of the C^*-algebras studied here will be group C^*-algebras. As well as group C^*-algebras, we shall also examine some work [J1, BJ1, BJ3] which relates wavelets to C^*-algebras known as Cuntz algebras. C^*-algebras are related to other fields of mathematics including dynamical systems, K-theory, topology, and noncommutative geometry.

We shall relate wavelets to C^*-algebras by using Hilbert C^*-modules, which we shall usually abbreviate as Hilbert modules. A Hilbert module is a generalisation of a Hilbert space for which the inner product takes its values in a C^*-algebra instead of the complex numbers. Hilbert modules can also be thought of as a generalisation of vector bundles [Sw, Hi], and as such they play an important role in noncommutative geometry. We shall use Hilbert modules to study wavelets by using methods which are very closely related to a construction announced in 1997 by Marc A. Rieffel of a Hilbert module over a group C^*-algebra associated with wavelets (see [R6, PR1, PR2]). A large amount of this thesis is devoted to understanding this construction.

Most of the background material that we require is contained in Chapter 0, where we study the Fourier transform, involutive algebras, group representations, group algebras, Hilbert modules, and bases and frames for Hilbert spaces and Hilbert modules. The reader who is already familiar with this material may wish to directly proceed to Chapter 1.

The classical definition of a dyadic orthonormal wavelet is a function ψ such that the family

$$\left\{\psi_{j,k}(x) := 2^{-j/2}\psi(2^j x - k)\right\}_{j,k\in\mathbf{Z}}, \quad \text{for } x \in \mathbf{R}$$

is an orthonormal basis for the Hilbert space of square integrable functions, $L^2(\mathbf{R})$. The functions $\psi_{j,k}$ are obtained from ψ by acting on it by translations and dilations. The translations and dilations are unitary operators on this Hilbert space, so they preserve inner products. In Chapter 1 we generalise the classical definition of a wavelet to an arbitrary Hilbert space in a manner similar to what has been done in [BCMO]. Associated with every wavelet is what is known as a generalised multiresolution analysis. Roughly speaking, a generalised multiresolution analysis (GMRA) of a Hilbert space is an increasing sequence of subspaces $(V_n)_{n\in\mathbf{Z}}$ of the Hilbert space, which approximate the Hilbert space more closely as n approaches infinity. If the Hilbert space has an element φ such that translations of φ span the subspace V_0, we call the generalised multiresolution analysis a multiresolution analysis (MRA), and we call φ a scaling function. We will prove in Theorem 1.1.11 that we can obtain wavelets when we have a multiresolution analysis; we use von Neumann algebras to prove this theorem. The projections onto the subspaces V_n are closely related to an important numerical algorithm known as the fast wavelet transform. The investigation of the fast wavelet transform was what originally lead to the development of the notion of a multiresolution analysis, and is also closely related to the study of filter banks. We shall show in Chapter 1 that the fast wavelet transform still makes sense in this more general setting. We will mainly be looking at the case that the Hilbert space is a space of square integrable functions defined on a locally compact Abelian group. When this is the case it is possible to define the Fourier transform, and we shall often make use of the Fourier transform as a tool for examining wavelets.

Most of the author's new results are contained in Chapter 2 and Chapter 3. Chapter 2 is where we shall introduce the construction that relates wavelets to Hilbert C^*-modules. This construction is the main tool and object that is examined in this thesis. It is one of the aims of this thesis to demonstrate the importance and utility of this tool for understanding wavelets. The author's work on this construction was partially inspired by results announced in [R6]. The construction described here is very similar to a construction described in [PR2], which was released as an eprint not long before the submission of this thesis. In order to take into account the dilation, we define a chain of Hilbert modules $(X_n)_{n\in\mathbf{Z}}$ over the C^*-algebra of the translation group, $C^*(\mathbf{Z}^d)$. The $C^*(\mathbf{Z}^d)$-valued inner products used by these Hilbert modules are sometimes known as "bracket products". As well as studying the properties of bracket products on X_n, we shall also study the properties of bracket products on $L^2(\mathbf{R}^d)$. We shall work out the details of this construction on both the Fourier and non-Fourier domains. We show in Corollary 2.2.7 that the dilation is an adjointable operator which maps between the elements of the above chain of Hilbert $C^*(\mathbf{Z}^d)$-modules. In Chapter 2 we shall also define some Hilbert modules $(Y_n)_{n\in\mathbf{Z}}$ which are over the larger C^*-algebra $L^\infty(\mathbf{T}^d)$, and whose $L^\infty(\mathbf{T}^d)$-valued inner products are Fourier transformed bracket products. These Hilbert $L^\infty(\mathbf{T}^d)$-modules are similar to ones described in [CaLa, CoLa], which are used to study Gabor systems.

If a wavelet corresponds to a multiresolution analysis, there exist functions on $\mathbf{Z}^d$ whose Fourier transform is contained in $L^\infty(\mathbf{T}^d)$ which correspond to scaling functions and wavelets, and are known as scaling and wavelet filters. We examine wavelets from this perspective in Chapter 3. Associated with these filters are some operators from the C^*-algebra to itself associated with the fast wavelet transform. We shall examine the convergence properties of an algorithm for obtaining the scaling function from the scaling filter known as the cascade algorithm. It is then possible to obtain the wavelets from the scaling function using the wavelet filters. The cascade algorithm (Theorem 3.4.10 and Theorem 3.4.11) gives us necessary and sufficient conditions for elements of $C_c(\mathbf{Z}^d)$ to be scaling filters. We demonstrate that the cascade algorithm converges in the topology given by the Hilbert module norm, as well as in the norm topology on $L^2(\mathbf{R}^d)$. We shall investigate wavelet matrices in this chapter and see that they correspond to Hilbert modules over matrix C^*-algebras. Our results on wavelet matrices are encapsulated in Theorem 3.5.4, which also tells us necessary and sufficients conditions for elements of C^*-algebras to be wavelet filters, when we have a corresponding set of scaling functions.

Part of the aim of this thesis is to show how results in operator algebra theory are useful for studying wavelets. We want to in particular demonstrate the importance of the construction in Chapter 2 to wavelet theory. Because of the wide variety of applications of wavelet theory, this represents an interesting application of the theory of C^*-algebras and Hilbert C^*-modules.

Chapter 0

Preliminary Material

This chapter contains most of the preliminary material that we shall need, most of which is related to operator algebras, Harmonic analysis, or both. We shall first examine the Fourier transform. We will then introduce C^*-algebras, von Neumann algebras and Hilbert C^*-modules. We shall describe the necessary background material on representations of groups and representations of C^*-algebras. The main Hilbert C^*-module that we introduce in Chapter 2 will be associated with a group C^*-algebra, so we shall introduce those. Wavelets are a specific tool for constructing frames and bases so in Section 0.5 we shall introduce frames and bases for both Hilbert modules and Hilbert spaces.

We shall assume a certain amount of knowledge on behalf of the reader. Most of the analysis that we assume is contained in [Ru1]. We shall occasionally use tensor products. The reader is referred to Appendix T of [W-O] or Section 7.3 of [Fo] for background information on them.

0.1 Analysis and the Fourier Transform

The following theorem tells us about a "completion process" that we shall often make use of. In order to apply C^*-algebras and Hilbert C^*-modules to wavelet theory we shall be constructing both the relevant Hilbert C^*-module and the C^*-algebra by taking the completion of a dense subspace. Recall that a Banach space is a normed linear space which is complete with respect to the topology induced by its norm.

Theorem 0.1.1 ([KR], Theorem 1.5.1) *If X is a normed linear space, there is a Banach space Y that contains X as an everywhere-dense subspace (and such that the norm on X is the restriction of the norm on Y). If Y_1 is another Banach space with these properties, the identity mapping on X extends to an isometric isomorphism from Y onto Y_1.*

We shall now introduce some of the norms and spaces that we shall use.

Definition 0.1.2 Let X be an arbitrary measure space with positive measure μ. If

$1 < p < \infty$, and f is a complex measurable function on X, define the norm

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}$$

and let $L^p(\mu)$ consist of all f for which

$$\|f\|_p < \infty.$$

We define $\|f\|_\infty$ to be the essential supremum of $|f|$. We then let $L^\infty(\mu)$ consist of all f for which $\|f\|_\infty < \infty$.

The reader is referred to Chapter 3 of [Ru1] for a proof that $\|\cdot\|_p$ is a norm and discussion of the properties of L^p-spaces. When it is obvious what measure the space X is equipped with, we shall often write $L^p(X)$ instead of $L^p(\mu)$. The space $L^p(\mu)$ is metric space with the norm $\| \|_p$ when it is thought of as a space of equivalence classes of functions any two of which differ only on a set of measure zero. The space $L^2(\mu)$ is a Hilbert space with inner product given by $\langle f, g\rangle = (\int_X f\overline{g} d\mu)^{1/2}$.

Definition 0.1.3 Let X be a topological space. We define $C(X)$ to be the space of continuous complex functions on X. We define $C_c(X)$ to be the space of continuous complex functions on X with compact support.

We say that a topological space X is *locally compact* if every point in X has a compact neighbourhood. If X is a locally compact Hausdorff space, a complex function f on X is said to *vanish at infinity* if for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ for all x not in K. We define $C_0(X)$ to be the space of continuous functions on X which vanish at infinity. Note that when X is compact, $C_c(X) = C_0(X) = C(X)$.

Because we are interested in generalising wavelets, we shall need to understand the Fourier transform on an arbitrary locally compact Abelian group. The Fourier transform is a very important tool for studying wavelets. Both the Fourier transform and wavelets are used to decompose spaces of functions into a basis, and are useful for applications such as signal processing. Most of the background information on the Fourier transform and locally compact Abelian groups in this section has been obtained from [Ru2].

On any locally compact Abelian group G we can define the *Haar measure* μ_G, which satisfies the property that it is translation invariant, ie $\mu_G(E + x) = \mu_G(E)$ for every $x \in G$, and measurable $E \subseteq G$. The Haar measure is unique up to multiplication by constants.

A *character* on G is a continuous homomorphism $\xi : G \to \mathbf{T}$ from G to the unit circle $\mathbf{T}$. The set of all characters of G forms a group $\hat{G}$, called the *Pontryjagin dual group* of G, where the group operation is defined by $(\xi_1\xi_2)(x) = \xi_1(x)\xi_2(x)$ for $x \in G$, $\xi_1, \xi_2 \in \hat{G}$. The space $\hat{G}$ is also a locally compact Abelian group. It can be shown that the dual of $\hat{G}$ is G. We will often write $(x, \xi) := \xi(x)$.

For $f \in L^1(G)$, we define the *Fourier transform* $\mathcal{F}f$ of f to be a function on $\hat{G}$ given by

$$(\mathcal{F}f)(\xi) \equiv \hat{f}(\xi) := \int_G f(x)\xi(x)dx,$$

where $\xi \in \hat{G}$. The Fourier transform has the following properties (see Chapter 1 of [Ru2]):

1. The Fourier transform maps $L^1(G)$ into a dense subalgebra of $C_0(\hat{G})$;

2. For $f, g \in L^1(G)$ the Fourier transform of $f * g$ is $\hat{f}\hat{g}$, with multiplication in the Fourier domain being pointwise, where $f * g$ is defined by

   $$(f * g)(x) = \int_G f(x-y)g(y)dy;$$

   We call $f * g$ the convolution of f and g .

3. The Fourier transform is a continuous map of $L^1(G)$ into $C_0(\hat{G})$, and for $f \in L^1(G)$, $\|\hat{f}\|_\infty \leq \|f\|_1$;

4. If f is continuous and $\hat{f} \in L^1(\hat{G})$, then $\mathcal{F}(\mathcal{F}(f))(x) = f(-x)$;

5. The Fourier transform is an isometry of $(L^1 \cap L^2)(G)$ onto a dense linear subspace of $L^2(\hat{G})$, and may be extended uniquely to an isometry of $L^2(G)$ onto $L^2(\hat{G})$;

6. We have that $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$ for all $f, g \in L^2(G)$ (Plancharel's identity).

0.2 Involutive Algebras

In order to study Hilbert modules we will need some background material on involutive algebras, especially C^*-algebras. The information in this section has been obtained from [DB], [Dv], [Dx1] and [Pe].

Definition 0.2.1 A linear space $\mathcal{A}$ over a field F is an *associative algebra* over F if for each $a, b, c \in \mathcal{A}$, $\lambda \in F$:

1. $a(bc) = (ab)c$;

2. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$;

3. $\lambda(ab) = (\lambda a)b = a(\lambda b)$.

The algebra is *real* if $F = \mathbf{R}$, and *complex* if $F = \mathbf{C}$. We say that $\mathcal{A}$ is *commutative* if $ab = ba$ for all $a, b \in \mathcal{A}$. If there exists an element $\mathbf{1} \in \mathcal{A}$ such that $\mathbf{1}a = a = a\mathbf{1}$ for all $a \in \mathcal{A}$ we say that $\mathcal{A}$ is *unital*.

The algebras in which we are interested shall almost always be complex algebras.

Definition 0.2.2 A *complex $*$-algebra* (or an *involutive algebra*) is an associative algebra with a mapping $a \to a^*$ of $\mathcal{A}$ into itself which satisfies for all $a, b \in \mathcal{A}$, $\lambda \in \mathbf{C}$:

1. $(a+b)^* = a^* + b^*$;
2. $(\lambda a)^* = \bar{\lambda} a^*$;
3. $(ab)^* = b^* a^*$;
4. $a^{**} = a$.

We call the map $a \to a^*$ an *involution*. Note that it is bijective.

Definition 0.2.3 An algebra $\mathcal{A}$ with a norm which satisfies for all $a, b \in \mathcal{A}$,

$$\|ab\| \leq \|a\|\|b\|$$

is called a *normed algebra*. A normed algebra which is also a Banach space (so it is complete with repect to the topology induced by its norm) is called a *Banach algebra*.

Definition 0.2.4 A *C^*-algebra* is a Banach $*$-algebra whose norm satisfies the C^*-condition:

$$\|a^* a\| = \|a\|^2$$

for $a \in \mathcal{A}$. A *pre-C^*-algebra* is a normed $*$-algebra with norm satisfying the C^*-condition but which is not necessarily complete. A *real C^*-algebra* is a Banach $*-$algebra over $\mathbf{R}$ with identity $\mathbf{1}$ which satisfies the C^*-condition and also the condition that $\mathbf{1} + a^* a$ is invertible.

We remark remark that the term "pre-C^*-algebra" sometimes has a slightly different meaning in some of the literature, where it also "is stable under the holomorphic functional calculus". This meaning is different to the definition above.

An important example of a C^*-algebra is the algebra of bounded operators $B(\mathcal{H})$ on a Hilbert space. The involution is the adjoint operation and the C^*-condition is satisfied because

$$\|A^* A\| = \sup_{\|x\|=\|y\|=1} |\langle A^* A x, y\rangle| = \sup_{\|x\|=1} |\langle Ax, Ax\rangle| = \|A\|^2.$$

Another important example of a C^*-algebra is $C_0(X)$, the space of continuous functions vanishing at infinity on a locally compact Hausdorff space X. The involution is given by complex conjugation and the norm is given by $\|f\| = \sup_{x \in X} |f(x)|$.

Any subalgebra of a C^*-algebra which is closed under involutions and is norm closed is also a C^*-algebra. One interesting closed subalgebra of $B(\mathcal{H})$ is $K(\mathcal{H})$, the algebra of compact operators on a separable Hilbert space $\mathcal{H}$.

To study Hilbert modules we will need a notion of positivity. We say that an element a of a C^*-algebra $\mathcal{A}$ is *positive* if we can write $a = b^* b$ for some $b \in \mathcal{A}$. We denote the positive elements of $\mathcal{A}$ by $\mathcal{A}^+$.

Two important theorems about C^*-algebras are the Gelfand-Naĭmark theorems, which states that every C^*-algebra is isomorphic to a subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$; and that every commutative C^*-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.

Theorem 0.2.5 (Gelfand-Naĭmark Theorems) *1. Let $\mathcal{A}$ be a commutative C^*-algebra. There is a locally compact Hausdorff space X such that $\mathcal{A}$ is isometrically $*$-isomorphic to $C_0(X)$.*

2. Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}$ is isometrically $*$-isomorphic to a norm-closed $*$-subalgebra of the bounded linear operators on some Hilbert space.

Definition 0.2.6 Let $\mathcal{A}$ be an involutive algebra and $\mathcal{H}$ be a Hilbert space. A *representation* of $\mathcal{A}$ in $\mathcal{H}$ is a $*$-homomorphism of the algebra $\mathcal{A}$ into $B(\mathcal{H})$.

We say that two representations $\rho : \mathcal{A} \to B(\mathcal{H})$ and $\tilde{\rho} : \mathcal{A} \to B(\tilde{\mathcal{H}})$ are *unitarily equivalent* if there is a unitary isomorphism U of $\mathcal{H}$ onto $\tilde{\mathcal{H}}$ such that

$$\rho(a) = U\tilde{\rho}(a)U^*, \quad \text{for all } a \in \mathcal{A}$$

where $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are the representation spaces of ρ and $\tilde{\rho}$. A representation ρ of $\mathcal{A}$ in $\mathcal{H}$ is *topologically irreducible* if $\mathcal{H}$ is nontrivial and the only closed subspaces of $\mathcal{H}$ invariant under ρ are 0 and $\mathcal{H}$.

An important property of topologically irreducible representations is given by Proposition 0.2.7. It is worth remarking that this proposition is a consequence of 0.2.5. We shall use it in the proof of Theorem 0.3.6.

Proposition 0.2.7 ([Dv], Corollary I.9.11, page 33) *For a C^*-algebra $\mathcal{A}$, and $a \in \mathcal{A}$, there is a topologically irreducible representation ρ of $\mathcal{A}$ such that*

$$\|\rho(a)\| = \|a\|.$$

Let S be a subset of $B(\mathcal{H})$, the *commutant* of S, denoted S' is the set of elements of $B(\mathcal{H})$ which commute with every element of S.

Proposition 0.2.8 ([Dv], Lemma I.9.1, page 26) *A representation $\rho : \mathcal{A} \to B(\mathcal{H})$ is topologically irreducible if and only if the commutant $\rho(\mathcal{A})' = \mathbf{C}\mathbf{1}$.*

PROOF: If $\rho(\mathcal{A})'$ is larger than the scalars, it will contain a proper projection P (for example a projection onto a one-dimensional subspace), and $P\mathcal{H}$ will be an invariant subspace for $\rho(\mathcal{A})$ and ρ will not be topologically irreducible.

Suppose $\rho(\mathcal{A})' = \mathbf{C}\mathbf{1}$, suppose K is a closed subspace of $\mathcal{H}$ invariant under $\rho(\mathcal{A})$. The projection P_K will commute with $\rho(\mathcal{A})$, and so either $P_K = 0$ or $P_K = \mathbf{1}$. Thus $K = 0$ or $K = \mathcal{H}$. □

In this thesis we shall make use of the following types of elements of involutive algebras.

Definition 0.2.9 For an involutive algebra $\mathcal{A}$, a *projection* $p \in \mathcal{A}$ is a self-adjoint idempotent: $p = p^* = p^2$. We say that two projections are *orthogonal* when $pq = 0$. We will denote the orthogonal sum of two projections by $p \oplus q$.

We say that $v \in \mathcal{A}$ is a *partial isometry* when v^*v is a projection. If $\mathcal{A}$ is unital and $v^*v = 1$, we say that v is an *isometry*.

We say that two projections p and q in a C^*-algebra $\mathcal{A}$ are *equivalent*, when there is a partial isometry $v \in \mathcal{A}$ such that $p = v^*v$ and $q = vv^*$, we write $p \sim q$;

Let us now introduce the concept of a von Neumann algebra. This concept will enable us to prove a theorem about the existence of wavelets given a multiresolution analysis (Theorem 1.1.11).

Definition 0.2.10 A *von Neumann algebra* $\mathcal{A}$ in $\mathcal{H}$ is an involutive subalgebra of $B(\mathcal{H})$ such that $\mathcal{A}$ is equal to the commutant of its commutant, so $\mathcal{A} = \mathcal{A}''$. For M a subset of $B(\mathcal{H})$, the *von Neumann algebra generated by* M is the commutant of $M \cup M^*$.

An important class of von Neumann algebras are the *finite* von Neumann algebras. The following definition is based on [Dx2, I.6.1, Definition 1; I.6.7, Definition 5; III.2.1].

Definition 0.2.11 Let $\mathcal{A}$ be a von Neumann algebra and let $\mathcal{A}^+$ be the positive elements of $\mathcal{A}$. A *finite normal trace* on $\mathcal{A}^+$ is a function τ defined on $\mathcal{A}^+$ taking non-negative real values which satisfies

1. If $S \in \mathcal{A}^+$ and $T \in \mathcal{A}^+$, then $\tau(S + T) = \tau(S) + \tau(T)$;
2. If $S \in \mathcal{A}^+$ and if λ is a non-negative real number, then $\tau(\lambda S) = \lambda\tau(S)$;
3. If $S \in \mathcal{A}^+$ and if U is a unitary operator of $\mathcal{A}$, then $\tau(USU^{-1}) = \tau(S)$;
4. For each increasing net $\mathcal{F} \subset \mathcal{A}^+$ with supremum $S \in \mathcal{A}^+$, $\tau(S)$ is the supremum of $\tau(\mathcal{F})$.

A von Neumann algebra $\mathcal{A}$ is said to be *finite* if for every non-zero $T \in \mathcal{A}^+$, there exists a finite normal trace τ on $\mathcal{A}^+$ such that $\tau(T) \neq 0$. We say that a projection in a von Neumann algebra is finite if its range is a finite von Neumann algebra.

Definition 0.2.12 Let $\mathcal{A}$ be a von Neumann algebra and let E be a projection in $\mathcal{A}$. Let $\mathcal{A}_E$ be the set of all elements B of $\mathcal{A}$ which can be written in the form $B = EAE$ for some $A \in \mathcal{A}$. We know from [Dx2, I.2.1, Proposition 1] that $\mathcal{A}_E$ is a von Neumann algebra. We say that a projection E is *finite* if the algebra $\mathcal{A}_E$ is finite.

We are interested in finite von Neumann algebras because their properties are useful for proving Theorem 1.1.11, which tells us that we can obtain multiwavelets from a multiresolution analysis. In order to prove Theorem 1.1.11, we need the following result about von Neumann algebras.

Proposition 0.2.13 ([Dx2] III.2.3, Proposition 6, page 261) *Let $\mathcal{A}$ be a von Neumann algebra and let E, F be two equivalent finite projections of $\mathcal{A}$, and G a projection of $\mathcal{A}$ majorising E and F (in other words, $G-E$ and $G-F$ are positive). There exists a unitary operator U of $\mathcal{A}$ such that $UEU^{-1} = F$ and $UGU^{-1} = G$. In particular, $G - E \sim G - F$.*

0.3 Group Representations and Group Algebras

We use the notation $U(\mathcal{H})$ to denote the unitary operators on a Hilbert space $\mathcal{H}$.

Definition 0.3.1 A *unitary representation* of a locally compact group G on a nonzero Hilbert space $\mathcal{H}_\pi$ is a homomorphism $\pi : G \to U(\mathcal{H}_\pi)$ such that $s \mapsto \pi_s h$ is continuous from G to $\mathcal{H}_\pi$ for every $h \in \mathcal{H}_\pi$. We call $\mathcal{H}_\pi$ the *representation space* of π.

If E is a closed subspace of $\mathcal{H}_\pi$, we call E an *invariant subspace* for π if $\pi(g)E \subseteq E$ for all $g \in G$. If E is invariant and nonzero, the restriction of π to E,

$$\pi^E(g) = \pi(g)|_E$$

is also a representation of π on E, which we call a *subrepresentation* of π.

We say that two representations $\pi : G \to U(\mathcal{H}_\pi)$ and $\tilde{\pi} : G \to U(\mathcal{H}_{\tilde{\pi}})$ are *equivalent* if there is a unitary operator $U : \mathcal{H}_\pi \to \mathcal{H}_{\tilde{\pi}}$ such that for all $g \in G$, $\tilde{\pi}(g) = U\pi(g)U^*$.

If $\{\pi_i\}_{i \in I}$ is a family of unitary representations, their direct sum $\oplus_i \pi_i$ is the representation π on $\mathcal{H} = \oplus_i \mathcal{H}_{\pi_i}$, defined by $\pi(x)(\sum_i v_i) = \sum_i \pi_i(x) v_i$, where each $v_i \in \mathcal{H}_{\pi_i}$.

When $\pi = \oplus_i \pi_i$, each $\mathcal{H}_{\pi_i}$ is an invariant subspace of $\mathcal{H}_\pi$, and each π_i is a subrepresentation of π (see [Fo], page 70).

Example 0.3.2 Suppose that a locally compact group G acts on a locally compact Hausdorff space X. The group G also acts on functions on X by

$$(\pi(g)f)(x) = f(g^{-1}x) \tag{1}$$

for $g \in G$, $x \in X$. If X has a G-invariant Radon measure μ, then π defines a unitary representation on $L^2(\mu)$ (see page 68 of [Fo]).

The following proposition relates unitary representations of groups to projections in a von Neumann algebra.

Proposition 0.3.3 ([Co] V.1.α, Proposition 3, page 450) *Let π be a unitary representation of a locally compact group G on a Hilbert space $\mathcal{H}_\pi$, consider the commutant*

$$R(\pi) = \{T \in B(\mathcal{H}_\pi) : T\pi(g) = \pi(g)T \quad \textit{for all } g \in G\ \}$$

which is by construction a von Neumann algebra. $R(\pi)$ satisfies

1. *If E is a closed subspace of $\mathcal{H}_\pi$ and P_E is the projection onto E, then E is an invariant subspace for π if and only if $P_E \in R(\pi)$.*

2. *If E_1 and E_2 are invariant subspaces for π, then the subrepresentations π^{E_1} and π^{E_2} are equivalent if and only if $P_{E_1} \sim P_{E_2}$.*

PROOF: The reader is referred to page 70 of [Fo] for a proof that $R(\pi)$ is a von Neumann algebra.

1. Let g be an arbitrary element of G. Suppose that $P_E \in R(\pi)$ and that $v \in E$, then $\pi(g)v = \pi(g)P_E v = P_E\pi(g)v \in E$, and so E is an invariant subspace for π. Suppose now that E is an invariant subspace for π, let v be an element of E, and let u be an element of $E^\perp$. We have that $\pi(g)P_E v = \pi(g)v = P_E\pi(g)v$. We also have that $\langle \pi(g)u, v\rangle = \langle u, \pi(g)^{-1}v\rangle = 0$, so $\pi(g)u \in E^\perp$. We therefore have that $\pi(g)P_E u = 0 = P_E\pi(g)u$, and so $\pi(g)P_E = P_E\pi(g)$.

2. Suppose that π^{E_1} is equivalent to π^{E_2}. Let U be the unitary operator $U : \mathcal{H}_{\pi^{E_1}} \to \mathcal{H}_{\pi^{E_2}}$ for which $\pi^{E_2}(g) = U\pi^{E_1}(g)U^*$. Let $W = UP_{E_1} = P_{E_2}UP_{E_1}$, then $W^* = P_{E_1}U^*P_{E_2} = U^*P_{E_2}$. We then have that
$$\begin{aligned} WW^* &= P_{E_2}UP_{E_1}P_{E_1}U^*P_{E_2} = P_{E_2}, \\ W^*W &= P_{E_1}U^*P_{E_2}P_{E_2}UP_{E_1} = P_{E_1}. \end{aligned}$$
So W is the partial isometry which provides an equivalence between P_{E_1} and P_{E_2}.

   Suppose now that $P_{E_1} \sim P_{E_2}$. There then exists a partial isometry W such that $W^*W = P_{E_1}$ and $WW^* = P_{E_2}$. The restriction of W to E_1 is a unitary mapping onto E_2, with inverse W^*. So $\pi^{E_2}(g) = W\pi^{E_1}(g)W^*$.

□

Let G be a locally compact group, recall that $C_c(G)$ is the space of all complex valued continuous functions on G with compact support. We make $C_c(G)$ into an algebra over the complex numbers $\mathbf{C}$ with multiplication given by convolution:
$$(a * b)(s) = \int_G a(st^{-1})b(t)dt = \int_G a(t)b(t^{-1}s)dt$$
for $a, b \in C_c(G)$, (recall that we defined the convolution for functions on Abelian groups in Section 0.1). We define a norm on $C_c(G)$ by
$$\|a\|_1 = \int_G |a(t)|d\mu_G(t)$$
for $a \in C_c(G)$. The completion of $C_c(G)$ with respect to this norm is the Banach algebra $L^1(G)$. It is not hard to show that $L^1(G)$ and $C_c(G)$ are commutative if and only if G is Abelian.

Let μ_G be left Haar measure on G. There exists a continuous homomorphism Δ (see [Fo, page 46]) from G into $\mathbf{R}_+$ known as the *modular function* such that

$$\mu_G(Es) = \Delta(s)\mu_G(E), \quad s \in G,\ E \subseteq G.$$

It is shown in [Fo, Proposition 2.24] that the modular function Δ is a continuous homomorphism from G into the multiplicative group of positive real numbers. Hence for $s \in G$, $\Delta(s^{-1}) = (\Delta(s))^{-1}$. The spaces $L^1(G)$ and $C_c(G)$ are involutive algebras with involution given by

$$f^*(s) = \Delta(s^{-1})\overline{f(s^{-1})}. \tag{2}$$

Abelian groups have the property that for all $s \in G$, $\Delta(s) = 1$. We shall make use of the modular function in Remark 2.2.10.

When G is an Abelian group, $C_c(G)$ is a commutative $*$-algebra and $L^1(G)$ is a commutative Banach $*$-algebra with respect to this involution. However, $L^1(G)$ is not a C^*-algebra because the norm does not satisfy the C^*-condition. Commutative Banach algebras have some properties which are useful.

Definition 0.3.4 We define a *multiplicative linear functional* on a commutative Banach algebra $\mathcal{A}$ to be a non-zero algebra homomorphism of $\mathcal{A}$ into $\mathbf{C}$. We denote the set of all multiplicative functionals by $M_{\mathcal{A}}$.

We define the *Gelfand transform* $\wedge : \mathcal{A} \to C_0(M_{\mathcal{A}})$ of a commutative Banach algebra into $C_0(M_{\mathcal{A}})$ by

$$\hat{a}(\varphi) = \varphi(a)$$

where $a \in \mathcal{A}$ and $\varphi \in M_{\mathcal{A}}$.

The Gelfand transform is used in the proof of the commutative Gelfand-Naimark theorem to construct the isomorphism between $\mathcal{A}$ and $C_0(X)$ when $\mathcal{A}$ is a C^*-algebra and X is a Hausdorff space. The Gelfand transform is the abstract analogue of the usual Fourier transform, because of the following result.

Proposition 0.3.5 *Suppose G is a locally compact Abelian group, and $\xi \in \hat{G}$, then the Fourier transform defines a multiplicative linear functional of $L^1(G)$ by $f \to \hat{f}(\xi)$. Every multiplicative linear functional is obtained in this way, and distinct characters induce distinct multiplicative linear functionals. Thus the Fourier transform of a function $f \in L^1(G)$ is precisely the Gelfand transform of f.*

PROOF: See Theorem 1.2.2 of [Ru2], and the remarks in Section 1.2.3 of [Ru2]. □

Suppose that π is a unitary representation of a locally compact group G. Associated with π there is a $*$-representation $\rho_\pi : L^1(G) \to B(\mathcal{H}_\pi)$ such that

$$\langle \rho_\pi(f)h, k\rangle = \int_G f(s)\langle \pi_s h, k\rangle ds \tag{3}$$

for all $f \in L^1(G), h, k \in \mathcal{H}_\pi$ (see [Fo], page 73). We will construct a C^*-algebra by defining another norm on $L^1(G)$ as

$$\|f\|_{C^*(G)} = \sup\{\|\rho(f)\| : \ \rho \text{ is a } *\text{-representation of } L^1(G)\ \}. \tag{4}$$

We know that the set of $*$-representations of $L^1(G)$ is non-empty because there is a $*$-representation of $L^1(G)$ associated with every unitary representation of G. The algebra $L^1(G)$ is a pre-C^*-algebra with respect to this norm, and we call the completion of $L^1(G)$ with respect to this norm the *group C^*-algebra* $C^*(G)$ of G. The algebra $C_c(G)$ is also a pre-C^*-algebra with repect to this norm, and is dense in $C^*(G)$. We shall make extensive use of group C^*-algebras throughout this thesis. For example, we can construct a group C^*-algebra from the group of translations that act on a wavelet. We make use of this in Chapter 2.

There is another C^*-algebra associated with G, the reduced C^*-algebra, which we shall now define. The *left regular representation* of G is the unitary representation on $L^2(G)$ defined by

$$\lambda_t(h)(s) = h(t^{-1}s) \tag{5}$$

where $s, t \in G$, $h \in L^2(G)$. The left regular representation of $L^1(G)$ is a $*$-representation $\lambda : L^1(G) \to B(L^2(G))$ and is defined by

$$\lambda_a(h)(s) = (a * h)(s) = \int_G a(st)h(t^{-1})dt \tag{6}$$

where $a \in L^1(G)$, $h \in L^2(G)$, $s, t \in G$. It can be shown (see [Fo, Page 73]) that for $k \in L^2(G)$,

$$\langle \lambda_a(h), k\rangle = \int_G a(t)\langle \lambda_t(h), k\rangle dt.$$

We now define the reduced C^*-algebra of G to be the norm closure $C^*_r(G) := \overline{\lambda(L^1(G))}$.

It is also possible to define the group C^*-algebra and the reduced group C^*-algebra for an arbitrary locally compact group G. The following theorem gives us an explicit formula for the norm of both the group C^*-algebra and the reduced group C^*-algebra when G is Abelian. Because of the relevance of this theorem to the Hilbert C^*-modules that we will study in this thesis, we include a proof. The proof is based on the proof in [Dv] and on some arguments in Example C.20 of [RW].

Theorem 0.3.6 ([Dv] Proposition VII.1.1, page 184) *If G is a locally compact Abelian group, then*

$$C^*(G) \cong C^*_r(G) \cong C_0(\hat{G}).$$

The C^*-norm of an element a of $C^*(G)$ is given by

$$\|a\| = \sup_{\xi\in\hat{G}} |\hat{a}(\xi)|. \tag{7}$$

Proof (Outline): By Proposition 0.2.7,

$$\|a\| = \sup\{\|\rho(a)\| : \ \rho \text{ is an irreducible representation}\} \tag{8}$$

for $a \in C^*(G)$. Let ρ be an irreducible representation of $L^1(G)$ on a Hilbert space $\mathcal{H}_\rho$. The representation $\rho(L^1(G))$ is commutative, so $\rho(L^1(G)) \subseteq \rho(L^1(G))'$. Proposition 0.2.8 therefore tells us that $\rho(L^1(G)) \subseteq \rho(L^1(G))' = \mathbf{C1}$. Thus every irreducible representation of $L^1(G)$ (and so of $C^*(G)$) is one dimensional. This means that the irreducible representations of $L^1(G)$ correspond to multiplicative linear functionals. By Proposition 0.3.5 the Gelfand map sends $f \in L^1(G)$ to its Fourier transform $\hat{f} \in C_0(\hat{G})$.

Equation (8) tells us that

$$\|a\| = \sup\{\|\rho(a)\| : \ \rho \text{ is an irreducible representation}\} = \|\hat{a}\|_\infty = \sup_{\xi \in \hat{G}} |\hat{a}(\xi)|$$

The range of the Fourier transform $\mathcal{F}$ is self-adjoint and separates points and so is dense in $C_0(\hat{G})$ by the Stone-Weierstrass theorem. The Fourier transform converts convolution to pointwise multiplication and the involution to complex conjugation. The map $a \mapsto \hat{a}$ extends to an isomorphism from $C^*(G)$ onto $C_0(\hat{G})$.

Now the Fourier transform extends to a unitary operator from $L^2(G)$ onto $L^2(\hat{G})$. If we conjugate the left regular representation by the Fourier transform we obtain for $f \in L^1(G)$ and $g \in L^2(G) \cap L^1(G)$

$$\mathcal{F}\lambda f \mathcal{F}^* \hat{g} = \mathcal{F}\lambda(f) g = \mathcal{F}(f * g) = \hat{f}\hat{g} = M_{\hat{f}}\hat{g}.$$

where $M_{\hat{F}}$ is the operator consisting of multiplication by $\hat{f}$. Each $\hat{f} \in C_0(\hat{G})$ is sent to the multiplication operator $M_{\hat{f}}$. This map is an isometric isomorphism and so $C_r^*(G) \cong C_0(\hat{G})$. □

0.4 Hilbert C^*-Modules

We now define Hilbert C^*-modules, the main tool we will use to study wavelets.

Definition 0.4.1 Suppose $\mathcal{A}$ is a C^*-algebra or a pre-C^*-algebra. A *right inner product* $\mathcal{A}$*-module* is a complex linear space E which is a right $\mathcal{A}$-module with compatible scalar multiplication: $\alpha(xa) = (\alpha x)a = x(\alpha a)$ for $x \in E$, $a \in \mathcal{A}$, and $\alpha \in \mathbf{C}$; and which has an $\mathcal{A}$-valued inner product $[\ ,\ ]_E : E \times E \to \mathcal{A}$ satisfying for all $x, y, z \in E$, $a \in \mathcal{A}$ and $\alpha, \beta \in \mathbf{C}$,

1. $[x, \alpha y + \beta z]_E = \alpha[x, y]_E + \beta[x, z]_E$
2. $[x, ya]_E = [x, y]_E a$
3. $[x, y]_E^* = [y, x]_E$
4. $[x, x]_E \geq 0$ (in the completion of $\mathcal{A}$ if $\mathcal{A}$ is a pre-C^*-algebra)
5. $[x, x]_E = 0 \Rightarrow x = 0$.

We can define a norm on E by $\|x\|_E = \|[x,x]_E\|^{\frac{1}{2}}$, we call this norm the *Hilbert module norm* on E. When $\mathcal{A}$ is a C^*-algebra, a right inner product $\mathcal{A}$-module which is complete with respect to its norm is called a *right Hilbert C^*-module over $\mathcal{A}$*, or a *right Hilbert $\mathcal{A}$-module.*

We call a right Hilbert $\mathcal{A}$-module E *full* if $[E,E]_E$ is dense in $\mathcal{A}$.

The reader is referred to Corollary 2.7 of [RW] for a proof that $\|\cdot\|_E$ is a norm.

We will often abbreviate a right inner product $\mathcal{A}$-module as an *inner product $\mathcal{A}$-module* and a right Hilbert $\mathcal{A}$-module as a *Hilbert $\mathcal{A}$-module*, or just a *Hilbert module.* It is also possible to define a *left Hilbert module* where the algebra acts on the left and the inner product is conjugate linear with respect to the second variable. A left Hilbert module is essentially the same as a right Hilbert module.

We shall show in Lemma 0.4.5 that because we can obtain a C^*-algebra $\mathcal{A}$ by completing $\mathcal{A}_0$, we can extend the module action of $\mathcal{A}_0$ on E_0 to a module action of $\mathcal{A}$ on the completion E of E_0, and obtain a Hilbert $\mathcal{A}$-module.

Useful references on Hilbert modules include [L], [R1], [R2], [RW], and [W-O].

Examples 0.4.2

1. Every Hilbert space is a full left Hilbert $\mathbf{C}$-module with the usual operations.

2. If $\mathcal{A}$ is a C^*-algebra, then $\mathcal{A}$ is a full Hilbert $\mathcal{A}$-module with module action given by C^*-algebraic multiplication, and $\mathcal{A}$-valued inner product $[a,b] = a^*b$, for $a, b \in \mathcal{A}$.

3. Let $\mathcal{A}^n$ be the direct sum of n copies of a C^*-algebra $\mathcal{A}$. We make $\mathcal{A}^n$ into a full Hilbert $\mathcal{A}$-module with the module action given by $(a_1, \ldots, a_n) \circ b = (a_1 b, \ldots, a_n b)$. The inner product is given by
$$[(a_1, \ldots, a_n), (b_1, \ldots, b_n)] = \sum_{i=1}^{n} a_i^* b_i.$$

4. For a C^*-algebra $\mathcal{A}$, the *standard Hilbert $\mathcal{A}$-module* is defined to be
$$\mathcal{H}_{\mathcal{A}} := \left\{ \mathbf{a} = (a_i) \in \prod_{i=1}^{\infty} \mathcal{A} : \sum_{i=1}^{\infty} a_i^* a_i \text{ converges in } \mathcal{A} \right\}$$
with the scalar multiplication and $\mathcal{A}$-valued inner product defined to be
$$\mathbf{a}a := (a_i a), \quad [\mathbf{a}, \mathbf{b}] := \sum_{i=1}^{\infty} a_i^* b_i.$$
The standard Hilbert module is important because of the Kasparov Stabilisation Theorem, which is stated below.

Lemma 0.4.3 (The Polarisation Identity) *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are complex vector spaces and $B : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is a map which is conjugate linear in the first variable and linear in the second variable. The map B satisfies the following identity:*

$$B(f,g) = \frac{1}{4}\left(B(g+f, g+f) + iB(g+if, g+if) - iB(g-if, g-if) - B(g-f, g-f)\right)$$

for $f, g \in \mathcal{X}$. Consequently, if E is a right inner product $\mathcal{A}$-module with inner product $[\ ,\ ]_E$, for a complex normed $*$-algebra $\mathcal{A}$, then for $f, g \in E$,

$$[f,g]_E = \frac{1}{4}\left([g+f,g+f]_E + i[g+if,g+if]_E - i[g-if,g-if]_E - [g-f,g-f]_E\right).$$

PROOF: Expanding the right hand side of the above identity verifies the result. □

We shall sometimes need to make use of the following result.

Lemma 0.4.4 (The Hilbert module Cauchy-Schwarz inequality) *Suppose that $\mathcal{A}$ is a C^*-algebra. If E is an inner product $\mathcal{A}$-module, then for $x, y \in E$,*

$$[x,y]_E^*[x,y]_E \le \|[x,x]_E\|[y,y]_E. \tag{9}$$

For a proof to this Lemma, the reader is referred to Lemma 2.5 of [RW].

The following result is very useful for constructing Hilbert C^*-modules from inner product pre-C^*-modules. It is a slightly less general version of [RW, Lemma 2.16, p15], the proof has been adapted from the proof contained in [RW]. In Chapter 2 we shall make use of this Lemma to prove Theorem 2.1.21.

Lemma 0.4.5 *Suppose that $\mathcal{A}_0$ is a dense $*$-subalgebra of a C^*-algebra $\mathcal{A}$. Suppose that E_0 is a right inner product $\mathcal{A}_0$-module. Let the linear space E be the completion of E_0 with respect to the Hilbert module norm. Then the module action of $\mathcal{A}_0$ on E_0 can be extended to a module action of $\mathcal{A}$ on E and the $\mathcal{A}_0$-valued inner product on E_0 can be extended to an $\mathcal{A}$-valued inner product on E in such a way that E is a right Hilbert $\mathcal{A}$-module. We call the Hilbert module E the completion of the inner product module E_0.*

PROOF: Let the inner product on E_0 be denoted by $[\ ,\ ]_0$, and the module action of $a \in \mathcal{A}_0$ on $x \in E_0$ be denoted by $x \cdot a$. As stated in Definition 0.4.1, the Hilbert module norm on E_0 is given by $\|x\|_{E_0} = \|[x,x]_0\|^{\frac{1}{2}}$ for $x \in E_0$. We calculate

$$\begin{aligned} \|x\cdot a\|_{E_0}^2 &= \|[x\cdot a, x\cdot a]_0\| \\ &= \|a^*[x,x]_0 a\| \end{aligned}$$

Now we know from Definition 0.4.1 that $[x,x]_0$ is positive in $\mathcal{A}$, so there exists $b \in \mathcal{A}$ such that $[x,x]_0 = b^*b$. From the previous calculation and the inequality

$$a^*b^*ba \le \|b\|^2 a^*a, \quad a, b \in \mathcal{A},$$

we deduce that

$$\|x\cdot a\|_{E_0}^2 \le \|a\|^2\|x\|_{E_0}^2. \tag{10}$$

We therefore have that right multiplication by $a \in \mathcal{A}_0$ is a bounded operator on E_0. It follows from the Hahn-Banach Theorem (see [KR, Theorem 1.6.1]) that right multiplication by $a \in \mathcal{A}_0$ extends to a bounded linear operator on E. Define

$$x\cdot a := \lim_{a_n \to a} x\cdot a_n, \quad a \in \mathcal{A},\ a_n \in \mathcal{A}_0,\ x \in E, \tag{11}$$

we want to show that the above limit does not depend on the choice of (a_n) and converges to an element of E. From equation (10) it follows that the operator of right multiplication by a_n has norm less than or equal to $\|a_n\|$. It therefore follows that if (a_n) converges to a in $\mathcal{A}$, then right multiplication by a_n converges to right multiplication by a in the norm topology on the Banach space dual of E. It therefore follows by uniform continuity that the above limit does not depend on the choice of (a_n) and converges to an element of E because it is complete.

We now extend the inner product to E. If x_n and y_n are sequences in E_0 that converge to $x, y \in E$, then we define

$$[x, y]_E := \lim_n [x_n, y_n]_0. \tag{12}$$

Now from Lemma 0.4.4 it follows that $\|[x_n, y_n]\| \leq \|x_n\|_{E_0}\|y_n\|_{E_0}$. We therefore have that taking the inner product with another element of E_0 is a bounded operator on E_0 and by uniform continuity the above limit is well defined and contained in $\mathcal{A}$. The space E satisfies the first three properties of Definition 0.4.1 because E_0 does. The fourth property of Definition 0.4.1 holds because the positive cone $\mathcal{A}^+$ is in fact always closed by [KR, Theorem 4.2.2 (i)]. The fifth property of Definition 0.4.1 holds because if $[x, x]_E = 0$, then there is a sequence $x_n \to x$ for which $\|x\|_E \to 0$, and x must be the zero element of E. □

There are some useful analogues of bounded and compact operators for Hilbert modules.

Definition 0.4.6 Suppose E, F are Hilbert $\mathcal{A}$-modules. We define $\mathcal{L}(E, F)$ to be the set of all maps $t : E \to F$ for which there exists a map $t^* : F \to E$ such that

$$[tx, y]_F = [x, t^*y]_E$$

for all $x \in E, y \in F$. We call $\mathcal{L}(E, F)$ the set of *adjointable operators* from E to F. We abbreviate $\mathcal{L}(E, E)$ as $\mathcal{L}(E)$.

It can be shown that every element of $\mathcal{L}(E, F)$ is a bounded $\mathcal{A}$-linear map (see Lemma 2.18 of [RW]).

For $t \in \mathcal{L}(E, F)$ we define

$$\|t\| := \sup_{\|x\|_E \leq 1} \|tx\|_F = \sup_{\|x\|_E \leq 1} \|[tx, tx]_F\|.$$

Definition 0.4.7 Suppose E, F are Hilbert modules, with inner product in a C^*-algebra $\mathcal{A}$. For $x \in E, y, z \in F$, define

$$\Theta_{x,y}(z) = x[y, z]_F.$$

It can be shown that $\Theta_{x,y} \in \mathcal{L}(F, E)$, and $(\Theta_{x,y})^* = \Theta_{y,x}$ (see [L]). We define $\mathcal{K}(F, E)$ to be the closed linear subspace of $\mathcal{L}(F, E)$ spanned by $\{\Theta_{x,y} : x \in E, y \in F\}$. We call $\mathcal{K}(F, E)$ the *generalised compact operators* from E to F. $\mathcal{K}(F, E)$ is also known as the *imprimitivity algebra* of E and F. We abbreviate $\mathcal{K}(E, E)$ as $\mathcal{K}(E)$.

By applying the above definition one can show that for Hilbert $\mathcal{A}$-modules E, F, G the following relations hold (see [L]):

$$\Theta_{x,y}\Theta_{u,v} = \Theta_{x[y,u]_F,v} = \Theta_{x,v[u,y]_F}$$
$$t\Theta_{x,y} = \Theta_{tx,y}$$
$$\Theta_{x,y}s = \Theta_{x,s^*y}$$

where $x \in E$, $y \in F$, $u \in F$, $v \in G$, $t \in \mathcal{L}(E,G)$, $s \in \mathcal{L}(G,F)$.

In the case that E, F are Hilbert spaces (so the C^*-algebra is $\mathbf{C}$), $\mathcal{K}(E,F)$ is the space of compact operators between these Hilbert spaces. Note that the generalised compact operators on a Hilbert module may be different from the compact operators obtained from treating the Hilbert module as a Banach space. This somewhat confusing notation arises because of the way that Hilbert modules generalise Hilbert spaces.

It is in fact true that $\mathcal{L}(E)$ and $\mathcal{K}(E)$ are C^*-algebras with the norm defined above, and $\mathcal{K}(E)$ is a closed two-sided ideal in $\mathcal{L}(E)$. The reader is referred to [RW], Proposition 2.21 and Lemma 2.25 for proofs of these claims.

Definition 0.4.8 Suppose E, F are Hilbert modules, an operator $u \in \mathcal{L}(E,F)$ is called *unitary* if

$$u^*u = \mathbf{1}_E, \quad uu^* = \mathbf{1}_F.$$

We say that E and F are isomorphic if there exists a unitary u contained in $\mathcal{L}(E,F)$.

Definition 0.4.9 We say that two C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are *Morita equivalent* if there is a full Hilbert $\mathcal{A}_1$-module E such that $\mathcal{A}_2 \cong \mathcal{K}(E)$. We call E a *Morita equivalence bimodule.*

There are various equivalent descriptions of Morita equivalence. The term "bimodule" arises because one of the definitions of Morita equivalence involves constructing a right Hilbert $\mathcal{A}_1$-module which is also a left $\mathcal{A}_2$-module, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are C^*-algebras. A Morita equivalence bimodule is sometimes also known as an imprimitivity bimodule. Morita equivalence is an important part of the theory of Hilbert modules, is closely related to K-theory, and has important applications to the study of group representations [L, R1, R2, RW]. We won't directly make much use of Morita equivalence in this thesis, but it is worth keeping in mind when we construct Hilbert modules to study wavelets that a right-Hilbert module can be given an left inner product into a Morita equivalent C^*-algebra to construct a Morita equivalence bimodule.

A Hilbert $\mathcal{A}$-module E is *countably generated* if there exists a countable set $\Phi := \{\phi_i\}_{i\in\mathcal{I}} \subseteq E$ such that the submodule $E_\Phi := \{\sum_{i\in\mathcal{I}} \phi_i a_i : a_i \in \mathcal{A}\}$ is dense in E.

Theorem 0.4.10 (The Kasparov Stabilisation Theorem) *Suppose that $\mathcal{A}$ is a C^*-algebra and E is a countably generated Hilbert $\mathcal{A}$-module. Then $E \oplus \mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{A}$ are isomorphic as Hilbert modules.*

The Kasparov Stabilisation Theorem tells us that if a Hilbert module is not too big, it can be embedded in the standard Hilbert module. We remark that an interesting generalisation of Kasparov Stabilisation Theorem is described in [RT].

0.5 Frames and Bases

Recall from the introduction that a wavelet basis is a certain type of either an orthonormal basis or frame for a Hilbert space. In this section we shall study arbitrary orthonormal bases and frames for Hilbert spaces and then extend these concepts to Hilbert modules. We will use the convention that the Hilbert space inner product is linear in the first variable and conjugate linear in the second variable.

Hilbert spaces and Hilbert modules are both types of Banach spaces, so we shall begin by looking at bases for Banach spaces.

Definition 0.5.1 Let E be a Banach space with norm $\|\|$, and let $\{e_j\}_{j=1}^{\infty}$ be a sequence of elements of E. The sequence is a *Schauder basis* of E if for each $f \in E$, there exists a unique sequence of coefficients $\{\alpha_i : \alpha_i \in \mathbf{C}, i \in \mathbf{N}\}$ such that

$$\lim_{m \to \infty} \|f - (\alpha_1 e_1 + \ldots + \alpha_m e_m)\| = 0.$$

We can then write $f = \sum_{i=1}^{\infty} a_i e_i$, and we say that the basis is *unconditional* if the series converges to f after an arbitrary permutation of its terms.

Definition 0.5.2 An *orthonormal basis* for a Hilbert space $\mathcal{H}$ is a set $\{e_i\}_{i=1}^{\infty}$ of elements of $\mathcal{H}$, such that for all $i, j \in \mathbf{N}$, $\langle e_i, e_j \rangle = 0$ whenever $i \neq j$, and for all $f \in \mathcal{H}$, such that

$$f = \sum_{i=1}^{\infty} \langle f, e_i \rangle e_i.$$

A set of vectors $\{e_j\}_{j=1}^{\infty}$ in $\mathcal{H}$ is called a *frame* if there exists $A, B > 0$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_j |\langle f, e_j \rangle|^2 \leq B\|f\|^2, \tag{13}$$

we call a frame *tight* if $A = B$, and *normalised* if $A = B = 1$.

A *Riesz basis* for $\mathcal{H}$ is a set $\{e_i\}_{i=1}^{\infty}$ of elements of $\mathcal{H}$, which is also a frame for $\mathcal{H}$, and for which for any $f \in \mathcal{H}$, there exists a unique set of complex numbers $\{\alpha_i : \alpha_i \in \mathbf{C}, i \in \mathbf{N}\}$ such that

$$f = \sum_{i=1}^{\infty} \alpha_i e_i.$$

An equivalent definition of a Riesz basis is as the image of an orthonormal basis under an invertible operator. An equivalent definition of a frame for a Hilbert space $\mathcal{H}$ is as the image of an orthonormal basis for a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ under the projection from $\mathcal{K}$ to $\mathcal{H}$. It is also true that a set of elements of $\mathcal{H}$ is a Riesz basis if and only if it a bounded unconditional basis. The reader is referred to [HL1] for proofs of these assertions. Wavelet bases have additional properties which allow us to analyse a Hilbert space according to “resolution”.

Because an orthonormal basis is also a Riesz basis it is also an unconditional basis. Note that because the summands in equation (13) are positive, the sum does not depend on the order in which its terms are added. We can therefore define a set $\{e_i : i \in \mathcal{I}\}$ for $\mathcal{I}$ a countable or finite index set, to be a orthonormal basis, Riesz basis, or frame, without having to enumerate $\mathcal{I}$.

In [FL1], [FL2] and [FL3], the concept of a frame has been extended to Hilbert modules. First we introduce concepts analagous to an orthonormal basis.

Definition 0.5.3 Let $\mathcal{I}$ be a finite or countable index set, let $\mathcal{A}$ be a C^*-algebra. A subset $\{x_i : i \in \mathcal{I}\}$ of a Hilbert $\mathcal{A}$-module E is a *generating set* of E if the set $\{\sum_{i\in\mathcal{I}} x_i a_i : a_i \in \mathcal{A}\}$ is dense in E with respect to the Hilbert module norm. It is $\mathcal{A}$-*orthogonal* if $[x_i, x_j]_E = 0$ for all $i, j \in \mathcal{I}$ such that $i \neq j$. It is $\mathcal{A}$-*orthonormal* if it is $\mathcal{A}$-orthogonal and $\|x_i\|_E = 1$ for all $i \in \mathcal{I}$.

A generating set $\{x_i : i \in \mathcal{I}\}$ of E is a *Hilbert basis* of E if (i) $\mathcal{A}$-linear combinations $\sum_{i\in\mathcal{I}} x_i a_i$ with each a_i in $\mathcal{A}$ are equal to zero if and only if every summand $x_i a_i$ equals zero for $i \in \mathcal{I}$, and (ii) $\|x_i\| = 1$ for every $i \in \mathcal{I}$.

Definition 0.5.4 An $\mathcal{A}$-module M (not necessarily a Hilbert module) is *finitely (countably) generated* if there exists a finite (countable) set $\Phi := \{\phi_i\}_{i\in\mathcal{I}} \subseteq M$ such that the submodule $M_\Phi := \{\sum_{i\in\mathcal{I}} \phi_i a_i : a_i \in \mathcal{A}\}$ is dense in M. If $M_\Phi = M$, we say that M is *algebraically finitely (countably) generated*. An $\mathcal{A}$-module M is *projective* if it is a direct summand of a free module $\mathcal{A}^n$.

Before defining frames for Hilbert modules, we remark that if we have an infinite sum of elements of a C^*-algebra, the order of summation does not matter when the summands are positive, provided the sum converges (see [W-O], p21).

Definition 0.5.5 Suppose $\mathcal{A}$ is a unital C^*-algebra and E is a Hilbert $\mathcal{A}$-module. Let $\mathcal{I} \subset \mathbf{N}$ be an index set. A set $\{x_i : i \in \mathcal{I}\} \subset E$ is a *frame* for E if there exist positive real numbers $C, D > 0$ such that for all $x \in E$

$$C[x, x]_E \leq \sum_{i\in\mathcal{I}} [x, x_i]_E [x_i, x]_E \leq D[x, x]_E. \tag{14}$$

Note that the summands in the middle term of the above equation are positive, so the order of summation does not matter. If $C = D$ we call the frame *tight*; if $C = D = 1$ we call the frame a *normalised tight frame*. If the sum in (14) always converges in norm, we call the frame a *standard* frame. A *Riesz basis* for E is a frame for E which is also a generating set which has the property that for $S \subset \mathcal{I}$, $\sum_{i\in S} x_i a_i = 0$ if and only if for all $i \in S$, $x_i a_i = 0$.

If F is also a Hilbert $\mathcal{A}$-module and $\{y_i\}_i$ is a frame for F , then $\{x_i\}_i$ and $\{y_i\}_i$ are *unitarily equivalent* if there exists a unitary adjointable operator $T : E \to F$ such that $T(x_i) = y_i$ for all $i \in \mathcal{I}$.

The sum in equation (14) is sometimes only required to converge in a weaker topology than the norm topology, in which case the frame is not a standard frame.

The following theorems relate frames to Hilbert modules. The reader is referred to [FL1, FL2] for proofs of these theorems.

Theorem 0.5.6 ([FL2], Corollary 4.5) *Every standard frame of a finitely generated or countably generated Hilbert C^*-module is a generating set.*

Every generating set of an algebraically finitely generated Hilbert C^*-module is a standard frame.

Theorem 0.5.7 ([FL2], Theorem 3.2) *Let V be an $\mathcal{A}$-linear partial isometry V on $\mathcal{A}^n$ or $l^2(\mathcal{A})$. Let $\{x_j\}$ be an $\mathcal{A}$-orthonormal basis for $\mathcal{A}^n$ or $l^2(\mathcal{A})$. Then $\{V(x_j)\}$ is a normalised tight frame for $V(\mathcal{A}^n)$ (or $V(l^2(\mathcal{A}))$). Every algebraically finitely or countably generated Hilbert module therefore possesses a standard normalised tight frame.*

Just like with an orthonormal basis, it is possible to reconstruct an arbitrary element of a Hilbert $\mathcal{A}$-module from its $\mathcal{A}$-valued inner products with elements of the frame. This property can actually be used to characterise frames.

Theorem 0.5.8 ([FL2], Theorem 4.1; [FL1], Theorem 4.1) *Suppose $\mathcal{A}$ is a unital C^*-algebra and E is a Hilbert $\mathcal{A}$-module. Let $\{x_j\}$ be a standard normalised tight frame for E. The reconstruction formula*

$$x = \sum_j x_j [x_j, x]_E \tag{15}$$

holds for all $x \in E$. Any finite set $\{x_j\}_j \subset E$ satisfying (15) for every $x \in E$ is a normalised tight frame of E.

Chapter 1

Wavelets in Hilbert Space

In this chapter we shall introduce wavelets in the setting of a Hilbert space which is acted on by some unitary operators (the translations and dilations) and in the following chapter we will relate wavelets to C^*-algebras.

Let us first review some recent work which has been done on generalising wavelets. Some of the earlier work on generalising wavelets was done by Goodman, Lee and Tang in [GLT]. This was focused on developing the theory of multiwavelets, but also worked with the notion of a "wandering subspace" of a Hilbert space. The method of generalising wavelets described in this chapter is very similar to a situation introduced by Baggett, Carey, Moran and Ohring in [BCMO] and which has been subsequently elaborated by Baggett, Medina, and Merrill in [BM] and [BMM]. A similar generalisation has been developed by Han, Larson, Papadakis and Stavropoulos in [HLPS].

In Section 1.1 we introduce many of the main definitions that shall be used throughout the thesis. We shall introduce the concept of a multiresolution structure, which consists of some translations and dilations acting on a Hilbert space (Definition 1.1.1). Associated with a multiresolution structure we can define wavelets (Definition 1.1.2), and define a (generalised) multiresolution analysis (Definition 1.1.7). This section also contains Theorem 1.1.11, which states that we can obtain multiwavelets from a multiresolution structure. Theorem 1.1.11 is a generalisation of a result from [BCMO], to the setting of when we are dealing with more than one scaling function. Theorem 1.1.11 is a generalisation of a result from [BCMO] to the setting of more than one scaling function. In Section 1.2 we shall examine multiresolution structures for which we can define the Fourier transform. We prove some results about the Fourier transform of the translations and dilations, and examine wavelets on the Cantor group. In Section 1.3 we examine the main multiresolution structure that we shall work with, in which the Hilbert space consists of square-integrable functions on $\mathbf{R}^d$. Many of the results proved in this section are used in Chapter 2. In Section 1.4 we introduce the fast wavelet transform, which is studied further in Chapter 3.

1.1 Wavelets and Multiresolution Structures

The classical definition of a wavelet in $L^2(\mathbf{R})$ is a function ψ such that the family

$$U_\psi := \left\{2^{j/2}\psi(2^j x - k)\right\}_{j,k\in\mathbf{Z}}$$

is an orthonormal basis for $L^2(\mathbf{R})$. Each element of U_ψ is obtained from ψ by an integer translation $\psi(x) \mapsto \psi(x-k)$ followed by a dilation $\psi(x) \mapsto 2^{j/2}\psi(2^j x)$ by a power of 2. The space $L^2(\mathbf{R})$ is an example of a Hilbert space and the translations and dilations are unitary operators on this Hilbert space. This allows us to generalise the definition above.

Definition 1.1.1 A *multiresolution structure* $(\Gamma, \mathcal{D})$ on a separable Hilbert space $\mathcal{H}$ consists of a discrete group Γ of unitary operators on $\mathcal{H}$ and a unitary operator $\mathcal{D}$ on $\mathcal{H}$ such that $\mathcal{D}^{-1}\gamma\mathcal{D} \in \Gamma$ for all $\gamma \in \Gamma$. We call Γ the group of *translations*, and $\mathcal{D}$ the *dilation*. For a multiresolution structure the set $\mathcal{D}^{-1}\Gamma\mathcal{D}$ is a subgroup of Γ. If m is the index of the group $\mathcal{D}^{-1}\Gamma\mathcal{D}$ in Γ, we call m the *index* of the multiresolution structure (recall that the index of a subgroup in a group is the number of cosets of the subgroup in the larger group). We assume that m is finite.

Multiresolution structures were introduced in [BCMO] using the term “affine system”.

Definition 1.1.2 For a multiresolution structure $(\Gamma, \mathcal{D})$ with Hilbert space $\mathcal{H}$, an element $\psi \in \mathcal{H}$ is an *orthonormal wavelet* if $\{\mathcal{D}^n(\gamma(\psi))\}_{\gamma\in\Gamma, n\in\mathbf{Z}}$ is an orthonormal basis for $\mathcal{H}$.

A finite set of elements $\{\psi^1, \ldots, \psi^M\}$ of $\mathcal{H}$ is an *orthonormal multiwavelet* if the set $\{\mathcal{D}^n(\gamma(\psi^i))\}_{\gamma\in\Gamma, n\in\mathbf{Z}, i=1,\ldots,M}$ forms an orthonormal basis for $\mathcal{H}$.

It is important not to confuse a multiresolution structure with a multiresolution analysis which will be defined later. We will generally abbreviate the terms ‘orthonormal wavelet’ and ‘orthonormal multiwavelet’ as ‘wavelet’ and ‘multiwavelet’, respectively. It is worth mentioning that there are also ‘biorthogonal wavelets’, where the wavelets form a basis which is not quite as general as a frame, but more general than an orthonormal basis. Biorthogonal wavelets are quite useful for image compression because of their symmetry properties, and are described in [Da1].

It is worth noting that the set of unitary operators given by a multiresolution structure is an example of a unitary system. A *unitary system* on a Hilbert space $\mathcal{H}$ is any subset of the unitary operators on $\mathcal{H}$ which contains the identity. The unitary system here is given by

$$\mathcal{U}_{\mathcal{D},\Gamma} = \{\mathcal{D}^n\gamma\}_{n\in\mathbf{Z}, \gamma\in\Gamma}\,.$$

For a given unitary system $\mathcal{U}$, a *complete wandering vector* for $\mathcal{U}$ is a function ψ such that $\mathcal{U}\psi$ is an orthonormal basis for $\mathcal{H}$. For a unitary system $\mathcal{U}$, let $W(\mathcal{U})$ be the set of complete wandering vectors of $\mathcal{U}$. So given a multiresolution structure, the *space of all wavelets* is $W(\mathcal{U}_{\mathcal{D},\Gamma})$. Unitary systems and wandering vectors are studied in more detail in [DL] and [Lr1].

Example 1.1.3 The simplest example of a multiresolution structure corresponds to what are known as dyadic wavelets. In this case, the Hilbert space is $L^2(\mathbf{R})$. The translation group $\Gamma = \mathbf{Z}$, and acts by $(\gamma\psi)(x) = \psi(x-\gamma)$. The dilation is given by $(\mathcal{D}\psi)(x) = 2^{-\frac{1}{2}}\psi(2x)$, and so $(\mathcal{D}^{-1}\psi)(x) = 2^{\frac{1}{2}}\psi(\frac{x}{2})$. In this case $(\mathcal{D}^{-1}\gamma\mathcal{D}\psi)(x) = \psi(x-2\gamma)$, so $\mathcal{D}^{-1}\mathbf{Z}\mathcal{D} = 2\mathbf{Z}$. We therefore have a multiresolution structure.

The simplest wavelet for this multiresolution structure is known as the Haar wavelet, it was introduced by Haar in [Haa]. It is given by

$$\psi(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ -1 & \text{if } x \in [\frac{1}{2}, 1) \\ 0 & \text{otherwise} \end{cases}$$

The Haar wavelet has the advantage that it is supported on a small set but in many applications it is affected by the disadvantage that it is not continuous.

Example 1.1.4 An important example of a multiresolution structure is based on lattices in $\mathbf{R}^d$. We shall examine this example in detail in Section 2.1, most of the results which we shall prove in Chapters 2 and 3 will be within the framework of this example. It is given by the Hilbert space $L^2(\mathbf{R}^d)$, a discrete Abelian subgroup Γ of $\mathbf{R}^d$ which induces a translation on $\mathcal{H}$ by $(\gamma f)(x) = f(x-\gamma)$ where $\gamma \in \Gamma$; and a dilation $\mathcal{D}$ given by $(\mathcal{D}f)(x) = \sqrt{\det \tilde{\mathcal{D}}} f(\tilde{\mathcal{D}}x)$ where $\tilde{\mathcal{D}}$ is a linear mapping from $\mathbf{R}^d \to \mathbf{R}^d$ which also maps Γ onto a proper subgroup of itself. We do not lose any generality when $\Gamma = \mathbf{Z}^d$. We shall now show that $\tilde{\mathcal{D}}$ maps $\mathbf{Z}^d$ into a subgroup of itself if and only if $\tilde{\mathcal{D}} \in M_d(\mathbf{Z})$.

We know that $\tilde{\mathcal{D}} \in M_d(\mathbf{R})$, so we can write

$$\tilde{\mathcal{D}} = \begin{pmatrix} \tilde{\mathcal{D}}_{11} & \cdots & \tilde{\mathcal{D}}_{1d} \\ \vdots & \ddots & \vdots \\ \tilde{\mathcal{D}}_{d1} & \cdots & \tilde{\mathcal{D}}_{dd} \end{pmatrix}.$$

We therefore have that for $\gamma \in \mathbf{Z}^d$, if we write $\gamma = (\gamma_1, \ldots, \gamma_d)^T$, then

$$\tilde{\mathcal{D}}\gamma = \begin{pmatrix} \tilde{\mathcal{D}}_{11} & \cdots & \tilde{\mathcal{D}}_{1d} \\ \vdots & \ddots & \vdots \\ \tilde{\mathcal{D}}_{d1} & \cdots & \tilde{\mathcal{D}}_{dd} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_d \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^d \tilde{\mathcal{D}}_{1i}\gamma_i \\ \vdots \\ \sum_{i=1}^d \tilde{\mathcal{D}}_{di}\gamma_i \end{pmatrix}.$$

It therefore follows that if $\tilde{\mathcal{D}} \in M_d(\mathbf{Z})$, then $\tilde{\mathcal{D}}\gamma \in \mathbf{Z}^d$. Suppose now that $\tilde{\mathcal{D}} \notin M_d(\mathbf{Z})$, then there exists i, j such that $\tilde{\mathcal{D}}_{ij} \notin \mathbf{Z}$, if we define $\gamma \in \mathbf{Z}^d$ by $\gamma_k = \delta_{j,k}$ where δ is the Kronecker delta, we obtain that $(\tilde{\mathcal{D}}\gamma)_i = \tilde{\mathcal{D}}_{ij}\gamma_j$ and so is not contained in $\mathbf{Z}^d$. We have that $(\mathcal{D}^{-1}\gamma\mathcal{D}f)(x) = f(x - \tilde{\mathcal{D}}\gamma)$, and so $\mathcal{D}^{-1}\mathbf{Z}^n\mathcal{D} = \tilde{\mathbf{Z}}^n \subset \mathbf{Z}^n$. We therefore have a multiresolution structure.

One possible choice of $\tilde{\mathcal{D}}$ is multiplication by 2, in this case the index of the multiresolution structure is $m = 2^d$. Another possible choice when $d = 2$ is given by

$$\tilde{\mathcal{D}} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

In this case the sublattice $\tilde{\mathcal{D}}\mathbf{Z}^2$ is known as the *quincunx lattice.*

It is sometimes useful to think of multiresolution structures in terms of representation theory, see Definition 0.3.1. When Γ is the translation group for a multiresolution structure with Hilbert space $\mathcal{H}$, the action of Γ on $\mathcal{H}$ can be thought of as a unitary representation π of Γ on $\mathcal{H}$.

Most of the multiresolution structures that we are interested in will have the property that $\mathcal{D}^{-1}\Gamma\mathcal{D}$ is a proper subgroup of Γ. When this is the case define $\Delta \in \mathrm{Hom}(\Gamma, \Gamma)$ by $\Delta(\gamma) := \mathcal{D}^{-1}\gamma\mathcal{D}$. For $n \in \mathbf{Z}$, define $\Gamma^n := \mathcal{D}^n\Gamma\mathcal{D}^{-n}$, and note that $\Gamma^{-1} = \Delta(\Gamma)$. When $n \geq 1$, Γ^n is not a subgroup of Γ but is still a subgroup of $U(\mathcal{H})$.

Definition 1.1.5 Suppose that $(\Gamma, \mathcal{D})$ is a multiresolution structure on a Hilbert space $\mathcal{H}$. For $n \in \mathbf{N}$ we define the *nth-level translation group* to be the group of unitary operators on $\mathcal{H}$ given by

$$\Gamma^n := \mathcal{D}^n\Gamma\mathcal{D}^{-n}. \tag{1.1}$$

We define the *nth-level translation representation* to be the representation π^n of Γ on $\mathcal{H}$ given by

$$\pi^n_\gamma(f) := \mathcal{D}^n\gamma\mathcal{D}^{-n}(f), \qquad \text{for } f \in \mathcal{H},\ \gamma \in \Gamma. \tag{1.2}$$

Proposition 1.1.6 *Suppose that* $(\Gamma, \mathcal{D})$ *is a multiresolution structure on a Hilbert space* $\mathcal{H}$*. Then for all* $n \in \mathbf{Z}$*,* π^n *is a unitary representation of* Γ *on* $\mathcal{H}$*, and the nth-level translation group* Γ^n *is isomorphic to* Γ*.*

PROOF: Suppose $n \in \mathbf{Z}$, we define $\iota_n(\gamma) := \mathcal{D}^n\gamma\mathcal{D}^{-n}$, for $\gamma \in \Gamma$. If γ_1 and γ_2 are both elements of Γ, then $\iota_n(\gamma_1\gamma_2) = \mathcal{D}^n\gamma_1\gamma_2\mathcal{D}^{-n} = \mathcal{D}^n\gamma_1\mathcal{D}^{-n}\mathcal{D}^n\gamma_2\mathcal{D}^{-n} = \iota_n(\gamma_1)\iota_n(\gamma_2)$. So ι_n is a homomorphism from Γ to Γ^n. Since $\pi^n_\gamma f = \iota_n(\gamma)f$ for $f \in \mathcal{H}$ and $\gamma \in \Gamma$, π^n is a homomorphism from Γ to $U(\mathcal{H})$ and the mapping $\gamma \mapsto \pi^n_\gamma f$ is continuous for all $f \in \mathcal{H}$. Hence π^n is a unitary representation.

From the definition of Γ^n, ι_n is a surjective map from Γ to Γ^n. Suppose that $\iota_n(\gamma_1) = \iota_n(\gamma_2)$. We then have that $\mathcal{D}^n\gamma_1\mathcal{D}^{-n} = \mathcal{D}^n\gamma_2\mathcal{D}^{-n}$, so $\mathcal{D}^{-n}\mathcal{D}^n\gamma_1\mathcal{D}^{-n}\mathcal{D}^n = \mathcal{D}^{-n}\mathcal{D}^n\gamma_2\mathcal{D}^{-n}\mathcal{D}^n$ and therefore $\gamma_1 = \gamma_2$. This means that ι_n is injective, and so for all $n \in \mathbf{Z}$, $\Gamma^n \cong \Gamma$. □

A consequence of $\Delta(\Gamma)$ being isomorphic to Γ is that for all m, n, $\Gamma^m \cong \Gamma^n$.

We will now introduce the notions of a multiresolution analysis and a generalised multiresolution analysis. These are extremely important tools for constructing and studying wavelets. The notion of a multiresolution analysis is a well known tool in wavelet theory, see for example [Me1] or [Da1]. The notion of a generalised multiresolution analysis was introduced in [BMM].

Definition 1.1.7 Let $(\Gamma, \mathcal{D})$ be a multiresolution structure on a Hilbert space $\mathcal{H}$. A sequence $\{V_n\}_{n\in\mathbf{Z}}$ of closed subspaces of $\mathcal{H}$ is called a *generalised multiresolution analysis* (GMRA) of $\mathcal{H}$ if

1. $\forall n \in \mathbf{Z}$, $V_n \subset V_{n+1}$

2. $\bigcup_{n\in\mathbf{Z}} V_n$ is dense in $\mathcal{H}$ and $\bigcap_{n\in\mathbf{Z}} V_n = \{0\}$

3. $\forall n \in \mathbf{Z}, \mathcal{D}(V_n) = V_{n+1}$

4. V_0 is invariant under Γ.

A *multiresolution analysis* (MRA) of order r is a GMRA for which there exists a set of elements $\{\varphi^1, \ldots, \varphi^r\} \in \mathcal{H}$ such that $\{\gamma(\varphi^i)\}_{\gamma\in\Gamma, i=1\ldots r}$ is an orthonormal basis for V_0. We call $\{\varphi^1, \ldots, \varphi^r\}$ a set of *scaling functions*.

We use the term "scaling functions" because although they are actually elements of a Hilbert space, in classical wavelet theory they are complex or real valued functions on $\mathbf{R}^p$.

If we denote the orthogonal complement of V_n in V_{n+1} by W_n, then W_n satisfies the following properties:

1. $\mathcal{D}(W_n) = W_{n+1}$;

2. $\mathcal{H} = \bigoplus_{n\in\mathbf{Z}} W_n$.

We call $\{W_n\}_{n\in\mathbf{Z}}$ the *wavelet spaces* associated with a multiresolution analysis $\{V_n\}_{n\in\mathbf{Z}}$.

Example 1.1.8 Let V_n be the space of functions which are square integrable on $\mathbf{R}$ and piecewise constant on each interval $[2^{-n}k, 2^{-n}(k-1))$, where $k \in \mathbf{Z}$. Then $\{V_n\}$ is a multiresolution analysis for $L^2(\mathbf{R})$ with the same multiresolution structure as described in Example 1.1.3. There is a scaling function for $\{V_n\}$ which is given by $\varphi = \chi_{[0,1)}$. The wavelet corresponding to φ is the Haar wavelet and was described in Example 1.1.3.

Example 1.1.9 Suppose that $\varphi(x) = \frac{\sin(\pi x)}{\pi x}$, we then have that $\hat{\varphi}(\xi) = \chi_{[-\frac{1}{2},\frac{1}{2})}(\xi)$. If we let V_n be the closed linear span of $\{2^{n/2}\varphi(2^n \cdot -k) : k \in \mathbf{Z}\}$, then V_n is a multiresolution analysis for $L^2(\mathbf{R})$, with the same multiresolution structure as in Example 1.1.3. There are several possible choices of wavelet, which differ from each other by multiplication in the Fourier domain by a unimodular function. Some of these include $\hat{\psi}(\xi) = \chi_I(\xi)$, and $\hat{\psi}(\xi) = e^{i\xi/2}\chi_I(\xi)$, where

$$I = [-1, -1/2) \cup (1/2, 1].$$

The latter choice of ψ is known as the Shannon wavelet.

An example of a wavelet which for which there does not exist a corresponding scaling function in known as Journé's wavelet. The reader is referred to [Pa1, HW] for more discussion of this example.

We now show that associated with every multiwavelet there is a generalised multiresolution analysis.

Proposition 1.1.10 *Suppose $\{\psi^1, \ldots, \psi^M\}$ is a multiwavelet. Define*

$$V_n = \overline{span\{\mathcal{D}^k(\gamma(\psi^i))\}}_{k<n,\gamma\in\Gamma,1\le i\le M}.$$

Then V_n is a generalised multiresolution analysis.

PROOF: We show that V_n satisfies the four properties of Definition 1.1.7. Properties 1 and 4 follow immediately from the definition of V_n. Property 2 follows from the fact that the wavelets are a basis for $\mathcal{H}$. Property 3 is satisfied because

$$\mathcal{D}V_n = \overline{\operatorname{span}\{\mathcal{D}^{k+1}(\gamma(\psi^i))\}}_{k<n.\gamma\in\Gamma,1\leq i\leq n} = V_{n+1}.$$

□

The following theorem shows that we can obtain wavelets from a multiresolution analysis and a set of scaling functions. This is a slight generalisation of [BCMO, Theorem 1]. The difference is that here we deal with the case when there is more than one scaling function. The proof will make use of von Neumann algebras, which we defined in Section 0.2.

Theorem 1.1.11 *Let $(\Gamma, \mathcal{D})$ be a multiresolution structure on a separable Hilbert space $\mathcal{H}$. Suppose that $\{V_n\}_{n\in\mathbf{Z}}$ is a multiresolution analysis with a set of scaling functions-$\{\varphi^1, \ldots, \varphi^r\}$. Suppose that the subgroup $\Gamma^{-1} = \Delta(\Gamma)$ has finite index m in Γ. Then $m > 1$, and there exists a multiwavelet $\{\psi^1, \ldots, \psi^{(m-1)r}\}$ for $\mathcal{H}$.*

We shall prove some Lemmas before proving Theorem 1.1.11. Lemma 1.1.12 applies Proposition 0.2.13 to the setting of unitary representations of groups. It was stated without proof (except for a reference to Proposition 0.2.13) in [BCMO]. We prove it here for the sake of completeness.

Lemma 1.1.12 *Let G be a locally compact group, and let ρ be a unitary representation of G whose commutant is a finite von Neumann algebra. Suppose that ρ is equivalent to $\sigma_1 \oplus \sigma_2$, and ρ is also equivalent to $\sigma_1 \oplus \sigma_3$, where $\sigma_1, \sigma_2, \sigma_3$ are also unitary representations of G. Then σ_2 is equivalent to σ_3.*

PROOF: Let $\mathcal{H}_{\sigma_1\oplus\sigma_2}$ be the representation space of $\sigma_1 \oplus \sigma_2$, and let $\mathcal{H}_{\sigma_1\oplus\sigma_3}$ be the representation space of $\sigma_1 \oplus \sigma_3$. Let P_1 be the projection in $\mathcal{H}_{\sigma_1\oplus\sigma_2}$ whose image is the invariant subspace of $\mathcal{H}_{\sigma_1\oplus\sigma_2}$ corresponding to σ_1. Let P_2 be the projection in $\mathcal{H}_{\sigma_1\oplus\sigma_3}$ whose image is the invariant subspace of $\mathcal{H}_{\sigma_1\oplus\sigma_3}$ corresponding to σ_1. Because $\sigma_1 \oplus \sigma_2$ is equivalent to $\sigma_1 \oplus \sigma_3$, we can identify $\mathcal{H}_{\sigma_1\oplus\sigma_2}$ with $\mathcal{H}_{\sigma_1\oplus\sigma_3}$, and we shall write both as $\mathcal{H}_\rho$. We then set

$$R(\rho) = \{T \in B(\mathcal{H}_\rho) : T\rho(g) = \rho(g)T \text{ for all } g \in G \}.$$

By Proposition 0.3.3, P_1 and P_2 are both contained in $R(\rho)$ and P_1 is equivalent to P_2 in $R(\rho)$. We now apply Proposition 0.2.13 to obtain that $\mathbf{1} - P_1$ is equivalent to $\mathbf{1} - P_2$ in $R(\rho)$. But $\mathbf{1} - P_1$ is the projection onto the invariant subspace of $\mathcal{H}_\rho$ corresponding to σ_2, and $\mathbf{1} - P_2$ is the projection onto the invariant subspace of $\mathcal{H}_\rho$ corresponding to σ_3. We therefore have by Proposition 0.3.3 that σ_2 is equivalent to σ_3. □

We will also make use of the following result from [HL2] to show that certain von Neumann algebras are finite. We have rephrased it to fit in with our notation. Although there is a proof in [HL2], we have supplied a proof here.

Lemma 1.1.13 ([HL2] Proposition 1.1) *Let Γ be a unitary group acting on a Hilbert space V. Suppose that there is a finite set Φ such that $\{\gamma\Phi : \gamma \in \Gamma\}$ is an orthonormal basis for V. Then the commutant Γ' is a finite von Neumann algebra.*

PROOF: Write $\Phi = \{\phi_i\}_{i\in I}$, where I is a finite index set. Let $M = \text{span}\Phi$. Let Λ be the left regular representation of Γ, recall that the representation space of Λ is $L^2(\Gamma)$. For $\gamma \in \Gamma$, let $\chi_\gamma \in L^2(\Gamma)$ be the characteristic function of γ (so $\chi_\gamma(\gamma) = 1$, and $\chi_\gamma(\beta) = 0$ whenever $\beta \neq \gamma$). Any element $v \in V$ can be written as $v = \sum_{\gamma\in\Gamma,i\in I} v_{i,\gamma}\gamma\phi_i$, where each $v_{i,\gamma} \in \mathbf{C}$. Define an operator $W : V \to L^2(\Gamma) \otimes M$ by

$$Wv = \sum_{\gamma\in\Gamma,i\in I} v_{i,\gamma}\chi_\gamma \otimes \phi_i.$$

The reader is referred to Appendix T of [W-O] or Section 7.3 of [Fo] for background information on tensor products. If $v_1 \neq v_2 \in V$ then $Wv_1 \neq Wv_2$, and if $Wv = 0$ then $v = 0$ so W is bijective. The operator W satisfies $W\gamma\phi_i = \chi_\gamma \otimes \phi_i$, for all $i \in I$, $\gamma \in \Gamma$. The adjoint W^* satisfies

$$W^*\Lambda_\gamma\chi_0 \otimes \phi_i = \gamma\phi_i$$

so $W^*W = \mathbf{1}_V$ and $WW^* = \mathbf{1}_{\mathcal{H}_\Lambda\otimes M}$. So W is unitary and V is unitarily isomorphic to $\mathcal{H}_\Lambda \otimes M$. The action of Γ on $L^2(\Gamma) \otimes M$ is given by

$$\{\Lambda_\gamma \otimes \mathbf{1}_M : \gamma \in \Gamma\}$$

where $\mathbf{1}_M$ is the identity operator on M. By the commutation theorem for von Neumann algebras (see [KR], Section 11.2) the commutant of $\{\Lambda_\gamma \otimes \mathbf{1}_M : \gamma \in \Gamma\}$ is $\{\Lambda_\gamma : \gamma \in \Gamma\}' \otimes B(M)$. Now the commutant of the left regular representation is the right regular representation, and by [KR], Proposition 7.7.4, the right regular representation is a finite von Neumann algebra. Because M is finite dimensional, the commutant of $\{\Lambda_\gamma \otimes \mathbf{1}_M : \gamma \in \Gamma\}$ is therefore the product algebra of a finite set of finite von Neumann algebras and so is finite by Proposition 7 of [Dx2], Section I.6.7. We therefore have that the commutant of Γ is finite. □

We prove Theorem 1.1.11 by examining representations of Γ on the multiresolution analysis spaces and associated wavelet spaces, and comparing these representations to the left regular representation of Γ. Lemma 1.1.13 enables us to then use Lemma 1.1.12 to prove the result.

PROOF OF THEOREM 1.1.11: Let π be the unitary representation of Γ on $\mathcal{H}$ which makes it a group of unitaries on $\mathcal{H}$. Let W_0 be the orthogonal complement of V_0 in V_1. We have that V_0 and V_1 are invariant subspaces of $\mathcal{H}$ for π, which implies that W_0 is an invariant subspace of $\mathcal{H}$ for π. Therefore π has the subrepresentations π^{V_0}, π^{W_0} and π^{V_1}.

Let Λ be the left regular representation of Γ, which has representation space $L^2(\Gamma)$. For any natural number p, let $\oplus_p\Lambda$ be the direct sum of p copies of Λ. For $\gamma \in \Gamma$, let $\chi_\gamma \in \Lambda$ be the characteristic function of γ. For $\gamma \in \oplus_p\Lambda$, $i = 1, \ldots, p$, let $\chi_{\gamma,i} \in \oplus_p\Lambda$

take the value χ_γ on the ith copy of Λ and 0 on all of the other copies of Λ. Let $\oplus_p L^2(\Gamma)$ be the representation space of $\oplus_p \Lambda$.

We shall show the existence of a multiwavelet by showing that the representation π^{W_0} is equivalent to $\oplus_{(m-1)r}\Lambda$. To do this we shall first show that the representations π^{V_0} and $\oplus_r \Lambda$ are equivalent. The unitary $U_{V_0} : V_0 \to \oplus_r L^2(\Gamma)$ that realises this equivalence is defined by $U_{V_0}(\pi^{V_0}_\gamma(\varphi^i)) = \chi_{\gamma,i}$ where $i = 1, \ldots, r$, and $\gamma \in \Gamma$. This unitary operator satisfies $\Lambda_\gamma = U_{V_0}\pi^{V_0}_\gamma U^*_{V_0}$, for $\gamma \in \Gamma$, and so verifies the equivalence.

We now show that the representation π^{V_1} is equivalent to $\oplus_{mr}\Lambda$. Let $\alpha_0, \ldots, \alpha_{m-1}$ be a set of coset representatives of Γ in Γ^1. The set

$$\{\mathcal{D}\gamma\varphi^i\}_{\gamma\in\Gamma, i=1,\ldots,r}$$

is an orthonormal basis for V_1. We can therefore decompose $V_1 = \oplus_{j=1}^r \oplus_{i=0}^{m-1} N_{i,j}$, where

$$N_{i,j} := \left\{ \sum_{\gamma\in\Gamma} c_\gamma \gamma(\alpha_i(\mathcal{D}(\varphi^j))) : c_\gamma \in R \right\}.$$

We define the unitary operator $U_{V_1} : V_1 \to \mathcal{H}_{\oplus_{mr}\Lambda}$ (where $\mathcal{H}_{\oplus_{mr}\Lambda}$ is the representation space of the unitary representation $\oplus_{mr}\Lambda$) by

$$U_{V_1}(\gamma\alpha_i\mathcal{D}\varphi^j) = \chi_{\gamma, ir+j}$$

for $\gamma \in \Gamma$, $i = 0, \ldots, m-1$, $j = 1, \ldots, r$. This unitary demonstrates the equivalence $\pi^{V_1} \cong \oplus_{mr}\Lambda$.

Let us now examine the representation π^{W_0}. By Lemma 1.1.13 we have that the commutant of $\oplus_{mr}\Lambda$ is a finite von Neumann algebra. Because $V_1 = V_0 \oplus W_0$, the representation $\oplus_{mr}\Lambda$ is unitarily equivalent to $(\oplus_r\Lambda) \oplus \pi^{W_0}$. But $\oplus_{mr}\Lambda$ is obviously also unitarily equivalent to $(\oplus_r\Lambda) \oplus (\oplus_{(m-1)r}\Lambda)$. We therefore by Lemma 1.1.12 have that π^{W_0} is unitarily equivalent to $\oplus_{(m-1)r}\Lambda$.

Let $U_{W_0} : W_0 \to \oplus_{(m-1)r}L^2(\Gamma)$ be a unitary realising this equivalence. For each $i = 1, \ldots, (m-1)r$, let $\psi^i = U^*_{W_0}(\chi_{0,i})$. We then have that $\{\gamma\psi^i\}_{\gamma\in\Gamma, i=1,\ldots,(m-1)r}$ is an orthonormal basis for W_0, and therefore $\{\psi^1, \ldots, \psi^{(m-1)r}\}$ is a multiwavelet for the multiresolution structure $(\Gamma, \mathcal{D})$. □

Remark 1.1.14 Let us now compare the proof of Theorem 1.1.11 to the proof of [BCMO, Theorem 1]. In the proof of Theorem 1.1.11, we have elaborated on the arguments in [BCMO] proving the equivalence $\pi^{W_0} \cong \oplus_{(m-1)r}\Lambda$. We have in particular demonstrated why Lemma 1.1.12 is a consequence of Proposition 0.2.13, and supplied a proof that the commutant of $\oplus_{mr}\Lambda$ is a finite von Neumann algebra. In [BCMO], it is always assumed that $r = 1$, so we have generalised the proof to more than one scaling function.

Example 1.1.15 Let us examine the multiresolution structure corresponding to dyadic wavelets in $L^2(\mathbf{R})$, with $\Gamma = \mathbf{Z}$, as described in Example 1.1.3. Suppose that we have a multiresolution analysis $\{V_n\}_{n\in\mathbf{Z}}$ with a single scaling function φ. By Theorem 1.1.11, we know that there exists a single wavelet ψ corresponding to φ. A possible choice for ψ is described in Theorem 5.1 of [Da1], and is given by

$$\psi = \sum_{\gamma\in\mathbf{Z}} (-1)^{\gamma-1} \langle \mathcal{D}\pi_{-\gamma-1}\varphi, \varphi\rangle \mathcal{D}\pi_\gamma \varphi.$$

In this example, the left regular representation Λ of Γ has representation space $\mathcal{H}_\Lambda = L^2(\Gamma) = L^2(\mathbf{Z})$.

1.2 Wavelets and the Fourier Transform

Most of the wavelets that we are interested in will span the Hilbert space $\mathcal{H} = L^2(G)$, where G is a locally compact group. If we impose some extra structure on the multiresolution structure, we are able to use Fourier analysis. We therefore introduce what we call a 'harmonic' multiresolution structure. This section also contains some more background material on Fourier analysis, and we shall prove some results about the behaviour of the translation and dilation in the 'Fourier domain' associated with a harmonic multiresolution structure. We also describe examples of wavelets defined on groups which are somewhat more unusual than Euclidean space, $\mathbf{R}^d$.

Both the wavelet transform and the Fourier transform are useful for studying properties of functions, and for applications such as signal analysis. The Fourier transform is not only an alternative to using wavelets, it is also very important for studying the properties of wavelets themselves, which is what we are interested in.

Definition 1.2.1 Suppose $(\Gamma, \mathcal{D})$ is a multiresolution structure on a Hilbert space $\mathcal{H}$. We say that the multiresolution structure is *harmonic* if

- The Hilbert space satisfies $\mathcal{H} = L^2(G, \mu_G)$ where G is a locally compact Abelian group with Haar measure μ_G;
- The translation group Γ is a closed discrete subgroup of G and we can write $\gamma f = f(x - \gamma)$ for $f \in \mathcal{H}$;
- There is a homomorphism $\tilde{\mathcal{D}} : G \to G$ such that $(\mathcal{D}f)(x) = \sqrt{m} f(\tilde{\mathcal{D}}x)$ for $f \in \mathcal{H}$, $x \in G$, where m is the index of the multiresolution structure.

We call a wavelet with a harmonic multiresolution structure a *harmonic wavelet.*

Recall that for $n \in \mathbf{Z}$, the group $\Gamma^n := \mathcal{D}^n \Gamma \mathcal{D}^{-n}$ is a group of unitaries acting on $L^2(G)$, and is isomorphic to Γ by Proposition 1.1.6. The following calculation demonstrates that for a harmonic multiresolution structure, Γ^n acts on $L^2(G)$ by translations. Like many calculations in wavelet theory, this calculation involves some tricky applications of translations and dilations, and we shall use functional notation in order to attempt to make it more clear what is going on.

Lemma 1.2.2 *Suppose that $(\Gamma, \mathcal{D})$ is a harmonic multiresolution structure on $L^2(G)$, for a locally compact Abelian group G. For $f \in L^2(G)$, $x \in G$, and $\gamma \in \Gamma$, we have that*

$$(\mathcal{D}^n \gamma \mathcal{D}^{-n} f)(x) = f(x - \tilde{\mathcal{D}}^{-n}\gamma) \tag{1.3}$$

and so elements of Γ^n act on $L^2(G)$ by translations.

Proof: We calculate

$$\begin{aligned}
f : x &\mapsto f(x); \\
\text{so } \mathcal{D}^{-n} f : x &\mapsto m^{-n/2} f(\tilde{\mathcal{D}}^{-n} x); \\
\text{so } \gamma\mathcal{D}^{-n} f : x &\mapsto (\mathcal{D}^{-n} f)(x - \gamma) \\
&= m^{-n/2} f(\tilde{\mathcal{D}}^{-n}(x-\gamma)) \\
&= m^{-n/2} f(\tilde{\mathcal{D}}^{-n}x - \tilde{\mathcal{D}}^{-n}\gamma); \\
\text{so } \mathcal{D}^n\gamma\mathcal{D}^{-n} f : x &\mapsto m^{n/2} m^{-n/2} f(\tilde{\mathcal{D}}^{-n}(\tilde{\mathcal{D}}^n x) - \tilde{\mathcal{D}}^{-n}\gamma) \\
&= f(x - \tilde{\mathcal{D}}^{-n}\gamma),
\end{aligned}$$

verifying equation (1.3). □

We therefore have that elements of Γ^n act on $L^2(G)$ by translations. This means that Γ^n is isomorphic to an additive subgroup of G, which we also denote by Γ^n, and is given by $\Gamma^n = \tilde{\mathcal{D}}^{-n}\Gamma$.

In order to study the properties of the dual $\widehat{\Gamma}$ of the translation group in a harmonic multiresolution structure, we will look at the *annihilator* $\mathrm{Ann}\Gamma$ of Γ. The annihilator of a closed subgroup Γ of a locally compact Abelian group G is the set of all $\lambda \in \widehat{G}$ such that $(\gamma, \lambda) = 1$ for all $\gamma \in \Gamma$.

The reader is referred to [Ru2, Proposition 2.1.2] for a proof to the following Proposition.

Proposition 1.2.3 *The annihilator $\mathrm{Ann}\Gamma \cong \widehat{G/\Gamma}$ and $\widehat{G}/\mathrm{Ann}\Gamma \cong \widehat{\Gamma}$. The annihilator of $\mathrm{Ann}\Gamma$ is Γ.*

Let us now look at what the translations and dilation do to the Fourier transform of some arbitrary $f \in \mathcal{H}$. We define $\hat{\mathcal{D}} : L^2(\widehat{G}) \to L^2(\widehat{G})$ and $\hat{\gamma} : L^2(\widehat{G}) \to L^2(\widehat{G})$ by

$$\hat{\mathcal{D}}p = \mathcal{F}\mathcal{D}\mathcal{F}^* p$$
$$\hat{\gamma}p = \mathcal{F}\gamma\mathcal{F}^* p$$

for $p \in L^2(\widehat{G})$, and $\gamma \in \Gamma$. By definition we have that $\hat{\mathcal{D}}\hat{f} = \widehat{\mathcal{D}f}$ and $\hat{\gamma}\hat{f} = \widehat{\gamma f}$. We will now obtain explicit formulae for $\hat{\mathcal{D}}$ and $\hat{\gamma}$.

Consider two locally compact Abelian groups G_1, G_2 and suppose we have a homomorphism $\alpha : G_1 \to G_2$. Then there exists a homomorphism $\hat{\alpha} : \widehat{G_2} \to \widehat{G_1}$ which satisfies

$$(\alpha x, \xi) = (x, \hat{\alpha}\xi)$$

for $x \in G_1, \xi \in \widehat{G_2}$ (see [Ru2, Chapter 2]). We shall call $\hat{\alpha}$ the *dual homomorphism* of α.

Proposition 1.2.4 *Suppose that $(\Gamma, \mathcal{D})$ is a harmonic multiresolution structure for a Hilbert space $\mathcal{H} = L^2(G)$, where G is an Abelian group. Then for $f \in L^2(G)$, $\gamma \in \Gamma$ and $\xi \in \widehat{G}$,*

$$(\hat{\gamma}\hat{f})(\xi) = \xi(\gamma)\hat{f}(\xi). \tag{1.4}$$

PROOF: Taking the Fourier transform we obtain:

$$\begin{aligned}
(\hat{\gamma}\hat{f})(\xi) = (\widehat{\gamma f})(\xi) &= \int_G (\gamma f)(x)\xi(x)dx \\
&= \int_G f(x-\gamma)\xi(x)dx \\
&= \int_G f(y)\xi(y+\gamma)dy \\
&= \int_G f(y)\xi(y)\xi(\gamma)dy \\
&= \xi(\gamma)\hat{f}(\xi)
\end{aligned}$$

□

Lemma 1.2.5 *Suppose that G_1 and G_2 are Abelian groups and $\alpha : G_1 \to G_2$ is an isomorphism. Suppose also that for all $E \subseteq G_1$, $\mu_{G_2}(\alpha(E)) = k_1\mu_{G_1}(E)$ where μ_{G_1}, μ_{G_2} are Haar measure on G_1 and G_2, and $k_1 \in \mathbf{C}$ is a constant. Define $\beta_{k_2} : L^2(G_2) \to L^2(G_1)$ by $(\beta_{k_2}f)(x) = k_2 f(\alpha(x))$ for some constant $k_2 \in \mathbf{C}$, and where $f \in L^2(G_2)$. Define $\hat{\beta_{k_2}} : L^2(\widehat{G_2}) \to L^2(\widehat{G_1})$ to be $\hat{\beta_{k_2}} = \mathcal{F}\beta\mathcal{F}^*$ so that $\hat{\beta_{k_2}}(\hat{f}) = \widehat{\beta(f)}$. Then $(\hat{\beta_{k_2}}\hat{f})(\xi) = \frac{k_2}{k_1}\hat{f}(\hat{\alpha}^{-1}(\xi))$.*

PROOF: When we calculate the Fourier transform we obtain:

$$\begin{aligned}
(\widehat{\beta f})(\xi) &= \int_{G_1} (\beta f)(x)\xi(x)dx \\
&= k_2 \int_{G_1} f(\alpha(x))\xi(x)dx
\end{aligned}$$

We now change the variable of integration to $y = \alpha(x)$, and we calculate

$$\begin{aligned}
(\widehat{\beta f})(\xi) &= \frac{k_2}{k_1}\int_{G_2} f(y)\xi(\alpha^{-1}(y))dy \\
&= \frac{k_2}{k_1}\int_{G_2} f(y)\hat{\alpha}^{-1}(\xi)(y)dy \\
&= \frac{k_2}{k_1}\hat{f}(\hat{\alpha}^{-1}(\xi)).
\end{aligned}$$

□

Corollary 1.2.6 *If $\mathcal{D}$ is the dilation for a harmonic multiresolution structure, then for $f \in L^2(G)$,*

$$(\hat{\mathcal{D}}\hat{f})(\xi) = \frac{1}{\sqrt{m}}\hat{f}((\hat{\hat{\mathcal{D}}})^{-1}(\xi)) \tag{1.5}$$

where m is the index of the multiresolution structure, and $\hat{\tilde{\mathcal{D}}} : \widehat{G} \to \widehat{G}$ is the dual homomorphism of $\tilde{\mathcal{D}}$.

PROOF: Because $\tilde{\mathcal{D}}$ is an isomorphism from G onto itself, the result follows from Proposition 1.2.5. The constant $1/\sqrt{m}$ can be calculated from the requirement that $\hat{\mathcal{D}}$ must be unitary. □

Example 1.2.7 Let us examine the multiresolution structure of Example 1.1.3. Recall that $G = \mathbf{R}$, $\Gamma = \mathbf{Z}$, and for $f \in L^2(\mathbf{R})$, $\mathcal{D} : f(x) \mapsto 2^{-1/2} f(2x)$, so $\tilde{\mathcal{D}}$ is multiplication by 2. The duals are $\widehat{G} = \widehat{\mathbf{R}} = \mathbf{R}$, and $\widehat{\Gamma} = \widehat{\mathbf{Z}} = \mathbf{T} \cong \widehat{G}/\mathrm{Ann}\Gamma$, where $\mathbf{T}$ is the unit circle. Recall from Section 0.1 that elements of $\widehat{\mathbf{R}}$ are characters, ie. homomorphisms of $\mathbf{R}$ to the circle. An explicit description of these homomorphisms is given by

$$(x, \xi) = e^{2\pi i x \xi}$$

where $x \in \mathbf{R}$ and $\xi \in \mathbf{R}$. We therefore have that $\mathrm{Ann}\mathbf{Z} = \mathbf{Z}$. We can identify $\widehat{\Gamma}$ with the interval $[-\frac{1}{2}, \frac{1}{2})$, when we do this the character is given by

$$(\gamma, \zeta) = e^{2\pi i \gamma \zeta}$$

for $\gamma \in \Gamma, \zeta \in [-\frac{1}{2}, \frac{1}{2})$. From Corollary 1.2.6,

$$(\hat{\mathcal{D}}\hat{f})(\xi) = 2^{-\frac{1}{2}} \hat{f}\left(\frac{\xi}{2}\right)$$

for $f \in L^2(\mathbf{R})$, $\xi \in \mathbf{R}$.

Example 1.2.8 Let us now examine the multiresolution structure of Example 1.1.4. We describe this example in detail in Section 1.3. In this case $G = \mathbf{R}^d$ and $\Gamma = \mathbf{Z}^d$. The duals are $\widehat{G} = \mathbf{R}^d$, and $\widehat{\Gamma} = \mathbf{T}^d \cong [-\frac{1}{2}, \frac{1}{2})^d$. Elements of $\widehat{\mathbf{R}^d}$ are characters on $\mathbf{R}^d$, and given by

$$(x, \xi) = e^{2\pi i \sum_j x_j \xi_j} = \prod_{j=1}^{d} e^{2\pi i x_j \xi_j}$$

where $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$ and $\xi = (\xi_1, \ldots, \xi_d) \in \mathbf{R}^d$. The character on $\mathbf{Z}^d$ is given by

$$(\gamma, \zeta) = e^{2\pi i \sum_j \gamma_j \xi_j} = \prod_j e^{2\pi i \gamma_j \xi_j}$$

where $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbf{Z}^d$ and $\zeta = (\zeta_1, \ldots, \zeta_d) \in [-\frac{1}{2}, \frac{1}{2})^d$.

We know from Example 1.1.4 that $\tilde{\mathcal{D}} \in M^d(\mathbf{Z})$, so we can write

$$\tilde{\mathcal{D}} = \begin{pmatrix} \tilde{\mathcal{D}}_{11} & \cdots & \tilde{\mathcal{D}}_{1d} \\ \vdots & \ddots & \vdots \\ \tilde{\mathcal{D}}_{d1} & \cdots & \tilde{\mathcal{D}}_{dd} \end{pmatrix}.$$

Now for $x \in \mathbf{R}^d, \xi \in \mathbf{R}^d$, $(\tilde{\mathcal{D}}x, \xi) = (x, \hat{\tilde{\mathcal{D}}}\xi)$, and

$$(\tilde{\mathcal{D}}x, \xi) = \prod_{j=1}^{d}\prod_{k=1}^{d} e^{2\pi i \tilde{\mathcal{D}}_{jk} x_k \xi_j}.$$

But

$$(x, \hat{\tilde{\mathcal{D}}}\xi) = \prod_{j=1}^{d}\prod_{k=1}^{d} e^{2\pi i (\hat{\tilde{\mathcal{D}}})_{kj} x_k \xi_j}.$$

We therefore have that for all $j, k = 1, \ldots, d$, $(\hat{\tilde{\mathcal{D}}})_{kj} = \tilde{\mathcal{D}}_{jk}$. Because $\tilde{\mathcal{D}} \in M^d(\mathbf{Z})$, $\hat{\tilde{\mathcal{D}}} = \tilde{\mathcal{D}}^*$.

Example 1.2.9 Let $p \in \mathbf{Z}$, and define

$$G = \{(g_n)_{n \in \mathbf{Z}} : g_n \in \mathbf{Z}_p, \exists N \in \mathbf{Z} \text{ such that } n > N \Rightarrow g_n = 0\}.$$

where $\mathbf{Z}_p$ is the cyclic group of order p. Then G is an Abelian group with the operation given by

$$(g^1 + g^2)_n = g^1_n + g^2_n \mod \text{p}.$$

The group G can be thought of as consisting of doubly infinite sequences of elements of $\mathbf{Z}^p$ which eventually end in zeros at one end, with the group operation being componentwise addition in $\mathbf{Z}^p$. We shall use the symbol "+" to denote the group operation on G.

We will be interested in the following subgroups:

$$\Gamma = \{g \in G : g_j = 0 \text{ for } j < 0\};$$

and

$$D = G/\Gamma = \{g \in G : g_j = 0 \text{ for } j \geq 0\}.$$

When $p = 2$, D is known as the *Cantor group*. The group $\mathbf{Z}_p$ can be equipped with the discrete topology, and counting measure. We equip G and its various subgroups with the product topology of each $\mathbf{Z}_p$. We then have that Γ is countable, closed, and discrete, and that $D = G/\Gamma$ is compact. To see that Γ is closed, examine the set $G - \Gamma$ of elements og G which are not elements of Γ. The set $G - \Gamma$ is open because for all $x \in G - \Gamma$ and for all $\varepsilon > 0$, there exists $x' \in G - \Gamma$ such that $|x - x'| < \varepsilon$. The group Γ is discrete because for $\gamma \in \Gamma$, there exists an open neighbourhood U in G of γ such that $U \cap \Gamma = \{\gamma\}$. The group D is compact because it is the quotient of G with a discrete group.

We are interested in the Hilbert space $\mathcal{H} = L^2(G, \mu_G)$, we define the translation to be the unitary operator

$$(\gamma f)(x) = f(x - \gamma)$$

for $f \in \mathcal{H}$, $x \in G$ and $\gamma \in \Gamma$. We define the dilation to be the unitary operator

$$((\mathcal{D}f)(x))_j = \frac{1}{\sqrt{p}}(x_{j-1})$$

for $f \in \mathcal{H}$, $x = (x_j) \in G$ and $j \in \mathbf{Z}$. We can write the dilation as

$$(\mathcal{D}f)(x) = \frac{1}{\sqrt{p}}(\tilde{\mathcal{D}}x) \text{ where } (\tilde{\mathcal{D}}x)_j = x_{j-1}$$

for $f \in \mathcal{H}$, $x = (x_j) \in G$ and $j \in \mathbf{Z}$. We have for $x = (x_j) \in G$, $f \in \mathcal{H}$, $\gamma = (\gamma_j) \in \Gamma$, and $j \in \mathbf{Z}$ that

$$(\Delta(\gamma)x)_j = (\mathcal{D}^{-1}\gamma\mathcal{D}f)(x_j) = f(x_j - \gamma_{j-1})$$

and so $(\mathcal{H}, \Gamma, \mathcal{D})$ is a multiresolution structure with index p. We also have

$$\Delta(\Gamma) = \{g \in G : g_j = 0 \text{ for } j < 1\};$$

which is isomorphic to Γ, so $(\mathcal{H}, \Gamma, \mathcal{D})$ is a harmonic multiresolution structure.

For the rest of this example, let us consider the case that $p = 2$. It is known (see [EG, Lg1, Lg2]) that the dual group of G is isomorphic to G, and that characters are given by

$$(x, \xi) = \prod_{j \in \mathbf{Z}} (-1)^{\xi_{-1-j} x_j}$$

for $x \in G$, $\xi \in \widehat{G}$. Let us now examine the Fourier transforms of the translations and dilation. We have for all $x \in G$, $\xi \in \widehat{G}$, that $(\tilde{\mathcal{D}}x, \xi) = (x, \hat{\tilde{\mathcal{D}}}\xi)$. When we expand this out we get

$$\prod_{j \in \mathbf{Z}} (-1)^{\xi_{-1-j}(\tilde{\mathcal{D}}x)_j} = \prod_{j \in \mathbf{Z}} (-1)^{\xi_{-1-j}x_{j-1}} = \prod_{j \in \mathbf{Z}} (-1)^{(\hat{\tilde{\mathcal{D}}}\xi)_{-1-j}x_j}.$$

So for all $x \in G$ and $\xi \in \widehat{G}$ we have $\xi_{-j}x_j = (\hat{\tilde{\mathcal{D}}}\xi)_{-1-j}x_j$ and so $((\hat{\tilde{\mathcal{D}}})(\xi))_j = \xi_{j+1}$. Thus by Corollary 1.2.6,

$$(\hat{\mathcal{D}}\hat{f})(\xi)_j = \frac{1}{\sqrt{2}}\hat{f}(\xi_{j-1}).$$

We now describe a simple example of a wavelet and scaling function in this setting. Let the scaling function be $\varphi(x) = \chi_D(x)$, the characteristic funcion of D. We then have that $(\mathcal{D}^{-1}\varphi)(x) = \varphi(x) + \varphi(x+1)$. The wavelet associated with φ is given by

$$\psi(x) = \mathcal{D}(\varphi(x) + 1) - \mathcal{D}(\varphi(x)).$$

Other more complex examples of wavelets and scaling functions on the Cantor group are described in [Lg1, Lg2].

1.3 The Standard Multiresolution Structure on $L^2(\mathbf{R}^d)$

Let us begin by defining what we call the standard multiresolution structure on $L^2(\mathbf{R}^d)$, this is a fairly standard way of formulating wavelets on $L^2(\mathbf{R}^d)$ and the main example that we shall work with. We shall then show that it is not only a multiresolution structure, but also a harmonic multiresolution structure (see Definitions 1.1.1, 1.2.1). We briefly examined the standard multiresolution structure in Examples 1.1.4 and 1.2.8. We shall need many of the results from this section in Chapter 2.

Definition 1.3.1 Let $\tilde{\mathcal{D}} \in M^d(\mathbf{Z})$ bs a $d \times d$ matrix with integer entries such that all of the eigenvalues of $\tilde{\mathcal{D}}$ are greater than 1. We shall call a matrix with these properties a *dilation matrix*. We define the *standard multiresolution structure* on $L^2(\mathbf{R}^d)$ associated with $\tilde{\mathcal{D}}$ to be the multiresolution structure with Hilbert space $L^2(\mathbf{R}^d)$ and with translation group $\Gamma = \mathbf{Z}^d$ (as an additive group), and dilation given by

$$(\mathcal{D}f)(x) = \sqrt{m} f(\tilde{\mathcal{D}}x)$$

for $x \in \mathbf{R}^d$, $f \in L^2(\mathbf{R}^d)$, and where $m \in \mathbf{N}$ is the index of the multiresolution structure.

We remark that because all of the eigenvalues of $\tilde{\mathcal{D}}$ are nonzero, $\tilde{\mathcal{D}}$ is invertible.

Lemma 1.3.2 *The standard multiresolution structure on $L^2(\mathbf{R}^d)$ is a harmonic multiresolution structure.*

PROOF: We showed in Example 1.1.4 that because $\tilde{\mathcal{D}} \in M^d(\mathbf{Z})$, $\tilde{\mathcal{D}}$ maps $\mathbf{Z}^d$ into a subgroup of itself, and that the standard multiresolution structure on $L^2(\mathbf{R}^d)$ is a multiresolution structure. It can be verified that the standard multiresolution structure on $L^2(\mathbf{R}^d)$ satisfies all of the required properties of Definition 1.2.1 and so is a harmonic multiresolution structure. □

The following result is due to Gröchenig and Madych, see [GM, Lemma 2].

Lemma 1.3.3 *Let $\tilde{\mathcal{D}}$ be as defined as in Definition 1.3.1, and let m be the index of the multiresolution structure corresponding to $\tilde{\mathcal{D}}$. Then $m = |\det \tilde{\mathcal{D}}|$.*

PROOF: From the definition of a multiresolution structure, m is the number of cosets of $\tilde{\mathcal{D}}\mathbf{Z}^d$ in $\mathbf{Z}^d$. Let $\alpha_0, \ldots, \alpha_{m-1}$ be a set of coset representatives of $\tilde{\mathcal{D}}\mathbf{Z}^d$ in $\mathbf{Z}^d$. We can then write the cosets as $\alpha_0 + \tilde{\mathcal{D}}\mathbf{Z}^d, \ldots, \alpha_{m-1} + \tilde{\mathcal{D}}\mathbf{Z}^d$. Let $Q_0 := [0,1]^d$. Consider the following computation,

$$\begin{aligned} \cup_{\gamma \in \mathbf{Z}^d}\{\tilde{\mathcal{D}}\gamma + \cup_{i=0}^{m-1}(\alpha_i + Q_0)\} &= \cup_{\gamma \in \mathbf{Z}^d}\{\cup_{i=0}^{m-1}(\alpha_i + \tilde{\mathcal{D}}\gamma + Q_0)\} \\ &= \cup_{\gamma \in \mathbf{Z}^d}\{\gamma + Q_0\} \\ &= \mathbf{R}^d. \end{aligned}$$

Because $\tilde{\mathcal{D}}$ is invertible, $\tilde{\mathcal{D}}^{-1}$ is also invertible, and so $\tilde{\mathcal{D}}^{-1}\mathbf{R}^d = \mathbf{R}^d$. Applying $\tilde{\mathcal{D}}^{-1}$ to the result of the previous computation, we obtain

$$\cup_{\gamma \in \mathbf{Z}^d}\{\gamma + \cup_{i=0}^{m-1}\tilde{\mathcal{D}}^{-1}(\alpha_i + Q_0)\} = \mathbf{R}^d.$$

Now let $Q := \cup_{i=0}^{m-1}\tilde{\mathcal{D}}^{-1}(\alpha_i + Q_0)$, then the above calculation tells us that $\mathbf{R}^d = \cup_{\gamma \in \mathbf{Z}^d}\{\gamma + Q\}$. Now for all $\gamma \in \mathbf{Z}^d$,

$$\gamma + Q = \cup_{i=0}^{m-1}\tilde{\mathcal{D}}^{-1}(\tilde{\mathcal{D}}\gamma + \alpha_i + Q_0).$$

This calculation tells us that the sets $(\gamma + Q)_{\gamma \in \mathbf{Z}^d}$ intersect in sets of measure zero, because $\tilde{\mathcal{D}}\gamma + \alpha_i$ ranges over $\mathbf{Z}^d$ as γ ranges over $\mathbf{Z}^d$ and $i = 0, \ldots, m-1$. Because the sets $(\gamma + Q)_{\gamma \in \mathbf{Z}^d}$ cover $\mathbf{R}^d$, the Lebesgue measure of Q is equal to 1. Because Q is a union of m null-intersecting subsets $\{\tilde{\mathcal{D}}^{-1}(\alpha_i + Q_0)\}_{i=0}^{m-1}$, each of these subsets has measure $1/|\det \tilde{\mathcal{D}}|$. It therefore follows that $m = |\det \tilde{\mathcal{D}}|$. □

Let us now examine the Harmonic analysis of Γ^n, and the role of the Fourier transform. It is well known that the dual of $\mathbf{R}^d$ is $\mathbf{R}^d$, the annihilator of $\mathbf{Z}^d$ as a subgroup of $\mathbf{R}^d$ is $\mathbf{Z}^d$, and the dual of $\mathbf{Z}^d$ is $\mathbf{T}^d$, where $\mathbf{T}$ is the unit circle (see [Ru2, 2.2.2, 2.5.7] for proofs of these statements). Although $\widehat{\mathbf{R}^d} \equiv \mathbf{R}^d$, we shall sometimes use the notation $\widehat{\mathbf{R}^d}$ to indicate that we are working in the Fourier domain. The dual of Γ^n is given by $\widehat{\Gamma^n} = \mathbf{R}^d/\mathrm{Ann}\Gamma^n$. As mentioned in Section 1.2, the elements of $\widehat{\mathbf{R}^d}$ can be thought of as homomorphisms from $\mathbf{R}^d$ to the unit circle which are given by

$$(x, \xi) = e^{2\pi i \sum_j x_j \xi_j} = \prod_{j=1}^{d} e^{2\pi i x_j \xi_j}$$

for $x = (x_1, \ldots, x_d) \in \mathbf{R}^d, \xi = (\xi_1, \ldots, \xi_d) \in \widehat{\mathbf{R}^d}$.

We remark that functions of $\widehat{\mathbf{Z}^d} = \mathbf{T}^d$ can also be thought of as $\mathrm{Ann}\mathbf{Z}^d$-periodic functions on $\widehat{\mathbf{R}^d}$ because $\mathbf{T}^d \cong \mathbf{R}^d/\mathbf{Z}^d$. We will now describe in detail how this works. We can define an injective homomorphism $\iota : \mathbf{Z}^d \to \mathbf{R}^d$ by $\iota\gamma = \gamma$, so in other words ι is the natural embedding of $\mathbf{Z}^d$ in $\mathbf{R}^d$. We know from Section 0.1 that there exists a homomorphism $\hat{\iota} : \widehat{\mathbf{R}^d} \equiv \mathbf{R}^d \to \widehat{\mathbf{Z}^d} \equiv \mathbf{T}^d$ (the "quotient map") which satisfies $(\iota\gamma, \xi) = (\gamma, \hat{\iota}\xi)$ for $\gamma \in \mathbf{Z}^d$, $\xi \in \mathbf{T}^d$, and where $(\ ,\ )$ represents the mapping to the unit circle of $\mathbf{Z}^d$ given by elements of $\mathbf{T}^d$ as defined in Section 1.2 (i.e. the character). Now if a is a function of $\mathbf{T}^d$ we can treat it as a $\mathbf{Z}^d$-periodic function on $\mathbf{R}^d$ by setting $a(\xi) = a(\hat{\iota}\xi)$ for $\xi \in \mathbf{R}^d$.

We shall now examine the Harmonic analysis of the groups Γ^n. For each $n \in \mathbf{Z}$, we can define an injective group homomorphism $\iota_n : \mathbf{Z}^d \to \mathbf{R}^d$ by

$$\iota_n(\gamma) = \tilde{\mathcal{D}}^{-n}\gamma \tag{1.6}$$

for $\gamma \in \mathbf{Z}^d$. Note that $\iota_n(\gamma) = \tilde{\mathcal{D}}^{-n}\iota\gamma$, and $\iota_0 = \iota$. We have that the image of ι_n is the group $\tilde{\mathcal{D}}^{-n}\mathbf{Z}^d = \Gamma^n$. Define another homomorphism $\hat{\iota_n} : \widehat{\mathbf{R}^d} \to \mathbf{T}^d$ by

$$\hat{\iota_n}\xi = \hat{\iota}\tilde{\mathcal{D}}^{*-n}\xi$$

where $\xi \in \widehat{\mathbf{R}^d}$, and $\tilde{\mathcal{D}}^*$ is the adjoint of $\tilde{\mathcal{D}}$. We then calculate that for $\gamma \in \mathbf{Z}^d$, $\xi \in \widehat{\mathbf{R}^d}$,

$$\begin{aligned}
(\iota_n\gamma, \xi) &= (\tilde{\mathcal{D}}^{-n}\iota\gamma, \xi) \\
&= (\iota\gamma, \tilde{\mathcal{D}}^{*-n}\xi) \\
&= (\gamma, \hat{\iota}\tilde{\mathcal{D}}^{*-n}\xi) \\
&= (\gamma, \hat{\iota_n}\xi).
\end{aligned}$$

Let us now examine the annihilator of Γ^n. We calculate

$$\begin{aligned}\mathrm{Ann}\Gamma^n &= \{\xi \in \widehat{\mathbf{R}^d} \ : \ (\gamma^n, \xi) = 1 \text{ for } \gamma^n \in \Gamma^n\} \\ &= \{\xi \in \mathbf{R}^d \ : \ (\tilde{\mathcal{D}}^{-n}\gamma, \xi) = 1 \text{ for } \gamma \in \mathbf{Z}^d\} \\ &= \{\xi \in \mathbf{R}^d \ : \ (\gamma, \tilde{\mathcal{D}}^{*-n}\xi) = 1 \text{ for } \gamma \in \mathbf{Z}^d\} \\ &= \{\xi \in \mathbf{R}^d \ : \ \tilde{\mathcal{D}}^{*-n}\xi \in \mathrm{Ann}\mathbf{Z}^d\} \\ &= \tilde{\mathcal{D}}^{*n}\mathbf{Z}^d.\end{aligned}$$

We furthermore have that $\xi \in \mathrm{Ann}\Gamma^n$ if and only if for all $\gamma \in \mathbf{Z}^d$, $(\gamma, \hat{\iota_n}\xi) = 1$. This means that if a is a function on $\mathbf{T}^d$, then $a \circ \hat{\iota_n}$ is an $\mathrm{Ann}\Gamma^n$-periodic function on $\widehat{\mathbf{R}^d}$. The annihilator of Γ^n will be needed in Chapter 2 for us to examine the image under the Fourier transform of the Hilbert modules that we shall construct.

1.4 The Fast Wavelet Transform

The fast wavelet transform was discovered by S. Mallat in 1986 and led to the development of the notion of a multiresolution analysis by S. Mallat and Y. Meyer. The fast wavelet transform is closely related to algorithms in computer vision and signal processing such as pyramid algorithms and subband coding schemes. The concepts introduced in this section will be developed further in Chapter 3, where will shall also make use of the wavelet Hilbert module construction that is introduced in Chapter 2. The calculations involved in the proofs contained in this section are not new (see for example [Da1] or [HW]). The only possible originality in the following theorems in this section (as far as the author is aware) is that the calculations are done in the groups G and Γ rather than $\mathbf{Z}^d$ and $\mathbf{R}^d$ for $d \in \mathbf{N}$. The reader is referred to [KL, Me2] for more detailed accounts of the history of the development of the fast wavelet transform.

Because of its discrete nature, the fast wavelet transform is useful for applications which use numerical algorithms involving wavelet theory. The analysis part of the fast wavelet transform allows us to obtain the scaling and wavelet coefficients $\langle f, \mathcal{D}^n\varphi\rangle_n$ and $\langle f, \mathcal{D}^n\psi^i\rangle_n$ of a function f at a level n from the scaling coefficients at the next finer level $n+1$. We can iterate this process to obtain the wavelet coefficients at coarser levels. The synthesis part of the fast wavelet transform goes in the other direction and allows us to obtain the scaling coefficients at a particular level from the scaling and wavelet coefficients at the next coarser level.

To motivate the constructions in this section, let us begin with an example.

Motivating Example 1.4.1 Let us examine the harmonic multiresolution structure corresponding to dyadic wavelets as described in Examples 1.1.3 and 1.1.15. In this example we shall describe scaling and wavelet filters, and the fast wavelet transform. Suppose that we have a multiresolution analysis $\{V_n\}_{n\in\mathbf{Z}}$ with scaling function φ. The "scaling filter" h is a function on $\mathbf{Z}$ defined by

$$h(\gamma) = \langle \varphi, \mathcal{D}\gamma\varphi \rangle$$

for $\gamma \in \mathbf{Z}$. Because $V_0 \subset V_1$, it follows that $\varphi = \sum_{\gamma\in\mathbf{Z}} h(\gamma)\mathcal{D}\gamma\varphi$. In order to obtain a wavelet ψ which corresponds to φ, it is sufficient to obtain a "wavelet filter" g which satisfies $\psi = \sum_{\gamma\in\Gamma} g(\gamma)\mathcal{D}\gamma\varphi$. This is because $\psi \in V_1$. When this is the case g is given by

$$g(\gamma) = \langle \psi, \mathcal{D}\gamma\varphi \rangle.$$

It is shown in [Da1, Chapter 5] that if g is given by

$$g(\gamma) = (-1)^{\gamma-1}\overline{h(-\gamma-1)} \tag{1.7}$$

then g will define a wavelet ψ which corresponds to a scaling function φ. This choice of g leads to the formula for ψ described in Example 1.1.15.

It is a consequence of Proposition 1.4.2 that h and g satisfy

$$\sum_{\gamma\in\mathbf{Z}} h(\gamma)h(\gamma-2\alpha) = \delta_{\alpha,0}$$
$$\sum_{\gamma\in\mathbf{Z}} g(\gamma)g(\gamma-2\alpha) = \delta_{\alpha,0}$$
$$\sum_{\gamma\in\mathbf{Z}} h(\gamma)g(\gamma-2\alpha) = 0$$

where $\alpha \in \mathbf{Z}$ and δ is the Kronecker delta.

The fast wavelet transform is a method for calculating $\langle f, \mathcal{D}^n\gamma\varphi\rangle$ and $\langle f, \mathcal{D}^n\gamma\psi\rangle$ for $n < N$ if we know $\langle f, \mathcal{D}^N\gamma\varphi\rangle$ for each $\gamma \in \mathbf{Z}$, when f is an arbitrary element of $\mathcal{H} = L^2(\mathbf{R})$. In Theorem 1.4.3 it is shown that for $\alpha \in \mathbf{Z}$ and $n \in \mathbf{Z}$,

$$\begin{aligned} \langle f, \mathcal{D}^n\alpha\varphi\rangle &= \sum_{\gamma\in\mathbf{Z}} h(\gamma-2\alpha)\langle f, \mathcal{D}^{n+1}\gamma\varphi\rangle \\ \text{and} \quad \langle f, \mathcal{D}^n\alpha\psi\rangle &= \sum_{\gamma\in\mathbf{Z}} g(\gamma-2\alpha)\langle f, \mathcal{D}^{n+1}\gamma\varphi\rangle. \end{aligned}$$

The above two equations are known as the analysis part of the fast wavelet transform. It is also shown in Theorem 1.4.3 that for $\alpha \in \mathbf{Z}$ and $n \in \mathbf{Z}$,

$$\langle f, \mathcal{D}^{n+1}\alpha\varphi\rangle = \sum_{\gamma\in\mathbf{Z}} h(\alpha-2\gamma)\langle f, \mathcal{D}^n\gamma\varphi\rangle + g(\alpha-2\gamma)\langle f, \mathcal{D}^n\gamma\psi\rangle.$$

The above equation is known as the synthesis part of the fast wavelet transform.

We shall now define some functions on Γ associated with the wavelets and scaling functions. These functions on Γ are known as "filters" because they are related to applications of wavelets to electrical engineering (see [SN]). We shall need these sequences to describe the fast wavelet transform. Many of the properties of the wavelets and scaling functions can be described by properties of these sequences and operators defined from them. Let $\phi^1, \ldots, \phi^r$, and $\psi^1, \ldots, \psi^{(m-1)r}$ be scaling functions and MRA-multiwavelets associated with a multiresolution structure $(\Gamma, \mathcal{D})$. We define:

$$h^{i,j}(\gamma) = \langle \varphi^i, \mathcal{D}\gamma\varphi^j\rangle, \quad i,j = 1,\ldots,r \tag{1.8}$$
$$g^{i,j}(\gamma) = \langle \psi^i, \mathcal{D}\gamma\varphi^j\rangle, \quad i = 1,\ldots,(m-1)r,\ j = 1,\ldots,r \tag{1.9}$$

where $\gamma \in \Gamma$. We call each $h^{i,j}$ a *scaling filter* and call each $g^{i,j}$ a *wavelet filter*. Scaling filters are also known as *low pass filters* and wavelet filters are also known as *high pass filters*. Because $V_0 \subset V_1$ and $W_0 \subset V_1$, where $(V_n)_{n\in\mathbf{Z}}$ is the multiresolution analysis and $(W_n)_{n\in\mathbf{Z}}$ are the associated wavelet spaces, we obtain:

$$\varphi^i = \sum_{\gamma,j} h^{i,j}(\gamma)\mathcal{D}\gamma\varphi^j, \tag{1.10}$$

$$\psi^i = \sum_{\gamma,j} g^{i,j}(\gamma)\mathcal{D}\gamma\varphi^j \tag{1.11}$$

The following Proposition describes some conditions on the scaling and wavelet filters that follow from the fact that the translations of the scaling functions and wavelets are an orthonormal set. These conditions are known as the *shifted orthogonality conditions*. We shall relate scaling and wavelet filters to Hilbert modules in Sections 3.1 and 3.2. In Sections 3.4 and 3.5 we shall look at necessary and sufficient conditions for arbitrary functions on Γ to be scaling and wavelet filters.

Proposition 1.4.2 *Suppose that $h^{i,j}$ and $g^{k,j}$ are functions on Γ for $i = 1, \ldots, r$, $j = 1, \ldots, r$, $k = 1, \ldots, (m-1)r$. If there exist scaling functions $\phi^1, \ldots, \phi^r$ and wavelets $\psi^1, \ldots, \psi^{(m-1)r}$ for which (1.8) and (1.9) are satisfied, then $h^{i,j}$ and $g^{k,j}$ satisfy*

$$\sum_{\gamma\in\Gamma,i} h^{p,i}(\gamma)h^{q,i}(\Delta(\alpha^{-1})\gamma) = \delta_{\alpha,0}\delta_{p,q}, \tag{1.12}$$

$$\sum_{\gamma\in\Gamma,i} g^{p,i}(\gamma)g^{q,i}(\Delta(\alpha^{-1})\gamma) = \delta_{\alpha,0}\delta_{p,q}, \tag{1.13}$$

$$\sum_{\gamma\in\Gamma,i} h^{p,i}(\gamma)g^{q,i}(\Delta(\alpha^{-1})\gamma) = 0, \tag{1.14}$$

where δ is the Kronecker delta.

Proof: Using (1.10), (1.11), and orthogonality we obtain

$$\begin{aligned}
\delta_{\alpha,0}\delta_{p,q} &= \langle\varphi^p, \alpha\varphi^q\rangle \\
&= \langle\sum_{\gamma,i} h^{p,i}(\gamma)\mathcal{D}\gamma\varphi^i, \sum_{\beta,j} h^{q,j}(\Delta(\alpha^{-1})\beta)\mathcal{D}\beta\varphi^j\rangle \\
&= \sum_{\gamma,\beta,i,j} h^{p,i}(\gamma)h^{q,j}(\Delta(\alpha^{-1})\beta)\langle\gamma\varphi^i, \beta\varphi^j\rangle \\
&= \sum_{\gamma,i} h^{p,i}(\gamma)h^{q,i}(\Delta(\alpha^{-1})\gamma), \\
\delta_{\alpha,0}\delta_{p,q} &= \langle\psi^p, \alpha\psi^q\rangle \\
&= \langle\sum_{\gamma,i} g^{p,i}(\gamma)\mathcal{D}\gamma\varphi^i, \sum_{\beta,j} g^{q,j}(\Delta(\alpha^{-1})\beta)\mathcal{D}\beta\varphi^j\rangle \\
&= \sum_{\gamma,\beta,i,j} g^{p,i}(\gamma)g^{q,j}(\Delta(\alpha^{-1})\beta)\langle\gamma\varphi^i, \beta\varphi^j\rangle \\
&= \sum_{\gamma,i} g^{p,i}(\gamma)g^{q,i}(\Delta(\alpha^{-1})\gamma), \\
\delta_{\alpha,0}\delta_{p,q} &= \langle\varphi^p, \alpha\psi^q\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle \sum_{\gamma,i} h^{p,i}(\gamma)\mathcal{D}\gamma\varphi^i, \sum_{\beta,j} g^{q,j}(\Delta(\alpha^{-1})\beta)\mathcal{D}\beta\varphi^j \rangle \\
&= \sum_{\gamma,\beta,i,j} h^{p,i}(\gamma) g^{q,j}(\Delta(\alpha^{-1})\beta)\langle \gamma\varphi^i, \beta\varphi^j\rangle \sum_{\gamma,i} h^{p,i}(\gamma) g^{q,i}(\Delta(\alpha^{-1})\gamma)
\end{aligned}
$$

proving the desired result. □

The following theorem allows us to calculate each $\langle f, \mathcal{D}^n\gamma\varphi^i\rangle$ and $\langle f, \mathcal{D}^n\gamma\psi^j\rangle$ for $n < N$ if we know each $\langle f, \mathcal{D}^N\gamma\varphi^i\rangle$, where f is an arbitrary element of $\mathcal{H}$. This process is known as the fast wavelet transform. In Section 3.3 we shall investigate the fast wavelet transform in more detail.

Theorem 1.4.3 *Let $(\Gamma, \mathcal{D})$ be a multiresolution structure for a Hilbert space $\mathcal{H}$ and suppose $f \in \mathcal{H}$. For $n \in \mathbf{Z}$, $\gamma \in \Gamma$, $p = 1 \ldots r$, let*

$$c^p_{n,\gamma}(f) := \langle f, \mathcal{D}^n\gamma\varphi^p\rangle.$$

And for $n \in \mathbf{Z}$, $\gamma \in \Gamma$, $p = 1 \ldots s$, let

$$d^p_{n,\gamma}(f) := \langle f, \mathcal{D}^n\gamma\psi^p\rangle.$$

Suppose that $\phi^1, \ldots, \phi^r$, and $\psi^1, \ldots, \psi^{(m-1)r}$ are a set of scaling functions and MRA-multiwavelets associated with $(\Gamma, \mathcal{D})$ and let h, g be scaling and wavelet filters defined by equations (1.8) and (1.8). Then it follows that

$$c^p_{n,\alpha}(f) = \sum_{\gamma,j} h^{p,j}(\Delta(\alpha^{-1})\gamma) c^j_{n+1,\gamma}(f) \tag{1.15}$$

$$d^p_{n,\alpha}(f) = \sum_{\gamma,j} g^{p,j}(\Delta(\alpha^{-1})\gamma) c^j_{n+1,\gamma}(f) \tag{1.16}$$

$$c^p_{n+1,\gamma}(f) = \sum_{\alpha,j} h^{p,j}(\Delta(\alpha^{-1})\gamma) c^j_{n,\alpha}(f) + \sum_{\alpha,j} g^{p,j}(\Delta(\alpha^{-1})\gamma) d^j_{n,\alpha}(f). \tag{1.17}$$

PROOF: Because $V_n \subset V_{n+1}$ we have

$$
\begin{aligned}
c^p_{n,\alpha} &= \langle f, \mathcal{D}^n\alpha\varphi^p\rangle \\
&= \sum_{\gamma,j} \langle f, \mathcal{D}^{n+1}\gamma\varphi^j\rangle\langle \mathcal{D}^n\alpha\varphi^p, \mathcal{D}^{n+1}\gamma\varphi^j\rangle \\
&= \sum_{\gamma,j} c^j_{n+1,\gamma}\langle \alpha\varphi^p, \mathcal{D}\gamma\varphi^i\rangle \\
&= \sum_{\gamma,j} c^j_{n+1,\gamma}\langle \varphi^p, \alpha^{-1}\mathcal{D}\gamma\varphi^i\rangle \\
&= \sum_{\gamma,j} c^j_{n+1,\gamma} h^{p,j}_{\Delta(\alpha^{-1})\gamma}
\end{aligned}
$$

proving equation (1.15). The calculation to prove equation (1.16) is almost exactly the same.

We now prove equation (1.17). Let $P_{V_{n+1}}f$ be the projection of f on V_{n+1}. Because $V_{n+1} = V_n + W_n$ we have that $P_{V_{n+1}}f = P_{V_n}f + P_{W_n}f$, so

$$\sum_{\alpha,p} c^p_{n+1,\alpha}\mathcal{D}^{n+1}\alpha\varphi^p = \sum_{\alpha,p} c^p_{n,\alpha}\mathcal{D}^n\alpha\varphi^p + \sum_{\alpha,q} d^q_{n,\alpha}\mathcal{D}^n\alpha\psi^q$$

Taking inner products we obtain for all $\gamma \in \Gamma, i$ that

$$\begin{aligned}
\sum_{\alpha,p} c^p_{n+1,\alpha}\langle\mathcal{D}^{n+1}\alpha\varphi^p, \mathcal{D}^{n+1}\gamma\varphi^i\rangle &= \sum_{\alpha,p} c^p_{n,\alpha}\langle\mathcal{D}^n\alpha\varphi^p, \mathcal{D}^{n+1}\gamma\varphi^i\rangle \\
&\quad + \sum_{\alpha,q} d^q_{n,\alpha}\langle\mathcal{D}^n\alpha\psi^q, \mathcal{D}^{n+1}\gamma\varphi^i\rangle \\
c^p_{n+1,\gamma} &= \sum_{\alpha,p} c^p_{n,\alpha}\langle\alpha\varphi^p, \mathcal{D}\gamma\varphi^i\rangle + \sum_{\alpha,q} d^q_{n,\alpha}\langle\alpha\psi^q, \mathcal{D}\gamma\varphi^i\rangle \\
&= \sum_{\alpha,j} h^{p,j}(\Delta(\alpha^{-1})\gamma)c^j_{n,\alpha} + \sum_{\alpha,j} g^{p,j}(\Delta(\alpha^{-1})\gamma)d^j_{n,\alpha}
\end{aligned}$$

which proves the result. □

Remark 1.4.4 When there is only one scaling function the fast wavelet transform and the scaling and wavelet operators are considerably simplified. The scaling and wavelet filters become

$$h(\gamma) = \langle\varphi, \mathcal{D}\gamma\varphi\rangle,$$
$$g^i(\gamma) = \langle\psi^i, \mathcal{D}\gamma\varphi\rangle.$$

We can then write

$$\begin{aligned}
\varphi &= \sum_{\gamma\in\Gamma} h(\gamma)\mathcal{D}\gamma\varphi, \\
\psi^i &= \sum_{\gamma\in\Gamma} g^i(\gamma)\mathcal{D}\gamma\varphi.
\end{aligned}$$

Now from Proposition 1.4.2 we have that the filters $h(\gamma)$ and $g^i(\gamma)$ define orthonormal scaling functions and wavelets then they satisfy

$$\sum_{\gamma\in\Gamma} h(\gamma)h(\Delta(\alpha^{-1})\gamma) = \delta_{\alpha,0}, \tag{1.18}$$

$$\sum_{\gamma\in\Gamma} g^i(\gamma)g^j(\Delta(\alpha^{-1})\gamma) = \delta_{\alpha,0}\delta_{i,j}, \tag{1.19}$$

$$\sum_{\gamma\in\Gamma} h(\gamma)g^i(\Delta(\alpha^{-1})\gamma) = 0, \tag{1.20}$$

where δ is the Kronecker delta.

Example 1.4.5 We now describe an example which is due to Daubechies (see [Da1]), which is probably the simplest example of a continuous and compactly supported wavelet. The scaling filter for the Daubechies wavelet is given by

$$h(0) = \frac{1-\sqrt{3}}{4\sqrt{2}}, \quad h(1) = \frac{3-\sqrt{3}}{4\sqrt{2}}, \quad h(2) = \frac{3+\sqrt{3}}{4\sqrt{2}}, \quad h(3) = \frac{1+\sqrt{3}}{4\sqrt{2}},$$

with h taking the value zero everywhere else. A possible choice for the wavelet filter g, obtained from h by using (1.7), is given by

$$g(-1) = \frac{1-\sqrt{3}}{4\sqrt{2}}, \quad g(-2) = -\frac{3-\sqrt{3}}{4\sqrt{2}}, \quad g(-3) = \frac{3+\sqrt{3}}{4\sqrt{2}}, \quad g(-4) = -\frac{1+\sqrt{3}}{4\sqrt{2}},$$

with g taking the value zero everywhere else. One way of obtaining the scaling function and wavelet is to use the cascade algorithm, which shall be discussed in Section 3.4. The reader is referred to [Da1] for proofs that that h and g define a continuous compactly supported scaling function and a continuous compactly supported wavelet.

Examples of multiwavelets are generally somewhat more tricky to construct. The interested reader is referred to [DGH] for some such examples.

Chapter 2

Wavelets and Hilbert Modules

This chapter is probably the most important chapter in this thesis. Here we introduce the construction that relates wavelets to Hilbert modules. We are interested in Hilbert modules because we shall show that Hilbert modules over the C^*-algebra $C^*(\mathbf{Z}^d)$, where d is a natural number, are useful for studying wavelets. The construction in this chapter is the main tool that we will use to study wavelets in the rest of this thesis.

The C^*-algebra valued inner product which is used to construct the Hilbert modules that we shall describe is sometimes known as the "bracket product". The bracket product has been used to study wavelets before, see for example [BDR], [Fi] and [BCMO]. The Hilbert modules that we construct are a special case of the main example described in [R1]. This example is constructed from a closed subgroup of a locally compact group and is also described in Example 1 of [R2] and Appendix C of [RW] (see Remark 2.2.10).

The connection between Hilbert module theory and wavelet theory was described in a talk given by M. A. Rieffel in 1997 [R6]. Some of the results in this talk were also mentioned in [FL2]. The ideas mentioned in [R6] have been elaborated on in two papers by J. A. Packer and M. A. Rieffel [PR1, PR2]. The Hilbert module constructed in [PR1] uses the same C^*-algebra as is used here, the group C^*-algebra $C^*(\mathbf{Z}^d)$, but has a different linear space than the one used in the construction that we shall describe. The linear space in [PR1] consists of functions on $\mathbf{Z}^d$, and it is used to study continuous low pass filters and the wavelet matrix completion problem. The paper [PR2] was placed on arXiv.org in August 2003 and describes Hilbert modules that are very similar to the Hilbert modules that we shall introduce. This paper contains some very interesting work which uses these Hilbert modules to generalise wavelets to arbitrary projective modules over $C(\mathbf{T}^d)$. In this thesis we shall be interested in using Hilbert modules to study ordinary wavelet theory.

An interesting construction of a Hilbert module which also uses bracket products has been used by P. G. Casazza, M. Coco, and M. C. Lammers and is described in [CaLa] and [CoLa]. In these papers, the bracket product is used to study Gabor systems. The Hilbert module described in [CaLa] and [CoLa] is over the C^*-algebra $L^\infty([0,1])$. In Section 2.3 we shall investigate how these Hilbert modules relate to wavelets. Our construction is related to work on frames for Hilbert modules from [FL1, FL2, FL3], which we briefly described in Section 0.5. Other results which relate wavelets to C^*-algebras and other operator algebras have been obtained in [BJ2, BJ1, DL, HL2, Lr1, RT].

Throughout this chapter, we shall be working in the setting of the standard multiresolution structure on the Hilbert space $L^2(\mathbf{R}^d)$ corresponding to a dilation matrix $\tilde{\mathcal{D}}$ which we described in Section 1.3, and Examples 1.1.4 and 1.2.8. In Section 2.1 we shall consider an embedding $\theta : \mathbf{Z}^d \to \mathbf{R}^d$, and use it to construct a Hilbert $C^*(\mathbf{Z}^d)$-module X_θ, which we shall show is contained in $L^2(\mathbf{R}^d)$. We shall also construct a Hilbert $C(\mathbf{T}^d)$-module $\hat{X}_\theta$, which is the image of X_θ under the Fourier transform. The C^*-algebra valued inner product that we shall use is known as the "bracket product". In Lemmas 2.1.5, 2.1.7, 2.1.8 and 2.1.9 we shall investigate the convergence properties of the bracket product in both the Fourier and non-Fourier domains and also of the associated module action. We study these operations in $L^2(\mathbf{R}^d)$, as well as X_θ and $\hat{X}_\theta$. This is important because we shall see in Chapter 3 that when using these operations to study wavelet theory, we often need to be able to use them in $L^2(\mathbf{R}^d)$. In Lemma 2.1.12 we analyse the role of the Fourier transform in detail. In Lemma 2.1.13 we show that the Hilbert modules that we construct can be embedded in $L^2(\mathbf{R}^d)$. The main result in this section is Theorem 2.1.21 which verifies that X_θ and $\hat{X}_\theta$ are full Hilbert modules, and that the Fourier transform defines a Hilbert module isomorphism.

In Section 2.2 we shall then incorporate the dilation by constructing a chain of Hilbert $C^*(\mathbf{Z}^d)$-modules $(X_n)_{n\in\mathbf{Z}}$. For each $n \in \mathbf{Z}$, the Hilbert $C^*(\mathbf{Z}^d)$-module X_n is associated with the action of Γ^n on $L^2(\mathbf{R}^d)$. We shall construct Hilbert $C(\mathbf{T}^d)$-modules $(\hat{X}_n)_{n\in\mathbf{Z}}$ which are the images of $(X_n)_{n\in\mathbf{Z}}$ under the Fourier transform on $\mathbf{R}^d$. The main result of this section in Theorem 2.2.6, which uses Theorem 2.1.21 to verify that X_n and $\hat{X}_n$ are full Hilbert modules. We then prove Corollary 2.2.7, which applies some basic Hilbert module theory to our construction. Corollary 2.2.7 demonstrates that the translations and dilation can be thought of as adjointable operators. Proposition 2.2.8 is based on [PR2, Proposition 1.11], and shows that each of the Hilbert $C^*(\mathbf{Z}^d)$-modules $(X_n)_{n\in\mathbf{Z}}$ share the same linear space. We conclude this section with Proposition 2.2.9, which is a formulation of necessary and sufficient conditions for a set of elements of the Hilbert module X_0 to be a multiwavelet.

The construction in Sections 2.1 and 2.2 are quite similar to Hilbert modules constructed in [PR2], but there are some differences. One difference is in how the dilation is incorporated. The Hilbert modules constructed in [PR2] can be thought of as being over the algebra $\theta(\mathbf{Z}^d)$, where θ is an embedding of $\mathbf{Z}^d$ in $\mathbf{R}^d$. This leads to a chain of isomorphic algebras each of which can be embedded in the next. An advantage to our approach is that because our algebras are identical, it is easier to formulate results about when mappings between Hilbert modules are adjointable, such as Corollary 2.2.7. In [PR2], while they describe their construction in detail in the Fourier domain, they only sketch the construction in the non-Fourier domain. We attempt to describe the Hilbert modules in Sections 2.1 and 2.2 in both the Fourier and non-Fourier domain. Another difference between the work here and that in [PR2] is that here we also examine the convergence properties of the bracket product in $L^2(\mathbf{R}^d)$. This is partially motivated by our desire to use the Hilbert modules described here to understand "ordinary" wavelet theory, which is defined in $L^2(\mathbf{R}^d)$.

When working with wavelets, it is sometimes useful to work with Hilbert modules over a "larger" C^*-algebra than $C(\mathbf{T}^d)$. In Section 2.3 we define a chain of Hilbert $L^\infty(\mathbf{T}^d)$-modules $(Y_n)_{n\in\mathbf{Z}}$ which are based on the construction described in [CaLa] and [CoLa]. Each Y_n contains $\hat{X}_n$ as a subset (here we only work in the Fourier domain). The main result in this section is Theorem 2.3.4 which is used to verify that each Y_n is a Hilbert $L^\infty(\mathbf{T}^d)$-module. This section also contains Proposition 2.3.6 and Proposition 2.3.7, which are slightly modified versions of Proposition 2.2.8 and Proposition 2.2.9.

Apart from Proposition 2.2.8 and Proposition 2.3.6, the results in this chapter were proved before the author had read [PR2].

Motivating Example 2.0.1 Before we develop the Hilbert module construction in detail, let us describe how this construction works when we are considering the classical setting of dyadic wavelets in $L^2(\mathbf{R})$ (see 1.1.3). In this setting we are considering the multiresolution structure with translation group being given by the integers, and dilation given by

$$(\mathcal{D}f)(x) = \sqrt{2}f(2x).$$

In order to construct a Hilbert module associated with this multiresolution structure, we shall make use of the "completion process" that was described in Section 0.4, specifically in Lemma 0.4.5. Recall that in this process we first construct an inner product module over a (pre)-C^*-algebra, and then we take the completion of the inner-product module with respect the Hilbert module norm (see Definition 0.4.1) to construct a Hilbert module.

Recall from Lemma 1.1.10 that associated with a multiwavelet $\psi^1, \ldots, \psi^r$ is a generalised multiresolution analysis $(V_n)_{n\in\mathbf{Z}}$. Recall from page 22 that associated with each V_n is a representation π^n of the translation group $\mathbf{Z}$ on $L^2(\mathbf{R})$, and V_n is an invariant subspace for π^n. In this setting the representation is given by

$$(\pi^n_k f)(x) = f(x - 2^{-n}k), \quad x \in \mathbf{R},\ k \in \mathbf{Z}.$$

Associated with the above representation is an embedding $\iota_n : \mathbf{Z} \to \mathbf{R}$ (where $\mathbf{Z}$ and $\mathbf{R}$ are thought of as additive groups), which is given by

$$\iota_n(k) = 2^{-n}k, \quad k \in \mathbf{Z}.$$

The representation π^n and embedding ι_n satisfy

$$(\pi^n_k f)(x) = f(x - \iota_n(k)), \quad x \in \mathbf{R},\ k \in \mathbf{Z}.$$

This motivates us to consider an arbitrary embedding $\theta : \mathbf{Z} \to \mathbf{R}$ (recall that an embedding is a mapping which is homeomorphic to its image). Associated with θ we define a representation π^θ of $\mathbf{Z}$ on $L^2(\mathbf{R})$ by

$$(\pi^\theta_k f)(x) = f(x - \theta(k)), \quad x \in \mathbf{R},\ k \in \mathbf{Z}.$$

The fact that θ is an embedding implies that there exists $r_\theta \in \mathbf{R}$ such that $\theta(k) = r_\theta k$, for $k \in \mathbf{Z}$.

We shall now define an inner product $C_c(\mathbf{Z})$-module by considering the linear space of continuous complex valued real functions, $C_c(\mathbf{R})$. For $f, g \in C_c(\mathbf{R})$ and $k \in \mathbf{Z}$, define a $C_c(\mathbf{Z})$-valued inner product by

$$[f,g]_\theta(k) := \int_{\mathbf{R}} \overline{f(x-\theta(k))} g(x)dx = \int_{\mathbf{R}} \overline{(\pi_k^\theta f)(x)} g(x)dx. \tag{2.1}$$

We make $C_c(\mathbf{R})$ into a right $C_c(\mathbf{Z})$-module by defining for $f \in C_c(\mathbf{R})$, $a \in C_c(\mathbf{Z})$,

$$(f \circ_\theta a)(x) := \sum_{k\in\mathbf{Z}} (\pi_k^\theta f)(x) a(k) \tag{2.2}$$

We show in Proposition 2.1.16 that the above operations make $C_c(\mathbf{R})$ into a right-inner product $C_c(\mathbf{Z})$-module.

It is possible to define the above operations in other spaces than $C_c(\mathbf{R})$ and $C_c(\mathbf{Z})$. We show in Lemma 2.1.7 that if $f, g \in L^2(\mathbf{R})$ then $[f,g]_\theta \in C_0(\mathbf{Z})$, where $[f,g]_\theta$ is given by equation (2.1). We show in Lemma 2.1.5 that if $f \in L^2(\mathbf{R})$ and $a \in l^1(\mathbf{Z})$ then $f \circ_\theta a \in L^2(\mathbf{R})$, where $\circ_\theta$ is given by equation (2.2).

We shall also define a Hilbert module in the Fourier domain. Recall that the Fourier transform on $\mathbf{R}$ defines a unitary map from $L^2(\mathbf{R})$ onto $L^2(\mathbf{R})$ and the Fourier transform on $\mathbf{Z}$ maps $l^1(\mathbf{Z})$ into $C(\mathbf{T})$. It is a consequence of Lemma 2.1.12 that if $f, g \in C_c(\mathbf{R})$, then

$$(\widehat{[f,g]_\theta})(\xi) = \frac{1}{r_\theta} \sum_{\beta\in\mathbf{Z}} \overline{\hat{f}(\xi + \beta/r_\theta)} \hat{g}(\xi + \beta/r_\theta), \quad \xi \in \mathbf{R}. \tag{2.3}$$

It also follows from Lemma 2.1.12 that if $f \in C_c(\mathbf{R})$ and $a \in C_c(\mathbf{Z})$, then

$$(\widehat{f \circ_\theta a})(\xi) = \hat{f}(\xi)\hat{a}(r_\theta \xi), \quad \xi \in \mathbf{R}. \tag{2.4}$$

This leads us to define for $p, q \in L^2(\widehat{\mathbf{R}})$,

$$[\![p,q]\!]_\theta(\xi) := \frac{1}{r_\theta} \sum_{\beta\in\mathbf{Z}} \overline{p(\xi + \beta/r_\theta)} q(\xi + \beta/r_\theta), \quad \xi \in \mathbf{R}. \tag{2.5}$$

If we identify the circle $\mathbf{T}$ with the quotient $\mathbf{R}/\mathbf{Z}$, then the above equation defines a function on $\mathbf{T}$. And for $p \in L^2(\widehat{\mathbf{R}})$, $b \in C(\mathbf{T})$,

$$(p \widehat{\circ_\theta} b)(\xi) := p(\xi) b(r_\theta \xi), \quad \xi \in \mathbf{R}. \tag{2.6}$$

In Lemma 2.1.13 we show that we can define a norm on $C_c(\mathbf{R}^d)$ by

$$\|f\|_{X_\theta} := sup_{\zeta\in\mathbf{T}} \widehat{[f,f]}_\theta(\zeta), \quad f \in C_c(\mathbf{R}).$$

Keeping in mind that elements of $C_c(\mathbf{Z})$ are also contained in $C^*(\mathbf{Z})$, we define a Hilbert $C^*(\mathbf{Z})$-module X_θ by taking the completion of $C_c(\mathbf{R})$ with respect to the norm $\|\cdot\|_{X_\theta}$. We show in Theorem 2.1.21 that X_θ is a Hilbert module. We also show in Lemma

2.1.13 that for $f \in C_c(\mathbf{R})$, $\|f\|_2 \leq \|f\|_{X_\theta}$, which implies that we can embed X_θ in $L^2(\mathbf{R})$.

Because the Fourier transform on $\mathbf{R}$ is a unitary mapping from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$, we can consider the image $\hat{X}_\theta$ of the Hilbert module X_θ under the Fourier transform on $\mathbf{R}$. We show in Theorem 2.1.21 that $\hat{X}_\theta$ is also a Hilbert $C(\mathbf{T})$-module with inner product and module action given by equations (2.3) and (2.4).

Let us now apply the Hilbert module associated with θ to wavelet theory. For each $n \in \mathbf{Z}$, consider the Hilbert module X_{ι_n} constructed from the embedding ι_n using the process described above. We shall abbreviate X_{ι_n} as X_n, and we call $(X_n)_{n\in\mathbf{Z}}$ a *wavelet chain of Hilbert $C^*(\mathbf{Z})$-modules*. The $C^*(\mathbf{Z})$-valued inner product in X_n is given by

$$[f, g]_n(k) := \int_{\mathbf{R}} \overline{f(x)} g(x - 2^{-n}k)dx, \quad f, g \in X_n,\ k \in \mathbf{Z}. \tag{2.7}$$

The module action of $C^*(\mathbf{Z})$ on X_n is given by the following equation

$$(f \circ_n a)(x) := \sum_{k\in\mathbf{Z}} f(x - 2^{-n}k)a(k), \quad f \in X_n,\ a \in l^1(\mathbf{Z}),\ x \in \mathbf{R}. \tag{2.8}$$

Note that the above equation only calculates $f \circ_n a$ when $a \in l^1(\mathbf{Z})$. Not all elements of $C^*(\mathbf{Z})$ are represented by functions on $\mathbf{Z}$. We do know that the Gelfand transform maps $C^*(\mathbf{Z})$ onto $C(\mathbf{T})$. Thus if we wish to use an arbitrary element of $C^*(\mathbf{Z})$, it is sometimes best to work in the Fourier domain and we define for $p, q \in L^2(\widehat{\mathbf{R}})$,

$$[\![p, q]\!]_n(\xi) := 2^n \sum_{\beta\in\mathbf{Z}} \overline{p(\xi + 2^n\beta)} q(\xi + 2^n\beta), \quad \xi \in \mathbf{R}. \tag{2.9}$$

And for $p \in L^2(\widehat{\mathbf{R}})$, $b \in C(\mathbf{T})$,

$$(p \widehat{\circ_n} b)(\xi) := p(\xi)b(2^{-n}\xi), \quad \xi \in \mathbf{R}. \tag{2.10}$$

We let $\hat{X}_n$ be the image of X_n under the Fourier transform. We show in Theorem 2.2.6 that X_n and $\hat{X}_n$ are Hilbert modules.

There is also a "larger" Hilbert module that we define in the Fourier domain. We set Y_n to be the set of functions on $\mathbf{R}$ for which

$$\operatorname{ess\,sup}_{\zeta\in\mathbf{T}} [\![p, p]\!]_n(\zeta) < \infty.$$

We shall show in Lemma 2.3.2 that for $p \in Y_n$, the sum in $[\![p, p]\!]_n$ converges in the weak* topology to an element of $L^\infty(\mathbf{T})$. We show in Theorem 2.3.4 that Y_n is a Hilbert $L^\infty(\mathbf{T})$-module when equipped with the operations given by equations (2.9) and (2.10).

2.1 Constructing a Hilbert Module from the Translations

In this section we shall consider an arbitrary embedding $\theta : \mathbf{Z}^d \to \mathbf{R}^d$ and construct the Hilbert $C^*(\mathbf{Z}^d)$-modules X_θ and $\hat{X}_\theta$. We shall define a $C^*(\mathbf{Z}^d)$-valued inner product

$[\ ,\ ]_\theta$ on X_θ and also the associated module action $\circ_\theta$. We do this by defining a $C_c(\mathbf{Z}^d)$-valued inner product on $C_c(\mathbf{R}^d)$ and a module action of $C_c(\mathbf{Z}^d)$ on $C_c(\mathbf{R}^d)$. We shall construct the Hilbert module X_θ by taking the completion of $C_c(\mathbf{R}^d)$ with respect to the Hilbert module norm. We shall define a $C(\mathbf{T}^d)$-valued inner product $[\![\ ,\ ]\!]_\theta$ on $\hat{X}_\theta$ by defining an $L^1(\mathbf{T}^d)$-valued inner product on $L^2(\widehat{\mathbf{R}^d})$. We shall also define a module action $\widehat{\circ_\theta}$ of $C(\mathbf{T}^d)$ on $\hat{X}_\theta$. We shall prove that these inner products make X_θ and $\hat{X}_\theta$ into isomorphic Hilbert $C^*(\mathbf{Z}^d)$-modules, keeping in mind that $C^*(\mathbf{Z}^d)$ and $C(\mathbf{T}^d)$ are isomorphic C^*-algebras.

We begin by examining the group C^*-algebra $C^*(\mathbf{Z}^d)$. Much of what we now mention about $C^*(\mathbf{Z}^d)$ is rather standard, but is mentioned here because there are many C^*-algebraic subtleties which affect the way that our Hilbert module construction works.

Suppose that a is a function on $\mathbf{Z}^d$. We can define the involution a^* of a to be

$$a^*(\gamma) = \overline{a(-\gamma)} \tag{2.11}$$

for $\gamma \in \mathbf{Z}^d$. We know from Section 0.3 that $C_c(\mathbf{Z}^d)$ and $l^1(\mathbf{Z}^d)$ are $*$-algebras with involution given by the above equation and multiplication given by convolution

$$(a * b)(\gamma) = \sum_{\alpha \in \mathbf{Z}^d} a(\alpha) b(\gamma - \alpha) \tag{2.12}$$

where a and b are elements of either $C_c(\mathbf{Z}^d)$ or $l^1(\mathbf{Z}^d)$, and $\gamma \in \mathbf{Z}^d$. It is shown in [Fo, Proposition 2.39] that if $a \in l^\infty(\mathbf{Z}^d)$ and $b \in l^1(\mathbf{Z}^d)$ then the convolution $a * b$ is well defined and is an element of $l^\infty(\mathbf{Z}^d)$.

Recall from Section 0.3 that the reduced group C^*-algebra $C^*_r(\mathbf{Z}^d)$ is the completion of the left regular representation of $l^1(\mathbf{Z}^d)$ with respect to the norm given by

$$\|a\|_{C^*(\mathbf{Z}^d)} := \sup_{\zeta \in \mathbf{T}^d} |\hat{a}(\zeta)|$$

where $a \in l^1(\mathbf{Z}^d)$, and $\hat{a} \in C(\mathbf{T}^d)$ is the Fourier transform of a. We can also define $C^*(\mathbf{Z}^d)$ to be the closure in $B(l^2(\mathbf{Z}^d))$ of the left regular representation of $l^1(\mathbf{Z}^d)$ with respect to the operator norm.

From the definition of $C^*(\mathbf{Z}^d)$, the algebras $C_c(\mathbf{Z}^d)$ and $l^1(\mathbf{Z}^d)$ are dense subalgebras of $C^*(\mathbf{Z}^d)$. These dense subalgebras have the property that they consist of functions on $\mathbf{Z}^d$. Any element of $C^*(\mathbf{Z}^d)$ can be thought of as an element of $C(\mathbf{T}^d)$ via the isomorphism of Theorem 0.3.6. It is worth noting that Theorem 0.3.6 maps $l^1(\mathbf{Z}^d)$ into $C(\mathbf{T}^d)$ using the Fourier transform and then constructs the isomorphism from the Fourier transform by using the Stone-Weierstrass Theorem. When a is an element of $l^1(\mathbf{Z}^d)$ this isomorphism maps a to its Fourier transform.

The Hilbert modules associated with wavelets will depend on the embeddings $\iota_n : \mathbf{Z}^d \to \mathbf{R}^d$, for $n \in \mathbf{Z}$, that were described in Section 1.3. We shall first describe how to construct a Hilbert module using an arbitrary embedding $\theta : \mathbf{Z}^d \to \mathbf{R}^d$. We shall also be interested in what happens to θ in the Fourier domain. Although the dual of $\mathbf{R}^d$ is $\mathbf{R}^d$, we shall use sometimes use the notation $\widehat{\mathbf{R}^d}$ to indicate that we are working in the Fourier domain.

Lemma 2.1.1 *Suppose that* $\theta : \mathbf{Z}^d \to \mathbf{R}^d$ *is an embedding. There exists a unique nonsingular linear transformation* $A_\theta : \mathbf{R}^d \to \mathbf{R}^d$ *which satisfies* $\theta = A_\theta \iota$*, where* ι *is the natural embedding of* $\mathbf{Z}^d$ *in* $\mathbf{R}^d$*. If* $\hat{\theta} : \widehat{\mathbf{R}^d} \to \mathbf{T}^d$ *is a homomorphism which satisfies*

$$(\theta\gamma, \xi) = (\gamma, \hat{\theta}\xi)$$

for all $\gamma \in \mathbf{Z}^d$ *and* $\xi \in \widehat{\mathbf{R}^d}$*, then* $\hat{\theta} = \hat{\iota} A_\theta^*$*. The annihilator of* $\theta(\mathbf{Z}^d)$ *is given by* $\mathrm{Ann}\ \theta(\mathbf{Z}^d) = (A_\theta^*)^{-1}\mathbf{Z}^d$*.*

Proof: Note that because θ is an embedding it is also a group homomorphism.

Let $\{e_i\}_{i=1}^d$ be the standard basis for $\mathbf{R}^d$. Consider the set $\{\theta(e_i)\}_{i=1}^d$. For $i, j = 1, \ldots, d$ there exists $a_{ij} \in \mathbf{R}$ such that $\theta(e_i) = \sum_{j=1}^d a_{ij} e_j$. If $\gamma = (\gamma_1, \ldots, \gamma_d)$ is an arbitrary element of $\mathbf{Z}^d$, then

$$\theta(\gamma) = \sum_{i=1}^d \gamma_i \theta(e_i) = \sum_{i=1}^d \gamma_i \sum_{j=1}^d a_{ij} e_j = \sum_{j=1}^d e_j \sum_{i=1}^d \gamma_i a_{ij}.$$

The elements a_{ij} therefore define a coordinate transformation matrix A_θ. This matrix is nonsingular because otherwise there will be linear dependence among the vectors $\{\theta(e_i)\}_{i=1}^d$, which is not allowed because θ is an embedding.

Let us now show that $\hat{\theta} = \hat{\iota} A_\theta^*$. We calculate

$$\begin{aligned} (\theta\gamma, \xi) &= (A_\theta \iota\gamma, \xi) \\ &= (\iota\gamma, A_\theta^* \xi) \\ &= (\gamma, \hat{\iota} A_\theta^* \xi) \end{aligned}$$

so $\hat{\theta} = \hat{\iota} A_\theta^*$.

Let us now examine the annihilator of $\theta(\mathbf{Z}^d)$. We calculate

$$\begin{aligned} \mathrm{Ann}\ \theta(\mathbf{Z}^d) &= \{\xi \in \widehat{\mathbf{R}^d} \ :\ (\theta\gamma, \xi) = 1 \text{ for } \gamma \in \mathbf{Z}^d\} \\ &= \{\xi \in \mathbf{R}^d \ :\ (\gamma, \hat{\iota} A_\theta^* \xi) = 1 \text{ for } \gamma \in \mathbf{Z}^d\} \\ &= \{\xi \in \mathbf{R}^d \ :\ A_\theta^* \xi \in \mathrm{Ann}\ \mathbf{Z}^d\} \\ &= (A_\theta^*)^{-1}\mathbf{Z}^d. \end{aligned}$$

□

It is a consequence of the above Lemma that $\xi \in \mathrm{Ann}\ \theta(\mathbf{Z}^d)$ if and only if for all $\gamma \in \mathbf{Z}^d$, $(\gamma, \hat{\theta}\xi) = 1$. This means that if a is a function on $\mathbf{T}^d$, then $a \circ \hat{\iota_n}$ is an $\mathrm{Ann}\ \theta(\mathbf{Z}^d)$-periodic function on $\widehat{\mathbf{R}^d}$, where here by $\circ$ we mean composition.

Notation 2.1.2 Throughout this section, when we are given an embedding $\theta : \mathbf{Z}^d \to \mathbf{R}^d$, we shall use A_θ to denote the linear transformation given by Lemma 2.1.1 and use $\hat{\theta}$ to denote the dual homomorphism given by Lemma 2.1.1.

The following operations will be used to define the Hilbert module that we shall eventually construct.

Definition 2.1.3 Let θ be a embedding of $\mathbf{Z}^d$ in $\mathbf{R}^d$. We define a representation π^θ of $\mathbf{Z}^d$ on $L^2(\mathbf{R}^d)$ by

$$(\pi^\theta_\gamma f)(x) := f(x-\theta(\gamma)) \quad x \in \mathbf{R}^d,\ \gamma \in \mathbf{Z}^d,\ f \in L^2(\mathbf{R}^d). \tag{2.13}$$

Note that if $f \in C_c(\mathbf{R}^d)$, then $\pi^\theta_\gamma f \in C_c(\mathbf{R}^d)$. For $f, g \in C_c(\mathbf{R}^d)$, $\gamma \in \mathbf{Z}^d$, we define the *bracket product associated with* θ to be the function on $\mathbf{Z}^d$ given by

$$[f,g]_\theta(\gamma) := \int_{\mathbf{R}^d} \overline{f(x-\theta(\gamma))} g(x) dx. \tag{2.14}$$

For $f \in C_c(\mathbf{R}^d)$, $a \in C_c(\mathbf{Z}^d)$, we define the *module action associated with* θ to be the function on $\mathbf{R}^d$ given by

$$(f \circ_\theta a) := \sum_{\gamma \in \mathbf{Z}^d} a(\gamma) f(x-\theta(\gamma)) = \sum_{\gamma \in \mathbf{Z}^d} a(\gamma)\pi^\theta_\gamma(f). \tag{2.15}$$

Note that it follows immediately from the above definition that for $f, g \in C_c(\mathbf{R}^d)$,

$$\begin{aligned} [f,g]_\theta(\gamma) &= \int_{\mathbf{R}^d} \overline{(\pi^\theta_\gamma f)} g(x) dx & (2.16)\\ &= (f^* * g)(\theta(\gamma)) & (2.17)\end{aligned}$$

where in equation (2.16) we embed $C_c(\mathbf{R}^d)$ in $L^2(\mathbf{R}^d)$.

Lemma 2.1.4 *It is the case that* $[f,g]_\theta \in C_c(\mathbf{Z}^d)$ *when* $f, g \in C_c(\mathbf{R}^d)$*; and* $f \circ_\theta a \in C_c(\mathbf{R}^d)$ *when* $f \in C_c(\mathbf{R}^d)$ *and* $a \in C_c(\mathbf{Z}^d)$.

PROOF: We first show that if $f \in C_c(\mathbf{R}^d)$ and $a \in C_c(\mathbf{Z}^d)$, then $f \circ_\theta a \in C_c(\mathbf{R}^d)$. In this case the sum in equation (2.15) is finite and therefore converges. Because the supports of f and a are both compact, the set of all $x \in \mathbf{R}^d$ such that there exists $\gamma \in \mathbf{Z}^d$ satisfying $a(\gamma)f(x-\theta(\gamma)) \neq 0$ is compact. We therefore have that $f \circ_\theta a \in C_c(\mathbf{R}^d)$.

We now show that if $f, g \in C_c(\mathbf{R}^d)$, then $[f,g]_\theta \in C_c(\mathbf{Z}^d)$. To see this consider the set of all $y \in \mathbf{R}^d$ such that there exists $x \in \mathbf{R}^d$ satisfying $\overline{f(x-y)}g(x) \neq 0$. Because the supports of both f and g are compact, this set is compact. This means that the set of $\gamma \in \mathbf{Z}^d$ such that there exists $x \in \mathbf{R}^d$ satisfying $\overline{f(x-\theta(\gamma))}g(x) \neq 0$ is also compact (and hence finite). □

We shall now see how the bracket product and associated module action can be defined on spaces other than $C_c(\mathbf{R}^d)$ and $C_c(\mathbf{Z}^d)$.

Lemma 2.1.5 *If* $f \in L^2(\mathbf{R}^d)$, $a \in l^1(\mathbf{Z}^d)$, *and* θ *is an embedding of* $\mathbf{Z}^d$ *in* $\mathbf{R}^d$, *then the function* $f \circ_\theta a$ *on* $\mathbf{R}^d$ *given for almost everywhere by*

$$(f \circ_\theta a) := \sum_{\gamma \in \mathbf{Z}^d} a(k)\pi^\theta_\gamma(f) \tag{2.18}$$

is measurable and is contained in $L^2(\mathbf{R}^d)$.

PROOF: Suppose that $f \in L^2(\mathbf{R}^d)$ and $a \in l^1(\mathbf{Z}^d)$, and let $S \subset \mathbf{Z}^d$ be a finite subset. In this case the sum $\sum_{\gamma \in S} a(\gamma)f(x-\gamma)$ is a finite sum and for $x \in \mathbf{R}^d$ we calculate

$$\begin{aligned} \left\| \sum_{\gamma \in S} a(\gamma)f(x-\theta(\gamma)) \right\|_2 &= \sqrt{\int_{\mathbf{R}^d} |\sum_{\gamma \in \mathbf{Z}^d} a(\gamma)f(x-\theta(\gamma))|^2 dx} \\ &\leq |\sum_{\gamma \in \mathbf{Z}^d} a(\gamma)| \sqrt{\int_{\mathbf{R}^d} |f(x-\theta(\gamma))|^2 dx} \\ &\leq \sum_{\gamma \in \mathbf{Z}^d} |a(\gamma)| \sqrt{\int_{\mathbf{R}^d} |f(x-\theta(\gamma))|^2 dx} \\ &= \|a\|_1 \|f\|_2 \end{aligned}$$

Now if $(S_n)_{n \in \mathbf{N}}$ is a sequence of finite subsets of $\mathbf{Z}^d$ for which $S_n \subset S_{n+1}$ and $\cup_{n \in \mathbf{N}} S_n = \mathbf{Z}^d$, then

$$(f \circ_\theta a)(x) = \lim_{n \to \infty} \sum_{\gamma \in S_n} a(\gamma)f(x-\theta(\gamma)).$$

This series is absolutely bounded by the bound $\|a\|_1\|f\|_2$ and it therefore follows that $\|f \circ_\theta a\|_2 \leq \|a\|_1\|f\|_2 < \infty$ and the sum in equation (2.18) converges unconditionally in $L^2(\mathbf{R}^d)$. □

Lemma 2.1.6 *Suppose that θ is an embedding of $\mathbf{Z}^d$ in $\mathbf{R}^d$, $a, b \in l^1(\mathbf{Z}^d)$, and $f, g \in L^2(\mathbf{R}^d)$, then*

$$(f+g) \circ_\theta a = f \circ_\theta a + g \circ_\theta a, \tag{2.19}$$

$$(f \circ_\theta a) \circ_\theta b = f \circ_\theta (a * b), \tag{2.20}$$

$$f \circ_\theta (a+b) = f \circ_\theta a + f \circ_\theta b. \tag{2.21}$$

It therefore follows that the module action associated with the bracket product makes $L^2(\mathbf{R}^d)$ into a right $l^1(\mathbf{Z}^d)$-module (where $L^1(\mathbf{Z}^d)$ has the convolution product) and also makes $C_c(\mathbf{R}^d)$ into a right $C_c(\mathbf{Z}^d)$-module.

PROOF: To prove equation (2.19) we make the following calculation

$$\begin{aligned} ((f+g) \circ_\theta a)(x) &= \sum_{\gamma \in \mathbf{Z}^d} (f+g)(x-\theta(\gamma))a(\gamma) \\ &= \sum_{\gamma \in \mathbf{Z}^d} f(x-\theta(\gamma))a(\gamma) + \sum_{\gamma' \in \mathbf{Z}^d} g(x-\theta(\gamma'))a(\gamma') \\ &= (f \circ_\theta a + g \circ_\theta b)(x). \end{aligned}$$

The following calculation proves equation (2.20)

$$((f \circ_\theta a) \circ_\theta b)(x) = \sum_{\gamma \in \mathbf{Z}^d} (f \circ_\theta a)(x-\theta(\gamma))b(\gamma)$$

$$\begin{aligned} &= \sum_{\gamma \in \mathbf{Z}^d} \sum_{\gamma' \in \mathbf{Z}^d} f(x - \theta(\gamma + \gamma'))a(\gamma')b(\gamma) \\ &= \sum_{\beta \in \mathbf{Z}^d} f(x - \theta(\beta)) \sum_{\gamma \in \mathbf{Z}^d} a(\beta - \gamma)b(\gamma) \\ &= (f \circ_\theta (a * b))(x). \end{aligned}$$

Equation (2.20) is verified by the following calculation

$$\begin{aligned} (f \circ_\theta (a + b))(x) &= \sum_{\gamma \in \mathbf{Z}^d} f(x - \theta(\gamma))(a + b)(\gamma) \\ &= \sum_{\gamma \in \mathbf{Z}^d} f(x - \theta(\gamma))a(\gamma) + \sum_{\gamma' \in \mathbf{Z}^d} f(x - \theta(\gamma'))b(\gamma') \\ &= (f \circ_\theta a + f \circ_\theta b)(x). \end{aligned}$$

□

Lemma 2.1.7 *If $f, g \in L^2(\mathbf{R}^d)$, $\gamma \in \mathbf{Z}^d$, and θ is an embedding of $\mathbf{Z}^d$ in $\mathbf{R}^d$, then the function on $\mathbf{Z}^d$ which is denoted by $[f, g]_\theta$ and given by*

$$[f, g]_\theta(\gamma) := \int_{\mathbf{R}^d} \overline{f(x - \theta(\gamma))} g(x) dx,$$

is contained in $C_0(\mathbf{Z}^d)$.

Proof: For $f, g \in L^2(\mathbf{R}^d)$ and $\gamma \in \mathbf{Z}^d$ we use the Cauchy-Schwartz inequality to obtain

$$\begin{aligned} |[f, g]_\theta(\gamma)| &= \left| \int_{\mathbf{R}^d} \overline{f(x - \theta(\gamma))} g(x) dx \right| \\ &\leq \int_{\mathbf{R}^d} |\overline{f(x - \theta(\gamma))} g(x)| dx \\ &\leq \left(\int_{\mathbf{R}^d} |f(x - \theta(\gamma))|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^d} |g(x)|^2 dx \right)^{\frac{1}{2}} \\ &= \|f\|_2 \|g\|_2 \end{aligned}$$

and so $[f, g]_\theta \in l^\infty(\mathbf{Z}^d)$. Let $(f_n)_{n \in \mathbf{N}}$ and $(g_n)_{n \in \mathbf{N}}$ be sequences of elements of $C_c(\mathbf{R}^d)$. Since $C_c(\mathbf{R}^d)$ is dense in $L^2(\mathbf{R}^d)$ and $[\ ,\ ]_\theta$ is linear in the second variable and conjugate linear in the first variable, it follows that if $f_n \to f$ in $L^2(\mathbf{R}^d)$ and $g_n \to g$ in $L^2(\mathbf{R}^d)$, then $[f_n, g_n]_\theta \to [f, g]_\theta$ in $l^\infty(\mathbf{Z}^d)$. Now $C_0(\mathbf{Z}^d)$ is the completion of $C_c(\mathbf{Z}^d)$ with respect to the supremum norm, so $[f, g]_\theta \in C_0(\mathbf{Z}^d)$ by [Ru1, Theorem 3.17]. □

Later in this section we shall use the above operations to construct a Hilbert module X_θ which can be embedded in $L^2(\mathbf{R}^d)$. Before doing so, we shall define some operations in the Fourier domain which will eventually be used to construct a Hilbert module $\hat{X}_\theta$. These operations shall involve some infinite sums for which some tricky convergence issues arise. We shall examine the convergence issues first.

Lemma 2.1.8 *Suppose $p \in L^1(\mathbf{R}^d)$. Identify $\mathbf{T}^d$ with the cube $[-\frac{1}{2}, \frac{1}{2})^d$, fix $\zeta \in \mathbf{T}^d$ and consider the function $p(\zeta + \cdot)$ on $\mathbf{Z}^d$. Then for almost every $\zeta \in \mathbf{T}^d$, $p(\zeta + \cdot) \in l^1(\mathbf{Z}^d)$, and $\sum_{\beta \in \mathbf{Z}^d} p(\cdot + \beta) \in L^1(\mathbf{T}^d)$. Moreover,*

$$\|p\|_{L^1(\mathbf{R}^d)} = \Big\| \|p(\zeta + \cdot)\|_{l^1(\mathbf{Z}^d)} \Big\|_{L^1(\mathbf{T}^d)} \tag{2.22}$$

$$= \Big\| \|p(\cdot + \beta)\|_{L^1(\mathbf{T}^d)} \Big\|_{l^1(\mathbf{Z}^d)}. \tag{2.23}$$

Note that in equation (2.22) the norm $\|p(\zeta + \cdot)\|_{l^1(\mathbf{Z}^d)}$ defines a function on $\mathbf{T}^d$ by varying $\zeta \in \mathbf{T}^d$. In equation (2.23) the norm $\|p(\cdot + \beta)\|_{L^1(\mathbf{T}^d)}$ defines a function on $\mathbf{Z}^d$ by varying $\beta \in \mathbf{Z}^d$. When we taken the second norm in these two equations, we are taking the norm of the functions just described.

PROOF: We can define a group isomorphism $I : \mathbf{Z}^d \times \mathbf{T}^d \to \mathbf{R}^d$ by setting $I(\beta, \zeta) = \beta + \zeta$ for $\beta \in \mathbf{Z}^d$, $\zeta \in \mathbf{T}^d$.

Let $\mu_{\mathbf{R}^d}$ be Lebesgue measure on $\mathbf{R}^d$, let $\mu_{\mathbf{Z}^d}$ be counting measure on $\mathbf{Z}^d$, and let $\mu_{\mathbf{T}^d}$ be Lebesgue measure on $\mathbf{T}^d$ normalised so that $\mu_{\mathbf{T}^d}(\mathbf{T}^d) = 1$. The group isomorphism I extends to a measure space isomorphism from $\mathbf{Z}^d \times \mathbf{T}^d$ equipped with the product measure $\mu_{\mathbf{Z}^d} \times \mu_{\mathbf{T}^d}$ onto the measure space $(\mathbf{R}^d, \mu_{\mathbf{R}^d})$.

Since $p \in L^1(\mathbf{R}^d)$, it follows that

$$p \circ I \in L^1(\mathbf{Z}^d \times \mathbf{T}^d, \mu_{\mathbf{Z}^d} \times \mu_{\mathbf{T}^d}).$$

The Lemma now follows directly from Fubini's Theorem. □

We remark that if $p, q \in L^2(\mathbf{R}^d)$, then $\bar{p}q \in L^1(\mathbf{R}^d)$. An application of Lemma 2.1.8 therefore tells us that the function on $\mathbf{T}^d$, given almost everywhere by $\sum_{\beta \in \mathbf{Z}^d} \bar{p}q(\cdot + \beta)$ is contained in $L^1(\mathbf{T}^d)$ and hence the series

$$\sum_{\beta \in \mathbf{Z}^d} \overline{p(\zeta + \beta)} q(\zeta + \beta)$$

is absolutely convergent for almost all $\zeta \in \mathbf{T}^d$.

Lemma 2.1.9 *Suppose that $p, q \in L^2(\widehat{\mathbf{R}^d})$ and $\theta : \mathbf{Z}^d \to \mathbf{R}^d$ satisfies $\theta(\gamma) = A_\theta \iota \gamma$, for a linear transformation $A_\theta : \mathbf{R}^d \to \mathbf{R}^d$. Let $[\![p, q]\!]_\theta$ be the function on $\mathbf{T}^d$ given by*

$$[\![p, q]\!]_\theta(\zeta) := \frac{1}{\det(A_\theta)} \sum_{\hat{\theta}(\xi) = \zeta} \overline{p(\xi)} q(\xi) \tag{2.24}$$

for almost every $\zeta \in \mathbf{T}^d$. Then the above sum converges absolutely almost everywhere and $[\![p, q]\!]_\theta \in l^1(\mathbf{T}^d)$. We furthermore have that $\{\xi \in \widehat{\mathbf{R}^d} : \hat{\theta}(\xi) = 0\} = (A_\theta^*)^{-1}\mathbf{Z}^d =$ Ann$\theta(\mathbf{Z}^d)$. We hence can write

$$[\![p, q]\!]_\theta(\hat{\theta}(\xi)) = \frac{1}{\det(A_\theta)} \sum_{\beta \in \text{Ann } \theta(\mathbf{Z}^d)} \overline{p(\xi + \beta)} q(\xi + \beta) \tag{2.25}$$

$$= \frac{1}{\det(A_\theta)} \sum_{\beta \in (A_\theta^*)^{-1}\mathbf{Z}^d} \overline{p(\xi + \beta)} q(\xi + \beta) \tag{2.26}$$

for almost every $\xi \in \mathbf{R}^d$.

PROOF: We know from the remarks following Lemma 2.1.8 that the series

$$\sum_{\beta \in \mathbf{Z}^d} \overline{p(\xi+\beta)} q(\xi+\beta)$$

is absolutely convergent for almost every $\xi \in \widehat{\mathbf{R}^d}$. It follows that the series

$$\sum_{\beta \in \mathrm{Ann}\theta(\Gamma)} \overline{p(\xi+\beta)} q(\xi+\beta) \tag{2.27}$$

is also absolutely convergent for almost every $\xi \in \widehat{\mathbf{R}^d}$.

For any $\beta \in \mathrm{Ann}\ \theta(\mathbf{Z}^d)$ and $\xi \in \widehat{\mathbf{R}^d}$, we have that $\hat{\theta}(\xi+\beta) = \hat{\theta}(\xi)\hat{\theta}(\beta) = \hat{\theta}(\xi)$. We therefore have that $\{\xi \in \widehat{\mathbf{R}^d} : \hat{\theta}(\xi) = 0\} = \mathrm{Ann}\theta(\mathbf{Z}^d)$, verifying equation (2.25). Because (2.27) converges absolutely, it follows from (2.25) that $[\![p,q]\!]_\theta(\zeta)$ converges absolutely for almost every $\zeta \in \mathbf{T}^d$. We know from Lemma 2.1.1 that $\mathrm{Ann}\ \theta(\mathbf{Z}^d) = (A_\theta^*)^{-1}\mathbf{Z}^d$, and this gives us the equality between equations (2.25) and (2.26). □

Definition 2.1.10 Suppose that $p, q \in L^2(\widehat{\mathbf{R}^d})$ and $\theta : \mathbf{Z}^d \to \mathbf{R}^d$ satisfies $\theta(\gamma) = A_\theta \iota \gamma$, for a linear transformation $A_\theta : \mathbf{R}^d \to \mathbf{R}^d$. We call $[\![p,q]\!]_\theta$ the *Fourier transformed bracket product* associated with θ, where $[\![p,q]\!]_\theta$ is defined by equation (2.24). Suppose $b \in C(\mathbf{T}^d)$, we define the *Fourier transformed module action* associated with θ to be the function on $\widehat{\mathbf{R}^d}$ denoted by $p\widehat{\circ_\theta}b$ and given by

$$(p\widehat{\circ_\theta}b)(\xi) := p(\xi)b(\hat{\theta}(\xi)) \tag{2.28}$$

for almost every $\xi \in \widehat{\mathbf{R}^d}$.

Lemma 2.1.11 *Suppose that $p \in L^2(\widehat{\mathbf{R}^d})$, $b \in C(\mathbf{T}^d)$ and $\theta : \mathbf{Z}^d \to \mathbf{R}^d$ is an embedding. Then $p\widehat{\circ_\theta}b \in L^2(\widehat{\mathbf{R}^d})$ and $\widehat{\circ_\theta}$ makes $L^2(\widehat{\mathbf{R}^d})$ into a right $C(\mathbf{T}^d)$-module.*

PROOF: Because b is contained in $C(\mathbf{T}^d)$, it follows that $\sup_{\xi \in \mathbf{R}^d} b(\hat{\theta}\xi) < \infty$. Because $|(p\widehat{\circ_\theta}b)(\xi)| \leq |p(\xi) \sup_{\xi' \in \mathbf{R}^d} b(\hat{\theta}(\xi'))|$, it follows that $p\widehat{\circ_\theta}b \in L^2(\mathbf{R}^d)$.

The fact that $\widehat{\circ_\theta}$ makes $L^2(\widehat{\mathbf{R}^d})$ into a right $C(\mathbf{T}^d)$-module follows from that fact that $\widehat{\circ_\theta}$ consists of pointwise multiplication. □

Lemma 2.1.12 *For $f, g \in L^2(\mathbf{R}^d)$ and $a \in l^1(\mathbf{Z}^d)$, suppose that $[f,g]_\theta \in l^1(\mathbf{Z}^d)$, then*

$$\hat{f}\widehat{\circ_\theta}\hat{a} = \mathcal{F}_{\mathbf{R}^d}(f \circ_\theta a) \tag{2.29}$$

$$\text{and} \quad [\![\hat{f}, \hat{g}]\!]_\theta = \mathcal{F}_{\mathbf{Z}^d}([f,g]_\theta) \tag{2.30}$$

where $\mathcal{F}_{\mathbf{R}^d}$ is the Fourier transform on $\mathbf{R}^d$, and $\mathcal{F}_{\mathbf{Z}^d}$ is the Fourier transform on $\mathbf{Z}^d$. Furthermore, if $p, q \in L^2(\widehat{\mathbf{R}^d})$, then for $\gamma \in \mathbf{Z}^d$

$$\mathcal{F}_{\mathbf{T}^d}([\![p,q]\!]_{\hat{\theta}})(\gamma) = [\check{p}, \check{q}]_\theta(\gamma) \tag{2.31}$$

where $\check{p}, \check{q} \in L^2(\mathbf{R}^d)$ are the inverse Fourier transforms of p, q and $\mathcal{F}_{\mathbf{T}^d}$ is the Fourier transform on $\mathbf{T}^d$.

Proof: We know from equation (2.17) that for $\gamma \in \mathbf{Z}^d$, $(\check{p}^* * \check{q})(\theta(\gamma)) = [\check{p}, \check{q}]_\theta(\gamma)$, so we make use of Lemma 2.1.1 to calculate

$$\begin{aligned}
(\mathcal{F}_{\mathbf{T}^d}[\![p,q]\!]_\theta)(\gamma) &= \int_{\zeta\in\mathbf{T}^d} [\![p,q]\!]_\theta(\zeta)(-\gamma,\zeta)d\zeta \\
&= \int_{\zeta\in\mathbf{T}^d} \frac{1}{\det(A_\theta)} \sum_{\hat{\theta}\xi=\zeta} \overline{p(\xi)}q(\xi)(-\gamma,\zeta)d\zeta \\
&= \int_{\zeta\in\mathbf{T}^d} \frac{1}{\det(A_\theta)} \sum_{\hat{\imath}A_\theta^*\xi=\zeta} \overline{p(\xi)}q(\xi)(-\gamma,\hat{\imath}A_\theta^*\xi)d\zeta \\
\text{now set } \eta := A_\theta^*\xi, & \\
&= \int_{\zeta\in\mathbf{T}^d} \frac{1}{\det(A_\theta)} \sum_{\hat{\imath}\eta=\zeta} \overline{p((A_\theta^*)^{-1}\eta)}q((A_\theta^*)^{-1}\eta)(-\gamma,\hat{\imath}\eta)d\zeta \\
&= \int_{\eta\in\mathbf{R}^d} \frac{1}{\det(A_\theta)} \overline{p((A_\theta^*)^{-1}\eta)}q((A_\theta^*)^{-1}\eta)(-\gamma,\hat{\imath}\eta)d\eta \\
&= \int_{\xi\in\mathbf{R}^d} \overline{p(\xi)}q(\xi)(-\gamma,\hat{\imath}A_\theta^*\xi)d\xi \\
&= \int_{\xi\in\mathbf{R}^d} \overline{p(\xi)}q(\xi)(\theta(\gamma),\xi)d\xi \\
&= (\check{p}^* * \check{q})(\theta(\gamma)) \\
&= [\check{p},\check{q}]_\theta(\gamma).
\end{aligned}$$

Now because we assumed that $[f,g]_\theta \in l^1(\mathbf{Z}^d)$, the Fourier transform of $[f,g]_n$ is defined. Therefore $[\![\hat{f},\hat{g}]\!]_\theta = \mathcal{F}_{\mathbf{Z}^d}([f,g]_\theta)$.

Now we also have

$$\begin{aligned}
(\mathcal{F}_{\mathbf{R}^d}(f \circ_\theta a))(\xi) &= \int_{\mathbf{R}^d} \sum_{\gamma\in\mathbf{Z}^d} a(\gamma)f(x-\theta(\gamma))(x,\xi)dx \\
&= \int_{\mathbf{R}^d} \sum_{\gamma\in\mathbf{Z}^d} a(\gamma)f(x)(x+\theta(\gamma),\xi)dx \\
&= \sum_{\gamma\in\mathbf{Z}^d} a(\gamma)(\theta(\gamma),\xi) \int_{\mathbf{R}^d} f(x)(x,\xi)dx \\
&= \sum_{\gamma\in\mathbf{Z}^d} a(\gamma)(\gamma,\hat{\theta}(\xi)) \int_{\mathbf{R}^d} f(x)(x,\xi)dx \\
&= \hat{a}(\hat{\theta}(\xi))\hat{f}(\xi) \\
&= (\widehat{\hat{f}\circ_\theta\hat{a}})(\xi).
\end{aligned}$$

□

Lemma 2.1.13 *For $f \in C_c(\mathbf{R}^d)$ and an embedding $\theta : \mathbf{Z}^d \to \mathbf{R}^d$, let*

$$\|f\|_{X_\theta} := \sup_{\zeta\in\mathbf{T}^d} \sqrt{\widehat{[f,f]}_\theta(\zeta)}. \tag{2.32}$$

Then $\|\cdot\|_{X_\theta}$ is a norm on $C_c(\mathbf{R}^d)$ which can also be expressed as

$$\|f\|_{X_\theta} = \sup_{\zeta\in\mathbf{T}^d}\sqrt{[\![\hat f,\hat f]\!]_\theta(\zeta)} \tag{2.33}$$

$$= \sup_{\zeta\in\mathbf{T}^d}\sqrt{\frac{1}{\det(A_\theta)}\sum_{\beta\in(A_\theta^*)^{-1}\mathbf{Z}^d}\overline{\hat f(\zeta+\beta)}\hat f(\zeta+\beta)}. \tag{2.34}$$

Furthermore, for all $f\in C_c(\mathbf{R}^d)$, $\|f\|_2\le\|f\|_{X_\theta}$, implying that the completion of $C_c(\mathbf{R}^d)$ with respect to the norm $\|\cdot\|_{X_\theta}$ is contained in $L^2(\mathbf{R}^d)$.

Proof: From equation (2.30) of Lemma 2.1.12, $\widehat{[f,f]}_\theta = [\![\hat f,\hat f]\!]_\theta$, so

$$\|f\|^2_{X_\theta} = \sup_{\zeta\in\mathbf{T}^d}[\![\hat f,\hat f]\!]_\theta(\zeta)$$

$$= \sup_{\zeta\in\mathbf{T}^d}\frac{1}{\det(A_\theta)}\sum_{\beta\in(A_\theta^*)^{-1}\mathbf{Z}^d}\overline{\hat f(\zeta+\beta)}\hat f(\zeta+\beta),$$

proving equation (2.34). We shall now show that $\|f\|_{X_\theta}$ is finite. Because $f\in C_c(\mathbf{R}^d)$, $[f,f]_\theta$ is contained in $C_c(\mathbf{Z}^d)$ and so is also contained in $l^1(\mathbf{Z}^d)$. The Fourier transform on $\mathbf{Z}^d$ maps $l^1(\mathbf{Z}^d)$ into $C(\mathbf{T}^d)$, so $\sup_{\zeta\in\mathbf{T}^d}|\widehat{[f,f]}_\theta(\zeta)|<\infty$, which implies that $\|f\|_{X_\theta}<\infty$.

Consider the function on $\mathbf{Z}^d$ which is given by $\hat f(\zeta+\cdot)$, where $\zeta\in\mathbf{T}^d$ and we identify $\mathbf{T}^d$ with the cube $[-\frac12,\frac12)^d$. We have from equation (2.34) that

$$\sqrt{\det(A_\theta)}\ \|f\|_{X_\theta} = \sup_{\zeta\in\mathbf{T}^d}\|\hat f(\zeta+\cdot)\|_{l^2(\mathbf{Z}^d)}.$$

To show that $\|\cdot\|_{X_\theta}$ is a norm, we need to verify the triangle inequality and verify that $\|f\|_{X_\theta}=0$ implies $f=0$. To verify the triangle inequality, let f and g be elements of $C_c(\mathbf{R}^d)$, we compute

$$\begin{aligned}\sqrt{\det(A_\theta)}\ \|f+g\|_{X_\theta} &= \sup_{\zeta\in\mathbf{T}^d}\|(\hat f+\hat g)(\zeta+\cdot)\|_2\\ &\le \sup_{\zeta\in\mathbf{T}^d}\|\hat f(\zeta+\cdot)\|_2+\|\hat g(\zeta+\cdot)\|_2\\ &\le \sup_{\zeta\in\mathbf{T}^d}\|\hat f(\zeta+\cdot)\|_2+\sup_{\zeta'\in\mathbf{T}^d}\|\hat g(\zeta'+\cdot)\|_2\\ &= \sqrt{\det(A_\theta)}\ \left(\|f\|_{X_\theta}+\|g\|_{X_\theta}\right).\end{aligned}$$

The triangle inequality is therefore satisfied. Suppose now that $\|f\|_{X_\theta}=0$, we then obtain that

$$\sup_{\zeta\in\mathbf{T}^d}\sum_{\beta\in(A_\theta^*)^{-1}\mathbf{Z}^d}\overline{\hat f(\zeta+\beta)}\hat f(\zeta+\beta)=0.$$

This implies that for all $\zeta\in\mathbf{T}^d$,

$$\sum_{\beta\in(A_\theta^*)^{-1}\mathbf{Z}^d}\overline{\hat f(\zeta+\beta)}\hat f(\zeta+\beta)=0.$$

From this it follows that $[\![\hat{f}, \hat{f}]\!]_\theta = 0$ and so $[f, f]_\theta = 0$. So in particular, $[f, f]_\theta(0) = 0$, implying that $f = 0$. Thus $\| \cdot \|_{X_\theta}$ is a norm.

Now we have for $f \in C_c(\mathbf{R}^d)$,

$$\begin{aligned}
\|f\|_2^2 &= \int_{\mathbf{R}^d} f(x)\overline{f(x)}dx \\
&= \int_{\mathbf{R}^d} \hat{f}(x)\overline{\hat{f}(x)}dx \\
&= \int_{\mathbf{T}^d} \frac{1}{\det(A_\theta)} \sum_{\beta \in (A_\theta^*)^{-1}\mathbf{Z}^d} \overline{\hat{f}(\zeta+\beta)}\hat{f}(\zeta+\beta)d\zeta \\
&= \int_{\mathbf{T}^d} [\![\hat{f}, \hat{f}]\!]_\theta(\zeta)d\zeta \\
&\leq \sup_{\zeta \in \mathbf{T}^d} [\![\hat{f}, \hat{f}]\!]_\theta(\zeta) \\
&= \|f\|_{X_\theta}^2
\end{aligned}$$

proving the assertion. □

We previously stated that the operations defined in Definitions 2.1.3 and 2.1.10 shall be used for constructing Hilbert modules. Lemmas 2.1.14 and 2.1.15 verify that these operations satisfy Properties 1, 2 and 3 of the definition of a Hilbert module (Definition 0.4.1).

Lemma 2.1.14 *Suppose that $f, g, h \in L^2(\mathbf{R}^d)$, $a \in l^1(\mathbf{Z}^d)$, $\alpha, \beta \in \mathbf{C}$, and θ is an embedding of $\mathbf{Z}^d$ in $\mathbf{R}^d$, then*

1. $[f, \alpha g + \beta h]_\theta = \alpha[f, g]_\theta + \beta[f, h]_\theta$;
2. $[f, g \circ_X a]_\theta = [f, g]_\theta * a$, *where the binary operation $*$ is convolution on $l^1(\mathbf{Z}^d)$;*
3. $[f, g]_\theta^* = [g, f]_\theta$, *where the unary operation $\cdot^*$ is involution on $l^1(\mathbf{Z}^d)$.*

Proof: Suppose that $\gamma \in \mathbf{Z}^d$.

1. We have that

$$\begin{aligned}
[f, \alpha g + \beta h]_\theta(\gamma) &= \int_{\mathbf{R}^d} \overline{f(x-\theta(\gamma))}(\alpha g(x) + \beta h(x))dx \\
&= \alpha \int_{\mathbf{R}^d} \overline{f(x-\theta(\gamma))}g(x)dx + \beta \int_{\mathbf{R}^d} \overline{f(x-\theta(\gamma))}h(x)dx \\
&= \alpha[f, g]_\theta(\gamma) + \beta[f, h]_\theta(\gamma).
\end{aligned}$$

2. We have

$$[f, g \circ_\theta a]_\theta(\gamma) = \int_{\mathbf{R}^d} \overline{f(x-\theta(\gamma))} \sum_{\delta \in \mathbf{Z}^d} a(\delta)g(x-\theta(\delta))dx$$

$$\begin{aligned}
&= \sum_{\delta\in\mathbf{Z}^d} a(\delta)\int_{\mathbf{R}^d}\overline{f(x-\theta(\gamma))}g(x-\delta)dx\\
&= \sum_{\delta\in\mathbf{Z}^d} a(\delta)\int_{\mathbf{R}^d}\overline{f(y-\theta(\gamma+\delta))}g(y)dy\\
&= \sum_{\delta\in\mathbf{Z}^d} a(\delta)[f,g]_\theta(\gamma-\delta)\\
&= ([f,g]_\theta * a)(\gamma).
\end{aligned}$$

3. We calculate

$$\begin{aligned}
[f,g]_\theta^*(\gamma) &= \int_{\mathbf{R}^d} f(x+\theta(\gamma))\overline{g(x)}dx\\
&= \int_{\mathbf{R}^d} f(y)\overline{g(y-\theta(\gamma))}dy\\
&= [g,f]_\theta(\gamma).
\end{aligned}$$

□

Lemma 2.1.15 *Suppose that $p,q,r \in L^2(\widehat{\mathbf{R}^d})$, $b \in C(\mathbf{T}^d)$, $\alpha,\beta \in \mathbf{C}$, and θ is an embedding of $\mathbf{Z}^d$ in $\mathbf{R}^d$, then*

1. *$[\![p,\alpha q+\beta r]\!]_\theta = \alpha[\![p,q]\!]_\theta + \beta[\![p,r]\!]_\theta$;*
2. *$[\![p, q\widehat{\circ_\theta} b]\!]_\theta = [\![p,q]\!]_\theta b$, where b is a function on $\mathbf{T}^d$ for which $q\widehat{\circ_\theta} b \in L^2(\mathbf{R}^d)$ whenever $q \in L^2(\mathbf{R}^d)$ (this is the case when $b \in C(\mathbf{T}^d)$ or $b \in L^\infty(\mathbf{T}^d)$);*
3. *$[\![p,q]\!]_\theta^* = [\![q,p]\!]_\theta$, where the unary operation $\cdot^*$ is complex conjugation on $C(\mathbf{T}^d)$.*

Proof: Let $\zeta \in \mathbf{T}^d$, we calculate

$$\begin{aligned}
\det(A_\theta)\ [\![p,\alpha q+\beta r]\!]_\theta &= \sum_{\delta\in(A_\theta^*)^{-1}\mathbf{Z}^d}\overline{p(\zeta+\delta)}(\alpha q+\beta r)(\zeta+\delta)\\
&= \alpha\sum_{\delta\in(A_\theta^*)^{-1}\mathbf{Z}^d}\overline{p(\zeta+\delta)}q(\zeta+\delta) + \beta\sum_{\delta'\in(A_\theta^*)^{-1}\mathbf{Z}^d}\overline{p(\zeta+\delta')}r(\zeta+\delta')\\
&= \det(A_\theta)\ \left(\alpha[\![p,q]\!]_\theta(\zeta) + \beta[\![p,r]\!]_\theta(\zeta)\right).
\end{aligned}$$

We also calculate

$$\begin{aligned}
\det(A_\theta)\ [\![p,q\widehat{\circ_\theta} b]\!]_\theta(\zeta) &= \sum_{\delta\in(A_\theta^*)^{-1}\mathbf{Z}^d}\overline{p(\zeta+\delta)}(q\widehat{\circ_\theta} b)(\zeta+\delta)\\
&= \sum_{\delta\in(A_\theta^*)^{-1}\mathbf{Z}^d}\overline{p(\zeta+\delta)}q(\zeta+\delta)b(\zeta)\\
&= \det(A_\theta)\ ([\![p,q]\!]_\theta * b)(\zeta).
\end{aligned}$$

We finally calculate

$$[\![p,q]\!]_\theta^*(\zeta) = \frac{1}{\det(A_\theta)}\sum_{\delta\in(A_\theta^*)^{-1}\mathbf{Z}^d}\overline{q(\zeta+\delta)}p(\zeta+\delta) = [\![q,p]\!]_\theta(\zeta).$$

□

Proposition 2.1.16 *Let $\theta : \mathbf{Z}^d \to \mathbf{R}^d$ be an embedding. The space $C_c(\mathbf{R}^d)$ is an inner product $C_c(\mathbf{Z}^d)$-module with operations $[\ ,\ ]_\theta$ and $\circ_\theta$, and $\|\cdot\|_{X_\theta}$ is equal to the Hilbert module norm on $C_c(\mathbf{R}^d)$.*

PROOF: In Lemma 2.1.6, we showed that $C_c(\mathbf{R}^d)$ is a right $C_c(\mathbf{Z}^d)$ module with the operation $\circ_\theta$.

From Lemma 2.1.14, properties 1, 2 and 3 of Definition 0.4.1 are satisfied for $C_c(\mathbf{R}^d)$ with the operations $\circ_\theta$ and $[\ ,\ ]_\theta$. Suppose that $f \in C_c(\mathbf{R}^d)$, then by Lemma 2.1.12,

$$(\mathcal{F}_{\mathbf{Z}^d}[f,f]_\theta)(\zeta) = [\![\hat{f},\hat{f}]\!]_\theta(\zeta) = \frac{1}{\det(A_\theta)} \sum_{\beta \in (A_\theta^*)^{-1}\mathbf{Z}^d} \overline{\hat{f}(\zeta+\beta)}\hat{f}(\zeta+\beta)$$

which is non-negative for all $\zeta \in \mathbf{T}^d$. So $\mathcal{F}_{\mathbf{Z}^d}[f,f]_\theta$ is a positive element of the C^*-algebra $C(\mathbf{T}^d)$. We know from Theorem 0.3.6 that $C^*(\mathbf{Z}^d)$ and $C(\mathbf{T}^d)$ are isomorphic and that this isomorphism is given by the Fourier transform on the subalgebra $l^1(\mathbf{Z}^d)$ which is dense in $C^*(\mathbf{Z}^d)$. This implies that $[f,f]_\theta$ is a positive element of $C^*(\mathbf{Z}^d)$, verifying Property 4 of Definition 0.4.1. Now if $[f,f]_\theta = 0$, then for all $\gamma \in \mathbf{Z}^d$, $\int_{\mathbf{R}^d} \overline{f(x-\gamma)}f(x)dx = 0$, so in particular $\int_{\mathbf{R}^d} \overline{f(x)}f(x)dx = 0$, so $\|f\|_2 = 0$, implying that $f = 0$. This verifies Property 5 of Definition 0.4.1. It therefore follows that $C_c(\mathbf{R}^d)$ is a right inner product $C_c(\mathbf{Z}^d)$-module. It is an immediate consequence of Definition 0.4.1 and the definition of $\|\cdot\|_{X_\theta}$ that $\|\cdot\|_{X_\theta}$ is the Hilbert module norm. □

Remark 2.1.17 Note that we have not used Lemma 2.1.13 in the proof of Proposition 2.1.16. Proposition 2.1.16 gives us an alternative proof to the fact that $\|\cdot\|_{X_\theta}$ is a norm, as proved in Lemma 2.1.13. Because we know that $C_c(\mathbf{R}^d)$ is an inner-product $C_c(\mathbf{Z}^d)$-module, the fact that $\|\cdot\|_{X_\theta}$ is a norm directly follows from [RW, Corollary 2.7].

Remark 2.1.18 Proposition 2.1.16 is a special case of one of the main results in [R1] (see also [RW]). The results in these references are more general in that $\mathbf{Z}^d$ and $\mathbf{R}^d$ are replaced by a closed subgroup H of a locally compact group G. The fact that $\mathbf{Z}^d$ is abelian enables us to use the Fourier transform in the proof for Proposition 2.1.16 to provide a simpler proof of the positivity of the $C_c(\mathbf{Z}^d)$-valued inner product.

We shall now define the Hilbert modules X_θ and $\hat{X}_\theta$, we will then show that they are Hilbert modules.

Definition 2.1.19 Let $\theta : \mathbf{Z}^d \to \mathbf{R}^d$ be an embedding. We define the *bracket product Hilbert $C^*(\mathbf{Z}^d)$-module* X_θ as follows: Let the linear space X_θ be the completion of $C_c(\mathbf{R}^d)$ with respect to the norm $\|\cdot\|_{X_\theta}$ that was defined in Lemma 2.1.13. We equip X_θ with the bracket product $[\ ,\ ]_\theta : X_\theta \times X_\theta \to C^*(\mathbf{Z}^d)$ and module action $\circ_\theta : X_\theta \times C^*(\mathbf{Z}^d) \to X_\theta$ as defined in Definition 2.1.3.

We define the *Fourier transformed bracket product Hilbert $C(\mathbf{T}^d)$-module* $\hat{X}_\theta$ as follows: Let the linear space $\hat{X}_\theta$ be the image of X_θ under the Fourier transform on $\mathbf{R}^d$. We equip $\hat{X}_\theta$ with the Fourier transformed bracket product $[\![\, , \,]\!]_\theta : \hat{X}_\theta \times \hat{X}_\theta \to C(\mathbf{T}^d)$ and module action $\widehat{\circ_\theta} : \hat{X}_\theta \times C(\mathbf{T}^d) \to \hat{X}_\theta$, as defined in Definition 2.1.10.

Remark 2.1.20 We know from Lemma 2.1.13 that we can continuously embed X_θ in $L^2(\mathbf{R}^d)$, this means that it makes sense to talk about the image of X_θ under the Fourier transform on $\mathbf{R}^d$. In Lemma 2.1.7 we defined the bracket product on $L^2(\mathbf{R}^d)$, so we can use this to describe the bracket product on X_θ. We show in Theorem 2.1.21 that a module action of $C^*(\mathbf{Z}^d)$ on X_θ exists, without providing an explicit formula for what it is (unless the element of $C^*(\mathbf{Z}^d)$ consists of a function on $\mathbf{Z}^d$). It is very useful that we have also constructed a Hilbert module $\hat{X}_\theta$ in the Fourier domain, because in practice we can use the Gelfand transform to represent an element of $C^*(\mathbf{Z}^d)$ as a continuous function on $\mathbf{T}^d$, and use the Fourier transformed module action $\widehat{\circ_\theta}$.

Theorem 2.1.21 is the main theorem of this section and states that Definition 2.1.19 defines a Hilbert module. The main idea of the proof is to use the completion process described in Lemma 0.4.5 to obtain a Hilbert $C^*(\mathbf{Z}^d)$-module from the inner product $C_c(\mathbf{Z}^d)$-module $C_c(\mathbf{R}^d)$.

Theorem 2.1.21 *Let $\theta : \mathbf{Z}^d \to \mathbf{R}^d$ be an embedding. Let $X_\theta \subset L^2(\mathbf{R}^d)$ be the completion of $C_c(\mathbf{R}^d)$ with respect to the norm $\| \cdot \|_X$ as defined in Definition 2.1.19. Let $\hat{X}_\theta \subset L^2(\widehat{\mathbf{R}^d})$ be the image of X_θ under the Fourier transform as defined in Definition 2.1.19. We have that*

1. *The space X_θ is a full Hilbert $C^*(\mathbf{Z}^d)$-module with the inner product $[\, , \,]_\theta$. The module action of $C^*(\mathbf{Z}^d)$ on X_θ has the property that for $f \in X_\theta$ and $a \in l^1(\mathbf{Z}^d)$ (note that $l^1(\mathbf{Z}^d) \subset C^*(\mathbf{Z}^d)$), the module action of a on f is given by the module action associated with the bracket product $f \circ_\theta a$ as defined in Definition 2.1.19. We furthermore have that if $(f_n)_{n\in\mathbf{Z}}$ is a convergent sequence in X_θ, then it is a convergent sequence in $L^2(\mathbf{R}^d)$.*

2. *The operations $[\![\, , \,]\!]_{\hat{\theta}}$ and $\widehat{\circ_\theta}$ make $\hat{X}_\theta$ into a full Hilbert $C(\mathbf{T}^d)$-module;*

3. *If we identify the isomorphic C^*-algebras $C^*(\mathbf{Z}^d)$ and $C(\mathbf{T}^d)$, then the Fourier transform on $\mathbf{R}^d$ is a unitary isomorphism between X_θ and $\hat{X}_\theta$.*

PROOF:

1. Lemma 0.4.5 tells us that the completion of an inner product module over a pre-C^*-algebra with respect to the Hilbert module norm is a Hilbert module. We have defined the linear space X_θ to be the completion of $C_c(\mathbf{R}^d)$ with respect to the Hilbert module norm $\| \cdot \|_{X_\theta}$. It therefore follows from Lemma 0.4.5 Because Lemma 0.4.5 states that we obtain the completion of an inner product module by taking the completion of the linear space with respect to the Hilbert module norm, it therefore follows that the space X_θ is a Hilbert $C^*(\mathbf{Z}^d)$-module with the inner

product $[\ ,\ ]_\theta$. It also follows from Lemma 0.4.5 that the module action of $C^*(\mathbf{Z}^d)$ on X_θ has the property that for $f \in X_\theta$ and $a \in l^1(\mathbf{Z}^d)$, the module action of a on f is given by the module action associated with the bracket product $f \circ_\theta a$. We know from Lemma 2.1.13 that if $f \in C_c(\mathbf{R}^d)$, then $\|f\|_2^2 \leq \|f\|_{X_\theta}^2$. It follows that if $f, g \in C_c(\mathbf{R}^d)$ and $\varepsilon > 0$, then $\|f-g\|_{X_\theta} < \varepsilon$ implies $\|f-g\|_2 < \varepsilon$. Therefore if $(f_n)_{n\in\mathbf{N}}$ is a convergent sequence in X_θ, then it is a convergent sequence in $L^2(\mathbf{R}^d)$.

To see that X_θ is full, choose an element $\phi \in X_\theta$ for which $[\phi, \phi]_\theta = \mathbf{1}$. A possible choice for ϕ would be any continuous compactly supported wavelet or scaling function corresponding to the translations $\theta(\mathbf{Z}^d)$. For example, let φ be the Daubechies scaling function that was described in Example 1.4.5. Recall that φ is a continuous compactly supported function on $\mathbf{R}$ satisfying $[\varphi, \varphi]_\theta = \mathbf{1}$. We can define a continuous compactly supported function ϕ on $\mathbf{R}^d$ or which $[\phi, \phi]_\theta = \mathbf{1}$ as follows. We let

$$\phi(\theta(x_1, x_2, \ldots, x_d)) = \varphi(x_1)\varphi(x_2)\ldots\varphi(x_d).$$

Then $[\phi, \phi]_\theta = \mathbf{1}$. Therefore for all $a \in C^*(\mathbf{Z}^d)$, it is the case that

$$\begin{aligned} [\phi, \phi \circ_\theta a]_\theta &= [\phi, \phi]_\theta \circ_\theta a \\ &= a. \end{aligned}$$

Because $\phi \circ_\theta a \in X_\theta$, it follows that $C^*(\mathbf{Z}^d) = [X_\theta, X_\theta]_\theta$, verifying that X_θ is full.

2. Let $\widehat{C_c(\mathbf{R}^d)}$ be the image of $C_c(\mathbf{R}^d)$ under the Fourier transform. Let $\widehat{C_c(\mathbf{Z}^d)} \subset C(\mathbf{T}^d)$ be the image of $C_c(\mathbf{Z}^d)$ under the Fourier transform. By Lemma 2.1.12, if $p, q \in \widehat{C_c(\mathbf{R}^d)}$, then $[\![p, q]\!]_\theta \in \widehat{C_c(\mathbf{Z}^d)}$. It also follows from Lemma 2.1.12 that $\widehat{C_c(\mathbf{R}^d)}$ is a right $\widehat{C_c(\mathbf{Z}^d)}$-module with module action $\widehat{\circ_\theta}$.

   From Lemma 2.1.15, Properties 1, 2 and 3 of Definition 0.4.1 are satisfied for $\widehat{C_c(\mathbf{R}^d)}$ with the operations $\widehat{\circ_\theta}$ and $[\![\ ,\ ]\!]_\theta$. Properties 4 and 5 of Definition 0.4.1 are satisfied by the same argument as for $C_c(\mathbf{R}^d)$. Therefore the operations $[\![\ ,\ ]\!]_\theta$ and $\widehat{\circ_\theta}$ make $\widehat{C_c(\mathbf{R}^d)}$ into an inner product $\widehat{C_c(\mathbf{Z}^d)}$-module. The completion of $\widehat{C_c(\mathbf{R}^d)}$ with respect to the Hilbert module norm is equal to $\hat{X}_\theta$. So by Lemma 0.4.5, the operations $[\![\ ,\ ]\!]_\theta$ and $\widehat{\circ_\theta}$ make $\hat{X}_\theta$ into a Hilbert $C(\mathbf{T}^d)$-module, and if $p, q \in \hat{X}_\theta$, then $[\![p, q]\!]_\theta \in C(\mathbf{T}^d)$. The fact that $\hat{X}_\theta$ is full follows from the fact that X_θ is full.

3. By Definition 0.4.8, we need to show that the Fourier transform $\mathcal{F}_{\mathbf{R}^d}$ is a unitary adjointable operator from X_θ to $\hat{X}_\theta$. In the notation of Section 0.4, we want to show that $\mathcal{F}$ is a unitary element of $\mathcal{L}(X_\theta, \hat{X}_\theta)$. From Lemma 2.1.12, if $f \in X_\theta$ and $p \in \hat{X}_\theta$, then

$$[\![\mathcal{F}_{\mathbf{R}^d}(f), p]\!]_\theta = \mathcal{F}_{\mathbf{Z}^d}\left([f, \mathcal{F}^*_{\mathbf{R}^d}(p)]_\theta\right). \tag{2.35}$$

   We therefore have that $\mathcal{F}_{\mathbf{R}^d}$ is adjointable with Hilbert module adjoint $\mathcal{F}^*_{\mathbf{R}^d}$. We have that $\mathcal{F}_{\mathbf{R}^d}\mathcal{F}^*_{\mathbf{R}^d} = \mathbf{1}_X$ and $\mathcal{F}^*_{\mathbf{R}^d}\mathcal{F}_{\mathbf{R}^d} = \mathbf{1}_{\hat{X}}$, so $\mathcal{F}_{\mathbf{R}^d}$ is unitary.

□

2.2 Incorporating the Dilation

In this section we shall use Theorem 2.1.21 to construct Hilbert modules which are related to wavelets. In order to study wavelets, we also want to incorporate the dilation. We do this by defining a chain of Hilbert $C^*(\mathbf{Z}^d)$-modules $(X_n)_{n\in\mathbf{Z}}$. Each Hilbert module X_n corresponds to the nth level of a multiresolution analysis. All of these Hilbert modules will have as their linear space a space which is the completion of $C_c(\mathbf{R}^d)$ with respect to a norm which depends on n. For each Hilbert module X_n, we shall define an "nth level bracket product" $[\ ,\ ]_n$, and an "nth level module action" $\circ_n$. We shall also define Hilbert $C(\mathbf{T}^d)$-modules $(\hat{X}_n)_{n\in\mathbf{Z}}$ in the Fourier domain. Throughout this section we shall work with the standard multiresolution structure with index m on $L^2(\mathbf{R}^d)$ associated with a dilation matrix $\tilde{\mathcal{D}} \in M^d(\mathbf{Z})$.

In Theorem 2.1.21 we have shown how to construct a Hilbert module X_θ from an embedding $\theta : \mathbf{Z}^d \to \mathbf{R}^d$. Recall from Lemma 2.1.1 that associated with this embedding is a linear transformation $A_\theta : \mathbf{R}^d \to \mathbf{R}^d$ for which $\theta = A_\theta \iota$, where ι is the natural embedding of $\mathbf{Z}^d$ in $\mathbf{R}^d$. We also showed in Lemma 2.1.1 that there is a dual homomorphism $\hat{\theta} : \widehat{\mathbf{R}^d} \to \mathbf{T}^d$ given by $\hat{\theta} = \hat{\iota} A_\theta^*$.

Notation 2.2.1 In equation (1.6) of Section 1.3 we defined some homomorphisms $(\iota_n)_{n\in\mathbf{Z}} : \mathbf{Z}^d \to \mathbf{R}^d$ by $\iota_n = \tilde{\mathcal{D}}^{-n}\iota$. In the situation that $\theta = \iota_n$, we have that $A_{\iota_n} = \tilde{\mathcal{D}}^{-n}$, and $\hat{\iota_n} = \hat{\iota}(\tilde{\mathcal{D}}^*)^{-n}$. In Definition 1.1.5 we defined representations $(\pi^n)_{n\in\mathbf{Z}}$ of $\mathbf{Z}^d$ on $L^2(\mathbf{R}^d)$ by

$$(\pi^n_\gamma f)(x) := (\pi^{\iota_n}_\gamma f)(x) = f(x - \iota_n(\gamma)) = f(x - \tilde{\mathcal{D}}^{-n}\iota(\gamma)).$$

We shall make use of the above notation throughout this section.

Definition 2.2.2 Suppose that $\tilde{\mathcal{D}}$ is a dilation matrix, and $n \in \mathbf{Z}$. For $f, g \in L^2(\mathbf{R}^d)$, $a \in l^1(\mathbf{Z}^d)$, $\gamma \in \mathbf{Z}^d$, and $x \in \mathbf{R}^d$, define the *nth level bracket product* $[\ ,\ ]_n$ and *nth level module action* $\circ_n$ to be

$$[f,g]_n(\gamma) \;:=\; [f,g]_{\iota_n}(\gamma) = \int_{\mathbf{R}^d} \overline{f(x - \tilde{\mathcal{D}}^{-n}\iota(\gamma))}\, g(x)dx, \tag{2.36}$$

$$f \circ_n a \;:=\; f \circ_{\iota_n} a = \sum_{\gamma\in\mathbf{Z}^d} a(\gamma)\pi^n_\gamma f. \tag{2.37}$$

For $p, q \in L^2(\widehat{\mathbf{R}^d})$ and $b \in C(\mathbf{T}^d)$, we define the nth level Fourier transformed bracket product $[\![p,q]\!]_n$ and *nth level Fourier transformed module action* $\widehat{\circ_n}$ to be

$$[\![p,q]\!]_n \;:=\; [\![p,q]\!]_{\iota_n}, \tag{2.38}$$

$$p\widehat{\circ_n}b \;:=\; p\widehat{\circ_{\iota_n}}b. \tag{2.39}$$

Lemma 2.2.3 *For $p, q \in \widehat{\mathbf{R}^d}$, $\zeta \in \mathbf{T}^d$, if we identify $\mathbf{T}^d$ with the cube $[-\frac{1}{2}, \frac{1}{2})^d$, then*

$$[\![p, q]\!]_n(\zeta) = m^n \sum_{\beta \in (\tilde{\mathcal{D}}^*)^n \mathbf{Z}^d} \overline{p\left((\tilde{\mathcal{D}}^*)^n \zeta + \beta\right)} q\left((\tilde{\mathcal{D}}^*)^n \zeta + \beta\right). \tag{2.40}$$

PROOF: Because $A_{\iota_n} = \tilde{\mathcal{D}}^{-n}$, it follows that $\det(A_{\iota_n}) = m^{-n}$ and that $(A^*_{\iota_n})^{-1}\mathbf{Z}^d = (\tilde{\mathcal{D}}^*)^n \mathbf{Z}^d$. Now from Lemma 2.1.9, $(\tilde{\mathcal{D}}^*)^n \mathbf{Z}^d = \{\xi \in \widehat{\mathbf{R}^d} : \hat{\theta}(\xi) = 0\}$. It is therefore the case that $\{\xi : \hat{\iota_n}(\xi) = \zeta\} = \{(\tilde{\mathcal{D}}^*)^n \zeta + \beta : \beta \in (\tilde{\mathcal{D}}^*)^n \mathbf{Z}^d\}$. The result now follows from equation (2.24). □

The following Lemma tells us how the operations we have just defined relate to each other when n changes.

Lemma 2.2.4 *Suppose that $n \in \mathbf{Z}$. We have*

1. *If $f, g \in L^2(\mathbf{R}^d)$, then*
$$[f, g]_n = [\mathcal{D}^{-n} f, \mathcal{D}^{-n} g]_0. \tag{2.41}$$
2. *If $p, q \in L^2(\widehat{\mathbf{R}^d})$, then*
$$[\![p, q]\!]_n = [\![\hat{\mathcal{D}}^n p, \hat{\mathcal{D}}^n q]\!]_0. \tag{2.42}$$
3. *If $f \in L^2(\mathbf{R}^d)$, $a \in l^1(\mathbf{Z}^d)$, then*
$$\mathcal{D}^{-n}(f \circ_n a) = (\mathcal{D}^{-n} f) \circ_0 a. \tag{2.43}$$
4. *If $p \in L^2(\widehat{\mathbf{R}^d})$, $b \in C(\mathbf{T}^d)$, then*
$$\hat{\mathcal{D}}^{-n}(p \widehat{\circ_n} b) = (\hat{\mathcal{D}}^{-n} p) \widehat{\circ_0} b). \tag{2.44}$$

PROOF:

1. We calculate
$$\begin{aligned}
[\mathcal{D}^{-n} f, \mathcal{D}^{-n} g]_0(\gamma) &= \int_{\mathbf{R}^d} \overline{(\mathcal{D}^{-n} f)(x - \iota(\gamma))} (\mathcal{D}^{-n} g)(x) dx \\
&= \int_{\mathbf{R}^d} m^{-n} \overline{f(\tilde{\mathcal{D}}^{-n} x - \tilde{\mathcal{D}}^{-n} \iota(\gamma))} g(\tilde{\mathcal{D}}^{-n} x) dx \\
&= \int_{\mathbf{R}^d} \overline{f(y - \tilde{\mathcal{D}}^{-n} \iota(\gamma))} g(y) dy \\
&= [f, g]_n.
\end{aligned}$$

2. We calculate
$$\begin{aligned}
[\hat{\mathcal{D}}^n p, \hat{\mathcal{D}}^n q]_{\hat{n}}(\hat{\iota_n} \xi) &= m^n \sum_{\beta \in \mathbf{Z}^d} \overline{(\hat{\mathcal{D}}^n p)(\xi + \tilde{\mathcal{D}}^{*n} \beta)} (\hat{\mathcal{D}}^n q)(\xi + \tilde{\mathcal{D}}^{*n} \beta) \\
&= \sum_{\beta \in \mathbf{Z}^d} \overline{p(\tilde{\mathcal{D}}^{*-n} \xi + \beta)} q(\tilde{\mathcal{D}}^{*-n} \xi + \beta) \\
&= \sum_{\beta \in \mathbf{Z}^d} \overline{p(\hat{\iota_n} \xi + \beta)} q(\hat{\iota_n} \xi + \beta) \\
&= [p, q]_{\hat{0}}(\hat{\iota_n} \xi).
\end{aligned}$$

3. We calculate

$$\begin{aligned}\mathcal{D}^{-n}(f \circ_n a) &= \sum_{\gamma \in \mathbf{Z}^d} a(\gamma)\mathcal{D}^{-n}\pi^n_\gamma(f) \\ &= \sum_{\gamma \in \mathbf{Z}^d} a(\gamma)\pi^0_\gamma(\mathcal{D}^{-n} f) \\ &= (\mathcal{D}^{-n} f) \circ_0 a.\end{aligned}$$

4. This is a consequence of equation (2.29) of Lemma 2.1.12, and the previous calculation.

□

Definition 2.2.5 For $n \in \mathbf{Z}$, let $X_n := X_{\iota_n}$ and let $\hat{X}_n := \hat{X}_{\iota_n}$ (see Definition 2.1.19). Equip X_n with the nth level bracket product $[\ ,\ ]_n$ and nth level module action $\circ_n$. We call X_n the *nth level wavelet Hilbert $C^*(\mathbf{Z}^d)$-module*, and $(X_n)_{n\in\mathbf{Z}}$ a *wavelet chain of Hilbert $C^*(\mathbf{Z}^d)$-modules.*

Equip $\hat{X}_n$ with the Fourier transformed nth level bracket product $[\![\ ,\ ]\!]_n$ and nth level Fourier transformed module action $\widehat{\circ_n}$. We call $\hat{X}_n$ the *nth level Fourier transformed wavelet Hilbert $C^*(\mathbf{Z}^d)$-module*, and $(X_n)_{n\in\mathbf{Z}}$ a *Fourier transformed wavelet chain of Hilbert $C^*(\mathbf{Z}^d)$-modules.*

Theorem 2.2.6 *Let $\tilde{\mathcal{D}} \in M^d(\mathbf{Z})$ be a dilation matrix and consider the standard multiresolution structure with index m on $L^2(\mathbf{R}^d)$ corresponding to $\tilde{\mathcal{D}}$ with translation group $\Gamma = \mathbf{Z}^d$ and dilation $\mathcal{D}$. Let $n \in \mathbf{Z}$ and let $X_n \subset L^2(\mathbf{R}^d)$ and $\hat{X}_n \subset L^2(\widehat{\mathbf{R}^d})$ be defined as in Definition 2.2.5. We have that*

1. *The space $C_c(\mathbf{R}^d)$ is an inner product $C_c(\mathbf{Z}^d)$-module with operations $[\ ,\ ]_n$ and $\circ_n$, and $\|\cdot\|_{X_n}$ is equal to the Hilbert module norm on $C_c(\mathbf{R}^d)$.*

2. *The space X_n is a full Hilbert $C^*(\mathbf{Z}^d)$-module with the inner product $[\ ,\ ]_n$. The module action of $C^*(\mathbf{Z}^d)$ on X_n has the property that for $f \in X_n$ and $a \in l^1(\mathbf{Z}^d)$ (note that $l^1(\mathbf{Z}^d) \subset C^*(\mathbf{Z}^d)$), the module action of a on f is given by $f \circ_n a$. We furthermore have that if $(f_n)_{n\in\mathbf{N}}$ is a convergent sequence in X_n, then it is a convergent sequence in $L^2(\mathbf{R}^d)$.*

3. *The operations $[\![\ ,\ ]\!]_n$ and $\widehat{\circ_X}$ make $\hat{X}_n$ into a full Hilbert $C(\mathbf{T}^d)$-module.*

4. *If we identify the isomorphic C^*-algebras $C^*(\mathbf{Z}^d)$ and $C(\mathbf{T}^d)$, then the Fourier transform on $\mathbf{R}^d$ is a unitary isomorphism between X_n and $\hat{X}_n$.*

PROOF: Part 1 is a direct result of Proposition 2.1.16. Parts 2,3 and 4 follow directly from Theorem 2.1.21. □

Recall from Definition 0.4.6 that we introduced adjointable operators, the main morphisms between Hilbert modules. The following corollary demonstrates that the translations and dilations are unitary adjointable operators. Recall from Definition 0.4.8 that two Hilbert modules are isomorphic if there is a unitary adjointable operator from one to the other.

Corollary 2.2.7 *For $n \in \mathbf{Z}$, $\gamma \in \mathbf{Z}^d$, the translation π_γ^n (when restricted to X_n) is a unitary element of $\mathcal{L}(X_n)$. The dilation $\mathcal{D}$ is a unitary element of $\mathcal{L}(X_n, X_{n+1})$, and $\hat{\mathcal{D}}$ is a unitary element of $\mathcal{L}(\hat{X_n}, \hat{X_{n+1}})$. Hence the Hilbert $C^*(\mathbf{Z}^d)$-modules $(X_n)_{n\in\mathbf{Z}}$ are all isomorphic (as Hilbert $C^*(\mathbf{Z}^d)$-modules) to each other and the Hilbert $C(\mathbf{T}^d)$-modules $(\hat{X_n})_{n\in\mathbf{Z}}$ are also isomorphic to each other.*

Proof: The translation π_γ^n maps X_n to itself. It is the case that for all $f, g \in X_n$,

$$[\pi_\gamma^n f, g]_n = [f, (\pi_\gamma^n)^{-1} g]_n$$

for all $\gamma \in \Gamma$, and so π_γ^n is an adjointable operator on X_n. The translation satisfies $\gamma^* = \gamma^{-1}$ and hence is a unitary operator on Hilbert modules.

We now verify that the dilation $\mathcal{D}$ maps X_n onto X_{n+1} and that $\hat{\mathcal{D}}$ maps $\hat{X_n}$ onto $\hat{X_{n+1}}$. By Definition 2.2.5, X_n is the completion of $C_c(\mathbf{R}^d)$ with respect to the norm $\|\cdot\|_{X_n}$ and $\hat{X_n}$ is the Fourier transform of X_n. For $f \in C_c(\mathbf{R}^d)$, we have

$$\begin{aligned}
\|f\|_{X_{n+1}} &= \sup_{\xi\in\mathbf{T}^d} |[\![\hat{f}, \hat{f}]\!]_{n+1}(\xi)|^{\frac{1}{2}} \\
&= \sup_{\xi\in\mathbf{T}^d} |[\![\mathcal{D}^{-1}\hat{f}, \mathcal{D}^{-1}\hat{f}]\!]_{n}(\xi)|^{\frac{1}{2}} \\
&= \|\mathcal{D}^{-1} f\|_{X_n}.
\end{aligned}$$

We therefore have that $\mathcal{D}$ maps X_n onto X_{n+1}. It follows that $\hat{\mathcal{D}}$ maps $\hat{X_n}$ onto $\hat{X_{n+1}}$ because $\hat{\mathcal{D}} = \mathcal{F}^*\mathcal{D}\mathcal{F}$.

From equation (2.41) it follows that for $f \in X_n$, $g \in X_{n+1}$,

$$[\mathcal{D}f, g]_{n+1} = [f, \mathcal{D}^{-1}g]_n$$

so $\mathcal{D}^{-1} = \mathcal{D}^*$ (where in this case $\mathcal{D}^*$ is the adjoint of $\mathcal{D}$ as an adjointable operator between Hilbert modules). Hence $\mathcal{D}$ is a unitary element of $\mathcal{L}(X_n, X_{n+1})$. From equation (2.42) we have that for $p \in \hat{X_n}$, $q \in \hat{X_{n+1}}$,

$$[\![\hat{\mathcal{D}}p, q]\!]_{n+1} = [\![p, \hat{\mathcal{D}}^{-1}q]\!]_n$$

so $\hat{\mathcal{D}}^{-1} = \hat{\mathcal{D}}^*$ (again $\hat{\mathcal{D}}^*$ is the adjoint of $\hat{\mathcal{D}}$ as an adjointable operator between Hilbert modules). Hence $\hat{\mathcal{D}}$ is a unitary element of $\mathcal{L}(\hat{X_n}, \hat{X_{n+1}})$. □

The next proposition demonstrates that each X_n shares the same linear space and is partially based on [PR2, Proposition 1.11]. The difference between each X_n is therefore in how the $C^*(\mathbf{Z}^d)$-valued inner product and the module action are defined within the linear space.

Recall from page 22 that we defined $\Delta \in \mathrm{Hom}(\mathbf{Z}^d)$ to be $\Delta(\gamma) = \mathcal{D}^{-1}\gamma\mathcal{D}$ (as a unitary operator on $\mathcal{H}$), where the translation $\gamma \in \mathbf{Z}^d$ is thought of as a unitary operator on $\mathcal{H}$. We shall be interested in the dual homomorphism $\hat{\Delta} \in \mathrm{Hom}(\mathbf{T}^d)$. This satisfies

$$(\Delta(\gamma), \zeta) = (\gamma, \hat{\Delta}(\zeta)), \quad \gamma \in \mathbf{Z}^d, \zeta \in \mathbf{T}^d.$$

It follows that if we think of $\mathbf{T}^d$ as being the quotient $\widehat{\mathbf{R}^d}/\mathbf{Z}^d$, then

$$\hat{\Delta}(\zeta) = \iota(\tilde{\mathcal{D}}^*(\zeta)), \quad \zeta \in \mathbf{T}^d. \tag{2.45}$$

In the above equation ι is the quotient map from $\mathbf{R}^d$ onto $\mathbf{T}^d$.

Proposition 2.2.8 *For $p, q \in \hat{X}_n$, we have that*

$$[\![p,q]\!]_{n-1}(\zeta) = \frac{1}{m} \sum_{\hat{\Delta}(\omega)=\zeta} [\![p,q]\!]_n(\omega). \tag{2.46}$$

This implies that

$$m^{-1/2}\|p\|_{X_n} \le \|p\|_{X_{n-1}} \le \|p\|_{X_n}. \tag{2.47}$$

We therefore have that the Hilbert $C^*(\mathbf{Z}^d)$-modules $(X_n)_{n\in\mathbf{Z}}$ all share the same linear space, and that the Hilbert $C(\mathbf{T}^d)$-modules $(\hat{X}_n)_{n\in\mathbf{Z}}$ also all share the same linear space.

Proof: Recall that for $f \in C_c(\mathbf{R}^d)$, $[f,f]_{n-1}(\gamma) = [f,f]_n(\Delta(\gamma))$ for $\gamma \in \mathbf{Z}^d$. We calculate

$$\begin{aligned}
[\![p,q]\!]_{n-1}(\zeta) &= m^{n-1} \sum_{\hat{\iota_{n-1}}(\xi)=\zeta} \overline{p(\xi)}q(\xi) \\
&= m^{n-1} \sum_{\hat{\Delta}(\hat{\iota_n}(\xi))=\zeta} \overline{p(\xi)}q(\xi) \\
&= m^{n-1} \sum_{\hat{\Delta}(\omega)=\zeta} \sum_{\hat{\iota_n}(\xi)=\omega} \overline{p(\xi)}q(\xi) \\
&= \frac{1}{m} \sum_{\hat{\Delta}(\omega)=\zeta} [\![p,q]\!]_n(\omega).
\end{aligned}$$

Now let

$$F := \{\zeta \in \mathbf{T}^d : \hat{\Delta}(\zeta) = 0\}.$$

Then F is a subgroup of $\mathbf{T}^d$ with m elements (because the index of $\Delta(\mathbf{Z}^d)$ in $\mathbf{Z}^d$ is equal to m). Because $\hat{\Delta}$ is a homomorphism, it follows that if $\omega, \omega' \in \mathbf{T}^d$ satisfy $\hat{\Delta}(\omega) = \hat{\Delta}(\omega')$, then there exists $\beta \in F$ such that $\omega = \omega' + \beta$. We therefore have that

$$[\![p,q]\!]_{n-1}(\zeta) = \frac{1}{m} \sum_{\beta\in F} [\![p,q]\!]_n(\tilde{\mathcal{D}}^{*-1}(\zeta) + \beta). \tag{2.48}$$

Therefore

$$\begin{aligned}
\|p\|_{X_{n-1}} &= \sqrt{\sup_{\zeta\in\mathbf{T}^d} [\![p,p]\!]_{n-1}(\zeta)} \\
&= \sqrt{\sup_{\zeta\in\mathbf{T}^d} \frac{1}{m} \sum_{\beta\in F} [\![p,q]\!]_n(\tilde{\mathcal{D}}^{*-1}(\zeta) + \beta)}
\end{aligned}$$

$$\begin{aligned} &\leq \sqrt{\frac{1}{m}\sum_{\beta\in F}\sup_{\zeta\in\mathbf{T}^d}[\![p,q]\!]_n(\tilde{\mathcal{D}}^{*-1}(\zeta)+\beta)} \\ &= \sqrt{\frac{1}{m}\sum_{\beta\in F}\|p\|_{X_n}} \\ &= \|p\|_{X_n}. \end{aligned}$$

We now use equation (2.40) and the fact that $(\tilde{\mathcal{D}}^*)^n\mathbf{Z}^d \subset (\tilde{\mathcal{D}}^*)^{n+1}\mathbf{Z}^d$ to obtain that

$$\begin{aligned} [\![p,p]\!]_{n-1}(\zeta) &= m^{n-1}\sum_{\beta\in(\tilde{\mathcal{D}}^*)^{n-1}\mathbf{Z}^d}\overline{p\left((\tilde{\mathcal{D}}^*)^{n-1}\zeta+\beta\right)}q\left((\tilde{\mathcal{D}}^*)^{n-1}\zeta+\beta\right) \\ &\geq m^{n-1}\sum_{\beta\in(\tilde{\mathcal{D}}^*)^{n}\mathbf{Z}^d}\overline{p\left((\tilde{\mathcal{D}}^*)^{n-1}\zeta+\beta\right)}q\left((\tilde{\mathcal{D}}^*)^{n-1}\zeta+\beta\right). \end{aligned}$$

This implies that

$$[\![p,p]\!]_{n-1}\left(\iota(\tilde{\mathcal{D}}^*\zeta)\right) \geq m^{-1}[\![p,p]\!]_n(\zeta)$$

where ι is the quotient map from $\widehat{\mathbf{R}^d}$ to $\mathbf{Z}^d$. Therefore

$$\begin{aligned} \|p\|_{X_{n-1}} &= \sqrt{\sup_{\zeta\in\mathbf{T}^d}[\![p,p]\!]_{n-1}(\zeta)} \\ &= \sqrt{\sup_{\zeta\in\mathbf{T}^d}[\![p,p]\!]_{n-1}\left(\iota(\tilde{\mathcal{D}}^*\zeta)\right)} \\ &\geq \sqrt{\sup_{\zeta\in\mathbf{T}^d}m^{-1}[\![p,p]\!]_n(\zeta)} \\ &= m^{-1/2}\|p\|_{X_n}. \end{aligned}$$

It therefore follows that the norms $(\|\cdot\|_{X_n})_{n\in\mathbf{Z}}$ are all equivalent. Because the linear space X_n is the completion of $C_c(\mathbf{R}^d)$ with respect to the norm $\|\cdot\|_{X_n}$, it follows that the Hilbert $C^*(\mathbf{Z}^d)$-modules $(X_n)_{n\in\mathbf{Z}}$ all share the same linear space, and that the Hilbert $C(\mathbf{T}^d)$-modules $(\hat{X}_n)_{n\in\mathbf{Z}}$ also all share the same linear space.

□

We will now demonstrate the utility of the Hilbert module approach with Proposition 2.2.9. Proposition 2.2.9 is a formulation of necessary and sufficient conditions for a set of elements of X_0 to be a multiwavelet. We shall show in Example 3.2.3 that not all wavelets are elements of X_0. But in the next section we shall prove a result which is similar to Proposition 2.2.9, but which holds in more generality.

Proposition 2.2.9 *Suppose that $(X_n)_{n\in\mathbf{Z}}$ is a wavelet chain of Hilbert $C^*(\mathbf{Z}^d)$-modules. Suppose that $\psi^1,\dots,\psi^M$ are elements of X_0. Then $\{\psi^1,\dots,\psi^M\}$ is an orthonormal multiwavelet if and only if*

1. *For $i,j=1,\dots,M$, and $m,n\in\mathbf{Z}$,*

$$[\mathcal{D}^n\psi^i,\mathcal{D}^m\psi^j]_n = \delta_{i,j}\delta_{m,n}\mathbf{1} \tag{2.49}$$

where δ_i,j is the Kronecker delta and $\mathbf{1}$ is the unit in $C^*(\mathbf{Z}^d)$.

2. *For all* $f \in L^2(\mathbf{R}^d)$,

$$f = \sum_{i=1}^{M} \sum_{n \in \mathbf{Z}} \mathcal{D}^n \psi^i \circ_n [\mathcal{D}^n \psi^i, f]_n \tag{2.50}$$

and the above sum converges in $L^2(\mathbf{R}^d)$.

PROOF: Suppose that $\{\psi^1, \ldots, \psi^M\}$ is an orthonormal multiwavelet. Then from orthogonality it holds that for each $i = 1, \ldots, M$, $[\psi^i, \psi^i]_0 = \mathbf{1}$ and so $[\mathcal{D}^n\psi^i, \mathcal{D}^n\psi^i]_n = \mathbf{1}$ for all $n \in \mathbf{Z}$. Equation (2.49) also follows from orthogonality. Equation (2.50) is a direct consequence of the fact that the wavelets form an orthonormal basis for $L^2(\mathbf{R}^d)$, with the sum converging because the L^2-norm of f is finite.

Now suppose that equations (2.49) and (2.50) hold. Then equation (2.49) implies that $\{\mathcal{D}^n(\gamma(\psi^i))\}_{\gamma \in \mathbf{Z}^d, n \in \mathbf{Z}, i=1,\ldots,M}$ is an orthonormal set and equation (2.50) implies that $\{\mathcal{D}^n(\gamma(\psi^i))\}_{\gamma \in \mathbf{Z}^d, n \in \mathbf{Z}, i=1,\ldots,M}$ is an orthonormal basis for $L^2(\mathbf{R}^d)$. Therefore $\{\psi^1, \ldots, \psi^M\}$ is an orthonormal multiwavelet. □

Remark 2.2.10 Some of the theory described above can be generalised considerably. Consider a multiresolution structure $(\Gamma, \mathcal{D})$ with index m acting on the Hilbert space $L^2(G)$, for G a locally compact group. Assume that Γ is a subgroup of G. Suppose that $\theta : \Gamma \to G$ is an embedding (we know that such a theta exists because Γ is assumed to be a subgroup of G). Then $\theta(\Gamma)$ is also a subgroup of G. Associated with θ we define a representation π^θ of Γ on $L^2(G)$ by

$$(\pi^\theta_\gamma f)(x) = f(x\theta(\gamma^{-1})), \quad \text{for } x \in G,\ \gamma \in \Gamma,\ f \in L^2(G).$$

Let Δ be the modular function on G and let δ be the modular function on Γ. For $f, g \in C_c(G)$, $a \in C_c(\Gamma)$, define

$$(f \circ_\theta a)(s) := \int_\Gamma f(s\theta(t^{-1}))\delta(t^{-1})a(t)d\mu_\Gamma(t); \tag{2.51}$$

$$[f, g]_\theta(t) := \sqrt{\frac{\Delta(t)}{\delta(t)}} \int_G \overline{f(r)} g(r\theta(t))d\mu_G(r). \tag{2.52}$$

Then from [RW, Theorem C.23], the completion of $C_c(G)$ with respect to the Hilbert module norm

$$\|f\|_X = \|[f, f]_\theta\|_*^{\frac{1}{2}}, \quad f \in C_c(G); \tag{2.53}$$

is a Hilbert $C^*(\Gamma)$-module.

Suppose now that there is an automorphism $\tilde{\mathcal{D}} : G \to G$ for which $(\mathcal{D}f)(x) = \sqrt{m} f(\tilde{\mathcal{D}}x)$ for $f \in \mathcal{H}$, $x \in G$. And that the translations $\gamma \in \Gamma$ act on $\mathcal{H}$ by $\gamma f(x) = f(x\gamma^{-1})$ for $f \in \mathcal{H}$. Then by a calculation that is very similar to the calculation verifying equation (1.3), we obtain that for $n \in \mathbf{Z}$, $f \in \mathcal{H}$, $\gamma \in \Gamma$,

$$(\mathcal{D}^n \gamma \mathcal{D}^{-n} f)(x) = f(x\tilde{\mathcal{D}}^n \gamma^{-1}), \quad \text{for } x \in G. \tag{2.54}$$

So consider the embedding $\iota_n : \Gamma \to G$ given by $\iota_n(\gamma) = \tilde{\mathcal{D}}^n(\gamma)$, and for $f, g \in C_c(G)$, $a \in C_c(\Gamma)$, write $(f \circ_n a) := (f \circ_{\iota_n} a)$ and $[f, g]_n := [f, g]_{\iota_n}$. We then have that

$$(f \circ_n a)(s) = \int_\Gamma f(s\tilde{\mathcal{D}}^n(t^{-1}))\delta(t^{-1})a(t)d\mu_\Gamma(t); \tag{2.55}$$

$$[f, g]_n(t) = \sqrt{\frac{\Delta(t)}{\delta(t)}} \int_G \overline{f(r)} g(r\tilde{\mathcal{D}}^n(t))d\mu_G(r). \tag{2.56}$$

At this level of generality there is no longer an obvious way of defining the Fourier transform. If we were to work with Harmonic multiresolution structures as described in 1.2, there are still some difficulties in constructing a Fourier transformed bracket product. There are impediments to proving an analogue of Lemma 2.1.8 because it identifies $\mathbf{T}^d$ with the cube $[-1/2, 1/2)^d$, and in the more general situation it is difficult to come up with an analogue of the cube which has all of the required properties (for example $[-1/2, 1/2)^d$ is a measurable subset of $\mathbf{R}^d$ for which 0 is contained in a neighbourhood of $[-1/2, 1/2)^d$, and we can tile $\mathbf{R}^d$ with translations of $[-1/2, 1/2)^d$ by elements of $\mathbf{Z}^d$). There are some results in Chapter 3 for which we encounter similar problems, including Lemma 3.4.4 and Lemma 3.4.6.

2.3 Some Hilbert modules over $L^\infty(\mathbf{T}^d)$

Consider the setting of the standard multiresolution structure in $L^2(\mathbf{R}^d)$ with dilation matrix $\tilde{\mathcal{D}}$ (see Definition 1.3.1), and let θ be an embedding of $\mathbf{Z}^d$ in $\mathbf{R}^d$. We shall now construct a Hilbert $L^\infty(\mathbf{T}^d)$-module which we denote by Y_θ, and is based on a Hilbert module described in [CoLa] and [CaLa]. We shall construct this Hilbert module in the Fourier domain, and show that it can be embedded in $L^2(\widehat{\mathbf{R}^d})$.

Just like the Hilbert $C(\mathbf{T}^d)$-module X_θ, the Hilbert $L^\infty(\mathbf{T}^d)$-module Y_θ shall have the same operations $[\![\ ,\ ]\!]_\theta$ (see equation (2.24)), and $\widehat{\circ_\theta}$ (see Definition 2.1.10). Recall that we showed that the sum in (2.24) converges absolutely almost everywhere, this time we shall look at convergence in the weak* topology. Recall that $L^\infty(\mathbf{T}^d)$ is the continuous dual space of $L^1(\mathbf{T}^d)^*$ in the sense that elements of $L^\infty(\mathbf{T}^d)$ are bounded linear functionals of elements of $L^1(\mathbf{T}^d)$. We make the following definition.

Definition 2.3.1 If $(f_n)_{n\in\mathbf{N}}$ is a sequence in $L^\infty(\mathbf{T}^d)$, then $(f_n)_{n\in\mathbf{N}}$ converges to f in the *weak* topology* if for all $a \in L^1(\mathbf{T}^d)$, $\lim_{n\to\infty} f_n(a) = f(a)$.

Let the linear space Y_θ be the set of measurable functions p on $\widehat{\mathbf{R}^d}$ for which the norm

$$\|p\|_{Y_\theta} := \sqrt{\operatorname{ess\,sup}_{\zeta\in\mathbf{T}^d} \frac{1}{\det(A_\theta)} \sum_{\beta\in(A_\theta^*)^{-1}\mathbf{Z}^d} \overline{p(\zeta+\beta)}p(\zeta+\beta)} \tag{2.57}$$

is finite. We shall show in Example 2.3.3 that the above series does not necessarily converge in norm in $L^\infty(\mathbf{T}^d)$, so we instead require weak* convergence. Compare this norm with $\|\cdot\|_{X_\theta}$, which we defined in equation (2.34). The difference between the two norms is that $\|\cdot\|_{Y_\theta}$ involves taking the essential supremum instead of the supremum and we are working in the Fourier domain. We can thus write

$$\|p\|_{Y_\theta} = \sqrt{\text{ess sup}_{\zeta\in\mathbf{T}^d}[\![p,p]\!]_\theta(\zeta)},$$

recalling that

$$[\![p,p]\!]_\theta(\zeta) = \frac{1}{\det(A_\theta)} \sum_{\beta\in(A_\theta^*)^{-1}\mathbf{Z}^d} \overline{p(\zeta+\beta)}p(\zeta+\beta). \tag{2.58}$$

Lemma 2.3.2 *Suppose that $p \in Y_\theta$. Then the sum in $[\![p,p]\!]_\theta$ converges in the weak* topology to an element of $L^\infty(\mathbf{T}^d)$.*

Proof: To simplify our notation we shall assume without any loss of generality that θ is the natural embedding of $\mathbf{Z}^d$ in $\mathbf{R}^d$. We therefore want to show that the sum defined for almost every $\zeta \in \mathbf{T}^d$ by

$$\sum_{\beta\in\mathbf{Z}^d} \overline{p(\zeta+\beta)}p(\zeta+\beta) \tag{2.59}$$

converges in the weak* topology to an element of $L^\infty(\mathbf{T}^d)$. Suppose that $(S_n)_{n\in\mathbf{N}}$ is a sequence of finite subsets of $\mathbf{Z}^d$ for which $S_n \subset S_{n+1}$ and $\cup_{n\in\mathbf{N}}S_n = \mathbf{Z}^d$. Then

$$\sum_{\beta\in\mathbf{Z}^d} \overline{p(\zeta+\beta)}p(\zeta+\beta) = \lim_{n\to\infty} \sum_{\beta\in S_n} \overline{p(\zeta+\beta)}p(\zeta+\beta).$$

Note that the summands in the above sum are positive so if it converges, it converges unconditionally. Because $p \in Y_\theta$, it is the case that the set of all $\zeta \in \mathbf{T}^d$ for which $\sum_{\beta\in\mathbf{Z}^d} \overline{p(\zeta+\beta)}p(\zeta+\beta)$ does not converge has measure zero. Define a function L on $\mathbf{T}^d$ by

$$L(\zeta) = \begin{cases} \sum_{\beta\in\mathbf{Z}^d} \overline{p(\zeta+\beta)}p(\zeta+\beta) & \text{when it converges,} \\ 0 & \text{otherwise.} \end{cases}$$

Because

$$\text{ess sup}_{\zeta\in\mathbf{T}^d} \sum_{\beta\in\mathbf{Z}^d} \overline{p(\zeta+\beta)}p(\zeta+\beta) < \infty,$$

it follows that $L \in L^\infty(\mathbf{T}^d)$.

We shall now show that the sum in (2.59) converges to L in the weak* topology. We want to show that for all $a \in L^1(\mathbf{T}^d)$,

$$\lim_{n\to\infty} \int_{\mathbf{T}^d} \sum_{\beta\in S_n} \overline{p(\zeta+\beta)}p(\zeta+\beta)a(\zeta)d\zeta = \int_{\mathbf{T}^d} L(\zeta)a(\zeta)d\zeta.$$

Because all of the terms in the above sum except for $a(\zeta)$ are positive, it is sufficient to show that the above relation holds for $|a(\zeta)|$ (instead of $a(\zeta)$). Suppose that E is a measurable subset of $\mathbf{T}^d$, let $\phi_a(E) = \int_E |a(\zeta)|d\zeta$, then by [Ru1, Theorem 1.29], ϕ_a is a measure and for all measurable $g : \mathbf{T}^d \to [0,\infty]$,

$$\int_{\mathbf{T}^d} g(\zeta)d\phi_a(\zeta) = \int_{\mathbf{T}^d} g(\zeta)a(\zeta)d\zeta.$$

It now follows from Lebesgue's monotone convergence theorem (see [Ru1, Theorem 1.26]) that

$$\lim_{n\to\infty}\int_{\mathbf{T}^d}\sum_{\beta\in S_n}\overline{p(\zeta+\beta)}p(\zeta+\beta)d\phi_a(\zeta)=\int_{\mathbf{T}^d}L(\zeta)d\phi_a(\zeta).$$

This verifies that the sum (2.59) converges to L in the weak* topology. □

We note that it follows from the polarisation identity (Lemma 0.4.3) that if $p, q \in Y_\theta$, then $[\![p,q]\!]_\theta \in L^\infty(\mathbf{T}^d)$.

Example 2.3.3 This example is from page 5 of [CoLa], it is an example of a situation where the sum in (2.58) does not converge in the norm topology in $L^\infty(\mathbf{T}^d)$. Assume that $d = 1$ (so we are looking at functions on $\mathbf{R}$), and that θ is the natural embedding of $\mathbf{Z}$ in $\mathbf{R}$. As usual, identify $\mathbf{T}$ with $[-1/2, 1/2)$. We define

$$\begin{aligned} p &= \sum_{k=0}^{\infty}\chi_{[k+\frac{1}{2^{k+1}},k+\frac{1}{2^k})} \\ \text{i.e.}\quad p &= \chi_{[\frac{1}{2},1)}+\chi_{[1\frac{1}{4},1\frac{1}{2})}+\chi_{[2\frac{1}{8},2\frac{1}{4})}+\dots. \end{aligned}$$

It then follows that for all $\zeta \in [-1/2, 1/2)$, $[\![p,p]\!]_\theta(\zeta) = 1$. However, the sum in (2.58) clearly does not converge in norm (the L^∞-norm of each term equals 1).

We have that for $p \in Y_\theta$,

$$\|p\|_2^2=\int_{\mathbf{T}^d}[\![p,p]\!]_\theta(\zeta)d\zeta\le \text{ess sup}_{\zeta\in\mathbf{T}^d}[\![p,p]\!]_\theta(\zeta)=\|p\|_{Y_\theta}^2.$$

and so Y_θ is contained in $L^2(\widehat{\mathbf{R}^d})$.

Theorem 2.3.4 *The space Y_θ is a Hilbert $L^\infty(\mathbf{T}^d)$-module when equipped with the Fourier transformed bracket product $[\![\ ,\ ]\!]_\theta$ with convergence in the weak* topology on $L^\infty(\mathbf{T}^d)$, and associated module action $\widehat{\circ_\theta}$.*

PROOF: The space Y_θ satisfies Properties 1,2 and 3 of Definition 0.4.1 by Lemma 2.1.15. For $p \in Y_\theta$, the sum

$$\sum_{\beta\in(A_\theta^*)^{-1}\mathbf{Z}^d}\overline{p(\zeta+\beta)}p(\zeta+\beta)$$

is nonnegative for all $\zeta \in \mathbf{T}^d$. Therefore $[\![p,p]\!]_\theta$ is a positive element of $L^\infty(\mathbf{T}^d)$, verifying property 4 of Definition 0.4.1. If $[\![p,p]\!]_\theta = 0$, then $\|p\|_{Y_\theta} = 0$ and hence $\|p\|_2 = 0$ since $\|p\|_2 \le \|p\|_{Y_\theta}$. This implies that $p = 0$, verifying property 5 of Definition 0.4.1.

We now show that the space Y_θ is complete. Suppose that (p_j) is a Cauchy sequence in Y_θ. Then for all $\varepsilon > 0$, there exists $J \in \mathbf{N}$ such that

$$j,k>J\Rightarrow \|p_j-p_k\|_{Y_\theta}^2<\varepsilon.$$

This implies that for almost every $\zeta \in \mathbf{T}^d$,

$$\begin{aligned} [\![p_j - p_k, p_j - p_k]\!]_\theta(\zeta) &< \varepsilon \\ \text{i.e.} \sum_{\beta \in (A_\theta^*)^{-1}\mathbf{Z}^d} \overline{(p_j - p_k)(\zeta+\beta)}(p_j - p_k)(\zeta+\beta) &< \varepsilon \det A_\theta \\ \text{so for all } \beta \in (A_\theta^*)^{-1}\mathbf{Z}^d, \quad \overline{(p_j - p_k)(\zeta+\beta)}(p_j - p_k)(\zeta+\beta) &< \varepsilon \det A_\theta. \end{aligned}$$

So for almost every $\xi \in \mathbf{R}^d$, there exists a scalar $p(\xi)$ for which $\lim_{j\to\infty} p_j(\xi) = p(\xi)$. This defines a function p almost everywhere on $\mathbf{R}^d$.

We shall now show that $\lim_{k\to\infty} \|p-p_k\|_{Y_\theta} = 0$. Suppose that $(S_n)_{n\in\mathbf{N}}$ is a sequence of finite subsets of $(A_\theta^*)^{-1}\mathbf{Z}^d$ for which $S_n \subset S_{n+1}$ and $\cup_{n\in\mathbf{N}} S_n = (A_\theta^*)^{-1}\mathbf{Z}^d$. Now $p - p_k = \lim_{j\to\infty} p_j - p_k$, so

$$\begin{aligned} \|p-p_k\|_{Y_\theta} &= \sqrt{\operatorname{ess\,sup}_{\zeta\in\mathbf{T}^d} \frac{1}{\det A_\theta} \sum_{\beta\in(A_\theta^*)^{-1}\mathbf{Z}^d} \lim_{j\to\infty} \overline{(p_j-p_k)(\zeta+\beta)}(p_j-p_k)(\zeta+\beta)} \\ &= \sqrt{\operatorname{ess\,sup}_{\zeta\in\mathbf{T}^d} \frac{1}{\det A_\theta} \lim_{n\to\infty} \sum_{\beta\in S_n} \lim_{j\to\infty} \overline{(p_j-p_k)(\zeta+\beta)}(p_j-p_k)(\zeta+\beta)} \\ &= \lim_{j\to\infty} \sqrt{\operatorname{ess\,sup}_{\zeta\in\mathbf{T}^d} \frac{1}{\det A_\theta} \lim_{n\to\infty} \sum_{\beta\in S_n} \overline{(p_j-p_k)(\zeta+\beta)}(p_j-p_k)(\zeta+\beta)} \\ &= \lim_{j\to\infty} \sqrt{\operatorname{ess\,sup}_{\zeta\in\mathbf{T}^d} \frac{1}{\det A_\theta} \sum_{\beta\in(A_\theta^*)^{-1}\mathbf{Z}^d} \overline{(p_j-p_k)(\zeta+\beta)}(p_j-p_k)(\zeta+\beta)} \end{aligned}$$

Because (p_j) is Cauchy in Y_θ, it follows that for all $\varepsilon > 0$, there exists a natural number k such that $\|p - p_k\|_{Y_\theta} < \varepsilon$. Since $p = (p - p_k) + p_k$, it follows that $p \in Y_\theta$. Therefore Y_θ is complete. □

Definition 2.3.5 For $n \in \mathbf{Z}$, let Y_n be the set of measurable functions p on $\widehat{\mathbf{R}^d}$ for which the norm

$$\|p\|_{Y_n} = \sqrt{\operatorname{ess\,sup}_{\zeta\in\mathbf{T}^d} [\![p,p]\!]_n(\zeta)} \tag{2.60}$$

is finite. Again we only require weak* convergence in the above series. Equip Y_n with the nth level Fourier transformed bracket product and the associated module action. By Theorem 2.3.4 it is a Hilbert $L^\infty(\mathbf{T}^d)$-module. Note that $\widehat{X_n} \subset Y_n$. We call $(Y_n)_{n\in\mathbf{Z}}$ a *wavelet chain of Hilbert* $L^\infty(\mathbf{T}^d)$*-modules.*

It is the case that for $p \in Y_n$, $\|p\|_2 \le \|p\|_{Y_n}$ and that $Y_n \subset L^2(\mathbf{R}^d)$, because these properties hold for any Y_θ.

The proof of the following proposition is very similar to the proof of Proposition 2.2.8.

Proposition 2.3.6 *For $p, q \in Y_n$, we have that*

$$[\![p,q]\!]_{n-1}(\zeta) = \frac{1}{m} \sum_{\hat{\Delta}(\omega)=\zeta} [\![p,q]\!]_n(\omega). \tag{2.61}$$

This implies that

$$m^{-1/2}\|p\|_{Y_n} \leq \|p\|_{Y_{n-1}} \leq \|p\|_{Y_n}. \tag{2.62}$$

We therefore have that the Hilbert $L^\infty(\mathbf{T}^d)$-modules $(Y_n)_{n\in\mathbf{Z}}$ all share the same linear space.

PROOF: The proof that (2.61) is satisfied is the same as the proof that (2.46) is satisfied. Let

$$F := \{\zeta \in \mathbf{T}^d : \hat{\Delta}(\zeta) = 0\}.$$

By the same argument as in Proposition 2.2.8, we have that F is a subgroup of $\mathbf{T}^d$ with m elements, and

$$[\![p,q]\!]_{n-1}(\zeta) = \frac{1}{m}\sum_{\beta\in F}[\![p,q]\!]_n(\tilde{\mathcal{D}}^{*-1}(\zeta)+\beta). \tag{2.63}$$

Therefore

$$\begin{aligned}
\|p\|_{Y_{n-1}} &= \sqrt{\text{ess sup}_{\zeta\in\mathbf{T}^d}[\![p,p]\!]_{n-1}(\zeta)}\\
&= \sqrt{\text{ess sup}_{\zeta\in\mathbf{T}^d}\frac{1}{m}\sum_{\beta\in F}[\![p,q]\!]_n(\tilde{\mathcal{D}}^{*-1}(\zeta)+\beta)}\\
&\leq \sqrt{\frac{1}{m}\sum_{\beta\in F}\text{ess sup}_{\zeta\in\mathbf{T}^d}[\![p,q]\!]_n(\tilde{\mathcal{D}}^{*-1}(\zeta)+\beta)}\\
&= \sqrt{\frac{1}{m}\sum_{\beta\in F}\|p\|_{Y_n}}\\
&= \|p\|_{Y_n}.
\end{aligned}$$

By the same argument as in Proposition 2.2.8, we have that

$$[\![p,p]\!]_{n-1}\left(\iota(\tilde{\mathcal{D}}^*\zeta)\right) \geq m^{-1}[\![p,p]\!]_n(\zeta)$$

where ι is the quotient map from $\widehat{\mathbf{R}^d}$ to $\mathbf{Z}^d$. And thus

$$\begin{aligned}
\|p\|_{Y_{n-1}} &= \sqrt{\text{ess sup}_{\zeta\in\mathbf{T}^d}[\![p,p]\!]_{n-1}(\zeta)}\\
&= \sqrt{\text{ess sup}_{\zeta\in\mathbf{T}^d}[\![p,p]\!]_{n-1}\left(\iota(\tilde{\mathcal{D}}^*\zeta)\right)}\\
&\geq \sqrt{\text{ess sup}_{\zeta\in\mathbf{T}^d}m^{-1}[\![p,p]\!]_n(\zeta)}\\
&= m^{-1/2}\|p\|_{Y_n}.
\end{aligned}$$

It therefore follows that the norms $(\|\cdot\|_{Y_n})_{n\in\mathbf{Z}}$ are all equivalent. Because the linear space Y_n is the set of functions on $\widehat{\mathbf{R}^d}$ for which the norm $\|\cdot\|_{Y_n}$ is finite, it follows that the Hilbert $L^\infty(\mathbf{T}^d)$-modules $(Y_n)_{n\in\mathbf{Z}}$ all share the same linear space.

□

Proposition 2.3.7 can be thought of as a slightly stronger version of Proposition 2.2.9. It gives neccessary and sufficient conditions for a set of elements of $L^2(\mathbf{R}^d)$ to be a multiwavelet. It also demonstrates that any wavelet will have it's Fourier transform contained in Y_0. The proof to Proposition 2.3.7 is almost exactly the same as the proof to Proposition 2.2.9.

Proposition 2.3.7 *Suppose that $(Y_n)_{n\in\mathbf{Z}}$ is a wavelet chain of Hilbert $L^\infty(\mathbf{T}^d)$-modules. Suppose that $\psi^1, \ldots, \psi^M$ are elements of $L^2(\mathbf{R}^d)$. Then $\{\psi^1, \ldots, \psi^M\}$ is an orthonormal multiwavelet if and only if*

1. *For $i, j = 1, \ldots, M$, and $m, n \in \mathbf{Z}$,*

$$[\![\hat{\mathcal{D}^n}\hat{\psi^i}, \hat{\mathcal{D}^m}\hat{\psi^j}]\!]_n = \delta_{i,j}\delta_{m,n}\mathbf{1} \tag{2.64}$$

where δ_i, j is the Kronecker delta and $\mathbf{1}$ is the function on $\mathbf{T}^d$ which takes the value 1 everywhere.

2. *For all $f \in L^2(\mathbf{R}^d)$,*

$$\hat{f} = \sum_{i=1}^{M}\sum_{n\in\mathbf{Z}} \hat{\mathcal{D}^n}\hat{\psi^i}\hat{\circ}_n[\![\hat{\mathcal{D}^n}\hat{\psi^i}, \hat{f}]\!]_n \tag{2.65}$$

and the above sum converges in $L^2(\mathbf{R}^d)$.

Furthermore, if $\{\psi^1, \ldots, \psi^M\}$ is a multiwavelet, then for all i, $\hat{\psi^i} \in Y_0$. If $\{\varphi^1, \ldots, \varphi^r\}$ is a set of scaling functions, then for all i, $\hat{\varphi^i} \in Y_0$.

PROOF: Suppose that $\{\psi^1, \ldots, \psi^M\}$ is an orthonormal multiwavelet. Then from orthogonality it holds that for each $i = 1, \ldots, M$, $[\![\hat{\psi^i}, \hat{\psi^i}]\!]_0 = \mathbf{1}$ and so $[\![\widehat{\mathcal{D}^n\psi^i}, \widehat{\mathcal{D}^n\psi^i}]\!]_n = \mathbf{1}$ for all $n \in \mathbf{Z}$. This implies that $\|\widehat{\mathcal{D}^n\psi^i}\|_{Y_n} = 1$. We therefore have that for all $n \in \mathbf{Z}$, $\widehat{\mathcal{D}^n\psi^i} \in Y_n$ and in particular $\hat{\psi^i} \in Y_0$. We have that $\hat{\varphi^i} \in Y_0$, because $[\![\hat{\varphi^i}, \hat{\varphi^i}]\!]_0 = \mathbf{1}$. Equation (2.64) also follows from orthogonality. Equation (2.65) is a direct consequence of the fact that the wavelets form an orthonormal basis for $L^2(\mathbf{R}^d)$, with the sum converging because the L^2-norm of f is finite.

Now suppose that equations (2.64) and (2.65) hold. Then equation (2.64) implies that $\{\mathcal{D}^n(\gamma(\psi^i))\}_{\gamma\in\mathbf{Z}^d, n\in\mathbf{Z}, i=1,\ldots,M}$ is an orthonormal set and equation (2.65) implies that $\{\mathcal{D}^n(\gamma(\psi^i))\}_{\gamma\in\mathbf{Z}^d, n\in\mathbf{Z}, i=1,\ldots,M}$ is an orthonormal basis for $L^2(\mathbf{R}^d)$. Therefore $\{\psi^1, \ldots, \psi^M\}$ is an orthonormal multiwavelet. □

Remark 2.3.8 In Chapter 3 we shall provide an example of a wavelet which is not contained in X_0. This example, and the above proposition, suggest that results about $(Y_n)_{n\in\mathbf{Z}}$ can be thought of as being more relevant to wavelets in general, while results about $(X_n)_{n\in\mathbf{Z}}$ are more relevant to "nicely behaved" wavelets.

Chapter 3

Filters and the Cascade Algorithm

In this chapter we shall explore in detail the concept of a "filter". There are quite a few standard results in wavelet theory that relate filters to wavelet theory in various ways. In this chapter we shall review quite a few of these results, but do so in the setting introduced in Chapter 2. These results will include the fast wavelet transform (that we briefly described in Section 1.4), and how it relates to the Cuntz relations [J1, BJ1, BJ3]; the cascade algorithm [Lw1, Lw2, St, BJ2], see also [Coh]; and wavelet matrices [Gr, SN, Tu].

In so doing we shall illustrate the utility of the bracket product notation. In Theorem 3.4.10 we shall prove that the cascade algorithm converges in the topology of the Hilbert module X_0. As far as the author is aware, this is the first time that the convergence of the cascade algorithm has been investigated with respect to this topology. In Lemma 3.4.1 we shall show demonstrate that an operator associated with the cascade algorithm is an adjointable operator between Hilbert modules.

In the first four sections of this chapter we shall be assuming the wavelets correspond to a single scaling function φ, while in the fifth section we will investigate multiwavelets which have more than one scaling function. In Section 3.1, we shall investigate an operator called the downsampling operator, and its Fourier transform. In Section 3.2 we will relate wavelets to certain elements of $C^*(\mathbf{Z}^d)$ known as scaling and wavelet filters. In this section we shall also show that the Shannon wavelet is not contained in the Hilbert $C^*(\mathbf{Z}^d)$-module X_0, but does have it's Fourier transform contained the Hilbert $L^\infty(\mathbf{T}^d)$-module Y_0, see Example 3.2.3. In Section 3.3 we shall relate the fast wavelet transform to representations of Cuntz algebras. In Section 3.4 we shall study the convergence properties of the cascade algorithm. In Section 3.5 we shall investigate the properties of multiwavelets by constructing Hilbert modules over a matrix C^*-algebra.

Wavelet theory partially emerged from the study of operators and filters and their applications to engineering and numerical algorithms [SN]. In this chapter we attempt to demonstrate that our construction relating wavelets to Hilbert modules fits in nicely with this perspective.

Notation 3.0.1 Unless otherwise stated, throughout this chapter we shall be working with the standard multiresolution structure (introduced in Section 1.3) that is associated with a dilation matrix $\tilde{\mathcal{D}}$ with index m. We shall be working with the Hilbert modules described in the previous chapter.

We shall be making extensive use of the operations $[\ ,\ ]_n$, $\circ_n$, $[\![\ ,\ ]\!]_n$ and $\widehat{\circ_n}$ which we defined in Definition 2.2.2, for $n \in \mathbf{Z}$. Recall from Lemmas 2.1.5, 2.1.7, 2.1.9, 2.1.11, 2.2.6 that these operations map between the following spaces:

$$\begin{aligned} \circ_n &: L^2(\mathbf{R}^d) \times l^1(\mathbf{Z}^d) \to L^2(\mathbf{R}^d); \\ [\ ,\ ]_n &: L^2(\mathbf{R}^d) \times L^2(\mathbf{R}^d) \to C_0(\mathbf{Z}^d); \\ [\![\ ,\ ]\!]_n &: L^2(\widehat{\mathbf{R}}^d) \times L^2(\widehat{\mathbf{R}}^d) \to L^1(\mathbf{T}^d); \\ \widehat{\circ_n} &: L^2(\widehat{\mathbf{R}}^d) \times C(\mathbf{T}^d) \to L^2(\widehat{\mathbf{R}}^d). \end{aligned}$$

We shall use the notation $\widehat{\mathbf{R}^d}$ to indicate that we are working in the Fourier domain. Recall that Lemma 2.1.12 relates $[\ ,\ ]_n$ and $\circ_n$ to $[\![\ ,\ ]\!]_n$ and $\widehat{\circ_n}$ via the Fourier transform. Theorem 2.2.6 tells us that we can use the above operations to construct isomorphic Hilbert $C^*(\mathbf{Z}^d)$-modules $X_n \subset L^2(\mathbf{R}^d)$, and Hilbert $C(\mathbf{T}^d)$-modules $\hat{X}_n \subset L^2(\widehat{\mathbf{R}^d})$. Recall from Proposition 2.2.8 that the Hilbert modules $(X_n)_{n\in\mathbf{Z}}$ all share the same linear space, so the difference between them is in how the inner product and module action are defined. In Definition 2.3.5 we construct a Hilbert $L^\infty(\mathbf{T}^d)$-module Y_n which contains $\hat{X}_n$ as a subset, and whose $L^\infty(\mathbf{T}^d)$-valued inner product is given by $[\![\ ,\ ]\!]_n$, with module action given by $\widehat{\circ_n}$.

3.1 The Downsampling Operator, P

Let us consider a multiresolution structure $(\Gamma, \mathcal{D})$ acting on a Hilbert space $\mathcal{H}$. Recall from page 22 that we defined $\Delta \in \mathrm{Hom}(\Gamma)$ to be $\Delta(\gamma) = \mathcal{D}^{-1}\gamma\mathcal{D}$ (as a unitary operator on $\mathcal{H}$), where the translation $\gamma \in \Gamma$ is thought of as a unitary operator on $\mathcal{H}$. Recall from page 64 that we defined a dual homomorphism $\hat{\Delta} \in \mathrm{Hom}(\mathbf{T}^d)$, satisfying $(\Delta(\gamma), \zeta) = (\gamma, \hat{\Delta}(\zeta))$, for $\gamma \in \mathbf{Z}^d, \zeta \in \mathbf{T}^d$.

Definition 3.1.1 Suppose that a is a function on the group of translations Γ. The *downsampling operator* P maps a to another function on Γ which is given by

$$(Pa)(\gamma) = a(\Delta\gamma), \quad \gamma \in \Gamma. \tag{3.1}$$

Suppose that $a \in C^*(\Gamma)$. Then there exists a net (a_i) in $C_c(\Gamma)$ such that $a_i \to a$. We define $P(a)$ to be the limit of $P(a_i)$.

For the rest of this section, let us assume that the above multiresolution structure is the standard multiresolution structure associated with dilation matrix $\tilde{\mathcal{D}}$ as described in 3.0.1. It follows from equation (1.3) that $\Delta(\gamma) = \tilde{\mathcal{D}}(\gamma)$, for $\gamma \in \mathbf{Z}^d$.

Lemma 3.1.2 *For $f, g \in X_n$, $P[f,g]_n = [f,g]_{n-1}$.*

PROOF: We compute for $\gamma \in \mathbf{Z}^d$,

$$(P[f,g]_n)(\gamma) = [f,g]_n(\Delta(\gamma)).$$

Now from the defintion of $[f,g]_n$,

$$\begin{aligned}
[f,g]_n(\Delta(\gamma)) &= \int_{\mathbf{R}^d} \overline{f(x - \tilde{\mathcal{D}}^{-n}\iota(\Delta(\Gamma)))}g(x)dx \\
&= \int_{\mathbf{R}^d} \overline{f(x - \tilde{\mathcal{D}}^{-(n-1)}\iota(\Gamma))}g(x)dx \\
&= [f,g]_{n-1}(\gamma).
\end{aligned}$$

□

Lemma 3.1.3 *Suppose that $\alpha_0, \ldots, \alpha_{m-1}$ are a set of coset representatives of $\Delta(\mathbf{Z}^d)$ in $\mathbf{Z}^d$, and $a, b \in C^*(\mathbf{Z}^d)$, then*

$$\sum_{i=0}^{m-1} (P\alpha_i a)(P\alpha_i b)^* = P(ab^*). \tag{3.2}$$

PROOF: Suppose that $a, b \in C_c(\mathbf{Z}^d)$, then for $\gamma \in \mathbf{Z}^d$,

$$\begin{aligned}
\left(\sum_{i=0}^{m-1} (P\alpha_i a)(P\alpha_i b)^*\right)(\gamma) &= \sum_{i=0}^{m-1} \sum_{\beta \in \mathbf{Z}^d} (P\alpha_i a)(\beta)(P\alpha_i b)^*(\gamma - \beta) \\
&= \sum_{i=0}^{m-1} \sum_{\beta \in \mathbf{Z}^d} a(\Delta\beta - \alpha_i)\overline{b(\Delta\gamma - \Delta\beta + \alpha_i)} \\
&= \sum_{\beta \in \mathbf{Z}^d} a(\beta)\overline{b(\beta - \Delta\gamma)} \\
&= P(ab^*)(\gamma).
\end{aligned}$$

Equation (3.2) is satisfied for arbitrary $a, b \in C^*(\mathbf{Z}^d)$ by continuity. □

We shall now examine the Fourier transform of P, which is given by

$$\hat{P} = \mathcal{F} P \mathcal{F}^*.$$

Because the Fourier transform maps $l^2(\mathbf{Z}^d)$ to $L^2(\mathbf{T}^d)$ and maps $l^1(\mathbf{Z}^d)$ to $C(\mathbf{T}^d)$, $\hat{P}$ can be thought of as a mapping from $L^2(\mathbf{T}^d)$ to itself, or as a mapping from $C(\mathbf{T}^d)$ to itself.

Proposition 3.1.4 *For all $a \in C(\mathbf{T}^d)$,*

$$(\hat{P}a)(\zeta) = \frac{1}{m} \sum_{\hat{\Delta}(\omega) = \zeta} a(\omega), \qquad \text{for } \zeta \in \mathbf{T}^d. \tag{3.3}$$

It follows that P is a positive map from $C^*(\mathbf{Z}^d)$ to itself.

Proof: For $\zeta \in \mathbf{T}^d$, there exist m choices of $\omega \in \mathbf{T}^d$ for which $\hat{\Delta}(\omega) = \zeta$. So for $\gamma \in \mathbf{Z}^d$,

$$\begin{aligned}
\left(\mathcal{F}_{\mathbf{T}^d}(\hat{P}a)\right)(\gamma) &= \int_{\zeta \in \mathbf{T}^d} (\hat{P}a)(\zeta)(-\gamma, \zeta) d\zeta \\
&= \int_{\zeta \in \mathbf{T}^d} a(\zeta)(-\Delta(\gamma), \zeta) d\zeta \\
&= \int_{\zeta \in \mathbf{T}^d} \frac{1}{m} \sum_{\hat{\Delta}(\omega)=\zeta} a(\omega)(-\Delta(\gamma), \omega) d\zeta \\
&= \int_{\zeta \in \mathbf{T}^d} \frac{1}{m} \sum_{\hat{\Delta}(\omega)=\zeta} a(\omega)(\gamma, \zeta) d\zeta.
\end{aligned}$$

It follows that $(\hat{P}a)(\zeta) = \frac{1}{m}\sum_{\hat{\Delta}(\omega)=\zeta} a(\omega)$. To see that P is positive, suppose that a is a positive element of $C^*(\mathbf{Z}^d)$. Then for all $\zeta \in \mathbf{T}^d$, $\hat{a}(\zeta) \geq 0$. It follows from (3.3) that $\hat{P}\hat{a}(\zeta) \geq 0$, verifying the result. □

We remark that one can also verify (3.3) by using equation (2.46) and noting that X_n is full (which is a consequence of Theorem 2.2.6).

The above proposition motivates us to define for $a \in L^\infty(\mathbf{T}^d)$,

$$(\hat{P}a)(\zeta) := 1/m \sum_{\hat{\Delta}(\omega)=\zeta} a(\omega).$$

The following calculation verifies that for $p, q \in Y_n$, $\hat{P}[\![p,q]\!]_n = [\![p,q]\!]_{n-1}$ (as one would expect).

$$\begin{aligned}
[\![p,q]\!]_{n-1}(\zeta) &= m^{n-1} \sum_{\hat{\iota_{n-1}}(\xi)=\zeta} \overline{p(\xi)} q(\xi) \\
&= m^{n-1} \sum_{\hat{\Delta}(\hat{\iota_n}(\xi))=\zeta} \overline{p(\xi)} q(\xi) \\
&= m^{n-1} \sum_{\hat{\Delta}(\omega)=\zeta} \sum_{\hat{\iota_n}(\xi))=\omega} \overline{p(\xi)} q(\xi) \\
&= \hat{P}[\![p,q]\!]_n.
\end{aligned}$$

Example 3.1.5 Consider the multiresolution structure of Example 1.1.3. In this examnple $G = \mathbf{R}$, $\Gamma = \mathbf{Z}$ and $\mathcal{D}$ is multiplication by 2. In this case we have that $\Delta\gamma = 2\gamma$, and so for $a \in C^*(\mathbf{Z})$, $(Pa)(\gamma) = a(2\gamma)$. In the Fourier domain we have

$$(\gamma, \hat{\Delta}\zeta) = (\Delta\gamma, \zeta) = (2\gamma, \zeta) = e^{4\pi i \gamma\zeta}$$

and so $\hat{\Delta}\zeta = 2\zeta$. We therefore have that

$$(\hat{P}\hat{a})(\zeta) = \frac{1}{2}\left(a(\frac{\zeta}{2}) + a(\frac{\zeta}{2} + \frac{1}{2})\right).$$

We will now examine the behaviour of the downsampling operator P on the Hilbert space $l^2(\mathbf{Z}^d)$. We have that $P \in B(l^2(\mathbf{Z}^d))$ and has norm equal to 1. The adjoint P^* of P as an operator on $l^2(\mathbf{Z}^d)$ is the operator that satisfies $\langle Pa, b\rangle = \langle a, P^*b\rangle$ where $a, b \in l^2(\mathbf{Z}^d)$. We shall call P^* an *upsampling operator*. It can be verified by a routine calculation that it is given by

$$(P^*a)(\gamma) = \begin{cases} a(\alpha) & \text{if there exists } \alpha \text{ such that } \gamma = \Delta\alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

We use (3.4) to define the upsampling operator P^* on $C_c(\mathbf{Z}^d)$, we can then use continuity to extend its definition to $C^*(\mathbf{Z}^d)$.

Lemma 3.1.6 *Suppose $a, b \in C^*(\mathbf{Z}^d)$, and $\alpha_0, \ldots, \alpha_{m-1}$ is a set of coset representatives of $\Delta\mathbf{Z}^d$ in $\mathbf{Z}^d$. Then $a = (P^*b)c$ if and only if for all i, $P\alpha_i a = b(P\alpha_i c)$.*

PROOF: We have

$$\begin{aligned} ((P^*b)c)(\gamma) &= \sum_{\beta\in\mathbf{Z}^d} (P^*b)(\beta)c(\gamma-\beta) \\ &= \sum_{\beta\in\mathbf{Z}^d} b(\beta)c(\gamma-\Delta\beta) \end{aligned}$$

and

$$\begin{aligned} (b(P\alpha_i c))(\gamma) &= \sum_{\beta\in\mathbf{Z}^d} b(\beta)(P\alpha_i c)(\gamma-\beta) \\ &= \sum_{\beta\in\mathbf{Z}^d} b(\beta)c(\Delta\gamma-\Delta\beta-\alpha_i) \\ &= ((P^*b)c)(\Delta\gamma-\alpha_i). \end{aligned}$$

This calculation tells us that if $a = (P^*b)c$ then for all i, $P\alpha_i a = b(P\alpha_i c)$. This converse to this result follows from the above calculation and the fact that $\alpha_0, \ldots, \alpha_{m-1}$ is a set of coset representatives of $\Delta\mathbf{Z}^d$ in $\mathbf{Z}^d$. □

3.2 Filters and Wavelet Matrices

Let us now examine multiwavelets for which there exists a multiresolution analysis with a single scaling function φ. These wavelets are often known as *higher multiplicity wavelets* when m is greater than 2. In this case the operators associated with the fast wavelet transform and the cascade algorithm are much easier to define than in the general case. We shall return to the case that there is more than one scaling function in Section 3.5.

We shall now define some important functions on $\mathbf{Z}^d$ associated with the scaling function and wavelets.

Definition 3.2.1 Let $\psi_1, \ldots \psi_{r-1}$ be a multiwavelet with scaling function φ. We define

$$h = [\varphi, \mathcal{D}^{-1}\varphi]_0 = [\mathcal{D}^n\varphi, \mathcal{D}^{n-1}\varphi]_n \tag{3.5}$$
$$g^i = [\varphi, \mathcal{D}^{-1}\psi^i]_0 = [\mathcal{D}^n\varphi, \mathcal{D}^{n-1}\psi^i]_n \tag{3.6}$$

for $i = 1, \ldots, m-1$. We call h and g^i the *scaling filter* and the *wavelet filters.*

In much of the wavelet literature, scaling filters are known as *low pass filters* and wavelet filters are known as *high pass filters.* The term *filter* is also used to refer to the Fourier transforms of h and g^i.

Because V_{n-1} and W_{n-1} are contained in V_n, we can write

$$\mathcal{D}^{n-1}\varphi = \mathcal{D}^n\varphi \circ_n h = \mathcal{D}^n\varphi \circ_n [\mathcal{D}^n\varphi, \mathcal{D}^{n-1}\varphi]_n \tag{3.7}$$
$$\mathcal{D}^{n-1}\psi^i = \mathcal{D}^n\varphi \circ_n g^i = \mathcal{D}^n\varphi \circ_n [\mathcal{D}^n\varphi, \mathcal{D}^{n-1}\psi^i]_n \tag{3.8}$$

The above equations are the Hilbert module versions of the scaling and wavelet equations. We shall now show that $\|h\|_2 = \|g^i\|_2 = 1$. We know that $\|\mathcal{D}^{-1}\varphi\|_2 = 1$, and so by (3.7), $\|\sum_{\gamma\in\mathbf{Z}^d} h(\gamma)\varphi(\cdot-\gamma)\|_2 = 1$. Now because the translations of φ by $\mathbf{Z}^d$ form an orthonormal set, it follows from Pythagoras' theorem that $\sum_{\gamma\in\mathbf{Z}^d}|h(\gamma)|^2 = \|\sum_{\gamma\in\mathbf{Z}^d} h(\gamma)\varphi(\cdot-\gamma)\|_2$. Therefore $\|h\|_2 = \sum_{\gamma\in\mathbf{Z}^d}|h(\gamma)|^2 = 1$. The proof that $\|g^i\|_2 = 1$ is the same.

Now we know from Proposition 2.3.7 that φ and ψ^i are contained in Y_0. So by Proposition 2.3.6, $\mathcal{D}^{-1}\varphi$ and $\mathcal{D}^{-1}\psi^i$ are also contained in Y_0. It therefore follows that $\hat{h}$ and $\hat{g^i}$ are contained in $L^\infty(\mathbf{T}^d)$.

Example 3.2.2 Recall we described the Haar wavelet $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)} \in L^2(\mathbf{R})$ in Example 1.1.3. The Haar wavelet corresponds to the scaling function $\varphi = \chi_{[0,1)}$. We have that $\mathcal{D}^{-1}\varphi = 2^{-1/2}\chi_{[0,2)}$ and $\mathcal{D}^{-1}\psi = 2^{-1/2}(\chi_{[0,1)} - \chi_{[1,2)})$. We calculate the scaling filter

$$\begin{aligned} h(k) = [\varphi, \mathcal{D}^{-1}\varphi]_0(k) &= \int_{\mathbf{R}} \overline{\varphi(x)}\mathcal{D}^{-1}\varphi(x-k)dx \\ &= 2^{-1/2}\int_{\mathbf{R}} \overline{\chi_{[0,1)}(x)}\chi_{[0,2)}(x-k)dx \\ &= 2^{-1/2}\int_{\mathbf{R}} \overline{\chi_{[0,1)}(x)}\chi_{[k,2+k)}(x)dx \end{aligned}$$

for $k \in \mathbf{Z}$. It therefore follows that $h = 2^{-1/2}(e_0 + e_{-1})$, where e_i is the element of $C_c(\mathbf{Z})$ for which $e_i(i) = 1$ and $e_i(j) = 0$ when $j \neq i$. We similarly calculate the wavelet filter to be

$$g = [\varphi, \mathcal{D}^{-1}\psi]_0 = 2^{-1/2}(e_0 - e_{-1}).$$

Note that in this case both h and g are contained in $C_c(\mathbf{Z})$, and in particular are contained in $C^*(\mathbf{Z})$.

Example 3.2.3 Let us now examine the Shannon wavelet that we described in Example 1.1.9. In this case $\hat{\varphi} = \chi_{[-1/2,1/2)}$, and $\hat{\psi}(\xi) = e^{i\xi/2}\chi_{[-1,1/2)\cup(1/2,1]}(\xi)$ for $\xi \in \mathbf{R}$. It follows from Corollary 1.2.6 that $\widehat{\mathcal{D}^{-1}\varphi} = \sqrt{2}\chi_{[-1/4,1/4)}$ and $\widehat{\mathcal{D}^{-1}\psi}(\xi) = \sqrt{2}e^{i\xi}\chi_{[-1/2,1/4)\cup(1/4,1/2]}(\xi)$ for $\xi \in \mathbf{R}$. We make use of Lemma 2.1.12 to calculate

$$\begin{aligned}\hat{h}(\zeta) = [\![\hat{\varphi}, \widehat{\mathcal{D}^{-1}\varphi}]\!]_0 &= \sqrt{2}\sum_{k\in\mathbf{Z}} \overline{\chi_{[-1/2,1/2)}(\zeta+k)}\chi_{[-1/4,1/4)}(\zeta+k) \\ &= \sqrt{2}\chi_{[-1/4,1/4)}(\zeta).\end{aligned}$$

And

$$\begin{aligned}\hat{g}(\zeta) = [\![\hat{\varphi}, \widehat{\mathcal{D}^{-1}\psi}]\!]_0 &= \sqrt{2}\sum_{k\in\mathbf{Z}} \overline{\chi_{[-1/2,1/2)}(\zeta+k)}e^{i\zeta}\chi_{[-1/2,-1/4)\cup(1/4,1/2]}(\zeta+k) \\ &= \sqrt{2}e^{i\zeta}\chi_{[-1/2,-1/4)\cup(1/4,1/2]}.\end{aligned}$$

Note that in this case $\hat{h}$ and $\hat{g}$ are contained in $L^2(\mathbf{T})\cap L^\infty(\mathbf{T})$, but are not contained in $C(\mathbf{T})$. So in particular, h and g are not contained in $C^*(\mathbf{Z})$. We know from Propositon 2.2.8 that if $\varphi \in X_0$, then $\mathcal{D}^{-1}\varphi \in X_0$ (because $\mathcal{D}^{-1}\varphi \in X_{-1}$ and X_0 shares the same linear space as X_1). So since $\hat{h} \notin C^*(\mathbf{Z})$, $\varphi \notin X_0$. This example is important because it illustrates that not all wavelets are contained in X_0, and hence not all MRA-wavelets correspond to filters contained in $C^*(\mathbf{Z}^d)$. However, most wavelets in applications are compactly supported and have finitely supported filters (see e.g. [Da1]).

The equations in the following proposition are the Hilbert module version of the *shifted orthogonality conditions*. This result is actually a special case of Proposition 1.4.2. We prove it again to illustrate the simplicity of the Hilbert module notation (especially for performing calculations).

Proposition 3.2.4 *Suppose that for $i = 1, \ldots, m-1$, h and g^i are scaling and wavelet filters for the scaling function φ and wavelets ψ^i, then they satisfy:*

$$P(hh^*) = \mathbf{1}, \tag{3.9}$$

$$P(g^i g^{j*}) = \delta_{i,j}\mathbf{1}, \tag{3.10}$$

$$P(g^i h^*) = 0 \tag{3.11}$$

where $\delta_{i,j}$ is the Kronecker delta, and $\mathbf{1}$ is the unit function on $\mathbf{Z}^d$ (i.e. $\mathbf{1}$ takes the value 1 at the origin, and zero everywhere else).

Proof: Using the fact that $[\varphi,\varphi]_0 = \mathbf{1}$, and using Equations (3.7) and (3.8), we have

$$[\mathcal{D}^{-1}\varphi, \mathcal{D}^{-1}\varphi]_0 = [\varphi\circ h, \varphi\circ h]_0 = [\varphi,\varphi]_0 hh^* = hh^*;$$

$$[\mathcal{D}^{-1}\psi^j, \mathcal{D}^{-1}\psi^i]_0 = [\varphi\circ g^j, \varphi\circ g^i]_0 = [\varphi,\varphi]_0 g^i g^{j*} = g^i g^{j*};$$

$$[\mathcal{D}^{-1}\varphi, \mathcal{D}^{-1}\psi^i]_0 = [\varphi\circ h, \varphi\circ g^i]_0 = [\varphi,\varphi]_0 g^i h^* = g^i h^*.$$

From orthogonality we obtain

$$\begin{aligned} P[\mathcal{D}^{-1}\varphi, \mathcal{D}^{-1}\varphi]_0 = [\mathcal{D}^{-1}\varphi, \mathcal{D}^{-1}\varphi]_{-1} &= \mathbf{1}, \\ P[\mathcal{D}^{-1}\psi^j, \mathcal{D}^{-1}\psi^i]_0 = [\mathcal{D}^{-1}\psi^j, \mathcal{D}^{-1}\psi^i]_{-1} &= \mathbf{1}\delta_{i,j}, \\ P[\mathcal{D}^{-1}\varphi, \mathcal{D}^{-1}\psi^i]_0 = [\mathcal{D}^{-1}\varphi, \mathcal{D}^{-1}\psi^i]_{-1} &= 0. \end{aligned}$$

And so by substituting the previous set of equations into the above set of equations we get

$$\begin{aligned} P(hh^*) &= \mathbf{1}; \\ P(g^i g^{j*}) &= \delta_{i,j}\mathbf{1}; \\ P(g^i h^*) &= 0. \end{aligned}$$

□

Using Proposition 3.1.4, we can write the shifted orthogonality conditions in the following way. For all $\zeta \in \mathbf{T}^d$,

$$\frac{1}{m}\sum_{\hat{\Delta}(\omega)=\zeta} \hat{h}(\omega)\overline{\hat{h}(\omega)} = 1, \tag{3.12}$$

$$\frac{1}{m}\sum_{\hat{\Delta}(\omega)=\zeta} \hat{g^i}(\omega)\overline{\hat{g^j}(\omega)} = \delta_{i,j}, \tag{3.13}$$

$$\frac{1}{m}\sum_{\hat{\Delta}(\omega)=\zeta} \hat{g^i}(\omega)\overline{\hat{h}(\omega)} = 0. \tag{3.14}$$

Example 3.2.5 In the multiresolution structure of Example 1.1.3 (where $G = \mathbf{R}$ and $\Gamma = \mathbf{Z}$) we can write the shifted orthogonality conditions as

$$\begin{aligned} \sum_{\alpha\in\mathbf{Z}} h(\alpha)h(\alpha - 2\gamma) &= \delta_{\gamma,0} \\ \sum_{\alpha\in\mathbf{Z}} g^i(\alpha)g^j(\alpha - 2\gamma) &= \delta_{i,j}\delta_{\gamma,0} \\ \sum_{\alpha\in\mathbf{Z}} g^i(\alpha)h(\alpha - 2\gamma) &= 0 \end{aligned}$$

for scaling and wavelet filters h, g^i contained in $C_c(\mathbf{Z}^d)$, and $\gamma \in \mathbf{Z}$. The Fourier transformed shifted orthogonality conditions can be written as

$$\begin{aligned} \hat{h}(\zeta/2)\overline{\hat{h}(\zeta/2)} + \hat{h}(\zeta/2+1/2)\overline{\hat{h}(\zeta/2+1/2)} &= 2 \\ \hat{g^i}(\zeta/2)\overline{\hat{g^j}(\zeta/2)} + \hat{g^i}(\zeta/2+1/2)\overline{\hat{g^j}(\zeta/2+1/2)} &= 2\delta_{i,j} \\ \hat{g^i}(\zeta/2)\overline{\hat{h}(\zeta/2)} + \hat{g^i}(\zeta/2+1/2)\overline{\hat{h}(\zeta/2+1/2)} &= 0 \end{aligned}$$

for all $\zeta \in [-1/2, 1/2) \cong \mathbf{T}$.

We now define the *scaling operator* H and *wavelet operators* $(G^i)_{i=1,\dots,m-1}$ which map functions on $\mathbf{Z}^d$ to functions on $\mathbf{Z}^d$ by

$$H(a) = P(ah^*), \tag{3.15}$$
$$G^i(a) = P(ag^{i*}); \tag{3.16}$$

with multiplication being convolution on $\mathbf{Z}^d$. Note that if h (or g^i) is an element of $C^*(\mathbf{Z}^d)$ then because P maps $C^*(\mathbf{Z}^d)$ to itself, H (or G^i) map elements of $C^*(\mathbf{Z}^d)$ to $C^*(\mathbf{Z}^d)$. Let us also investigate how these operators act on $l^2(\mathbf{Z}^d)$. It follows from the Fourier transformed shifted orthogonality conditions that for all ω, $\hat{h}(\omega) < m$. It therefore follows from the Plancharel identity that $\|H(a)\|_2 \leq m\|a\|_2$ for $a \in l^2(\mathbf{Z}^d)$, and so $H \in B(l^2(\mathbf{Z}^d))$. By the same argument $G^i \in B(l^2(\mathbf{Z}^d))$ for $i = 1, \dots, m-1$.

We can write the shifted orthogonality conditions in terms of these operators as

$$H(h) = \mathbf{1}, \tag{3.17}$$
$$G^i(g^j) = \delta_{i,j}\mathbf{1}, \tag{3.18}$$
$$H(g^i) = G^i(h) = 0. \tag{3.19}$$

We can use Proposition 3.1.4 to write the Fourier transform of H and G^i as

$$(\hat{H}a)(\zeta) = \frac{1}{m}\sum_{\hat{\Delta}(\omega)=\zeta} a(\omega)\overline{\hat{h}(\omega)}, \tag{3.20}$$
$$(\hat{G^i}a)(\zeta) = \frac{1}{m}\sum_{\hat{\Delta}(\omega)=\zeta} a(\omega)\overline{\hat{g^i}(\omega)}, \tag{3.21}$$

where $a \in C(\mathbf{T}^d)$.

The operators H and G^i are examples of what we shall call *filtering operators*. If $b \in C^*(\mathbf{Z}^d)$, we define the filtering operator F_b associated with b, $F_b : C^*(\mathbf{Z}^d) \to C^*(\mathbf{Z}^d)$ to be

$$F_b a = P(ab^*) \tag{3.22}$$

The filtering operators that we have defined consist of a convolution followed by a downsampling. We can write the scaling and wavelet filters as $H = F_h$ and $G^i = F_{g^i}$.

Lemma 3.2.6 *For $a, b \in C^*(\mathbf{Z}^d)$,*

$$\|F_b a\|_{C^*(\mathbf{Z}^d)} \leq \frac{1}{m}\sup_{\zeta\in\mathbf{T}^d}\sum_{\hat{\Delta}(\omega)=\zeta}|\hat{b}(\omega)|\|a\|_{C^*(\mathbf{Z}^d)}.$$

PROOF: By definition we have

$$\|a\|_{C^*(\mathbf{Z}^d)} = \sup_{\zeta\in\mathbf{T}^d}\hat{a}(\zeta).$$

So

$$\begin{aligned}
\|F_b a\|_{C^*(\mathbf{Z}^d)} &= \sup_{\zeta\in\mathbf{T}^d} \frac{1}{m} \sum_{\hat{\Delta}(\omega)=\zeta} \hat{a}(\omega)\overline{\hat{b}(\omega)} \\
&\leq \frac{1}{m} \sup_{\zeta\in\mathbf{T}^d} \sum_{\hat{\Delta}(\omega)=\zeta} \sup_{\xi\in\mathbf{T}^d} \hat{a}(\xi)\hat{b}(\omega) \\
&= \frac{1}{m} \sup_{\zeta\in\mathbf{T}^d} \sum_{\hat{\Delta}(\omega)=\zeta} |\hat{b}(\omega)| \|a\|_{C^*(\mathbf{Z}^d)}.
\end{aligned}$$

□

It is sometimes also possible to define a filtering operator F_b when $b \notin C^*(\mathbf{Z}^d)$, using (3.22). When $\hat{b} \in L^\infty(\mathbf{T}^d)$, F_b will be a bounded operator on $l^2(\mathbf{Z}^d)$. Let us now calculate the adjoint of F_b as an operator on $l^2(\mathbf{Z}^d)$.

Lemma 3.2.7 *Suppose that $F_b \in B(l^2(\mathbf{Z}^d))$. The adjoint of the filtering operator F_b as an operator on $l^2(\mathbf{Z}^d)$ is given by*

$$F_b^* a = (P^* a) b^*. \tag{3.23}$$

PROOF: We need to check that $\langle F_b a, c\rangle = \langle a, F_b^* c\rangle$, for $a, b, c \in l^2(\mathbf{Z}^d)$, and F_b defined by Equation 3.23. We have

$$\begin{aligned}
\langle F_b a, c\rangle &= \langle P(ab^*), c\rangle \\
&= \langle ab^*, P^* c\rangle \\
&= \langle a, (P^* c) b\rangle \\
&= \langle a, F_b^* c\rangle
\end{aligned}$$

proving the assertion. □

It is interesting to formulate the shifted orthogonality conditions in terms of what are known as *wavelet matrices*. Wavelet matrices are extensively used in the the study of wavelets from an algebraic perspective [Tu], and in the study of subband coding [SN]. An important development in the theory of wavelet matrices was in a proof by Gröchenig in 1987 [Gr] of an m-dimensional version of the theorem of the existence of wavelets given a scaling function (see Theorem 1.1.11). To simplify notation, write $g^0 := h$, the shifted orthogonality conditions can then be written as

$$P(g^i g^{j*}) = \delta_{ij}\mathbf{1}, \text{ where } i, j = 0, \ldots, m-1. \tag{3.24}$$

Let $\alpha_0, \ldots, \alpha_{m-1}$ be a set of coset representatives of $\Delta\mathbf{Z}^d$ in $\mathbf{Z}^d$, and set $\alpha_0 = 0$. For $\gamma \in \mathbf{Z}^d$, define $A(\gamma) \in M_m(\mathbf{C})$ by

$$A_\gamma = \begin{pmatrix} g^0(\Delta\gamma+\alpha_0) & g^0(\Delta\gamma+\alpha_1) & \cdots & g^0(\Delta\gamma+\alpha_{m-1}) \\ g^1(\Delta\gamma+\alpha_0) & g^1(\Delta\gamma+\alpha_1) & \cdots & g^1(\Delta\gamma+\alpha_{m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ g^{m-1}(\Delta\gamma+\alpha_0) & g^{m-1}(\Delta\gamma+\alpha_1) & \cdots & g^{m-1}(\Delta\gamma+\alpha_{m-1}) \end{pmatrix}. \tag{3.25}$$

We have just defined a "sequence" of matrices indexed by $\mathbf{Z}^d$, because this is a tensor product construction, it is the same as defining a matrix of sequences which are indexed by $\mathbf{Z}^d$. With this discussion in mind we define our *wavelet matrix* A as follows

$$A = \begin{pmatrix} P(\alpha_0 g^0) & P(\alpha_1 g^0) & \cdots & P(\alpha_{m-1} g^0) \\ P(\alpha_0 g^1) & P(\alpha_1 g^1) & \cdots & P(\alpha_{m-1} g^1) \\ \vdots & \vdots & \ddots & \vdots \\ P(\alpha_0 g^{m-1}) & P(\alpha_1 g^{m-1}) & \cdots & P(\alpha_{m-1} g^{m-1}) \end{pmatrix}. \tag{3.26}$$

This definition of A is equivalent to the previous definition of A (3.25).

Remark 3.2.8 Recall from Examples 0.4.2 that for an arbitrary C^*-algebra $\mathcal{A}$, $\mathcal{A}^m$ is a Hilbert $\mathcal{A}$-module. It is known (see Chapter 15 of [W-O]) that it satisfies

$$\mathcal{L}(\mathcal{A}^m) \cong M_m(\mathcal{M}(\mathcal{A})), \text{ and } \mathcal{K}(\mathcal{A}^m) \cong M_m(\mathcal{A})$$

where $\mathcal{M}(\mathcal{A})$ is the multiplier algebra of $\mathcal{A}$. The multiplier algebra of a C^*-algebra $\mathcal{A}$ is a certain unital C^*-algebra which contains $\mathcal{A}$, and when $\mathcal{A}$ is unital, $\mathcal{M}(\mathcal{A}) = \mathcal{A}$. So if $\mathcal{A}$ is unital, $\mathcal{L}(\mathcal{A}^m) = \mathcal{K}(\mathcal{A}^m) \cong M_m(\mathcal{A})$. From now on assume that $\mathcal{A}$ is unital. The isomorphism between $\mathcal{K}(\mathcal{A}^m)$ and $M_m(\mathcal{A})$ is given by

$$\Theta_{(a_0,\ldots,a_{m-1}),(b_0,\ldots,b_{m-1})} \mapsto \begin{pmatrix} a_0 b_0^* & \cdots & a_0 b_m^* \\ \vdots & \ddots & \vdots \\ a_{m-1} b_0^* & \cdots & a_{m-1} b_{m-1}^* \end{pmatrix}.$$

We have that $\mathcal{A}^m$ is a right Hilbert module over $\mathcal{A}$, and a left Hilbert module over the C^*-algebra $\mathcal{L}(\mathcal{A}^m) = \mathcal{K}(\mathcal{A}^m)$. The space $\mathcal{A}^m$ is a left Hilbert $\mathcal{K}(\mathcal{A})$-module with the inner product $[\mathbf{a}, \mathbf{b}]_{\mathcal{K}(\mathcal{A}^m)} = \Theta_{\mathbf{a},\mathbf{b}}$. It is worth noting that this construction shows that the C^*-algebra $\mathcal{K}(\mathcal{A}^m)$ is Morita equivalent to $\mathcal{A}$. In the situation that $h, g^i \in C^*(\mathbf{Z}^d)$ we then have that $A \in \mathcal{K}(C^*(\mathbf{Z}^d)^m)$.

Lemma 3.2.9 *Let $\alpha_0, \ldots, \alpha_{m-1}$ be a set of coset representatives of $\Delta\mathbf{Z}^d$ in $\mathbf{Z}^d$, and also let $\alpha_0 = 0$. Let $\omega_0, \ldots, \omega_{m-1}$ be separate elements of $\mathbf{T}^d$ which satisfy $\hat{\Delta}(\omega_i) = 0$ for all i, and set $\omega_0 = 0$. The following three statements are equivalent:*

1. *The shifted orthogonality conditions are satisfied, i.e.*

$$P(g^i g^j) = \delta_{ij}\mathbf{1}, \text{ where } i, j = 0, \ldots, m-1.$$

2. *The operator given by*

$$A = \begin{pmatrix} P(\alpha_0 g^0) & P(\alpha_1 g^0) & \cdots & P(\alpha_{m-1} g^0) \\ P(\alpha_0 g^1) & P(\alpha_1 g^1) & \cdots & P(\alpha_{m-1} g^1) \\ \vdots & \vdots & \ddots & \vdots \\ P(\alpha_0 g^{m-1}) & P(\alpha_1 g^{m-1}) & \cdots & P(\alpha_{m-1} g^{m-1}) \end{pmatrix} \tag{3.27}$$

is unitary.

3. The operator B given by

$$B(\zeta) = m^{-1/2} \begin{pmatrix} \hat{g^0}(\zeta+\omega_0) & \hat{g^0}(\zeta+\omega_1) & \cdots & \hat{g^0}(\zeta+\omega_{m-1}) \\ \hat{g^1}(\zeta+\omega_0) & \hat{g^1}(\zeta+\omega_1) & \cdots & \hat{g^1}(\zeta+\omega_{m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{g^{m-1}}(\zeta+\omega_0) & \hat{g^{m-1}}(\zeta+\omega_1) & \cdots & \hat{g^{m-1}}(\zeta+\omega_{m-1}) \end{pmatrix} \tag{3.28}$$

is unitary.

PROOF: The operator A is unitary if and only if $AA^* = \mathbf{1} = A^*A$. So A is unitary if and only if

$$\begin{pmatrix} \sum_{i=0}^{m-1} P(\alpha_i g^0)P(\alpha_i g^0)^* & \cdots & \sum_{i=0}^{m-1} P(\alpha_i g^0)P(\alpha_i g^{m-1})^* \\ \vdots & \ddots & \vdots \\ \sum_{i=0}^{m-1} P(\alpha_i g^{m-1})P(\alpha_i g^0)^* & \cdots & \sum_{i=0}^{m-1} P(\alpha_i g^{m-1})P(\alpha_i g^{m-1})^* \end{pmatrix} = \mathbf{1}.$$

This is the case if and only if

$$\sum_{\gamma\in\mathbf{Z}^d} g^i(\gamma)\overline{g^j(\gamma+\Delta\lambda)} = \delta_{i,j}\delta_{\lambda,0} \quad \text{where } i = 0,\ldots,m-1, \text{ and } \lambda \in \mathbf{Z}^d,$$

which is equivalent to the shifted orthogonality conditions.

Now examine the operator B. The Fourier transform of the shifted orthogonality conditions is given by

$$\frac{1}{m}\sum_{\hat{\Delta}\omega=\zeta} \hat{g^i}(\omega)\overline{\hat{g^j}(\omega)} = \delta_{i,j}. \tag{3.29}$$

We can write this as

$$\frac{1}{m}\sum_{k=0}^{m-1} \hat{g^i}(\zeta+\omega_k)\overline{\hat{g^j}(\zeta+\omega_k)} = \delta_{i,j} \tag{3.30}$$

for all $\zeta \in \mathbf{T}^d$. By inspection the above equation is equivalent to B being unitary. □

Remark 3.2.10 In this remark we shall study for the sake of comparison some work done by J.A. Packer and M.A. Rieffel from [PR1] to study wavelet theory. We shall not prove any new results in this remark. The notation that we shall use will be based on the notation used in the rest of this thesis, rather than on [PR1]. The main difference between the Hilbert modules constructed in this thesis and the one constructed in [PR1] is that the Hilbert module they construct consists of functions on $\mathbf{Z}^d$ rather than on $\mathbf{R}^d$. This leads to filters being elements of the Hilbert module that they construct (rather than wavelets). This gives a different (but similar) perspective on filters to the one described above. We shall now describe how this Hilbert module is constructed.

Let $\tilde{\mathcal{D}}$ be a dilation matrix and consider the groups $\Gamma := \mathbf{Z}^d$ and $\Gamma^{-1} := \tilde{\mathcal{D}}(\mathbf{Z}^d)$. We are interested in the pre-C^*-algebras $C_c(\mathbf{Z}^d)$ and $C_c(\Gamma^{-1})$. The embedding of Γ^{-1} in $\mathbf{Z}^d$ makes $C_c(\Gamma^{-1})$ a subalgebra of $C_c(\mathbf{Z}^d)$. Let $\alpha_0,\ldots,\alpha_{m-1}$ be a set of coset representatives for cosets of Γ^{-1} in $\mathbf{Z}^d$. For each $i = 0,\ldots,m-1$ let e_i be the function on $\mathbf{Z}^d$ which takes the value 1 at α_i and 0 elsewhere. Each element f of $C_c(\mathbf{Z}^d)$ is the sum of its restrictions to the cosets of Γ^{-1} in $\mathbf{Z}^d$, and so there exist unique $(f_i)_{i=0,\ldots,m-1}$ each contained in $C_c(\Gamma^{-1})$ such that

$$f = \sum_i f_i * e_i.$$

By definition, $C_c(\mathbf{Z}^d)$ is a free $C_c(\Gamma^{-1})$-module of rank m, with the e_i's being a module basis. The mapping $f \mapsto (f_i)_i \in C_c(\Gamma^{-1})^q$ is a $C_c(\Gamma^{-1})$-module isomorphism from $C_c(\mathbf{Z}^d)$ to $C_c(\Gamma^{-1})$. Define a $C_c(\Gamma^{-1})$-valued inner product on $C_c(\mathbf{Z}^d)$ by setting

$$\begin{aligned} \langle f, g\rangle_{C_c(\Gamma^{-1})}(\gamma) &= (f^* * g)(\gamma) \\ &= \sum_{\alpha\in\mathbf{Z}^d} \overline{f(\alpha)} g(\alpha-\gamma). \end{aligned}$$

We then have that

$$f = \sum_i e_i \langle e_i, f\rangle_{C_c(\Gamma^{-1})}.$$

Let us now examine what happens in the Fourier domain. Both $\hat{\mathbf{Z}^d}$ and $\hat{\Gamma^{-1}}$ are isomorphic to $\mathbf{T}^d$. We shall only identify $\hat{\mathbf{Z}^d}$ with $\mathbf{T}^d$. Let F be the finite subgroup of $\mathbf{T}^d$ consisting of the characters of $\mathbf{Z}^d$ which take the value 1 on all of Γ^{-1}. The group F is the annihilator of Γ^{-1} in $\hat{\mathbf{Z}^d}$. It is the case that F is the dual of $\mathbf{Z}^d/\Gamma^{-1}$ and is finite of order m. The elements of F give us automorphisms of $C_c(\mathbf{Z}^d)$.

The Fourier transform of $C_c(\mathbf{Z}^d)$ embeds in $C(\mathbf{T}^d)$ which is a C^*-algebra. The group F acts on $C(\mathbf{T}^d)$ by translation. Elements of $C_c(\Gamma^{-1})$ are mapped by the Fourier transform to elements of $C(\mathbf{T}^d)$ which are invariant under translation by elements of F. We let $\mathcal{A}$ be the completion in $C(\mathbf{T}^d)$ of $C_c(\hat{\Gamma}^{-1})$. The $\mathcal{A}$-valued inner product on $C(\mathbf{T}^d)$ is given by

$$\langle f, g\rangle_{\mathcal{A}}(\zeta) := m^{-1} \sum_{w\in F} \overline{f(x-w)} g(x-w).$$

In [PR1] a *low-pass filter* is defined to be an element $\hat{h} \in C(\mathbf{T}^d)$ for which

$$\hat{h}(0) = m, \tag{3.31}$$
$$\langle \hat{h}, \hat{h}\rangle_{\mathcal{A}} = m, \tag{3.32}$$

where $m = |\det \tilde{\mathcal{D}}|$. A set of elements $\hat{g}^1, \ldots, \hat{g}^{m-1} \in C(\mathbf{T}^d)$ is defined to be a *high-pass filter family* corresponding to the low pass filter $\hat{h}$ if it satisifes the shifted orthogonality conditions.

If we compare this definiton to Definition 3.2.1 several differences are apparant. The main difference is that in [PR1], filters are examined from the perspective of them being arbitrary continuous functions on $\mathbf{T}^d$. We showed in Example 3.2.3 that not all scaling functions have continuous filters. We shall examine how to construct scaling functions and wavelets from arbitrary filters in Section 3.4, where we shall also prove an analogue of (3.31). The filters in [PR1] are also normalised differently (by a factor of $\sqrt{m}$). It is interesting that [PR1] demonstrates that the action of convoluting two functions on $\mathbf{Z}^d$ and then downsampling is in fact a form of $\mathcal{A}$-valued inner product. In [PR1] the correspondence given by Swan's theorem between vector bundles and Hilbert modules over commutative C^*-algebras is used to prove some interesting results about continuous filters.

3.3 The Fast Wavelet Transform and Representations of Cuntz Algebras

The filtering operators that we have just defined are used for the *fast wavelet transform.* We described the fast wavelet transform in Section 1.4. We return to the fast wavelet transform because the Hilbert module construction makes the proofs simpler and more illuminating, and because of its relation to representations of Cuntz algebras. We shall study some work done by Bratteli and Jorgensen relating the fast wavelet transform to representations of Cuntz algebras. We shall also look at how the fast wavelet transform relates to wavelet matrices.

The analysis part of the fast wavelet transform allows us to obtain the scaling and wavelet coefficients $[f, \mathcal{D}^n\varphi]_n$ and $[f, \mathcal{D}^n\psi^i]_n$ of a function f at a level n from the scaling coefficients at the next finer level $n+1$. We can iterate this process to obtain the wavelet coefficients at coarser levels. The synthesis part of the fast wavelet transform goes in the other direction and allows us to obtain the scaling coefficients at a particular level from the scaling and wavelets at the next coarser level.

Although we already more or less proved Propositions 3.3.1 and 3.3.2 in Section 1.4, when we proved Theorem 1.4.3, we prove them again to illustrate how our Hilbert module construction can simplify calculations. It is also interesting to state these results using our Hilbert module notation.

Proposition 3.3.1 (Fast Wavelet transform - Analysis Part) *Suppose that*

$$\{\psi^i\}_{i=1,\ldots,m-1} \in X_0$$

is a multiwavelet with a single scaling function φ. Then for arbitrary $f \in X_{n+1}$,

$$[\mathcal{D}^n\varphi, f]_n = H[\mathcal{D}^{n+1}\varphi, f]_{n+1}, \tag{3.33}$$

$$[\mathcal{D}^n\psi^i, f]_n = G^i[\mathcal{D}^{n+1}\varphi, f]_{n+1}. \tag{3.34}$$

PROOF: We have that

$$\begin{aligned}
[\mathcal{D}^n\varphi, f]_n &= P\left([\mathcal{D}^n\varphi, f]_{n+1}\right) \\
&= P\left([\mathcal{D}^{n+1}\varphi \circ_{n+1} [\mathcal{D}^{n+1}\varphi, \mathcal{D}^n\varphi]_{n+1}, f]_{n+1}\right) \\
&= P\left([\mathcal{D}^{n+1}\varphi, f]_{n+1}[\mathcal{D}^n\varphi, \mathcal{D}^{n+1}\varphi]_{n+1}\right) \\
&= H[\mathcal{D}^{n+1}\varphi, f]_{n+1}.
\end{aligned}$$

The proof that $[\mathcal{D}^n\psi^i, f]_n = G^i[\mathcal{D}^{n+1}\varphi, f]_{n+1}$ is the same. □

Proposition 3.3.2 (Fast Wavelet transform - Synthesis Part) *Suppose that*

$$\{\psi^i\}_{i=1,\dots,m-1} \in X_0$$

is a multiwavelet with a single scaling function φ*. Then for* $f \in X_{n+1}$*,*

$$[\mathcal{D}^{n+1}\varphi, f]_{n+1} = H^*[\mathcal{D}^n\varphi, f]_n + \sum_{i=1}^{m-1} G^{i*}[\mathcal{D}^n\psi^i, f]_n$$

where $H^* = F_h^*$ *and* $G^{i*} = F_{g^i}^*$ *as defined in Lemma 3.2.7.*

PROOF: We have that $P_{V_{n+1}}f = P_{V_n}f + P_{W_n}f$, so

$$\mathcal{D}^{n+1}\varphi \circ_{n+1} [\mathcal{D}^{n+1}\varphi, f]_{n+1} = \mathcal{D}^n\varphi \circ_n [\mathcal{D}^n\varphi, f]_n + \sum_{i=1}^{m-1} \mathcal{D}^n\psi^i \circ_n [\mathcal{D}^n\psi^i, f]_n.$$

Taking an inner product on the left with $\mathcal{D}^{n+1}\varphi$ in X_{n+1} we obtain

$$\begin{aligned} [\mathcal{D}^{n+1}\varphi, \mathcal{D}^{n+1}\varphi]_{n+1}[\mathcal{D}^{n+1}\varphi, f]_{n+1} &= \Big[\mathcal{D}^{n+1}\varphi, \mathcal{D}^n\varphi \circ_n [\mathcal{D}^n\varphi, f]_n\Big]_{n+1} \\ &\quad + \sum_{i=1}^{m-1} \Big[\mathcal{D}^{n+1}\varphi, \mathcal{D}^n\psi \circ_n [\mathcal{D}^n\psi, f]_n\Big]_{n+1}. \end{aligned}$$

From orthogonality of the translates of φ we have that $[\mathcal{D}^{n+1}\varphi, \mathcal{D}^{n+1}\varphi]_{n+1} = \mathbf{1}$, and we get

$$\begin{aligned} [\mathcal{D}^{n+1}\varphi, f]_{n+1} &= \Big[\mathcal{D}^{n+1}\varphi, \mathcal{D}^n\varphi \circ_{n+1} P^*[\mathcal{D}^n\varphi, f]_{n+1}\Big]_n \\ &\quad + \sum_{i=1}^{m-1} \Big[\mathcal{D}^{n+1}\varphi, \mathcal{D}^n\psi^i \circ_{n+1} P^*[\mathcal{D}^n\varphi, f]_{n+1}\Big]_{n+1} \\ &= [\mathcal{D}^{n+1}\varphi, \mathcal{D}^n\varphi]_{n+1}P^*[\mathcal{D}^n\varphi, f]_n \\ &\quad + \sum_{i=1}^{m-1} [\mathcal{D}^{n+1}\varphi, \mathcal{D}^n\psi^i]_{n+1}P^*[\mathcal{D}^n\psi^i, f]_n \\ &= H^*[\mathcal{D}^n\varphi, f]_n + \sum_{i=1}^{m-1} G^{i*}[\mathcal{D}^n\psi^i, f]_n \end{aligned}$$

□

The fast wavelet transform is closely related to a family of C^*-algebras known as Cuntz algebras. For $n \in \mathbf{N}$, the *Cuntz algebra* $\mathcal{O}_n$ is generated by a set of elements $S_0, \dots, S_{n-1}$ which satisfy

$$S_i^* S_i = \mathbf{1} \tag{3.35}$$

$$\sum_{i=0}^{n-1} S_i S_i^* = \mathbf{1}. \tag{3.36}$$

The represention of $\mathcal{O}_n$ that we are about to describe have been described by Bratelli and Jorgensen ([J1, BJ1, BJ3]). These representations have been used for studying both fractals and wavelets. This is an interesting example of how C^*-algebras are relevent to wavelet theory.

Theorem 3.3.3 *We have that*

$$HH^* = \mathbf{1}, \text{ and } G^i G^{i*} = \mathbf{1} \text{ for } i = 1, \ldots, m-1 \tag{3.37}$$

and that

$$H^*H + \sum_{i=1}^{m-1} G^{i*}G^i = \mathbf{1}. \tag{3.38}$$

So the mapping $\pi : \mathcal{O}_m \to B(l^2(\mathbf{Z}^d))$ *which is given by*

$$\begin{aligned} \pi(S_0) &= H^* \\ \pi(S_i) &= G^{i*} \quad \text{for } i = 1, \ldots, m-1 \end{aligned}$$

is a $*$*-representation of* $\mathcal{O}_m$.

Proof: On page 81 we used the shifted orthogonality conditions to show that $H, G^i \in B(l^2(\mathbf{Z}^d))$. We shall now demonstrate that $H^*H + \sum_{i=1}^{m-1} G^{i*}G^i = \mathbf{1}$. Suppose that $a \in l^2(\mathbf{Z}^d)$, let φ, ψ^i be a scaling function and some wavelets. If we set $f = \varphi \circ_0 a$ then we have that $a = [\varphi, f]_0$. Now we know from Proposition 3.3.2 that

$$a = [\varphi, f]_0 = H^*[\mathcal{D}^{-1}\varphi, f]_{-1} + \sum_{i=1}^{m-1} G^{i*}[\mathcal{D}^{-1}\psi^i, f]_{-1}$$

But we know from Proposition 3.3.1 that

$$\begin{aligned} [\mathcal{D}^{-1}\varphi, f]_{-1} &= H[\varphi, f]_0 \\ \text{and } [\mathcal{D}^{-1}\psi^i, f]_{-1} &= G^i[\varphi, f]_0. \end{aligned}$$

We therefore have that $a = H^*Ha + \sum_{i=1}^{m-1} G^{i*}G^i a$ for all $a \in l^2(\mathbf{Z}^d)$.

Suppose that a and b are contained in $l^2(\mathbf{Z}^d)$, and $\hat{b} \in L^\infty(\mathbf{T}^d)$, then we have for $\gamma \in \mathbf{Z}^d$,

$$\begin{aligned} (F_b F_b^* a)(\gamma) &= P(P^*(a)bb^*)(\gamma) \\ &= (P^*(a)bb^*)(\Delta\gamma) \\ &= \sum_{\alpha \in \mathbf{Z}^d} (P^*a(\alpha))(bb^*)(\Delta\gamma - \alpha) \\ &= \sum_{\alpha \in \Delta\mathbf{Z}^d} a(\Delta^{-1}\alpha)(bb^*)(\Delta\gamma - \alpha) \\ &= \sum_{\alpha \in \mathbf{Z}^d} a(\alpha)((bb^*)(\Delta\gamma - \Delta\alpha)) \\ &= (aP(bb^*))(\gamma). \end{aligned}$$

So if $P(bb^*) = \mathbf{1}$, then $F_b F_b^* a = a$. Since h, g^i satisfy the shifted orthogonality conditions, $HH^*a = a$ and $G^iG^{i*}a = a$. We therefore have that π is a representation of $\mathcal{O}_m$. $\square$

The fast wavelet transform can be conveniently expressed in terms of wavelet matrices.

Corollary 3.3.4 *Suppose that A is a wavelet matrix as defined on page 83, and corresponds to scaling function φ and wavelets $\psi^1, \ldots, \psi^{m-1}$. Let $\alpha_0, \ldots, \alpha_{m-1}$ be a set of coset representatives of $\Delta \mathbf{Z}^d$ in $\mathbf{Z}^d$. We can write*

$$\begin{pmatrix} [\mathcal{D}^n\varphi, f]_n \\ [\mathcal{D}^n\psi^1, f]_n \\ \vdots \\ [\mathcal{D}^n\psi^{m-1}, f]_n \end{pmatrix} = A^{*T} \begin{pmatrix} P\alpha_0[\mathcal{D}^{n+1}\varphi, f]_{n+1} \\ P\alpha_1[\mathcal{D}^{n+1}\varphi, f]_{n+1} \\ \vdots \\ P\alpha_{m-1}[\mathcal{D}^{n+1}\varphi, f]_{n+1} \end{pmatrix}. \tag{3.39}$$

Proof: Equation (3.39) can be expressed as

$$\begin{aligned} [\mathcal{D}^n\varphi, f]_n &= \sum_{j=0}^{m-1} (P\alpha_j h)^* (P\alpha_j [\mathcal{D}^{n+1}\varphi, f]_{n+1}) \\ [\mathcal{D}^n\psi^i, f]_n &= \sum_{j=0}^{m-1} (P\alpha_j g^i)^* (P\alpha_j [\mathcal{D}^{n+1}\varphi, f]_{n+1}) \end{aligned}$$

where $i = 1, \ldots, m-1$. Now by Lemma 3.1.3, these equations are equivalent to

$$\begin{aligned} [\mathcal{D}^n\varphi, f]_n &= P([\mathcal{D}^{n+1}\varphi, f]_{n+1} h^*) \\ [\mathcal{D}^n\psi^i, f]_n &= P([\mathcal{D}^{n+1}\psi^i, f]_{n+1} g^{i*}) \end{aligned}$$

where $i = 1, \ldots, m-1$. This equation is exactly the analysis part of the fast wavelet transform. □

We shall now demonstrate that the synthesis part of the fast wavelet transform is a direct consequence of the analysis part and the matrix A being unitary (in other words the shifted orthogonality conditions). Because A is unitary, Equation (3.39) can be written as

$$\begin{pmatrix} P\alpha_0[\mathcal{D}^{n+1}\varphi, f]_{n+1} \\ P\alpha_1[\mathcal{D}^{n+1}\varphi, f]_{n+1} \\ \vdots \\ P\alpha_{m-1}[\mathcal{D}^{n+1}\varphi, f]_{n+1} \end{pmatrix} = A^{T} \begin{pmatrix} [\mathcal{D}^n\varphi, f]_n \\ [\mathcal{D}^n\psi^1, f]_n \\ \vdots \\ [\mathcal{D}^n\psi^{m-1}, f]_n \end{pmatrix}. \tag{3.40}$$

and this is equivalent to saying that for $i = 0, \ldots, m-1$,

$$P\alpha_i[\mathcal{D}^{n+1}\varphi, f]_{n+1} = (P\alpha_i h)[\mathcal{D}^n\varphi, f]_n + \sum_{j=1}^{m-1} (P\alpha_i g^j)[\mathcal{D}^n\psi^j, f]_n$$

which is equivalent to the synthesis part of the fast wavelet transform by Lemma 3.1.6.

3.4 The Cascade Algorithm

In this section we are interested in wavelets for which there exists a single scaling function φ. The cascade algorithm allows us to obtain a scaling function φ from the scaling filter h associated with it. It also gives us some necessary and sufficient conditions for an element of $C_c(\mathbf{Z}^d)$ to be the scaling filter for some scaling function. The main results in this section are Theorem 3.4.10 and Theorem 3.4.11. Theorem 3.4.10 describes some sufficient conditions for an element of $C^*(\mathbf{Z}^d)$ to be a scaling filter. Theorem 3.4.11 describes some necessary conditions for h to be a scaling filter. Theorem 3.4.10 and Theorem 3.4.11 are quite similar to results in [Lw1, St, BJ2], except Theorem 3.4.10 also demonstrates that the cascade algorithm also converges in X_0.

We note that in this section we are investigating the convergence of the cascade algorithm in X_0, with the scaling filter candidate $h \in C^*(\mathbf{Z}^d)$. It may perhaps be possible to strengthen the results described here if one works with scaling filter candidates for which $\hat{h} \in L^\infty(\mathbf{T}^d)$, and one investigates the convergence of the cascade algorithm in Y_0.

Recall from (3.7) that the scaling equation can be written as

$$\varphi = \mathcal{D}\varphi \circ_1 h = \mathcal{D}(\varphi \circ_0 h),$$

this leads us to define a *cascade approximation operator*

$$^hM_n f := \mathcal{D}f \circ_{n+1} h = \mathcal{D}(f \circ_n h). \tag{3.41}$$

If $f \in X_n$, then because $\mathcal{D} \in \mathcal{L}(X_n, X_{n+1})$, $^hM_n f \in X_{n+1}$. It is also the case that for any scaling filter h, $^hM_n \in B(L^2(\mathbf{R}^d))$, because $\hat{h} \in L^\infty(\mathbf{T}^d)$. We shall mainly be interested in what happens at the core subspace of the multiresolution analysis so we write

$$^hMf := {}^hM_0 f = \mathcal{D}(f \circ_0 h).$$

More generally, for arbitrary $b \in C^*(\mathbf{Z}^d)$, we define the *cascade approximation operator associated with* b to be

$$^bM_n f := \mathcal{D}(f \circ_n b), \tag{3.42}$$

and

$$^bMf := {}^bM_0 f = \mathcal{D}(f \circ_0 b).$$

The cascade algorithm consists of choosing an initial estimate $\phi^{(0)}$ for φ, and repeatedly applying ^hM to $\phi^{(0)}$ until it is sufficiently close to φ. Before we examine the convergence properties of the cascade algorithm, let us examine the properties of cascade approximation operators in general.

Lemma 3.4.1 *If $b \in C^*(\mathbf{Z}^d)$, then $^bM_n \in \mathcal{L}(X_n, X_{n+1})$, and for $g \in X_{n+1}$,*

$$(^bM_n)^* g = (\mathcal{D}^{-1}g) \circ_n b^*.$$

PROOF: Let $f \in X_n$ and $g \in X_{n+1}$, then using Corollary 2.2.7, we obtain

$$\begin{aligned} [^bM_n f, g]_{n+1} &= [\mathcal{D}(f \circ_n b), g]_{n+1} \\ &= [f \circ_n b, \mathcal{D}^{-1}g]_n \\ &= [f, (\mathcal{D}^{-1}g) \circ_n b^*]_n \end{aligned}$$

□

Lemma 3.4.2 *For $f, g \in X_n$, $b \in C^*(\mathbf{Z}^d)$,*

$$[{}^bM_n f, {}^bM_n g]_n = P([f,g]_n bb^*).$$

PROOF: Let $f, g \in X_n$, then

$$\begin{aligned} [{}^bM_n f, {}^bM_n g]_n &= [f, ({}^bM_n)^* \, {}^bM_n g]_{n-1} \\ &= [f, \mathcal{D}^{-1}({}^bM_n g) \circ_n b^*]_{n-1} \\ &= [f, \mathcal{D}^{-1}\mathcal{D}(g \circ_n b) \circ_n b^*]_{n-1} \\ &= [f, g \circ_n (bb^*)]_{n-1} \\ &= P[f, g \circ_n (bb^*)]_n \\ &= P([f,g]_n bb^*) \end{aligned}$$

□

Lemma 3.4.3 *Suppose that $b \in C^*(\mathbf{Z}^d)$ and $f \in X_0$, then*

$$\|{}^bMf\|_{X_0} \leq \frac{1}{m} \sup_{\zeta \in \mathbf{T}^d} \sum_{\hat{\Delta}(\omega)=\zeta} \hat{b}(\omega)\overline{\hat{b}(\omega)} \|f\|_{X_0}.$$

Hence if $h \in C^*(\mathbf{Z}^d)$ is a scaling filter, then $\|{}^hMf\|_{X_0} \leq \|f\|_{X_0}$.

PROOF: We calculate

$$\begin{aligned} \|{}^bMf\|_{X_0} &= \|[{}^bMf, {}^bMf]_0\|_{C^*(\mathbf{Z}^d)}^{\frac{1}{2}} \\ &= \|[P([f,f]_0 bb^*)\|_{C^*(\mathbf{Z}^d)}^{\frac{1}{2}} \quad \text{(by Lemma 3.4.2)} \\ &= \frac{1}{m} \sup_{\zeta \in \mathbf{T}^d} \sum_{\hat{\Delta}(\omega)=\zeta} [\![\hat{f}, \hat{f}]\!]_0(\omega)\hat{b}(\omega)\overline{\hat{b}(\omega)} \quad \text{(by Proposition 3.1.4)} \\ &\leq \frac{1}{m} \sup_{\zeta \in \mathbf{T}^d} \sum_{\hat{\Delta}(\omega)=\zeta} \hat{b}(\omega)\overline{\hat{b}(\omega)} \sup_{\xi \in \mathbf{T}^d} [\![\hat{f}, \hat{f}]\!]_0(\xi) \\ &= \frac{1}{m} \sup_{\zeta \in \mathbf{T}^d} \sum_{\hat{\Delta}(\omega)=\zeta} \hat{b}(\omega)\overline{\hat{b}(\omega)} \|f\|_{X_0}. \end{aligned}$$

The fact that $\|{}^hMf\|_{X_0} \leq \|f\|_{X_0}$ now follows from equation (3.12). □

Lemma 3.4.2 leads us to define the *transition operator* ${}^bT : C^*(\mathbf{Z}^d) \to C^*(\mathbf{Z}^d)$ to be for $a, b \in C^*(\mathbf{Z}^d)$,

$${}^bTa = P(abb^*) = F_{bb^*}a. \tag{3.43}$$

by Lemma 3.4.2 the transition operator satisfies

$$[{}^bM_n f, {}^bM_n g]_n = {}^bT[f,g]_n$$

for $f, g \in X_n$. We can sometimes also define the transition operator for $b \notin C^*(\mathbf{Z}^d)$. For example, if $\hat{b} \in L^\infty(\mathbf{T}^d)$, then (3.43) defines a bounded operator on $L^2(\mathbf{Z}^d)$. The transition operator tells us what happens to inner products when acted on by a cascade approximation operator. We shall later see that the transition operator ${}^hT$ tells us about the convergence properties of the cascade algorithm. The use of the transition operator was first suggested by W. Lawton in [Lw1, Lw2].

In order to examine the convergence properties of the cascade algorithm as well as necessary and sufficient conditions for a filter to be the scaling filter, we will first prove some standard results about scaling functions and scaling filters.

The proofs to the following two Lemmas are similar to the proof of [HW, Chapter 2, Theorem 1.7], a similar result is proved in [Coh, Theorem 1(c)].

Lemma 3.4.4 *Suppose that φ is a scaling function for which $\hat{\varphi}$ is continuous at 0. Then $\hat{\varphi}(0) = 1$.*

Proof: Let $C \subset \widehat{\mathbf{R}^d}$ be the cube $[-1/2, 1/2)^d$. Let $f = \chi_C$, the characteristic function of C. Let $(V_j)_{j\in\mathbf{Z}}$ be the multiresolution analysis corresponding to φ, and let P_{V_j} be the projection onto V_j.

Because $\overline{\cup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R}^d)$, $\lim_{j\to\infty} P_{V_j} f = f$, and in particular $\lim_{j\to\infty} \|P_{V_j} f\|_2 = \|f\|_2$. We have that

$$P_{V_j} f = \mathcal{D}^j\varphi \circ_j [\mathcal{D}^j\varphi, f]_j,$$

so

$$\begin{aligned}
\|P_{V_j} f\|_2^2 &= \int_{\mathbf{R}^d/(\tilde{\mathcal{D}}^*)^{-j}\mathbf{Z}^d} \sum_{\beta\in(\tilde{\mathcal{D}}^*)^{-j}\mathbf{Z}^d} \overline{P_{V_j} f(\xi+\beta)} P_{V_j} f(\xi+\beta) d(\xi) \\
&= m^{-j} \int_{\mathbf{T}^d} [\![\widehat{P_{V_j} f}, \widehat{P_{V_j} f}]\!]_j(\xi) d(\xi) \\
&= m^{-j} \int_{\mathbf{T}^d} ([\![\widehat{\mathcal{D}^j\varphi}, \widehat{\mathcal{D}^j\varphi}]\!]_j [\![\widehat{\mathcal{D}^j\varphi}, \hat{f}]\!]_j)(\xi) d(\xi) \\
&= m^{-j} \int_{\mathbf{T}^d} [\![\widehat{\mathcal{D}^j\varphi}, \hat{f}]\!]_j(\xi) d(\xi) \\
&= m^{-j} \int_C |\widehat{\mathcal{D}^j\varphi}(\xi)| d(\xi)
\end{aligned}$$

because $f = \chi_C$. So

$$\lim_{j\to\infty} \|P_{V_j} f\|_2^2 = |\hat{\varphi}(0)|.$$

But

$$\lim_{j\to\infty} \|P_{V_j} f\|_2^2 = \|f\|_2^2 = 1,$$

so $|\hat{\varphi}(0)| = 1$. □

The condition that $\hat{\varphi}$ is continuous at zero is fairly weak, for example if $\varphi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$, then $\hat{\varphi}$ is continuous everywhere.

Lemma 3.4.5 *Suppose that φ is a solution of the equation ${}^hM\varphi = \varphi$, where $h \in C^*(\mathbf{Z}^d)$ satisfies $P(hh^*) = \mathbf{1}$ and that $\hat{h}(0) = \sqrt{m}$, where m is the index of the multiresolution structure. If we define $V_j = \overline{\operatorname{span}\{\mathcal{D}^j\gamma\varphi\}_{\gamma\in\mathbf{Z}^d}}$, then $\overline{\cup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R}^d)$.*

PROOF: Let $W = \cup_{j\in\mathbf{Z}} V_j$, we shall first show that W is translation invariant. If $f \in W$, then for all $\varepsilon > 0$, there exists $j_0 \in \mathbf{Z}$ and $h \in V_{j_0}$ such that $\|f - h\| < \varepsilon$. We then have that $h \in V_j$ for all $j \geq j_0$, we can therefore write

$$h = \mathcal{D}^j\varphi \circ_j [\mathcal{D}^j\varphi, h]_j.$$

Suppose that $l \leq j$, and let $\alpha \in \Gamma^l$, we then have

$$\alpha h = \alpha\mathcal{D}^j\varphi \circ_j [\mathcal{D}^j\varphi, h]_j \in V_j.$$

Since $\|\alpha f - \alpha h\|_2 = \|f - h\|_2 < \varepsilon$, we have that W is invariant under translations by elements of Γ^l for all $l \in \mathbf{Z}$. Since $\cup_{j\in\mathbf{Z}}\Gamma^j$ is dense in $\mathbf{R}^d$, it follows that W is invariant under all translations by elements of $\mathbf{R}^d$.

Suppose that there exists a g such that $g \perp f$ for all $f \in W$. Since W is translation invariant, it follows that for $\alpha \in \mathbf{R}^d$, $f \in W$,

$$\int_{\mathbf{R}}^{d} f(x-\alpha)\overline{g(x)}dx = 0.$$

Taking the Fourier transform we have

$$\int_{\widehat{\mathbf{R}^d}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\xi(\alpha)d\xi = 0$$

and so $\hat{f}(\xi)\hat{g}(\xi) = 0$ for almost every $\xi \in \mathbf{R}^d$. Now choose $f = \mathcal{D}^j\varphi \in V_j \subseteq W$. Then because $\hat{\varphi}$ is continuous at 0 and $\hat{\varphi}(0) = 1$, there is a neighbourhood of 0 for which $\hat{f} \neq 0$. We then have that $\hat{g}(\xi) = 0$ on this neighbourhood. As $j \to \infty$, this neighbourhood can be made arbitrarily large, and thus $\hat{g} = 0$ almost everywhere. We therefore have that $g = 0$, and so $\overline{\cup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R}^d)$. □

The following Lemma is also a standard result of wavelet theory. The proof presented here is similar to the proof of [HW, Chapter 2, Theorem 1.6], a similar result is proved in [Coh, Theorem 1(b)].

Lemma 3.4.6 *Suppose we have a sequence $\{V_j\}_{j\in\mathbf{Z}}$ of closed subspaces of $L^2(\mathbf{R}^d)$ such that*

1. *For all j, $V_j \subset V_{j+1}$;*
2. *$f \in V_n$ if and only if $\mathcal{D}f \in V_{n+1}$;*
3. *there exists $\varphi \in V_0$ such that $\{\gamma\varphi\}_{\gamma\in\mathbf{Z}^d}$ is an orthonormal basis for V_0.*

It then follows that $\cap_{j\in\mathbf{Z}} V_j = \{0\}$.

PROOF: Suppose there exists $f \in \cap_{j\in\mathbf{Z}} V_j$ such that $f \neq 0$, we can without loss of generality assume that $\|f\|_2 = 1$. For all $j \in \mathbf{Z}$, f is contained in V_{-j}, so $\mathcal{D}^j f \in V_0$. We also have that $\|\mathcal{D}^j f\|_2 = 1$. Because $\{\gamma\varphi\}_{\gamma\in\mathbf{Z}^d}$ is an orthonormal basis for V_0, we can write

$$\mathcal{D}^j f = \varphi \circ_0 a^j$$

for some $a^j \in l^2(\mathbf{Z}^d)$ satisfying $\|a^j\|_2 = 1$. Taking the Fourier transform we have

$$\begin{aligned} \widehat{\mathcal{D}^j f}(\xi) &= \varphi(\xi)\hat{a^j}(\xi) \\ \text{so} \quad \hat{f}(\xi) &= m^{j/2}\hat{a^j}(\hat{\tilde{\mathcal{D}}}^{-j}\xi)\hat{\varphi}(\hat{\tilde{\mathcal{D}}}^{-j}\xi). \end{aligned}$$

Now let C be the cube $[-1/2, 1/2)^d$, then for $j \geq 1$,

$$\begin{aligned} \int_{\hat{\tilde{\mathcal{D}}}^{-1}C\backslash C} |\hat{f}(\xi)|^2 d\xi &\leq m^j \int_{\hat{\tilde{\mathcal{D}}}^{-1}C\backslash C} |\hat{\varphi}(\hat{\tilde{\mathcal{D}}}^{-j}\xi)|^2 d\xi \int_{\hat{\tilde{\mathcal{D}}}^{-1}C\backslash C} |\hat{a^j}(\hat{\tilde{\mathcal{D}}}^{-j}\xi)|^2 d\xi \\ &= m^{-j} \int_{\hat{\tilde{\mathcal{D}}}^{-j-1}C\backslash \hat{\tilde{\mathcal{D}}}^{-j}C} \|\hat{\varphi}(\mu)\|^2 d\mu \int_{\hat{\tilde{\mathcal{D}}}^{-j-1}C\backslash \hat{\tilde{\mathcal{D}}}^{-j}C} \|\hat{a^j}(\mu)|^2 d\mu \\ &\leq m^{-j} \int_{\mathbf{R}^d\backslash \hat{\tilde{\mathcal{D}}}^{-j}C} |\hat{\varphi}(\mu)|^2 d\mu \int_{\hat{\tilde{\mathcal{D}}}^{-j-1}C\backslash \hat{\tilde{\mathcal{D}}}^{-j}C} |\hat{a^j}(\mu)|^2 d\mu \\ &\leq m^{-j} \int_{\mathbf{R}^d\backslash \hat{\tilde{\mathcal{D}}}^{-j-1}C} |\hat{\varphi}(\mu)|^2 d\mu \; m^j \int_{\mathbf{T}^d} |\hat{a^j}(\mu) d\mu|^2 \\ &\leq \int_{\mathbf{R}^d\backslash \hat{\tilde{\mathcal{D}}}^{-j-1}C} |\hat{\varphi}(\mu)|^2 d\mu \end{aligned}$$

Therefore if we let j approach ∞, we obtain $\int_{\hat{\tilde{\mathcal{D}}}^{-1}C\backslash C} |\hat{f}(\xi)|^2 d\xi = 0$, for $\hat{f}(\xi) = 0$ for almost every $\xi \in \hat{\tilde{\mathcal{D}}}^{-1}C\backslash C$. The above argument applies for all j, we obtain that $\hat{f}(\xi) = 0$ for almost every $\xi \in \hat{\tilde{\mathcal{D}}}^{-j-1}C\backslash \hat{\tilde{\mathcal{D}}}^{-j}C$. This implies that $\hat{f} = 0$ almost everywhere. □

The following Lemma gives us some necessary conditions for a function h on $\mathbf{Z}^d$ to be a scaling filter.

Lemma 3.4.7 *Let φ be a scaling function for a multiresolution analysis. Suppose that $\hat{\varphi}$ is continuous at 0, and $\alpha_0, \ldots, \alpha_{m-1}$ is a set of coset representatives of $\Delta(\mathbf{Z}^d)$ in $\mathbf{Z}^d$. Then for all $i = 0, \ldots, m-1$,*

$$\sum_{\gamma \in \alpha_i \Delta \mathbf{Z}^d} h(\gamma) = m^{-\frac{1}{2}}$$

where m is the index of the multiresolution structure. We can also state this as

$$\hat{h}(\omega) = 0$$

whenever $\omega \neq 0$ but $\hat{\Delta}(\omega) = 0$.

We also have that $\sum_{\gamma \in \mathbf{Z}^d} h(\gamma) = m^{\frac{1}{2}}$, in other words $\hat{h}(0) = m^{\frac{1}{2}}$.

Proof: From the previous lemma, $\hat{\varphi}(0) = 1$, so by the Fourier transformed version of the scaling equation, we have $\hat{h}(0) = m^{\frac{1}{2}}$, so $\sum_{\gamma \in \mathbf{Z}^d} h(\gamma) = m^{\frac{1}{2}}$ (because $\hat{h}(0) = \sum_{\gamma \in \mathbf{Z}^d} h(\gamma)$) .

From the definition of the downsampling operator P, we have that

$$\sum_{\gamma \in \alpha_i \Delta \mathbf{Z}^d} h(\gamma) = m^{-\frac{1}{2}} \text{ if and only if } \sum_{\gamma \in \mathbf{Z}^d} (P(\alpha_i h))(\gamma) = m^{-\frac{1}{2}},$$

and this is true if and only if $(\hat{P}(\hat{\alpha_i h}))(0) = m^{-\frac{1}{2}}$. Now from the shifted orthogonality conditions,

$$\frac{1}{m} \sum_{\hat{\Delta}(\omega)=\zeta} \hat{h}(\omega)\overline{\hat{h}(\omega)} = 1$$

for all $\zeta \in \mathbf{T}^d$. So if $\omega \neq 0$, but $\hat{\Delta}(\omega) = 0$, $\hat{h}(\omega)\overline{\hat{h}(\omega)} = 0$ and thus $\hat{h}(\omega) = 0$. So we have that

$$(\hat{P}(\hat{\alpha_i h}))(\zeta) = \frac{1}{m} \sum_{\hat{\Delta}(\omega)=\zeta} (\hat{\alpha_i h})(\omega) = \frac{1}{m} \sum_{\hat{\Delta}(\omega)=\zeta} \hat{h}(\omega)\alpha_i(\omega),$$

so

$$\begin{aligned}
(\hat{P}(\hat{\alpha_i h}))(0) &= \frac{1}{m} \sum_{\hat{\Delta}(\omega)=0} \hat{h}(\omega)\alpha_i(\omega) \\
&= \frac{1}{m}\hat{h}(0)\alpha_i(0) \\
&= \frac{1}{m}\hat{h}(0) \\
&= m^{-\frac{1}{2}}.
\end{aligned}$$

□

Definition 3.4.8 Suppose that $\mathcal{A}$ is a unital Banach algebra, and that $a \in \mathcal{A}$. The *spectrum* of a is defined to be the set

$$\sigma(a) := \{\lambda \in \mathbf{C} : \lambda\mathbf{1} - a \text{ is not invertible } \}.$$

The complement of the spectrum is known as the *resolvent*. The *spectral radius* of a is defined to be

$$\rho(a) := \sup_{\lambda\in\sigma(a)} |\lambda|.$$

The spectrum is a generalisation of the set of eigenvalues of a matrix. We will make use of this concept to study the convergence of the cascade algorithm. The following proposition tells us the main properties of the spectrum of an element of a Banach algebra.

Proposition 3.4.9

- *In any unital Banach algebra $\mathcal{A}$, the spectrum of each $a \in \mathcal{A}$ is a non-empty compact set;*
- *Suppose that p is a polynomial, and $a \in \mathcal{A}$, then*

$$\sigma(p(a)) = p(\sigma(a));$$

- *The spectral radius of a satisfies*

$$\rho(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

PROOF: The reader is referred to ([Dv], Theorem I.2.1, Lemma, I.2.2, and Proposition I.2.3) for proofs of the the above results. □

The transition operator ${}^hT$ is contained in the Banach algebra of bounded operators on $C^*(\mathbf{T}^d)$, and so it is possible to analyse its spectral theory.

Theorem 3.4.10 differs from analagous results in [Lw1, St, BJ2] in two ways. Firstly, the analagous results assume that the scaling filter h is finitely supported. Secondly, we also show that the cascade algorithm converges in the topology of the Hilbert module X_0. Although most applications of wavelet theory involve finitely supported filters, it is still important to study wavelets in a "pure mathematical" context. For example, the Shannon wavelet (see Examples 3.2.3) has a filter which is not finitely supported but is an important example in wavelet theory. It is important that we have convergence in X_0 because this demonstrates that the cascade algorithm fits in nicely within the Hilbert module framework.

Theorem 3.4.10 *Suppose that $h \in C^*(\mathbf{Z}^d)$ satisfies $P(hh^*) = \mathbf{1}$ and the condition that $\hat{h}(0) = m^{\frac{1}{2}}$. Suppose that if $\lambda \in \sigma({}^hT)$, then either $|\lambda| < 1$, or $\lambda = 1$ and is simple and corresponds to the eigenvector $\mathbf{1}$.*

Then if $\phi^{(0)} \in X_0$ is a function for which $\sum_{\gamma\in\mathbf{Z}^d} \phi^{(0)}(x-\gamma) = 1$ almost everywhere, there exists $\varphi \in X_0$ such that $\lim_{n\to\infty} {}^hM^{(n)}\phi^{(0)} = \varphi$ in both $L^2(\mathbf{R}^d)$ and X_0, and φ is the scaling function for a multiresolution analysis.

PROOF: Consider the sequence $\phi^{(n)} := {}^hM^{(n)}\phi^{(0)}$, we shall show that this sequence is Cauchy. In order to do this, we will first show that for all $i, j \in \mathbf{N} \cup \{0\}$,

$$\lim_{n\to\infty} {}^hT^{(n)}[\phi^{(i)}, \phi^{(j)}]_0 = \mathbf{1}.$$

We have that for all n, l,

$$\begin{aligned}
[\phi^{(n)} - \phi^{(n+l)}, \phi^{(n)} - \phi^{(n+l)}]_0 &= [\phi^{(n)}, \phi^{(n)}]_0 - [\phi^{(n+l)}, \phi^{(n)}]_0 \\
&\quad - [\phi^{(n)}, \phi^{(n+l)}]_0 + [\phi^{(n+l)}, \phi^{(n+l)}]_0 \\
&= {}^hT^{(n)}\Big([\phi^{(0)}, \phi^{(0)}]_0 - [\phi^{(l)}, \phi^{(0)}]_0 \\
&\quad - [\phi^{(0)}, \phi^{(l)}]_0 + [\phi^{(l)}, \phi^{(l)}]_0\Big) \\
&= {}^hT^{(n)}[\phi^{(0)} - \phi^{(l)}, \phi^{(0)} - \phi^{(l)}]_0.
\end{aligned}$$

We also have that for all $l \in \mathbf{N} \cup \{0\}$,

$$\begin{aligned}
\widehat{[\phi^{(l)}, \phi^{(0)}]}_0(0) &= \sum_{\gamma\in\mathbf{Z}^d} [\phi^{(l)}, \phi^{(0)}]_0(\gamma) \\
&= \int_{\mathbf{R}^d} \phi^{(l)}(x) \sum_{\gamma\in\mathbf{Z}^d} \phi^{(0)}(x-\gamma)dx \\
&= \int_{\mathbf{R}^d} \phi^{(l)}(x)dx \\
&= \int_{\mathbf{R}^d} {}^hM^{(l)}\phi^{(0)}(x)dx
\end{aligned}$$

$$\begin{aligned} &= ({}^hM^{(l)}\hat{\varphi}^{(0)})(0) \\ &= \hat{\varphi^{(0)}}(0) \end{aligned}$$

It is the case that $\int_{\mathbf{R}^d} \phi^{(0)}(x)dx = \int_{[-1/2,1/2)^d} \sum_{\gamma\in\mathbf{Z}^d} \phi^{(0)}(x-\gamma)dx = 1$. So $\hat{\phi}^{(0)}(0) = 1$, and $\widehat{[\phi^{(l)}, \phi^{(0)}]}_0(0) = 1$ for all l. Since we have that $[\phi^{(n+l)}, \phi^{(n)}]_0 = {}^hT^{(n)}[\phi^{(l)}, \phi^{(0)}]_0$, and so

$$\begin{aligned} \widehat{[\phi^{(n+l)}, \phi^{(n)}]}_0(0) &= ({}^h\hat{T}^{(n)}\widehat{[\phi^{(l)}, \phi^{(0)}]}_0)(0) \\ &= ({}^hT\widehat{[\phi^{(l+n-1)}, \phi^{(n-1)}]}_0)(0) \\ &= (\hat{P}(\widehat{[\phi^{(l+n-1)}, \phi^{(n-1)}]}_0 hh^*)(0) \\ &= \frac{1}{m}\sum_{\hat{\Delta}(\omega)=0} \widehat{[\phi^{(l+n-1)}, \phi^{(n-1)}]}_0(\omega)\hat{h}(\omega)\overline{\hat{h}(\omega)} \quad \text{by Lemma 3.4.7} \\ &= \frac{1}{m}\widehat{[\phi^{(l+n-1)}, \phi^{(n-1)}]}_0(0)\hat{h}(0)\overline{\hat{h}(0)} \\ &= \widehat{[\phi^{(l+n-1)}, \phi^{(n-1)}]}_0(0) \end{aligned}$$

And so by induction on n,

$$\widehat{[\phi^{(i)}, \phi^{(j)}]}_0(0) = 1$$

for all $i, j \in \mathbf{N} \cup \{0\}$.

Now we have that

$$|[\phi^{(0)}, \phi^{(l)}]_0(\gamma)| = |\langle \phi^{(0)}, \gamma\phi^{(l)}\rangle| \le \|\phi^{(0)}\|_2\|\phi^{(l)}\|_2 \le C$$

for some constant $C > 0$. So from the spectral properties of ${}^hT$, there exists $k \in \mathbf{C}$ such that $\lim_{n\to\infty} {}^hT^{(n)}[\phi^{(i)}, \phi^{(j)}]_0 = k\mathbf{1}$ for all $i, j \in \mathbf{N}$. But since $\hat{\mathbf{1}}(0) = 1$, and $\widehat{[\phi^{(i)}, \phi^{(j)}]}_0(0) = 1$ for all $i, j \in \mathbf{N} \cup \{0\}$, $\lim_{n\to\infty} {}^hT^{(n)}[\phi^{(i)}, \phi^{(j)}] = \mathbf{1}$ with convergence in the topology of $C^*(\mathbf{Z}^d)$.

If we examine the Hilbert module norm we obtain

$$\begin{aligned} \|\phi^{(n)} - \phi^{(n+l)}\|_{X_0}^2 &= \sup_{\zeta\in\mathbf{T}^d} [\![\hat{\phi}^{(n)} - \hat{\phi}^{(n+l)}, \hat{\phi}^{(n)} - \hat{\phi}^{(n+l)}]\!]_0(\zeta) \\ &= \sup_{\zeta\in\mathbf{T}^d} {}^hT^{(n)}[\![\hat{\phi}^{(0)}, \hat{\phi}^{(0)}]\!]_0(\zeta) + {}^hT^{(n)}[\![\hat{\phi}^{(l)}, \hat{\phi}^{(l)}]\!]_0(\zeta) \\ &\quad -{}^hT^{(n)}[\![\hat{\phi}^{(0)}, \hat{\phi}^{(l)}]\!]_0(\zeta) - {}^hT^{(n)}[\![\hat{\phi}^{(l)}, \hat{\phi}^{(0)}]\!]_0(\zeta) \end{aligned}$$

and again since $\lim_{n\to\infty} {}^hT^{(n)}[\phi^{(i)}, \phi^{(j)}] = \mathbf{1}$, it follows that for all $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that when i and j are greater that N, $\|\phi^{(i)} - \phi^{(j)}\|_X < \varepsilon$, and so the sequence is Cauchy in X_0. Since X_0 in complete, there exists $\varphi \in X_0$ such that

$$\lim_{n\to\infty} \|\phi^{(n)} - \varphi\|_{X_0} = 0.$$

Now recall that in Theorem 2.2.6 (2) it is stated that if a sequence converges in X_n, then it converges in $L^2(\mathbf{R}^d)$. It therefore follows from Theorem 2.2.6 (2) that

$$\lim_{n\to\infty} \|\phi^{(n)} - \varphi\|_2 = 0.$$

We shall now check that φ is the scaling function for a multiresolution analysis. We know that $\varphi = {}^hM\varphi$, in other words, $\varphi = \mathcal{D}(\varphi \circ h)$, and so $h = [\mathcal{D}\varphi, \varphi]$. We define $V_j = \overline{\operatorname{span}\{\mathcal{D}^j\gamma\varphi\}_{\gamma\in\mathbf{Z}^d}}$. So by definition $f \in V_j$ if and only if $\mathcal{D}f \in V_{j+1}$, and because $\varphi = \mathcal{D}(\varphi \circ h)$, $V_j \subset V_{j+1}$. Because $\hat{h}$ is continuous, $\hat{\varphi}$ is continuous at zero, and so by Lemma 3.4.5, $\cup_{j\in\mathbf{Z}}V_j$ is dense in $L^2(\mathbf{R}^d)$. By Lemma 3.4.6 $\cap_{j\in\mathbf{Z}}V_j = \{0\}$. Therefore φ is the scaling function for a multiresolution analysis. □

Theorem 3.4.11 *Let $h \in C^*(\mathbf{Z}^d)$. Suppose that whenever $\sum_{\gamma\in\mathbf{Z}^d}\phi^{(0)}(x-\gamma) = 1$ for almost every $x \in \mathbf{R}^d$, there exists φ such that*

$$\lim_{n\to\infty} \|\varphi - {}^hM^{(n)}\phi^{(0)}\|_2 = 0,$$

and φ is the scaling function for a multiresolution analysis. Then h satisfies $P(hh^*) = \mathbf{1}$, $\hat{h}(0) = \sqrt{m}$, ${}^hT$ has a simple eigenvalue 1 corresponding to the eigenvector $\mathbf{1}$, and if $\lambda \neq 1$ is another eigenvalue of ${}^hT$, then $|\lambda| < 1$. Furthermore, the spectral radius of ${}^hT = 1$.

Proof: Since ${}^hM\varphi = \varphi$, h satisfies the shifted orthogonality conditions, and $\hat{h}(0) = \sqrt{m}$ by Lemma 3.4.4. Because h satisfies the shifted orthogonality conditions, ${}^hT(\mathbf{1}) = P(hh^*) = \mathbf{1}$, so 1 is an eigenvalue of ${}^hT$, with eigenvector $\mathbf{1}$.

Now $\|{}^hTa\| \leq \frac{1}{m}\operatorname{esssup}_{\zeta\in\mathbf{T}^d}\hat{h}(\omega)\overline{\hat{h}(\omega)}\|a\|$, so by the shifted orthogonality conditions, $\|{}^hT\|_{*\to*} = 1$, and so the spectral radius of ${}^hT$ is equal to 1.

Suppose that λ is an eigenvalue of ${}^hT$ for which $|\lambda| = 1$, and λ has an eigenvector $v \neq \mathbf{1}$. Let $a \in C^*(\mathbf{Z}^d)$ be defined as $a = \mathbf{1} + cv$, where $c \in \mathbf{C}\backslash\{0\}$. We have that

$$\begin{aligned}\lim_{n\to\infty} {}^hT^{(n)}[f, f\circ a]_0 &= \lim_{n\to\infty}[{}^hM^{(n)}f, {}^hM^{(n)}(f\circ a)]_0 \\ &= [\varphi, k\varphi]_0 \quad \text{for some } k \in \mathbf{C}, \\ &= k\mathbf{1},\end{aligned}$$

and so the sequence converges. But ${}^hT^{(n)}a = \mathbf{1} + \lambda^n cv$ which diverges if $\lambda \neq 1$, and when $\lambda = 1$, we can choose v such that $\hat{v}(0) = 0$, in which case ${}^hT^{(n)}a$ will converge to $\mathbf{1}$. In either case we have a contradiction so $|\lambda| < 1$. □

If Theorem 3.4.11 can be strengthened slightly so that it is in terms of the spectrum of ${}^hT$ rather than just the eigenvalues of ${}^hT$, and Theorems 3.4.10 and 3.4.11 are formulated in terms of $\hat{h} \in L^\infty$ instead of $h \in C^*(\mathbf{Z}^d)$ then we will have necessary and sufficient conditions for h to be the scaling filter for a multiresolution analysis. In the case that $h \in C_c(\mathbf{T}^d)$ the spectrum of ${}^hT$ coincides with the eigenvalues and we do have necessary and sufficient conditions. It is worth mentioning that there is also a result called Cohen's theorem [Coh, Theorem 2] which also gives necessary and sufficient conditions for h to be a scaling filter which (loosely speaking) are related to the behaviour of $\hat{h}$ on fundamental neighbourhoods of $\operatorname{Ann}\mathbf{Z}^d$.

3.5 Multiwavelets and Wavelet Matrices

Let us consider the situation where we have wavelets which correspond to a multiresolution analysis of order r. This means that we have r scaling functions $\varphi^0, \dots, \varphi^{r-1}$, and $(m-1)r$ multiwavelets $\psi^1, \dots, \psi^{(m-1)r}$ where m is the index of the multiresolution structure. So far we have been working with Hilbert modules X_n which are contained in our Hilbert space $L^2(\mathbf{R}^d)$. It will be convenient for us to work with finite vectors of elements of $L^2(\mathbf{R}^d)$. In other words we will work with elements of $(L^2(\mathbf{R}^d))^p$ where p is a natural number. We are especially interested in the cases that $p = r$, and $p = mn$. In the same way that we constructed Hilbert modules contained in $L^2(\mathbf{R}^d)$, we can construct Hilbert modules contained in $(L^2(\mathbf{R}^d))^p$.

For $p \in \mathbf{N}$ we let X_n^p be the space of column vectors containing p elements of the Hilbert module X_n. We let $M^p(C^*(\mathbf{Z}^d))$ be the C^*-algebra of $p \times p$ matrices with elements in $C^*(\mathbf{Z}^d)$. We will make X_n^p into a left-Hilbert $M^p(C^*(\mathbf{Z}^d))$-module. We are going to make X_n^p into a left module rather that a right module because this will simplify calculations involving matrices (this way we have matrices on the left multiplying by column vectors). For $n \in \mathbf{Z}$, define a module action $\circ_n^p : M^p(C^*(\mathbf{Z}^d)) \times X_n^p \to X_n^p$ by

$$(a \circ_n^p \mathbf{f})_i = \sum_{j=1}^{p} f^j \circ_n a_{ij}; \tag{3.44}$$

and define a $M^p(C^*(\mathbf{Z}^d))$-valued inner product by

$$\left({}_{X_n^p}[\mathbf{f}, \mathbf{g}]\right)_{i,j} = [g^j, f^i]_{X_n} \tag{3.45}$$

where $\mathbf{f}, \mathbf{g} \in X_n^p$, $a \in M^p(C^*(\mathbf{Z}^d))$, and $i, j = 1, \dots, p$. We can write these operations in matrix notation as

$$\begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{p1} & \cdots & a_{pp} \end{pmatrix} \begin{pmatrix} f^1 \\ \vdots \\ f^p \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^p f^i \circ_n a_{1i} \\ \vdots \\ \sum_{i=1}^p f^i \circ_n a_{pi} \end{pmatrix} \tag{3.46}$$

and

$${}_{X_n^p}[\mathbf{f}, \mathbf{g}] = \begin{pmatrix} [g^1, f^1]_n & \cdots & [g^p, f^1]_n \\ \vdots & \ddots & \vdots \\ [g^1, f^p]_n & \cdots & [g^p, p^p]_n \end{pmatrix}. \tag{3.47}$$

We can also define the module action and $M^p(C^*(\mathbf{Z}^d))$-valued inner product on other spaces, if we are careful. It is a consequence of Lemma 2.1.5 that if $\mathbf{f} \in (L^2(\mathbf{R}^d))^p$ and $a \in M^p(l^1(\mathbf{Z}^d))$ then $\mathbf{f} \circ_n^p a \in (L^2(\mathbf{R}^d))^p$. It is a consequence of Lemma 2.1.7 that if $\mathbf{f}, \mathbf{g} \in (L^2(\mathbf{R}^d))^p$, then ${}_{X_n^p}[\mathbf{f}, \mathbf{g}] \in M^p(C_0(\mathbf{Z}^d))$. We remark that it is probably possible to construct Hilbert modules over $M^p(L^\infty(\mathbf{T}^d))$ in a similar way.

Lemma 3.5.1 *With the above operations X_n^p is a full left Hilbert $M^p(C^*(\mathbf{Z}^d))$-module.*

PROOF: We shall verify each of the axioms of Definition 0.4.1. Let $\alpha, \beta \in \mathbf{C}$, $\mathbf{f}, \mathbf{g}, \mathbf{h} \in X_n^p$, $a \in M^p(C^*(\mathbf{Z}^d))$, we calculate

1.

$$\left({}_{X_n^p}[\alpha\mathbf{f}+\beta\mathbf{g},\mathbf{h}]\right)_{i,j} = [h^j,\alpha f^i+\beta g^i]_n = \alpha[h^j,f^i]_n+\beta[h^j,g^i]_n = \left(\alpha_{X_n^p}[\mathbf{f},\mathbf{h}]+\beta_{X_n^p}[\mathbf{g},\mathbf{h}]\right)_{i,j};$$

2.

$${}_{X_n^p}[a\circ_n^p\mathbf{f},\mathbf{g}] = \begin{pmatrix} \sum_{k=1}^p a_{1k}[g^1,f^k]_n & \cdots & \sum_{k=1}^p a_{1k}[g^p,f^k]_n \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^p a_{pk}[g^1,f^k]_n & \cdots & \sum_{k=1}^p a_{pk}[g^p,f^k]_n \end{pmatrix}$$

and

$$\begin{aligned} a_{X_n^p}[\mathbf{f},\mathbf{g}] &= \begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{p1} & \cdots & a_{pp} \end{pmatrix} \begin{pmatrix} [g^1,f^1]_n & \cdots & [g^p,f^1]_n \\ \vdots & \ddots & \vdots \\ [g^1,f^p]_n & \cdots & [g^p,f^p]_n \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=1}^p a_{1k}[g^1,f^k]_n & \cdots & \sum_{k=1}^p a_{1k}[g^p,f^k]_n \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^p a_{pk}[g^1,f^k]_n & \cdots & \sum_{k=1}^p a_{pk}[g^p,f^k]_n \end{pmatrix} \\ &= {}_{X_n^p}[a\circ_n^p\mathbf{f},\mathbf{g}]; \end{aligned}$$

3.

$$\begin{aligned} \left({}_{X_n^p}[\mathbf{g},\mathbf{f}]\right)_{ij} &= [f^j,g^i]_n \\ &= [g^i,f^j]_n^* \\ &= \left({}_{X_n^p}[\mathbf{f},\mathbf{g}]\right)_{ji}^* \\ &= \left({}_{X_n^p}[\mathbf{f},\mathbf{g}]^*\right)_{ij}; \end{aligned}$$

4. We shall now show that ${}_{X_n^p}[\mathbf{f},\mathbf{f}]$ is a positive element of $M^p(C^*(\mathbf{Z}^d))$. We shall make use of [Pas1, Proposition 6.1], which states that for a C^*-algebra $\mathcal{A}$, a matrix $\mathbf{a}=(a_{ij})\in M^p(\mathcal{A})$ is positive if and only if

$$\sum_{i,j=1}^p b_i^* a_{ij} b_j \geq 0 \quad \text{for all } b_1,\ldots,b_p\in\mathcal{A}$$

So suppose that $b_1,\ldots,b_p\in C^*(\mathbf{Z}^d)$, we then calculate that

$$\begin{aligned} \left[\sum_{i=1}^p f_i\circ_n b_i, \sum_{j=1}^p f_j\circ_n b_j\right]_n &= \sum_{i=1}^p\sum_{j=1}^p [f_i\circ_n b_i, f_j\circ_n b_j]_n \\ &= \sum_{i=1}^p\sum_{j=1}^p b_i^*[f_i,f_j]_n b_j. \end{aligned}$$

Now from Definition 0.4.1, $[\sum_{i=1}^p f_i \circ_n b_i, \sum_{j=1}^p f_j \circ_n b_j]_n \geq 0$, and so

$$\sum_{i=1}^p \sum_{j=1}^p b_i^* [f_i, f_j]_n b_j \geq 0.$$

But

$$[f_i, f_j]_n = \left({}_{X_n^p}[\mathbf{f}, \mathbf{f}] \right)_{ji},$$

so it follows from [Pas1, Proposition 6.1] that ${}_{X_n^p}[\mathbf{f}, \mathbf{f}]$ is positive.

5. Suppose ${}_{X_n^p}[\mathbf{f}, \mathbf{f}] = 0$, then for all i, j, $[f^j, f^i]_n = 0$ and in particular $[f^i, f^i]_n = 0$, which implies $f^i = 0$, and so $\mathbf{f} = 0$.

We need to show that X_n^p is complete with respect to the norm

$$ {}_{X_n^p}\|\mathbf{f}\| = \|{}_{X_n^p}[\mathbf{f}, \mathbf{f}]\|_*^{1/2}. \tag{3.48}$$

Suppose that $(\mathbf{f}^j)_{j \in \mathbf{N}}$ is a Cauchy sequence in X_n^p, then for all $\varepsilon > 0$, there exists $J \in \mathbf{N}$ such that if $j, k > J$, then ${}_{X_n^p}\|\mathbf{f}^j - \mathbf{f}^k\|^2 < \varepsilon^2$. For $(i_1, i_2) \in \{1, \ldots p\}^2$ we define $e_{i_1 i_2} \in M^p(C^*(\mathbf{Z}^d))$ to be

$$(e_{i_1 i_2})_{j_1 j_2} = \begin{cases} \mathbf{1} & \text{if } (i_1, i_2) = (j_1, j_2) \\ 0 & \text{otherwise} \end{cases} \tag{3.49}$$

From positivity we have the inequality

$$\begin{aligned} {}_{X_n^p}\|\mathbf{f}^j - \mathbf{f}^k\|^2 &= {}_{M^p(C^*(\mathbf{Z}^d))}\| \sum_{i_1, i_2} [f_{i_2}^j - f_{i_2}^k, f_{i_1}^j - f_{i_1}^k]_n e_{i_1 i_2} \| \\ &\geq {}_{M^p(C^*(\mathbf{Z}^d))}\| [f_{i_2}^j - f_{i_2}^k, f_{i_1}^j - f_{i_1}^k]_n e_{i_1 i_2} \| \quad \text{for all } i_1, i_2 \\ &= \| [f_{i_2}^j - f_{i_2}^k, f_{i_1}^j - f_{i_1}^k]_n \|_{C^*(\mathbf{Z}^d)} \\ &= \| f_{i_2}^j - f_{i_2}^k \|_n^2. \end{aligned}$$

So for every i_1, i_2, $\|f_{i_2}^j - f_{i_2}^k\|_n < \varepsilon$. In other words, for all i, $(f_i^j)_j$ is Cauchy, and so converges by the completeness of X_n. We set $f_i := \lim_{j \to \infty} f_i^j$. We then have that $\lim_{j \to \infty} \mathbf{f}^j = \mathbf{f}$, where $(\mathbf{f})_i = f_i$. We therefore have that X_n^p is complete.

We finally verify that X_n^p is full. Because X_n is full, for all $b \in C^*(\mathbf{Z}^d)$, there exists $f, g \in X_n$ such that $[f, g]_n = b$. It then follows that if $a \in M^p(C^*(\mathbf{Z}^d))$, we can choose $\mathbf{f}, \mathbf{g}$ such that for all i, j $[g^j, f^i]_n = a_{ij}$. This means that X_n^p is a full Hilbert $M^p(C^*(\mathbf{Z}^d))$-module. □

We can define the dilation $\mathcal{D}$ in the natural way acting componentwise on X_n^p, we then have for $\mathbf{f}, \mathbf{g} \in X_n^p$

$$ {}_{X_n^p}[\mathcal{D}\mathbf{f}, \mathcal{D}\mathbf{g}] = {}_{X_{n-1}^p}[\mathbf{f}, \mathbf{g}].$$

We can also define a *matrix downsampling operator* P_p which acts on matrices of functions on $\mathbf{Z}^d$ by

$$P_p(a_{ij}(\gamma)) = a_{ij}(\Delta\gamma). \tag{3.50}$$

As with P, we can use continuity to extend P_p to an operator on $M^p(C^*(\mathbf{Z}^d))$. Because P is a mapping from $C^*(\mathbf{Z}^d)$ to itself it follows that P_p is a mapping from $M^p(C^*(\mathbf{Z}^d))$ to itself. It follows from Lemma 3.1.2 that $P_p({}_{X_n^p}[\mathbf{f},\mathbf{g}]) = {}_{X_{n-1}^p}[\mathbf{f},\mathbf{g}]$. There is a corresponding upsampling operator which is given by

$$(P_p^* a)(\gamma) = \begin{cases} a(\alpha) & \text{if there exists } \alpha \text{ such that } \gamma = \Delta\alpha \\ 0 & \text{otherwise.} \end{cases} \tag{3.51}$$

Lemma 3.5.2 *Suppose $\alpha_0, \dots, \alpha_{m-1}$ is a set of coset representatives of $\Delta\mathbf{Z}^d$ in $\mathbf{Z}^d$. For $\mathbf{f} \in X_n^p$, we set $(\alpha_i \mathbf{f})(\gamma) = \mathbf{f}(\gamma - \alpha_i)$. Let $a, b \in M^p(C^*(\mathbf{Z}^d))$, the following identity is satisfied:*

$$\sum_{i=0}^{m-1} (P_p \alpha_i a)(P_p \alpha_i b)^* = P_p(ab^*). \tag{3.52}$$

Proof: We have

$$(P_p\alpha_i a)(P_p\alpha_i b)^* = \begin{pmatrix} \sum_{k=1}^p (P_p\alpha_i a)_{1k}(P_p\alpha_i b)^*_{1k} & \cdots & \sum_{k=1}^p (P_p\alpha_i a)_{1k}(P_p\alpha_i b)^*_{pk} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^p (P_p\alpha_i a)_{pk}(P_p\alpha_i b)^*_{1k} & \cdots & \sum_{k=1}^p (P_p\alpha_i a)_{pk}(P_p\alpha_i b)^*_{pk} \end{pmatrix}.$$

Therefore

$$\begin{aligned} \left(\sum_{i=0}^{m-1} (P_p\alpha_i a)(P_p\alpha_i b)^*\right)_{j_1,j_2} (\gamma) &= \left(\sum_{i=0}^{m-1}\sum_{k=1}^{p} (P\alpha_i a_{j_1k})(P\alpha_i b_{j_2k})^*\right)(\gamma) \\ &= \sum_{i=0}^{m-1}\sum_{k=1}^{p}\sum_{\beta\in\mathbf{Z}^d} (P\alpha_i a_{j_1k})(\beta)(P\alpha_i b_{j_2k})^*(\gamma-\beta) \\ &= \sum_{i=0}^{m-1}\sum_{k=1}^{p}\sum_{\beta\in\mathbf{Z}^d} a_{j_1k}(\Delta\beta-\alpha_i)\overline{b_{j_2k}(\Delta\beta-\Delta\gamma-\alpha_i)} \\ &= \sum_{k=1}^{p}\sum_{\beta\in\mathbf{Z}^d} a_{j_1k}(\beta)\overline{b_{j_2k}(\beta-\Delta\gamma)} \\ &= \sum_{k=1}^{p} (a_{j_1k} * b^*_{j_2k})(\Delta\gamma) \\ &= (P_p(ab^*))_{j_1j_2})(\gamma). \end{aligned}$$

□

Lemma 3.5.3 *Suppose $a, b, c \in M^p(C^*(\mathbf{Z}^d))$ and $\alpha_0, \dots, \alpha_{m-1}$ is a set of coset representatives of $\Delta\mathbf{Z}^d$ in $\mathbf{Z}^d$. Then $a = (P_p^* b)c$ if and only if for all i, $P_p\alpha_i a = b(P_p\alpha_i c)$.*

Proof: We have that

$$\Big((P_p^* b)c\Big)_{ij} = \sum_{k=1}^{p} (P_p^* b)_{ik} c_{kj} = \sum_{k=1}^{p} P^* b_{ik} c_{kj}$$

and

$$(bP_p\alpha_i c)_{ij} = \sum_{k=1}^{p} b_{ik}(P\alpha_i c)_{kj}.$$

The result now follows from Lemma 3.1.6. □

We shall use the Hilbert modules X_n^r and X_n^{mr} to prove some results about multiwavelets, where r is the order of the multiresolution analysis, and m is the index of the multiresolution structure. Recall from Theorem 1.1.11 that corresponding to the r scaling functions $\{\varphi^1, \ldots, \varphi^r\}$ there exists a multiwavelet with $(m-1)r$ elements $\{\psi^1, \ldots, \psi^{(m-1)r}\}$. We now make some more definitions.

To simplify our notation we write $\psi^{i,j} := \psi^{r(i-1)+j}$, where $i = 1, \ldots, m-1$ and $j = 1, \ldots, r$. We now define $\Phi \in X_0^r$ and $\Psi^1, \ldots, \Psi^{m-1} \in X_0^r$ by

$$\Phi := \begin{pmatrix} \varphi^1 \\ \vdots \\ \varphi^r \end{pmatrix}$$

and

$$\Psi^i := \begin{pmatrix} \psi^{r(i-1)+1} \\ \vdots \\ \psi^{ri} \end{pmatrix} = \begin{pmatrix} \psi^{i,1} \\ \vdots \\ \psi^{i,r} \end{pmatrix}.$$

To further simplify notation we also write $\Psi^0 := \Phi$, and $\psi^{0,j} := \varphi^j$. For $i = 1, \ldots, m-1$, we define

$$g^i := {}_{X_0^r}[\mathcal{D}^{-1}\Psi^i, \Phi] = \begin{pmatrix} [\varphi^1, \mathcal{D}^{-1}\psi^{i,1}]_0 & \cdots & [\varphi^r, \mathcal{D}^{-1}\psi^{i,1}]_0 \\ \vdots & \ddots & \vdots \\ [\varphi^1, \mathcal{D}^{-1}\psi^{i,r}]_0 & \cdots & [\varphi^r, \mathcal{D}^{-1}\psi^{i,r}]_0 \end{pmatrix}. \tag{3.53}$$

We call g^0 the *scaling filter*, and when $i = 1, \ldots, m-1$, we call g^i the *wavelet filter*. We now define the *wavelet matrix* $A \in M^{mr}(C^*(\mathbf{Z}^d))$ to be

$$A := \begin{pmatrix} P_r\alpha_0 g^0 & \cdots & P_r\alpha_{m-1} g^0 \\ \vdots & \ddots & \vdots \\ P_r\alpha_0 g^{m-1} & \cdots & P_r\alpha_{m-1} g^{m-1} \end{pmatrix}. \tag{3.54}$$

Theorem 3.5.4 *Suppose that $\Phi \in X_0^r$ is a vector of scaling functions for a multiresolution analysis $(V_n)_{n\in\mathbf{Z}}$ of degree r which corresponds to a harmonic multiresolution structure of index m. Suppose that $\psi^1, \ldots, \psi^{(m-1)r}$ are elements of X_0, the following conditions are equivalent:*

- *$\{\psi^1, \ldots \psi^{(m-1)r}\}$ is a multiwavelet with scaling functions $\varphi^1, \ldots, \varphi^r$;*
- *the wavelet matrix A is a unitary element of $M^{mr}(C^*(\mathbf{Z}^d))$;*

- *the shifted orthogonality conditions are satisfied, which means that for all* $i, j = 0, \ldots, m-1$,

$$P_r(g^i g^{j*}) = \delta_{ij} \mathbf{1}_r. \tag{3.55}$$

If the above conditions are true, let $\mathbf{f}$ *be an arbitrary element of* $X_0^r \cap X_1^r$, *we then have that the analysis part of fast wavelet transform holds*

$$_{X_0^r}[\mathbf{f}, \Psi^i] = P_r \left(_{X_1^r}[\mathbf{f}, \mathcal{D}\Phi] g^{i*} \right); \tag{3.56}$$

and the synthesis part of the fast wavelet wavelet transform holds

$$_{X_1^r}[\mathbf{f}, \mathcal{D}\Phi] = \sum_{i=0}^{m-1} \left(P_r^* {}_{X_0^r}[\mathbf{f}, \Psi^i] \right) g^i. \tag{3.57}$$

Proof: Let

$$W = \left\{ \sum_{i=1}^{m-1} \psi^i \circ_0 a_i \ : \ a_i \in C^*(\mathbf{Z}^d) \right\}$$

be the space spanned by $\psi^1, \ldots, \psi^{(m-1)r}$. To prove this theorem we first will show that if $\psi^1, \ldots, \psi^{(m-1)r}$ is a multiwavelet with scaling functions Φ, then the shifted orthogonality conditions hold. We then show that the shifted orthogonality conditions are equivalent to A being unitary. We next show that orthogonality implies the analysis part of the fast wavelet transform holds. We use this together with A being unitary to show that the synthesis part of the fast wavelet transform holds. We will then use the synthesis part of the fast wavelet transform to show that $\{\psi^1, \ldots \psi^{(m-1)r}\}$ is a multiwavelet with scaling functions $\varphi^1, \ldots, \varphi^r$.

So suppose $\{\psi^1, \ldots \psi^{(m-1)r}\}$ is a multiwavelet with scaling functions $\varphi^1, \ldots, \varphi^r$. We have that for $j = 0, \ldots, m-1$,

$$\begin{aligned}
_{X_0^r}[\mathcal{D}^{-1}\Psi^i, \Phi] \circ_0^r \Phi &= \begin{pmatrix} [\varphi^1, \mathcal{D}^{-1}\psi^{j,1}]_0 & \cdots & [\varphi^r, \mathcal{D}^{-1}\psi^{j,1}]_0 \\ \vdots & \ddots & \vdots \\ [\varphi^1, \mathcal{D}^{-1}\psi^{j,r}]_0 & \cdots & [\varphi^r, \mathcal{D}^{-1}\psi^{j,r}]_0 \end{pmatrix} \begin{pmatrix} \varphi^1 \\ \vdots \\ \varphi^r \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=1}^r \varphi^k \circ_0 [\varphi^k, \mathcal{D}^{-1}\psi^{j,1}]_0 \\ \vdots \\ \sum_{k=1}^r \varphi^k \circ_0 [\varphi^k, \mathcal{D}^{-1}\psi^{j,r}]_0 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{D}^{-1}\psi^{j,1} \\ \vdots \\ \mathcal{D}^{-1}\psi^{j,r} \end{pmatrix} \\
&= \mathcal{D}^{-1}\Psi^j.
\end{aligned}$$

The above equation is equivalent to the wavelets being contained in V_1^X. For all $i, j = 0, \ldots, m-1$, we obtain

$$\begin{aligned}
_{X_0^r}[\mathcal{D}^{-1}\Psi^i, \mathcal{D}^{-1}\Psi^j] &= {}_{X_0^r}\left[_{X_0^r}[\mathcal{D}^{-1}\Psi^i, \Phi] \circ_0^r \Phi, {}_{X_0^r}[\mathcal{D}^{-1}\Psi^j, \Phi] \circ_0^r \Phi \right] \\
&= {}_{X_0^r}[g^i \circ_0^r \Phi, g^j \circ_0^r \Phi] \\
&= g^i g^{j*} {}_{X_0^r}[\Phi, \Phi].
\end{aligned}$$

The following equation is equivalent to the orthonormality of the translations of the scaling functions

$$_{X_0^r}[\Phi, \Phi] = \begin{pmatrix} [\varphi^1, \varphi^1]_0 & \cdots & [\varphi^r, \varphi^1]_0 \\ \vdots & \ddots & \vdots \\ [\varphi^1, \varphi^r]_0 & \cdots & [\varphi^r, \varphi^r]_0 \end{pmatrix} = \mathbf{1}_r.$$

The following equation is equivalent to the orthonormality of the translations of the wavelets

$$_{X_0^r}[\Psi^i, \Psi^j] = \begin{pmatrix} [\psi^{i,1}, \psi^{j,1}]_0 & \cdots & [\psi^{i,r}, \psi^{j,1}]_0 \\ \vdots & \ddots & \vdots \\ [\psi^{i,1}, \psi^{j,r}]_0 & \cdots & [\psi^{i,r}, \psi^{j,r}]_0 \end{pmatrix} = \delta_{ij}\mathbf{1}_r.$$

This means that

$$_{X_0^r}[\mathcal{D}^{-1}\Psi^i, \mathcal{D}^{-1}\Psi^j] = g^i g^{j*}.$$

And so

$$P_r(g^i g^{j*}) = \delta_{ij}\mathbf{1}_r.$$

We have also proved that the shifted orthogonality conditions imply the orthogonality of the translations of the functions which correspond to the wavelet filters, and that they are contained in V_1. We now show that the shifted orthogonality conditions hold if and only if A is unitary. Note that we are now no longer assuming that $\{\psi^1, \ldots \psi^{(m-1)r}\}$ is a multiwavelet with scaling functions $\varphi^1, \ldots, \varphi^r$. The wavelet matrix A is unitary if and only if $AA^* = A^*A = \mathbf{1}_{mr}$. We now calculate AA^* by treating it as a "block matrix",

$$\begin{aligned} AA^* &= \begin{pmatrix} P_r\alpha_0 g^0 & \cdots & P_r\alpha_{m-1}g^0 \\ \vdots & \ddots & \vdots \\ P_r\alpha_0 g^{m-1} & \cdots & P_r\alpha_{m-1}g^{m-1} \end{pmatrix} \begin{pmatrix} (P_r\alpha_0 g^0)^* & \cdots & (P_r\alpha_0 g^{m-1})^* \\ \vdots & \ddots & \vdots \\ (P_r\alpha_{m-1}g^0)^* & \cdots & (P_r\alpha_{m-1}g^{m-1})^* \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{m-1}(P_r\alpha_k g^0)(P_r\alpha_k g^0)^* & \cdots & \sum_{k=0}^{m-1}(P_r\alpha_k g^0)(P_r\alpha_k g^{m-1})^* \\ \vdots & \ddots & \vdots \\ \sum_{k=0}^{m-1}(P_r\alpha_k g^{m-1})(P_r\alpha_k g^0)^* & \cdots & \sum_{k=0}^{m-1}(P_r\alpha_k g^{m-1})(P_r\alpha_k g^{m-1})^* \end{pmatrix} \\ &= \begin{pmatrix} P_r(g^0 g^{0*}) & \cdots & P_r(g^0 g^{m-1*}) \\ \vdots & \ddots & \vdots \\ P_r(g^{m-1}g^{0*}) & \cdots & P_r(g^{m-1}g^{m-1*}) \end{pmatrix}. \end{aligned}$$

The last equality was because of Lemma 3.5.2. We therefore have that

$$AA^* = \mathbf{1}_{mr} \text{ if and only if } P_r(g^i g^{j*}) = \delta_{ij}\mathbf{1}_r \text{ for all } i, j.$$

Now assume that A is unitary, we shall show that this means that $\{\psi^1, \ldots \psi^{(m-1)r}\}$ is a multiwavelet with scaling functions $\varphi^1, \ldots, \varphi^r$. We already know that because A is unitary, the functions $\{\psi^1, \ldots \psi^{(m-1)r}\}$ and their translations by elements of $\mathbf{Z}^d$ are an orthonormal set. We already know that the space W is contained in V_1, because we know that for all i, $\mathcal{D}^{-1}\psi^i \in V_0$. It is therefore sufficient to show that $V_1^X \subseteq V_0^X \oplus W$.

Suppose that $\mathbf{f}$ is an arbitrary element of X_0^r, we have that

$$\begin{aligned}
{}_{X_0^r}[\mathbf{f}, \Psi^i] &= P_{r\,X_1^r}[\mathbf{f}, \Psi^i] \\
&= P_{r\,X_1^r}\left[\mathbf{f}, {}_{X_1^r}[\Psi^i, \mathcal{D}\Phi] \circ_1^r \mathcal{D}\Phi\right] \\
&= P_{r\,X_1^r}\left[{}_{X_1^r}[\Psi^i, \mathcal{D}\Phi] \circ_1^r \mathcal{D}\Phi, \mathbf{f}\right]^* \\
&= P_r\left({}_{X_1^r}[\Psi^i, \mathcal{D}\Phi]{}_{X_1^r}[\mathcal{D}\Phi, \mathbf{f}]\right)^* \\
&= P_r\left({}_{X_1^r}[\mathbf{f}, \mathcal{D}\Phi]{}_{X_1^r}[\mathcal{D}\Phi, \Psi^i]\right) \\
&= P_r\left({}_{X_1^r}[\mathbf{f}, \mathcal{D}\Phi] g^{i*}\right)
\end{aligned}$$

which proves (3.56).

Using Lemma 3.5.2 we can write this as

$$\begin{pmatrix} {}_{X_0^r}[\mathbf{f}, \Psi^0] & \cdots & {}_{X_0^r}[\mathbf{f}, \Psi^{m-1}] \\ \vdots & \ddots & \vdots \\ {}_{X_0^r}[\mathbf{f}, \Psi^0] & \cdots & {}_{X_0^r}[\mathbf{f}, \Psi^{m-1}] \end{pmatrix} = \begin{pmatrix} P_r\alpha_{0\,X_1^r}[\mathbf{f}, \mathcal{D}\Phi] & \cdots & P_r\alpha_{m-1\,X_1^r}[\mathbf{f}, \mathcal{D}\Phi] \\ \vdots & \ddots & \vdots \\ P_r\alpha_{0\,X_1^r}[\mathbf{f}, \mathcal{D}\Phi] & \cdots & P_r\alpha_{m-1\,X_1^r}[\mathbf{f}, \mathcal{D}\Phi] \end{pmatrix} A^*.$$

This can be rewritten as

$$\begin{pmatrix} P_r\alpha_{0\,X_1^r}[\mathbf{f}, \mathcal{D}\Phi] & \cdots & P_r\alpha_{m-1\,X_1^r}[\mathbf{f}, \mathcal{D}\Phi] \\ \vdots & \ddots & \vdots \\ P_r\alpha_{0\,X_1^r}[\mathbf{f}, \mathcal{D}\Phi] & \cdots & P_r\alpha_{m-1\,X_1^r}[\mathbf{f}, \mathcal{D}\Phi] \end{pmatrix} = \begin{pmatrix} {}_{X_0^r}[\mathbf{f}, \Psi^0] & \cdots & {}_{X_0^r}[\mathbf{f}, \Psi^{m-1}] \\ \vdots & \ddots & \vdots \\ {}_{X_0^r}[\mathbf{f}, \Psi^0] & \cdots & {}_{X_0^r}[\mathbf{f}, \Psi^{m-1}] \end{pmatrix} A.$$

This is equivalent to saying that for all $i = 0, \ldots, m-1$,

$$P_r\alpha_{i\,X_1^r}[\mathbf{f}, \mathcal{D}\Phi] = \sum_{k=0}^{m-1} {}_{X_0^r}[\mathbf{f}, \Psi^k](P_r\alpha_i g^i).$$

By Lemma 3.5.3 this is equivalent to

$${}_{X_1^r}[\mathbf{f}, \mathcal{D}\Phi] = \sum_{i=0}^{m-1} \left(P_r^*\,{}_{X_0^r}[\mathbf{f}, \Psi^i]\right) g^i$$

which proves (3.57).

Suppose now that $f \in V_1^X$, and that $\mathbf{f} \in X_1^r$, is given by

$$\mathbf{f} = \begin{pmatrix} f \\ \vdots \\ f \end{pmatrix}.$$

We can then write

$$\mathbf{f} = {}_{X_1^r}[\mathbf{f}, \mathcal{D}\Phi] \circ_1^r \mathcal{D}\Phi.$$

Consider the projection given by

$$S\mathbf{f} := \sum_{i=0}^{m-1} {}_{X_0^r}[\mathbf{f}, \Psi^i] \circ_0^r \Psi^i.$$

This projection has the property that $(S\mathbf{f})_j = P_{V_0^X \oplus W} f_j = P_{V_0^X \oplus W} f$. We have that

$$\begin{aligned}
X_1^r[S\mathbf{f}, \mathcal{D}\Phi] &= X_1^r\left[\sum_{i=0}^{m-1} X_0^r[\mathbf{f}, \Psi^i] \circ_0^r \Psi^i, \mathcal{D}\Phi\right] \\
&= \sum_{i=0}^{m-1} X_1^r\left[(P_r^* X_0^r[\mathbf{f}, \Psi^i]) \circ_1^r \Psi^i, \mathcal{D}\Phi\right] \\
&= \sum_{i=0}^{m-1} (P_r^* X_0^r[\mathbf{f}, \Psi^i]) X_1^r[\Psi^i, \mathcal{D}\Phi] \\
&= \sum_{i=0}^{m-1} (P_r^* X_0^r[\mathbf{f}, \Psi^i]) g^i.
\end{aligned}$$

Now using (3.57) we obtain

$$X_1^r[S\mathbf{f}, \mathcal{D}\Phi] = X_1^r[\mathbf{f}, \mathcal{D}\Phi]$$

and so $X_1^r[S\mathbf{f}, \mathcal{D}\Phi] \circ_1^r \mathcal{D}\Phi = X_1^r[\mathbf{f}, \mathcal{D}\Phi] \circ_1^r \mathcal{D}\Phi$ which means that $S\mathbf{f} = \mathbf{f}$. This proves that $V_1^X = V_0^X \oplus W$. Therefore $\{\psi^1, \dots \psi^{(m-1)r}\}$ is a multiwavelet with scaling functions $\varphi^1, \dots, \varphi^r$. □

We remark that the proof of Theorem 3.5.4 made no use of the Fourier transform, so it could quite possibly be generalised to the case that $\mathbf{Z}^d$ and $\mathbf{R}^d$ are not are replaced by more general groups.

We shall use the above theorem to relate the fast wavelet transform for multiwavelets to representations of Cuntz algebras. Let us define the Hilbert space which will be our representation space. We let $M^p(l^2(\mathbf{Z}^d))$ be the $p \times p$ matrices with elements in $l^2(\mathbf{Z}^d)$, where p is any natural number. We have that $M^p(l^2(\mathbf{Z}^d))$ is a Hilbert space with inner product given by

$$\langle a, b\rangle = \sum_{j=1}^{p}\sum_{k=1}^{p}(a_{jk}b_{jk}^*)(0) = \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{\gamma\in\mathbf{Z}^d} a_{jk}(\gamma)\overline{b_{jk}(\gamma)}. \tag{3.58}$$

It is easy to verify that the inner product does make $M^p(l^2(\mathbf{Z}^d))$ into an inner product space. The fact that it is complete follows from $l^2(\mathbf{Z}^d)$ being complete.

We define the downsampling operator P_p and upsampling operator P_p^* on $M^p(l^2(\mathbf{Z}^d))$ in exactly the same way as they are defined on $M^p(C^*(\mathbf{Z}^d))$, in other words so they satisfy equations (3.50) and (3.51). We then have for $a, b \in M^p(l^2(\mathbf{Z}^d))$ that

$$\begin{aligned}
\langle P_p a, b\rangle &= \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{\gamma\in\mathbf{Z}^d} a_{jk}(\Delta\gamma)\overline{b_{jk}(\gamma)} \\
&= \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{\gamma\in\Delta\mathbf{Z}^d} a_{jk}(\gamma)\overline{b_{jk}(\Delta^{-1}\gamma)} \\
&= \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{\gamma\in\mathbf{Z}^d} a_{jk}(\gamma)\overline{P^* b_{jk}(\gamma)} \\
&= \langle a, P_p^* b\rangle.
\end{aligned}$$

We also have that P_p and P_p^* are contained in $B(M^p(l^2(\mathbf{Z}^d)))$ and have operator norm equal to 1.

For $a, b \in M^r(C^*(\mathbf{Z}^d))$, define the *filtering operator* $F_b : M^r(C^*(\mathbf{Z}^d)) \to M^r(C^*(\mathbf{Z}^d))$ by

$$F_b(a) = P_r(ab^*) \tag{3.59}$$

Equation (3.59) also defines an operator $F_b : M^r(l^2(\mathbf{Z}^d)) \to M^r(l^2(\mathbf{Z}^d))$ in exactly the same way. We shall abuse notation and write F_b for both of these operators. Now for $a, b, c \in M^r(l^2(\mathbf{Z}^d))$, we have that

$$\begin{aligned} \langle F_b a, c\rangle &= \langle P_r(ab^*), c\rangle \\ &= \langle ab^*, P_r^* c\rangle \\ &= \langle a, (P_r^* c)b\rangle. \end{aligned}$$

So if we define $F_b^* = (P_r^* c)b$, then $\langle F_b a, c\rangle = \langle a, F_b^* c\rangle$.

Lemma 3.5.5 *For $i = 0, \ldots, m-1$, let g^i be defined as in (3.53). The filtering operator F_{g^i} is a bounded operator on $M^r(l^2(\mathbf{Z}^d))$.*

PROOF: We have from the shifted orthogonality conditions (3.55) that for all $i = 1, \ldots, m-1$, $P_r(g^i g^{i*}) = \mathbf{1}_r$. For all i, write $g^i = (g^i_{k_1,k_2})$. A routine calculation shows that

$$(g^i g^{i*})_{k_1,k_2} = \sum_{k_3=1}^{r} g^i_{k_1,k_3} g^{i*}_{k_1,k_3}.$$

And so

$$\begin{aligned} (P_r(g^i g^{i*}))_{k_1,k_2} &= P(\sum_{k_3=1}^{r} g^i_{k_1,k_3} g^{i*}_{k_1,k_3}) \\ &= \sum_{k_3=1}^{r} P(g^i_{k_1,k_3} g^{i*}_{k_1,k_3}). \end{aligned}$$

Taking the Fourier transform it follows from (3.55) that for almost every $\zeta \in \mathbf{T}^d$,

$$\left(\sum_{k_3=1}^{r} \hat{P}(\hat{g^i}_{k_1,k_3} \overline{\hat{g^i}_{k_1,k_3}})\right)(\zeta) = 1. \tag{3.60}$$

Now for all k_1, k_3, and almost every $\zeta \in \mathbf{T}^d$, $\hat{g^i}_{k_1,k_3}(\zeta)\overline{\hat{g^i}_{k_1,k_3}}(\zeta) \geq 0$, and because $\hat{P}$ is positive (by Proposition 3.1.4) it follows that for all k_1, k_3,

$$\left(\hat{P}(\hat{g^i}_{k_1,k_3} \overline{\hat{g^i}_{k_1,k_3}})\right)(\zeta) \geq 0.$$

It therefore follows from (3.60) that

$$\left(\hat{P}(\hat{g^i}_{k_1,k_3} \overline{\hat{g^i}_{k_1,k_3}})\right)(\zeta) \leq 1.$$

And thus by (3.3),

$$\frac{1}{m}\sum_{\hat{\Delta}(\omega)=\zeta} \hat{g^i}_{k_1,k_3}(\omega)\overline{\hat{g^i}_{k_1,k_3}(\omega)} \leq 1.$$

Therefore $\hat{g^i}_{k_1,k_3}(\zeta)\overline{\hat{g^i}_{k_1,k_3}(\zeta)} \leq m$, and so for all k_1, k_3, i, it is true that $\hat{g^i}_{k_1,k_3} \in L^\infty(\mathbf{T}^d)$. We thus have that $g^i \in M^r(L^\infty(\mathbf{T}^d))$, and because $P_r \in B(M^r(l^2(\mathbf{Z}^d)))$, $F_{g^i} \in B(M^r(l^2(\mathbf{Z}^d)))$. □

We can express the fast wavelet transform in terms of filtering operators. Theorem 3.5.4 tells us that for $i = 0, \ldots, m-1$, and $f \in X_0^r \cap X_1^r$

$$X_0^r[f, \Psi^i] = F_{g^i} X_1^r[f, \mathcal{D}\Phi], \tag{3.61}$$

$$X_1^r[f, \mathcal{D}\Phi], = \sum_{J=0}^{m-1} F^*_{g^j} X_0^r[f, \Psi^j]; \tag{3.62}$$

where g^i are filters corresponding to scaling functions and multiwavelets Φ and Ψ^i.

Corollary 3.5.6 *Suppose that $\{\psi^1, \ldots \psi^{(m-1)r}\}$ is a multiwavelet with scaling functions $\varphi^1, \ldots, \varphi^r$, suppose that g^i for $i = 0, \ldots, m-1$ are corresponding scaling and wavelet filters as defined by (3.53). We have that*

$$F_{g^i} F^*_{g^i} = \mathbf{1}_r \quad \textit{for } i = 0, \ldots, m-1$$

and that

$$\sum_{i=0}^{m-1} F^*_{g^i} F_{g^i} = \mathbf{1}_r.$$

So the mapping $\pi : \mathcal{O}_m \to B(M^r(l^2(\mathbf{Z}^d)))$ which is defined by $\pi(S_i) = F^*_{g^i}$ is a $*$-representation of $\mathcal{O}_m$.

Proof: It follows from Lemma 3.5.5 that $F_{g^i} \in B(M^r(l^2(\mathbf{Z}^d)))$. So suppose $a \in M^p(l^2(\mathbf{Z}^d))$, let $f = a \circ_0^r \Phi$, we have that

$$\begin{aligned} a &= X_0^r[f, \Phi] \\ &= \sum_{i=0}^{m-1} F^*_{g^i} X_{-1}^r[f, \mathcal{D}^{-1}\Psi^i] \quad \text{by (3.62)} \\ &= \sum_{i=0}^{m-1} F^*_{g^i} F_{g^i} X_0^r[f, \Phi] \quad \text{by (3.61)} \\ &= \sum_{i=0}^{m-1} F^*_{g^i} F_{g^i} a. \end{aligned}$$

Now suppose that $a, b \in M^r(l^2(\mathbf{Z}^d))$, $\gamma \in \mathbf{Z}^d$, then we have

$$\begin{aligned} (F_b F_b^* a)(\gamma) &= P_r(P_r^* abb^*)(\gamma) \\ &= (P_r^* abb^*)(\Delta\gamma) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha \in \mathbf{Z}^d} (P^* a)(\alpha)(bb^*)(\Delta\gamma - \alpha) \\
&= \sum_{\alpha \in \mathbf{Z}^d} a(\alpha)(bb^*)(\Delta\gamma - \Delta\alpha) \\
&= (aP_r(bb^*))(\gamma).
\end{aligned}
$$

So if $P(bb^*) = \mathbf{1}$, then $F_b F_b^* a = a$. Since $(g^i)_{i=0,\ldots,m-1}$ satisfy the shifted orthogonality conditions, $F_{g^i} F_{g^i}^* a = a$ for $i = 0, \ldots, m-1$.

We therefore have that π is a $*$-representation of $\mathcal{O}_m$. □

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