# Wavelets and $C^*$-algebras

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# Abstract

A wavelet is a function which is used to construct a specific type of orthonormal basis. We are interested in using $C^*$-algebras and Hilbert $C^*$-modules to study wavelets. A Hilbert $C^*$-module is a generalisation of a Hilbert space for which the inner product takes its values in a $C^*$-algebra instead of the complex numbers. We study wavelets in an arbitrary Hilbert space and construct some Hilbert $C^*$-modules over a group $C^*$-algebra which will be used to study the properties of wavelets.

We study wavelets by constructing Hilbert $C^*$-modules over $C^*$-algebras generated by groups of translations. We shall examine how this construction works in both the Fourier and non-Fourier domains. We also make use of Hilbert $C^*$-modules over the space of essentially bounded functions on tori. We shall use the Hilbert $C^*$-modules mentioned above to study wavelet and scaling filters, the fast wavelet transform, and the cascade algorithm. We shall furthermore use Hilbert $C^*$-modules over matrix $C^*$-algebras to study multiwavelets.

**Key Words and Phrases.** Wavelet, filter, $C^*$-algebra, Hilbert $C^*$-module, cascade algorithm.

# Declarations

I certify that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge and belief it does not contain any material previously published or written by another person except where due reference is made in the text.

Peter Wood, Candidate

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# Contents

# List of Notation

| | | |
|---|---|---|
| $\mathbf{N}$ | Natural numbers | |
| $\mathbf{Z}$ | Integers | |
| $\mathbf{Z}_p$ | Integers mod $p$ | |
| $\mathbf{R}$ | Real numbers | |
| $\mathbf{C}$ | Complex numbers | |
| $\mathbf{T}$ | Unit circle | |
| | | |
| $\mathcal{H}$ | Hilbert space | |
| $\mathcal{F}, \wedge$ | Fourier transform | 1 |
| $\mathcal{A}$ | Involutive algebra | 4 |
| $\mathbf{1}$ | Unit in a unital involutive algebra | 4 |
| $B(\mathcal{H})$ | Algebra of bounded operators on a Hilbert space $\mathcal{H}$ | 4 |
| $U(\mathcal{H})$ | Unitary operators on a Hilbert space $\mathcal{H}$ | |
| $G$ | Locally compact group | 1 |
| $C(X)$ | Continuous functions on $X$ | |
| $C_c(X)$ | Compactly supported continuous functions on $X$ | |
| $C_0(X)$ | Continuous functions on $X$ which vanish at infinity | |
| $C^*(G)$ | Group $C^*$-algebra of $G$ | 9 |
| $\langle\ ,\ \rangle$ | Inner product in Hilbert space | |
| $[\ ,\ ]_E$ | $C^*$-algebra valued inner product in right Hilbert module $E$ | 11 |
| ${}_E[\ ,\ ]$ | $C^*$-algebra valued inner product in left Hilbert module $E$ | |
| $\mathcal{H}_{\mathcal{A}}$ | Standard Hilbert $\mathcal{A}$-module | 12 |
| $\mathcal{L}(E)$ | Adjointable operators on Hilbert module $E$ | 14 |
| $\mathcal{K}(E)$ | Generalised compact operators on Hilbert module $E$ | 14 |
| | | |
| $\Gamma$ | Group of translations for a multiresolution structure | 20 |
| $\mathcal{D}$ | Dilation for a multiresolution structure | 20 |
| $m$ | Index of a multiresolution structure | 20 |
| $\psi$ | Orthonormal wavelet | 20 |
| $\{\psi, \ldots, \psi^M\}$ | Orthonormal multiwavelet | 20 |
| $\Delta$ | $\Delta(\gamma) = \mathcal{D}^{-1}\gamma\mathcal{D}$ | 22 |
| $\Gamma^n$ | $\Gamma^n = \mathcal{D}^n\Gamma\mathcal{D}^{-n}$ | 22 |
| $\{V_n\}_{n\in\mathbf{Z}}$ | Generalised multiresolution analysis | 22 |
| $\varphi$ | Set of scaling functions for a multiresolution analysis of degree 1 | 22 |
| $\{\varphi^1, \ldots, \varphi^r\}$ | Set of scaling functions for a multiresolution analysis of degree $r$ | 22 |
| $\tilde{\mathcal{D}}$ | Group homomorphism used to generate dilation for a harmonic multiresolution structure | 27 |
| $\pi^n$ | Representation of $\Gamma$ on $\mathcal{H}$ which corresponds to action of $\Gamma^n$ on $\mathcal{H}$ for a harmonic multiresolution structure | 27 |
| | | |
| $X_\theta$ | Hilbert $C^*(\mathbf{Z}^d)$-module which corresponds to embedding $\theta : \mathbf{Z}^d \to \mathbf{R}^d$ | 57 |
| $[\ ,\ ]_\theta$ | $C^*(\mathbf{Z}^d)$-valued inner product for $X_\theta$ | 57 |

# Introduction

In this thesis we shall use some constructions employing $C^*$-algebras to prove results about wavelet theory. The main way that we shall do this is by constructing a Hilbert $C^*$-module using the $C^*$-algebra which is generated by a set of translations associated with a multiresolution analysis.

Wavelets are a tool that can be used to analyse an arbitrary function in terms of resolution and frequency. They do this by decomposing spaces of functions into an orthonormal basis, or more generally a Riesz basis or a frame. An orthonormal basis is a basis where each element is orthogonal to the others and has norm equal to one. A Riesz basis is the image of an orthonormal basis under an invertible operator. A frame is a set which spans the space but need not be linearly independent. Both orthonormal bases and Riesz bases are also frames.

Wavelets have numerous applications including image compression, artificial vision, telecommunications, denoising, seismic signal processing, and medical signal processing including tomography, computer aided mammography, and analysis of both ECG and EEG signals, to mention a few. More applications are described in [Me2], [Da1], and [KL]. Wavelet theory is relatively new, beginning in the early 1980's. Since then there have been literally thousands of papers published on the subject. Although modern wavelet theory began quite recently, there are deep connections between wavelet theory and earlier research, such as Littlewood-Paley theory [EG, LP1, LP2], Calderon-Zygmund operators, pyramid algorithms, and subband coding schemes.

$C^*$-algebras are normed Banach algebras which have an involution. They also have the property that they can be realised as bounded operators on a separable Hilbert space. Any commutative $C^*$-algebra can also be realised as an algebra of continuous functions on a compact Hausdorff space. Associated with any group there is a group $C^*$-algebra, and most of the $C^*$-algebras studied here will be group $C^*$-algebras. As well as group $C^*$-algebras, we shall also examine some work [J1, BJ1, BJ3] which relates wavelets to $C^*$-algebras known as Cuntz algebras. $C^*$-algebras are related to other fields of mathematics including dynamical systems, K-theory, topology, and noncommutative geometry.

We shall relate wavelets to $C^*$-algebras by using Hilbert $C^*$-modules, which we shall usually abbreviate as Hilbert modules. A Hilbert module is a generalisation of a Hilbert space for which the inner product takes its values in a $C^*$-algebra instead of the complex numbers. Hilbert modules can also be thought of as a generalisation of vector bundles [Sw, Hi], and as such they play an important role in noncommutative geometry. We shall use Hilbert modules to study wavelets by using methods which are very closely related to a construction announced in 1997 by Marc A. Rieffel of a Hilbert module over a group $C^*$-algebra associated with wavelets (see [R6, PR1, PR2]). A large amount of this thesis is devoted to understanding this construction.

Most of the background material that we require is contained in Chapter 0, where we study the Fourier transform, involutive algebras, group representations, group algebras, Hilbert modules, and bases and frames for Hilbert spaces and Hilbert modules. The reader who is already familiar with this material may wish to directly proceed to Chapter 1.

The classical definition of a dyadic orthonormal wavelet is a function $\psi$ such that the family

$$\left\{\psi_{j,k}(x) := 2^{-j/2}\psi(2^j x - k)\right\}_{j,k\in\mathbf{Z}}, \quad \text{for } x \in \mathbf{R}$$

is an orthonormal basis for the Hilbert space of square integrable functions, $L^2(\mathbf{R})$. The functions $\psi_{j,k}$ are obtained from $\psi$ by acting on it by translations and dilations. The translations and dilations are unitary operators on this Hilbert space, so they preserve inner products. In Chapter 1 we generalise the classical definition of a wavelet to an arbitrary Hilbert space in a manner similar to what has been done in [BCMO]. Associated with every wavelet is what is known as a generalised multiresolution analysis. Roughly speaking, a generalised multiresolution analysis (GMRA) of a Hilbert space is an increasing sequence of subspaces $(V_n)_{n\in\mathbf{Z}}$ of the Hilbert space, which approximate the Hilbert space more closely as $n$ approaches infinity. If the Hilbert space has an element $\varphi$ such that translations of $\varphi$ span the subspace $V_0$, we call the generalised multiresolution analysis a multiresolution analysis (MRA), and we call $\varphi$ a scaling function. We will prove in Theorem 1.1.11 that we can obtain wavelets when we have a multiresolution analysis; we use von Neumann algebras to prove this theorem. The projections onto the subspaces $V_n$ are closely related to an important numerical algorithm known as the fast wavelet transform. The investigation of the fast wavelet transform was what originally lead to the development of the notion of a multiresolution analysis, and is also closely related to the study of filter banks. We shall show in Chapter 1 that the fast wavelet transform still makes sense in this more general setting. We will mainly be looking at the case that the Hilbert space is a space of square integrable functions defined on a locally compact Abelian group. When this is the case it is possible to define the Fourier transform, and we shall often make use of the Fourier transform as a tool for examining wavelets.

Most of the author's new results are contained in Chapter 2 and Chapter 3. Chapter 2 is where we shall introduce the construction that relates wavelets to Hilbert $C^*$-modules. This construction is the main tool and object that is examined in this thesis. It is one of the aims of this thesis to demonstrate the importance and utility of this tool for understanding wavelets. The author's work on this construction was partially inspired by results announced in [R6]. The construction described here is very similar to a construction described in [PR2], which was released as an eprint not long before the submission of this thesis. In order to take into account the dilation, we define a chain of Hilbert modules $(X_n)_{n\in\mathbf{Z}}$ over the $C^*$-algebra of the translation group, $C^*(\mathbf{Z}^d)$. The $C^*(\mathbf{Z}^d)$-valued inner products used by these Hilbert modules are sometimes known as "bracket products". As well as studying the properties of bracket products on $X_n$, we shall also study the properties of bracket products on $L^2(\mathbf{R}^d)$. We shall work out the details of this construction on both the Fourier and non-Fourier domains. We show in Corollary 2.2.7 that the dilation is an adjointable operator which maps between the elements of the above chain of Hilbert $C^*(\mathbf{Z}^d)$-modules. In Chapter 2 we shall also define some Hilbert modules $(Y_n)_{n\in\mathbf{Z}}$ which are over the larger $C^*$-algebra $L^\infty(\mathbf{T}^d)$, and whose $L^\infty(\mathbf{T}^d)$-valued inner products are Fourier transformed bracket products. These Hilbert $L^\infty(\mathbf{T}^d)$-modules are similar to ones described in [CaLa, CoLa], which are used to study Gabor systems.

If a wavelet corresponds to a multiresolution analysis, there exist functions on $\mathbf{Z}^d$ whose Fourier transform is contained in $L^\infty(\mathbf{T}^d)$ which correspond to scaling functions and wavelets, and are known as scaling and wavelet filters. We examine wavelets from this perspective in Chapter 3. Associated with these filters are some operators from the $C^*$-algebra to itself associated with the fast wavelet transform. We shall examine the convergence properties of an algorithm for obtaining the scaling function from the scaling filter known as the cascade algorithm. It is then possible to obtain the wavelets from the scaling function using the wavelet filters. The cascade algorithm (Theorem 3.4.10 and Theorem 3.4.11) gives us necessary and sufficient conditions for elements of $C_c(\mathbf{Z}^d)$ to be scaling filters. We demonstrate that the cascade algorithm converges in the topology given by the Hilbert module norm, as well as in the norm topology on $L^2(\mathbf{R}^d)$. We shall investigate wavelet matrices in this chapter and see that they correspond to Hilbert modules over matrix $C^*$-algebras. Our results on wavelet matrices are encapsulated in Theorem 3.5.4, which also tells us necessary and sufficients conditions for elements of $C^*$-algebras to be wavelet filters, when we have a corresponding set of scaling functions.

Part of the aim of this thesis is to show how results in operator algebra theory are useful for studying wavelets. We want to in particular demonstrate the importance of the construction in Chapter 2 to wavelet theory. Because of the wide variety of applications of wavelet theory, this represents an interesting application of the theory of $C^*$-algebras and Hilbert $C^*$-modules.

# Bibliography


[Au] Auscher, P.
*Solution of two problems on wavelets*
J. Geom. Anal. 5 (2) (1995), 181-236

[BCMO] Baggett, L., Carey, A., Moran, W., Ohring, P.
*General Existence Theorems for Orthonormal Wavelets, an Abstract Approach*
Publ. RIMS, Kyoto Univ. 31 (1995), 95-111

[BDR] de Boor, C., Devore, R., Ron, A.
*Approximation from Shift-Invariant Subspaces of* $L_2(\mathbf{R}^d)$
Trans. Amer. Math. Soc. 341 (2) (1994), 787-806

[BJ1] Bratteli, O., Jorgensen, P. E. T.
*Isometries, Shifts, Cuntz Algebras and Multiresolution Wavelet Analysis of Scale N*
Integr. equ. oper. theory 28 (1997), 382-443

[BJ2] Bratteli, O., Jorgensen, P. E. T.
*Convergence of the Cascade Algorithm at Irregular Scaling Functions*
Contemp. Math. 247 (1999), 93-130

[BJ3] Bratteli, O., Jorgensen, P. E. T.
*Wavelets through a looking glass : the world of the spectrum*
Birkhäuser, Boston, (2002)

[BM] Baggett, L., Merrill, K.
*Abstract Harmonic Analysis and Wavelets in* $\mathbf{R}^n$
Contemp. Math. 247 (1999), 17-27

[BMM] Baggett, L., Medina, H., Merrill, K.
*Generalised Multiresolution Analyses, and a Construction Procedure for all Wavelet Sets in* $\mathbf{R}^n$
J. Fourier Anal. Appl. 5 (6) (1999), 563-573

[Ca] Calogero, A.
*Wavelets on general lattices, associated with general expanding maps of* $\mathbf{R}^n$
Elec. Res. Ann. Amer. Math. Soc. 5 (1999), 1-10

[CG] Calogero, A., Garrigós, G.
*A characterization of wavelet families arising from biorthogonal MRA's of multiplicity d*
J. Geom. Anal. 11 (2) (2001), 187-217

[CaLa] Casazza, P.G., Lammers, M.C.
*Bracket products for Weyl-Heisenberg frames*
Advances in Gabor Analysis, eds.: Feichtinger, H.G., Strohmer, T., Birkhäuser, Boston (2001),

[CoLa] Coco, M., Lammers, M.C.
*A Hilbert C∗-module for Gabor systems*
preprint math.FA/0102165 at www.arxiv.org (2001),

[Coh] Cohen, A.
*Ondellettes, analyses multirésolutions et filtres miroir en quadrature*
Ann. Inst. H. Poincaré, Anal non linéaire 7 (1990), 439-459

[Co] Connes, A.
*Noncommutative Geometry*
Academic Press, (1994)

[Da1] Daubechies, I.
*Ten Lectures on Wavelets*
Society for Industrial and Applied Mathematics, (1992)

[DB] Doran, R. S., Belfi, V. A.
*Characterisations of $C^*$-algebras*
Dekker, New York, (1986)

[DL] Dai, X., Larson, D. R.
*Wandering Vectors for Unitary Systems and Orthogonal Wavelets*
Memoirs. Amer. Math. Soc. 640 (1998),

[Dv] Davidson, K.
*$C^*$-algebras by Example*
American Mathematical Society, (1996)

[Dx1] Dixmier, J.
*$C^*$-algebras*
North-Holland Publishing Co., (1977)

[Dx2] Dixmier, J.
*von Neumann Algebras*
North Holland, (1981)

[DGH] Donovan, G.C., Geronimo, J. S., Hardin D. P.
*Orthogonal multiwavelet constructions: 101 things to do with a hat function*
Advances in Wavelets, Ka-Sing Lau (ed.), Springer-Verlag Singapore (1999),

[DS1] Duffin, R.J., Schaeffer, A.C.
*A class of nonharmonic Fourier series*
Trans. AMS 72 (1952), 341-366

[EG] Edwards, R. E., Gaudry, G. I.
*Littlewood-Paley and Multiplier Theory*
Springer-Verlag, New York, (1985)

[EsGa] Esteban, D., Galand, C.
*Application of QMF's to split band voice coding schemes*
Proc. IEEE Int. Conf. ASSP 34 (1986), 434-441

[Fi] Fischer, A.
*On Wavelets and Prewavelets with Vanishing Moments in Higher Dimensions*
J. Approx Theory 90 (1997), 46-74

[Fo] Folland, G. B.
*A course in abstract harmonic analysis*
CRC Press, Boca Raton, Florida, (1995)

[FL1] Frank, M., Larson, D. R.
*Frames in Hilbert $C*$-modules and $C^*$-algebras*
arXiv:math.OA/0010189 (1998),

[FL2] Frank, M., Larson, D. R.
*A module frame concept for Hilbert $C^*$-modules*
Contemp. Math. 247 (1999), 207-233

[FL3] Frank, M., Larson, D. R.
*Modular frames for Hilbert $C^*$-modules and symmetric approximation of frames*
arXiv:math.OA/0010015 (2000),

[Gb] Gabor, D
*Theory of Communication*
J. Inst. Elect. Eng. 93 (1946), 429-457

[Ga] Garrigós, G.
*Connectivity, homotopy degree, and other properties of $\alpha$-localised wavelets on* $\mathbf{R}$
Publ. Mat. 43 (1) (1999), 303-340

[GS] Garrigós, G., Speegle, D.
*Completeness in the set of wavelets*
Proc. Amer. Math. Soc. 128 (4) (2000), 1157-1166

[GLT] Goodman, T. N. T., Lee, S. L., Tang, W. S.
*Wavelets in wandering subspaces*
Trans. Amer. Math. Soc. 338 (2) (1993), 639-654

[Gr] Gröchenig, K.
*Analyse multi-échelles et bases d'ondelettes*
C. R. Acad. Sci. Paris, 305, Série I (1987), 13-15

[GM] Gröchenig, K., Madych, W. R.
*Multiresolution Analysis, Haar Bases, and Self-Similar Tilings of* $\mathbf{R}^n$
IEEE Trans. Info. Thy. 38 (2) (1992), 556-568

[Haa] Haar, A.
*Zur theorie der orthogonalen funktionen systems*
Math. Ann. 69 (1910), 331-371

[Han] Han, D.
*Wandering vectors for irrational rotation unitary systems*
Trans. Amer. Math. Soc. 350 (1998), 309-320

[HL1] Han, D., Larson, D. R.
*Frames, bases, and group representations*
Mem. Amer. Math. Soc. 147 (2000),

[HL2] Han, D., Larson, D. R.
*Wandering vector multipliers for unitary groups*
Trans. Amer. Math. Soc. 353 (2001), 3347-3370

[HLPS] Han, D., Larson, D. R, Papadakis, M., Stavropoulos, Th.
*Multiresolution analyses of abstract Hilbert spaces and wandering subspaces*
Contemp. Math 247 (1999), 259-283

[HR] Hewitt, E., Ross, K. A.
*Abstract Harmonic Analysis I*
Springer-Verlag, Berlin, (1963)

[HW] Hernández, E., Weiss, G.
*A first course on wavelets*
CRC Press, Florida, (1996)

[Hi] Higson, N.
*A Primer on $KK$-theory*
Proc. Sym. in Pure Math. 512 (1) (1990), 239-283

[J1] Jorgensen, O
*Harmonic Analysis of Fractal Processes via $C^*$-algebras*
Math. Nachr. 200 (1999), 77-117

[KL] Kahane, J.P., Lemarié-Rieusset, P.G.
*Fourier Series and Wavelets*
CRC Press, (1996)

[KR] Kadison, R. V., Ringrose, J. R.
*Fundamentals of the theory of operator algebras, Vols I and II*
Academic Press, New York, (1986)

[L] Lance, E. C.
*Hilbert $C^*$-Modules: A Toolkit for Operator Algebraists*
Cambridge University Press, (1995)

[Lr1] Larson, D. R.
*von Neumann Algebras and Wavelets*
Operator Algebras and Applications (1997), 267-312

[Lg1] Lang, W. C.
*Orthogonal Wavelets on the Cantor Dyadic Group*
SIAM J. Math. Anal. 27 (1) (1996), 305-312

[Lg2] Lang, W. C.
*Wavelet Analysis on the Cantor Dyadic Group*
Houston J. Math. 24 (3) (1998), 533-544

[Lw1] Lawton, W. M.
*Tight frames of compactly supported affine wavelets*
J. Math. Phys. 31 (8) (1990), 1898-1901

[Lw2] Lawton, W. M.
*Necessary and sufficient conditions for constructing orthonormal wavelet bases*
J. Math. Phys. 32 (1) (1991), 57-61

[Lw3] Lawton, W. M.
*Conjugate Quadrature Filters*
Advances in Wavelets (Springer, 1999), 103-119

[LP1] Littlewood, J. E., Paley, R. E. A. C.
*Theorems on Fourier series and power series I*
J. London Math. Soc. 9 (1931), 230-233

[LP2] Littlewood, J. E., Paley, R. E. A. C.
*Theorems on Fourier series and power series II*
Proc. London Math. Soc. 42 (1936), 52-89

[Me1] Meyer, Y.
*Wavelets and operators*
Cambridge University Press, (1992)

[Me2] Meyer, Y.
*Wavelets, algorithms and applications*
Society for Industrial and Applied Mathematics, (1992)

[PR1] Packer, J. A., Rieffel, M. A.
*Wavelet filter functions, the matrix completion problem, and projective modules over* $C(\mathbf{T}^n)$
arXiv:math.FA/0107231 v1, (2001)

[PR2] Packer, J. A., Rieffel, M. A.
*Projective multi-resolution analyses for* $L^2(\mathbf{R}^2)$
arXiv:math.FA/0308132, (2003)

[Pa1] Papadakis, M.
*On the dimension function of orthonormal wavelets*
Proc. Amer. Math. Soc. 128 (7) (2000),

[Pa2] Papadakis, M.
*Generalised Frame Multiresolution Analysis of Abstract Hilbert Spaces*
Preprint, http://www.math.uh.edu/ mpapadak/gfmra26.ps.gz (2001),

[Pas1] Paschke, W. L.
*Inner product modules over* $B^*$*-algebras*
Trans. Amer. Math. Soc. 182 (1973), 443-468

[Pe] Pedersen, G.
$C^*$*-algebras and their Automorphism Groups*
London Mathematical Society, (1979)

[RW] Raeburn, I., Williams, D.
*Morita Equivalence and Continuous-trace* $C^*$*-Algebras*
American Mathematical Society, (1998)

[RT] Raeburn, I., Thompson, S. J.
*Countably generated Hilbert modules, the Kasparov stabilisation theorem, and frames in Hilbert modules*
Proc. Amer. Math. Soc. 131 (5) (2003), 1557-1564

[R1] Rieffel, M. A.
*Induced Representations of* $C^*$*-Algebras*
Advances in Math. 13 (1974), 176-257

[R2] Rieffel, M. A.
*Morita Equivalence for Operator Algebras*
Proc. Sym. in Pure Math. 38 (1) (1982), 285-298

[R3] Rieffel, M. A.
*The cancellation theorem for projective modules over irrational rotation $C^*$-algebras*
Proc. London Math. Soc. 47 (2) (1983), 285-302

[R4] Rieffel, M. A.
*Projective modules over higher-dimensional non-commutative tori*
Can. J. Math. 60 (2) (1988), 257-338

[R5] Rieffel, M. A.
*Noncommutative tori - a case study of noncommutative differentiable manifolds*
Geometric and topological invariants of elliptic operators (Brunswick, ME, 1988), Contemp. Math. 105 (1990), 191-211

[R6] Rieffel, M. A.
*Multiwavelets and Operator Algebras*
Talk given at AMS special session (January 1997),

[Ru1] Rudin, W.
*Real and Complex Analysis*
Mcgraw-Hill, New York, (1987)

[Ru2] Rudin, W.
*Fourier Analysis on Groups*
Interscience Publishers, New York, (1962)

[SB] Smith, M. J. T., Barnwell III, T. P.
*Exact reconstruction techniques for tree-structured subband codes*
IEEE Trans. ASSP 34 (1986), 434-441

[SN] Strang, G., Nguyen, T
*Wavelets and Filter Banks*
Wellesley - Cambridge Press, (1996)

[St] Strang, G.
*Eigenvalues of $(\downarrow 2)H$ and convergence of the cascade algorithm*
IEEE Transactions on Signal Processing 44 (1996), 233-238

[Sw] Swan, R.
*Vector bundles and projective modules*
Trans. Amer. Math. Soc. 105 (1962), 264-277

[Tu] Turcajová, R.
*Compactly Supported Wavelets and Their Generalisations: An Algebraic Approach*
PhD Thesis, The Flinders University of South Australia, (1995)

[Vi1] Villemoes, L. M.
*Energy Moments in Time and Frequency for Two-Scale Difference Equation Solutions and Wavelets*
SIAM J. Math. Anal. 23 (6) (1992), 1519-1543

[W-O] Wegge-Olsen, N. E.
*K-theory and $C^*$-algebras*
Oxford University Press, Oxford, (1993)

[Wut] The Wutam Consortium
*Basic Properties of Wavelets*
J. Fourier Anal. Appl. 4 (4-5) (1998), 575-594