A Philosophy for the Powerful Learning of Mathematics

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To the Glory of God in Jesus' Name;

In loving memory of my parents, Richard and Gwendoline,

and

With great respect for John P. Keeves

Declaration

I certify that the thesis does not contain any material published or written previously by another person except where due reference is made in the text or end notes.

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Prologue (Abstract)

Human philosophy is fundamentally about Being-human. In contrast to the process−relational philosophy of Whitehead however, research into mathematics education has been almost exclusively analytical or meta-analytical. As a result the holistic and complex notion of Being-mathematical is largely ignored. Consequently the interaction between the philosophical and the practical in mathematics education remains limited, misdirected, and sometimes inappropriate. Therefore mathematical processes continue to be conceptually inaccessible for many individuals; understanding instrumental, and the difficulties encountered only partially overcome by rote or procedural learning.

The current study proposes a way forward through a dialogic complementarity of symbol processing and situated action that is ethical, informed by Dialogical Self Theory, and which promotes creativity and problem solving. In these terms the learning of mathematics is referred to as powerful mathematical learning, which is expounded as a phenomenological argument of eidetic intuitions that includes the development, the uses, and the meaningfulness of mathematics. Necessarily as a creative work, the first three stages of Wallas' process of creativity, namely, Preparation, Incubation, and Illumination are executed. The fourth stage of the creative process is a combination of Verification and Validation. As a precursor to a possible future research study, the essential ideas of powerful mathematical learning are conveyed as a systemic basis together with how the system can be examined logically and empirically, through the use of measurement principles and the employment of multi-level modelling strategies and causal structures.

Chapter One

Introduction: A Convoluted Scene

The Problem and Possible Solution: A Brief Outline

If Being-mathematical is a system then the teaching and learning of mathematics is a complex social system (Davis & Simmt, 2003; Hurford, 2010; Stigler & Hiebert, 1999). In these terms mathematics education is currently poorly understood, because in contrast to Whitehead's (1962, 1963) holistic understanding of education for example, mathematics education research has been almost exclusively 'part-specific' in a multitude of factors and different epistemologies (e.g., Begle, 1979; Hattie, 2009, 2012). Therefore mathematics education research has not grappled with the whole that is Being-mathematical. Consequently the interaction between scholarship and practice has been limited, misdirected, or even inappropriate (Coburn & Stein, 2010; Kilpatrick, 1988; Klein, 2003; Tirosh & Graeber, 2003; Wiliam, 2003, 2008). Thus for many students the processes that are mathematics remain conceptually inaccessible; understanding is often instrumental, and the difficulties that students face are relieved only partially by rote or procedural learning (Gray & Tall, 2007; Skemp, 1976; Stacey, 2010). Within the ambit or enworldedness of mass education (Chattopadhyaya, Embree, & Mohanty, 1992; Letteri, 2009), it is the exception rather than the norm for the individual to develop "informed powers of mind and a sense of potency in action" so that the person is able to learn mathematics meaningfully "across the transformations of time and circumstance" (Bruner, 1979, p. 122).

Therefore a piecemeal approach to mathematics education is no longer a viable option if this field of inquiry is to be sustained meaningfully in a dynamically changing world (Kilpatrick, 2010). However, if focused understandings of mathematics education are philosophized and theorized into wholes that make sense, then it is likely that the teaching and learning of mathematics can be enhanced globally because the world is globalizing.

Being and Being-mathematical

Being is a complex holism of interactive movements **—** actual, potential, historical, and socio-political, particularly in the sense that Being **is** communication, or at the very least, Being is inseparably intertwined with the locus of activity that is communication (Maheux & Roth, 2014; Nancy, 2000). If the movements relate to mathematics then a sub-complexity of Being is Being-mathematical. However, Being-mathematical cannot be known absolutely in either analytic or relational terms, because the whole is always more or less than the sum of the parts, and the gap between what is known and that which is unseen is continually open to interpretation and an imaginative dynamic. It is perhaps possible to experience the fullness of 'Being-human', at least in the moment, because of the human ability to grasp a sense of the whole through intuition, including the presence of the individual which may surpass or exceed the physicality of the individual. Consequently the intent of this study is to develop an intuitive understanding of the essence and possibilities of Being-mathematical, but not independently of a pegagogy that intentionally seeks to optimize, or enhance the teaching and learning of mathematics through a systems approach that is phenomenological.

Phenomenology is fundamentally a philosophy of intentional activity, together with the essences that emerge in-mind as a result of the intentionality associated with 'Being-there' (Husserl, 1927; Merleau−Ponty, 1962, 1964). An 'essence' is an eidetic intuition, or mental imagery experienced vividly and always with a feeling of certainty. Essentially therefore, that which has been intuited in the mind can be expressed accurately and in great detail through speech acts, thereby mediating the *noumenon* which is a thing-in-itself, and the *phainomenon*, which is to experience the thing-in-itself in terms of an embodied mind that exprsesses itself through an 'enactive' body. This implies a specific kind of bodily activity, namely, the activity of 'betweenness' that interrelates the *noumenon* and the *phainomenon*. That is the interrelating of a body and a mind with the self through the activity that occurs between the body and other bodies, or between the body and things, or between an embodied mind and things in-mind (Davis, 1996; Merleau−Ponty, 1962, 1968). The purpose of such activity is for the mind to grasp the invisible in relation to the visible which is made possible through sense perception and the intuitive functioning of an enactive body.

Therefore the potency of Being-mathematical is crucially dependent upon the integrity and the quality of the eidetic *phainomena* in relation to their corresponding *noumena*, as well as other *phainomena*. The *phainomena* are **things** generated in, or into consciousness, but such a substantiation in consciousness would not be possible without the person's five senses; the modality of Being that is intuition, and the existence of the things that are *noumena*, which have an independent reality, or are real independently from the mind and the body of the individual knower (Churchill & Richer, 2000; Dall'Alba, 2009; Mautner, 2005). In this regard a 'thing' has at least five defining attributes (Heidegger, 1967; Kurzweil, 2012; Marcus, 1993; Mautner, 2005):

- **(a)** it gives shape to a void, but cannot fill the void completely, and therefore the void that remains also shapes the thing;
- **(b)** it has at least one distinguishing characteristic;
- **(c)** an identity relationship can be specified meaningfully between a *phainomenon* and a *noumenon*;
- **(d)** a Being can draw near to it, or be part of it; and
- **(e)** it can be described and named.

Therefore Being-mathematical rests upon a 'psychology of the eidetic', or a focus on how intuition can be activated experientially so that the learner develops an ongoing sense of mathematical wholeness, or completeness as a result of '*noumena* encounters' that facilitate mathematical betweenness, insight, or 'seeing clearly mathematically' through the essences that are mathematical *phainomena* (Bruner, 1966, 1979, 1986; Husserl, 1927, 2002). However, the emergence of the things that are *phainomena* would not be possible if it were not for intuition as a synthesis of mind, or a modality of Being that

> fulfils, at the intellectual level, the function fulfilled by perception at the sensorial level: intuition is the direct, cognitive prelude to action (mental or practical). It organizes

> information in a behaviourally meaningful and intrinsically credible structure. (Fischbein, 1987, p. 56)

Importantly therefore, a broad overview of phenomenological writing suggests that an eidetic intuition is a 'mind−body social event' that can empower Being-mathematical with a culturally, historically, and technologically informed intentionality, or agentic feeling to act in terms of a future and goal oriented mental structure (Bakhtin, Liapunov, & Holquist, 1993; diSessa, 1983; Ihde, 1979, 1990; Levinas, 1973; Merleau−Ponty, 1962, 1964; Hegel, 1967; Ray, 1994; Roth, 2011; Sartre, 1947, 1957; Schutz, 1970; Shank, 2006; Tieszen, 2005; Van Manen, 2014). Therefore the *phainomena* of mind upon which this study is based represent a complex 'text for action', which is linked inextricably to the *noumenon* realities of the past−present and the present−future of the individual who wrote the narrative text, and in so being, Being-mathematical has been expressed anew (Ricoeur, 1991, 2002).

In general terms however, Being-mathematical is fuelled by an intentionality of consciousness that evokes, provokes, or invokes comportments of Being that are essentially backwards and forwards movements between Being and that which interrelates Being, including the object and the subject; the question and the inquiry, and the politics of society and the problem and its solution (Dreyfus, 1991; Engeström, 1995, 1999; Heidegger, 1970). In *Being and Time* for example, Heidegger (1927) grappled with

> the question of the meaning of Being [and concluded] there is no 'circular reasoning' but rather a remarkable 'relatedness backward or forward' which what we are asking about (Being) bears to the inquiry itself as a mode of Being of an entity. (p. 28)

Therefore an inquiry into Being is always situated, or grounded because 'Beingness', namely, the betweenness of the entity in his or her Being-there (*Da-Sein*) is constrained by being (*seiende*), which refers essentially to the ten categories that are Aristotle's physical substantiality. However, Being-there is not limited entirely to being because Being (*Sein*) is "bound up in a unique way with the awareness and unity proper to psychic life" (Heidegger, 1970, p. 332) — in part as a result of the human ability to abstract from, or beyond the situation through the use of symbols (Dewey, 1929b). Therefore although *Sein* reflects *seiende*, or is constituted in terms of *seiende*, most mathematics teaching and learning has not attained the essence of Being-mathematical through holistic eidetic or primitive understandings interrogated, or analysed for that which is repeatable and constant over time (Bakhtin, Liapunov, & Holquist, 1993; diSessa, 1983; Heidegger, 1967; Husserl, 1927).

Consequently, many individuals have not developed strongly as mathematical Beings or in the processes of mathematics; nor as a mathematical *Da-Sein* which requires fundamentally not only inquiry, but also creativity and other modalities of Being for the purpose of actualizing human mathematical potential. If however, students are to grasp the object that is Mathematics through mathematical processes, and ideally in the parsimonious and elegant sense of Ockham's razor (Fearn, 2001), then it is crucial that the individual engages with, or communicates relationally with the things that are

> mathematics as an expression of the human mind [which] reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality. Though different traditions may emphasize different aspects, it is only the interplay of these antithetic forces and **the struggle** [for emphasis] for their synthesis that constitute the life, usefulness, and supreme value of mathematical science. (Courant & Robbins, 1941, p. xv; also see Whitehead, 1911)

The human mind. Arguably Being-human cannot create, discover, construct, or even know what it means to know independently of mind, and the latter certainly has no essence without its brain, which is complex as part of a complexity of bodily systems. Moreover, if Being-mathematical is a system then the mind is blind to that system if the individual is not illuminated sufficiently in the essence of 'Being-mathematical', because Being-mathematical is the very vitality of a mathematical *Da-Sein*, which in turn is dependent upon an embodied mind to reveal its essence. However, Being requires a mind to be, but without an illuminated essence of Being, the mind is also blind to any possible system that may be experienced or utilized as Being-mathematical (Dreyfus, 1992; Heidegger, 1927).

Nevertheless, despite the meaning of ‘mind’ being “surprisingly elusive, so much so that one might wonder if there is any meaning conveyed by the term at all,” (McCarthy, 2010, p. 307) a neuroscientific insight has suggested that if the brain is an organ of the body then the mind is a personalised brain (Greenfield, 2003). Consequently in an intentional or goal-directed embodied sense, the “brain is what we have; the mind is how we use it” (Jensen, 2000a, p. 77). Therefore phenomenologically the humanity of Being, or Being-mathematical is inseparable from embodied minds that

> necessarily *includes* our brains but also are necessarily *not restricted* to our brains. This entails that minds are irreducible to our brains, *not* because they are in any way immaterial properties or facts, but instead because they are necessarily and wholly spatially spread throughout our living organismic bodies and belong to their complete neurobiological constitution. (Hanna & Maiese, 2009, p. 2)

Thus Being-mathematical must include all bodily systems which assist in giving rise to a complexity that can be referred to as a ‘mathematical mind’. It is feasible therefore that Being-mathematical can be enhanced by a physical system like an artificial intelligence, even though the potentiality of the human mind currently far exceeds the potentiality of any known non-human physical system (Carter, 2007). However, the field of Artificial Intelligence (AI) has become relevant to almost every intellectual task, because it involves a universality of being that “encompasses a huge variety of subfields, ranging from the general (learning and perception) to the specific, such as playing chess, proving mathematical theorems, writing poetry, driving a car on a crowded street, and diagnosing diseases” (Russell & Norvig, 2014, p. 1).

Popper’s Three Worlds. Being and Being-mathematical are situated because the body and its brain are both part of the real world. Although the mind is inextricably linked to the body, phenomenologically it is not reducible to the body. It is essentially more than the body because it includes an intuitive imagination that is capable of exceeding the realities of the real world, especially through mathematics and notions of the infinite. Consequently, the mind is subject to a reality of Being that may be comprehended as an embodied world that is inextricably linked to, but is nonetheless apart from the real world. Therefore the mind and the body involve two different worlds, but there is a third world, namely, the knowledge products and artefacts of human culture and creativity that have arisen as an interaction of the real and embodied worlds. These are Popper's (1978) **Three Worlds**, namely, three superordinate emerging complexities that are not only distinguishable, but also inseparable in making life possible for Being-in-the-world, or to be more accurate, Being-in-the-Three-Worlds.

Therefore the complex modality of Being-human, namely, Being-mathematical exists interrelationally in terms of Popper's (1978) Three Worlds. In essence Being-mathematical is a human expression involving the Three Worlds, which might be systemic because they can interrelate bi-directionally in ways that are goal directed. Thus Being-mathematical means to be part of a tripartite complexity which is essentially **Being-in-the-world**. However, Being-human is not only fuelled in the Three Worlds, but also plays a fundamental role in sustaining and changing life through these interrelating Worlds. Consequently, Popper's (1978) Three Worlds could not exist without the notion of Being, and vice versa. Therefore Being-human, or Being-mathematical is an exogenous and an endogenous variable in the complexity of the Three Worlds.

In the tradition of synthetic philosophy therefore, of which phenomenology is a manifestation, if Being-mathematical is to be understood, then the consciousness of Being-in-the-world must be illuminated in its fundamentals and of man's place in it. It is necessarily a combinatorial endeavour (Phillips, 2010). This means giving an "account of space, time and the world as we 'live' them" (Merleau−Ponty, 1962, p. vii).

At its core then, Being-mathematical is an integrative ontology of Popper's (1978) Three Worlds that involves bi-directional movements (actions) in *seiende* and *Sein* (Bakhtin, Liapunov, & Holquist, 1993; Ricoeur, 1991, 2002). As indicated in **Figure 0·1** (see p. xiii), each World mediates the other two Worlds through acts or modalities of Being that include minds, bodies, and technologies (Bakhtin, Liapunov, & Holquist, 1993). If however, the intent of the individual is to create mathematically, then phenomenologically these actions are in "the quest for original experiences" (Held, 2010, p. 92) that must involve mathematical things. In *Critique of Pure Reason*, Kant "introduced into philosophy the idea of the thing in itself. Things in themselves consist of the way the world really is, independently of how we may happen to think of or experience it" (Landesman, 1997, p. 2). Yet Being-mathematical as

> that which is necessarily hidden might be a Kantian thing-in-itself, a sort of thing, like the measles, that never shows itself except in its effects. But this cannot be what phenomenology deals with. The subject of phenomenology must be something that does not show itself but can be made to show itself. (Dreyfus, 1991, pp. 31−32)

Therefore the challenge in understanding Being-mathematical is for the latter 'to show itself' essentially and originally (Merleau−Ponty, 1962) *in toto* The Natural−Physical World (World 1), The Mind (World 2), and World 3, namely, Knowledge: The Cultural History and Creativity of Diverse Human Groups and Societies (Bruner, 1996; Keeves & Lakomski, 1999; Popper & Eccles, 1977). Consequently, although apparently constrained by time and space, World 1 includes "human 'originary' movements [that] arise from the incarnate capacities of our living body" (Bautista, Roth, & Thom, 2012, p. 367). These bodily capacities are crucial for the development of World 2 and the manifestations of World 3. In particular if it were not for the bodily capacities of the individual human mind, different mind−bodies would not be able to interrelate in terms of a 'group mind', or a collective knowing that was based on "consent and consensus" (Partridge, 1971). Furthermore, the latent construct that is 'The Mind' would be incapable of facilitating the growth and development of World 3 through 'extrapersonal' manifestations. These are the processes and products of new knowledge that are "sloughed off" by the creators and generators of the new knowledge, which subsequently "becomes independent of them. It is like a spider's web —

something that is created by and continues to be part of an organism, but is nevertheless an independent entity" (Almond, 2010, p. 299).

Thus a mathematical education in the Three Worlds implies an epistemological approach to teaching and learning that is energized by socio-cultural "corporeal-kinetic forms" (Bautista, Roth, & Thom, 2012, p. 367). These forms co-occur in an interactive society (or societies) whose products of individual and group minds are rooted in 'their' natural−physical worlds and the "knowledge creation metaphor," (Paavola & Hakkarainen, 2005, p. 535) because minds and bodies conjoin to establish new learning, which is essentially limitless through the process of creativity and diverse symbol systems that include artificial intelligences (Dewey, 1929b; Russell & Norvig, 2014; Wallas, 1926). Hence, if the purpose of learning is ultimately for the individual to affirm, expand, or introduce change into World 3, especially by giving rich expression to the bodily and mental actions of World 1 and World 2 respectively, then "education and training are the keys to the future. A key can be turned in two directions. Turn it one way and you lock resources away; turn it the other way and you release resources and give people back to themselves" (Robinson, 2011, p. 285).

Historically however, human potential and creativity have often been limited or underutilized (Peddiwell, 1939). Consequently, if Being-mathematical is to be optimized in the Three Worlds for a twenty-first century world, then it is essential to grasp the diversity and potentiality of time and space in the Three Worlds. In World 1 if time is linear and unidirectional then space appears to be confined to three dimensions, and vice versa. World 2 is very different. Time exists as a historical and future present. The idea of time is Aristotle's series of **nows** in a present that "can be understood on the basis of the 'withdrawal' which determines the mutual relation between the arrival as authentic future and the having-been as authentic past" (Held, 2010, pp. 91−92). Similarly mind-space has the capacity to be multi-dimensional in the present tense that is working memory. In this regard even infinity can be conceived intuitively and in unusual ways (e.g., consider infinity as a number to the right of the real number line). Although World 3 is a three dimensional reality it too is not static, because creativity tends to change the way in which human beings relate to one another in time and space through new knowledge, products and artefacts. Thus time and space are understood differently when that which exists is transcended by that which did not exist. As an example consider how innovative technologies have influenced the human dynamic since the beginning of the twenty-first century (e.g., the growth and development of complex virtual realities like social media).

Therefore comprehending time and space in the Three Worlds has important implications for Being-mathematical. In particular, Being-mathematical in the Three Worlds is empowered by the definitive property of the brain which is diverse movements and actions in time and space (Jensen, 2000b; Oaklander, 2008; Popper, 1979; Popper & Eccles, 1977; Sylwester, 2006; Zull, 2002). For example, consider Kant-based Arithmetic (and therefore basic Algebra) in relation to the intuition of time, or Geometry which is dependent upon the intuition of space (Fidelman, 1985; Kant, 1950). It is possible to make the invisible in World 1, visible in World 2, that is for example by expanding 1-, 2-, and 3-dimensional patterns into *n*-dimensional patterns which can be grasped logically and spatially through the imaginative use of mind-space. Consider for example the case of Poincaré's (1963) continuum of *n* dimensions, where the *nth* dimension can be divided into many regions through one or more cuts which are themselves continua of *n-1*dimensions.

Therefore an embodied mind in action in the time−space of the Three Worlds is essential if any individual in Being-there is to 'Be-mathematical', because mathematics is fundamentally a 'doing word with symbols' that unfolds as a relational activity in the communicative expression and situatedness that is Being-human (Wittgenstein & Diamond, 1989). In terms of neuroscience however, such action and interaction is made possible, or reinforced because of the automatic activation of at least two neural response mechanisms that facilitate intuitive functioning, namely, the mirror neuron system (MNS) and another system which engages the viscera-motor centres (VMC) of the body (Del Giudice, Manera, & Keysers, 2009; Gallese & Goldman, 1998). The MNS allows individuals to experience the phenomena that are embodied minds in action, and thus to develop an understanding of the intents, actions and goals of others. In turn, or simultaneously the VMC-system relates (or relays) the corresponding emotions and bodily feelings of the active person. Encouragingly for mass education therefore, Being-mathematical as a complex modality of Being-human can probably be taught and learned, because

> there is something shared between our first- and third-person experience of these [bodily] phenomena: the observer and the observed are both individuals endowed with a similar brain-body system. A crucial element of social cognition is the brain's capacity to directly link the first- and third-person experiences of these phenomena (i.e., link 'I do and I feel' with 'he does and he feels'). We will define this mechanism 'simulation'. (Gallese, Keysers, & Rizzolatti, 2004, p. 396)

The Context of Being and Being-mathematical

Globalization. In the twenty-first century individuals-in-society (Vygotsky, 1978) are being exposed to a plethora of meaning making, imagination, and reason. Thus 'to Be' in Popper's (1978) Three Worlds is to live in a globalizing world, and to "exist in the manner of having a world," (Varela, Thompson, & Rosch, 1991, p. 150) which can mean that "we are our world" (Letteri, 2009, p.13). That is in relation to an "increased networking and connectivity between peoples and knowledge, on the one hand, and the imposition of hierarchy and potentially exploitive power relations on the other hand" (Ernest, 2008, p. 34). However, from the perspective of philosopher "Rousseau to contemporary critical theory and deconstructionism," (Martin & Martin, 2010, p. 95) the 'globalizers' have 'globalized' the masses who are not really "free to discover their own talents and abilities," (Higgs & Smith, 1997, p. 154; also see Robinson, 2011) but to a large degree have become a heterogeneity of of discontents (Caruana, 2010; Stiglitz, 2003). In fact the consequences of globalization are highly uneven and inequitable across the earth (Eriksen, 2007; Giddens, 2011).

It is therefore not surprising that the major driving forces (in the time−space of Three Worlds) behind globalization are asymmetric power relations, advancing technologies, global media corporations, and free market capitalism (Derudder, Hoyler, Taylor, & Witlox, 2012). In particular, Taylor (2012) stated that there are hierarchies to be climbed as a result of competition, and networks to be developed through mutual exclusion and cooperation (Taylor, 2012). As a result the activities of people have diversified under the influence of an increasingly entangled urban system of key cities (e.g., New York, London, and Hong Kong). These cities have configured as a connective of transnational and variable actors, for example, capital, information, and a homogeneity of social, cultural, political and ideological aspects of humankind (Derudder, De Vos, & Witlox, 2012; Maringe, 2010).

However, even though humanity is divided by “belief systems, citizenship status, class, ethnicity, gender, nation, race and sexuality,” (Derudder, Hoyler, Taylor, & Witlox, 2012, p. 1) increased globalization (Ball & Forzani, 2007; Bandura, 2001; Derudder, Hoyler, Taylor, & Witlox, 2012; McMichael, 2000; Skovsmose, 2008) has meant that greater and greater numbers of people can “freely exercise their talents, decide where they want to live, and fashion their own identities” (Lechner & Boli, 2004, p. 8). For this reason human society as a global–local phenomenon is shaped by highly interconnected and fluid socio-cultural systems that self-organize and emerge as structures having “new properties, new functions and ... new elements” (Higgs & Smith, 1997, p. 299). Consequently a new field of inquiry, namely, Sociology and Complexity Science (Castellani & Hafferty, 2009), as well as the ontogenesis of ‘Being-dialogical’ (Bertau, 2004) suggest that “never in the history of humankind have global connections had such a broad reach and deep impact on the selves and identities of an increasing number of people” (Hermans & Hermans−Konopka, 2010, p. 21). While international flows can be traced back to antiquity “what is new is that, as the entire world becomes compressed, so human awareness of the world as an entity is heightened" (Caruana, 2010, p. 52). Thus the 'Beingness of Being-there' is increasingly dependent upon World 1 and World 3 for its identity, especially as mediated by advancing technologies.

Moreover, globalization as a transnational phenomenon (Eriksen, 2007; Yasukawa, 2010) has been described as a "runaway world" — "that package of changes" (Giddens, 1999, p. 3) which ideates the "speed and spontaneity of human action and events; intercultural fusion; great fascination with celebrity news; economic interconnectedness; labour export and exploitation of labour from poor countries" (Maringe, 2010, p. 19; also see Andersson, 1997; Costello, 2011; Engel, Rutkowski, & Rutkowski, 2009; Keeves & Darmawan, 2010). However, if education is to effect stable and desirable change over time (Bloom, 1964; Critchley, 1998), future oriented educators have argued for the **glocalization** of curricula (Atweh et al., 2008; Tien & Talley, 2012).

The term "glocalization" is a composite derived from the Japanese word *dochakuka* (Robertson, 1995) which encapsulates the notion that all teaching and learning should be globally informed, and inform locally, but always with reference to a historically and socially situated dialectic of values (Bruner, 1996; Greer, Mukhopadhyay, Powell, & Nelson–Barber, 2009; Maringe, 2010; Munck, 2009). For example, the motto of Victor Harbor High School in South Australia is **Local values, Global perspectives**.

Thus the complexity of Being that is Being-mathematical has no meaning or relevance apart from Popper's (1978) Three Worlds, because Being-human implies Being-in-the-world, or at the very least enworlded by Three Worlds that includes the local and the global. Hence, in 'Being-educated' for a globalizing–localizing world the individual needs to: (1) grapple with new situations of risk; (2) relate differently to uncertainty in the wake of external and manufactured risk (e.g., the world stock markets); (3) democratize democracy by questioning tradition and fundamentalism in open dialogue between equals; and (4) allow him or herself to be reshaped as a result of networking with various others (Giddens, 2011). If these principles are incorporated into mathematics education however, then such an approach to the teaching and learning of mathematics might help to undo the "dense web of power" (Brown & Walshaw, 2012, p. 2) that characterizes the socio-political dynamics of mathematics education in the Three Worlds. In other words Being-mathematical means to empower rather than disempower, or disenfranchise the individual from the possibility of contributing meaningfully and creatively to the present and future wellbeing of his or her community or society.

It is in this context that the teaching and learning of mathematics is of deep concern internationally (Archer, DeWitt, Osborne, Dillon, Willis, & Wong, 2012; Kinsler, 2010; Richland, Zur, & Holyoak, 2007; West, 2012; Williams & Lemons−Smith, 2009). Mathematics curricula are failing to enable young people to function sensitively and effectively in the order and chaos associated with a globalizing world (Waldrop, 1993). Therefore universities, colleges, schools, and other institutions of learning have a moral and societal obligation through education, and mathematics education in particular,

> to ensure that they supply young citizens from around the world with the deep understanding and the intellectual tools which they will need to become wise leaders of commerce, industry and politics in a world that is at once conceptually borderless and yet in some ways more fraught than ever by national conflicts. (UK/US Study Group, 2009, p. 2 as cited in Foskett & Maringe, 2010, p. 307; also see Brown, 1972)

Individual and Societal Wellbeing

The increasing belief, or World-view (*Welttanschauung*) that "societies that enable all citizens to play a full and useful role in the social, economic and cultural life of their society will be healthier than those where people face insecurity, exclusion and deprivation," (Wilkinson & Marmot, 2003, p. 11; also see Wilkinson & Pickett, 2009) has resulted in "the ascent of man" (Bronowski, 1973) into a world that demands constant change (Clark, 2008;

Critchley, 1998). In particular this world comprises knowledge-based communities and technocratic societies (Gardner, 2006a; Kolb, Boyatzis, & Mainemelis, 2001; Mautner, 2005; Runco, 2004; Trilling & Fadel, 2009). In these terms, Popper (1978) does not adequately describe complexity in the twenty-first century. In essence because the Three Worlds have developed sub-Worlds. In particular, collective World-views (e.g., the need for a paradigm shift in global−local ethics) and artificial intelligences are part of the World that is The Mind (World 2). These sub-Worlds interact with World 1 and The Knowledge Culture of Creative and Diverse Human Societies (World 3), especially through information and communications technology including virtual realities, for the purpose of enriching or influencing individuals as Beings-in-the-world (Allen, 2004; Lama, 2001; Mansheng & Keeves, 2003; Russell & Norvig, 2014; Simon, 1979).

The result is increasingly complex societies in which the meaningful learning of mathematics (and science) by all students is considered integral if the world is to develop in wellbeing (Holt & Marjoram, 1973; D. Siegel, 2010, 2012), or where equity of opportunity actually benefits the whole (Jurdak, 2009; Rubinstein et al., 2006; United States House of Representatives Bill 2170, 2009; Wieman, 2007; Wieman, Perkins, & Gilbert, 2010; Williams & Lemons−Smith, 2009). However, an epistemological fact in a world that has 'survived' post-modernism is the ubiquitousness of mathematics (D'Ambrosio, 2007), which indicates that even though the position of mathematics

> is rivalled by science, medicine, computing or English, unlike these subjects mathematics is taught universally from the beginning of schooling, its symbolism is universal, and its uses underpin the functioning of all modern societies. (Ernest, 2008, p. 23)

As a consequence the past 60 years has seen unprecedented numbers of students engaging in the study of mathematics at the primary, secondary and tertiary levels of education — hence, the term **mass mathematics education** (Connell, 1980; Fauvel & Van Maanen, 2000; Keeves, 1999; Keeves & Aikenhead, 1995; OECD, 2006a; Schoenfeld, 2004). It is therefore a view widely held that mathematics is an indispensable "filter to a variety of education and career opportunities," (Ingram, 2009, p. 233) and perhaps even a "draconian filter to the pursuit of further technical and quantitative studies," (Confrey, as cited in Vinner, 2007, p. 2) which are key if the world is to glocalize in a manner which enhances humanity.

Thus retrospectively it was predictable that the teaching and learning of mathematics (and science) would experience tremendous reform efforts post-World War II (Atweh, 2004; Begle, 1970; Keeves & Aikenhead, 1995; Kline, 1973; Senger, 1999; Tyack & Cuban, 1995; White−Fredette, 2010), because a healthy society, a vibrant democracy, and the schools of tomorrow require progressive citizens (Dewey, 1897, 1916; Dewey & Dewey, 1915; Piaget, 1973). Therefore if Being-mathematical is to add or multiply value in the complexity of Three Worlds and sub-Worlds, the individual must learn to embody a potency of action that includes the necessary skill and knowledge; attitudes and values, and the desire to act in a manner that can facilitate change for the better, and resist change for the worse (Hoskins, 2013). The learner needs to appreciate the difference between the two kinds of change. This means essentially that Being-mathematical in, and for a glocalizing world is fundamentally a question of ethics. Being-mathematical should facilitate the growth of an enlightened society (Pais & Valero, 2012) through the development of a civic literacy, a social conscience, a toleration and respect for diversity, as well as the ability to make and communicate sensible decisions (Bandura, 1986, 1997, 2001; Newell & Davis, 1988; Rilling & Sanfey, 2011). Consequently ethics need to play a basic role in **how** the Three Worlds interact as a positive complexity for both the person and the society in which the individual participates.

A poignant example pertains to 'climate change'. Climate science is complex; involves diverse modelling, and the scientific community has often not presented a unified or systematic position on the matter (Rasch, 2012). As a result the 'non-scientific' community is divided on the causes of global warming and what interventions are plausible and necessary.

However, most individuals do not have the conviction or the confidence to be ‘active citizens’ in this matter, because of a lack of scientific and values-based knowledge regarding the core issues (Holt & Marjoram, 1973; McCright, 2012). Although “understanding the complexity of global environmental problems, such as climate change, and proposed solutions, such as sustainability, usually requires collaboration across disciplinary boundaries by a range of scholars and stakeholders,” (McCright, 2012, p. 86) a grasp of complexity science and the ‘learning that transforms’ those who participate in it is a key for ‘life in our times’ (Downey, 2012; Goldstein, Hazy, & Lichtenstein, 2010; Leithwood, 1992; D. Siegel, 2010; Trilling & Fadel, 2009). Consistent with transformative learning and the conservation and sustainability of ecological systems, De Leo (2012a, 2012b) advocated ‘educating the whole person’ through an integrated interdisciplinary approach that can empower the individual to make informed judgements, and demonstrate leadership towards shared values for peace, justice, and human rights. Notably, in the *Adelaide Declaration on National Goals for Schooling in the Twenty-First Century*, it was stated that when all Australian students left

> school, they should have the capacity to exercise judgement and responsibility in matters of morality, ethics and social justice, and the capacity to make sense of their world, to think about how things got to be the way they are, to make rational and informed decisions about their own lives, and to accept responsibility for their own actions. (MCEETYA, 1999)

Crucially though, complexity science has come to the fore over the past two decades with the intent to understand and influence phenomena that emerge at the intersection of order and chaos (Johnson, 2009; Mitchell, 2009; Waldrop, 1993). It is an interdisciplinary field that uses advanced mathematics, computer science, and the natural and social sciences to model systems that comprise many interacting components (Castellani & Hafferty, 2009; Downey, 2012), including the modelling of **liminal spaces** (Le Ann, 2006; Munck, 2009). These are the spaces between people or objects, especially when interaction occurs (Goldstein, Hazy, & Lichtenstein, 2010).

In these terms the 'climate change' debate and the mathematics that underpins complex modelling can be related to Lundin's (2012) notion of *Hating School, Loving Mathematics*. This perception implies that it is not mathematics that limits most students from Being-mathematical in the Three Worlds, but rather the nature, pedagogy, and curriculum associated with the mathematics taught. If students however, learned mathematics that was particularly germane to climate science and sustainability for example, namely, the basics of mathematical complexity and dynamical systems (Meyers, 2011), and in ways that were transformative for them in the Three Worlds and sub-Worlds, then the majority of students would probably enjoy Being-mathematical.

Still no matter what mathematics is taught, it has been argued that "the route via mathematical thinking, in which we currently invest so much, is **a dead end** [for emphasis] and that we thus need to look for other ways forward" (Lundin, 2012, p. 83). The point is that Being-mathematical in the Three Worlds should involve significantly more than just conscious thinking. If students are to be mathematical learners who are relevant in their society (Bruner, 1971), then these individuals need to be afforded the opportunity to influence powerfully, or to even control their social and physical reality (Brown & Walshaw, 2012).

Dying a slow death. In terms of a postmodernist philosophy of 'the death of man, or the death of the subject' (Mautner, 2005, p. 484), a substantial and pertinent education in mathematics is lacking in many schools. Nonetheless, the global shift from "elite to mass education" (Hourigan & O'Donoghue, 2007, p. 463; also see Resnick, 1987, 2010; Tirosh & Graeber, 2003) has been driven by the notions that "modern society has very little place for unthinking manual labour," (Adey & Shayer, 2002, p.1) and "critical thinking is, in fact, a survival imperative in the twenty-first century" (Jensen, 2008, p. 143; also see McPeck, 1981). However, even though increasing proportions of students are undertaking tertiary studies (OECD, 2006a) — a trend which is likely to continue for at least the next two decades

(Foskett & Maringe, 2010) — of enduring concern internationally has been the under-preparedness of many students to engage effectively with mathematics when entering tertiary institutions or the world of work (Adey & Shayer, 2002; Hourigan & O'Donoghue, 2007; Hoyles, Noss, Kent, & Bakker, 2010). In Australia over the past decade as a case in point,

> students are abandoning higher–level mathematics in favour of elementary mathematics, that not enough trained mathematics teachers are entering the high school system, and that many university courses such as engineering that should include a strong mathematics and statistics component, no longer do. (Jourdan & Cretchley, 2007, p. 464; also see Nardi, 2010; Nzekwe−Excel, 2010; Tytler, Symington, & Smith, 2011; Vale, 2010)

It would appear therefore that society is circumventing the need for the individual in mass education to understand mathematics conceptually for him or herself. The structure of tertiary level learning is such that a large proportion of Australian students are either avoiding careers in the mathematical sciences altogether, or engaging with courses that only require an understanding of mathematics that is peripheral and instrumental. This social dynamic is troubling since "worldwide demand for new mathematical solutions to complex problems is unprecedented" (Rubinstein et al., 2006, p. 3).

Consequently, the pressing challenge for mathematics educators in Australia is not only to empower all students in terms of a changing construct that is basic numeracy (Kalantzis & Harvey, 2003; Perso, 2007; Stacey, 2010), but also to emphasize and role model mathematical insight and creativity as a basis for lifelong learning in a country whose tools of functioning are becoming increasingly embedded in mathematics (OECD, 2006a; Stillman & Brown, 2009). In fact all science and technology are grounded epistemologically in a developmental "control of the tools of mathematical structures, and not enough young people even become aware of their existence" (Dienes, 1964, p. 7). This statement is as germane today as it was when first written five decades ago in Adelaide.

Therefore if many more students in Australia do not learn mathematics in-and-for a globalizing world, then it is probable that by "2020 Australia will not be where we aspire to be," (Bradley, Noonan, Nugent, & Scales, 2008, p. xii) that is, having a strong workforce which is geared to compete globally as a consequence of informed risk taking, strategic planning, and performance which is characterized by innovation. This prediction would be supported unequivocally by Bruner (1979), because he could not "imagine an educated man a century from now who will not be largely bilingual ... in both a natural language and mathematics. For these two are the tools essential to the unlocking of new experience and the gaining of new powers" (p. 122).

Hence, in broad terms the meaningful learning of mathematics by significantly more students than is currently the case is essential if the aspirations of a developing twenty-first century world are to be realized (Australian Education Council, 1990; Lips & McNeill, 2009; West, 2012). Knowledge is expanding exponentially and both schools and universities are faced with the challenge of preparing students for "jobs that do not yet exist, to use technologies that have not yet been invented, and to solve problems that we don't even know are problems yet" (Darling−Hammond, 2008, p. 2). In an address to the National Press Club of Australia, a former Vice-Chancellor of the University of Queensland argued that

> to be competitive in a global knowledge economy, all nations must widen participation in higher education, while concentrating excellence at the top end of their university systems, that is both broadening the base and strengthening the top. Achieving both aims will help to guarantee Australia's competitiveness and prosperity into the future, create a fairer and more inclusive society and enable us to deal with future challenges, including those we have not yet imagined. (Greenfield, 2011)

Therefore if the majority of individuals are to become influential citizens with a global perspective, "Education is the only key to their future. Mathematics is in this sense a major tool to allow this key to turn in the keyhole of our society" (Alsina, 2002, p. 239). Many students however, especially in Western contexts do not take the learning of mathematics seriously (Sullivan, 2008). In part because of the increasing, or relative decline in mathematics teachers who have an awareness of how to be mathematical in the Three Worlds and sub-Worlds (Beswick, 2009; Lehmann, 2014; Long, Meltzer, & Hilton, 1970).

A Whole New Mind

In his book *A whole new mind: Why right-brainers will rule the future*, Pink (2005) suggests that a new age is currently evolving and emerging out of, or in flux with the Information Age, scilicet, the **Conceptual Age**. This is an age that goes beyond postmodernist philosophy (Baudrillard, 1989; Frie & Orange, 2009; Lyotard, 1984; Rosenthal, 1992) which "emphasises the elusiveness of meaning and knowledge," (Kirby, 2006) and "the absence of any ultimate bedrock of rationality, and of any ultimate foundations for science and ethics" (Mautner, 2005, p. 484). However, a new paradigm of authority and knowledge has come to the fore in global complexities that are new technologies and *au courant* social forces (Frie & Orange, 2009; Gardner, 1993; Kirby, 2006). Consequently, meaning making not only involves local accounts, and cultural and historical sensibilities, universal structures, essences, and overarching theories (Johnson, 1987; Shank, 2006), but also 'flow moments' when in "being digital" (Negroponte, 1996)

> you click, you punch the keys, you are 'involved', engulfed, deciding. You are the text, there is no-one else, no 'author'; there is nowhere else, no other time or place. You are free: you are the text: the text is superseded. (Kirby, 2006)

This is an example of how "digital age" (ISTE, 2012) technology; social and online media facilitates the '**create**−activity' of twenty-first century creators and empathizers (Pink, 2005; Shriki, 2010). It is through this kind of global interactive activity that humankind has been experiencing an emerging and largely undetected shift

> from an economy and a society built on the logical, linear, computerlike capabilities of the Information Age to an economy and a society built on the inventive, empathic, big-picture capabilities of what's rising in its place, the Conceptual Age. (Pink, 2005, pp. 1−2)

The 'New Age' (Malloch & Porter−O'Grady, 2009) and the 'Global Age' (Bell & de-Shalit, 2011; Berdan, 2011; Chiu, Tuan, Wu, Lin, & Chou, 2013) refer to attributes of the

Conceptual Age, which has been influenced conceptually by a postmodernist legacy that is largely existential rather than epistemological. In particular postmodernism has provided the Conceptual Age with an impetus for the global rethinking of “assumptions that had not kept pace with the lived experience of social life or with the conduct of oppositional political practice” (Rosenthal, 1992, pp. 104−105). Consequently, ‘whole new minds’ should develop in response to “an evocation to live with change, diversity and uncertainty” in a manner that is non-judgemental until ‘after’ the event (Biesta, 2012, p. 582). This might involve a Habermasian-like focus on social practices, institutions, and theories of cognition and formal linguistics but not as autonomous domains (Aylesworth, 2005; Higgs & Smith, 1997). Hence the future of human nature, conduct, or cognition ought to be located fundamentally within a paradigm that safeguards each person’s right to choose his or her destiny, but in relation to an ‘ethical freedom’ which respects an ‘equality of asymmetry’, namely, that of individual differences (Habermas, 2003; Stanovich, 1999). Therefore the future is not to fashion the political will of others, but to empower the other in the formation of his or her own political will as an active citizen who benefits society (Foucault, 1989).

However, throughout history ‘balance’ has been a key to human success (Clark, 2008). Therefore a successful and holistic thinker in the Conceptual Age is analytical, creative, and a wise practitioner, or a consummate balancer (Sternberg, 1985, 1999, 2007) who enables his or her mind systemically (Sternberg, 1990) for the express purpose of empowering his or her Being. It was the educational psychologist Gardner (2006b) who articulated different systems of mind, namely, *Five Minds for the Future*:

(1) The disciplined mind is proficient in at least one way of thinking — failure to do so implies that the individual is destined to be a follower not a leader.
(2) The synthesizing mind identifies information from disparate sources, and then learns to understand and evaluate the information contextually so that new information can be constructed, or generated in flows that make sense not only to the individual, but also to that person’s socio-cultural relations.
(3) Discipline and synthesis are prerequisites if **the creating mind** is to break new ground. New ideas come forth and unfamiliar questions are posed, which in turn

excite fresh ways of thinking with the intention of manifesting unexpected answers.

(4) The respectful mind appreciates differences among individuals and groups; attempting to interact with their diversity by working meaningfully with them.

(5) Although **the ethical mind** is more abstract than the respectful mind, it too is respectful as it ponders the complexity and nature of one's own industry in relation to self and a globalizing world.

System I and System II. A superficial understanding of Pink's (2005) Conceptual Age might lead to the premature conclusion that the logical−analytical abilities of the brain's left hemisphere are inferior to, or are less important than the global−intuitive abilities of the right hemisphere, if different and necessary possibilities of mind are to function optimally in the New Age (Gardner, 2006b; Sperry, 1983; Stanley, 1995; Trevarthen, 1990; Zull, 2002). Such a conclusion would be an error in judgement. Although right brain capability is "a powerful metaphor for interpreting our present and guiding our future," (Pink, 2005, p. 3) Nobel laureate Kahneman (2011) offered a more integrated notion of mind than the left brain−right brain division of human mental abilities (Jensen, 2008; Sousa, 2001, 2003, 2008).

Two systems of mind were described in detail by Kahneman: System I and System II. The two systems are not necessarily systems as defined in a relatively straghtforward sense by Meadows and Wright (2008), and consequently "there is no one part of the brain that either of the systems would call home" (Kahneman, 2011, p. 29). However, the use of familiar terminology makes the discussion more readable. The terms System I and System II have also been described as Type I and Type II processes respectively (Kahneman, 2011; Stanovich, 2011; Woods, 2012).

When faced with a (novel) problem the role of System I is to generate the intuitive idea; in turn System II interrogates the integrity of the idea as best it can: "The automatic operations of System I generate surprisingly complex patterns of ideas, but only the slower System II can construct thoughts in an orderly series of steps" (Kahneman, 2011, p. 21). In Heideggerian terms the interaction between System I and System II is perpetuated by an intuition of Being that is not easily satisfied. The outcome of 'Being-intuitive' is interpretation and re-interpretation, or the possibility of re-interpretation in order to attain an increasing feeling of certainty. In other words a single interpretation is 'never' complete, because an intentionality of consciousness 'knows no depth' phenomenologically, and therefore the problem solver cannot reach a 'final' or absolute understanding in the wake of a 'change dynamic' that is psychologically essential to Being-human, even though he or she is situated in Being-there (Heidegger, 1927, 1967, 1970).

Gödel's (1906−1978) incompleteness theorems, and G. E. Moore's (1873−1958) proofs of an external world suggested that the mind could not know itself completely nor could it be known absolutely (Landesman, 1997; Mautner, 2005). However, the mind as a personalized brain desires to know completely, if possible, which is perhaps an evolutionary driver in the survival of Being-human (Howells, 1973). By way of example, System I and System II can be influenced by the law of small numbers because the human mind favours certainty over doubt (Kahneman, 2011). Moreover, the bodily related feelings and intuitions (Aldous, 2006; Damasio, 2005; Niedenthal, 2007) of System I are

> not prone to doubt. It suppresses ambiguity and spontaneously constructs stories that are as coherent as possible. Unless the message is immediately negated, the associations that it evokes will spread as if the message were true. System II is capable of doubt, because it can maintain incompatible possibilities at the same time. However, sustaining doubt is harder work than sliding into certainty. (Kahneman, 2011, p. 114)

The agency and persona-like nature of System I and System II, or Type I and Type II processes are so different that they have been called separate 'minds', or interrelate as different 'selves' (Kahneman, 2011; Stanovich, 2011). The Type I process can be named Autonomous Storyteller (do not let the facts stand in the way of a good story — a quote that has been attributed to the famous American author Mark Twain); the Type II process, Reflective Skeptic —calculating, algorithmic, metacognitive, and the consciousness of 'Being-aware' in Three Worlds that is called 'I' (Kahneman, 2011; McPeck, 1981; Stanovich, 2009, 2011).

***I*-consciousness**. The Russian psychologist Vygotsky (1896−1934) as an activity theorist contended that "understanding between minds is impossible without some mediating expression" (Vygotsky, 1986, p. 7). In an enactivist or symbolic interactionist sense (Blumer, 1969, 1972; Davis, 1996; Engeström, 1987; Mead & Morris, 1962; Proulx, 2009), the expression of "bodily behaviour, inter-bodily resonance, and intentions that are made visible in action and the shared situational context" (Fuchs, 2012) are all mediating signs which are necessary to facilitate understanding between minds. However, the complexity of mediating signs is culturally situated and therefore with respect to "every child, we are justified in asking not only what his chronological age is, what his intellectual age is, but also at what stage of cultural development he is," (Vygotsky, 1997, p. 231) because

> every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (interpsychological), and then inside the child (intrapsychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relations between human individuals. (Vygotsky, 1978, p.57)

Thus the embodied mind in action (Hanna & Maise, 2009; Lakoff & Johnson, 1999), or the 'dancing brain' (Kalbfleisch, 2010), namely, "the body has a mind of its own" (Blakeslee & Blakeslee, 2008) develops intrapersonally as an emergent social structure reflecting, constituted, and grounded in its interpersonal relations. In other words the mind of the individual develops a society of mind as a consequence of being a mind in society (Vygotsky, 1978; Minsky, 1985). That is an "embodied interaction" (Fuchs, 2012) — interpersonal, intrapersonal, and extrapersonal (Sternberg, 2003a) — the "foundation of thought and willed action, the underlying mechanism by which the physical and psychological coordinates of the self come into being" (Wilson, 1998, p. 291).

In particular, the social world of the individual is variegated and dynamic in the extraordinary relationship that is hand and brain; hearing, gesture and thought (Goldin−Meadow, 2003;

McNeill, 1992, 2005; Napier, 1980; Wilson, 1999). Therefore, in 'Being and Becoming' Whitehead proposed that "mathematics should be an integral part of a new kind of liberal education, incorporating science, the humanities, and 'technical education' (making things with one's hands), thereby integrating 'head-work and hand-work'" (Woodhouse, 2012, p. 1). This view of education is very much consistent with Whitehead's (1861−1947) critical realist **process−relational philosophy,** namely, that the choices and actions of each individual affect the interrelated processes that constitute the world of humanity, including The Natural−Physical World of which physical bodies are an integral part (Mesle, 2008).

The denouement of Vygotsky's (1978) principle of mediation however, — rooted in the dialectical philosophy of Hegel and Marx (Hegel, 1967; Mautner, 2005; Mepham & Reuben, 1979; Paavola & Hakkarainen, 2005; Wegerif, 2011) — was also to overcome "the split between the Cartesian individual and the untouchable societal structure" (Engeström, 1999) in the sensibility that

> child development is a complex dialectical process characterized by periodicity, unevenness in the development of different functions, metamorphosis or qualitative transformation of one form into another, intertwining of external and internal factors, and adaptive processes which overcome impediments that the child encounters. (Vygotsky, 1978, p. 73)

Moreover, Vygotsky (1991) argued that as minds in different societies

> we become ourselves through others and that this rule applies not only to the personality as a whole, but also to the history of every individual function. This is the essence of the progress of cultural development expressed in a purely logical form. The personality becomes for itself what it is in itself through what it is for others. (p. 39)

However, Vygotsky's framework for a social formation of mind is limited by his monological and dialectical thinking (Wegerif, 2011). Interestingly though, Plato was also a dialectical thinker, but not monological. He had a penetrating mind that was highly intuitive and logical: "For he who can view things in their connexion is a dialectition, but he who cannot, is not" (Ulich, 1961, p. 60). Nonetheless to incorporate a dialogical nature of mind, the Vygotskyan framework was expanded by Wertsch (1985a, 1985b, 1991) through the use of Bakhtin's

(1981, 1984, 1986) ideas on dialogicality and the dialogic imagination. In other words the expansion was underpinned by the notion that the social life of the mind is characterized by social dialectics; contradictions, and "should not be viewed as a monologue in which only one voice, theme, or perspective is heard. Instead, social life should be conceived as a dialogue in which multiple opposing themes are given voice" (Baxter & Braithwaite, 2007, p. 287).

However, the notion of a dialogical mind is not new in human relations. In a Sanskrit epic of ancient India, *The Mahābhārata* (the other being *The Ramayana*), "different voices, often of a markedly different character and representing a multiplicity of relatively independent worlds interact to create a self-narrative" (Hermans & Kempen, 1993, p. 208). Thus through a dialogical structure the poetic composition "is presented as a series of nested conversations, many of which are consciously presented as lenses through which other conversations at other narrative levels might be reinterpreted" (Brodbeck & Black, 2007, p. 23).

Furthermore, although Activity Theory underwent a dialogical turn in the decade of the 1990s so that voice could be understood as a communicative mediated action (R. Engeström, 1995; Y. Engeström, 1999; Wertsch, 1991), it was Dialogical Self Theory (DST) that emerged strongly to bridge the 'Vygotskean gap' (Fernyhough, 2008, 2009) between the interpersonal and intrapersonal dimensions of 'Being-psychological'. Also drawing on neuroscientific research, it is noteworthy that the "frontoparietal mirror-neuron areas provide the basis for bridging the gap between the physical self and others through motor-simulation mechanisms" (Uddin, Iacoboni, Lange, & Keenan, 2007, p. 153).

Nevertheless, DST is a high level abstraction and "bridging theory in which a larger diversity of theories, research traditions and practices meet, or will meet, in order to create new and unexpected linkages" (Hermans & Gieser, 2012). Therefore DST is fast becoming a Conceptual Age dynamic in that it has brought together two disparate and fundamental concepts, namely, **self** and **dialogue**, both of which have different conceptual histories or traditions in psychology and philosophy. The 'self' has its roots in American pragmatism through the writings of James (1890); Mead (1932, 1938); Mead and Morris(1962); and Peirce (Goudge, 1950; Otte, 2006), and Dewey (1929a, 1929b, 1933, 1943, 1997). In contrast 'dialogue' has its essence in the ideas, reflections, and essays of Russian dialogist and philosophical activist Bakhtin (1981, 1984, 1986), and "the philosophical articulation of the dialogic principle (*das dialogische Prinzip*)" by Buber (1947) as descibed by Zank (2007). It is in particular however, the vivid and confronting phenomenology of Bakhtin that inspired the 'dialogue' in Dialogical Self Theory (Bakhtin, Holquist, & Liapunov; 1990; Bakhtin, Liapunov, & Holquist, 1993; Hermans & Gieser, 2012). Moreover, results from neuroscientific research are used to substantiate the understanding of 'self' in DST. For example, "the neural systems of midline structures and mirror neurons show that self and other are two sides of the same coin, whether their physical interactions or their most internal mental processes are examined" (Uddin, Iacoboni, Lange, & Keenan, 2007, p. 153). Consequently, in Dilaogical Self Theory

> the self is considered, at least in Western traditions, as a reflexive concept that deals with the question of which processes take place 'internally', that is, *within* the person, dialogue is taking place 'externally', that is *between* person and other. By bringing the two concepts together in the combined notion of 'dialogical self', **the between is interiorized into the within and reversibly, the within is exteriorized into the between** [for emphasis]. As a consequence, the self does not have an existence separate from society but is part of the society; that is, the self becomes a 'mini-society' or, to borrow a term from Minsky (1985), a 'society of mind'. Society, from its side, is not 'surrounding' the self, influencing it as an external 'determinant', but there is a society-of-selves, that is, the self is in society and functions as an intrinsic part of it. The consequence is that changes and developments in the self automatically imply changes and developments in society at large and reversed. In other words, self and society are not mutually exclusive but inclusive (Hermans, 2001). (Hermans & Gieser, 2012)

Therefore in the language of de Saint-Exupéry (French writer and aviator, 1900–1944), "Man is but a network of relationships, and these alone matter to him" (as cited in Merleau−Ponty, 1962, p. 456). Therefore **learning** implies growth or development in the network of relationships that constitutes Being-human. In terms of a dialogical self, mathematical learning means learning **how** to dialogue mathematically and relationally in terms of the Three Worlds. In learning to be dialogical therefore, it is not possible for the self to remain static over time (Critchley, 1998).

The dialogical self involves a dynamic complexity of embodied ***I*-positions**, where the *I*-complexity, or the agentic *I*-self, or simply the *I*, is not only the **subject** of its Being, but also corresponds to the *Me* as **object,** or the point of its Being. Moreover, each *I*-position is linked inextricably to relationships and activities within the person, and between the person and other persons or things. Thus *I*-self is intrinsic to a socially and historically situated 'inside–outside' world in which the Three Worlds interconnect.

The embodied *I* as an intentional and conscious self (Churchill & Richer, 2000) can move or fluctuate between similar or different positions, or positions in opposition. The positions, repositions and counter-positions of the *I* can manifest "within the self and between self and perceived or imagined others" (Hermans & Gieser, 2012). Therefore each of these positions has the potential to inter- or intra-relate through multiple exchanges that reflect dominance and social power as part of sign-mediated or embodied social relations. The *I*-positions for example, can language like characters in a narrative play, opera, soliloquy, or a Tarkovsky-like science fiction sequence (Bird, 2008). Furthermore, each *I*-position can represent one or more of Gardner's (2006b) 'five minds', thereby faciliting a rich dialogue in different 'minds'. That is the dialogical self has the potential through its *I*-positions to articulate the viewpoint of a single mind; multiple minds, or even compositions of minds. Nevertheless, the self is socialized epistemologically because it has

> no ontology that is independent of the methods used to describe it or show it or think it: it is coterminous with the terminology and iconography used in each situation and context in which it is presented, described or conceived. Such methods, however, and the terminologies that issue from them are *reflexively potent*: they become signs that project inwards to the mind of the initiating subject and outwards into the thinking process of others. The fundamental status of self, then, is that of a sign that produces a double interpretant — one by a thinking process and the other by the thinking process of another person. (Perinbanayagam, 1991, p. 317)

Therefore the *I*-processes of Being-mathematical might involve *alter ego* different forms of (Socratic) dialogue and reflection (Knezic, Wubbels, Elbers, & Hajer, 2010); together with the assimilation and accommodation of new knowledge that was stimulated by cognitive equilibrium and disequilibrium (Inhelder & Piaget, 1958; Gruber & Vonèche, 1977), as well as discursive and intuitive question and answer performances

> that include productively challenging colleagues [or peers], paraphrasing, and interpreting presentations by others. And although individual performances still matter, much 'knowledge work' is 'distributed', involving collaboration with others. (Resnick, 2010, p. 186)

Thus although each voice communicates a story from a personal perspective, it is essentially the dialogic activity that structures the self in terms of a self-organizing or systems-based community (Csikszentmihalyi, 1990, 1994; Hermans & Hermans−Konopka, 2010; Iiskala, Vauras, Lehtinen, & Salonen, 2011). The dialogic activity between and within people is fuelled by the complexity science principles of **internal diversity** (e.g., different *I*-positions, as well as various human potentialities, abilities, beliefs, interests, and prior learning); **redundancy** (overlap of ideas or actions); **decentralized control** (in the Conceptual Age it is not prudent for the *I*-self to centre in terms of a single *I*-position); **organized randomness** (e.g., learning spaces that allow individuals to engage in structured activities freely), and **neighbour interactions** that involve people and things.

These principles are learning principles because if applied holistically, they can facilitate growth and development in the self as a result of *I*-interactions in-mind and between the person and others. Consequently, the functioning *I*s constitute the complex *I* in the entity referred to as *Me*, namely, a society of mind that facilitates bodily activity in the space and time of the Three Worlds (Hermans & Gieser, 2012). It is in these terms that the self is agentic (Bandura, 1997, 2001; Davis & Simmt, 2003; James, 1890).

Therefore when co-learning mathematics in a community of inquiry, it is the agency of the individual self, that is, the *I* in *Me* who facilitates interpersonal relations between active bodies; resulting in the ‘making of mathematics’ which is referred to as mathematization (Jaworski, 1996, 2004). As a consequence then of the process and product that is mathematization, the *I*s develop in their complexity which is a goal of education, or an increasingly complex self (Csikszentmihalyi, 1990, 1997). In effect the self as agent “is synonymous with the ‘I’ of William James,” (VandenBos, 2007, p. 828) namely, “the subjective relation between I-as-subject and me-as-object” (Roeser, Peck, & Nasir, 2006, p. 394). In essence therefore this relation is enriched, or empowered if the individual gets “back to the naked immediacy of experience as it is felt from within the utmost particularity of a specific life” (Bakhtin, Liapunov, & Holquist, p. *x*, 1993; also see Husserl, 1927). That is by intentionally ‘making mathematics’, or Being-mathematical within a community of inquiry

> things happen by one’s actions. Agency embodies the endowments, belief systems, self-regulatory capabilities and distributed structures and functions through which personal influence [is] exercised, rather than residing as a discrete entity in a particular place. The core features of agency enable people to play a part in their self-development, adaptation, and self-renewal with changing times. (Bandura, 2001, p. 2)

Summary insights: A Whole New Mind. A mind for the future is a social complexity that includes a body that thinks (Kahneman, 2011) and acts (Bandura, 2001) through Type I and Type II processes. Consequently, Being-in-the-world (Heidegger, 1927; Dreyfus, 1991) while Being-to-the-world (Davis, 1996; Merleau−Ponty, 1962; Meurs, 2012) is feasible through an embodied mind that is agentic. The essence of Being-mathematical therefore is to enrich and enable the relationship between Being-in-the-world and Being-to-the-world. Excitingly so, this is possible through an extended self that is dialogical in intentionality and agentic through *I*-consciousness (Chisholm, 2005; Jacquette, 2004; Prinz, 2012). In other words the dialogical self is constituted intentionally and agentically through thinking and affective dispositions (Goldin, 2000; Stanovich, 2011; Woods, 2012) that include the ‘global−intuitive’ (Fischbein, 1987; Noddings & Shore; Sinclair, 2010); the ‘linear−analytic’ (Goldin−Meadow, 2003; McNeill, 2005); the ‘dialectical’ (Merleau−Ponty, 1974; Riegel, 1975), and persona-like *I*-positions that can articulate as a disciplined, synthesizing, creating, respectful, and ethical community-in-mind.

Thus the self, extending beyond the physical body of the individual (Hermans & Hermans−Konopka, 2010), is reflected in a mind−body as "an oral society in which the present is currently running a dialogue with the past and the future inside of one skin" (Antin as cited in Hermans & Gieser, 2012). This is only possible because minds "are made and molded in and through the mirror of others, thus designing themselves after others;" (Prinz, 2012, p. xvi) concomitant with the language forms that are metaphor, metonymy, and synecdoche which underpin the "imaginative dynamics of conceptual blending" (Fauconnier & Turner, 2002, p. xii; also see Panther, Thornburg, & Barcelona, 2009; Peddiwell, 1939). With the aid of different language forms, the student conceptualizes himself as "a second Plato, a sort of re-author of his Dialogues, and thus and only thus he understands those Dialogues," (Ryle, 1949, p. 57) especially in terms of a mind that

> is defined by its membership in a collective of other minds. The paradigmatic example of Enlightenment neuroscience is the study of vision (the isolated mind that looks out on the world); the paradigmatic example of pragmatist neuroscience is the study of the brain's social responsiveness (minds created by community) ... . (Brothers, 1997, p. 108)

Essentially therefore, the 'Vygotskyan gap' between the interpersonal and intrapersonal planes of the mind is bridged through an ontology of self that is dialogical — in the sense that a society of mind 'socializes' in relation to a mind in society.

Creators and Empathizers

The Conceptual Age is propelled and energized by creators and empathizers who not only solve problems in unfamiliar contexts in standard ways, but also associate apparently unrelated objects and events intuitively, or perhaps even systemically, yielding new and diverse constructions that can be interrogated analytically or tested empirically (Aldous, 2007; Nonaka & Takeuchi, 1995; Peddiwell, 1939; Pehkonen, 1997; Shriki, 2010). For example, in the "world of industrial mathematics there may be no textbook to map the path," (Pollak, 1987, p. 254) and consequently creators and empathizers are needed in mathematics, because

> the traditional reductionist methods of the physical sciences and engineering are no longer adequate to answer many of the questions raised in an industrial environment. Today's problems are complex and nonlinear, they involve phenomena on multiple length and time scales, and their analysis can extend well beyond the realm of textbook mathematics. (OECD, 2008, p. 6; also see English & Sriraman, 2010)

The Japanese thinker, Masuda (1985) conjectured that *homo sapiens* was evolving into a more complex human species, namely, *homo intelligens* as a result of (meta-) activity in the left and right frontal lobes (Fluellen, 2005; Goswami, 2004, 2008; Hartman, 2001; Iiskala, Vauras, Lehtinen, & Salonen, 2011); finger dexterity (Wilson, 1998), and language ability (Albert, Connor, & Obler, 2000). Although with a different intent, McNeill (2005) hypothesized a complex Brain Model that systemized a dedicated thought−language−hand link, which if Masuda (1985) is correct, can play a crucial role in the emergence of *homo intelligens*. Moreover, if the dedicated thought−language−hand link is compared to an orchestral recital, then the conductor of the symphony is thought to be Broca's area, and in close support is Wernicke's area (McNeill, 2005, 2012). **Figure 1·1** depicts Broca's area and Wernicke's area in proximity to other functional areas of the complex human brain.

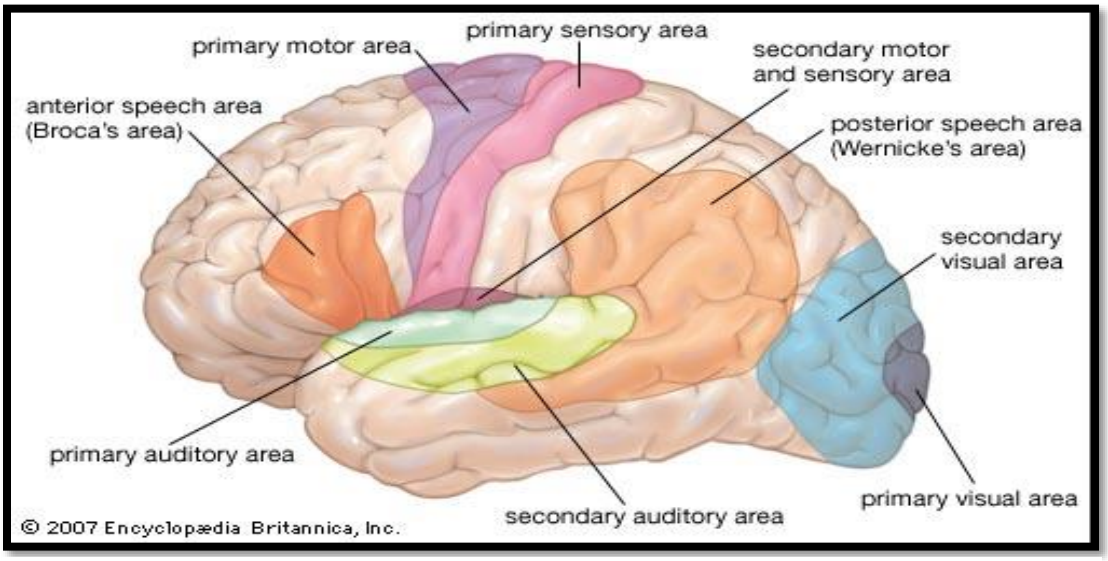


Figure 1·1. Functional areas of the human brain (Source: http://www.britannica.com/EBchecked/topic/135877/Broca-area)

In empathizing with the ‘human condition’, Hawking (2008) maintained that the twenty-first century would be the century of ‘complexity’ (Davis & Simmt, 2003), and if humankind were to survive beyond the next 100 years humanity needed to begin a sustained effort to move into space. The point being that human existence involves interdependent complexities, and if people become ‘too interconnected’ in terms of the Three Worlds then the wellbeing and viability of the respective Worlds will be increasingly compromised. Therefore humanity’s footprint in World 1 should be expanded (Ballou, 2007; Popper, 1978). In line with the **Educate to Innovate** STEM (Science, Technology, Engineering, and Mathematics) campaign (United States White House, 2009; U.S. Office of Science and Technology Policy, 2013), United States President Obama expressed his vision for the future when he addressed a gathering of eminent thinkers at the John F. Kennedy Space Centre, Merritt Island, Florida.

> We’re no longer competing to achieve a singular goal like reaching the Moon. In fact, what was once a global competition has long since become a global collaboration.
>
> …Fifty years after the creation of NASA [National Aeronautics and Space Administration], our goal is no longer just a destination to reach. Our goal is the capacity for people to work and learn and operate and live safely beyond the Earth for extended periods of time, ultimately in ways that are more sustainable and even indefinite. (United States Office of Science and Technology Policy, 2010)

Creativity. Although ‘Being-creative’ is a sub-complexity of Being, it is not necessarily less complex than Being because it is the primary human modality for the conceptualization and manifestation of the Conceptual Age. From history it is evident that to survive and flourish, “creating or discovering something ‘new’ is a fundamental aspect of being human,” (Sriraman, 2010, p. 593) namely, through the constituents of Being that are passion, intuition, logic and objectivity (Bataille, 1982; Brinck, 1997; Grinnell, 2009).

The American essayist and poet, Emerson (1803−1882) and the German educationist, Froebel (1782−1852) both valued the creative process as the driving force in the sustainability of humanity. The creator who empathizes has “an appetite for wonder,” (Dawkins, 2013) and at his or her most formidable is driven by the “creative power of child−life” (Froebel as cited in

Ulich, 1961, p. 576). In education however, "if conformity is used to replace man's creative consciousness of values, he may not be able to bring to realization the great laws of existence on which human evolution depends" (Emerson as cited in Ulich, 1961, p. 577). Instead, if a teacher in Being-creative, caring, and encouraging 'touches' the imagination of the young person towards greatness, the consequences for the individual and others is likely to be profound (Emerson in Ulich, 1961; Rigsbee, 2010).

In an interview on the purpose of education, the American linguist and philosopher Chomsky articulated (in the tradition of the Enlightenment) that the highest form of living was to produce something exciting for oneself and others through inquiry, creativity, and by seeking out the riches of the past in order to develop an understanding of the present for the future (Chomsky, 2012; also see Chomsky & Arnove, 2008). In this vein, advocates for a 'learning revolution' in schools and society have contended that human abilities must be reconceptualised in relation to a broad new educational framework (Dryden & Vos, 2005; Robinson, 2011). The purpose being that individuals need to be empowered, and consequently can empower those around them to cope with, as well as influence the nature and direction of 'mass changes' that have been brought about by the drivers of globalization (Robinson, 2011). As creatures of intent therefore (Goswami, 2004), Being-human (as a global complexity) should currently involve 'transformative learning', namely, learning how to be creative for the purpose of thinking and acting differently in dialogic response to our talents, passions, and a changing world (Robinson & Aronica, 2013).

It is however, in and through the dialogical self that Being-creative is the origin of new ideas with which others can empathize (Aldous, 2005; Wegerif, 2011; Wegerif, Boero, Andriessen, & Forman, 2009). Since Beig-creative is a sub-complexity of Being it is constituted in terms of thinking–feeling: 'I think to feel and I feel to think', often in terms of that which pleases or delights 'Being-aesthetic' (Aldous, 2006; Damasio, 2005; Robinson, 2011; Sinclair, 2004;

2010). It is noteworthy that such an approach to Being is at the core of social-emotional intelligence (Goleman, 1999, 2006) which articulates consciously through a ‘diffusivity’ of thinking−feeling *I*-positions. These *I*-positions can represent a particular stance, concept, or point of view in relation to the Being of the other person. It is as if the Being of the *I* becomes infused with the Being of the other person, especially if the other person is enacting an embodied mind. However, to develop a psychology of *I*-consciousness in these terms it is necessary to learn to be ‘mindful’ by focusing on points of commonality and difference (Langer, 1997, 2000), as well as by

> paying attention to the present moment from a stance that is nonjudgmental and nonreactive. It teaches self-observation; practitioners are able to describe with words the internal seascape of the mind. At the heart of the process, I believe is a form of internal “tuning in” to oneself that enables people to become “their own best friend.” (D. Siegel, 2010, p. 86)

Therefore Being-creative requires an empathy for the *Other*, where the *Other* may be a person, a persona-like *I*-position, or even a thing. In all instances however, the creative outcome is an enhanced conceptual horizon (Bussi, 2009; Cobb, 2006; Heidegger, 1927, 1967; Husserl, 2002; Johnson, 1987; Leont’ev, 1978) that enriches the complexity of the self, because dialogically there is a forging of ‘us’ through a sustained bodily interaction as well as an imaginative dynamic; out of which emerges the different *I*-positions. For example as a particular educator noted, “I began to grasp that teaching requires a plural pronoun. The best teaching is never so much about ‘me’ as about ‘us’” (Tomlinson as cited in SA TfEL, 2010, p. 29). Hence, the Conceptual Age requires a learned ability, or abilities to be *I*-capable through teaching and learning moments that are

> **animated** [for emphasis] by a different form of thinking and a new approach to life—one that prizes aptitudes that I call “high concept” and “high touch.” High concept involves the capacity to detect patterns and opportunities, to create artistic and emotional beauty, to craft a satisfying narrative, and to combine seemingly unrelated ideas into something new. High touch involves the ability to empathize with others, to understand the subtleties of human interaction, to find joy in one’s self and to elicit it in others, and to stretch beyond the quotidian in pursuit of purpose and meaning. (Pink, 2005, pp. 2−3)

In short, the Conceptual Age necessitates that people in mass education learn more diversely than probably ever before (Ball & Forzani, 2007, p. 529). In particular, all students should be given genuine opportunities to develop creative and empathetic powers of mind and body in the learning of Being-mathematical (Bruner, 1960, 1979, 1986; Bruner & Anglin, 1973; Fischbein, 1987, 1999; Noddings & Shore, 1984). The purpose of such learning is to promote interactions between linear−analytic and global−synthetic, or even systemic ways of thinking, knowing, and decision making (Bagni, 2010; Freudenthal, 1973; Rilling & Sanfey, 2011; Schoenfeld, 2011; Schön, 1983), that inform, and are informed by embodied acts of holistic and intuitive performance (Davis, 1996; Dreyfus, 1992; Dreyfus & Dreyfus, 1986; Hauser & Wood, 2010; Resnick, 1986) which take place for their own sake, thereby optimizing human experience (Csikszentmihalyi, 1990, 2000).

Imagination. It is imagination however, that is essentially the vital element of creativity, empathy and inquiry because as was intimated by the Ancient Greek philosopher Aristotle (c. 384 BC), it is a prerequisite of thought in all its dynamics (Dewey, 1929b; Gardner, 1993; Johnston, 2010; McCarthy, 2010; Poincaré, 1952a, 1952b). Furthermore, the source of an imaginative ability to form, to juxtapose and to evaluate images in terms of the Three Worlds is the thinking−feeling body because the construction, generation, or creation of images is tied to sense perception which is a capacity of the body (Johnson, 1987; Kosslyn, 1983, 1994; Lakoff & Johnson, 1999; McCarthy, 2010). Consequently, an increasingly interconnected world "is in need of a self that transcends the limits of the modern encapsulated self and may learn from the experience of being part of a broader field of awareness," (Hermans & Hermans−Konopka, 2010, p. 192) namely, through a complex dialogical interaction which is essentially the embodied and extended self (Fuchs, 2012; Merleau−Ponty, 1962; Sartre, 1957).

Practical wisdom. The world is globalizing as a myriad or concatenation of global−local situations. In this context imaginative ideas that are constituted through 'practical wisdom' are key. This specific kind of wisdom is a form of balanced knowing that informs the intuitive choices and goal oriented decisions that people make in their personal and professional lives on a daily basis (Almond, 2010; Atkinson & Claxton, 2000; Clark, 2008; Schoenfeld, 2011). Therefore in learning to be a STEM innovator for example (Herbert, 2010; Hughes, 2010), the teacher and the student need to encounter the world authentically, spontaneously, and affectively for the purpose of developing a relational mind (Finlay & Evans, 2009) that involves discipline, synthesis, imagination and creativity, and a genuine empathy for all Three Worlds (Gardner, 2006b). That is if a STEM innovator engages in imaginative question−inquiry interactions which involve Gardner's (2006b) "five minds for the future," then practical wisdom is likely to arise as a consequence of a reflective and situated body that brings a

> clarity and form to this experience by finding an intellectual distance. He or she learns responsibility and self-direction by reflecting on these processes and fitting their unique contributions together in a complete, well-rounded story. (Csikszentmihalyi & Rathunde, 1990, p. 35)

Summary insights: Chapter One

Human beings exist in "active relationship" with one another; a society of relations (Sellars, 1916, pp. 75–76) that demands both constancy and change within a solar system that is but an "insignificant atom" (Rouse Ball, 1935, p. 497) in a universe that is accelerating in time and space (Dvoeglazov, 2010). It has been conjectured that "when the history of ideas is written four hundred years from now, the twentieth century will be known as the dawn of the great scientific idea of the origin of a changing universe that is still evolving" (Livio, 2000, p. v). However, in a critical realist sense nothing in the universe stands alone; every event is interconnected (Crotty, 2003; Sellars, 1916; Whitehead, 1911) and although "individuality involves distinctness and relative autonomy," (Sellars, 1916, p. 77) separation of mind and Being, or brain and self is not feasible in the sense of a Cartesian-like dualism (Damasio, 2005; Popper, 1978; Popper & Eccles, 1977; Smith, 2003). English poet, Donne (1572–1631) contemplated:

> *No man is an island,*
> *Entire of itself.*
> *Each is a piece of the continent,*
> *A part of the main.* (Donne, Meditation 17, 1624)

Therefore in the vast expanse of Three Worlds there exists a species of bipedal primates that are dependent upon language, tools and selves-in-local-communities (Dunbar, 2003; Suzuki, McConnell, & Mason, 2007; Wilson, 1998). Within these communities of selves there is a desire for a stronger society through the growth of equality between selves (Pickett & Wilkinson, 2010; Wilkinson & Pickett, 2010). Moreover there is a felt need to be educated for an uncertain future, because of an intricate socialization process that stretches back into a 'hierarchical history' of Being-human that was influenced by less affluence, technology, and globalization than is currently the case (Pinkard, 1996; Schutz, 1970, 1972). As depicted in **Figure 1·2**, the Agricultural Age (farmers of the eighteenth century), the Industrial Age (factory workers of the nineteenth century), and the Information Age (knowledge workers of the twentieth century) have emerged out of history into a moment of Being that Pink (2005) described as the Age of creators and empathizers. The cultural historian, Berry (2006) expressed the present time moment in *The Dream of the Earth* when he wrote,

> *It's all a question of story. We are in trouble just now*
> *because we do not have a good story. We are in between stories.*
> *The old story, the account of how we fit into it,*
> *is no longer effective. Yet we have not learned the new story.*
> (as cited in Suzuki, McConnell, & Mason, 2007, p. 19)

Fundamentally therefore, a globalizing world needs to develop a culture of mind that empowers the individual to live out a new story through a learned intelligence (Brown & Coles, 2006; Bruner, 1996; Shayer & Adey, 2002; Adey, Csapó, Demetriou, Hautamäki, & Shayer, 2007), and a narrative structure (Johnson, 1987) that goes "hand in hand with a joint

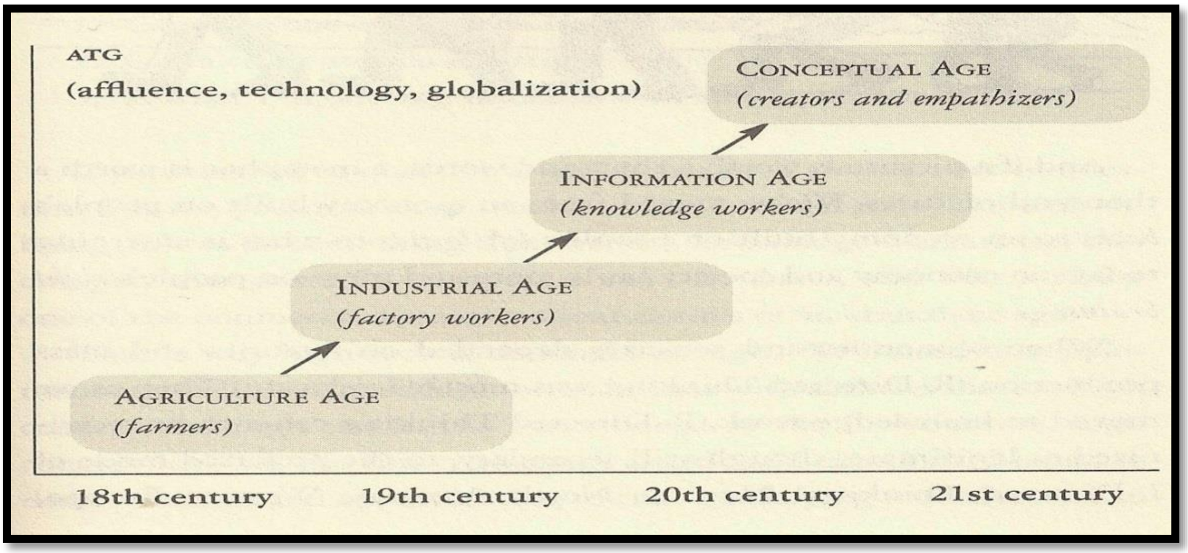


Figure 1·2. What 'Age' will the creators and empathizers of the Conceptual Age create? (adapted from Pink, 2005, p. 49)

concern for developing one's own and others' potential" (Scheibe, Kunzmann, & Baltes, 2009, p. 176). However, it has become evident through recent research in psychology, genetics, neuroscience, and educational interventions that "interventions of the right kind, including in schools, can make people smarter. And certainly schools can be made much better than they are now" (Nisbett, 2009, p. 2). Consequently all role players in mathematics education can be guardedly confident about the future of mass mathematics education, because intelligence is "learnable through social processes that include participation in certain forms of high-demand learning" (Resnick, 2010, p. 186).

Therefore the primary goal of mathematics education in the Conceptual Age is the need to learn how to inquire and problem solve creatively through multi-faceted dialogues. The purpose of which is to implement imaginative ideas effectively and wisely as part of Type I and Type II persona-processes of inquiry. Intelligence for such inquiry and problem solving (Skemp, 1979; Sternberg, 1999; Sternberg, Jarvin, & Grigorenko, 2009) can be developed in complex social situations, where dialogic modalities of Being are of the form that "one mode provides experiential richness and fluidity, the other logical coherence and stability" (Labouvie−Vief, 1990, p. 53).

Chapter Two

The Problem Conceptualized: What is Known, Not Known, and Needed

Most high school teachers don't have any idea that mathematics is a living, growing subject (Alex Gunning — Australian gold medallist at the 2014 International Mathematics Olympiad; Lehmann, 2014)

Since the early 1950s there have been many attempts in Western and Eastern nations to transform education practically, especially with respect to the teaching and learning of mathematics in schools (Bruner, 1960; Cockcroft, 1982; Connell, 1980; Dindyal, 2006; Dunkin & Biddle, 1974; Fraser, 1980; Keeves, 1999; Klein, 2003; Knapp, 1997; Leung, 2001; Leung, Graf, & Lopez−Real, 2006; Malaty, 1999; Ng, 2009; Schoenfeld, 2004; Shumway, 1980). The result has "been large-scale changes in education systems across the world," (Wiliam, 2003, p. 484) and a substantial body of knowledge as to 'what works' in terms of "how, where and why people learn or do not learn mathematics" (Begle & Gibb, 1980, p. 8) has been amassed (Barr & Emans, 1930; Begle, 1970; Lester, 2007a, 2007b; Wiliam, 2003).

Paradoxically however, mathematics education research has had limited influence in schools or tertiary institutions (Clifford, 1973; Grootenboer, 2010; Holton, 2001; Kilpatrick, 1988; Nardi, 2010; Stigler & Hiebert, 1999; Tirosh & Graeber, 2003; Wiliam, 2003, 2008). Therefore mathematics education "has not attained the status of a discipline, and it is not completely a profession;" (Kilpatrick, 2008, p. 36; also see Ball & Forzani, 2007; Lagemann, 2000) particularly in the wake of many 'failed or curtailed' reform efforts including the very public demise of the 'overly' abstract **New Math** (algebra) curricula in the early 1970s (Crockett, Liston, & Zeichner, 2008; Grant & Murray, 1999; Klein, 2003; Kline, 1973; Perso, 2007; Schoenfeld, 2004; Tyack & Cuban, 1995; Yates, 2009).

However, the action research cycle 'Plan (Look and Think)−Act−Reflect' (Creswell, 2008; Eikeland, 2008; Reason & Bradbury, 2004; Stringer, 2007) — has been effective in promoting some meaningful change in a broad spectrum of mathematics classrooms (Pegg, Lynch, & Panizzon, 2007; Schoenfeld, 2008a; Shayer & Adey, 2002; Sami, 2012; Stone, 2007). In particular, the cycle appears to result in positive outcomes when a sustained dialogue between teachers is promoted, and when quantitative research methods are employed to understand the nature of the educational outcomes (Adey & Shayer, 1994; Shayer & Adhami, 2007).

Nonetheless, facilitating appropriate change in schools and other learning institutions has been challenging. The complexity of mathematics learning has not been thoroughly understood and articulated simply (Even & Tirosh, 2008, p. 202; also see Borasi, 1996; Calais, 2008; Gilbert & Coomes, 2010; Hiebert, 1986; Hiebert & Carpenter, 1992; Moore−Russo, Conner, & Rugg, 2011; Nesher & Kilpatrick, 1990; Schoenfeld, Smith, & Arcavi, 1993; Thames & Ball, 2010). As a consequence many mathematics teachers have taught largely in the same manner that they were taught in school (Pehkonen, 1997; Shriki, 2010; Vinner, 2007; Wiliam, 2003). Therefore school activities have often positioned students "with little conceptual agency, teaching them instead how to perform algorithms correctly (disciplinary agency) or to set up apparatus to obtain known empirical results (material agency)" (Greeno, 2006, p. 88; also see Roschelle, Singleton, Sabelli, Pea, & Bransford, 2008; Sawyer, 2006; Stacey, 2008, 2010; Wiliam, 2003).

However, although mathematics education as a field of inquiry has substantial knowledge with respect to many of the individual factors, or component aggregates that constitute learning-for-understanding environments (Hiebert et al., 1997; Pegg, Lynch, & Panizzon, 2007; Schoenfeld, 2008a), the teaching and learning of mathematics is not a well-researched area (Chipman, 2005; English et al., 2008). For example, Goldman (2009) argued that as a "field we are still largely in the dark regarding cause−effect relationships among cognitive, social, and affective dimensions of learning" (p. 452). Consequently, many mathematics educators “beg the question of what constitutes a desirable future, or even a desirable or healthy system,” (Fenwick, 2009, p. 110) particularly because the learning of mathematics is socially, culturally, historically, and politically situated (Clarke, Keitel, & Shimizu, 2006; Cobb, 2006; Goos, 2004, 2005; Lave, 1988; Sriraman, Roscoe, & English, 2010). The psychology of student learning is therefore diverse in varying degrees of cognitive and socio-affective maturity, development, and values (Fullan, 2009; Hermans & Hermans−Konopka, 2010; Knapp, 1997; Nardi, 2010; Resnick, 2010; Tretter, 2010; Voyat, 1982; E. Wilson, 1998). In particular values are “local, specific and constructed, and hence, values are an integral component in the meaning systems that people generate in social action” (Brown & Walshaw, 2012, p. 2).

Therefore if teaching is a values-based social action “system, then each feature, by itself, doesn’t say much about the kind of teaching that is going on. What is important is how the features fit together to form a whole” (Stigler & Hiebert, 1999, p. 75). For this reason if meaningful change is to be realized at the level of mass mathematics education, then the teaching and learning of mathematics must be understood at least at the level of a system, and in different countries of the world. This implies being able to model systemically (Carroll, 1963; Keeves, 2002; Keeves & Sellin, 1997; Sellin & Keeves, 1997) in a glocalizing world that “is holistic, self-organizing, emergent, highly relational, dynamic, interconnected, non-linear, and evolving” (Castellani & Hafferty, 2009, p. 21). Hence through the lens of complexity science, a primary goal for mass mathematics education is to emerge beyond rote, procedure, and straightforward applications,[1] by grouping suitable entities or processes together (e.g., Carroll, 1963, 1989) in such a way that the teaching and learning of mathematics across nations

> is greater than the sum of the parts [(Hurford, 2010; Jaworski, 2004; Johnson, 2009)]. An immediate implication of this fact is that it will be difficult, if not impossible, to improve teaching by changing individual elements or features. In a system, all the features reinforce each other. If one feature is changed, the **system** will rush to “repair

> the damage', [cf., chemical equilibria and Le Chatelier's Principle (Atkins & De Paula, 2013; Moore, Davies, & Collins, 1978)] perhaps by modifying the new feature so it functions the way the old one did. (Stigler & Hiebert, 1999, p. 97)

The Need for a New Philosophy in Mathematics Education

A distinctive and exciting period in human history and education is currently unfolding. Resnick (1987) argued that "it is new to take seriously the aspiration of making thinking and problem solving a regular part of a school program for all of the population, even minorities, even non-English speakers, even the poor" (p. 7). The very notion of 'schooling for all' (at least in Western countries) was a "17th–century invention, born during the spread of Protestant Christianity in Northern Europe and then taken up in Southern Europe as part of the Catholic Reformation" (Resnick, 2010, p. 184). Throughout this period however, "almost all, who have ever fully understood arithmetic, have been obliged to learn it over again in their own way" (Colburn as cited in Hiebert et al., 1997). This is a key insight for the meaningful learning of mathematics.

Nevertheless, the Woods Hole Conference of the National Academy of Sciences held on Cape Cod, Massachusetts in 1959 supported the proposition that "any subject can be taught to anybody at any age in some form that is honest" (Bruner, 1979, p. 108). In this sense the ten-day conference was a landmark event in the perspective that mathematics and science learning in primary and secondary schools needed to be a **structured process** (Bruner, 1960). This overarching and fundamental belief was influenced by the 'teaching for understanding' ideas of Dewey (Dewey, 1897, 1916, 1933; Dewey & Dewey, 1915; McLellan & Dewey,1895a) and Whitehead (1911, 1943, 1948); Brownell (Brownell, 1944, 1945; Brownell & Chazal, 1935) and Tyler (Bloom, 1956; Finder, 2004; Tyler, 1949); Conant (1947) and Bruner (1956, 1957), as well as the School Mathematics Study Group (SMSG) led by Begle of Yale and Stanford Universities (Begle, 1954, 1970, 1979).

Within a decade of the Woods Hole Conference, the first 'international' journals dedicated solely to mathematics education were published. These journals were edited by the 'two Hans's' (Furinghetti, 2007; Kaiser, 2007), namely, Hans Freudenthal (Educational Studies in Mathematics, 1968−) and Hans Wäsche (ZDM [Zentralblatt für Didaktik der Mathematik]: The International Journal on Mathematics Education, 1969−). As a result research burgeoned[2] in an attempt to answer the vexing and fundamental questions: (1) How can the individual understand mathematics?, and (2) Why do so many people find the learning of mathematics so difficult? (Bloom, 1981; Hiebert, 1986; Hiebert & Carpenter, 1992; Hiebert et al., 1997; Llewellyn, 2012; Skemp, 1976; Schoenfeld, 2008a; Tucker, Singleton, & Weaver, 2013). In the sentiment of the French mathematician Poincaré (1854−1912),

> One … fact must astonish us, or rather would astonish us if we were not too much accustomed to it. How does it happen that there are people who do not understand mathematics? If the science invokes only the rules of logic, those accepted by all well-formed minds … how does it happen that there are so many people who are entirely impervious to it? (as cited in Sfard, 1991, p. 1; French original was published in 1908)

In the twentieth century there were two primary modes of learning mathematics, namely, symbol processing and situated action. The former is underpinned by the Physical Symbol System Hypothesis which is grounded epistemologically in the philosophy and practice of cybernetics (George, 1979; Sayre, 1976) and the computer metaphor, or the information processing human brain (Anderson, 1983; Jensen, 1989; Keeves, 2002; Kilpatrick, 1985; Lakomski, 1999; Newell & Simon, 1972; Simon, 1979; Sternberg, 1990). In these terms the mind is the 'software' of the brain that is thought to process information like

> a physical symbol system. A physical symbol system is a system capable of inputting, outputting, storing, and modifying symbol structures, and of carrying out some of these actions in response to the symbols themselves. 'Symbols' are any kinds of patterns on which these operations can be performed, where some of the patterns denote actions (that is, serve as commands or instructions).
>
> ...The physical symbol system hypothesis has been tested so extensively over the past 30 years that it can now be regarded as fully established, although over less than the whole gamut of activities that are called 'thinking'. (Simon, 1990, p. 3)

However, the computer metaphor is inadequate to understand the diverse capabilities of a thinking−feeling body (Almond, 2010). Being-in-the-world is not limited to process (Dreyfus, 1991; Heidegger, 1927; Husserl, 2002; Merleau−Ponty, 1962, 1964; Sartre, 1957; Schutz, 1970), because human Beings do not only process symbols, but respond to stimuli (Roth, 2011; Vygotsky, 1978); text and context, through bodies that enact Being-in-the-world by engaging in, or with specific situations (Greeno, 1997, 2006; Varela, 1995; Varela, Thompson, & Rosch, 1991). Consequently knowledge does not transfer easily (if at all) between different tasks, because the 'learning body' is situated. Thus the reification, or concretization of knowledge is dependent on intuition, which is a global−synthetic capability that enables the individual to make sense of the whole situation. This 'sense making' through intuition, occurs when 'knowing' is compressed into mental structures that are grounded phenomenologically in terms of the **how**, **where**, **when, why, and what** the body learned. Thus the premise for learning in a particular situation is the mind-body-in-action; change the situation and the mind−body learns differently (Johnson, 1987; also see Fischbein, 1987; Semadeni, 2008; Tall, 2008; Thurston, 1990).

Therefore learning mathematics through symbol processing alone limits Being-mathematical, because Being-human is situated — physically and socially in relation to the Three Worlds — which in an anti-reductionist, or holistic sense means that at its very core Being-human is an embodied social essence. This is because each society is formed or structured in response to the social formation of minds in that society (Adolphs, 1999; Brothers, 1997; Chambers, 2014; Ricoeur, 1991; Schutz, 1970). Necessarily then, the learning of mathematics in mass education needs to occur in complex social and physical environments (Anderson, Reder, & Simon, 1996; Clarke, Keitel, & Shimizu, 2006; Cobb, 2006).

Keeves (2002) argued however, that "the choice is not between a symbol processing approach or a situated action approach to learning, but rather the search for a further approach that encompasses both of these approaches, each to be used in appropriate situations" (pp. 122−123). But symbol processing occurs as part of a situation that is socially and culturally informed. In this sense symbol processing is not mutually exclusive from situated action because both learning approaches occur in terms of the same body. In other words if the two approaches occur in relation to the same embodied mind then it is feasible that the one approach can complement the other and vice versa. That is the brain can be engaged as a physical symbol system to represent situated realities symbolically and visually, and "through them, the universe of meanings that allow one to interpret and to organise the data collected from real experience in the world" (Lerman, 2001, p. 103). However, in order to abstract between situations, or to situate the abstraction, Being-mathematical is crucially dependent upon the symbolic, in the sense that

> the invention or discovery of symbols is doubtless by far the single greatest event in the history of man [the invention of symbols as a human construct occurred only within the last few thousand years (Nieder & Dehaene, 2009)]. Without them, no intellectual advance is possible; with them, there is no limit set to intellectual development except inherent stupidity. (Dewey, 1929b, p. 151)

But consistent with Vygotsky's (1978, 1997) ideas on socialization and culture, Lerman (2001) argued that the self was formed, or unfolded through a consciousness that was situated both temporally and culturally in relation to different symbols. Hence, the need in mathematics education for "a further approach that encompasses" (Keeves, 2002, p. 122) symbol processing and 'situated activity, situated social practice, situated learning, or distributed cognition' (Lakomski, 1999).

The fundamental goal of which is for a mathematical mind in society to develop "as elegant and powerful a level as possible between abstraction and concretization" (Kahn et al., 2007, p. 382). The **possible** between abstraction and concretization is to incorporate symbol processing and situated action in Being-mathematical. However, if the possible is to materialize it is necessary to not only teach **for** dialogue, but more importantly "to treat dialogue as an *end-in-itself*," (Wegerif, Boero, Andriessen, & Forman, 2009, p. 185) because in Aristotlean terms, "I become me and you become you, only in the context of dialogues"

(Wegerif, Boero, Andriessen, & Forman, 2009, p. 185). Being-mathematical therefore is a 'concrete−abstract' personalization, or mathematization of the self which occurs fundamentally by engaging symbolically, dialogically, and essentially with other people and with things that include mathematical objects (e.g., systems of equations).

A dialogical narrative of Being-mathematical. The following description represents a narrative on how the human modalities of symbol processing and situated action can emerge relationally into *I*-consciousness. The self of the individual learner extends beyond his or her body to include the other person and relevant objects, as well as the space and activity between the respective bodies. However, the *I*-consciousness of the individual learner is part of the self of the learner because an embodied mind is a part of the self. Therefore there is essentially no gap, or difference dialogically between inter-bodily dialogue and that which occurs between *I*-positions. Consequently, the individual learner can use the interpersonal and dialogical activity within the self of the learner to develop a symbol processing or situated action approach to the learning of mathematics. The *I*-positions are then necessarily structured with the dialogic activity that occurs, or occurred between individuals. This is possible because of an "intercorporeality of being" (Davis, 1996; Merleau−Ponty, 1962) that enables a person to engage with another person, or a person with a thing through sense perception, the betweenness of bodily activity, physical forces (e.g., Newton's law of universal gravitation), and comportments of Being.

Moreover, if the dialogic activity between persons is to relate the symbol processing and situated action approaches meaningfully, then this interpersonal dialogue between embodied minds will enable a similar dialogue within *I*-consciousness of the individual. It is this interpersonal dialogue that paves the way for the respective symbol processing and situated action *I*-positions to be related coherently. Through reflection, imagination, and the individuality and intentionality of *I*-consciousness however, the dialogue that interweaves the

I-positions might not be identical to the dialogue that occurs, or occurred between the different persons. Nonetheless, the dialogic principles that enable an intercorporeality of being between learners are identical for the dialogic activity that is necessary to engage the respective *I*-positions meaningfully and usefully. Therefore in an existentialist sense (Stokes, 2006) it is by Being-dialogical that the essence, viability, and coherence of the extended self makes Being-mathematical possible, namely, through a mind in society that provides a framework or structure for a society of mind, and vice versa.

It is this Vygotskyan (1978, 1997) informed perspective that implies a new philosophy in relation to a dialogical ontology of Being, namely, a forerunner to "a new and challenging theory of education" (Wegerif, Boero, Andriessen, & Forman, 2009, p. 185). It is noteworthy that at the inaugural *Mathematics Education and Contemporary Theory* conference (held in July 2011 at Manchester Metropolitan University), it was agreed that there was indeed "a place for theory in the future development of mathematics education research" (Brown & Walshaw, 2012, p. 1).

A new focus in mathematics education research. Over the past three decades the "theory of knowing" (Von Glasersfeld, 1990, p. 1), namely, constructivism has been a dominant factor in mathematics education discourse (Bussi, 2009; Lerman, 1989; Steffe & Thompson, 2000; Tobin, 2007; White−Fredette, 2010; Wiliam, 2003). However, constructivism has failed to address the challenges of mass mathematics education meaningfully and effectively because it provides a "post-epistemological perspective," (Dawson, 1991, p. 492) and as such it limits Being (Roth, 2011; Scanlon, 2012). Therefore at the secondary school level especially, educators have struggled to make sense of constructivism epistemologically (Gibbons, 2004; Splitter, 2009a; Meyer, 2009). This was indicated by Keeves (2002) when he stated that constructivism was "both incomplete and inadequate for the effective learning and teaching of mathematics and science at the upper secondary school level" (p. 114). In other words the 'constructivist metaphor' restricts 'Being-able' to go beyond the meaningful organization of concrete particulars, namely, towards hypothetico−deductive abstract learning that involves both Type I and Type II embodied processes of mind (Inhelder, de Caprona, & Cornu−Wells, 1987; Inhelder & Piaget, 1958; Kahneman, 2011; Lenman & Shemmer, 2012; Roth, 2011). Poignantly stated therefore, the 'constructivist graveyard' is being "marked by the corpses of the many ill-fated efforts to define the term 'constructivist teaching'" (Davis, 1996, p. 230).

Consequently, the demise of constructivism as a viable mediator between the stimulus that is symbol processing and the response that is situated action, or vice versa (Vygotsky, 1978, 1986, 1991), suggests that a new or different approach in mathematics education is required. Nonetheless, from a holistic or ontological point of view

> it seems, then, that whether in general, or simply in relation to education, **everything** [for emphasis] turns on how we judge the status of knowledge and of truth. The educational debate relates closely to what might be called the epistemological debate about knowledge — what it essentially *is*. (Almond, 2010, pp. 303−304)

Out of necessity therefore, a new research focus has arisen in mathematics education that "attempts to get beyond the scrutiny of separate elements of learning and to consider the 'nature' of the learner as a whole," (Ernest, 2009, p. 37; also see Hurford, 2010) because essentially more and more research with respect to the parts "can lead us to know less and less about a ... phenomenon, until finally we know much less than we did before we started doing research" (Sternberg, 2000a, p. 363). Thus epistemologically in mathematics education "in one way or another, we are forced to deal with complexities, with 'wholes' or 'systems'...[which] implies a basic re-orientation in scientific thinking" (Von Bertalanffy, 1968, p. 3; also see Castellani & Hafferty, 2009; Laszlo, 1972; Otte, 2007; Sutherland, 1973). It is noteworthy that over the past 50 years, the goal of international assessments in mathematics and science (e.g., IEA, PISA, and TIMMS) has been to build a body of knowledge that can be understood systemically (Darmawan & Aldous, 2013).

Epistemology as a creative process. A new vision for school learning was advocated by McPeck (1981) who argued that if teachers and students were to become critical thinkers in mathematics, it would be necessary to develop an epistemological framework on, or through which the practice of teaching and learning could be effected (cf., diSessa, 1993; Hauser & Spelke, 2004). Therefore “to get students thinking for themselves” (McPeck, 1981, p. 154) in mathematics would require an epistemic, or foundational construct — an artefact of understanding — that gave shape to the mathematics education research ideas of the time. This view reflects Whitehead’s maxim that education should empower both teachers and learners “to appreciate the current thought of their epoch” (as cited in Bereiter & Scardamalia, 2006, p. 696). That is through an epistemological inquiry that fuels movement between the mathematics education problem and its solution, namely, an inquiry that grapples with the nature and the possibility of mathematical and pedagogical knowledge, including the scope, depth and “limits of human knowledge, and with how it is acquired and possessed” (White, 2005a, p. 194; also see Sullivan, 2008).

In the spirit of Whitehead’s maxim therefore, if teachers are to “perform their duties well, they should have a clear picture of what the time in which they are working wants” (Genzwein, 1970, p. 419). Towards the goal of having a ‘clear picture’ for the teaching and learning of mathematics, Burton (2008) encouraged the mathematics education community to embrace an epistemology in which mathematics was “re-perceived as humane, responsive, negotiable and creative” (p. 527). Essentially therefore, ethics through a creative process **is** the epistemology that is necessary if individuals are to develop a personal mathematical identity, as well as a positive psychology for Being-mathematical in the Three Worlds (Erikson, 1980; Ernest, 2009; Hoffman, 2010; Papert, 2006; Peters, 1966; Radford, 2008a; Snyder & Lopez, 2009). However, such a psychology of Being-mathematical is only made possible through a self that extends beyond the limitations and situations of a physical body to include the Three Worlds as part of a situated, symbolic, and technological network of relationships, namely, the dialogical self.

Moreover, the motivation for a new epistemology was spurred by Burton's (1999) question as to why intuition was so important to mathematicians but missing from mathematics education (Gray & Tall, 2007; Harteis, Koch, & Morgenthaler, 2008; Malaspina & Font, 2009, 2010; Semadeni, 2008). Simply, any epistemology that is relevant and flexible with respect to an emerging and unfolding Conceptual Age — an "Age of Uncertainty" (Claxton, 1999, p. 243) — needs to unpack intuition as 'a felt certainty', and as a core and central feature of the creative process. Without intuition creativity is not possible, and as Hennessey and Amabile argued, creativity is

> essential to human progress. If strides are to be made in the sciences, humanities, and arts, we must arrive at a far more detailed understanding of the creative process, its antecedents, and its inhibitors. ... Deeper understanding requires more interdisciplinary research, based on a systems view of creativity that recognizes a variety of interrelated forces operating at multiple levels. (2010, p. 569)

However, many mathematics classrooms around the world — including Australia (Clarke, Goos, & Morony, 2007; Hollingsworth, Lokan, & McCrae, 2003; Sullivan, 2011) — do not actively encourage a culture of creativity and constructive thinking (Bishop, Seah, & Chin, 2003; Sriraman, Roscoe, & English, 2010; Wiliam, 2003). Both of which are essential to the development of Being-mathematical. The reasons are complex, but influenced by the 'cognitive revolution' in education (Gardner, 2005; Royer, 2005), many mathematics teachers have tended to focus on 'mathematical thinking' that involves declarative, long term and working memory (Kirschner, Sweller, & Clark, 2006), and not on the whole Being of the mathematics learner (Faure, 1972; Lundin, 2012).

It is noteworthy therefore, Aldous (2005, 2006, 2007, 2014) demonstrated in an empirical and measurement study involving more than 400 school students, that if an individual's problem solving was not grounded psychologically in a cognitive and non-cognitive interaction, then students' creative problem solving attempts tended to be unsuccessful (Damasio, 2005; Fischbein, 1987, 1999; Fischbein & Grossman, 1997; Krutetskii, 1976). In other words students who 'attended to a body that feels' during the problem solving process, were more likely to solve the novel problem than those students who relied only on a cognitive approach. Interestingly with respect to the successful problem solvers who also 'attended to feeling', a weak but statistically significant gender effect in favour of girls was found. Nonetheless, the Aldous (2005, 2006, 2007, 2014) study indicates that a brain is necessary to think mathematically, but an intentional embodied mind through a creative process is fundamental to Being-mathematical.

Why a new philosophy? An epistemology, namely, as a creative process is unlikely to apprise the teaching and learning of mathematics at the level of a complex system, which is essential if mathematics teachers are to appreciate mathematical learning holistically and essentially (Castellani & Hafferty, 2009; Wheeler, 1981). That is epistemology is foundational, and "mathematics educators need to bring research and practice together through an organized system of knowledge that will enable them to see beyond the specifics of each and explain how they can work together" (Kilpatrick, 2010, p. 5). Moreover, as an outcome of the New Math reform movement there was the realization that mathematics education needed to develop "a mathematical philosophy; all we have is mathematical epistemology," (Long, Meltzer, & Hilton, 1970, p. 457) and unfortunately,

> epistemology has turned out to be somewhat of a disappointment. First of all, epistemologists disagree with one another about the nature of the criteria. There are empiricists who want us to rely upon sense awareness alone. There are rationalists who favor clear and distinct ideas. And there are numerous disagreements within these camps as to the nature of the deliverances of sense awareness and intellectual intuition. Thus no single story has emerged in the epistemological marketplace. (Landesman, 1997, p. 191)

Nevertheless, epistemology is relevant to a philosophy of mathematics education, because it values "both inquiry and the *justification* that the evidence thereby produced offers to candidates beliefs, judgments, and actions" (H. Siegel, 2010, p. 283). Moreover, epistemology emphasizes questions as to "the nature of knowledge, its categorizations and representations, and the ways of evaluating which knowledge is of most worth" (Cunningham & Allen, 2010, p. 483). Within the framework of Aristotle's *Posterior Analytics* for example, epistemology dealt only with the necessary (*epistémé*) and not the contingent (*doxa*) when adding to knowledge (Marenbon & Mautner, 2005; Roberts & Wood, 2007). If the current challenges of mass mathematics education however, are to be addressed in epistemic terms alone, then the complexity of Being-mathematical cannot be understood by Beings-in-the-world without recognizing the 'dependencies', or backdrop of a world that is globalizing in a myriad of local situations.

It is particularly noteworthy that the mathematics education philosopher, Ernest recognized that if constructivism was to take root in classrooms it was not sufficient to argue only epistemologically, but it was crucial to establish (social) constructivism philosophically (Ernest, 1994, 1998). However, despite the 'best efforts' of Ernest (1994, 1998, 2009, 2010), and others like Steffe and Thompson (2000), the hegemony of constructivism in mathematics education has waned. The "concept of constructivism" has become increasingly problematic (Splitter, 2009, p. 135), not only philosophically (Gibbons, 2004; Meyer, 2009; Roth, 2011), but also psychologically (Clark,Nguyen, & Sweller, 2006; Kirschner, Sweller, & Clark, 2006), and practically (Keeves, 2002) — in no small measure, because "contemporary constructivists promote a variey of different views and often disagree about the goals and the shape of the constructivist project" (Lenman & Shemmer, 2012, p. 3). Nonetheless, any epistemology for the teaching and learning of mathematics needs to form part of an overall philosophy, or ontology of Being that implies "a quest undertaken for its own sake," (Mautner, 2005, p. 466) or from the Greek, 'for a love of wisdom' if a comprehensive and balanced view of the phenomenon under scrutiny, namely, Being-mathematical is 'to be made to show itself' phenomenologically in relation to the Three Worlds.

Furthermore, if any philosophy is to be realized in practice then such knowledge must be communicated convincingly as a "web of belief" (Quine & Ullian, 1970); an artifact of wisdom that can promote understanding, or a philosophical system that reflects bridging the gap between theory and practice (Griffiths, 1998; Schön, 1983). This implies Quine and Ullian's five doxastic virtues (Keeves, 2002):

> I) conservatism — the philosophy may have to conflict with some previous beliefs, but the fewer the better;
> II) generality — the plausibility of a philosophy depends largely on how compatible the philosophy is with our being observers placed at random in the world;
> III) simplicity — when there are philosophies to choose between, and their claims are equal except in respect of simplicity, the one that appears simpler to implement is preferred;
> IV) refutability — some imaginable event, recognizable if it occurs, must suffice to refute the philosophy; and
> V) modesty — the less story the better.
> (adapted from Quine & Ullian, 1970, pp. 43–51)

Therefore simplicity as an outcome of complexity is required to facilitate a dialogue between scholars and practitioners for the progressive learning of mathematics. Importantly however, if a meaningful dialogue is to occur between the many different role players in mathematics education, then as explained by the German philosopher Apel (1922−), that which is said should make sense and "when we engage in discourse with others, we implicitly acknowledge the notion of a *community* of participants in discourse — even if this is a regulative ideal rather than actual practice" (Christensen, 2005, p. 33). Although no individual's network of belief implies truth, it should be coherent at the time of communication (at least as far as the participating individual is concerned), because "communication *is* the relation with others" (Maheaux & Roth, 2014, p. 503).

Powerful mathematical learning. In this study, ethical learning that is essentially creative, dialogical and mathematical is termed powerful mathematical learning. In its essence powerful mathematical learning is a progressive inquiry through dialogue (Barrow,

2010) in a Deweyan dynamic sense (Dewey & Dewey, 1915), which means that fundamentally and holistically “inquiry is the tool that helps us to control and ultimately bring ourselves into constructive reciprocal **relationships** [for emphasis] with our environments” (Johnston, 2010, p. 106).

However, if a philosophy of powerful mathematical learning is to enable Being-mathematical in the Conceptual Age, especially in schools, then it should reflect or capacitate Aristotle’s three intellectual virtues or ‘excellences of mind’, namely, *epistémé*, *techné*, and *phronesis* (Kinsella & Pitman, 2012). Knowledge as *epistémé* is not absolute nor does it constitute truth, because truth in an absolute sense is unknowable in a changing dynamic that involves the Three Worlds. However, *epistémé* does relate to an increasing certitude of knowledge that is scientific, conceptual and grounded in theoretical notions, principles, and World 1 which is The Natural−Physical World.

Phronesis, as an intellectual and ethical virtue, relates to practical wisdom being applied in specific socio-cultural situations. Therefore *phronesis* refers to the knowledge of concrete particulars (Eikeland, 2008; Kessels & Korthagen, 1996) which are recognized, or are emergent in direct “confrontation with the situation itself, by a faculty that is suited to confront it as a complex whole,” (Nussbaum, 1986, p. 301), namely, intuition through the operation of System I. However, the specialized virtues that are *epistémé* and *phronesis* “have a bias in favour of the dialectic poles they protect. It remains for integrity and integrated knowing to rise above these biases for the truly integrated judgement” (Kolb, 1984, p. 228).

It is the role of *techné* to bring together, or moderate *epistémé* and *phronesis* in the Platonic sense of an integrated judgement that constitutes *eudaimonia* (well-being or happiness), because *techné* has a tool-like character. It is the the nature of *techné* however, that makes the philosophy of powerful mathematical learning more accessible to researchers and teachers, that is on a basis of universal *epistémé* which are applicable in local situations. The term *techné* features in English words such as technology and technique (Wiliam, 2003) and reflects the idea of a wise craftsperson performing a skilled act, with the aid of his or her tools, for the purpose of bringing

> into being those things that are **contingent and variable** [for emphasis]. Techné is variously translated as art, craft, or skill. It differs from epistémé in that epistémé is concerned with things that are the way they are of necessity (otherwise they would not be eternal truths), whereas techné deals with things that could be different from what they, in fact, are.
>
> ... Phronesis is also different from techné because it is designed to move people to action rather than to production. Aristotle's point here is that techné is product oriented because the aim of the production is not the production itself but the product, whereas action is process oriented — the end is doing *well*. (Wiliam, 2008, p. 434)

Thus at its most useful, a philosophy of powerful mathematical learning would facilitate an enhanced best practice for the teaching and learning of mathematics in a New Age where learning epistemologically to be creative is crucial (Lohmar & Yamaguchi, 2010; Malloch & Porter−O'Grady, 2009; Oaklander, 2008). However, any philosophy ought to be testable. Therefore a philosophical and practical basis for powerful mathematical learning must be structured essentially, or systematized as a model that can then be examined holistically for coherence and adequacy, with the "recognition that certainty and the so-called 'truth' of knowledge can never be fully satisfied" (Keeves, 2002, p. 114). In his transcendental pragmatics, Apel's (1980, 1984) sense of "truth is universal consensus in the long run. This is a limiting concept, like a Kantian regulative idea — we can move towards the goal, but never entirely reach it" (Christensen, 2005, p. 33). However, World I exists and although there are limitations on Being-human to examine it, the ability to understand the real world in a literal phenomenological sense is increasing quickly through advancing technologies (*techné*).

The Study and its Aim

The aim of the study is to develop a philosophy of powerful mathematical learning for Beings-in-the-world through a holistic conceptualization of Being-mathematical. This necessarily involves describing and justifying an essential structure, or basis that can facilitate powerful mathematical learning in schools and tertiary institutions through dialogue.

Epistemology addresses the essential question: What is the knowledge claim, and on what authority, or process is it substantiated? In this study the basis for powerful mathematical learning is grounded epistemologically in Wallas' (1926) creative process. However, without a suitable ontology of Being-mathematical, the individual is blind to powerful mathematical learning even though he or she might be aware of it, or even operate therein epistemologically. The reason is that 'ontology' deals fundamentally with the question: What or who does the individual see? It is when epistemology is related meaningfully to an ontology of Being that the discourse may be described as philosophical. However, the discourse is a philosophy if it is sufficiently comprehensive in 'comporting' the Being of the reader towards truth, or wisdom in relation to a particular field of knowledge.

In this study it is through an unfolding phenomenological philosophy that an epistemology for powerful mathematical learning is illuminated by an enfolding ontology of Being-mathematical. This implies a comportment of Being towards grasping powerful mathematical learning visually, holistically, and dialogically for the purpose of being 'practically wise' and creative in different socio-cultural situations.

The **history** of the powerful mathematical learner is constituted in the future in terms of his or her present. It is through the **dialogue** of *I* and *Other* that the self relates and interrelates meaningfully and optimally with the respective dimensions of time and self. Over time however, the **principles of quality teaching and learning** remain largely constant across different **communities-of-practice** that excel in what they do and achieve. Although each mathematical community-of-practice is a unique socio-cultural entity, the **aims of education** should always be informed by **protocols and taxonomies** of empirically substantiated educational knowledge, as well as a **model of school learning** that can unify the activities of the community at a global level. In so Being the community is likely to afford each individual the opportunity to develop his or her **intelligence** by engaging with mathematics in a powerful manner that is both **ethical** and **creative** for a ‘common good’.

The Significance of the Study

Mathematics education does not currently have a coherent philosophy that relates an epistemology and ontology of Being-mathematical so that students in mass education can learn mathematics substantially beyond the basics (Herbel−Eisenmann, Choppin, Wagner, & Pimm, 2012; Lundin, 2012; Skovsmose, 1994; Sriraman & English, 2010). Although “we know that learning in general is very complex, and acknowledgement of this complexity is critical to advance our understanding of problem solving,” (Grootenboer, 2010, p. 292; also see English, 2010; Goldin, 2010; Lesh, 2010; Presmeg, 2010) the research of complexity and complex systems in mathematics education is in its infancy (English & Sriraman, 2010). The significance of this study therefore, is the philosophical development of a dialogical basis for the teaching and learning of mathematics, namely, an ethical−creativity system that relates symbol processing and situated action dialectically.

More than three decades ago, Wheeler (1981) recognized that best practice was dependent upon teachers and educators understanding “learning in a way that at present they do not: that they would have to become students of learning, not merely practitioners of it” (as cited in Silver, 1987, p. 51). In this sense this study adds to knowledge philosophically and systemically. It has been stated that it is timely “to start the work of combining, merging and fusing our theories, and thus to make them more widely known, applicable and applied” (Dreyfus & Gray, 2002, p. 115; also see Radford, 2008b). In essence therefore, the significance of this synthesis is to address a key research question for mathematics education, namely, “Is it possible to embrace new ontological possibilities for the learner and teacher beyond established states of representation” (Brown & Walshaw, 2012, p. 3)?

Research Questions

The following two questions sustain this study's inquiry towards a philosophy that satisfies, or underpins the aim and significance of the study. In particular however, it is important to remember "Dewey's reminder that all legitimate problems have social import or connection, including the problems of philosophy;" (Johnston, 2010, p. 104) followed by the "hard dictum that Dewey set out, that one's philosophical conceptions must be based on the best science of the day" (McCarthy, 2010, p. 320).

1) Who are powerful learners of mathematics?

2) How can powerful mathematical learning be realized through an ontology of Being that is dialogical?

End Notes

1. A current meaning of 'the basics' lies in knowing "the concepts, the skills and how to use them in standard ways to solve problems that relate directly to real-world situations" (Stacey, 2010, p. 17).

2. In *Twenty-Five Years of Educational Practice and Theory: 1955−1979*, Husén (1980) wrote that "it would not be surprising to find that of all educational researchers in our history close to 100 percent are still alive and active in their field" (p. 200).

Chapter Three

A Present History of Powerful Mathematical Learners

Where is the Life we have lost in living?
Where is the wisdom we have lost in knowledge?
Where is the knowledge we have lost in information?
T. S. Eliot, *The Rock* (1934)

Philosophy in this study involves clarifying a sub-whole of Being, namely, Being-mathematical. It is through dialogue that the fullness of Being-mathematical is made possible, because the extended self is ‘held together’, or is constituted existentially through a human modality that is dialogue. It appears as if dialogue is a modality of Being — more than any other modality — that enables humans to comport towards a wholeness of Being through meaningful, creative, and ethical discourses that interconnect the past and the future with the ‘present tense’ of Being-there.

With reference to **Figure 3·1**, the present unfolds epistemologically in its past and future through an ontology or expression of Being that is cyclical and bidirectional in its discourse. Therefore a present history is temporally situated in Type I and Type II processes of mind; the emergence of a story and an analysis of that story. However, the present is always becoming the past, and the past gives way to the future. Thus the past influences the present in the future. This is essentially what a present history means, because the future is necessarily present in its history. Phenomenologically then, a present history is possible in the future because “historical epochs become ordered around a questioning of human possibility, of which each has its formula, rather than around an immanent solution, of which history would be the manifestation” (Merleau−Ponty, 1974, p. 24).

Two examples of powerful mathematical learners. It is useful at this stage to stimulate a mental picture of two individuals in the history of mathematics who were mathematical in the Three Worlds, namely, Archimedes of Syracuse (c. 287−212 BC) and the Renaissance polymath, Da Vinci (1452−1519). The standout characteristic of both men

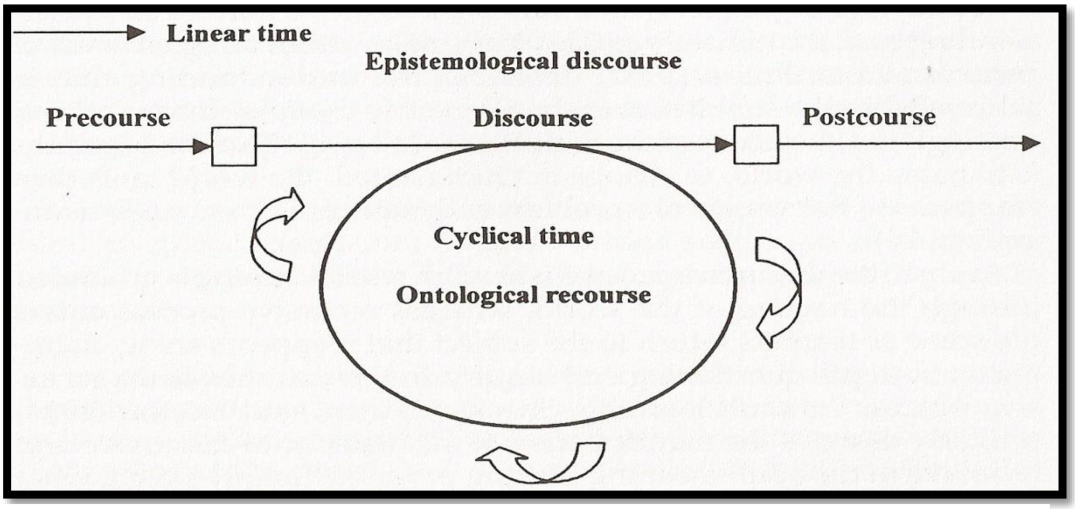


Figure 3·1. Powerful mathematical learners dialogue in the temporal dimensions of discourse and recourse (adapted from Kolb, Baker, & Jensen, 2002, p. 59)

was the ability to enable their minds through the creative use of the body, and vice versa. The result was the expansion of World 3 through new mathematical knowledge (Clark & Kemp, 1988; Netz & Noel, 2007). In so Being, although both Archimedes and Da Vinci interrelated the Three Worlds through mathematics, it was their creative use of mathematics that interconnected their respective selves in terms of the Three Worlds.

It is simply not known however, the role which an unfolding and enfolding dialogue played in their mathematical Beingness. But Archimedes did attend the university at Alexandria; engaged with the lectures of Conan, and addressed his mathematical writings to some of the finest mathematicians of the day (Rouse Ball, 1935). Da Vinci on the other hand was an extraordinary 'mathematical painter', as well as an applied mathematician who invented 'futuristic' machines (Suh, 2005). Interestingly, both individuals appear to have been motivated for the work's sake rather than that which motivates the self extrinsically (e.g., the applause of people or monetary gain). Their mathematical work was certainly an expression of their respective selves, and consequently reflected or signified who they were as people. Thus both Archimedes and Da Vinci 'self identified' through the growth and development of their mathematics (Csikszentmihalyi, 1990).

Although Archimedes and Da Vinci were high level STEM and STEAM (Science, Technology, Engineering, Arts, Mathematics) learners respectively, it is not being suggested that students in mass mathematics education can develop Being-mathematical to the same impressive levels of Being. Through a historical view of their lives however, a **wholeness** of Being-mathematical can reflect vividly in the minds of individual students and teachers towards the goal of powerful mathematical learning for an emerging twenty-first century.

Mathematics: A changing dynamic. The ideas in this section are informed by Stewart's (2012) *Taming the Infinite: The Story of Mathematics*. A relational understanding of number is fundamental to powerful mathematical learning, because sophisticated mathematical structures, patterns and forms are evidence of a deep understanding of number. In particular, individuals who learn arithmetic essentially and logically through symbolic and visual−spatial modalities of Being lay a meaningful and indispensable foundation for all future mathematical learning. It is these two modalities of sense making that pave the way for a systematic understanding of arithmetic through algebra and geometry.

Thus if any individual is to understand the algebra of arithmetic, the individual needs to **see** how and why arithmetic works computationally and geometrically. In effect it was through algebra and geometry that problems in arithmetic were systematised, and consequently provided mathematics with a key to abstract the underlying laws in nature. The German mathematician, theoretical physicist, and philosopher Weyl (1885−1955) felt that mathematics and physics were so interwined that the one without the other meant that neither mathematics nor the real world could be properly understood (Wittmann, 1969).

In terms of The Natural−Physical World (World 1) human beings were attracted by the unknown. Being-human meant to discover or 'conquer' the unknown. Algebra, and the visualization of algebra through geometry has played a very significant role in this regard (Hershkowitz, 1989). For example, creative and diligent mathematicians developed logarithms and trigonometry in terms of coordinate and differential geometry, which prepared the way for an increasingly quantitative and scientific understanding of World 1. In particular the algebra of geometry, and vice versa has influenced and made globalization possible. For example by enhancing navigation, map-making, and more recently the construction of GPS satellite systems.

Deeper questions about numbers earmarked the origins of number theory. However, certain number patterns associated with World 1 seemed to be irregular, and did not relate directly but rather through their ‘derivatives’ (e.g., the displacement, velocity, and acceleration of visible objects). Then as a result of an interaction between prior mathematical knowledge (World 3), imagination (World 2), and the real-world (World 1), Leibniz (1646−1716) and Newton (1642−1727) initiated (apparently) independently of each other, the single greatest advance in the history of mathematics, namely, the calculus. The key idea was to relate the infinitesimally small and the infinitely large by means of the same underlying process that was the limit. In effect the limiting process **was** the calculus, which meant that physical laws could be understood and formulated in relation to instantaneous change, especially when the limit was eventually placed on a rigorous mathematical footing.[1] Hitherto insoluble problems were meaningfully addressed through the creative use of differential equations. Twenty-first century civilization owes much to the differential equation, because essentially technology advances by constructing and solving new differential equations.

However, as a result of the calculus and the *zeitgeist* that was the Age of Enlightenment, mathematicians were adventurous and ambitious in their thinking. ‘Impossible quantities’ like the square root of minus one; triangles whose angle sum was more or less than 180° were used to inspire new algebras and geometries. Through these new ideas scientists changed the world. For example, without mathematically ‘impossible quantities’ Control Theory would not exist, and the space shuttle would probably have been too unstable to fly.

Interestingly therefore, mathematical creativity does not need to relate initially to the reality of the real-world to be useful in the long term. In the eighteenth and nineteenth centuries the mathematics of projective geometry, groups, rings, and fields were no longer viewed as processes but as things. The meaning of formulas, transformations, and highly abstract structures depended primarily on the rules that governed the manipulation of the symbols rather than on the meaning of the symbols themselves. Consequently, the long-standing rules of algebra were modified to accommodate the mathematical characteristics of these new structures including the cardinality of infinite sets which were treated like 'special' numbers. For example, the cardinality of an infinite set might be the same as the cardinality of a proper subset of itself.

Rubber-sheet geometry (topology) meant that circles could be squashed into triangles and squares even though the continuity of the object was preserved. Through the topology of knots for example, scientists have gained insight into the nature and functioning of the double-stranded DNA molecule. But geometry is not limited to three dimensions. Geometry 'out of this world' has been used to understand World 1, because higher-dimensional geometry enables Being-mathematical to see what is not initially visible at all. Therefore the relationship between World 1 and World 2 is bi-directional. Objects and things from both worlds are 'needed' to facilitate human understanding and progress (Bronowski, 1973).

As mathematical knowledge flourished so too did the development of World 3. However, a few mathematicians like Frege (1848−1948) and Russell (1872−1970) were concerned that the foundations of mathematics did not support the burgeoning superstructure that was Mathematics. Consequently, there were attempts to validate mathematics logically through a rigorous treatment of sets. However, Russell proved that the set of all sets was a logical impossibility (Enderton, 1997). This was a precursor to Gödel's proof that if mathematics was logically consistent then it was an impossibility to prove the property of consistency

(Hofstadter, 1979). Gödel's theorems altered the way mathematicians understood the logical foundations of mathematics. The new understanding implied that unsolved problems — past, present, or future — might not have a solution at all, but be in a perpetual state of 'limbo', namely, 'undecidability'. Many such problems have been shown to exist.

Nevertheless throughout the twentieth century mathematics became increasingly important. For example, consider the substantial role that Einstein's theory of general relativity and Bohr's quantum physics have played in the technocracy of globalising societies. However, there was a major change in perspective. Mathematics might not be 'true', and it certainly was not truth. Gödel had shown this to be the case. Therefore philosophers like Popper proposed a postpositivist worldview that was fundamentally probabilistic.

As a result, mathematical statistics as a rational approach to chance or probability has had an increasingly significant effect on scientific endeavour and on human decision making, especially post-World War II. For example the State of Qatar on the Arabian Gulf employed the services of the American and European-based company Rand Statistics Corporation, for the purpose of advising which educational system would best suit the very wealthy but developing nation. On the basis of their data collection and analysis, Rand proposed a number of options including the charter school model which Qatar ultimately adopted. Although top-down educational reforms have not been very successful in the multi-ethnic country, the Father Emir of Qatar remained resolute when he was quoted as saying that

> the new world educational system recognizes that education is a universal right and hence enables students wherever they might be to have access to the means of innovation, creativity, acquisition of knowledge and expertise and the practice of responsibility. (Source: http://www.qf.com.qa/about, 2014)

Nevertheless, companies like Rand Statistics Corporation have had a profound effect on decision making in our world through the use of calculating machines and computational mathematics. Powerful computer technologies have enabled mathematicians to solve, optimise, or model complex problems involving thousands of variables. The computer has also been an aid to proof. For example in Graph Theory, approximately 1200 hours of computing time was necessary in the 1990s to prove the four-colour theorem rigorously (Chartrand & Lesniak, 1996).

Furthermore, computers have given scientists the opportunity to understand systems in ways that have revolutionized science, particularly because more and more scientists have “felt the compartmentalization of science as an impediment to their work. More and more felt the futility of studying parts in isolation from the whole. For them, chaos was the end of the reductionist program in science” (Gleick, 1988, p. 304). Essentially chaos is the study of complex systems that exhibit irregular and unpredictable patterns, and because the complexity is highly sensitive to small disturbances it cannot be modelled effectively using systems of linear equations.

Nonetheless, chaos is influenced by deterministic laws because the complex system tends to hone in on a complex motion that is referred to as a strange attractor; in a sense like a never-ending self-similar fractal that is increasingly complex across smaller and smaller scales. In other words small perturbations might result in large effects, and large perturbations might result in small effects. As a consequence simple systems can give rise to complex behaviour; complex systems can give rise to simple behaviour, and it appears as if the laws of complexity are universal regardless of the constituent elemental components of the system. Thus an “order without periodicity” (Gleick, 1988, p. 306) has changed the way that many scientists view the world and develop knowledge. As a result modelling often no longer emphasizes a high-detailed approach but rather focuses on ‘what really matters’, or a holistic approach in relation to sensitive start-up and boundary conditions.

Summary insights: Mathematics is not static. Powerful mathematical learners have an understanding of number that evolves through increasingly abstract algebraic and geometric structures. The fundamental purpose of such learning is to facilitate the creative use of technology so that complex systems can be modelled probabilistically, especially in ways that are non-linear and dynamic. Therefore a K−12 education in mathematics needs to reflect, at least in principle, a present history of mathematics from the development of number through to chaos and complexity (and preferably beyond). Consequently, mathematics curricula around the world require revision so that an education in mathematics reflects what mathematics means currently as an expression of human creativity stretching back generations. Essentially therefore, learning the mathematics of World 3 should represent an educational journey of **how** Being-mathematical has changed, and is changing.

Major Social Forces in the Powerful Learning of Mathematics

The present history of a powerful mathematical learner is moulded by social forces that interact through ongoing discourse−recourse events. For more than a century the nature of mathematics education has been shaped internationally by five major social forces (Schoenfeld, 2004). **First**, humanists have viewed mathematics as a socio-cultural artefact, or expression of thousands of years of human civilization. In terms of the 'static−dynamics' of World 3 therefore, mathematics is prized for its logico−deductive reasoning in the Aristotlean tradition that 'all conclusions must be rigorously informed by a logical structure that is predicated upon sound premises' (Eikeland, 2008; Mautner, 2005; Pakaluk, 2005). As a consequence powerful mathematical learners need to empathize with the 'logicality' of an emerging

> culture and civilization. If the latter terms mean anything beyond honorifics, they signal the value of movement, growth, and cultivation. They disclose the meaning of ascension: to become educated is to rise to meet the challenges of life and [in so Being] to realize one's capacities as fully as possible. (Hansen & Laverty, 2010, p. 225)

The **second** societal force to impact on mathematics education has been 'education for social efficiency'. This meant the adoption of pragmatic and differentiated curricula (Shepard, 2000) so that students could prepare for their "predetermined social roles" (Schoenfeld, 2004, pp. 255−256). In *Plato's Republic* for example (Ferrari, 2007), a "society is stably organized when each individual is doing that for which he has aptitude by nature in such a way as to be useful to others" (Dewey, 1916, p. 88). This perspective of education's role in society, and vice versa is evident globally, because a complexity of culturally and socially differentiated power relations *a posteriori* the teaching and learning of mathematics "is not a 'natural' or 'inevitable' to human progress or enlightenment, but a socially constructed enterprise in which its status and selection is derived from the particular functions of schooling as an institution of upbringing and labor selection" (Popkewitz, 1988, p. 221). It would be naïve therefore to suggest that mathematics education occurred in a political vacuum, but "is greatly influenced by and must reflect or even anticipate changes in the educational system" (Howson, 1978, p. 183).

However, education for social efficiency has been ameliorated by the **third** social force, namely, the 'democracy in education movement' that continues to contend for social justice in all aspects and strata of education. In terms of a democratic worldview, no twenty-first century society should advocate an inequitable class system, but "must see to it that its members are educated to personal initiative and adaptability. Otherwise, they will be overwhelmed by the changes in which they are caught and whose significance or connections they do not perceive" (Dewey, 1916, p. 88). This is precisely what has happened to many individuals who have been referred to as the 'maladapted discontents' of globalisation (Stiglitz, 2003).

It is apparent from recorded history that forces of influence or change in mathematics education have not been mutually exclusive. In the case of the Aztecs (1400−1600 AD) of ancient Mexico for example, although their compulsory education system enforced a rigid class system, an element of democracy was permitted and this resulted in an emergent middle class (Smith, 2012). Through limited differentiation there was a "freedom to choose one's education based on a child's promise in a particular field" (Volante, 2012, p. 3).

The **fourth** and probably most potent societal force that influences the nature of mass mathematics and science education has occurred in relation to personal and national security. However, military and economic agency that is effected through advanced technologies can fuel or hinder the aspirations of an increasingly emulous globalizing world (Connell, 1980; D'Ambrosio, 2007; Hourigan & O'Donoghue, 2007; Ihde, 2009; Keeves & Aikenhead, 1995; Lips & McNeill, 2009; Schoenfeld, 2004; United States Office of Science and Technology Policy, 2013).

A poignant example of military and economic disempowerment in relation to education pertains to the plight of females in the Palestinian territories (the West Bank and the Gaza Strip). Many women and girls have experienced the gendered nature of (mathematics and science) education under siege, because "in conflict zones the educational front is closely related to the conflict front. Education in time of war and political conflict is not a neutral site, but rather a contested one" (Shalhoub−Kevorkian, 2008, p. 179).

Therefore Being-in-the-world will probably involve social forces that oppose the potential of the individual's Beingness. In spite of negative socio-cultural, political, or environmental conditions however, becoming a powerful mathematical learner implies developing the self in relation to the Three Worlds, or incorporating 'available' aspects of the Three Worlds creatively and pragmatically so that the self can expand, or re-express what Being-in-the-world can mean.

From a Piagetian developmental perspective, all learners commence with a self that is relatively egocentric and phenomenistic, namely, the individual focuses primarily on the superficial and the obvious in relation to an *I*-consciousness which is essentially the self. But through social development, cognition becomes increasingly progressive in the construction and realization of Being-in-the-world. The *I*-subject then has the choice to penetrate

> more deeply and more extensively into the object of his cognition. And egocentrism is replaced by *reflection*; the subject rethinks and restructures aspects of an object of thought “constructed” earlier, critically reanalyses his initial assumptions about these aspects, and in general, submits his earlier cognitions to a searching *prise de conscience*. (Flavell, 1963, p. 256)

The goal of such socio-cognitive development is formal operations through hypothetico–deductive reasoning (Husserl, 1927; Inhelder & Piaget, 1958; Keeves, 2002; Voyat, 1982) that necessarily involves both Type I and Type II processes of mind. If such cognitive development is to be attained then learning can be sparked and sustained by “an internal dissonance: Something is not settled. We realize that our ideas did not make as much sense as we thought, and that other ways of thinking would be more satisfying. This kind of conflict is created by communicating with others” (Hiebert et al., 1996, p. 46). As a consequence of such communication all ‘learning acts’ need to be in advance of the student’s development (Piaget, 1970, 1973, 1985; Shayer, 2002), because such learning was deemed “good learning” (Vygotsky, 1978, p. 89).

The **fifth** social force likely to influence powerful mathematical learners is to recognize that human development is fundamentally social in all its relations (human and non-human), and as Vygotsky (1986) argued, central to these relations was meaning and imagination as language acts of thought. In Vygotskyan terms the growth and development of the human mind is not a linear, sequential, or uniform event in the complexity of its social relations. The development of a mind in society through cognition and language occurs as the self experiences Being-in-the-world through a complex dialectic of socio-cultural interactions (Roth & Radford, 2011). Consequently, human development and embodied thought does not proceed like a chrysalis in “the gradual accumulation of separate changes” (Vygotsky, 1978, p. 73).

In terms of the Conceptual Age however, as many Beings-in-the-world as possible need to reach the level of formal mathematical thought, because such thinking is empowering through

"What if?" constructions that can be reflected upon in logical−analytical terms. This implies developmentally that the formal thinker no longer needs to deal "with objects directly but with verbal elements," (Inhelder & Piaget, 1958, p. 252) even though all mathematics originates "in empirics, although the genealogy is sometimes long and obscure" (Rees, 1962, p. 10; also see Collis, 2014). Correspondingly, powerful mathematical learners who function hypothetically and deductively in the following Piagetian operations, or 'verbal elements' are likely to develop an increasing propensity, or predilection for logical and formal thought:

(1) Exclusion of irrelevant variables;
(2) conjunctions, disjunctions, and implications;
(3) elementary combinatorics that involve permutations and combinations;
(4) notions of probability;
(5) notions of correlation;
(6) coordination of frames of reference including networks;
(7) multiplicative compensation (e.g., moments of force about a balance point);
(8) equilibrium of physics or vector systems that involve three or more variables;
(9) proportional thinking;
(10) modelling in one, two, and three dimensions; and
(11) thinking in terms of infinity (including limits and infinite sets).
(adapted from Shayer & Adhami, 2007, pp. 267–268)

Summary insights: Social forces. All learners of mathematics are subject to major social forces. Importantly therefore, powerful mathematical learners need to become increasingly aware of these forces in dialogue with their mathematics teachers and mentors, who should not only be aware of, but also reflect on the history of different cultures and civilizations; the asynchronous development of the individual; the dialectics of social efficiency and democracy in education, and the relative empowerment of the learner through innovative technologies. The purpose of such reflection is to enable a creative and ethical dialogue between Beings-in-the-world so that powerful mathematical learning can be promoted and sustained in classrooms.

Metaphor

Creativity involves connections between disparate ideas and contexts, or new relations (e.g., the antagonist Iago in Shakespeare's *Othello* (c. 1601−1604) has a 'curvilinear character'). Such connections and relations are often invoked through the creative use of metaphor. In this regard an important group of metaphors are those that facilitate 'the dialectical', namely, a field of oppositions with sufficient binding power so that the field acts as a unit through resolved dialectics, or complementarities (McNeill, 2005, 2012).

The term complementarity was coined by the Danish physicist, Bohr in 1927 (Gardner, 1993; Grinnell, 2009; Sfard, 1991). It implied a dialectical "conflict or opposition of some kind" (McNeill, 2005, p. 85) that was settled through a change of perspective which recognized "that an independent reality in the ordinary physical sense can neither be ascribed to the phenomena nor to the agencies of observation" (Otte, 2003, p. 203). In a complementarity sense therefore, the mathematical dynamics of Being-in-the-Three-Worlds is often dialectical because "dialectical thinkers accept contradiction as the basis of reality, and they possess the ability to synthesize their contradictory thoughts, feelings, and experiences into a more advanced and coherent cognitive organization" (Smith & Reio, 2006, p. 120). For example, certain infinite geometric sequences have a finite sum. But the inherent finite−infinite paradox is resolved metaphorically and intuitively by understanding the limiting process as a 'journey ever-closer to a destination but never actually arriving at the destination'.

However, mathematical learning should not only be serious or contradictory but also fun (Grinell, 2009). By Being-mathematical in relation to complementarities the powerful mathematical learner is an 'adventurer in dialectics' (Merleau−Ponty, 1974), or is a participant in the "dynamic interplay of unified opposites," (Baxter & Braithwaite, 2007, p. 276) which does not as

> Sartre [French existentialist philosopher, 1905–1980: Sartre, 1947,1957] claims, provide finality, that is to say, the presence of the whole in that which, by its nature, exists in separate parts; rather it provides the global and primordial cohesion of a field of experience wherein each element opens onto the others. (Merleau−Ponty, 1974, p. 204)

Although Being-mathematical requires "a complementarist approach, if its dynamics and meaning are to be properly understood," (Otte, 2003, p. 203) mathematical understandings can also be influenced by phenomenological or metaphorical primitives (diSessa, 1983; Schoenfeld, 2011) that constitute the self-binding essentials between the different *I*-positions. That is since the *I*-positions are embodied they necessarily exist in phenomenological or metaphorical relation to objects, persons, and diverse movements and activities that interconnect the extended self. Consequently, an embodied mind in society can be enacted as a complex metaphor, especially for a society of mind to emerge, because

> each age defines education in terms of the meanings it gives to teaching and learning, and those meanings arise in part from the metaphors used to characterize teachers and learners. In the ancient world, one of the defining technologies was the potter's wheel. The student became clay in the hands of the teacher. In the time of Descartes and Leibniz, the defining technology was the mechanical clock. The human being became a sort of clockwork mechanism whose mind either was an immaterial substance separate from the body (Descartes) or was itself a preprogrammed mechanism (Leibniz). The mind has also, at various times, been modeled as a wax tablet, a steam engine, and a telephone switchboard. (Kilpatrick, 1985, p. 1)

The age which preceded the Conceptual Age, namely, the Information Age was characterized by the computer information processing metaphor. Consequently, constructivist conversations around teaching for learning were influenced by the ideas of "programming," "assembly," or "debugging" (Hobart & Schiffman, 1998; Kilpatrick, 1985; Simon, 1979, 1990). Therefore metaphor not only empowers, but also limits and delimits human potential (Johnson, 1987; Kilpatrick, 1985; Schoenfeld, 1987; Skemp, 1976). In this regard globalisation has enriched humanity, because it has meant that minds in particular societies have become potentially more complex through engagement with minds from other societies that reflect different metaphorical and historical backgrounds.

However, the German mathematician Hilbert declared at the International Congress of Mathematicians in Paris that "history teaches the continuity of the development of science" by either addressing or casting aside the problems of previous ages (Hilbert, 1900, p. 437). Tolstoy (1828−1910) in considering the educational problems of previous ages, noted that the power brokers of society and the populous were both desirous of, and active in the goal of a worthwhile education for all. But in their interconnectedness there was a substantial conflict of ideas and emotions, because global and situated perspectives of mathematics, science and education tended to be inherently oppositional.

In Germany between the seventeenth and nineteenth centuries for example, the statistical records showed that the populous went to school, but were not particularly interested in school (Tolstoy, 1967). The circumstances in France, England, Russia, and the United States during the same time period were not that different; not to mention the voices of social dissent in schools of ancient cultures like India, Egypt, Greece, and even Rome (Bauer, 2007; Braudel, 1993; Freeman, 2004; Huskinson, 2000). More recently, America in the 1960s, and to a lesser degree other countries including Australia, have been characterized by a time of social upheaval. In the philosophy of Spinoza (Ratner, 1927; Scruton, 2002), there was a conatus that arose from within the younger members of society who demanded widespread change so that the 'true' nature of the self could be preserved through the forging of a new equality. As a result the educational system faced severe criticism, and thus as a system was counteracted in effecting meaningful and substantial societal change (Coleman et al., 1966; Bruner, 1977).

Not too dissimilarly therefore, the present history of mass mathematics education is dialectical in power relations, social forces, and various metaphors that substantiate Being-in-the-world. However, the 'problem of oppositions' cannot be simply cast aside by glocalizing cultures and societies, because the challenges of a rising integral civilization are highly interconnected (Ray, 1996). Nevertheless, cultural change is underway and learners need to be educated in the change that is the Conceptual Age, or **Web 2.0** (Barton & Lee, 2012; Lee & McLoughlin, 2011). Hence, in spite of differing metaphorical and historical influences as well as political rhetoric, mathematics education and social science curricula need to articulate metaphorically with the times if they are to be meaningful in classrooms around the world (Conant,1947). By engaging therefore with Conceptual Age curricula, powerful mathematical learners form part of 'the metaphorical' that is influencing the structure of a globalizing world, namely, that which is used "to explain 'how the world works' and today [the] metaphor is migrating from 'machine' to 'web' (interconnectedness)" (Goerner, 2000).

As the name suggests the Information Age was characterized by a voluminous amount of new information as a direct consequence of increased computer processing capability, ease of communication, and a host of postmodernist genres including deconstructionism, poststructuralism, and flows of thought that involved feminist epistemology and philosophy (Darling–Hammond, 2008; Mautner, 2005; Shank, 2006). Predictably then, a next stage in human progress is the conceptualization of newly generated knowledge into understandings that do not yet exist and that can benefit the individual in his or her society. Hence, a primary goal of the Conceptual Age is "not so much to see that which no one has seen, but to see that which everyone sees, in a totally different way" (Schopenhauer as cited in Goerner, 2000). History indicates that new metaphors are likely to play a fundamental role in this regard.

Consequently, powerful mathematical learners for the Conceptual Age are individuals with different and similar present histories to past students, which therefore positions them Being-there to network familiar mathematics into meaningful wholes that are surprising, or new by engaging dialogically with the ideas and metaphors of self and others (Francisco, 2013; Hermans & Gieser, 2012; Schilbach et al., 2006).

Progressive Education in Mathematics

The idea of progressive education is not new. Being-human involves change, and change tends to be "ubiquitous and relentless, forcing itself on us at every turn" (Fullan, 1993, p. vii). Nevertheless the timeless in education manifests through, or in contrast to change, because as an expression of humanity "to see what is general in what is particular and what is permanent in what is transitory is the aim of scientific thought" (Whitehead, 1911, p. 8). It is only from history however, or direct experiential learning (Husserl, 1927; Kolb, 1984) that science and education advance through the interweaving of dynamic conceptual schemes which not only correlate with the 'known' facts, but also give rise to fruitful discussion, research and practice (Conant, 1947).

By way of example, certain essentials of progressive educational reform are reflected in the literary sketch that follows. Importantly however, it is imperative that a progressive or futuristic curriculum be grounded in a valuing strand, because although "there is no such thing as a future, but many alternative futures from which to choose," (Silvernail, 1980, p. 17) it has been argued convincingly that humanity is "at a turning point in history, ready for the emergence of new values and the renewal of old ones, transmitted through education and communication technologies" (De Leo, 2012b, p. 5). The curriculum therefore is not a vehicle to clone the next generation in favour of, or in opposition to the past, but a means to empower lifelong learning, as well as to facilitate wise choices through a holistic understanding of the past, thereby enabling a wider and more detailed horizon as compared to 'past futures' (Thompson, 1978).

A vignette of progressive education. The ideas in this section are informed by Ulich's (1961) *Three Thousand Years of Educational Wisdom: Selections from Great Documents*. The integrity of a nation is preserved through its values (Confucius, c. 551–479 BC). However, values are not passed on automatically from one generation to the next. It is only through contemplation, or an intense form of reflection that society's values can be genuinely internalised, modified, or rejected by the individual (Ibn Khaldoun, c. 1332−1406; Lao−Tsu, c. 6th century BC Russell, 1872−1970). But the student who is not value oriented, and does not reflect is unable to meaningfully express his or her personalised values through the self (Chuang Tzu, c. 3rd century BC). Anti-values, or pseudo-values are often the product of authoritarianism which does not encourage the Beingness of the student, but instead limits or hinders his or her Being through fear and retribution. In contrast, values that are concomitant with a freedom of expression that encourages independent thought (Rousseau, 1712−1778) and an elasticity of mind (Bacon, 1561−1626), tend to be cherished and 'lived out' in terms of who the person has become as a consequence of the internalised values.

But if the direct experience of the child is authoritarianism and not nurture (Pestalozzi, 1746−1827), or empathy from the other person (Erasmus, 1466−1536), then the ability of the child to create for a common good may be undone (Froebel, 1782−1852). However, if the creative process is valued and "awakened in the child, [it] will seek form, guidance, discipline and loyalties," (Ulich, 1961, p. 577) because the uniqueness of the child has been respected by the adult (Emerson, 1803−1882). The consequence of a creative education coupled with the following values (Franklin, 1706−1790) is the likely emergence of a flexible and well balanced adult through childhood (Froebel, 1782−1852):

(1) Temperance — Self-control
(2) Listen more than you speak — Avoid trifling conversation
(3) Order — Let each part of your business have its time
(4) Resolution — Perform without fail what you resolve
(5) Frugality — Waste nothing
(6) Industry — Always be employed in something useful
(7) Sincerity — Be honest with yourself and with those in authority (Glückel Von Hamelin, 1644−1724)
(8) Justice — Wrong no one
(9) Moderation — Avoid extremes in intellect or bodily activity
(10) Hygiene — Cleanliness is next to godliness
(11) Tranquility — Pursue peace as much as possible
(12) Chastity — Never use venery to injure your own or another's peace or reputation
(13) Humility — Imitate Jesus and Socrates
(adapted from Ulich, 1961, pp. 434−435)

As a consequence, the creative self that reflects and understands a harmony of such values will probably make a considerable contribution to society (Herbart, 1776−1841). However, with the provisos that the person has learned to manage money and property responsibly; exercises the body regularly, and continues to develop and educate his or her Being creatively as an adult (Maimonides, 1135−1204). But to mention Tolstoy's antithesis on education and the sentiment of Pestalozzi, that adult "education is ineffective unless it grows out of the initiative of the people themselves, unless it speaks their language, and unless it influences not only isolated individuals, but the life of the whole community" (Ulich, 1961, p. 480). Education in these terms however, is particularly beneficial if the culture of learning is to develop a penetrating intuition (Plato, 428−348 BC) and a reflective imagination that interact with the inductive and the deductive (Aristotle, 384−322 BC) dialogically (Socrates, 469−399 BC). The purpose of which is to inquire beyond the superficial and the assumed for the express purpose of facilitating a wiser and less gullible society (Galileo, 1564−1642).

In addition, if values are to assist the learner in realizing his or her potential through the lifelong learning (Descartes, 1596−1650) of the body and the mind (Aenea Silvio, 1405−1464), then the self of the individual needs to learn discipline as a result of repeated practice from a young age (Locke, 1632−1704). But a system of learning (Petty, 1632−1687) that encourages the assimilation and accommodation (De Montaigne, 1533−1592) of knowledge in a liberal arts tradition, requires practice in grammar, rhetoric, the dialectic, arithmetic, geometry, music, and astronomy (Maurus, 776−856). Thus a systems approach to education can empower the learner to grasp and appreciate the interrelatedness of his or her community, the world at large, and the universe (Petty; Sellars, 1880−1973). Importantly for the growth and development of a society therefore, all youth without exception, both male and female, need to be instructed in the sciences and with high expectations of personal achievement (Comenius, 1592−1670).

Furthermore, a holistic education needs to include both deliberate and spontaneous instruction in the classroom and in the family (Plutarch, c. 46−120 AD), especially by taking advantage of teachable moments (De Montaigne). At least in part however, teaching is a science and consequently must be structured, because "the totality of the studies ought to be classified so that each step prepares for the next one" (Comenius as cited in Ulich, 1961, p. 345). If possible, the different steps should be interrelated or underpinned by metaphor or analogy as informed by the natural order of things (The Jesuit Order, 1539−). It was the Roman philosopher and orator Cicero (c. 106−43 BC) who said, “If we follow nature as our guide we will never go astray” (Ulich, 1961, p. 344).

However, teaching is also an art form because the Being of the student, and the Being of his or her mathematics teacher are both unique compositions having different cognitive and non-cognitive abilities, backgrounds, and interests that need to be identified in relationship with each other (Dewey & Dewey, 1915; Skinner, 1954; Wallas, 1926). In response to the uniqueness of the student and the teacher therefore, it is primarily the responsibility of the teacher to differentiate the curriculum adequately, because a one-size-fits-all approach to education will limit the development of self, society, and the life-purpose of both the student and the teacher (Moses Hayyim Luzzatto, 1707−1747). There are many cases or instances of knowledge, but if the curriculum is appropriately differentiated then the student is likely to enlarge his or her horizon of Being, even though the student can ‘only see what that person’s knowledge allows them to see’ (Sluga & Stern, 2009; Wittgenstein, 2009). Therefore, the beginning and end of didactics is

> to seek and find a method by which the teachers teach less and the learners learn more, by which the schools have less noise, obstinacy, and frustrated endeavour, but more leisure, pleasantness, and definite progress (Comenius as cited in Ulich, 1961, p. 340).

In agreement, the primary intent of powerful mathematical learning is not for the student to endure hardship, punishment, or frustration but to be nurtured through a creative process that enriches and diversifies the self of the individual; culminating in practical wisdom, resilience, and the growth of freedom through experiential knowledge (Pestalozzi; Quintillian, c. 35−95 AD). But as Jefferson (1743−1826) recognized, “a free society must be able to encompass both the vision of equality and the vision of excellence. Equality without excellence degenerates into mediocrity; excellence without equality becomes privilege” (Ulich, 1961, p.

463).

Key Influencers of Mathematics Education (1895−1945)

Mathematics education in the first half of the twentieth century was largely influenced by two men, namely, Thorndike (1874−1949) and Dewey (1859−1952). The psychologist, Thorndike adopted a connections-based and mechanistic learning approach to mathematics education, whereas the philosopher and social critic, Dewey contended that although imitation and mechanical drills were likely to reap positive rewards more quickly, such an approach to the teaching and learning of mathematics would limit the reflective power of the student in meaning making and problem solving (Dewey, 1897, 1933; Hiebert et al, 1997; Thorndike, 1906, 1924). Retrospectively, 'Thorndike Learning' has been favoured in mathematics classrooms and lecture theatres around the world. Consequently, mathematical learning has by and large been monological and not taken the form of a developmental dialogue. After World War I (1918−) however,

> many scholars who belonged to the more "progressive" camp spoke out strongly for their points of views, and for a time, some found a relatively wide following. During the 1960s, that happened again. Beyond that, even though the history of educational scholarship has been filled with contests between and among different groups and individuals, it is always worth remembering that the story is not one in which the soldiers of darkness have been pitted against the soldiers of light. (Lagemann, 2000, p. xi)

Thorndike-type learning. The learning of mathematics through imitation, drill and practice is to learn the language, or pattern forms of mathematics for the purpose of 'making the invisible visible' when solving familiar or unfamiliar problems (Devlin, 1994, 2000; Resnik, 1997; Steen, 1988). Moreover, if a student becomes technically proficient in the use of a **pattern** it means that knowledge of the pattern is well organized in long term memory, because 'neurons that fire together wire together' (Blakemore & Frith, 2005; Wolfe, 2001, 2006). The taught pattern can then be related to the student's prior learning which is a key to his or her future learning (Aristotle, 1943; Miller & Keyt, 1991). In other words if a sequence of mind–body actions, or mathematical connections logically deduced (Thorndike, 1931) takes shape neurobiologically in the psychological structure of the student (Zull, 2002, 2004), then essentially a new skill set in positive affect, volitional tendency, and formal mathematics has enriched the Gestalt of the student–teacher relationship in the mind of the learner (Resnick & Ford, 1981; Thorndike, 1931; Vygotsky, 1986).

Pure mathematicians especially, take pride in their ability to manipulate seemingly intractable mathematical structures into equivalent (isomorphic) structures that are 'user friendly' in the crafting of elegant solutions. Therefore skill development in a wide array of pattern forms is an important element of Being-mathematical. Fundamentally however, and as acknowledged by Whitehead, although 'there is no royal road to learning',

> without a doubt, technical facility is a first requisite for valuable mental activity: we shall fail to appreciate the rhythm of Milton, or the passion of Shelley, so long as we find it necessary to spell the words and are not quite certain of the forms of the individual letters. (1911, p. 8)

For the purpose of learning different mathematical techniques fluently and thoroughly, Thorndike-type learning is an excellent facilitator of 'manipulatory pattern form enablement'. For example and with reference to **Figure 3·2**, through the imaginative use of symbols and the axioms of the real numbers, the pattern form *Solve for X:* $X^2 = Y^2$, can be developed simply and methodically into an increasingly complex algebraic structure. In this way the mathematical pattern can emerge in the mind of the student as the person's body repeats essentially the same sequence of mathematical actions over time. In this regard, Thorndike advocated the following connectionist approach to the teaching and learning of mathematics:

1. Identify for the learner the stimuli (or the situation to which he or she is to react);
2. Identify the reaction (or response) which he or she is to make; and
3. Have the learner make this response to the situation under conditions which reward success and which identify failure.
4. Repeat Step 3 until the connection has been deliberately and firmly established. (adapted from Brownell, 1944, pp. 26–27)

I. Example of a basic pattern form in step-by-step logical connections

Initial Stimulus

Solve for X: $X^2 = Y^2$, X, Y ε $\Re$

Sequence of Stimulus−Response Connections

$X^2 = Y^2$, X, Y ε $\Re$ (re-write in a familiar form)

$\therefore X^2 - Y^2 = 0$ (difference of two squares; factorize L.H.S.)

$\therefore (X - Y)(X + Y) = 0$ (two linear factors)

$\therefore X - Y = 0$ or $X + Y = 0$ ($\because \Re$ is a field. That is if a.b = 0, (a, b ε $\Re$) then a = 0 or b = 0)

$\therefore$ <u>X = Y or X = -Y (X is the subject)</u>

Check both answers by substituting back into the original equation:

L.H.S. = $X^2 = Y^2$ = R.H.S. and L.H.S. = $X^2 = (-Y)^2 = -Y.-Y = Y^2$ = R.H.S.

II. Simple algebraic abstraction of the pattern form

Let X = ax and Y = by

III. More advanced algebraic abstraction of the pattern form

Let X = ax + b and Y = cy + d

By proceeding algebraically within the pattern structure,

$ax + b = cy + d$ or $ax + b = -(Y) = -(cy + d)$

$\therefore ax = cy + d - b$ *or* $ax = -cy - d - b$ (make x the subject; $a \neq 0$)

$$\therefore\ x_1 = \frac{cy+d-b}{a} \ or\ x_2 = \frac{-cy-d-b}{a} = \frac{-(cy+d+b)}{a}$$ (take out -1 as a common factor)

Check: $x_1 = \dfrac{cy+d-b}{a}$ $(a \neq 0)$

L.H.S. = $X^2 = (a(\frac{cy+d-b}{a}) + b)^2$ [a cancels with a, or $\frac{a}{a} = 1\ remainder\ 0$]

= $(1(cy+d-b)+b)^2 = (cy+d-b+b)^2$ $[-b+b=0]$

= $(cy+d)^2 = Y^2$ = R.H.S. [check x_2 in a similar manner]

IV. Trigonometric abstraction of the pattern form

Let X = sin$x$ and Y = cos$x$ (continue the pattern)

Figure 3·2. Linear sequential learning through the development of a particular pattern form

Beyond procedural learning. Powerful mathematical learners however, are not limited to the accurate recognition and application of procedures. In Manchester, New Hampshire in the United States in the late 1920s the superintendant of primary schools, L. P. Benezet made the decision (because of poor results and an inability on the part of many students to express themselves coherently in arithmetic) that certain students would learn arithmetic, not through traditional methods, but through inquiry learning that gave students the opportunity to engagewith number meaningfully (Butterworth, 2002). That is class teachers would respond to students' interests and suggestions, and give each learner the opportunity to develop his or her understanding of number in both oral and written terms. This meant that students were allowed to grapple with problems through estimation, "personal engagement, argument and reflection. They feel free to hold their own opinions, change their minds, build on others' thinking, invent their own methods, and adopt the methods of others" (Hiebert et al., 1997, p. 127).Therefore, Benezet was clearly progressive in mathematics education, and his ideas appear to be consistent with the notion that Being-mathematical is essentially backwards and forwards movements between the question and an unfolding process that might result in a solution. The following problem is an example of the type of challenge that Benezet encouraged his inquiring students to address:

> Here is a wooden pole that is stuck in the mud at the bottom of a pond. There is some water above the mud and part of the pole sticks up into the air. One-half of the pole is in the mud; 2/3 of the rest is in the water; and one foot is sticking out into the air. Now, how long is the pole? (as cited in Butterworth, 2002, p. 21)

The superintendant then decided to conduct an experiment. He gave problems, like the one above, to other students in his schools (of similar age and background), but who had been taught arithmetic procedurally and formally. The outcome was most interesting. Benezet reported that those "students who had been taught to use their heads instead of their pencils" (as cited in Butterworth, 2002, p. 23) found authentic problems easy to solve and expressed themselves well, but the procedural learners often guessed and gave nonsense replies. If the latter group did not have a procedure that fitted a given problem, then they were at a loss mathematically.

Although Benezet does not appear to have realized the vital role of the body in cognition, he along with Dewey (1933), Brownell (1944, 1945), Wheat (1951), Wheeler (1935), and others recognized that the learning of mathematics in a mechanistic fashion alone would "dull and almost chloroform the child's reasoning faculties" (Benezet as cited in Butterworth, 2002, p. 20). Consequently, Brownell in particular developed a more balanced **Meaning Theory** that was underpinned by three tenets, namely, (a) learning should be complex; (b) the pace of instruction ought to be adapted to the level of challenge; and (c) the learning emphasis was to be placed on relationships and not the mastery of mathematical techniques (Brownell, 1935, 1944, 1945). However, for students to understand mathematics did not mean that rote learning, drill and practice should be outlawed, but to the contrary:

> There is no hesitation to recommend drill when those virtues are the ones needed in instruction. Thus, drill is recommended when ideas and processes, already understood, are to be practiced to increase proficiency, to be fixed for retention, or to be rehabilitated after disuse. But within the "meaning" theory there is absolutely no place for the view of arithmetic as a heterogeneous mass of unrelated elements to be trained through repetition. The "meaning" theory conceives of arithmetic [and therefore mathematics] as a closely knit system of understandable ideas, principles, and processes. (Brownell, 1935, reprinted in Bidwell & Clason, 1970, p. 520)

As discussed later in this study, it is indeed possible to develop a conceptual understanding when the 'root of learning' lies in the well practised procedure. Nevertheless, Brownell (1935, 1944) made the point that because the real numbers constitute a system, the learning and understanding thereof should be relational and systemic. However, not all scholars have appreciated the complexity of the psychology of mathematical learning in the Three Worlds. This is not surprising because in the history of thought there has been an 'intellectual struggle' between mechanistic (e.g., critical rationalists) and organic thinkers (e.g., critical theorists).

The former group have tended to value stable processes and activities that can be structured hierarchically or sequentially; the latter group viewing an active world in more fluid and decentralized terms (Braidotti, 2006; Fry, 2012; Higgs & Smith, 1997). However, Being-in-the-Three-Worlds is complex and a balanced view recognizes that both world views have their place, and neither are absolute depending on that which is being observed. For example in educational research, if the phenomenon under scrutiny is stable and structured essentially, then Activity Theory is likely to be a productive approach, but if the phenomenon involves more chaos than order; lateral movement than hierarchical structure, then complexity science is probably a better lens through which to understand the activity (Beswick, Watson, & De Geest, 2007). Therefore it is useful for powerful mathematical learners to incorporate into their present histories both the mechanistic and organismic metaphors. The mechanistic metaphor identifies

> with scientific atomism, political chaos, utilitarian morality, religious agnosticism and philosophical materialism. The latter is associated with idealism, religious faith, political harmony, teleological science, and moral law. The struggle between the two has led to recurrent cycles in which one or the other has dominated. (R. Wheeler, 1935, p. 335)

However, although Wheeler projected the possibility that the two disparate world views could be harmonized, he did maintain that “the whole purpose of arithmetic is to discover number relationships and to be able to reason with numbers. It is not to learn the tables” (as cited in Royer, 2005, p. 107). Thorndike (1931) himself acknowledged, that learning was probably more complex and probabilistic than the connectionist view that “learning is essentially the formation of connections or bonds between situations and responses, that the satisfyingness of the result is the chief force that forms them, and that habit rules in the realm of thought as truly and as fully as in the realm of action” (Thorndike, 1924, reprinted in Bidwell & Clason, 1970, p. 462). It was the early twentieth century Gestalt theorists and students of phenomenologist Stumpf (1848−1936), namely, Koffka (1886−1941), Wertheimer

(1880−1943), and Köhler (1887−1967) who probably influenced Thorndike to take seriously the possibility that mental activity was more than just the additive combination of elementary connections or bonds, but was multiplicative through interconnections.

From the phenomenologists' point of view therefore, student learning was not limited to the sum of the parts in the sense that the repetition of a sequence of mental actions would necessarily produce a similar outcome for every student, or the same outcome for the same student on different occasions. Thus phenomenologically student behaviour was not solely elemental in response to a sequence of thoughts (Petermann, 1932; Ellis, 1938). In particular, Wertheimer (1880−1943) understood that the core essence of

> Gestalt theory is resolved to penetrate the *problem* itself by examining the fundamental assumptions of science. It has long seemed obvious — and is, in fact, the characteristic tone of European science — that "science" means breaking up complexes into their elements. Isolate the elements, discover their laws, then reassemble them, and the problem is solved. All wholes are reduced to pieces and piecewise relations between pieces.
>
> The fundamental 'formula' of Gestalt theory might be expressed in this way: There are wholes, the behaviour of which is not determined by that of their individual elements, but where the part−processes are themselves determined by the intrinsic nature of the whole. It is the hope of Gestalt theory to determine the nature of such wholes. (Wertheimer, 1938, p. 2)

Brownell (1935, 1944, 1945) therefore was a remarkable scholar in mathematics education, because more so than Thorndike, he appreciated along with Whitehead (1911) that the road to mathematical pedantry is paved with step-by-step technical procedures that exclude consideration of the general ideas that are reflected in the technicalities. In this sense "mathematics had suffered from the general application or misapplication of connectivistic theory," (Brownell, 1944, p. 27) because mathematics "relies constantly upon the principle of rhythm, the regular breaking up and putting together of minor activities into a whole; a natural principle, and the basis of all easy, graceful, and satisfactory activity" (McLellan & Dewey, 1895b, reprinted in Bidwell & Clason, 1970, p. 162).

In other words the unity or 'oneness' of mathematical thought cannot be understood by the learner until he or she grasps "the union of two like impressions: the relation of two equal magnitudes. A child does not perceive this *one* until he sees the *equality* of two magnitudes" (Speer, 1897, reprinted in Bidwell & Clason, 1970, p. 170). Nonetheless, Thorndike's understanding of Being-mathematical as a complex phenomenon was limited philosophically. In particular, he appears not to have engaged seriously with the phenomenological ideas of Husserl, Heidegger, or Merleau−Ponty.

Consequently, his connectionist approach to the learning of mathematics, although

substantial and necessary, is insufficient for students to optimise their potential in Being-mathematical. If mathematical learning is linked solely to automatism, custom, and habit "as bone of their bone and flesh of their flesh," (Thorndike, 1931, p. 160) the learner is mathematically limited in the Three Worlds. Mathematics is a holistic expression of Being-human, and therefore Being-mathematical in these terms requires much more than a trained body.

Instead however, students **need** to be exposed to a culture of Being-mathematical that facilitates a 'certainty of knowing' through formal training in pattern forms, as well as a felt and experienced liberty to connect meaningfully with the mathematics curriculum through creativity and real-world problem solving (Dewey, 1916, 1929b; 1933; Dewey & Dewey, 1915; McLellan & Dewey, 1895a). Learning mathematics holistically is not unidimensional in either meaning making or skill development, but is simultaneously both, allowing the individual to construct, or create "all possible patterns" (Tucker et al., 1959, p. 676) that lie within the potentiality of an extended self. Thus, Brownell's Meaning Theory is consistent with powerful mathematical learning, because 'Brownell-type learning' is not only likely to increase retention and recall on the part of the student, but also

(1) that which is learned can be used in acts of authentic problem solving;
(2) the fluency of transfer between different mathematical contents may be enhanced;
(3) the time needed to master skills through drill and practice may decrease;

(4) students would be more attuned to absurd or incorrect solutions;
(5) the joy of learning mathematics would be experienced in eureka moments;
(6) students in the same class would solve problems in different ways;
(7) students would be less reliant on the teacher to learn; and
(8) there would be greater respect for the teacher and the subject.
(adapted from Hiebert et al., 1997)

In a phenomenological sense therefore, powerful mathematical learners learn organismically and mechanistically. The two 'isms' are not mutually exclusive but constitute one another fundamentally (Crain, 2005; Pugh, 1971; Yates, 2009). In Being-mathematical 'the whole' that is the pattern form is expressed in and through each part. Each part in relationship with the other parts implies that the whole can emerge in the embodied mind of the individual, thereby enhancing the organism and the mechanism, which is essentially the powerful mathematical learner. However, if the whole is to emerge in Being-mathematical then it is crucial for the mathematics teacher to allow

> full recognition of the value of children's experiences as means of enriching number ideas, of motivating the learning of **new** [for emphasis] abilities, and especially of extending the application of number beyond the confines of the textbook. (Brownell, 1935, reprinted in Bidwell & Clason, 1970, p. 520)

Key Influencers of Mathematics Education Post-World War II

Mathematical competence relates directly to what one **knows** in mathematics, namely, facts, procedures and conceptual understandings (Schoenfeld, 2004; Walmsley, 2003). However, knowing in these terms does not imply Being-mathematical for a particular time in history. In the Conceptual Age, the powerful mathematical learner in Being-mathematical prepares to "manage the demands of changing information, technologies, jobs, and social conditions" (Darling‒Hammond, 2008, p. 2). Therefore powerful mathematical learning is fundamentally a learned flexibility of Being that facilitates intelligent and cogent action in situations that have not yet been experienced directly (Bradley, Noonan, Nugent, & Scales, 2008; Skemp, 1979). This means that the individual learner can solve new problems, or similar problems in different contexts, because of the training or mathematics education that has been his or her experience (Schoenfeld, 2004). The whole person by learning to Be-mathematical is confident to address the unfamiliar through the familiar (Broadie, 1970; Clarke & McDonough, 1989; Damasio, 2005; Hermans & Hermans−Konopka, 2010).

If however, Being-mathematical is to become a powerful learning reality for many teachers and students in the Conceptual Age, then mathematics curricula around the world will need to change fundamentally. It would not be the first time that major curriculum change in mathematics was advocated. Against the backdrop of a complex interplay of social and technological forces, two major reform movements post-World War II attempted to empower students to learn new mathematics in new ways that could result in authentic and creative problem solving.

Curriculum reform in mathematics education (1945−1970). The first post-war reform movement came to be labelled **New Math** in the wake of the UICSM (University of Illinois Committee on School Mathematics) "New Mathematics Curriculum" of 1952. The new curriculum was followed by the launch of Sputnik in October 1957 (Malaty, 1999), which also spurred change in the teaching and learning of mathematics so that the "very obvious advances of Russian science and technology" could be redressed (Keeves & Aikenhead, 1995, p. 18).

Moreover, mathematics curriculum development was encouraged by "general agreement ... that the teaching of mathematics had been unsuccessful" (Kline, 1973, p. 15) and mathematics education had a "shameful record" (Kline, 1973, p. 170). In this context the Carnegie Corporation of New York funded the curriculum initiative of UICSM under the directorship of Beberman. The goal set by UICSM was to enable students to engage substantially with mathematics beyond the functional mathematics that was necessary for 'everyday living'. In order to achieve this goal the teaching and learning of mathematics needed to include discovery learning and the precise use of language (Beberman, 1959;

Lagemann, 2000).

However, educational reform in the teaching and learning of mathematics was not limited to the United States. Major changes in mathematics and science education occurred in the 1950s and the 1960s on both sides of the Atlantic (Duschl, 2008; Keeves & Aikenhead, 1995; Schoenfeld, 2004; Walmsley, 2003). The underlying premise was to avoid overemphasizing any particular aspect of learning and to describe “a pedagogy based on a use of all the dynamics of learning in proper proportion” (Knight et al., 1930, reprinted in Bidwell & Clason, 1970, p. 484; also see Keeves, 1965). In particular the National Science Foundation (NSF) in the United States, and the Nuffield Foundation in the United Kingdom sponsored change in the learning and teaching of STEM subjects with the goal that students would be empowered to “think like scientists” (Duschl, 2008; Garrett, 2008). Therefore through the impetus that was New Math, students were no longer viewed as ‘bodily instruments’ to carry out routine and disconnected mechanical processes (Kline, 1973; Rosenthal, 1965).

An effect of World War II was the development and release of new knowledge. A sense of freedom and increased self-belief compelled change in American and European educational systems. For example, the Mathematics Association in Great Britain published a visionary and influential document in the late 1950s (in the tradition of Dewey) rejecting the idea that sound teaching at the primary school level “could be produced by careful analysis of the logical steps involved,” instead embracing

> the view that children, developing at their own rates, learn through their active response to the experiences that come to them; through constructive play, experiment and discussion children become aware of relationships and develop mental structures which are mathematical in form and are in fact the only sound basis of mathematical techniques. (as cited in Howson, 1978, p. 185)

Moreover, groups like the School Mathematics Study Group (SMSG) developed new curriculum materials that included textbooks designed to ‘breathe life’ into classroom learning (Kline, 1973; Malaty, 1999). In particular the SMSG (out of Yale University and

Stanford University) pioneered the introduction and integration of calculus and analytic geometry in high schools (Begle, 1954). In 1961, Begle the director of the SMSG was honoured by the Mathematical Association of America. The Distinguished Service to Mathematics citation read in part,

> With the assistance of many individuals and components of the mathematical community…he has conducted a national experiment, unprecedented ... in its combination of depth, scope and size. He has done so with character and courage, with good judgment and balance, with understanding and endurance, and in a continual searching for the first rate. (as cited in Iverson, Eisner, & Gross, 1978)

Consequently many mathematics teachers abandoned traditional 'show and tell' methods of teaching that encouraged the learning of mathematics as a sequence of manipulations, in favour of "guided discovery with manipulatives and a stress on problem solving [so] that students could truly *understand* mathematics rather than simply solve problems with little meaning" (Walmsley, 2003, p. 28). Even American and Canadian television were employed from 1959 to assist teachers (and students) with topics that they had not encountered before like modern algebra and statistics (Potvin, 1970).

Thus New Math curricula raised the expectation of what was necessary and what was possible in schools. As a result numerous learning projects were launched in numerous countries including Australia. For example, in response to the general trend that was increased formalism and rigour, set theory and structure, one of the Australian States in the late 1960s attempted to underpin the learning of calculcus with $\varepsilon - \partial$ proofs. Even though substantial in-service training was provided in high schools, this curriculum change was probably over ambitious and less successful than expected (Blakers, 1978). Currently in Australian universities such analytic proofs are only likely to appear in the second year of a bachelor of science degree (if at all).

Overall however, the New Math failed to take effect in classrooms as a process to be understood, primarily because the different role players did not provide a workable classroom model that incorporated skills instruction and understanding through new content. For example, primary school teachers across the United States struggled with the unfamiliarity of set theory, logic and modular arithmetic (Schoenfeld, 2007), while in the high schools of French-speaking Canada, "there were general complaints that too much time was spent on sums and problems to the detriment of an understanding of basic ideas, that there were too many theorems to memorise, and that there was no attempt to show the unity of the diverse branches of mathematics taught at school" (Potvin, 1970, pp. 366−367). Unfortunately for mass mathematics education therefore, the New Math Movement (NMM) was not unified in addressing the many challenges of implementing a new system of mathematical learning. In fact the NMM had uncomfortable bedfellows who had profound disagreements. In particular, "meetings between mathematicians and psychologists resulted only in determining that the two had nothing to say to each other" (Klein, 2003).

Therefore due primarily to the influence of mathematicians, New Math curricula internationally proved to be too abstract and disconnected from the life-world of most students to be cognitively accessible and affectively meaningful (Keeves & Aikenhead, 1995; Kline, 1973; Rosenthal, 1965; Schoenfeld, 2004; Walmsley, 2003). Furthermore it was not only Australia that underestimated the challenge that New Math would present to teachers. In Nigeria for example, the Lagos experiment (initiated in January 1964) was designed to implement 'Modern Mathematics' in primary schools in the State. Although relatively successful in Lagos State, elsewhere in Nigeria, State Ministries of Education handled the in-service education in an erratic manner. As a result most teachers did not understand the new mathematics concepts, with the consequence that their students continued with, or reverted to rote learning (Ogbonna Ohuche, 1978).

Retrospectively it is apparent that the developers of the new curricula were not sufficiently aware of the significant differences associated with, and the marked neurological changes that occur in the frontal lobes between pre- and postpubescence, and adulthood (Blakemore & Frith, 2005; Haier & Jung, 2008; O'Boyle, 2005, 2008). That is in the 1950s and 1960s, the entity that was Mathematics Education essentially confused the child with the adult and did not "recognize that the transition to adulthood involves an introduction to new realms of experience, the discovery and exploration of new mysteries, the gaining of new powers" (Bruner, 1979, pp. 118−119). Thus the prior learning of teachers, students, and parents was inadequate to operationalize the aspirant program of mathematical learning in schools (Ausubel, 1968).

So "by the early 1970s New Math was dead" in its heartland, namely, the United States. The effect was felt in many other countries. Particularly in American school education however, the rallying cry was "back to basics" (Klein, 2003). This 'mantra for change' resulted in a return to traditional-type curricula of the pre-Sputnik years that involved the "mastery of core mathematical procedures" (Schoenfeld, 2004, p. 258). Nevertheless, the New Math reform movement coupled with the "cognitive science revolution of the 1960's and 1970's" (Piotrowski, 2004, p. 214) has had a lasting influence on the teaching and learning of mathematics around the world including Australia and New Zealand.

Especially in terms of content, a theoretical and rigorous approach to school mathematics has been de-emphasized. For example, a formal treatment of Euclidean geometry has declined almost completely in high schools, with the result that most students currently graduate from high school without any formal training in the fundamentals of logic, but from a positive perspective, most students who now complete Year 12 mathematics do so with at least an elementary understanding of the way in which science and the world currently operates, because they have encountered probability and statistics (Blakers, 1978; Egsgard, 1970; Grinnell, 2009; Phillips & Burbules, 2000; Popper, 1965, 1979; Zammito, 2004). Nevertheless, and importantly for powerful mathematical learning, New Math portrayed mathematics as a dynamic and sophisticated process that students needed to experience for themselves if they were to engage with mathematics productively in society (Bruner, 1960). In this regard however, not all countries and societies embraced the ideals of New Math.

In the early 1950s Communist China adopted a Soviet-style mathematics curriculum that emphasized rigour and logical deduction (Wang & Cai, 2007). But as part of the Great Cultural Revolution (1966−1976), Chairman Máo Zédōng in May 1966 gave a threefold directive, namely, (a) less school, more work (*pan tu, pan kung* which meant half work, half study), (b) students should not be exposed to any bourgeois influences, and (c) "barefoot" teaching was desirable in order to make mathematics curricula more practically oriented (Vogeli, 1970). As a consequence of these educational reforms, schools in Mainland China made the 'Great Leap Forward' in going back to basics!

Curriculum reform in mathematics education (1970−1979). Essentially, New Math opened the door for mathematics education to become meaningful in classrooms around the world. However, Mathematics Education had experienced a hard lesson. The teaching and learning of mathematics did not occur in a vacuum (Howson, 1978). In particular, if the prior learning and experience of both teachers and parents was insufficient to sustain the affect and volition of student learning, then any educational reform was likely to be unsuccessful in the long-term.

However, for the decade of the 1970s the consequences of New Math were not uniform globally. In the United States for example, the back to basics movement "swept most of the new math out of America's classrooms," (Schoenfeld, 2007, p. 542) but in Australia the response was more positive. In the 1950s and 1960s educational progress had been steady and at times outstanding, with parents showing a significant interest in schools (Blakers, 1978; Connell, 1993, 2002).

Nevertheless, the Director of the Australian Council for Educational Research from

1955−1976, Radford contended that the “proper handling of individual differences between children still remains the most challenging problem in education,” especially with respect to gifted students and the socially disadvantaged (Radford, 1961, p. 3; also see Keeves, 1999; McCann, 2005; Renzulli, 1988; Sternberg, 2010a, 2010b). Of specific interest in the 1970s however, was an interational focus on the “conceptualization, assessment, and study of classroom learning” (Fraser, 1980, p. 221). This focus included the notion of ‘individualization’ which was an attempt by teachers to try “something different for each of the 30-plus students in a single classroom” (Tomlinson, 2001, p. 2).

The success of the ‘individualized approach’ to classroom learning was ambivalent. A meta-analysis of research findings on individualized instruction in mathematics reported that “individualized approaches offer positive results in many instances” (Horak, 1981, p. 252). But it was argued that relatively few studies supported individualized programs (including self-paced mathematics instruction) in favour of traditional classroom learning (Schoen, 1976). Retrospectively however, individualization as an effective didactic approach was not sustainable in the long term because of the relatively high levels of stress that teachers tended to experience in individualized classrooms (Levin, 1980; Tomlinson, 2001).

Nonetheless individualization was an outflow didactically from the learning strategies of New Math, because it fostered a move away from the ‘teacher-centred’ classroom towards a learning environment that respected the right of each student to learn uniquely in relationship with the teacher. By implication therefore, many individualized classrooms were neither teacher-centred nor ‘student-centred’.

The Three Worlds are interconnected. Interestingly therefore, individualization of curricula occurred at the same time that individual differences in hemispheric processing, or bilateral activity in the human brain were being researched (Beaumont & Dimond, 1973; Dimond & Beaumont, 1974a, 1974b; Hardyck, 1977a; Harnad, Doty, Goldstein, Jaynes, & Krauthamer, 1977; Hellige, 1975; Myslobodsky & Rattock, 1977; Parker, 2009). Notably, brain functioning was not uniform across learners. For example, hemispheric functioning was highly compartmentalized for right-handers, but left-handers tended to dual process between both hemispheres. This meant that although asymmetric processing between hemispheres was evident in the brains of left-handed individuals, the difference between left and right brain processing was not as pronounced as in the case of right-handed individuals (Hicks & Kinsbourne 1978; Semmes, 1968). As a possible consequence, the need for increased 'cross-talk' between the hemispheres might have been the reason why left-handers showed small deficits on different spatial ability tests compared to right-handers (Levy, 1969; Miller, 1971; Nebes, 1971). However, the connection between hand preference, hemispheric laterality, and mental functioning was not well established (Hardyck, 1977a, 1977b).

Nevertheless, the notion of 'whole brain learning' appeared in self-help books in secular bookstores. For example, the highly popular "*Use Both Sides of Your Brain* by Tony Buzan (1974) and *Drawing on the Right Side of the Brain* by Betty Edwards (1979)," (Jensen, 2008, p. 3) encouraged individuals to develop an understanding of their lateral brain and nervous system functioning so that individuals could improve their thinking.

In addition, seminal works by Skemp (1972, 1976) highlighted the psychological dimensions of two different kinds of mathematical understanding. First, instrumental understanding meant 'possessing a rule and knowing how to use it correctly', and second, relational understanding meant knowing what to do and why. Skemp (1976) further emphasized that the instrumental and relational teaching of mathematics gave rise to mathematics classes that were essentially different. And the transfer of learning between the two kinds of mathematics class was likely to be problematic for both teachers and students. By making this distinction however, Skemp (1976) might have inadvertently deepened the 'rift' between teaching for skills, and teaching for understanding with the result that a false dichotomy in mathematics education was perpetuated (Ma, 1999; Yates, 2009).

In the late 1960s and 1970s researchers like Kilpatrick (1967), Lucas (1972) and Kantowski (1977), influenced by the writings of Pólya (1954, 1957), monitored students in acts of problem solving. The goal of this observation was to appreciate the subtle interplay between the skills and understandings, or 'heuristic practices' that students used to solve problems by themselves. Essentially, the different studies were an attempt to co-relate the problem−solving strategies to problem−solving success (Schoenfeld, 2007).

Therefore the individualization of curricula, brain differences, and a focus on understanding and problem solving in mathematics, channelled mathematics education research towards identifying, describing, and comprehending the many different facets of both students' and teachers' understanding of the subject. A strong focus on the different types of mathematical understanding continued unabated into the twenty-first century (e.g., Alsina & Nelsen, 2006; Beswick, 2005; Borgen & Manu, 2002; Confrey, 1991; Conradie & Frith, 2000; Ekenstam, 1977; Even & Tirosh, 2008; Herscovics & Bergeron, 1983; Hossain, Mendick, & Adler, 2013; Lin, 1988; Llewellyn, 2012; Ma, 1999; Maher, 2005; Martin, 2008; Nunokawa, 2005; Pirie & Kieren, 1992a, 1992b, 1994; Pirie & Schwarzenberger, 1988; Roh, 2008; Schoenfeld, 1989, 2008a; Sfard, 1991; Stacey & Vincent, 2009; Wieman, Adams, & Perkins, 2008; Zhou & Bao, 2009).

In broad terms however, three kinds of mathematical understanding were described, namely, "understanding as structured progress, understanding as forms of knowing, and understanding as process" (Mousley, 2005, p. 553). The challenge for powerful mathematical learners is to develop an understanding of mathematics that is enhanced and complex by interrelating the different kinds of understanding. In pursuit of this ambitious goal all mathematics teachers should teach those skills and processes that enable students to engage deeply with mathematical concepts, and in so Being to construct, generate, and create problem solving solutions (Schoenfeld, 2008a).

Summary insights: Post-World War II. Since the end of World War II there have been two major curriculum reform movements that have significantly influenced the teaching and learning of mathematics across the globe. The first reform movement that was New Math lost momentum by the mid-1970s (Keeves & Aikenhead, 1995) but did provide a new vision, at least in terms of possibility, that the teaching and learning of mathematics in classrooms was not inevitably bound to a compendium of disconnected facts and procedures that rested upon the sole authority of the teacher and the textbook. For example, in the conservative and newly independent People's Republic of Bangladesh (1971), the National Curriculum Committee in consultation with the President "agreed that new ways of teaching, learning and understanding must be found if the new generation is not to be intellectually smothered beneath a mountain of facts" (Sharfuddin, 1978, p. 167).

Internationally therefore, a primary outcome of 35 years (1945−1980) of dialogue, discussion, agreement and disagreement in Mathematics Education was the imperative that individual students not only use mathematics as a product, but understand the subject as a process. The challenge for the next reform movement in mathematics learning was to effect this vision in classrooms through "inspired teaching by broadly informed competent teachers" (Courant, 1962 as cited in Kline, 1973, p. 125). However, the decade of the 1970s ended with a concern, at least in Australia, that the place of mathematics in the curriculum was tenuous, because of

> recurrent suggestions from teachers of other disciplines, and from administrators, that there must be some easier (and shorter) route to the essential mathematical competencies of the modern world; the yearning for a "royal road" did not end with Euclid's rebuke to King Ptolemy! (Blakers, 1978, p. 158)

Curriculum reform in mathematics education (1980−1989). The back to basics movement of the 1970s was at least in part an overcorrection to the implementation of New

Math curricula in classrooms. But students who were 'trained' to be technically proficient in carrying out procedures were often neither adept at the basics, nor at problem solving in real world contexts (Schoenfeld, 2004). For example, the following question was posed on the third NAEP (National Assessment of Educational Progress) secondary mathematics examination to a representative sample of 45,000 13-year-olds across the United States:

> An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training site, how many buses are needed? (Carpenter, Lindquist, Matthews, & Silver, 1983, p. 656)

Approximately 30 per cent of the students were not able to carry out the long division correctly, and almost one in three students answered that the number of buses required would be "31 remainder 12" (Schoenfeld, 1987, p. 196). In the wake of such confronting evidence it was difficult to argue with social efficiency calls by parents, politicians, or the media for a back to basics curriculum in schools (Perso, 2007; Schoenfeld, 2007; Yates, 2009). Blakers' (1978) concern that future mathematics curricula, particularly in Australia, might be watered-down to accommodate the groundswell of mathematics students became apparent in the 1980s. It was both significant and confronting that although "nineteenth century educators would not recognise the contemporary conditions of culture, commerce, and technology, they would, however, still recognise much of the Australian school curriculum" (Kalantzis & Harvey, 2003, p. 1). As a lecturer of university mathematics (in the United States and possibly Great Britain as well) from 1965 to 1995, Ralston (1999) reported the general trend that increasingly his students knew less and less mathematics when entering university, and were mechanistic practitioners of mathematics rather than individuals who had developed a principle-based understanding of the subject.

Therefore in the 1970s and the 1980s the teaching and learning of mathematics in Australia, the United States, Great Britain and other countries saw the implementation of curricula, the outcome of which for many students was largely a superficial understanding of mathematics. In this context the National Council of Teachers of Mathematics (NCTM) in the United States argued that the primary goal in mathematics education was to afford students a genuine opportunity to become competent problem solvers. Therefore to maintain the integrity of the subject in mathematics classes, it was incumbent on mathematics teachers to enable all their students to solve meaningful problems. Consequently, for the decade of the 1980s the NCTM recommended an eight point *Agenda for Action* in school mathematics, namely,

(1) **problem solving** was to be the focus of school mathematics;
(2) **basic skills** should encompass more than computational facility;
(3) teachers and students should take full advantage of the power of **calculators and computers** at all grade levels;
(4) stringent **standards** of both effectiveness and efficiency ought to be applied to the teaching of mathematics;
(5) the success of mathematics programs and student learning should be evaluated by a **wider range of measures** than afforded by conventional testing;
(6) **more mathematics study** was required for all students, and therefore a more flexible curriculum with a greater range of mathematical options should be designed to accommodate the diverse needs of the student population;
(7) mathematics teachers should demand of themselves and their colleagues a **high level of professionalism**; and
(8) **public support** for mathematics instruction must be raised to a level commensurate with the importance of mathematical understanding to individuals and society. (adapted from NCTM, 1980)

Interestingly, the back to basics mantra of the 1970s was now countered by a 'problem solving basics', which meant that rote learning and the practice of procedures should be de-emphasized in favour of mathematical applications that required logical reasoning, information processing, and decision making, together with estimation activities, and the learning of proper communication skills (NCTM, 1980). However, during the 1980s the "deeper findings about the nature of thinking and problem solving were not generally known or understood," (Schoenfeld, 2004, p. 258) and as a result problem solving in classrooms tended to be shallow.

Nevertheless, the problem solving focus in mathematics education would not be denied in the United States, particularly in the wake of the American economy losing its degree of dominance in the world. For example, certain Asian economies like the Japanese economy began to strengthen significantly (Hsu, 1999; Ito, 1992; Nakamura, 2002). Against this background, the National Commission on Excellence in Education (1983) issued the following warning:

> Our Nation is at risk. Our once unchallenged preeminence in commerce, industry, science, and technological innovation is being overtaken by competitors throughout the world. This report is concerned with only one of the many causes and dimensions of the problem, but it is the one that undergirds American prosperity, security, and civility. We report to the American people that ... the educational foundations of our society are presently being eroded by a rising tide of mediocrity that threatens our very future as a Nation and a people. What was unimaginable a generation ago has begun to occur — others are matching and surpassing our educational attainments. (p. 1)

Unsurprisingly therefore, the **second major reform movement** in mathematics education since the Second World War came to the fore in the early 1980s as a result of the interaction between the ‘problem solving movement’ in the United States (Schoenfeld, 2004); political voices expressed through emotive documents like *A Nation at Risk*, as well as the ongoing cognitive revolution (Gardner, 1983, 1985; Schoenfeld, 2004; Veenema & Gardner, 1996) that can be

> traced back to a very particular point in space and time: September 11, 1956, at a ‘Symposium on Information Theory’ held in Cambridge at the Massachusetts Institute of Technology. On that day, George Miller, Noam Chomsky, Alan Newell and Herbert Simon presented papers in the apparently disparate fields of **psychology**, **linguistics and computer science** [for emphasis]. (Friesen & Feenberg, 2007, p. 720)

Moreover, the seminal works of Gardner (1983) and Sternberg (1985) that related to intelligence theory; the growth of the constructivistic metaphor (Kafai, 2006; Papert, 1980; Tobin, 2007; Wilenski, 1991; Wiliam, 2003); increased knowledge of metacognition (including managerial decision making), and how the mind becomes self-aware in learning (Johnson−Laird, 1983a, 1983b; Kilpatrick, 1985; Schoenfeld, 1983, 1987), stimulated a problem solving dynamic in mathematics classrooms.

In particular, the NCTM wanted students to develop mathematically by grappling with problems, the solution of which occurred through multiple perspectives, reflection, and the harnessing of different intellectual abilities including construction. This approach to learning was contiguous with New Math curricula that viewed students as active participants in the guided discovery, or learning process. However, the problem solving movement of the 1980s was a stronger and more informed counter to the "behaviorist perspective, as epitomized in the work of B. F. Skinner," (Veenema & Gardner, 1996, p. 69; also see Evans, 1981; Skinner, 1954) than had been the case during the time of New Math. As a consequence, the learning of mathematics was understood through a problem solving ability that included the "effective processing, representation and structuring of information by the student's cognitive apparatus" (Friesen & Feenberg, 2007, p. 721). In these terms the centroid of classroom focus was 'somewhere' between the teacher and the student, both of whom engaged in processes of learning that conceptualized mathematics (Hiebert, 1986).

However, many mathematics teachers were not able to look conceptually and pedagogically "below the instrumental or formal surface of mathematics in order to get clues about how to present it more effectively" (Wheeler, 1989, p. 283). The lack of self-awareness in this regard on the part of many teachers meant that their students' preconceptions, ways of thinking, and conceptions were often limited mathematically (Shulman, 1986). In this 'semi-stasis' of educational change, the United States National Research Council formed the Mathematical Sciences Education Board (MSEB), which was mandated to address mathematics education not ad hockery, but intentionally and continually (Schoenfeld, 2007). As a result, early in 1989 the MSEB published a report, *Everybody Counts*. In this seminal report the MSEB addressed the 'loss of human potential' that had been occurring in mathematics education across the nation, particularly with respect to the Latino community, African Americans, and Native Americans.

Soon after the publication of *Everybody Counts* the problem solving movement in the United States, essentially through the National Council of Teachers of Mathematics, expressed its goals and beliefs for mathematics curriculum development in the NCTM (1989) *Curriculum and Evaluation Standards for School Mathematics* (CESSM). This document became known as the **Standards**, and has become one of the most influential publications in mathematics education since the Woods Hole Conference publication *The Process of Education* (Bruner, 1960). Through the Standards, the NCTM emphasized that the understanding of mathematics occurred fundamentally through process, and that the Standards document "clearly sat in the education-for-democratic-equality and education-for-social-mobility camps" (Schoenfeld, 2004, p. 268).

In Australian mathematics education for example, the Standards has been a pointer towards the need for students in schools to understand mathematics conceptually, that is by becoming procedurally fluent, strategically competent, productive in outlook, and adaptive through logical and reflective reasoning (Clarke, 2007; Kilpatrick, Swafford, & Findell, 2001; Sheppard, 2009; Sullivan, 2011). In these terms the Standards were used to promote a social constructivist pedagogy that involved metalearning (Dorier, 1995; Hekimoglu & Sloan, 2005; Norton, McRobbie, & Cooper, 2002). This pedagogical approach required students to develop mathematical connections "through the interaction of communities of people" (as cited in Norton, McRobbie, & Cooper, 2002, p. 37) based upon the following

> five general goals for all students: (1) that they learn to value mathematics, (2) that they become confident in their ability to do mathematics, (3) that they become mathematical problem solvers, (4) that they learn to communicate mathematically, and (5) that they learn to reason mathematically. (NCTM, 1989, p. 5 as cited in Schoenfeld, 2004, p. 266)

However, effecting meaningful change in mass mathematics education often proved elusive because the Being of each student was such an idiosyncratic network of relationships and experiences (Clarke, 1985; Nehring, 1992). Nevertheless, successful problem solvers tended to grasp a problem literally, or in terms of a Gestalt-whole that was characterized by metaphorical understandings (Otte & Zawadowski, 1985). Moreover, the confidence of problem solvers increased when they were taught **how** to reflect on their informal figurings (Ter Heege, 1985), and then afforded the "opportunity to inform the teacher of difficulties experienced, success achieved, and sources of anxiety" (Clarke, 1985, p. 256). Dialogically this would mean understanding mathematics as a socio-cognitive event that could take the form of the following learning sequence:

> **(a)** form a view of the mathematical idea;
> **(b)** step back and reflect upon it;
> **(c)** use it appropriately and flexibly;
> **(d)** communicate it effectively to another;
> **(e)** reflect on another's perspective of the idea;
> **(f)** incorporate another's perspective into one's own framework, or challenge and logically reject this alternative view.
> (adapted from Hoyles, 1985, p. 212)

But a number of challenging questions were advanced:

> Why is it that so many intelligent, well-trained, well-intentioned teachers put such a premium on developing students' skill in the routines of arithmetic and algebra despite decades of advice to the contrary from so-called experts? What is it that teachers know that others do not? (Kilpatrick, 1988, p. 274; also cited in Sfard, 1991, p. 10)

Table 3·1 lists Bogen's (1969) relational dichotomies, or left and right hemisphere functioning of the human brain. The complex, dual and appositional nature of the human brain is surely a fundamental reason as to why powerful mathematical learning needs to occur as expressed by both Hoyles (1985) and Kilpatrick (1988). Being-human is not monological, but complex in dialogue through the interaction of two hemispheres. Therefore in terms of whole brain learning, it is pertinent to mention that neither System I nor System II thinking has been associated with any particular brain region (Kahneman, 2011).

Table 3·1. Bogen's table of left and right hemisphere dichotomies (Fidelman, 1985, p. 59)

Author	Left hemisphere	Right hemisphere
Weisenberg and McBride	lingual	visual
Milner	verbal	conceptual or non-verbal
Semmes, Weinstein, Ghent, Teuber	discrete	diffused
Zangwill	symbolic	visuo-spatial
Hacaen, Ajuriaguerra, Angelergues	lingual	pre-verbal
Bogen and Gazzangia	verbal	visuo-spatial
Levi-Agresti and Sperry	logical or analytical	synthetic conceptual

Decade of the Brain (1990−1999). The 1990s were proclaimed as the *Decade of the Brain* by President Bush (1989−1993) of the United States. He called upon "all public officials and the people of the United States to observe the decade with appropriate programs, ceremonies, and activities" (Presidential Proclamation 6158, 1990).

With respect to problem solving for example, neuroimaging devices such as PET (positron emission tomography) and fMRI (functional magnetic resonance imaging) allowed cognitive scientists to analyse the human brain while the person attempted to solve a particular problem, or sequence of problems activity (Brown & Wheatley, 1995; Jensen, 2008). However, by the end of the decade there was "almost no literature on the links between brain science and education" that allowed neuroscientific research to inform education either theoretically or practically (Blakemore & Frith, 2005). Bruer (1998) asserted that "well-founded educational applications of brain science may come eventually, but right now, brain science has little to offer educational practice or policy" (p. 14). The reason was that the 'learning brain' functioned in systems of interconnected processes that activated holistically in complex environments like classrooms (Zimmerman, 1986), and the available technology was insufficient to measure such complexity. Unfortunately, the gap between neuroscience and the social complexity of Being-human made the 'gullible and naïve' in mathematics education "vulnerable to pseudoscientific fads, inappropriate generalizations, and dubious programs" (Wolfe & Brandt, 1998, p. 10).

However, brain research of the 1990s did provide some neuroscientific support for the constructivist belief that different learners formed their understandings actively, especially when interacting in familiar environments in ways that were personally reasonable (Abbott & Ryan, 1999; Ernest, 1998, 2013). Interestingly and in contrast to Bruer (1998), Brown and Wheatley (1995) contended that the "burgeoning research on brain functioning ... supports a constructivist approach to designing learning environments for school mathematics" (p. 10).

Moreover, student learning was enhanced when the mathematics teacher accommodated the idea pedagogically that the brain of each individual was unique, and therefore should be allowed to develop on its 'own time schedule' in response to a rich external environment that encouraged meaning making, curiosity, and social collaboration (Rushton & Larkin, 2001; Stanley, 1995; Tomlinson, 2001; Wolfe & Brandt, 1998). This implied that the mathematics teacher's pedagogy needed to be "based on concepts and the principles that govern them, in contrast with teaching that is rooted solely or largely in facts" (Tomlinson & Kalbfleisch, 1998, p. 54).

Ideally therefore, but in most instances impractical, each mathematics student would develop optimally, if allowed to participate in a uniquely personal pedagogical process. But nevertheless, to meet the diverse learning needs of individual mathematics students because of "varying readiness levels, varying interests, and varying learning profiles," (Tomlinson & Kalbfleisch, 1998, p. 54) it was considered imperative to differentiate the curriculum in spite of teachers being under time pressure to complete overcrowded syllabi (Tomlinson & Kalbfleisch, 1998; Tomlinson, 2001). This meant affording students several equitable but different pathways of learning, with each pathway including a similar grouping of important mathematical ideas and principles. Thus 'differentiation' was not meant to be the same as 'individualization' of the 1970s which often resulted in teacher exhaustion and student frustration (Tomlinson, 2001).

Although the time and opportunity to learn in a context of individual differences and classroom organization was not a new issue in education (Keeves, 1999; Zimmerman, 2002), it was reported that brain studies provided evidence for a social constructivist approach to the learning of mathematics, where "the constitution and powers of the human cognizing subject depend heavily on the experiences and interpersonal relationships of the person during the course of development" (Ernest, 1998, p. 212).

However, perhaps the clearest message for education in the Decade of the Brain was that emotion and feelings were an indispensable part of human cognitive functioning (Damasio, 1999; Sousa, 1998; Wolfe & Brandt, 1998). In particular the interconnected neural fibres which proceeded from the emotional to the logical (rational) regions of the brain (Sylwester, 2000), promoted the formation of a "highly integrated representation of outcomes that were flexible, sensitive to previous and current contingencies, and supported goal-directed, voluntary choice in behaviour" (Killcross, 2000, p. 506). For example, the ventromedial prefrontal cortex guided deductive reasoning and decision making as part of social cognition which included "emotional states that serve to bias cognition" (Adolphs, 1999, p. 475).

This view of learning was not consistent with the view that "complex human cognition is just a simple reflection, once removed, of its environment," (Anderson, 1996, p. 364) because in the construction or development of understanding, the brain has the capacity to retrain or change itself in idiosyncratic ways (Doidge, 2007). The brain is highly plastic not only in response to its immediate external environment, but also in response to **itself** (Abbott & Ryan, 1999; Brown & Wheatley, 1995; Kosslyn, 1994).

Therefore the information processing computer metaphor was not sufficient to understand the brain as a self-regulating organism. The brain was found to be capable of thinking in relation to a living body that feels (Damasio, 1999), whereas a high level computer could only respond to instructions, or preprogrammed scenarios and corresponding stimuli. Furthermore a computer could not actively participate in its 'own learning' as was the case with self-regulated learners, which meant that they were "metacognitively, motivationally, and behaviorally active participants in their own learning" (Zimmerman, 1990). Essentially, because a computer did not have a human-like body it could not self-generate or self-interpret thoughts, feelings, and actions in pursuit of personal goals as was the case with self-regulated learners (Dreyfus, 1992; Zimmerman, 2002).

Summary insights: Decade of the Brain. During the 1990s neuroscience provided limited support for a constructivist pedagogy that acknowledged the individuality of the student in social learning environments that fostered self-regulated learning. By implication therefore, mathematical learning is not uniform across individuals — neither cognitively, affectively, volitionally, nor in relation to prior learning. Certain constructivists acknowledged that learning was more complex than previously thought, and involved a "balance between emotion and logic, the role of intuition, and the relationship between intrinsic and extrinsic motivation" (Abbott & Ryan, 1999, p. 68). By the end of the decade however, the importance of affect — especially with respect to mathematical problem solving — was not well known, understood, or accepted by many individuals in mathematics education (e.g., Burton, 1999).

Curriculum reform in mathematics education (1990−2000). The NCTM's (1989) Standards coupled with brain research (Cangelosi, 1996; Sousa, 1998, 2001) implied that the 'whole person' should learn within a community of problem solvers, if the individual was to make sense of mathematics for him or herself (Hiebert et al., 1996; Jaworski, 1994, 1996). However, what was not said is that the individual needs to learn in terms of the community if his or her Being is to personalise a community of self. If this does not occur in mass mathematics education then many students will probably continue to experience the subject as a "boring string of terms, symbols, facts, and algorithms, truly understood by rare geniuses" (Cangelosi, 1996, p. vii).

However, although Being-mathematical is always a socio-cultural event, different teaching cultures do emphasize different aspects of Being-mathematical at different times. For example in a comparative study that involved mathematics classrooms in Beijing, Hong Kong and London, it was noted that "Beijing teachers emphasized the content or concepts of mathematics, Hong Kong teachers emphasized mathematical skills, while London teachers emphasized experiencing and enjoying mathematics" (Leung, 1995, p. 313). But reforming a culture of mathematics education is possible if the cultural dynamics are respected albeit that the outcome is likely to be dialectical (Pinkard, 1996; Rosen, 1982). As discussed below, the growth and development of the Singapore education system is an example that other countries can learn from.

Education in the Asia−Pacific region was influenced by the 1997 Asian financial crisis (Mok, Lawler, & Hinsz, 2009). In the wake of increased globalisation, the broad view was adopted that learners and "employees need to be problem-solvers, multi-skilled to enable them to work across portfolios, team players, and capable of learning new skills and strategies as required" (Ng, 2009, p. 3). It was the nation of Singapore (Lion City) however, under the leadership of Prime Minister Goh Chok Tong, that embarked on system-wide reforms in June 1997, namely, *Thinking Schools, Learning Nation* (Bell & de-Shalit, 2011; Darling−Hammond, 2010; Ortmann, 2010). At the opening of the *7th International Conference on Thinking*, Tong argued that the future wellbeing of nations was dependent upon a nation's ability to learn cooperatively and adapt quickly to the pace of global change (MOE, 1997).

The Tong-led reforms can be traced back to the 1970s when visionary leaders in Singapore made the decision to transform the country from a myriad of fishing villages to an Asian economic powerhouse. Central to economic development was the goal of a world class education system that accentuated mathematics, science and technology. Within two decades, the *Singapore Mathematics Curriculum Framework Pentagon Model* for the holistic teaching and learning of mathematics, as illustrated in **Figure 3·3**, emerged under the aupices of the Ministry of Education. The Pentagon Model was unveiled in 1990 against a backdrop of changing mathematics curricula in many different countries (Dindyal, 2006). The purpose of the Pentagon Model, especially post-1997 was to focus the teaching and learning of mathematics in schools towards the future of the nation, which meant producing creative problem solvers.

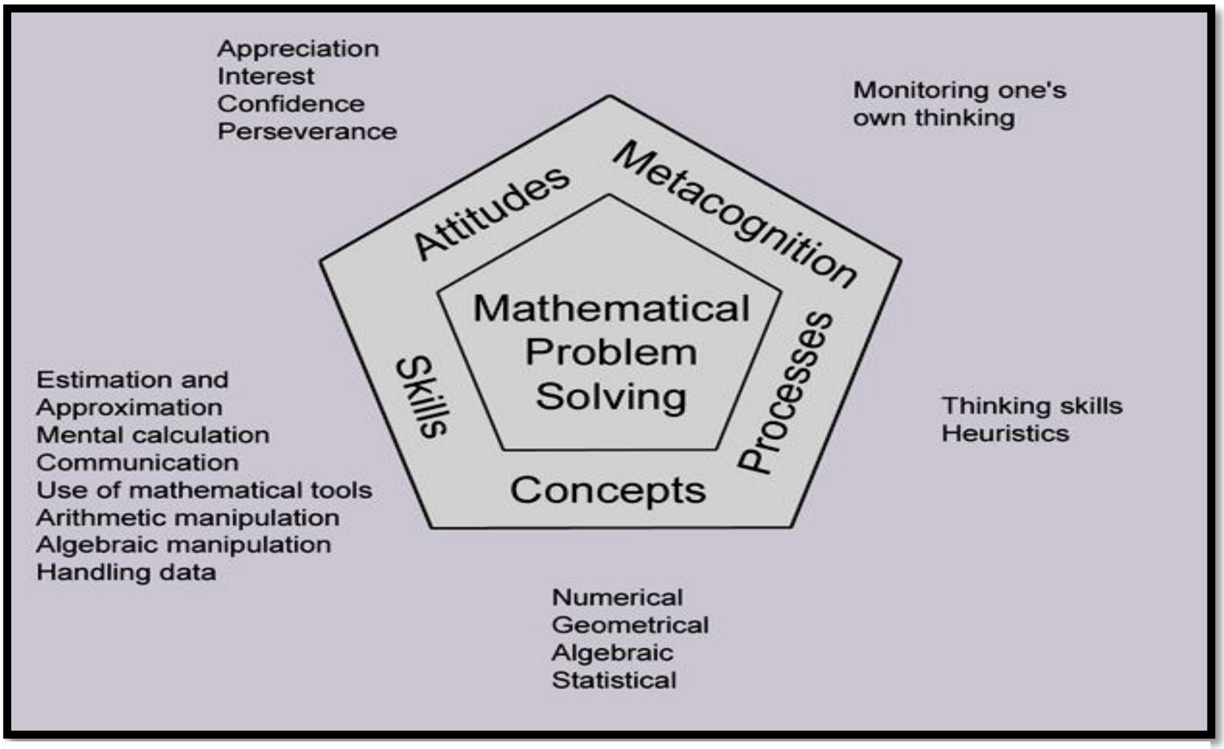


Figure 3·3. Singapore Mathematics Curriculum Framework Pentagon Model (Dindyal, 2006, p. 181)

In particular therefore, Singapore students were given the opportunity to grapple with "messy" and complex problems that reflected the nature of "actual real-world mathematics situations" (Ginsburg, Cooke, Leinwand, Noell, & Pollock, 2005, p. 32). In contrast however, many American mathematics students faced textbook problems that were relatively trivial and were designed to 'work out'. Singapore's complex problem solving approach to the teaching and learning mathematics was 'justified', when Singapore's eighth-grade students achieved the highest average mathematics scale scores on both the 1995 and 1999 TIMMS, or Trends in International Mathematics and Science Studies (Dindyal, 2006; National Center for Education Statistics, 2009).

Thus many Singapore students came to value mathematics through a problem solving approach that reflected mathematics as an authentic expression of their socio-cultural situation. *Mathematics Education: The Singapore Journey* is ongoing as the nation actualizes the five goals of the NCTM's (1989) Standards in relation to its unique Anglo−Asian heritage and globalisation (Yoong, Yee, Kaur, Yee, & Fong, 2009).

The Math Wars. Although the NCTM (1989) Standards precipitated or influenced substantial change in the world of mathematics education (Bossé, 2006), it also initiated the *Math Wars* of the 1990s particularly in the United States (Bossé, 2006; Schoenfeld, 2004, 2007). In effect, the (basic) skills–process debate in the teaching and learning of mathematics loomed large. Schoenfeld (2004) reported that,

> the seeds for battle were sown—not that anyone at the time could predict that the *Standards* would have much impact or that the battle would rage. The *Standards* were vague. This was part of their genius and part of what caused so much trouble. Because of their vagueness, they served as a Rorschach test of sorts — people tended to read much more into them than was there. …The genius is that the *Standards* set in motion a highly creative design process during the following decade, far transcending what the authors of the *Standards* could have produced in 1989. … Some of the materials produced would be considered pretty flaky. Some of the classroom practices employed in the name of the *Standards* would appear pretty dubious. And the *Standards* would be blamed for all of them. (p. 268)

Therefore vigorous "political and philosophical debate" (Hekimoglu & Sloan, 2005, p. 37) ensued within mathematics education circles. As a result the 1990s saw numerous attempts to refine, enhance and amplify the NCTM (1989) Standards statement. These included the *Professional Standards for Teaching Mathematics* (1991) document, the *Assessment Standards for School Mathematics* (1995) document, and the *Addenda Series* (1991−1995). However, the Math Wars did not abate and the NCTM "undertook to revise the standards into a more cohesive document supported by additional research and classroom experience" (Bossé, 2006, p. 4). The outcome was the publication of the *Principles and Standards for School Mathematics*, also labelled *Standards 2000*. This document outlined the "five strands of content that students should learn" (NCTM, 2000, p. 3) across the different grade bands. Notably, the **Content Strands** were not uniformly emphasized between or within the different bands as depicted in **Figure 3·4**. Especially however, Standards 2000 advocated that

Algebra should have an increasingly strong emphasis in the curriculum as students approached Grade 12, but always in meaningful relation to the five **Process Standards**, namely, problem solving, reasoning and proof, communication, connections, and representations. This implied that without Algebra the pattern forms of Number, Geometry, Measurement, and Data Analysis and Probability could not be expressed in general terms; interrelated, or used to solve real world problems and deduce logically mathematical theorems.

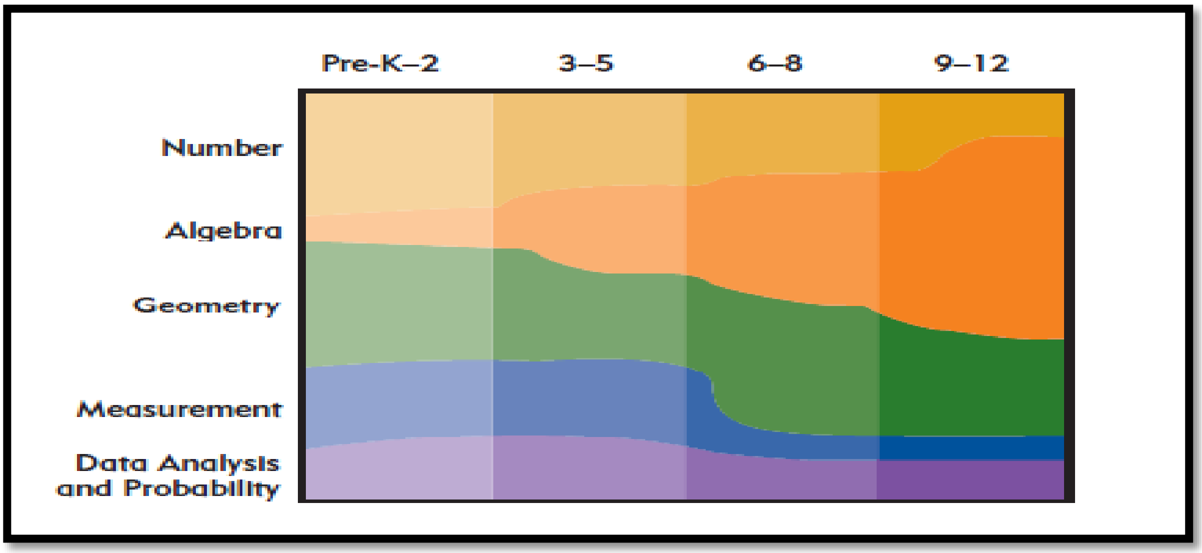


Figure 3·4. The Content Standards recommended different emphases across the grade bands so that the Process Standards could be actualized effectively (NCTM, 2000, p. 4).

In Australia the skills−process debate in mathematics education was less heated than in the United States, largely because Australia's experience with New Math in the 1950s and 1960s was more positive. However, Australia did experience the *Reading Wars* (particularly in the 1980s and the 1990s), namely, whether a child should be taught to read using phonics, whole language, or both (Duncan, 2006; Milburn, 2008; Nicholson, 1992; Welsh, 2007). Interestingly, Thorndike (1917, 1973) understood reading, at least at the secondary school level as a complex form of reasoning. He therefore might have advocated a dual approach that included 'part-specific phonics' as well as whole language. However, the reading debate that occurred in Australia is but another example of how difficult it appears to be for humans to reconcile an analytic and holistic, or systemic approach to teaching and learning.

The use of technology. Standards 2000 maintained that "technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (NCTM, 2000, p. 3). However, a number of caveats concerning the use of technology in education were forthcoming, because the use of technology did not necessarily enhance student learning. It was nonetheless argued that technology could "become a valuable education tool, but only if we use it to capitalize on our new understanding of how the human mind works" (Veenema & Gardner, 1996, p. 69).

Furthermore, Gardner (2000) concluded that a "marriage of education and technology could be consummated. But it will only be a happy marriage if those charged with education remain clear on what they want to achieve for our children and vigilant that the technology serves these ends" (p. 35). Hence, technology should not be introduced in mathematics education purely "for its own sake," (Lederman & Niess, 2000, p. 345) or because it was a bandwagon phenomenon in an increasingly technocratic society (Zelchenko, 1999). Ideally, learning mathematics through the use of appropriate technology should mean the growth of student understanding, creativity, and the development of self-regulated processes (Kaput, 1992; Tall, 2000). However, in the experience of mathematician Koblitz (1948−) of the University of Washington, a thoughtful approach to the integration of technology in mathematics curricula was not in evidence in most American classrooms, because technology was

> used in the classroom in a way that fosters a golly-gee-whiz attitude that sees science as a magical black box, rather than as an area of critical thinking. Instead of asking whether or not technology can support the curriculum, educators try to find ways to squeeze the curriculum into a mold so that computers and calculators can be used. (Koblitz as cited in Stoll, 1999, p. 6)

Summary: The Standards. In spite of Standards 2000, the skills–process debate was not resolved. There were two primary reasons, namely, (a) research findings were limited on how to meaningfully integrate skills, processes and concepts in different mathematics classes

(Schoenfeld, 2004, 2007), and (b) the lack of clarity as to what was meant by 'basic skills' (Hekimoglu & Sloane, 2005).

The traditionalists advocated vehemently that it was the teacher's responsibility to "say what is right and to make sure the students learn it. What the students feel is irrelevant and inappropriate for discussion in school" (Schoenfeld, 2004, p. 271). However, those who supported the different Standards' documents tended to argue for a balance between individual and group work; skills and process, and the teacher as a facilitator of conceptual understanding and an expositor of mathematical knowledge (Schoenfeld, 2004; Short, 2007). Therefore the Standards' documents emphasized conceptual understanding through both social and individual activity, but not at the expense of skill mastery (Goldsmith & Mark, 1999).

Towards Future Reform

Countries like Singapore, South Korea, and Japan have performed well on numerous comparative international tests in mathematics including creative problem solving (Ginsburg, Cooke, Leinwand, Noell, & Pollock, 2005; Loveless, 2013; OECD, 2014). However, even these countries have had limited success in enabling many of their students to learn mathematics significantly beyond the basics. In an international study that involved more than forty countries, PISA (Programme for International Student Assessment) described the most capable mathematics students as individuals who could communicate precisely in the sense of being able to

> conceptualise, generalize, and utilize information based on their investigations and modelling of complex problem situations. They can link different information sources and representations and flexibly translate among them. Students at this level are capable of advanced mathematical thinking and reasoning. These students can apply this insight and understandings along with a mastery of symbolic and formal mathematical operations and relationships to develop new approaches and strategies for attacking novel situations. (OECD, 2014)

Therefore the most capable PISA students in the large scale study 'embodied' the five general goals of the Standards (NCTM, 1989). Internationally however, the learning past of many students inhibits a more creative future in mathematics. Nevertheless, if individuals in mass mathematics education are to become powerful mathematical learners, then it is essential that students are educated in a progressive form of mathematical learning, namely, through a reflective and recursive dialogue that involves both skills and processes (Kilpatrick, 1985). A goal of which is to develop a conceptual understanding of mathematics that **embodies** "an ensemble of solving techniques" (Fischbein, Jehiam, & Cohen, 1995, p. 29) for the express purpose of Being-creative and Being-ethical.

However, in order to become a powerful mathematical learner the individual student in mass mathematics education requires a teacher to role model creative learning (Even & Tirosh, 1995; Vygotsky, 1978, 1997). But this is not possible if the teacher him or herself is not intentionally dialogical and patient in Being-mathematical, because it is essentially an 'ecology' of Being-dialogical, or relations and interactions between complex organisms and their total environment, including virtual realities that makes 'mathematical sense-making' plausible.

This view of mathematics education focuses on the mastery of content through deliberate practice, as well as through creative and ethical problem solving in the global whole that is the Three Worlds. If mathematics educators however, are to dialogue learning ecologies in classrooms, schools, and universities that involve both cognitive and non-cognitive ways of knowing, then

> the big challenge we face in education is the encounter of the old and the new. The old is present in the societal values, which were established in the past and are essential for life in a community. Since the modern state, this is intrinsic to the concept of citizenship, and the new is intrinsic to the promotion of creativity, which points to the future. (D'Ambrosio, 2007a, p. 174; also see Swetz, 1995)

End Note

1. The limit has been rigorously defined:

For all real numbers Ɛ > 0, there exists a real number ƌ > 0 such that if

$|x-a| < \partial \;\Rightarrow\; |f(x)-l| < \varepsilon, \text{ then}$

$\lim f(x) = l \;\; as \;\; x \to a.$

Chapter Four
Being-dialogical

All happy families are alike; each unhappy family is unhappy in its own way.
Leo Tolstoy, *Anna Karenina* (1873−1877)

Multitudes of students have experienced the learning of mathematics as needless, immoderate, or even inordinate. If this situation is to change in mass education then the relation between the process that is mathematics, the object that is Mathematics, and the teacher and the student need to be revisited in fundamentally human terms. Currently psychology, sociology and neuroscience are 'becoming' in the notion of humanness: An Aristotlean or Vygotskyean movement from a level of constrained or restrained potentiality to a complex level of Being-dialogical (Fernyhough, 2008, 2009; Hermans & Kempen, 1993).

In a neo-Vygotskyan sense, the *I* as subject and the *Me* as object are mediated by an *Other*. The being that is *Me* as a physical, or architectonic and socio-cultural substantiality relates the *I* to *Other* on the basis of a relationship that has its essence within the self of the *I*. The complexity of relationships that constitute the embodied and extended self, or Self[1] of the *I* are referred to as **in-relationships**. And it is this network, or framework of relationships that implies the betweenness, or potentiality of Being-mathematical. However, because the *I* and the *Me* have a living body that is simultaneously a part of the physical universe and a situated knowledge-based culture with a present history, the developing mind in society has the potential through dialogue to enable, or create in-relationships which emerge essentially as a complex society of mind.

Importantly for powerful mathematical learning however, the *Me* correlates or traverses numerous teaching and learning trajectories, or loci about two centres of the Self, namely, *I* and *Other.* Although the *Me* centres about *I−Other* dialogically, the relationship between *I* and *Other* is only subject to change as a result of a happenstance, an intervention, an insight, or growth and development that involves the *Me*. Therefore the learning of the mathematical

Me is not constant or necessarily continuous, but as intimated by Bakhtin, Liapunov, and Holquist (1993), is in flux with the 'concrete moments' of being that are distributed between, or interconnect the two centres or foci of Being-mathematical. Being consequently subsumes being, that is its physical substantiality through a *Me* that has an embodied mind. As a result the horizons of humanness can only emerge beyond the literal specifics of a real world and a particular culture, because of the symbolic and abstract dynamics of *I−Other* that facilitate change in-relation to the static components of the *Me*.

The *I* pertains to the individual. The *Other* can be more complex than the *I* and is interpersonal, intrapersonal, or extrapersonal in relation to the embodied *Me*. However, the *I* of Being is in-relation to that which the entity or individual (*Da-Sein* or Being-there) says, experiences through his or her senses, enacts, and by choice comports toward and about (Heidegger, 1927). Although the *I* is essentially conscious awareness and an ability to cope as experienced or believed by the individual, the *Other* in-relation to the *I* is a seen or unseen reality, and it is only in-relation to the complex *O*ther that the *I* has its essence concomitant with the *Me*.

For example, the *I* may refer to a student in his or her conscious awareness, but the *Other* may refer to the student's teacher, or family mentor in the entirety of that individual's *Da-Sein* which includes his or her potential by Being-in-the-world. Moreover, the *Other* may refer to a personified concrete or abstract object in-relationship with the dialogic *I*, as well as to an existential entity that is assumed to have free will and ability but no visible form in the real world (e.g., God or spirits). In broad terms therefore, *Other* may refer to a single *Other*, or to a complexity of *Others*, or all *Others* in-relation with the *I* including inanimate things.

Although the *I*, the *Me* and all *Others* in-relation with the *I* completely specify the Self of the individual, it is the nature of the different relationships that are *I−Other* that constitute the quintessential whole of powerful mathematical learning. Thus if the teaching and learning experience of the *I* in-relation to the *Other*, limits or disempowers the self who is the *Me*, then it is highly unlikely that the individual will actualize Being-mathematical substantially beyond the basics. This is precisely what happened in mathematics classrooms around the world in the twentieth century, largely because of the dominant learning psychologies that influenced the teaching and learning of mathematics.

Behaviourist teaching focused primarily on the words and 'bodily practices' of the *I*-student in-relation to a centred and at times authoritarian *Other* that was the teacher (Veenema & Gardner, 1996). The consequence for the *Me* of the self of the individual student was that his or her mind was limited mathematically to the being, or the concrete specifics of the *Other*. The result was that the *I−Other* of the student was patterned or conditioned according to schedules of reinforcement that did not allow the learner to develop in constructive and creative ways, or modes of Being that engaged purposefully with his or her emotions, interests, memories, and motives.

With respect to **cognitivism** in the classroom, teaching focused predominantly on the mind instead of on an interactive mind−body. The result in New Math for example, was that the mathematics was often too rigorous and abstract for the *I−Other* relationship that was the student and the teacher. In terms of **constructivism** however**,** the *I−Other* relationship in the classroom was described as student-centred, and in this relationship the role of the teacher was primarily to facilitate and guide. Consequently, neither the *I* nor the *Other* had sufficient means or liberty to sustain the *I−Other* relationship mathematically, cognitively or emotionally. In other words the *I−Other* interaction, particularly between the student and the teacher, undermined the confidence of the individual student as well as the teacher — both of whom felt inadequate in, or were frustrated by the teaching and learning relationship, especially when it came to assessment and high stakes testing.

Thus epistemologically the dominant psychologies all misconstrued the ontology of Being in-relation to the *I−Other* of the human Self and the object of being which is essentially the *Me*.

Being-Autotelic in the Dialogical Self

The world is not only globalizing, but is globalizing in its interconnectedness. Consequently, the possibilities for Self development and powerful mathematical learning are greater than at anytime in history.

However, if such learning is to be realized in classrooms and schools internationally then the focus of self development should be neither the *I* nor the *Other*, but the relationship that is *I−Other*. When the self of the individual is predominantly monological through *I* or *Other*, then the relationship that is *I* and *Other is* constrained by either *I* or *Other*. But when in dialogue, the growth and development of the self is optimized through a complexity of *I* and *Other*, which in an optimal sense is a 'multiplicative' rather than an 'additive' relationship. In other words when *Da-Sein* is interrogated as to the meaning of Being, the most advantageous answer emerges through the mode of Being that is dialogue. Because phenomenologically, dialogue is a mode of Being that allows the self the freedom and possibility to actualize powerful mathematical learning in a normative sense, namely, by way of a

> deep, challenging, responsive, enriching, disruptive encounter and conversation-in-context; and also a mutual and critical process of building shared understanding, meaning and creative action. Furthermore dialogue is understood historically as the interplay of social forces that shape the life we now live individually and collectively. Those social forces at times attest to the power of domination that arrests dialogue, but also at times attest to groups of people in dialogue encountering the other as profoundly different — which opens new possibilities for social transformation. (Westoby & Dowling, 2013, p. 5)

Powerful mathematical learning is a dialogue. Although Being-mathematical involves the entity *Da-Sein*, it is primarily an event in terms of the embodied and extended Self. However, Being-mathematical can be understood by Being-there through different modalities of Being. In particular, Being-human is optimized through dialogue when the dialogue promotes an 'end-in-the-self', namely, a primitive essence of Being that addresses the "hole of being at the heart of Being" (Sartre, 1957, p. 617). Essentially, dialogue offers possibilities for humanness that can bridge the gap between being and Being, or empower Being **in-relation** to its being; never absolutely, but only intuitively until the intuitive is interrogated for the logicality of its Being; the likely outcome being a deeper intuitive−analytical understanding. Therefore, Being has no essence without its being, and the potential of Being cannot be actualized beyond the realization or understanding of its being in World 1(The Natural−Physical World) and World 3 (Knowledge). However, if the substantiality of being is constrained, undermined, or curtailed in World 1 and World 3, then to that degree the individual is 'blind' to the possibilities, or potentiatlity of his or her Being in-relation to the systemic of World 2 which is the complex interrelational Mind.

Nevertheless, the driver behind a sustained two-way interaction between Being and being is the notion of Nothingness (Sartre, 1957). Although it is supported by Being, or owes its very existence to Being (and vice versa), it is the antithetic dialectical relationship between Being and Nothingness that gives Being its primary stimulus to Be. In other words humanity is promulgated towards higher and higher levels of Being, or understanding for the very purpose of Being without 'Nothingness', which paradoxically is not desirable or possible for a human being if he or she is to continue to grow and develop in terms of the Self. For example, Plato's dialogues did not take the form of definitive arguments, but intuitive insights that were imaginative and aesthetically pleasing in relation to rigorous logic and empirical observation (Armstrong, 2006). Stated differently, the essence of Plato's 'holistic' dialogues was motivated by an intrinsic need to 'free' his Being from the inert, or contradictory aspects of his being through intuitive−analytical essences that emerged in dialogue, especially with others.

Etymologically, dialogue is more than discussion, conversation, and debate. Dialogue is rooted in the Greek *diálogos.* The prefix *diá* together with *logos* is literally 'words passing between, through or completely'. Metaphorically, dialogue implies discourses of Being that eventuate 'a passing' between the *noumenon* (e.g., a mathematics problem) and the being, or the body of an individual, in such a way that the dialoguer grasps the *noumenon* eidetically, or as a *phainomenon* which is then developed coherently and analytically in-relation to the prior learning of the individual and the *Other*. This dialogic outcome, at least temporarily, closes the gap between the Beingness of the *I* and the *Other*, which is essentially what is meant by '*I* understand'.

***The essentials of dialogue*.** Dialogue for powerful mathematical learning is a relational and phenomenological event in terms of *I−Other*. Therefore the dialogic formations of Being-mathematical are contingent upon the *I* relating to the notion of **otherness** by embracing at least eight essential ideas or tenets (Bertau, Gonçalves, & Raggatt, 2012; Hermans & Hermans−Konopka, 2010).

The intent through affect is to **innovate (1)**, namely, the *I* draws close to the *Other* (or vice versa) for the purpose of engaging in a creative process that hopefully will result in a new product or object of mind. (The *Other* is usually another person, but an abstract *Other* is also a possibility if the *Other* is represented by a particular personified *I*-position, or what has been referred to as an *other-in-the-self* (Hermans & Hermans−Konopka, 2010)). In so Being the *Other* reciprocates by assisting the *I* to understand its own perspective in the context of the learning situation that includes the enablement made possible by the *Other*. The dialogical *I* then adapts, revises and develops its initial standpoint, or position of learning in-relation to the verbal or non-verbal 'communication' of the *Other*.

If however, dialogue is to be successful in these terms then the *I−Other* engagement must have a sufficiently **broad bandwidth (2)**. A narrow or dogmatic mindset on the part of the *I* or the *Other* does not facilitate good, or pragmatic dialogue which requires a disposition of mind, and an intentionality of consciousness that is open to a range of positions, especially different cultural positions (Hermans, 2001). A narrow bandwidth on the part of numerous teachers in Pakistan for example, was the main reason that the CAME (Cognitive Acceleration in Mathematics Education) intervention did not realize the same positive outcomes for students as was the case in the United Kingdom (Iqbal & Shayer, 2000; Shayer & Adhami, 2007). The CAME project is discussed at length in the next chapter, but if *I* and *Other* adopt **opposite positions**, or polar opposites of perspective without the intentionality of dialogical activity occurring between them, then powerful mathematical learning is not a possibility for the *I* in terms of the *Other* (Bertau, Gonçalves, & Raggatt, 2012; Hermans & Hermans−Konopka, 2010).

However, if the *I* wishes to engage with the *Other,* the *I* needs to be willing to change in terms of the *Other,* or through the essence and ideas of the *Other*. This implies the meaningful integration of ideas between the first and third person, which is likely to create an epistemological problem that the *I* needs to resolve through an extended self, or Self that is ontologically dialogical. The viewpoint of the *I* and the viewpoint, or essence of the *Other* do not match holistically, because the *I* as an agentic persona has a 'hole in its being' relative to the *Other*, which implies a social or relativistic application of Heisenberg's uncertainty principle (Bakhtin, Liapunov, & Holquist, 1993; Barresi, 2002; Lindley, 2008). In other words the *Other* apart from the *I* is not the same as when the *Other* is in-relation with the *I*. Therefore the *I* can never know the *Other* in terms of the *Other's* Being-there separate from the *I*.

Therefore **misunderstanding (3)** is intrinsic to the dialogue that is powerful mathematical learning. It is the role of System I thinking, or Being-human to develop a coherent narrative in a selective synthesis and intuition of ideas. In the analysis of such ideas misunderstanding is minimized. Consequently, powerful mathematical learners increase their understanding in dialogue with the *Other*, but also recognize that understanding, misunderstanding, and lack of understanding always co-exist epistemologically in terms of the network of relationships that constitute the Self of the individual.

Thus the Self of the individual can never know itself absolutely, and hence the deep human need for the *I* to be a lifelong learner in-relation with the growing complexity that is the *Other*. The *I* cannot learn without the *Other*. Unfortunately however, a modernist view of the self has been greatly influenced by Enlightenment philosophy. The ideas of Spinoza (1632−1677), Locke (1632−1704), Voltaire (1694−1778), and Newton (1643−1727) promoted a 'Descartes-type dualism' between self and *Other*. The separation of mind and body; self and *Other* has influenced the ontology of many teachers' pedagogy and instruction, with the result that the high level learning of mathematics in classrooms has been severely constrained. The main point is this — the degree to which teaching and learning is at odds with the human condition is the degree to which educational outcomes are limited or undermined.

A fundamental premise of powerful mathematical learning is that Being-mathematical is optimized through dialogue, because the Self is a **dialogical space (4)** that incorporates *I* and *Other* in terms of multiple mathematical narratives that are more or less logical. Consequently, the Self includes an embodiment of *I−Other* in an emergent mental space that corresponds to the physical space between the individual and the *Other*. In particular the affective dimension is considered a crucial part of the interphysical dynamic, because "the participants typically feel a strong sense of sharing and have the impression that the space is *between* them and connects them" (Hermans & Hermans−Konopka, 2010, p. 181). In the sense of Merleau−Ponty's (1962) intercorporeality of being, the bodily engagement or enactment of interpersonal space between *I* and *Other* is the source domain for the development of a dialogical space-in-mind (Bertau, Gonçalves, & Raggatt, 2012).

Therefore the dialogical self overcomes the egocentrism of the *I* as portrayed in a modernist philosophy of the self; the very perspective that gave rise to teacher-centred and student-centred conceptualizations of learning. The dialogical self however, respects the **alterity (5)** of the *Other* for the *I*-purpose of fostering meaningful dialogical relationships that acknowledge, accept, and even stimulate the differences between *I* and *Other*. The goal of which is to develop the *Me* of the embodied self into an increasingly complex and ethical 'reification' of *I-Other* relationships, which in terms of Being-dialogical and powerful mathematical learning is the *raison d'être* of the Self.

In addition, the notion of alterity in dialogical relationships is key for the Conceptual Age, because it introduces **risk and uncertainty (6)** into the relationship between *I* and *Other*, which is an essential element of Being-creative. However, this might not be comfortable for either *I* or *Other,* because humanness is predicated on traditions, customs and routines that have evolved or developed to bring stability through 'sameness', thereby actualizing the self by the meeting of basic needs (Vinner, 2007). A hierarchy of human needs includes physiological and safety needs; the need for love and to belong, and to be valued and respected. All of these characteristics of self influence a person's identity, self-respect, willingness to discover, and degree of creativity (Holzknecht, 2007; Maslow, 1970).

Therefore if the self is to be creative or innovative in its potentials then **power differences (7)** between *I* and *Other* need to be comparable or resolvable, that is, if a common dialogical space is to eventuate and a balanced relationship between *I* and *Other* is to be fostered. But if the relationship is not power-balanced through dialogue, then the outcome for the self of the *I* will likely be to perpetuate a mind in society that continues to conform to existing institutional structures and not also to create them (Mead & Morris, 1934/1962; Ricoeur, 2002).

However, since *I−Other* reflects, or informs a power differential relationship in the Self, the *I−Other* relationship is in effect a specific, or a combination of "speech genres" (Hermans &

Hermans−Konopka, 2010, p. 187). Thus if the dialogue is to realize powerful mathematical learning for a globalizing world then **the style, thematic content, and compositional structure of the speech acts (8)** should encourage or enable an increasingly stable but dynamic *I−Other* relationship.

As a result of the eight essentials of dialogue therefore, this modality of Being is significantly more than 'just talk'. It involves modes of Being which promote self-awareness in and through complex forms of communication that include interpersonal and intrapersonal interactions 'in moments of silence'. These interactions always involve non-verbal or non-cognitive forms of communication that complement, or inhibit the speech acts of *I−Other* (Hermans & Gieser, 2012; Hermans & Hermans−Konopka, 2010; McNeill, 2012).

Summary insights: The essentials of dialogue. Powerful mathematical learning has creative, or innovative potential only if the dialogue between *I* and *Other* has sufficient bandwidth to sustain a field of awareness, or a dialogical space where the otherness and sameness of both the *I* and the *Other* can interact meaningfully. However, even though the notions of sameness and otherness cannot be fully comprehended by *I*-consciousness, the interpersonal *Other* in particular, has a responsibility to facilitate a power differential that is in balance with a speech genre that reflects the language and possibilities of powerful mathematical learning. A globalizing world society is interconnecting in creative and innovative ways and is in need of a mathematical Self that has a broad horizon of Being-there (*Da-Sein*). This requires a philosophy of Being-mathematical that goes beyond the relative narrowness of an embodied self whose Being exists predominantly as a singularity in *I*-consciousness (Bertau, 2004; Nancy, 2000).

'Beginnings' of Being-mathematical. Powerful mathematical learning occurs through an essentiality of Being that is fundamentally the dialogic relationship between *I* and *Other*. The *I* unfolds epistemologically in terms of the *Other* and the *Other* enfolds ontologically in terms of the *I*. Therefore the *Other* is crucial in the expression of *I*-mathematical. However, the nature of the relationship between *I* and *Other* must be specified clearly if powerful mathematical learning is to eventuate in *I−Other*.

A "first philosophy" for powerful mathematical learning is ethics (Ernest, 2009). At the level of the individual, the *I* of the student needs to become Self-aware of the love that the *Other*, especially the mathematics teacher has for the Being who is the student. Simply put, the 'best' teachers love children (Robinson, 2011). However, the English word **love** does not capture the complexity of 'Being-loved' as is the case in the language of Ancient Greek. In this regard there are four Greek words that articulate the different dimensions of human love, namely, *agápe, éros, philía,* and *storgē* (Lewis, 1960; Strong, 1995). *Agápe* is the highest form of love: The *Other* is unconditionally committed (even at the expense of him or herself) to the *I* of the individual student becoming a powerful learner. Philosophically then, *agápe* is consistent with *agathon* of the *Nicomachean Ethics* in which Aristotle described the good as that which was sought or needed by the *Other* (Aristotle, 2006). In the ethics of Plato, the aim of the *Other* was for the student to increase in e*udaimonia*, or an inward disposition of 'Being-well' (contentment) through the influences of the *Other* towards the highest levels of moral thought and conduct (Frede, 2009).

Éros refers to intimate, passionate, or physical love. At a superficial level at least, *éros* has no place in the mathematics education relationship of *I−Other*. However, in a deeper embodied sense; in a Platonic sense the *Other* and the *I* could come to appreciate the inner beauty of each as a consequence of an intercorporeality of being that is reflected epistemologically in their discourse, and ontologically in their recourse (Kolb, Baker, & Jensen, 2002; Merleau−Ponty, 1962; Plato, 1991).

Philia describes a dispassionate, virtuous, or Aristotlean and Socratic-type intellectual friendship that is developed by Beings-in-dialogue, whereas *storgē* refers primarily to the sense of belonging that is afforded by a loving family, or a supportive community. Therefore in terms of mass mathematics education, the *Other* needs to love, or facilitate *agápe*, *éros*, *philía*, and *storgē* with respect to the student, if that individual is to be encouraged or drawn into a powerful learning relationship with the *Other*. Love is the essential beginning of powerful mathematical learning and is the antithesis of fear or angst. Although angst is symptomatic of the human condition and associated with self-awareness (Heidegger, 1927), the love of the *Other* has the potential to 'free' both the teacher and the student from negative affect in the teaching and learning of mathematics. It is through love that even the experience of Nothingness is viewed as a potentiality towards becoming increasingly mathematical in an *eudaimonian* sense (Sartre, 1947, 1957).

Thus the "first philosophy" of the *Other*, if powerful mathematical learning is to occur in terms of *I−Other*, is for the *Other* to love the *Da-Sein* of the student because his or her essential tendency is to comport towards 'closeness', if the 'prose of the text and the context' is considered beneficial by the *I*-conscious intercorporeality of being of the student (Heidegger, 1927; Merleau−Ponty, 1962, 1964; Merleau−Ponty & Lefort, 1974). A basic human response is such that "if we feel chosen by somebody, we will choose that person in return whether our feeling is correct or not. There is simply a human bias: feeling liked by someone begets liking him back" (Bruner, 1986, p. 58).

Choice. **'**To love or not to love'; 'in Being to be or not to be' a powerful mathematical learner is fundamentally a matter of choice, because consciousness is intentional through the *I* of experience; logic and emotion, as well as those elements of a world-view that might be fleeting, or indefinitely in mind (Ray, 1994; Shakespeare, Mowat, & Werstine, 2012; Sobchack, 1992; Tieszen, 2005). Therefore the *I* of the teacher has a choice to make on how to be in-relation to the student. From the perspective of the 'process philosopher and anti-materialist', Whitehead as well as certain existentialists like Kierkegaard (1813−1855), Nietzsche (1844−1900), and Camus (1913−1960), the mathematics teacher should not submit his or her professional life to that of mediocrity by conforming to the norm (Stokes, 2006). Instead, if the mathematics teacher is to make a substantial difference to the being of the student then the teacher needs to foster an excellence of Being-mathematical that is uniquely essential to the teacher. The mathematics teacher ought to draw on all Three Worlds to infuse his or her Being-mathematical with the necessary knowledge and passion that can inspire students to become powerful mathematical learners.

Consequently for powerful mathematical learning, the *Da-Sein* of an individual student as a Being-in-the-world is an issue for the self of the *Other*, preferably a 'fulfilled' teacher (Dreyfus, 1991; Heidegger, 1927). Necessarily therefore, engaging with different students requires meaning making and interpretation on the part of the *Other*, as he or she chooses to engage passionately with the uncertainties and the difficulties of as Beings-mathematical. In so Being the teacher needs to provide each student with a strong exemplar of Being-mathematical; an *übermensch* who is neither slave nor master in-relation to Mathematics (Bruner, 1979; Solomon, 2005; Stokes, 2006).

Without exception however, the Being of the *Other* is limited or empowered historically and politically by the socio-cultural value system of the teaching and learning situation. For example, Dan was a duke of China's Zhou dynasty (c. 11th century to 9th century BC). He was advised by an *Other* to "take his position in the primacy of virtue. The little people will then pattern themselves on him throughout the world. The king will then become illustrious" (as cited in Armstrong, 2006, p. 35). Similarly, each mathematics teacher needs to value and role model powerful mathematical learning in ways of Being, that his or her students can relate to, and are willing to choose for themselves relative to their socio-cultural values and prior learning.

The autotelic *Other* and the flow state. From both Western and Eastern perspectives there is substantial agreement that the teacher is a vital and indispensable social determinant in the process that is the student's education (Darling−Hammond, 1997, 2002, 2010; Tan, McInerney, Liem, & Tan, 2008). In particular it is the authenticity, or genuineness of the relationship between the *Other* as teacher and the *I* of the student that is causal in the teaching and learning of mathematics. As the adult and qualified educator in the teaching and learning situation, it is the responsibility of the *Other* to role model two basic kinds of social relationship, namely, heteronomous and autonomous (Kahn et al., 2007).

The heteronomous relationship between *I* and *Other* relates to a unilateral authority where choices are made to respect the rules and laws of the social order, namely, that neither the student nor the mathematics teacher is a law unto themselves (Piaget, 1932/1969). However, if the relationship between *I* and *Other* is to be solely heteronomous as was predominantly the case with the traditional self under behaviourism, powerful mathematical learning would be an impossibility, because it would mean the alienation of Being-human in the fullness of its potentiality. Essentially, if the student's learning is determined, or projected solely by the stimuli and reflexive responses of the teacher then "learning does not, cannot, go beyond these explicit events and the temporal parameters that relate them" (Plotkin, 1987, p. 144).

Therefore the teacher of powerful mathematical learning needs to role model an autonomy of Being that the *I* of the student can imitate towards higher and higher levels of subjective consciousness and self-awareness, resulting in the *I* deciding "for itself what it will be and what it will do" (Rae, 2011, p. 25). It is noteworthy that in the Axial Age of India (c. 1600 to 900 BC), the Brahman ritualists did not use imitation to conform slavishly to the rudiments of their heritage, but instead reflected on the deeper meaning and dynamics of their external rites and the gods. In so Being they created a perspective of self that was independent and autonomous (Armstrong, 2006), thereby developing 'being into Being' as a result of an inward focus that embodied outward bodily activity. In other words the Brahman caste enacted the interpersonal psychological plane, or mind in society into the intrapersonal psychological plane of the self.

Therefore the powerful mathematics teacher loves the Being of the student by enacting, or facilitating a heteronomous and autonomous *I−Other* relationship. The relationship develops as the teacher creates a proximal learning environment in which the rules and principles of the socio-cultural situation can be internalised by the *I*. Thus the *I* is not wholly restricted to the 'situatedness' of the *Other*, provided that the student is allowed to be 'imaginal' within zones of freedom of movement and zones of promoted activity (Goos, 2005; Hermans, Rijks, & Kempen, 1993; Valsiner, 1997). These respective zones of Being-mathematical, namely, ZFM and ZPA are neo-Vygotskyan concepts that have their origin in the zone of proximal development (ZPD), which was defined by Vygotsky (1978) as the

> distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers. (p. 86)

The creation of ZFM and ZPA by the mathematics teacher is a form of progressive education, or scaffolding that allows the *I* of the student to choose his or her Being-mathematical in relation to, or even in opposition to the constraints of a social efficiency agenda. Although the *Other* allows the *I* to choose his or her modes of Being, it cannot be to the detriment of *Others*. Therefore it is essential that all ZFM and ZPA activities are ethical in *I−Other*, which means that intrinsic moral values and a moral accountability underpin, or inform holistically all *I−Other* relationships that is *I−Other* (Aristotle, 2006). In other words a fundamental goal of ZFM and ZPA learning is for the *I* to develop a moral compass in-relation to the *Other*.

However, because the *I−Other* relationship is dialogic it implies that the *I* is not cloned when in-relation to the *Other*. Stated differently, although the 'subjective−objective' epistemologies of the *I* and the *Other* are inextricably linked, the ontology of Being-human that is the teacher and the student might vary markedly in Popper's (1978, 1979) Three Worlds. Only in the case of an omnipotent and omniscient *I−Other* would the possibility exist of no essential difference between the two Beings.

Nevertheless if powerful mathematical learning is to occur, a sufficiently intimate relationship is required if the *I* of the student is to appreciate the socio-cultural, historical and political customs and conventions of Being-mathematical as role modelled by the *Other*. But equally important is privacy, because it means that in an embodied sense the *I* and *Other* can develop uniquely and creatively, and as a consequence possibly enhance each other's Being-mathematical, perhaps even by interchanging roles.

However, although to be human has been "benchmarked" by **authenticity of relation**, **autonomy**, **imitation**, **intrinsic moral value**, **moral accountability**, **privacy**, **reciprocity**, **conventionality**, and **creativity** (Kahn et al., 2007), this does not mean that if the *Other* role models teaching and learning in these attributes, that powerful mathematical learning will necessarily emerge as part of the student's Self. In the challenging and pressured environments of mass mathematics education, if powerful mathematical learning is to materialize in terms of Popper's Three Worlds, then the *Other* needs to adopt an intent and an affect that unifies the benchmarks of Being-human through a psychology of Being-mathematical that may be described as **autotelic**.

The word autotelic is rooted in the Greek language, namely, *auto* and *telos* which when amplified includes a focused intent that 'the self is the goal and the goal is the self', in spite of contrariwise negativity, adversity, or mundaneness (Csikszentmihalyi, 1990). In this philosophical study, 'Being-autotelic' means being able to take a 'less than perfect' teaching and learning situation, and transform it into powerful mathematical learning through an ethical and creative dialogue. Moreover, it is fundamentally the dialogical self that enables Being-autelic through an intentionality of consciousnss that extends beyond the body to involve different and multiple *Others*; resulting potentially in a broader field of awareness, thereby avoiding the anti-social and closed system possibilities of solipsism, relativism, as well as any cultural restrictions that militates against the dialogical tenets, or benchmarks of Being-human.

All humans, at least to some degree, have the ability of Being-autotelic. From a philosophical and psychological perspective, the mathematical experience of *I−Other* is **optimized** when both *I* and *Other* choose Being-mathematical as a focused end-in-the-self rather than to achieve a future external goal (Csikszentmihalyi, 1990, 1997). In particular given the essentials of dialogue and the nature of the Conceptual Age, it is an autotelic dialogue over time that enables powerful mathematical learners through an intentional *I−Other* relationship. The heteronomous and autonomous duality of Being-mathematical is optimized when teachers and learners seek out, focus on, as well as generate and engage fully with mathematics in-relation to each teaching and learning moment.

The psychology of optimizing human experience has been called **flow**, because the metaphor seems to have captured the phenomenological essence of an embodied mind when it is engrossed in an activity (Nakamura & Csikszentmihalyi, 2009). When in a state of flow *Da-Sein* has (a) a sense that its Being is adequate to cope with the challenges at hand, (b) in a goal-directed, (c) rule-bound action system, (d) that provides clear clues as to how well the individual is performing (adapted from Csikszentmihalyi, 1990, p.71). However, operationalizing flow in classrooms has been difficult (Armstrong, 2008; Hollingsworth, & Lewis, 2006; Sedig, 2007), mainly because the majority of classrooms are not dialogical and interrelational in the benchmarks of Being-human.

Nevertheless, the concept of flow has been well established "from qualitative accounts of **how it feels** [for emphasis] when an activity is going well" (Nakamura & Csikszentmihalyi, 2009, p. 198). Therefore, Being-autotelic will likely action flow because the flow state is self-justifying (Nakamura & Csikszentmihalyi, 2009). Interestingly, eminent thinkers like

Buddha, Socrates, Confucius, and Goethe all put the joy of Being ahead of rewards extrinsic to the self (Armstrong, 2006; Bortoft, 1996; Csikszentmihalyi, 1990). However, as an example to powerful mathematical learners, a quintessential expression of flow in the history of mathematics and science was Archimedes of Syracuse, whose ‘unencumbered’ style of intuitive−practical thinking culminated in “elegant unanticipated juxtapositions” that delighted the self (Netz & Noel, 2007, p. 28).

The didactical contract. In every mathematics classroom around the world, the prior learning of the student and the teacher informs the habits and behaviour of *I* and *Other* respectively. As a consequence the student and the teacher have expectations of each other, and these expectations imply a didactical contract between *I* and *Other* that tends to operate at a tacit level of self-awareness (Brousseau, 1997). The notion of a didactical contract in classrooms came to the fore over the past two decades, especially in French educational research (Schoenfeld, 2008b).

Historically and biblically the idea of a contract is associated with a treaty or a covenant (Michener, 1980). The purpose of which was the ‘drawing in’ or ‘coming together’ of two parties in order to realize a mutually beneficial or altruistic outcome. Essentially however, if the didactical contract is at an implicit level of understanding then it is likely that the sense of agreement between *I* and *Other* will be limited. Stated differently, if the disposition of the self is not made explicit in classrooms then powerful mathematical learning is unlikely to occur, because the different types of self can oppose or nihilate in motivation, personality, or learning (Dweck, 2000). The act of nihilation allows the self to withdraw from a particular mode of Being without ‘fully annihilating’ its essence (Rae, 2011). In simple terms, a new psychological mindset is required if the teaching and learning of mathemtics mass education is to be ‘successful’ (Dweck, 2006).

Three selves. There are basically three models of the self in the speech genres of collective history (Hermans, 2012; Hermans & Hermans−Konopka, 2010). First, the **traditonal self** is subject to a socialization process that networks the individual into a hierarchy of authority. As an outcome the traditional self is enculturated into dogmatic ‘truths’, a moral telos, and a vital and aesthetic connection with the universe, the environment and a higher power called God. Second, and in contrast to the traditional self is the **modern self**. It is autonomous, individualistic, and values empirical observation, as well as inductive and deductive reasoning that can lead to universal truths and scientific laws. The modern self is not dialectical in its understanding of Being. For example, it is an embodied unified self; sharply distinct from the external *Other*. Moreover, Being is highly analytic and understood by a ‘separation of the parts’. For example, the facts of science and the values of faith distinguish politics from religion and theory from practice. Third, the **post-modern self** is a profound skeptic of master-narratives that emphasize totality and unity of Being. As an adversary of the modern self, the post-modern self focuses on local and situated accounts of reality through a perspective of self that is not unified but fragmented. Thus the post-modern self is opposed to symbolic hierarchies (e.g., the monarchy) and the social power that determines right from wrong, instead understanding truth in terms of language communities. Correspondingly, post-modernism decentralizes the *I* as subject and the *Me* as object in order to foster a critical theorist philosophy that is goal oriented in the deconstruction of power relations that maintain inequity and social injustice in society.

Summary insights: The autotelic *Other* and the flow state. The world is globalizing in terms of different selves. Thus a New Age needs to arise in which people are able to learn from the past in terms of a present history that is the future (Cox, 1987; D’Ambrosio, 2007a; Dice, 2010). However, mathematics education in classrooms has “changed very little over a considerable period” (Wiliam, 2003, p. 473). A dialogical world-view of the self can perhaps provide a stimulus for change, namely, by facilitating an autotelic flow whereby ‘elements’ of the traditional, modern, and post-modern selves can be related meaningfully albeit dialectically.

In classrooms it is the responsibility of the teacher to deliberately scaffold powerful mathematical learning through an intentional dialogue, especially one in which the self of the *I* commits to work with the ideas of the *Other* in terms of learning zones that promote freedom of activity and expression: valuing both sameness and difference. But in order to implement a new understanding of Being in mathematics classrooms, the teacher needs to 'negotiate' a simple and clear contract between *I* and *Other*. The contract ought to explain the essentials of an ethical dialogue; the characteristics of the dialogical self and flow, together with the requirement that the individual student and teacher commit to a problem solving inquiry of Being that is creative or innovative. For practical purposes and the development of an ecology of selves, or a community of powerful learners, it is highly desirable that the didactical contract between *I−Other* should be equitable for all dyadic pairs that involve the mathematics teacher.

An informed knowledge of the self can be equivalent to an increase of between 15 and 20 IQ points for the individual (Gardner, 2006a). To this end the didactical contract — signed by both *I* and *Other* — can mediate the relationship as a permanent but flexible feature of classroom activity, thereby providing stability and focus in the complex dynamics of 'auto' and 'telos'. It is a fundamental tenet of Being-human that both stability (non-change), and change are necessary to maintain the growth of the Self in terms of the *Me* that constitutes an increasingly complex complementarity about *I* and *Other*. Therefore Being-mathematical in an optimal sense is not ordinary, but requires a dialogic transformation of Being that this study refers to as powerful mathematical learning (Bertau, Gonçalves, & Raggatt, 2012; Schoenfeld, 2008a).

End Note

1. *I−Other* is essentially a psychological event that occurs in terms of an embodied and extended self. However, in this study the 'lowercase' self emphasizes the embodied dimension of the dialogical self, whereas the 'uppercase' Self includes not only the embodied dimension, but the dimension that extends beyond the body to include interpersonal and extrapersonal *Others* in-relation.

Chapter Five

Powerful Learning Principles, Quality Teaching, and Communities-of-Practice

Mathematics is the universal language of science. Therefore, Being-dialogical can inform and structure the teaching and learning of mathematics at a deeply human level through different language forms (Boehm, 1959; Conant, 1947; Hoffert, 2009; Veness, 1982). This is possible globally because human brains, bodies, cognition, and problem solving are readily comparable the world over (Törner, Schoenfeld, & Reiss, 2007). In an evolutionary developmental and globalising sense this is not surprising, because humans are the only species of the genus *Homo* still in existance (Goodman et al., 1990; Howells, 1973). Therefore the emergence of powerful mathematical learners for the Conceptual Age is a human possibility across the globe. But learning occurs in different cultures and always in relation to a complexity of social forces (Bruner, 1996). The teaching and learning of mathematics, the nature of school systems**,** the sequencing and choice of mathematical content in syllabi**,** as well as goal setting for individual students, all vary significantly between schools and nations (Törner, Schoenfeld, & Reiss, 2007).

However, human knowing and development as articulated by **Piaget** (Gruber & Vonèche, 1977; Inhelder & Piaget, 1958), **Chomsky** (Chomsky, 1965; Chomsky & Arnove, 2008), **Lévi−Strauss** (1969), and **Bruner** (1960, 1996), suggest that powerful mathematical learning needs to be structured in terms of universal principles (*epistémé*) that can inform, and be informed by, the cultural and societal context in which instruction takes place. Since powerful mathematical learning is embodied and therefore situated, the nature of learning and cognition in the situation will determine the viability of the learner to transfer his or her learning to other situations and contexts (Anderson, Reder, & Simon, 1996, 1997; Brown, Collins, & Duguid, 1989). Crucially therefore, the degree to which the individual can transfer learning to, or engage with new situations is dependent on the degree to which he or she has learned epistemically in different situations, because Being-mathematical in different learning situations demands a different tension (*phronesis*) between the universal and the particular (Cole & Scribner, 1974; Cole & Means, 1981; Lévi−Strauss, 1979).

Therefore it is essential that powerful mathematical learning develops through discourse−recourse experiences that unfold and enfold in manifold situations. The individual can then apply his or her learning through *epistémé*, and *in situ* enrich his or her universal understanding on how to learn in unfamiliar territory. The more experience that the learner is afforded in different mathematical situations, the more powerful his or her problem solving is likely to be in new situations, especially if his or her experience is underpinned by a didactical contract that encourages Being-dialogical.

The *Epistémé* of Powerful Mathematical Learning

By 1950, although the progressive teaching of mathematics was considered intricate, it was considered to be plausible through intentional goal setting and planning that reduced the tension in the teaching and learning situation (Anderson & Gates, 1950). In this regard, psychology broadened its perspective from predominantly neurophysiological perspectives to include behavioural and operational principles. Learning was no longer understood independently of the individuality of the learner and the teacher, because different teaching environments appeared to require a subjective approach that respected the specifics of the particular situation. Consequently, teaching for meaning in mathematics was reflected upon in broader terms than a mere concatenation of rigid habit and stimuli-response complexes (Brownell, 1939, 1944).

In particular, scholars had begun to grapple with general learning principles in relation to, or in abstraction from diverse theories and epistemological positions. This approach to the better understanding of teaching and learning occurred with an emphasis on process in the “totality of learning, including not only motor and verbal responses but also attitudes and affective (emotional) aspects of behavior, and recognition of the interrelatedness of these many aspects of learning" (Anderson & Gates, 1950, p. 13).

As a result learning was defined as a **change in behaviour** that was grounded in the personal experience and motivation of the individual student. The driver for learning was to resolve the 'new' in relation to the student's prior learning. If the student was limited by his or her responses then the situation became a context for problem solving. In order to address the problem meaningfully however, the student needed to develop an incentive or problem solving goal. The attainment of which implied a change in the person's behaviour over time that included a cognitive and affective process. In this sense learning was a principles-based product that was developmental in new meanings and understandings (Anderson & Gates, 1950).

Learning mathematics and science as a process: Woods Hole and beyond. The Woods Hole Conference was held in September 1959 under the directorship of Bruner, a Harvard University psychologist. The Conference was a seminal and unifying event in understanding mathematics and science education as a process. There were 35 delegates from different professional backgrounds and educational persuasions. Participating members included mathematicians, psychologists, historians, educators, cinematographers, as well as natural and medical scientists predominantly from the United States. Delegates of the Conference had been thinking extensively about, or doing exploratory research on how to operationalize process in the teaching and learning of mathematics and science in schools, and that also included the use of different media.

The aim of the Conference was not to provide an immediate solution to the challenges of making mathematics and science more meaningful in schools, but was rather a vigorous attempt to articulate the basic processes that would enable young people to grasp the essentials of scientific problem solving (Bruner, 1960). This necessarily involved **method** because implicit in the act of problem solving is the scientific method, which implies a thorough examination of the elements of a situation for the purpose of abstracting a pattern, or multiple patterns (inductive thinking) that can be used to logically deduce a solution.

Conservatively stated, delegates at the Woods Hole Conference agreed that the majority of school students could learn sophisticated scientific ideas, provided that learning meant the revisiting of ideas through a spiral curriculum that acknowledged the prior learning and growth of each individual student. The child was emerging into an adult. This view of mathematics and science education emphasized process, because mathematics and science were not 'still-life photographs' but dynamic events in time and space (Conant, 1947; Dewey, 1916, 1929a, 1929b). But most students according to the School Mathematics Study Group would probably need to practice procedurally on the way to mastery, and should not be exposed to formal ideas before the developing mind was ready to cope intellectually with the relative degree of abstractness (Begle, 1954, 1970; Bruner, 1960). Nonetheless, the most advanced mathematical ideas were thought to be intuitively accessible at both the primary and secondary school levels.

However, if teaching for process was to be successful in schools it would need to acknowledge that learning was both intuitive and analytical. In these terms students were likely to become self-motivated provided that the mathematics teacher was (a) an effective communicator of the fundamental and general ideas of the subject; (b) a confident problem solver who knew how to generate intuitive ideas and test them analytically; (c) an educator who inspired the student to self-identify with the teacher through an ongoing process of inquiry; (d) technologically proficient in modelling, dramatizing, or automatizing mathematics as a kinetic process, and (e) was competent to verify and confirm knowledge.

The major advance between Woods Hole and the problem solving focus of the 1980s and the 1990s was the realization that student's mathematical understandings were constituted, or constructed as a result of complex social interactions (Ernest, 1994, 1998, 2013; Vygotsky, 1978, 1986, 1991). Closely associated with the social dynamic was the notion of a system (Salomon, 1991; Schutz, 1970, 1972; Von Bertalanffy, 1969). However, although intuitive and analytical functioning involved guesses and following hunches, effective learning in these terms was not well understood but did appear to rely upon a 'social system' that promulgated an increasingly coherent and substantial knowledge of the subject (Burton, 1999). This was probably best achieved, not through a teacher-centred or student-centred approach, but by emphasizing and enriching the pedagogical relationship between the teacher and the student (Zull, 2002).

Twenty-five crucial ideas to facilitate powerful mathematical learning. The powerful learning of mathematics involves intuitive–analytical problem solving towards the realization of improved performance, achievement, and sense making and reasoning in mass mathematics education (NCTM, 2009). Since Thorndike's (1903) rendition of educational psychology, the field has advanced substantially through the publication of thousands of studies. Drawing on this vast and rich resource of human endeavour, approximately 35 scholars recently identified 25 empirically-based heuristics that encapsulate 'the way things are' didactically (Winne & Nesbit, 2010). There is no claim that the grouping of heuristics is exhaustive. If applied holistically however, instructional designs and meaningful learning would most probably be enhanced and enriched.

The word heuristic implies general features (Pólya, 1957), strategies (Schoenfeld, 1985), and principles (*epistémé*) that can be applied intuitively and directly (*techné*) to the teaching, learning, and problem solving process. Therefore as a consequence of 'Being-a-specific-life' in the teaching and learning situation, the Self of the mathematics teacher is likely to develop a felt immediacy-of-act, or *phronesis* that means essentially 'knowing how to best' apply general principles in the classroom (Bakhtin, Liapunov, & Holquist, 1993). However, if the

Self of the mathematics teacher is limited pedagogically in *epistémé* or *techné*, then acts of ‘classroom wisdom’ (*phronesis*) are restricted or undermined, because “wisdom is one of enormous heuristic significance ... [having] considerable analytic power and pertaining to many significant real-life adaptive criteria” (Labouvie−Vief, 1990, p. 79).

Necessary pedagogical and instructional principles. Associated mathematical ideas should be structured contiguously, or proximally in space and time (**1. Contiguity effects**). However, since learners are bodily-minded (Johnson, 1987), relational and conceptual learning is mediated by perceptual motor or haptic experiences, especially at the beginning stages of learning (**2. Perceptual-motor grounding**). Moreover in order to facilitate rich conceptual understandings, instructional designs should enable students to code or process ideas visually and verbally through the use of multimedia effects that appeal to all the senses (**3. Dual code and multimedia effects**). Consequently, real-world stories and examples that include case histories are likely to be better remembered than disconnected facts and abstract principles, because a ‘good story’ unifies ideas in a meaningful way (**4**. **Stories and example cases**). In addition, if a student develops his or her own story through multiple and varied examples, then abstracting principles from the story will be more achievable (**5**. **Multiple examples**). But if the story is not anchored in real-world problems that interest the learner, then the skills, understandings and motivation of the student are likely to be relatively shallow (**6**. **Anchored learning**). In other words learning is enhanced when learners generate their own responses, or story in relation to their prior learning and interests rather than simply recognizing answers (**7**. **Generation effect**). Nevertheless, the teacher needs to ensure that student learning is ultimately a well organized structure that outlines, integrates, and synthesizes information (**8**. **Organization effects**), because students’ abilities to self-regulate and monitor their understandings cognitively and affectively can be fickle depending on the task at hand and the mood of the individual (**9**. **Partial metacognition and meta-affect**).

Therefore importantly for future learning, students' ideas and concepts must not only be well organized but also cohere in relation to the social group of which the student is a part (Von Glasersfeld, 1991a, 2000). If technology is used to assist in this regard, then technology should be applied in such a way as to highlight related ideas and focus student attention on that which is important (**10**. **Coherence effect**). However, probably the most important factor influencing the quality of student learning is timely, accurate, and succinct feedback in-relation to learners' responses (**11**. **Feedback effects**). Thus, the excellent teacher is alert in the moment to valuable 'feedback−learning' opportunities (Woods & Jeffrey, 1996), especially if it means that the student will be prevented from internalising an incorrect idea (**12**. **Negative suggestion effects**). The acquisition of erroneous understandings can be difficult to undo especially if such learning occurs at the beginning of the learning process (Veenema & Gardner, 1996). In this regard teachers need to respect the knowings and expectations of secondary school students especially, because

> students are clear about the type of feedback they want and believe they need for improvement. They want honest and concise feedback focused on how to bridge the gap between where they are and where they need to be. A challenge for teachers is to provide sufficient information to achieve this goal, but not to overwhelm them with too much information so that they ignore it. (Peterson & Irving, 2008, p. 249)

Although feedback needs to be precise and relevant, the nature of the feedback may be an opportunity for the teacher to introduce meaningful challenges into the learning situation (**13**. **Desirable difficulties**). Students tend to thrive on difficulties that they feel are within their grasp, or zone of proximal development (Csikszentmihalyi, 1990; Vygotsky, 1978). Therefore teachers should engage students with material or assignments that are neither too easy nor too difficult, because 'stretching' students academically assists them to retain and retrieve their learning from long-term memory optimally and effectively (**14**. **Goldilocks principle**).

However, learning optimally in terms of a flow psychology means avoiding boredom and anxiety (Csikszentmihalyi, 2000). Therefore complicated material should be subdivided into meaningful subparts (**15**. **Segmentation principle**) that do not overload working memory (**16**. **Manageable cognitive load**), because being able to hold, or reflect on ideas in working memory is a key if the student is to make sense of mathematics for him or herself (Baddeley, 2007; Byers & Erlwanger, 1985). Although enhanced working memory capability is a characteristic of mathematically gifted students (Krutetskii, 1976), all sense-making in mathematics requires the use of working memory, and this can be enhanced if students are encouraged to explain, or give account of their understandings (**17**. **Explanation effects**).

Moreover, a requirement for the Conceptual Age is to reason and make sense of mathematics creatively. Teachers can foster creativity by promoting a classroom climate, or culture that is influenced by why, why not, how, and what-if questions (**18**. **Deep questions**). This type of questioning requires students to resolve cognitive conflicts, contradictions, paradoxes, anomalies, and hindrances to goals (**19**. **Cognitive disequilibrium**), which should characterize challenging mathematical questions for the purpose of facilitating flexible problem solving (**20**. **Cognitive flexibility**). However, the self-construction of mathematical principles, or novel solutions through discovery learning is too challenging for most students (**21**. **Discovery learning**) "without careful guidance, scaffolding, or materials with well-crafted affordances" (Winne & Nesbit, 2010, p. 656) on how to self-regulate the learning processes (**22**. **Self-regulated learning**).

Although the **22 principles** mentioned above are necessary for students to optimize their learning, as well as to achieve in high stakes testing they are not sufficient, but assessment that is formative and consistent over time (Black & Wiliam, 1998a, 1998b; Shepard, 2000; Wiliam, 2007) is likely to enhance both achievement and learning (**23**. **Testing effect**). In particular, assessment **for** learning strategies in the United Kingdom (Department for Children, Schools and Families, 2008) suggested that if tests were spaced at regular intervals then long-term retention and results would be superior to intensive one-off study sessions and tests (**24**. **Spacing effect**). Correspondingly, deliberate practice through repeated testing ahead of examinations is essential if students are to learn how to think and self-regulate efficiently and effectively under time pressure (**25**. **Examination expectations**).

If mathematics teachers draw holistically on the above mentioned principles to instruct, then the quality of student learning is likely to correlate positively with 'confidence in' and 'liking mathematics' — both of which are strong predictors of students' mathematics achievement (Winheller, Hattie, & Brown, 2013).

Quality Teaching

The implementation of epistemic learning principles in classrooms and schools is not straightforward. An understanding as to why this is the case is reflected in the total number of sub-groups of the 25 principles — in excess of 33 million, or 2^{25} combinations. This perspective however, does not include the complex notion of emergence, or the probability of multiplied numbers of interactions resulting in the whole being more than the sum of the parts. Therefore to make teaching both manageable and excellent in the 'order and disorder' of classrooms and schools, it is necessary to provide teachers with a structure, or the dimensions of good-quality knowledge that can be used to facilitate a holistic *epistémé*-based approach to pedagogy.

In previous research there has been broad agreement as to what constitutes high quality, or deep learning (Alexander & Winne, 2006; Even & Tirosh, 2008; Kirby & Lawson, 2012a). In particular a high quality learning environment is thought to be **constructivistic**, **collaborative**, **intentional**, **conversational**, **reflective**, and each student is taught how to mediate his or her deep learning by using a variety of **tools** or scaffolds (De Jong & Pieters, 2006). Towards the goal of quality teaching for quality learning in primary and secondary classrooms and schools therefore, Lawson and Askell−Williams (2012) "set out a systematic and parsimonious structure for considering the range of features of high-quality knowledge" (p. 145). The six dimensions considered in **Table 5·1** include Extent, Well-foundedness, Structure, Complexity, Generativity, and Representational format. Interestingly, the six dimensions appear to match well with Quine and Ullian's (1970) five interrelated beliefs, or virtues: Generality or Fruitfulness (Extent and Generativity), Refutability or Testability (Well-foundedness and different Representational formats), Simplicity (Structure), and Conservatism and Modesty (Simply Complexity).

Table 5·1. Dimensions of Knowledge Quality (Lawson & Askell−Williams, 2012, p. 145)

Dimension	Descriptions in literature
Extent	Extent, extensive, quantity of major ideas, scope of knowledge (deep)
Well-foundedness	Accurate, accord with reality, accord with the relevant knowledge community, relevant data, correctness of responses, thorough understanding (deep)
Structure	Structure, economy, capacity, well-integrated (deep), organization, shape
Complexity	Relating operation, (deep) understanding, complex, precise, adequacy of justification; adequacy of explanation; elaborated, degree of synthesis, logical coherence, consistency and closure, internal consistency, integration, coherence with prior beliefs
Generativity	Supportive of transfer, flexible, power, transfer-appropriate processing, robust encoding, extended abstract, availability, generativity
Representational format	Variety of types of memory element, imagery, knowing-in-action and personal practical knowledge, declarative, procedural, semantic, episodic, verbal/visual

Powerful mathematical learning is highly unlikely to occur by chance. The history of mathematics education attests to this. Both teachers and students need to act deliberately if powerful mathematical learning is to manifest in classrooms and schools. However, mathematics teachers who draw on the **25 principles** to facilitate the six dimensions specified in Table 5·1 are likely to empower their didactical contracts in classrooms.

Communities-of-Practice. Wenger (2009) argued that the quality of teaching in mathematics classrooms was enhanced by integral communities-of-practice who allowed learning to occur on the basis of human diversity and possibility, namely, (a) the individual contributed to the practices of his or her mathematical community, (b) the mathematical community refined and focused its practices in relation to other learning communities within the school or institution, and (c) the school or institution sustained the interconnectedness of the various communities of practice through a core vision of intent that was expressed through common learning principles and norms. However, communities-of-practice need to decide on how to select and implement teaching and learning strategies. In this regard no theory is complete because of the complexity that characterizes the potentiality of Being-human. Therefore the professional practitioner is encouraged to develop an intuitive−analytical understanding of Being-mathematical by drawing on diverse philosophical, theoretical and empirical emphases.

It is clear from the history of education that many mathematics teachers have had a very limited understanding of the human condition. Consequently knowing how to teach, learn, and motivate the subject of mathematics has by and large been ineffective and emotionally challenging. However, although educational knowledge is currently far from complete, it has been argued that "we know enough to continue making progress on translating this knowledge for classroom practitioners at all levels, and developing educational curricula and materials that support high-quality learning" (Kirby & Lawson, 2012b, pp. 373−374).

Hence in future, high quality mathematics teachers need to understand teaching and learning as a **personal narrative** of what Being-mathematical can mean, or should mean in relation to a particular present history. Toward this ambitious but vital understanding for powerful mathematical learning, all mathematics teachers need to reflect on (a) **neurophysiological theories** that relate to biological mechanisms of learning; the limits and rhythms of teacher and student physiology, as well as how to stimulate and optimize memory processes especially working memory (Clark, Nguyen, & Sweller, 2006; Edelman, 1993; Sylwester, 1995); (b) **behaviour modification** that occurs through the selective reinforcement of stimulus‒response pairs (Thorndike, 1922, 1931; Evans, 1981; Skinner, 1954, 1974); (c) the **dialogic transformation** of internal cognitive structures as a consequence of "communication, explanation, recombination, contrast, inference, and problem solving" (Wenger, 2009, p. 217); (d) task-oriented, or problem solving **constructivist theories** that describe situated conscious processes of mind (Hershkowitz, Schwarz, & Dreyfus, 2001; Inhelder & Piaget, 1958; Inhelder, de Caprona, & Cornu‒Wells, 1987; Papert, 1980); (e) learning theories that articulate the **social development of mental structures** through interpersonal relations that include imitation and the modelling of agentic behaviour (Bandura, 1986, 2001); (f) **activity-with-objects** and **activity-with-relationships** (Engeström, Miettinen, & Punamäki, 1999; Noss & Hoyles, 2006; Vygotsky, 1978; Wertsch, 1985b); (g) **socialization theories** that focus on the internalization of a group mind — perhaps through a socio-cultural framework of joint activity that involves the habituation and extension of norms (Parsons, 1962; Wenger, 2009); and (h) **organizational theories** that develop an understanding of how an organization can become 'part' of each individual member, and as each individual member expresses the organization differently through internalised *I*-positions, the organization learns to be different in relation to the individual (Nonaka & Takeuchi, 1995; Wenger, 2009).

Therefore teaching for powerful mathematical learning means having a knowledge of learning that is well-founded, wide-ranging, and consistent with the creativity metaphor that is essentially the Conceptual Age.

A quality teaching narrative. Pedagogically and didactically therefore, powerful mathematical learning should not be limited to 25 principles and six dimensions, because quality teaching does not occur in isolation of "contexts ranging from emotional states within the individual to classrooms and to institutions and beyond" (Kirby & Lawson, 2012b, p. 372). Consequently, teaching mathematics as a powerful learning dialogue requires teachers to be both students and practitioners of learning.

Currently, and as described above, there are at least eight major classes of 'learning theory' that mathematics teachers and educational leaders can draw on to enrich their textual and contextual dialogues. In this regard the ideas in this section are informed by Illeris' (2009) *Contemporary Theories of Learning*. It is pertinent to Being-mathematical that the contemporary theories all discuss learning as a bi-directional interaction between the internal (embodied) and external aspects of the self (Illeris, 2009). In phenomenological terms therefore, powerful mathematical learning occurs when Beings with bodies interact in terms of Selves so that each *I* can make sense of the different dialogues on the basis of a developing socio-cultural perspective (Bakhtin, Liapunov, & Holquist, 1993; Davis, 1996; Merleau−Ponty, 1962; Varela, Thompson, & Rosch, 1991; Vygotsky, 1997).

Consequently the meaningful learning of mathematics can be viewed as a dual landscape, expressed as an outward psychological movement from the person to the external, objectified culture and an inward sociological movement from the objectified culture to the individual person (Jarvis, 2009). In this sense mathematical learning is transformative through two cultures, namely, the mind in a cultural society, and an embodied mind that emerges as part of a self-developmental process — restless and creative because of the unfolding dialectic that are the two cultures (Hegel, 1967; Kegan, 2009). Necessarily therefore, it can be said that the powerful mathematical learner expands him or herself by questioning or modelling the external culture in relation to his or her mind−body culture; a possible outcome over time is an essentially new mathematical practice for the learner that involves both activity-with-objects and activity-with-relationships (Engeström, 2009).

However, the powerful teaching and learning of mathematics in classrooms needs to be pragmatic if it is to be successful in relation to the dialectic of two cultures. Pragmatism in the tradition of Dewey and Peirce (Moore, 1961; Shank, 2006) means learning for the future by teaching "a preparedness to respond in a creative way to difference and otherness. This includes an ability to act imaginatively in situations of uncertainty" (Elkjaer, 2009, p. 74). Consequently, learners who enculturate in these terms are likely to develop multiple approaches to understanding mathematics (Gardner, 2009).

Be that as it may, powerful mathematical learning is not limited to the classroom. A current and central idea in education is that of lifelong learning at different levels of society, namely, the social macro-level, the institutional meso-level, and the individual micro-level (Commission of the European Communities, 2000; European Commission, 2005). At its most powerful therefore, mathematical learning is a multi-level cultural event that facilitates an unfolding social and self-reflexive biography of the person (Alheit, 2009). Therefore learning to be *Me*, or the self-as-object is both a construct and a self-organizing dynamic of *I*-positions, where the *I*s are not only constituted in terms of other individuals or objects, but also in terms of societal structures and complex group minds that reflect global−local changes (Hermans & Hermans−Konopka, 2010; Hermans & Gieser, 2012; Wildemeersch & Stroobants, 2009).

In succinct terms however, the *I*s develop in dialogue with one another by adopting different modes of Being-mathematical including affective or feeling modes, conceptual modes, imaginal modes, and practical or behavioural modes (Heron, 2009). Although the different modes can be in operation simultaneously, the English poet and philosopher Coleridge (1772−1834) was of the opinion that most teachers and students in schools narrowed

> into a dull or resigned acceptance of a limited representative self and a disavowal or oblivion of the real self. Similarly, too much teaching offers insufficient opportunity and too feeble a provocation to enrich the image of self by imaginative participation in many modes of being; just as, all too frequently, it is, in the face of the helplessness of

> the child, an unjust invasion of the real self. But the mature adult — and this is what every teacher should be — is one who senses in others, because he has felt it in himself, beneath the image of the representative self the secret movements of a deeper self. For the image he has imaginative liberality, sympathy in feeling and tact in action; for the true self he has reverence. (as cited in Walsh, 1964, pp. 15−16)

Therefore the teacher, or student who wishes to become a powerful learner of mathematics needs to intentionally explore different mathematical discourses in *I*-development modes for the purpose of dialogic transformation. The Self that becomes entangled in this form of play, changes continuously into an increasingly stable Self (Csikszentmihalyi, 1990; Tennant, 2009). However, self-regulated and discovery learning discourses need to be checked for coherence. In particular, through negative suggestion, timely feedback, deep questions, explanation effects, and the meaningful resolution of cognitive disequilibria.

Importantly, feedback should be dialogical, because dialogue facilitates meaningful understanding (Hermans & Gieser, 2012). On the one hand, the mental structures of students need to incorporate an understanding of Being-in-the-world as “settled in relation to some pre-existing, rule-bound code that maps onto states of the world” (Bruner, 2009, p. 160). In other words the mind in-relation to its Being owes its very existence to a socio-cultural dynamic that is the Self (Hermans & Gieser, 2012; Vygotsky, 1978, 1997). As a result higher order cognitive, affective, and volitional functioning, especially in relation to frontal lobe development, cannot be interpreted apart from the cultural and social resources and activities that inform the superorganic, or neurophysiological and emergent mind of the self that learns from itself (Duncan, 1980; Kroeber, 1917; Stuss & Knight, 2013).

On the other hand, mathematics teachers need to give feedback in the context of two cultures, namely, a mind in society and a society of mind. Given the nature of complexity and the power of symbolism, the two cultures are interrelated but not identical. That is all students in a mathematics class may ‘belong’ outwardly to the same culture, but each embodied mind has the potential to be culturally idiosyncratic. Thus in a radical constructivist sense, meaning making in mathematics does not imply the discovery of an external ontological reality as would be the case if the mind was completely specified by the computer metaphor (Von Glasersfeld, 1991a, 1991b).

Consequently, if a mathematics teacher provides feedback within the cultural dynamics of his or her embodied mind alone, then the gap between the interpsychological plane of the extended self and the intrapsychological plane is likely to remain unbridged. However, if powerful mathematical learning is to occur then the planes can interrelate meaningfully through dialogue. When the mathematics teacher and student dialogue in terms of *I−Other*, or as 'we', then Usher (2009, p. 182) asserted that "we can examine how, as selves, we move back and forth between our own particular stories through which we construct our identities and the social production that is knowledge."

Therefore, becoming a powerful learner of mathematics is not an easy process. It requires an intentionality of consciousness that is motivational in the long term, namely, to set up 'feel good' intermediate aims which are attainable in the pursuit of powerful mathematical learning (Ziehe, 2009). In terms of human motivation however, Pink (2009) argued that if a person engaged in a creative process, and was encouraged to play an influential role in directing or empathizing with the creative process, then the expected outcome should be an enrichment or enhancement of the Self for that individual. But in this regard deliberate practice and intentionality are indispensable on the part of the teacher and the student (Anderson, 2010), because

> knowledgeability is routinely in a state of change rather than stasis, in the medium of socially, culturally, and historically ongoing systems of activity, involving people who are related in multiple and heterogeneous ways, whose social locations, interests, reasons, and subjective possibilities are different. (Lave, 2009, p. 207)

The CAME and CASE Projects. All teaching and learning involves a narrative of some sort (Kahneman, 2011; Pink, 2005). Historically however, facilitating a meaningful 'outside in' and 'inside out' mathematics teaching and learning narrative has not been straightforward. In the history of Mathematics Education however, the CAME (**Cognitive Acceleration in Mathematics Education**) intervention project produced very encouraging results in this regard. The CAME I intervention was funded by the Leverhulme Foundation and took place between 1993−1995; CAME II was funded jointly by the Economic and Social Research Council and the Esmée Fairbairn Trust for the years 1995–1997. The ‘Teaching for Thinking’ mathematical program involved a total of approximately 2500 students aged between 11 to 13 years (primarily in comprehensive schools in the United Kingdom). At the age of 16 years, the achievement of those students who had been taught to think for themselves was compared to similar groups of students that had not participated in a CAME intervention. On average, a large positive effect (0.8 S.D.) was reported in favour of the ‘Teaching for Thinking’ students (Shayer & Adhami, 2007).

CAME was preceded by the CASE (**Cognitive Acceleration through Science Education**) interventions of the 1980s and early 1990s. As a consequence the CAME project benefited greatly from the CASE I (1981−1983) feasibility study; the main intervention study that was CASE II (1984−1987), and CASE III (1989−1991) focused specifically on understanding the teaching skills that were deemed essential if students in mass science education were to be accelerated successfully (Adey & Shayer, 1990; Adey & Shayer, 1994; Iqbal & Shayer, 2000; Shayer & Adey, 2002).

The CAME project was grounded theoretically in the ‘inside out’ developmental theory of Piaget (1970, 1973, 1985); the ‘outside in’ socio-cultural psychology of Vygotsky (1978, 1986, 1997), and the generally accepted ideas on metacognition. Educationally therefore, the aim of the project was to accelerate students cognitively towards the goal of successfully assimilating and accommodating formal operational knowledge in mathematics. In other words the CAME intervention study was used to speed up and enhance the general intellectual development (intelligence) of children through mathematical problem solving

(Adey, 2006; Adey, Csapó, Demetriou, Hautamäki, & Shayer, 2007). The following five tenets formed the basis for the relatively unstructured teaching and learning intervention.

First, lessons were designed so that all students felt that they could engage with the task(s). This meant identifying students' prior learning to make the context of the problem clear, and to develop the type of language that the students could use to describe and think about the problem meaningfully. **Second**, the problem was framed in such a way that students were motivated to solve the problem, that is, the problem provoked a cognitive conflict within the minds of students that they wished to resolve. There was a feeling in the class that 'we want to do this and we can do this if we put our minds to it'. **Third**, students addressed the problem as a social process. It was authentic problem solving. There was 'no answer at the back of the book'. **Fourth** having engaged with the task, each student was encouraged to think about his or her thinking in relation to the problem as a whole, and then to make his or her thinking strategies explicit. Importantly for cognitive acceleration teachers to realize however, metacognitive activity on the part of students did not come easily or naturally (Adey, 2006). This is not surprising because metacognition is primarily a System II complexity of mind, and consequently requires more energy and analytical ability as compared to System I thinking (Kahneman, 2011). **Fifth**, CAME students were encouraged and given the opportunity to develop cognitively and metacognitively by applying their thinking abilities and skills to a wide variety of problems and situations, thereby facilitating learning transfer. Interestingly in relation to powerful mathematical learning and globalization, the five tenets of CAME were based on the notion that the

> ability to process many aspects of reality simultaneously is the key to high performance in any sphere, and conversely any context-related intervention is likely to affect the learning ability of a child generally, and not just in that specific context, provided the intervention activity is conducted 'with an eye on the Towers of the Eternal City' [St Augustine, City of God, and Dante, Purgatorio, XVI, 94−96] and not just on verbal learning or unthinking techniques. (Adey & Shayer, p. 7)

Summary insights: *Epistémé*, Quality Teaching, and Communities-of-Practice

Although learning narratives may differ depending on the socio-cultural and historical situation, the teaching and learning of powerful mathematics is viable globally because essentially, all creative and ethical learning dialogues reflect, or are informed by the *epistémé* pedagogical principles, the teaching-for-quality dimensions, and the 'inside out': 'outside in' community-in-practice approaches that are highlighted in this chapter. Moreover, powerful mathematical learning is consistent with a Piagetian−Vygotskyan ontogenetic and socio-cultural learning perspective. Educational research has shown that any teaching and learning narrative that has high expectations for student learning in terms of a Piagetian−Vygotskyan program was likely to result in encouraging student performance and achievement (Hattie, 2009, 2012).

The CAME and CASE projects were cases in point and the major lessons learned can be applied to Conceptual Age learning. First, the teaching and learning of mathematics and science should develop student intelligence and real-world problem solving capability. Second, it was dialogue — in particular between teachers and other teachers — that acted as a multiplier effect for the foundational tenets of 'Teaching for Thinking' to take root and flourish in classrooms and schools (Adey & Shayer, 1994, 2002; Shayer & Adhami, 2007). If however, dialogue was constrained in schools because of cultural traditions for example, then the CAME and CASE projects were not very successful in those situations. Third, dialogue fostered a sense of community, belonging, or 'ownership' of the learning program between the different participants, namely, the teachers, the students, and the researchers.

Chapter Six

Being-a-learner: Aims, Protocols, Taxonomies, and a Model of School Learning

Globalization is changing the world irreversibly (Darling‒Hammond, 2010; Friedman, 2005). Therefore 'Being-a-learner' in Popper's Three Worlds is potentially a transformative experience, but the future success, development of nations, and even the relevance of cultures in a Conceptual Age is largely dependent upon the majority of students' ability to learn powerfully. However, this type of learning is dependent upon teachers' capacity to effect collaborative and strategic learning partnerships that are ethical. As well as an ability to communicate dialogically, and take calculated risks so that new ideas can be analysed into useful processes and products (Darling‒Hammond, 1997, 2010; Piaget, 1973).

At the heart of globalization is humanity's desire to be **free** from narrow selves; constrained by socio-cultural power relations that limit potential, connectivity, and creativity. As a result teaching and learning is tending towards intellectual freedom, or a freedom of Being that is singular‒plural (Nancy, 2000), and therefore demands the right to pursue, understand, and generate knowledge through a reflective intelligence that is capable of expressing 'difference' without the fear of sanction or penalty — even though particular situations may be ambiguous and the issues and solutions obscure and equivocal (McMurrin, 1964).

Within the past two decades for example, despite the weight of history, culture, religion and ethnicity, nations in **Africa** (e.g., Burundi, Ghana, Nigeria, Rwanda, and South Africa), **Asia** (e.g., Indonesia, Mainland China, Singapore, South Korea, and Viet Nam), **South America** (e.g., Argentina and Brazil), and the **Arab World** (e.g., Oman, Qatar, and the United Arab Emirates) have begun to empower teachers and students into the global discourse of the twenty-first century. Notably, Brazil's primary and secondary school students benefited from an increase in expenditure per student by almost 150 per cent between 2005 and 2009 (OECD, 2012a). While in South Korea, the share of tertiary graduates doubled from 20 per cent in 1997 to 40 per cent in 2010, which represented the largest percentage increase among the OECD countries (OECD, 2012b).

However, if powerful mathematical learning is to manifest itself in twenty-first century classrooms, then Being-mathematical requires as discussed previously, that epistemic pedagogical principles are coupled with, or related to a quality of knowledge that includes Extent, Well-foundedness, Structure, Complexity, Generativity, and Representational format. The purpose is to 'give voice', or expression to the educational aims of diverse communities-of-practice.

The Aims of Education

Different teaching and learning communities will no doubt emphasize different aims at different times. However, Whitehead declared that meaningful education was to appreciate and implement the following goal-directed, or intentional ideas so that the lives of **all** who were involved in the educational process could prosper:

- **(1)** Students are alive. Consequently, the purpose of education is to guide and provide a teaching and learning structure so that each self can develop, especially through 'living' thoughts (the student makes them his or her own by testing, applying, proving, and intercombining).
- **(2)** Learning is to develop a culture of thought, through activity that is receptive to beauty and humane feeling.
- **(3)** Students need to be taught as if they have the potential to be special.
- **(4)** Throughout any school, the main ideas should be few and 'thrown together' in every possible combination and permutation.
- **(5)** Students do not live in the past or the future, because the present contains all there is — it is the past and the future, the complete sum of existence, backwards and forwards, that whole amplitude of time, which is eternity.
- **(6)** A curriculum that does not connect with, or inspire the mind of the individual will be inert in the life and culture of that person.
- **(7)** If learning is to be real and thorough, then it needs to involve the joy of discovery, not merely the execution of intellectual minuets.
- **(8)** The best education is to be found in gaining the utmost from the simplest apparatus.
- **(9)** The general culture, or essence of learning should pervade all school activities; specialist courses ought to elicit the notion of 'sameness in diversity'.
- **(10)** Style is the ultimate morality of mind, because it is an aesthetic sense, based on admiration for the direct attainment of a foreseen end, simply and without waste. (adapted from Whitehead, 1932, 1−23)

Importantly for powerful mathematical learning, not only are all ten aims consistent with Whitehead's process–relational philosophy of education, but also with the five virtues that underpin Quine and Ullian's (1970) "web of belief," namely, Conservatism, Generality, Simplicity, Refutability, and Modesty. In particular however, Whitehead's third aim resonates with the philosophy of the great American educator, Tyler (1902 – 1994), whose ideas on curriculum were influenced by Judd and Charters at the University of Chicago in the 1920s. Tyler expressed the view that every person was valuable because of their uniqueness, and consequently ought to be encouraged to participate in the social dynamics of the group. This view is in accord with Dewey's (1897, 1916) pedagogical creed. In so Being the participant is likely to spur the self-development of the *I* in-relation to other individuals and the group as a whole. However, although Tyler appeared to walk a 'middle-way' between the behaviourism of Skinner and the progressive education of Dewey, he did have 'faith' that young people could develop intellectually — not only through prescriptive pedagogical approaches to teaching and learning, but also as a result of didactics which facilitated authentic, relevant, and even important problem solving (Finder, 2004; Tyler, 1949).

The educational wisdom, or philosophies of Dewey, Tyler, and Whitehead inform and enrich the aims of modern education. As indicated in **Figure 6·1**, powerful teaching in schools involves at least nine different foci, or modes of Being that can foster high-quality learning, especially learning that is applicable in new situations through a 'smorgasbord' of problem solving and self-directed learning skills. However, the 'arrows' in Figure 6·1 are not specified or described. This is an indication that powerful learning environments are not well understood, because they have not been modelled systemically or statistically. Hence, the importance of this study is to provide a philosophical and comprehensive basis that can be modelled statistically, so that the aims and characteristics of powerful learning environments can be interlinked meaningfully for the purpose of facilitating Conceptual Age mathematics learning and knowledge development in mass education.

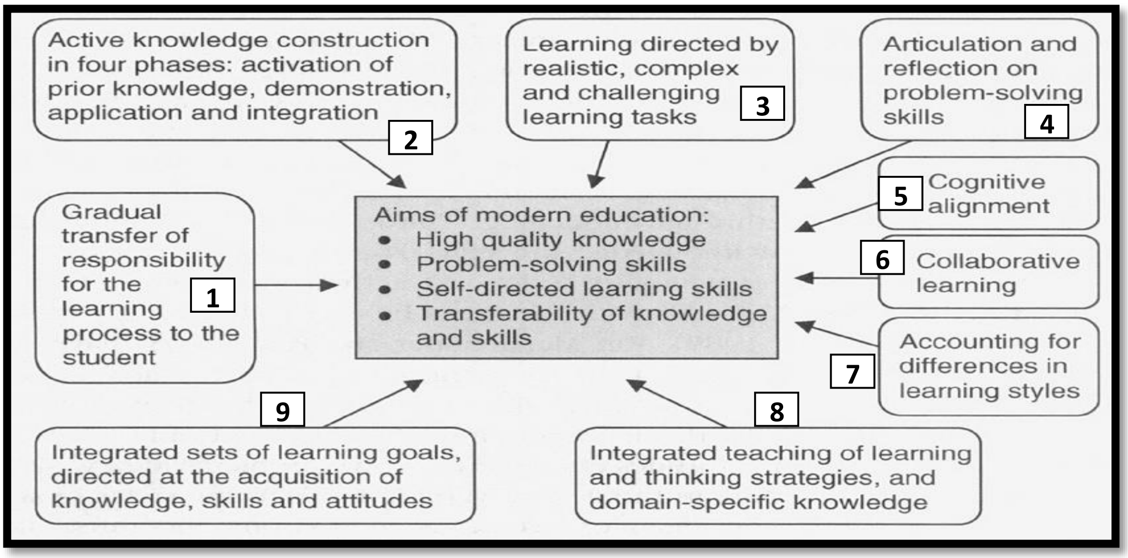


Figure 6·1. Characteristics of powerful learning environments that facilitate the main aims of modern education (adapted from Könings, Brand-Gruwel, & van Merriënboer, 2005, p. 648)

Teaching Protocols

In terms of a present history of mathematics education, the aims and activities, or foci of Conceptual Age learning environments need to be linked essentially, and particularly by teachers in classrooms. The following five questions have a tool-like character (*techné*) that can empower teachers' *phronesis* in locally situated classrooms.

(1) **Mathematical focus, coherence and accuracy**: Do students encounter a 'balanced diet' of facts, procedures, concepts and problem solving skills?
(2) **Cognitive demand**: Are students challenged by engaging in different or successive zones of proximal development?
(3) **Access**: Does each student encounter mathematics in some form that is honest?
(4) **Agency, authority and accountability**: Is each student developing a meaningful voice in the classroom discourse (agency and authority) which is underpinned by mathematical norms (accountability)?
(5) **Uses of assessment**: To what extent are understandings interrogated?
(adapted from Schoenfeld, 1985, 1987, 2008a, 2013)

Essentially, Conceptual Age learners need to 'know what to do when they do not know what to do' (Wineburg, 1998). Consistent with this sentiment, the CAME and CASE projects of the 1980s and 1990s facilitated a learning methodology that equipped students to think mathematically and scientifically for themselves. **Table 6·1** compares "normal good instructional teaching" and a CA-type (cognitive acceleration) flexible intervention that encouraged novel problem solving and discovery (Adey, 2006, p. 50). From the two 'protocols', a teacher can develop a balanced methodological approach, or scaffolding structure whereby the aims of modern education can be progressed through different modes, or foci of Being-mathematical.

Table 6·1. Comparison of CA-type intervention and normal good instructional teaching (Adey, 2006, p. 50)

Instruction	Intervention
Carefully ordered	Follow direction of argument
Specific objectives	Virtual objectives
Small packets, reinforced	Students often puzzled
Lots of stuff delivered	Not much stuff delivered
Students have notes to revise	Nothing obvious to show
You know what you have covered	Not sure what you have covered
Relatively easy	Seems dangerous

It is noteworthy that the **Instruction and Intervention** lessons were very different in the use of classroom time. Not only was the Intervention lesson not scripted in broad terms, but the culture and flow of the lesson was dependent on a student–teacher multilogue that was driven by "the individual and collective responses, moment by moment, of the students" (Shayer & Adhami, 2007, p. 288; also see Armstrong, 2008). Consequently, adept CAME or CASE-type teaching often had students-in-activity 70 per cent of the time; leaving the remaining 30 per cent for teacher comments and suggestions (Iqbal & Shayer, 2000; Rinaldi, 2013; Stone, 2007). In contrast, Instruction-type lessons tended to limit students' generative learning activity to a maximum of 50 per cent of available classroom time, and often significantly less (Bishop, 1989; Iqbal & Shayer, 2000; Stacey, 2010).

In terms of *phronesis* therefore, if the ambitious aims of modern education are to be realized effectively in diverse classrooms, then certain lessons need to be directed by the teacher, and other lessons are best progressed by the students themselves. Powerful mathematical learning is not an 'either–or' interaction, but a meaningful multilogue that assists both the teacher and the students to Be-mathematical.

S−R−O−C. Instruction-type protocols tend to be logical and linear, whereas Intervention-type protocols reflect, or encourage the non-linear dynamics of Being-human. In this regard a learning protocol has been developed and tested in schools that can promote, or mediate both kinds of student learning, depending upon the pedagogical aims selected by the teacher or students. The learning protocol is referred to as **Select−Relate−Organise−Check**, or S−R−O−C for short, and was specifically designed to scaffold, enhance, and expedite cognitive and metacognitive learning activity in classrooms (Askell−Williams, Lawson, & Skrzypiec, 2012).

First, from a S−R−O−C perspective it is the responsibility of the teacher to establish, or structure a zone of promoted activity (Valsiner, 1997) that can focus and motivate student attention towards the perceptual-motor grounding and identification of key ideas and objectives (Butler & Winne, 1995). The mode of Being that the teacher adopts determines whether the start of the lesson is to be Instruction-type, or Intervention-type learning. For example, if the teacher **selected** the key ideas in-relation to the prior learning of the class, then the first time-moment of the lesson would unfold predominantly in terms of instruction. However, if the teacher asked stimulating and Socratic-like goal-directed questions, with the expectation that on reflection students would **select**, construct, or generate the key ideas for themselves, then the first time-moment of the lesson would be consistent with Intervention-type learning.

The second time-moment of the lesson is for students to develop, or **relate** meaningful connections between the key ideas. The teacher could adopt an instruction-mode and still make use of the deep learning questions Why?, Why not?, How?, and What if?. Alternatively the students themselves could use the same questions to engage in forms of discovery learning.

An important aim of modern education is integrated teaching and learning. Consistent with this aim, the third time-moment of the lesson is for students to **organize**, or demonstrate their learning contiguously in space and time. This is an opportunity for students Being-there (*Da-Sein*) to organize their own learning, as well as the group's learning by valuing multiple intelligences and different representational formats.

The fourth and last time-moment of the lesson is often the most important, because studies in psychology have demonstrated that to a large degree this is how the lesson will be remembered by the learner (Kahneman, 2011). Consequently, it is essential for the learning group to **check** their learning for coherence and mathematical accuracy, because students' mathematical and metacognitive functioning is often not at the level of formal operations (Biggs & Collis, 1982; Mattick & Knight, 2007; Shayer & Adey, 2002). Importantly therefore, the Select−Relate−Organise−Check protocol was designed to facilitate and enable both the teacher and students to develop, and **check** for deep learning against the six Quality Teaching dimensions that are Extent, Well-foundedness, Structure, Complexity, Generativity, and Representational format.

Moreover, a significant strength of the protocol is that it can be embedded in the mathematics lesson. It need not take time away from the lesson, but instead can act as an integrated tool so that learning can be self-directed through feedback-type questions like: (a) Where am I going and what are my goals?; (b) How am I going and what progress has been made towards my goals?; and (c) Where to next? (Hattie, 2012). It is noteworthy that most student feedback is likely to come from peers, but should not be relied upon to be correct (Nuthall, 2007). Therefore pedagogical tools that facilitate sound learning are essential in mass mathematics education.

The Taxonomies

Internationally, progressive and high performing school systems have attracted teachers with pedagogical promise, and a willingness to engage in ongoing professional development towards the goal of realizing best practice for every student (Barber & Mourshed, 2007). Therefore excellent teaching is a profession in lifelong learning and action research. In particular, it means Being-able to initiate and engineer student and collaborative zones of proximal development that can foster the freedom, agency, and capability of Being-mathematical (Armstrong, 2008; Lesh, 2006; Shayer & Adhami, 2007).[1] However, teachers require *techné*, namely, instructional tools, routines and advancing technologies if dialogic teaching and learning ecologies are to emerge in classrooms and schools (Resnick, 2010).

Over the past 70 years, at least 20 taxonomies have been designed to reflect major changes in educational theory, research, and practice (Marzano & Kendall, 2007). Typically, a taxonomy is a highly informative learning protocol that summarizes, classifies, or networks important educational objectives *via* an underlying continuum of increasing complexity. The respective taxonomies discussed below can facilitate teachers' pedagogical practice in relation to powerful mathematical learning. In particular, the taxonomies are tools for the purpose of fostering students' mathematical focus, coherence, and accuracy as they manage cognitive demand, but still access important ideas and concepts.

Bloom's Taxonomy of Educational Objectives: The Cognitive Domain. The most well known of all taxonomies is Bloom's (1956) Taxonomy of Educational Objectives for the Cognitive Domain (Postlethwaite, 1994). It is strongly informed by the ideas that emerged out of, or were tested empirically during the Eight-Year Study between 1933−1941 (Aikin, 1942). The study was a large scale curriculum project that spanned 30 secondary school systems in the United States from Boston to Los Angeles. Tyler was the Director of Evaluation of the study, and he and his team sought to answer the following four questions, namely,

(1) What educational purposes should the school seek to obtain?
(2) What educational experiences can be provided that are likely to attain these purposes?
(3) How can these educational experiences be effectively organized?

(4) How can we determine whether these purposes are being attained?
(Madaus & Stufflebeam, 1989, p. 202)

The Eight-Year Study occurred in the wake of a substantial increase in the school going population during the time of the Great Depression (c. 1930s). In this context the prior learning and future expectations of many students did not connect meaningfully with school curricula, as these were often designed specifically for students who were bound for college (university). In contrast, Tyler and his team aspired to influence mass education across the United States equitably through considerable systemic research.

It was this research, coupled with input from approximately 40 scholars in the United States over a four year period from 1949–1953, that provided the impetus for Bloom's (1956) Taxonomy. For the next four decades the Taxonomy remained "a standard reference for discussions of testing and evaluation, curriculum development, and teaching and teacher education" in the United States (Anderson & Sosniak, 1994, p. vii).

The educational objectives were organized into a hierarchy of six major classes of behaviour. The notion of complexity was used to classify the objectives in terms of increasing learning difficulty. Furthermore, the construction of the Taxonomy occurred along the following lines, namely: (a) in order to bridge the gap between theory and practice the major distinctions between classes reflected the type of language that teachers used to differentiate between student behaviours; (b) the hierarchy was to make sense logically and be consistent internally; (c) all behaviours were compatible with well established principles of psychology; and (d) the classification was purely descriptive so that educational goals could be applied to most teaching and learning situations.

The first edition of the Taxonomy appeared in May 1956, just four months before the Symposium on Information Theory which retrospectively was a seminal moment in the history of the cognitive revolution (see p. 103). Therefore it is not surprising that the developers of Bloom's (1956) Taxonomy understood the goals of education as a process of learned behaviours in the sense that each successive behaviour was dependent on all previous behaviours. The Taxonomy was organized hierarchically as follows:

(1.0) **Knowledge**: The remembering, either by recognition or recall of ideas, material, or phenomena (p. 62).
(2.0) **Comprehension**: The objectives, behaviours, or responses that represent an understanding of the literal message contained in a communication (p. 89).
(3.0) **Application**: Given a problem that is new to the student, the individual correctly selects and applies an appropriate abstraction without having to be prompted (p. 120).
(4.0) **Analysis**: The breakdown of the material into its constituent parts, and detection of the relationships of the parts and the ways in which they are organized (p. 144).
(5.0) **Synthesis**: The putting together of elements and parts so as to form a whole. This is a process of combining the elements or parts into a new pattern or structure (p. 162).
(6.0) **Evaluation**: The use of quantitative or qualitative criteria (or standards) to appraise the extent to which particulars are accurate, effective, economical, and satisfying (p. 185). (adapted from Bloom, 1956)

Retrospectively, Bloom's (1956) Taxonomy influenced change in curriculum and testing internationally. Prior to the widespread publication and distribution of the Taxonomy, practitioners produced curricula and tests through an epistemology that emphasized facts and their accurate recall (Postlethwaite, 1994). However, the Taxonomy provided teachers, curriculum developers, and test designers with a common language and simple structure to expand the learning of students into meaningful, differentiated, and sequentially ordered educational objectives. Teachers in particular found the Taxonomy to be of great practical value, because it made intuitive and logical sense and was easily applied in the hither and thither of complex classrooms and schools. For example, **Figure 6·2** represents a clear and understandable generalized problem solving process. The first four steps relate essentially to Application (3.0), and Step 5 and Step 6 to Comprehension.

However, the authors of the Taxonomy recognized that an Aristotlean-type taxonomy "must be validated by demonstrating its consistency with the theoretical views in research findings of the field it attempts to order" (Bloom, 1956, p. 17). Prospectively and importantly for powerful mathematical learning, the linearity of the hierarchical structure has been questioned (Anderson & Sosniak, 1994; Anderson & Krathwohl, 2001; Marzano & Kendall, 2007). In particular, Madaus, Woods, and Nuttal (1973) could not establish a causal link between Analysis and Synthesis in their empirical study.

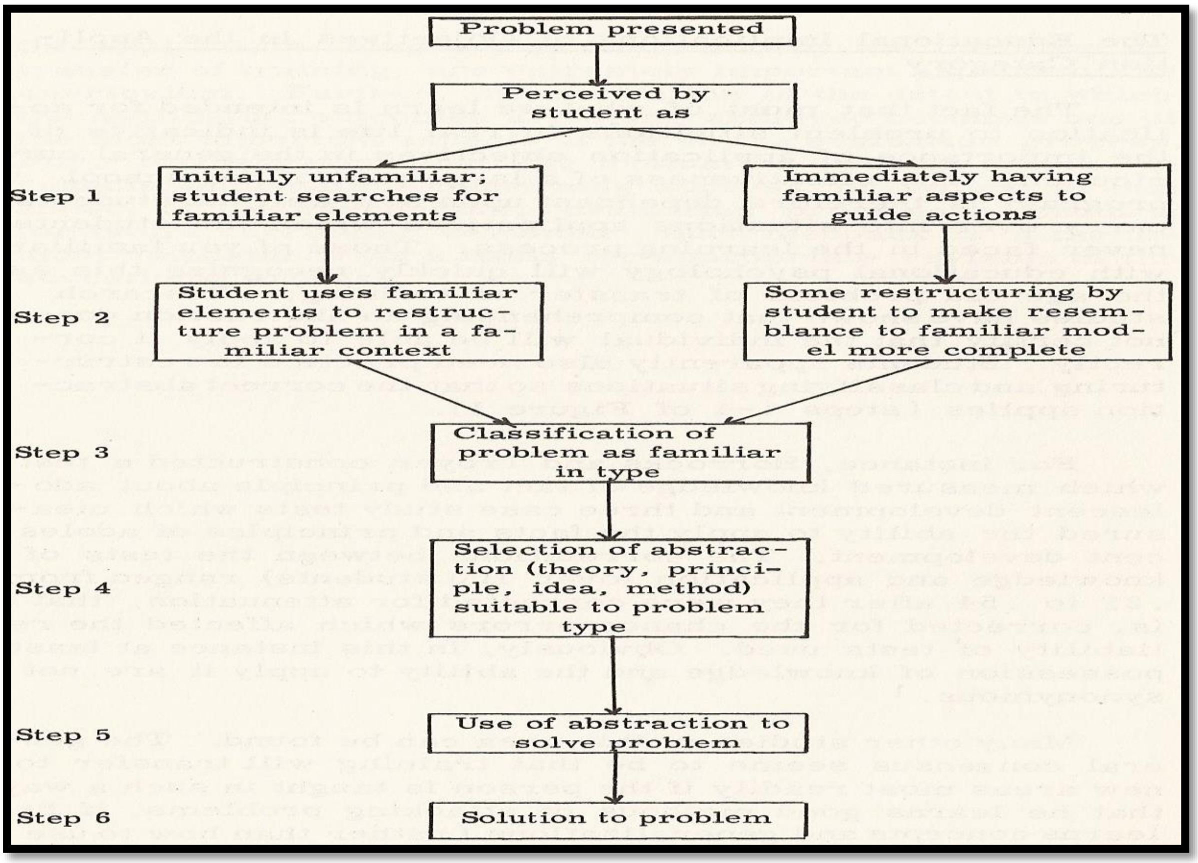


Figure 6·2. The problem solving process when answering questions in the Application category (Bloom, 1956, p. 121; Anderson & Sosniak, 1994, p. 22)

This 'finding' might indicate why the highly acclaimed Soviet mathematics teacher, Shatalov frequently oscillated between linear-analytic detail and global-synthetic perspective so that students could understand mathematics more meaningfully (Jensen, 2008). Moreover, a System I narrative is not automatically analysed for coherence and accuracy by System II thinking, because emotionally and intuitively the focus of interest is different, namely, Being-in-the-world is to 'live' primarily as an organism and not as a mechanism with separate parts. Therefore learning mathematics intuitively and analytically through left and right brain complementary functioning (Williams, 1983), appears to be more complex than is suggested by Bloom's (1956) linear hierarchical taxonomy.

The Revised Taxonomy for Learning, Teaching, and Assessing. By the end of the twentieth century it was evident to most educational scholars that Bloom's (1956) Taxonomy needed to be revised. The world had changed, and educating for the future was inconsistent with school systems that were based on the factory model made popular between 1900−1920 by the Ford Motor Company's assembly line of organized mass production (Darling−Hammond, 2010; Snow, 2013). In other words transmission oriented curricula that largely ignored individual differences and creativity were no longer in step with the way in which humanity was communicating and globalizing. Therefore if school systems were to be relevant to the majority of their constituents, then students had to learn problem solving that involved "incompleteness, anomaly, trouble, inequity, and contradiction" (Bruner, 1971, p. 111).

Consequently, Anderson and Krathwohl (2001) revised Bloom's (1956) Taxonomy of Educational Objectives. The contributors included cognitive psychologists, curriculum and instructional theorists, and testing and assessment specialists. The objectives were expounded as ends not means — intended results, outcomes, or changes — facilitated through a matrix structure that interrelated Knowledge as product and Cognition as process. The two-dimensional structure reflected the complexity of learning as an interaction between different kinds of sequential process activities and the holistic embodying of those activities.

Therefore students were viewed as active learners who had agency and authority in the learning situation, and were accountable for their own constructive understandings. In addition, meaningful learning occurred when the individual made sense of a problem situation in the context of a vibrant or robust social dynamic. This view of the active learner is consistent with a powerful learning environment where the development of **knowledge** is likely to take place in four phases. These include (a) the activation of prior knowledge and experiences; (b) the teacher is responsible for scaffolding a learning framework that does not pre-empt or limit learning as a relational or creative event; (c) cognition is embodied and therefore each student is considered a cognitive apprentice and given the opportunity to apply his or her knowledge and skills; and (d) the individual student is expected to integrate existing and new learning coherently by grappling purposefully with authentic real-world activities (Collins, Brown, & Newman, 1989; Könings, Brand−Gruwel, & van Merriënboer, 2005).

Consequently, Bloom's (1956) Knowledge category was expanded substantially beyond recognition and recall to emphasize different kinds of knowledge, especially the conceptualization of knowledge through procedures and processes (also referred to as reified or objectified knowledge). This is particularly important for the learning of mathematics, because from the eighteenth century forward the study of mathematics has not only involved process but also holistic structures that have been understood as **things**. As a result the Knowledge dimension was broadened through the use of four different adjectives, namely,

A. **Factual** Knowledge: The basic elements students must know to be acquainted with a discipline and solve problems in it;
B. **Conceptual** Knowledge: The interrelationships among the basic elements within a larger structure that enable them to function together;
C. **Procedural** Knowledge: How to do something, methods of inquiry, and criteria for using skills, algorithms, techniques, and methods; and
D. **Metacognitive** Knowledge: Knowledge of cognition in general as well as awareness and knowledge of one's own cognition, especially the control, monitoring, and regulation of cognitive processes.
(adapted from Anderson & Krathwohl, 2001, pp. 43, 46)

Bloom's (1956) Taxonomy was uni-dimensional in six classes of intended student behaviours. However, Anderson and Krathwohl (2001) recognized that this confounded process and product. Therefore the Revised Taxonomy included a second dimension, namely, the **Cognitive Process Dimension** which articulated process objectives along the following continuum, namely,

(1) **Remember**: Retrieve relevant information from long-term memory;
(2) **Understand**: Construct meaning from instructional messages, including oral, written, and graphic communication;

(3) **Apply**: Carry out or use a procedure in a given situation;
(4) **Analyze**: Break material into its constituent parts and determine how the parts relate to one another and to an overall structure or purpose;
(5) **Evaluate**: Make judgements based on criteria and standards; and
(6) **Create**: Put elements together to form a coherent or functional whole; reorganize elements into a new pattern or structure.
(Anderson & Krathwohl, 2001, p. 31)

Each of the six categories are verbs. This is consistent with the educational understanding that the Revised Taxonomy was designed to portray meaningful learning as an intentional and constructivistic act on the part of the student. However, interpreting the Revised Taxonomy in terms of powerful mathematical learning means that students should primarily attend school to engage with the process of creativity, which necessarily subsumes the cognitive processes of 'remembering, understanding, applying, analysing, and evaluating'.

Therefore to learn by **creating** is to learn holistically. This implies engaging with structured and unstructured tasks that require 'interpreting, exemplifying, classifying, summarizing, inferring, comparing, and explaining', towards the goal of 'differentiating, organizing, attributing, checking, and critiquing' the generation and product that is 'new' knowledge for the individual. However, this activity-in-process would not be possible apart from the four categories of the Knowledge dimension. Therefore, powerful mathematical learners move bidirectionally between product and process to address real world problems meaningfully or creatively. For example in **Table 6·2**, the development of conceptual knowledge is possible if the problem situation is understood, but to grasp the situation conceptually is to deepen one's understanding of the problem, thereby increasing the likelihood of a creative solution.

Moreover, the Revised Taxonomy matrix enhanced Bloom's (1956) Taxonomy by placing Understanding within a structure that augmented a literal comprehension of basic facts, together with an analytic evaluation of knowledge that fostered a metacognitive self-awareness, namely, through a reflective application of skills and concepts especially when solving novel problems. However, Kirby and Lawson (2012b) contended that in classes around the world it is assessment that drives teaching and learning rather than meaning making and understanding (Kirby & Lawson, 2012b).

Table 6·2: The Revised Taxonomy Knowledge−Process Matrix (adapted from Anderson & Krathwohl, 2001, p. 28)

The Knowledge Dimension	The Cognitive Process Dimension					
	1. Remember	2. Understand	3. Apply	4. Analyze	5. Evaluate	6. Create
A. Factual knowledge						
B. Conceptual knowledge						
C. Procedural knowledge						
D. Meta-cognitive knowledge						

But importantly for powerful mathematical learning, the Revised Taxonomy can be used as an assessment tool to promote 24 possible Knowledge−Process interactions, understandings, or *I*-positions with an emphasis on the Create dimension. Mathematics teachers can then make more holistic pedagogical decisions as to how student outcomes can be improved, enriched, or aligned cognitively. Thus phenomenologically each of the matrix entries need in some way to reflect the whole that is powerful mathematical learning, because "what is essential in all philosophical discourse is not found in the specific propositions of which it is composed but in that which, although unstated as such, is made evident through these propositions" (Heidegger, 1962, p. 206).

However, the framework of the Revised Taxonomy does not include an affective dimension and is therefore limited as an instructional and learning tool towards the goal of developing powerful learning environments, because as a mind in society "behind every thought there is an affective−volitional tendency" (Vygotsky, 1986, p. 252). Nonetheless, the matrix entries involving Meta-Cognitive Knowledge can be construed as meta-positions, namely, those positions that "take a broader array of specific *I*-positions into account and have an executive function. As mediated by higher cortical brain activity, they have the potential to influence the lower emotional circuits of the brain so that long-term planning becomes possible" (Hermans & Gieser, 2012).

Bloom's Taxonomy of Educational Objectives: The Affective Domain. In the Decade of the Brain (1990−1999), McLeod (1992) proposed in what was to become a seminal publication, that affect enabled the learner to progress "beyond the domain of cognition" (p. 576). If he was correct, then this would have significant pedagogical implications for higher levels of human reasoning and Being-mathematical. Consequently, numerous researchers broadened the ambit of mathematics education research by inquiring into

> anguish, anxiety, attitudes, autonomy, beliefs, confidence, curiosity, disaffection, dislike, emotions, enthusiasm, fear, feelings, frustration, hospitality, interest, intuition, moods, panic, perseverance, sadness, satisfaction, self-concept, self-efficacy, suffering, tension, viewpoint and worry. (Walshaw & Brown, 2012, p. 185)

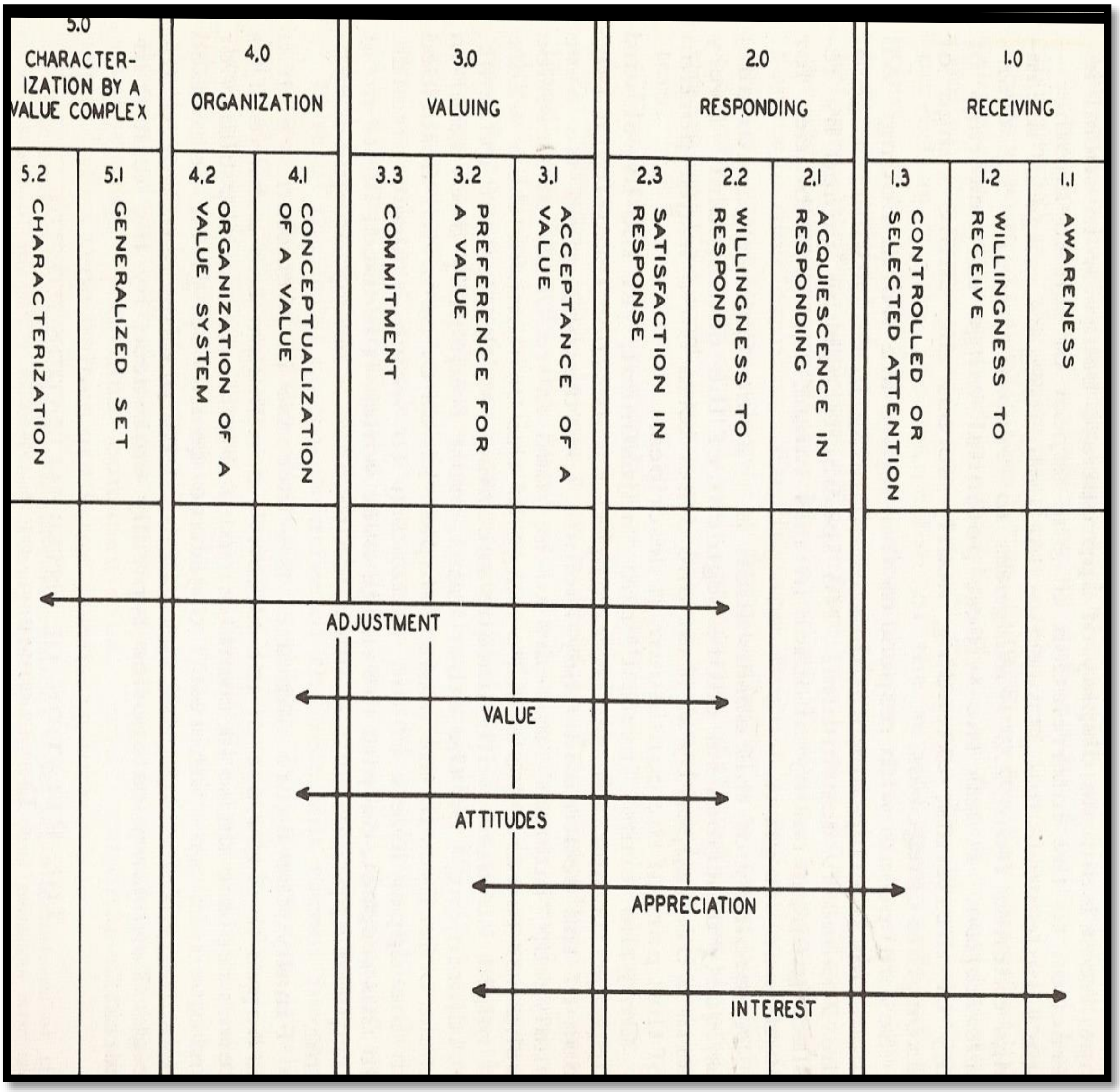
The research results are significant for powerful mathematical learners. In particular, affect and cognition appear to be inextricably linked and mutually causal in the development of high quality process and knowledge schemata, or mental models (Aldous, 2006; Anderman & Wolters, 2006; Damasio, 2005; DeBellis & Goldin, 2006; Goldin, 1998, 2004, 2007; Goldin & Kaput, 1996; Kahneman, 2011). But it is the teacher who is crucial in influencing the development of students' emotional structures towards the goal of successful novel problem solving in mathematics (Askell−Williams, Lawson, & Skrzypiec, 2012; Darling−Hammond, 2010; Fischbein, 1999; Wilson & Cooney, 2002). In essence, teaching for powerful mathematical learning means instructing students, or role modelling how to establish mental represesentations that are high in positive affect and capability to make connections between objects or ideas that on the surface might have little in common (Bruner, 1966, 1979).

In so Being, the teacher counters the internalization of values and self-representations by the student that might cause resistance or indifference to meaningful learning (Roeser, Peck, & Nasir, 2006). Then ultimately, that which the teacher values will influence the richness of the learning environment and the quality of the student's knowledge structures, because the mathematics teacher's values are in effect deeply rooted psycho-social beliefs that are acted upon repeatedly, consistently, and are invariably context independent in-relation to the being of the student (Bishop, Seah, & Chin, 2003).

However, the preeminent or most well known taxonomy that purports to articulate the complexity of affect in education remains Bloom's Taxonomy of Educational Objectives for the Affective Domain (Krathwohl, Bloom, & Masia, 1964). The Affective Domain Taxonomy (ADT) as portrayed in **Table 6.3**, provides a comprehensive framework of the categories and subcategories of the affective domain that can guide the mathematics teacher's instruction, pedagogy and dialogue in the classroom.

The ADT was developed in recognition of the 'classroom fact' that student learning is influenced by "a feeling tone, an emotion, or a degree of acceptance or rejection" (Krathwohl, Bloom, & Masia, 1964, p. 7) in relation to the Being of the teacher and that which is being taught or communicated. Consequently, the ADT represents an unfolding continuum of affect as the student **Receives**, **Responds**, **Values**, and **Organizes** new learning, to the stage where the internalization might hold such importance in the person's life that it pervades or underpins his or her Being-dialogical.

Table 6·3. The range of meaning typical of commonly used affective terms measured against the Affective Taxonomy continuum (Krathwohl, Bloom, & Masia, 1964, p. 37)



Importantly for powerful mathematical learners however, Bloom's (1956) Taxonomy of Cognitive Objectives can be interrelated with the Affective Domain Taxonomy, because cognition is substantially enhanced when intertwined with positive affect and intent (LeDoux, 1995, 1996; Goldin, 2004). **Table 6·4** relates the categories of the two taxonomies in terms of an increasing complexity.

Therefore the primary aim of powerful mathematical learning is to develop an ethical complexity of values that can enable the Being of the individual to be creative and dialogical in a globalizing world. Towards this aim, the Cognitive and Affective Taxonomies can be used by teachers to facilitate the progressive educational belief "that innate powers of youth should be released to provide the creative direction for schooling" (NCTM, 1970, p. 119).

Table 6·4. Relations between the categories of the Cognitive and Affective Taxonomies (adapted from Krathwohl, Bloom, & Masia, 1964, pp. 49−50)

Cognitive Categories of Bloom's (1956) Taxonomy	Affective Categories of Krathwohl, Bloom, and Masia's (1964) Taxonomy
1. The cognitive continuum begins with the student's recall and recognition of **Knowledge (1.0)**.	**1.** The affective continuum begins with the student **Receiving (1.0)** stimuli and passively attending to it. It extends through active attention, namely,
2. It extends through his or her **Comprehension (2.0)** of the knowledge;	**2.** his or her **Responding (2.0)** to stimuli on request, willingly responding to these stimuli, and taking satisfaction in this responding;
3. his or her skill in **Application (3.0)** of the knowledge that is comprehended;	**3.** his or her **Valuing (3.0)** the phenomenon or activity so that the individual voluntarily responds and seeks out ways to respond;
4. his or her skill in **Analysis (4.0)** of situations involving this knowledge;	**4.** his or her **Conceptualization (4.1)** of each value responded to;
5. his or her skill in **Synthesis (5.0)** of this knowledge into new organizations, and	**5.** his or her **Organization (4.2)** of these values into a value system, and
6. his or her skill in **Evaluation (6.0)** for the purpose of judging and valuing material and methods in a particular area or field of knowledge.	**6. Characterization (5.0)** of Being-human through a core complexity of values.

Thus in an abstract Vygotskyan sense, the Cognitive and Affective Taxonomies combine to represent the growth of student interest and appreciation, motivation, and Being from the interpersonal psychological plane of Receiving and Responding, to the intrapersonal psychological plane of Valuing. It is noteworthy that Receiving, Responding and Valuing are all present continuous verbs in relation to a procedure, process, or object which is described phenomenologically as “what one does to it,” (Bruner, 1966, p. 12) either actually or potentially.

However, what is crucial for the powerful learning of mathematics is that Receiving, Responding and Valuing need to develop into mental attitudes that are structured with insightful mathematical capacity (Fischbein, 1987, 1999). It is these affective–cognitive schemas of mind that empower Being-mathematical in both cognitive and non-cognitive terms. In other words the embodied affective–cognitive *I*-competencies constitute well organized dispositions of mind that are essentially transformative and visual. Although limited, their understanding of affect in relation to cognition was visionary on the part of Krathwohl, Bloom, and Masia (1964), but their vision is still in need of development and expansion.

The New Taxonomy of Educational Objectives. The developers of Bloom’s Cognitive and Affective Taxonomies recognized, as did ancient Greek philosophers, that learning is basically a tripartite interaction, or ‘alliance’ between the domains of cognition, affect and psychomotor skills. However, Bloom and his colleagues did not manage to unite the three domains into a coherent and comprehensive taxonomy, nor did they construct a well defined taxonomy for the psychomotor domain (Chiarelott, 2006). To date no single psychological theory adequately explains human learning (Anderson & Krathwohl, 2001; Kirby & Lawson, 2012a).

Nevertheless, any theory of learning that is holistic needs to address Being- human as a stimulus–response, question–inquiry social organism, because people do not function, engage, or respond in parts but as whole organisms (Krathwohl, Bloom, & Masia, 1964). From an enactivist, or embodied constructivist perspective for example, Reid (1996) argued that

> it is not a matter of an individual having a cognitive structure, which determines how the individual can think, or of there being conceptual structures which determine what new concepts can develop. The organism as a whole *is* its continually changing structure which **determines** [for emphasis] its own actions on itself and its world. This holistic vision of the cognitive entity is central. (as cited in Ernest, 2010, p. 43)

This view is consistent with Goethe's (1749−1832) organic vision of intellect and intuition (Bortoft, 1996). The German philosopher's perspective of mind and Being was influenced by his intuitive understanding of The Natural−Physical World (World 1) which for him, was a holistic network of relationships rather than a mechanistic focus on the individual parts. Similarly, Einstein's theory of general relativity is an expression of the interconnectedness of the universe in relation to him Being-in-the-world (Einstein, 1952; Gardner, 1993).

As part of this broad church of human sentiment, individuals like Hauenstein (1998) attempted to construct frameworks of educational objectives that were inclusive of the cognitive, affective, and psychomotor dimensions of Being. However, it is the New Taxonomy of Marzano and Kendall (2007) — 'as a work in progress' — that reflects a current theme in (Mathematics) Education, and that is to understand teaching and learning integrally and systemically rather than engaging primarily with specific aspects of human growth and development.

A healthy system. It should not be assumed that teaching and learning is a complex system. However, teaching and learning is always a socially, culturally, politically, and historically situated event. A complicated interaction between ordered and disordered behaviour that involves the Three Worlds.

However, for the purpose of differentiating between systems and non-systems the following five criteria were proposed as key identifiers as to whether the individual parts functioned systemically, or were 'just a bunch of stuff', namely,

(1) Can the parts be identified clearly?
(2) Do the parts influence each other?

(3) Do the parts together produce an effect that is different from the effect of each part on its own?
(4) Does the effect, the behaviour over time, persist in a variety of circumstances?
(5) If A causes B, is it possible that B also causes A?
(adapted from Meadows & Wright, 2008, pp. 13, 34)

Yet these five criteria do not imply a healthy system. Essentially if a World, or a complex system is to be healthy then (a) the many parts 'work together' and eventuate a whole that is more than the sum of the parts; (b) the parts are like 'agents' are influenced by memory or feedback; (c) although the whole is organic, or combines in a certain order, the agents can adapt their strategies according to their prior learning, or respective histories; (d) the whole is constituted in terms of achievable goals that are realized, at least in part, without a central controller; (e) the whole is open to new inputs and outputs that sustain and benefit the whole; (f) the whole becomes increasingly complex as it turns inputs into outputs, with the result that emergent outcomes might be surprising, or even extreme; and (g) fundamentally the whole absorbs and generates different kinds of energy that contribute to the overall health of the system (adapted from Higgs & Smith, 1996; Johnson, 2009).

Therefore since Being-mathematical is irreducibly complex, powerful mathematical learning is complex no matter the socio-cultural situation, and consequently, teaching for 'a healthy system' of powerful mathematical learning always entails implementing holistically and flexibly the (a) – (g) complexity science principles.

A taxonomy of learning systems**.** As indicated in **Figure 6·3**, the New Taxonomy comprises an interaction between three component systems or parts, namely, the Self-system, the Metacognitive system and the Cognitive system. The Self is the most complex of the three systems, because it is emergent in all the relationships that involve the individual person. Consequently when a New Task is presented to the Self, the person makes a decision to engage with the task by Being-there. The choice and intensity of response depends on at least three primary factors. First, the individual examines the importance of the task in relation to his or her personal needs, learning goals, and self-actualization (Holzknecht, 2007; Maslow, 1970). Second, the self-efficacy of the person answers the question, "Am I capable of meaningfully addressing the task or problem given the human, social, and technological resources at hand?" Third, the student responds emotionally based on the richness of his or her prior learning, experience, and interest in that particular moment. Taken together, the three factors intuit the student's level of motivation to proceed with his or her current behaviour, or to learn something new.

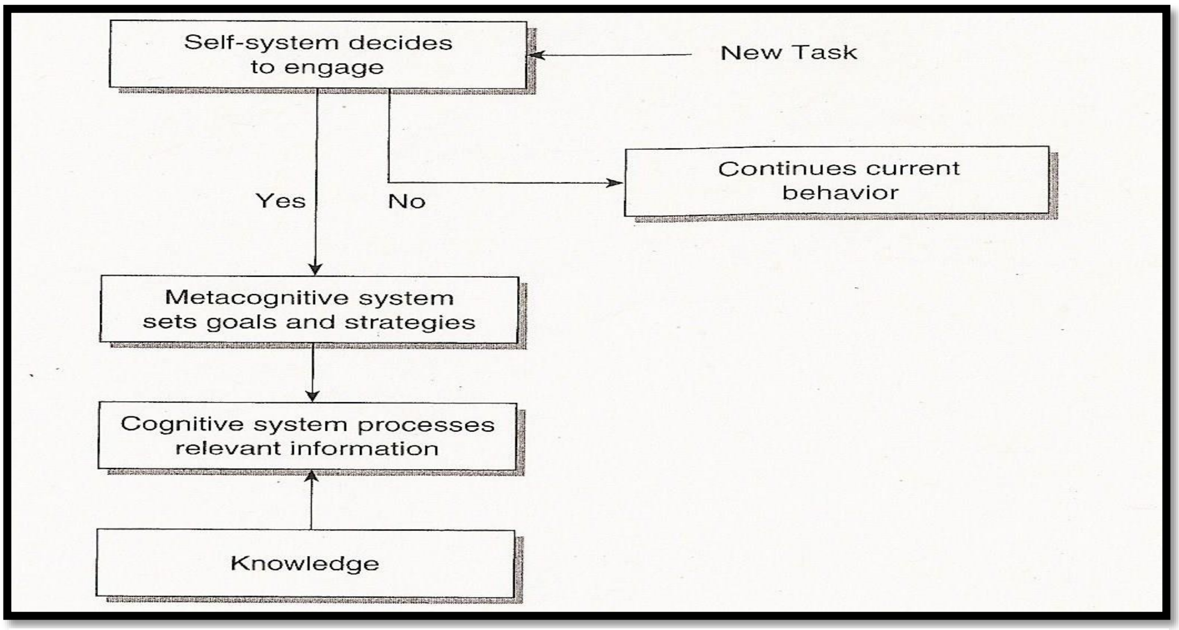


Figure 6·3. The New Taxonomy: A Systems Perspective of Learning (Marzano & Kendall, 2007, p. 11)

Therefore the Self-system is most important in influencing the amount of energy that the mind, or World 2 of the individual is willing to expend on the New Task. If the learner decides to engage with the task and on what basis, then it is through the Metacognitive system that the individual exercises executive control in monitoring, evaluating, and regulating his or her mind–body activity. In terms of metaphor or analogy, the Metacognitive system can act like a chemical catalyst to lower the activation energy required to develop, or juxtapose an intuitive–analytical solution, or 'new learning' which is a System I–System II embodied enaction. In contrast, or paradoxically brain behavioural and neuroimaging studies have indicated that mathematically gifted students appear to be capable of 'heightening'

> brain activation, approximating (or exceeding) that of an adult brain even though they are still adolescents, which is suggestive of enhanced processing power and may reflect highly developed attentional and executive functions that serve to fine-tune their unique form of cerebral organization. (O'Boyle, 2008, p. 184)

Nevertheless, it is the Metacognitive system that in effect systematizes the (efficient) working of the Cognitive system through the setting of goals and strategies, which then result in the Cognitive system drawing on the different Domains of Knowledge (informative or procedural). Thus with reference to **Figure 6·4**, it is the Metacognitive system that plays a key executive role in enabling the four Levels of Cognitive Processing, namely, Retrieval from long-term memory, Comprehension, Analysis, and Knowledge Utilization.

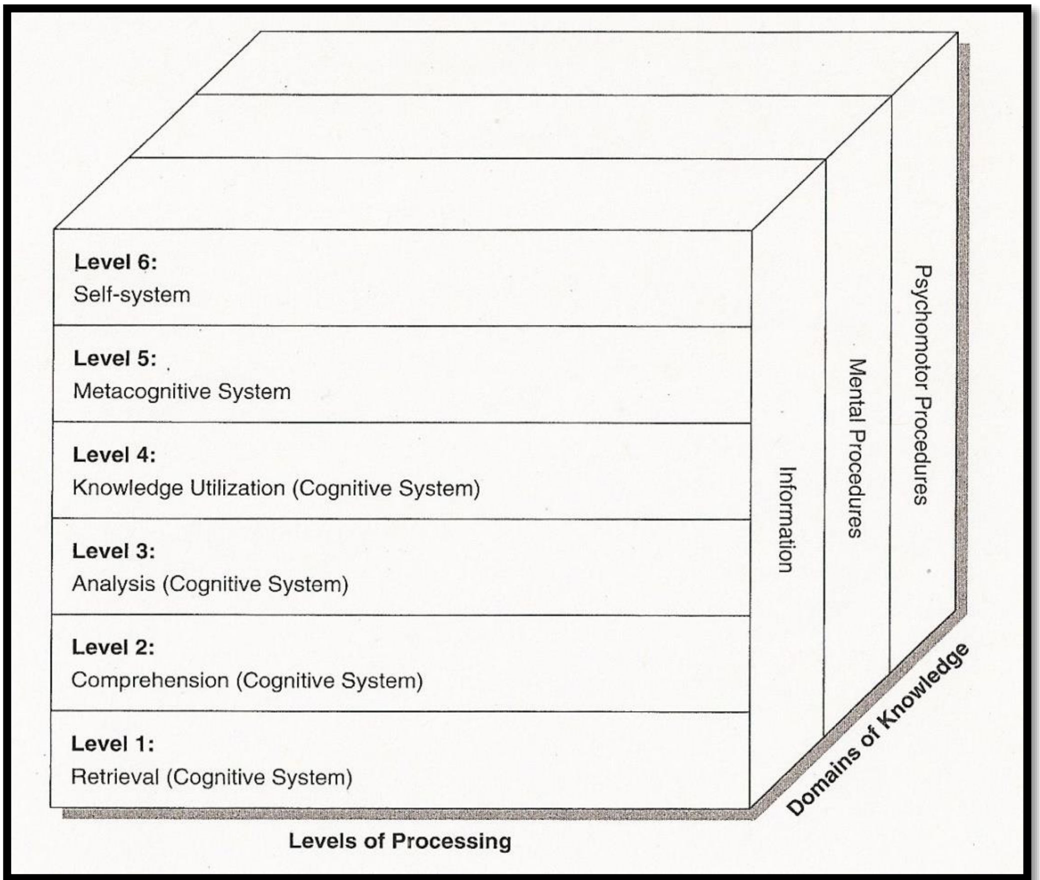


Figure 6·4. The New Taxonomy: A hierarchy of the Self that involves six levels of processing across three domains of knowledge (Marzano & Kendall, 2007, p. 13). In particular, Level 6 refers to the system that enables, generates, or influences the embodied and extended self, but this does not mean that the Self is itself a supersystem. It is certainly a complexity, but it may be too large and complex to function as a system.

From the systems' perspective of the New Taxonomy therefore, powerful mathematical learners are sufficiently and meta-cognitively self-aware in Being-able to conceptualize the utilization of knowledge for the purpose of Decision Making, Problem Solving, Experimenting, and Investigating. In particular however, the quality of the Level 4 Knowledge Utilization processes is dependent on (a) the amount of pertinent and correct information that is recognized and recalled accurately, including the fluent execution of procedures; (b) the integration, symbolization, and transformation of learning through psychomotor and intuitive functioning; and (c) the logical analysis of that which is intuited by correctly matching, classifying, analyzing for errors, generalizing, and specifying particular cases, consequences, and parameters. In short, the quality of the student's three Knowledge Domains determines to a large degree the capability that the individual has to engage mathematically in the Three Worlds of Being-mathematical.

But the arrows in Figure 6·3 are all uni-directional which denies the structure of static−dynamic complex systems (Camazine et al., 2001; Gazzaniga, Ivry, Mangun, & Steven, 2009; Rupert, 2009). Thus the New Taxonomy systems-view would likely limit powerful mathematical learning in classrooms, especially with respect to disparate Knowledge Utilization and creativity. Chaos theory and complexity science, as well as critical realist phenomenology all embrace the fundamental idea that the parts interact as an expression of the whole. In other words a healthy system is like a polyphony of voices, or even a symphony (Bakhtin, 1981, 1986; Pink, 2005). The Portuguese poet and philosopher Pessoa (1888−1935) wrote in *The Book of Disquiet*:

> My soul is like a hidden orchestra; I do not know which instruments grind and play away inside me, strings and harps, timbales and drums. I can only recognize myself as symphony. (as cited in Damasio, 2012)

Piaget as a developmental psychologist noted that human behaviour was never purely cognitive or purely affective, but rather behaviour was always cognitive in its affect, and vice versa (Rosenberg & Hovland, 1960; Tomei, 2005). Therefore if the New Taxonomy is to be a realistic systems model, then all arrows should be bi-directional but not necessarily of equal influence or importance depending on the New Task. Even the link between the New Task and the Self-system needs to be bi-directional, because in terms of an extended self both the individual self and the object can be developed, or modified relationally. This is especially the case when solving novel problems that require an objectified change in perspective or mathematical structure to be successful.

Furthermore, neurobiological evidence has indicated that the cognitive aspects of learning, focused attention, the different kinds of memory, as well as executive and social functioning are not only affected by emotion in the brain but are subsumed within processes called socio-emotional thought (Adolphs, 1999, 2004; Blakemore & Frith, 2005; Immordino−Yang & Damasio, 2007; LeDoux, 1995, 1996). Therefore the New Taxonomy is understood as an integrated social−emotional event that occurs interactively through three primary systems, namely, the Self-system, the Metacognitive system, and the Cognitive system.

However, the three systems have not been arranged in order of increasing complexity or difficulty, because which system is more complicated is not easily ascertained, nor is such information particularly relevant to powerful mathematical learning. What matters is how the different systems and processes interact with one another in order to facilitate an increasingly complex and knowledgeable whole that is Being-mathematical. From the perspective of the New Taxonomy however, cognition is subordinate to the executive functioning of the pre-frontal lobes, which in turn are largely under the control of the Self-system through an embodied mind that has internalized or self-developed beliefs, attitudes, values, and other affect such as moods, preferences, and mental predispositions.

The Technology Domain. None of the aforementioned taxonomies have foregrounded the **vital** relationship between humans and technology. In a phenomenological embodied sense, existence in a globalizing world is technologically textured because most humans live and move and have their essence in-relation to diverse technologies that are an expression of who we are and might become in terms of a rising tide of change (Ihde, 1990, 2009). In 1996 a special issue of *Scientific American* was published in which authors predicted certain key technologies that would have a global impact on life in the twenty-first century. **Table 6·5** highlights just eight of those key technologies — all of which have a presence in our world today.

Therefore recent history suggests that powerful mathematical learning needs to prepare students and teachers for a decidedly technological world. In particular however, technology has the potential to change the teacher‒student dynamic remarkably. A case in point involves the developing field of Artificial Intelligence. Language, logic, and learning are currently being incorporated into a broad cognitive science perspective that empowers robots to interact with teachers and students towards the realization of educational goals. In this regard for example, the **Talking‒Thinking‒Teaching‒Head** is an international project that is designed to enhance human‒machine communication (Powers, 2013).

Table 6·5. Technologies of the future

Key Technologies	Designing the Future
1. Microprocessors	Every 18 months micrprocessors double in speed. By the year 2020, one computer would be as powerful as all those in Silicon Valley (Northern California) today (Patterson, 1996, p. 1).
2. Virtual Reality (VR)	VR would transform computers into extensions of our whole bodies (Laurel, 1996, p. 19).
3. The Automobile: Clean and Customized	Built-in intelligence would let automobiles tune themselves to their drivers and cooperate to get through crowded traffic systems safely (Zetsche, 1996, p. 31).
4. Health and Wellbeing	Medical advances would challenge thinking on living, dying, and Being-human (Caplan, 1996, p. 77).
5. High-Temperature Superconductors	Electric current can be conducted without resistance (Chu, 1996, p. 105).
6. Robotics in the Twenty-first Century	Automotons are likely to find work as subservient household help (Engelberger, 1996).
7. **Fusion**	Energy derived from fused nuclei might become widely used by 2050 (Furth, 1996, p. 121).
8. The Emperor's New Workplace	Information technology would probably evolve more quickly than behaviour (Zuboff, 1996, p. 151).

Furthermore, Keeves (2004) called for curriculum reform in mathematics and science education by including technologies that involved recent modes of human thinking. Even though the world is globalizing technologically, the teaching and learning of mathematics and science in many classrooms has not kept pace with the **processes of mind** that have given rise to the new technologies (Keeves & Darmawan, 2010). In other words in Being-mathematical, it is important for Conceptual Age learners to experience through technology the processes of mind that in effect designed and built present day technologies, and in so Being, lay the foundation for future possibilities of 'Being-technological' and Being-human. However, because of globalisation, it is only the **wholeness** of the educative process as a result of global pedagogies and technologies that will "transform the thinking of large bodies of people to work together to provide the changes necessary to overcome the challenges that confront the human race during the twenty-first century and beyond" (Keeves & Darmawan, 2010, p. 27).

Learning technologies. Through the panorama of Whitehead's process−relational philosophy, 'learning technologies' are those that afford teachers and students the opportunity to understand and develop in the key ideas of their time, or epoch. Towards this goal, consider the TPACK (Technological Pedagogical and Content Knowledge) framework or model as depicted in **Figure 6·5**. Both Australian and American teachers have used the model in an attempt to expedite, enrich, and enhance their pedagogical practice and content knowledge with new technologies. However, the meaningful integration of technology in the teaching and learning of mathematics and science has not been straightforward.

In a factor analysis study of 596 teachers across the United States, the only TPACK domain that distinguished itself clearly was that of technology (Archambault & Barnett, 2010). Therefore technology rich environments might not facilitate learning, other than learning how to use the technologies themselves (Martin & Pirie, 2003). Thus in possibility, technology in classrooms might be counterproductive, or even counterintuitive to meaningful and goal-oriented learning, especially if the use of technology impedes or delimits human potential and movement in-relation to the Self (Wilson, 1998, 1999; Jensen, 2000b).

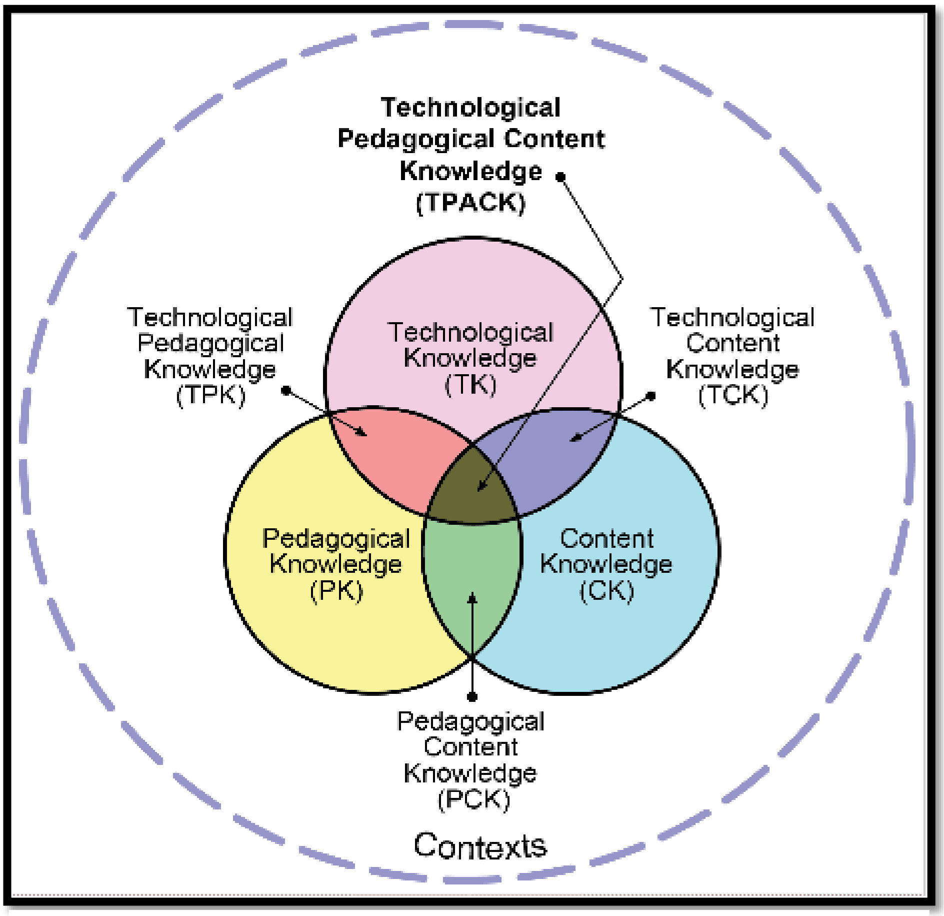


Figure 6·5. The TPACK Model for the meaningful integration of technology in classrooms and other learning environments (Source: http://tpack.org).

Nevertheless, new technologies that progress the differentiation and advancement of teaching and learning spaces are indispensable, if powerful learning environments are to be mediated for the Conceptual Age (Mukhopadhyay, 2009; Tomlinson & Kalbfleisch, 1998). However, if a mathematics teacher appropriates different technologies with the following four factors in mind, then students will probably learn what is intended, because the mentioned factors have been evidenced empirically, and are consistent with epistemic learning principles (Hattie, 2009, 2012; Winne & Nesbit, 2010). **First**, the teacher should make use of a variety of teaching strategies so that students have multiple opportunities to process the learning event cognitively and metacognitively. **Second**, the teacher is pre-trained in the use of the technology as a teaching and learning tool. **Third**, the teacher encourages the students to work as individuals and in small groups. **Fourth**, the teacher is alert to student attempts, comments, and body language for the purpose of Being-able to provide insightful and timely feedback.

Therefore if computer-based technologies (e.g., Autograph, Geometer's Sketchpad, MATLAB, engineering kits, 3-D printers, and internet and social media technologies) are integrated meaningfully with teachers' embodied processes-in-action, then it is plausible that learning environments can be created that promote the growth and development of sophisticated schema in the minds of students, for the express purpose of addressing new problems in new contexts powerfully (Schwartz & Schmid, 2012; Tall, 2000). In order to facilitate such outcomes in classrooms and schools around the world, the International Society for Technology in Education (ISTE) was established in 2000. Subsequently, the National Educational Technology Standards and Performance Indicators for students, teachers, and administrators were released (ISTE, 2012).

The National Educational Technology Standards for Students (NETS·S) encouraged all students to understand different technologies conceptually with the expectation that learners would then become functional Digital-Age, or Web 2.0 citizens. In so Being, individuals would most likely be inspired to learn for themselves and others. However, in an age of global and digital connectedness learning necessarily involves inter-cultural sensitivities, communication, collaboration, and problem solving. The problems being addressed are in many instances unfamiliar to the individual. Nonetheless, Being-able to think critically and insightfully through the use of technology requires efficient information gathering and processing so that project management and decision making can be timely and effective.

But if students are to learn authentically using diverse technologies, the National Educational

Technology Standards for Teachers (NETS·T) recognized the vital role that all teachers need to play. Especially, it is crucial for classroom teachers to engage in professional development programs that facilitate effective pedagogy through the use of contemporary digital tools and resources. It is however, the responsibility of each mathematics teacher to empower him or herself in these terms so that creative learning environments can be established.

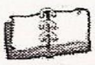
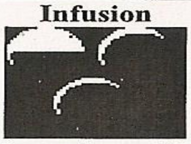
Realistically though, teachers do require support especially from administrators. Thus, the National Educational Technology Standards for Administrators (NETS·A) advocated that school administrators should adopt a visionary and systemic approach in the equipping of teachers and students with Digital-Age resources. In this regard, administrators are crucially situated to promote a learning culture in schools that is relevant to the life and times of the twenty-first century. However, finances in schools are often under strain. It is noteworthy therefore, an aim of powerful mathematical learning is to gain the most from the simplest apparatus (Whitehead, 1932; Quine & Ullian, 1970).

Tomei's Taxonomy. Although the ISTE Standards afford students, teachers, and administrators the freedom to sequence, focus, and develop their own understanding of technology in education, many teachers value the classification of technological objectives through a taxonomy, because "each step offers a progressive level of complexity by constructing increasingly multifaceted objectives addressing increasingly complex student learning outcomes" (Tomei, 2005, pp. 90−91).

With this intent the six levels of Tomei's (2005) Taxonomy for the Technology Domain (TTTD) are described in **Table 6·6**. In particular, if students are to dialogue and collaborate meaningfully using different technologies (Level 2.0), especially for the purpose of thinking critically in familiar and new situations (Level 3.0), then each student needs to be broadly computer literate so that the person can access, and interrelate different Digital-Age technologies synergistically (Level 1.0).

Furthermore, interactive technologies that interface the real world with the virtual world are very valuable for powerful mathematical learners, because these technologies can enhance 'self-teaching' as individuals or small groups test their ideas, or master concepts by engaging with standard or novel learning situations (Level 4.0). To foster self-directed learning in these terms there are extensive and free online resources available. For example, the PhET interactive simulations for the learning of mathematics, physics, chemistry and biology are likely to "train emotional self-confidence by assisting the person through learning experiences that build a sense of mastery" (Matthews, Zeidner, & Roberts, 2006, p. 171). The home of the PhET simulations is the University of Colorado in conjunction with Nobel physics laureate Wieman.

Table 6·6. Levels of the Taxonomy for the Technology Domain (Tomei, 2005, p. 90)

Taxonomy Classification	Defining the Level of the Technology Taxonomy
Literacy  **Understanding Technology**	**Level 1.0** The minimum degree of competency expected of teachers and students with respect to technology, computers, educational programs, office productivity software, the Internet, and their synergistic effectiveness as a learning strategy.
Collaboration  **Sharing Ideas**	**Level 2.0** The ability to employ technology for effective interpersonal interaction.
Decision-Making  **Solving Problems**	**Level 3.0** Ability to use technology in new and concrete situations to analyze, assess, and judge.
Infusion  **Learning with Technology**	**Level 4.0** Identification, harvesting, and application of existing technology to unique learning situations.
Integration  **Teaching With Technology**	**Level 5.0** The creation of new technology-based materials, combining otherwise disparate technologies to teach.
Tech-ology  **The Study of Technology**	**Level 6.0** The ability to judge the universal impact, shared values, and social implications of technology use and its influence on teaching and learning.

The many PhET simulations have been used across the world (translatable into any language using the Translation Utility) and at all levels — from primary school to tertiary level courses (Adams et al., 2012; Wieman, Adams, & Perkins, 2008). The PhET acronym stood for Physics Education Technology but is currently a well known acronym without a specific elaboration.

Furthermore, teachers can assist learners in mass education to become powerful mathematical learners by bringing together disparate technologies to address challenging problems creatively. However, Level 5.0 of Tomei's (2005) Taxonomy should not be taught apart from Level 6.0, because powerful learners need to appreciate that high level learning carries an ethical and social responsibility to benefit and to protect, principally when it comes to the use and development of new technologies. For example, although the arms and sex industries are of the most lucrative and forward thinking industries in the world today, both industries leave much to be desired when it comes to ethical and social responsibility.

A Model of School Learning

Students are taught and learn in classrooms that are situated in schools. The complexity of such an event in time and space reflects a present history that is unfathomable to a single human mind without the use of high level modelling procedures. However, Carroll of Harvard University constructed a systemic model of school learning with the claim "that this conceptual model probably contains, at least at a superordinate level, every element required to account for an individual's success or failure in school learning" (Carroll, 1963, p. 733).

The scope of Carroll's (1963) Model is particularly relevant to powerful mathematical learning because it relates to a specific learning task, namely, the understanding of an unfamiliar concept. In this regard there are five important factors; three of which are expressed solely in terms of time. First, the **aptitude** of the student to understand the concept is equated to the amount of time that the individual requires to master the learning task if the instruction received is consistent with best practice. Second, **perseverance** — the amount of time that the learner is willing to commit to the task at hand, and third, **opportunity** — the time available or allowed for learning. The other two factors are mutually dependent, namely, that the **quality of instruction** influences how efficiently the student learns in relation to his or her **ability to understand the instruction**. Consequently, the five factor model was expressed by Carrol (1963) using the following mathematical function:

$$\boxed{Degree\,of\,learning = f\left(\frac{time\ actually\ spent}{time\ needed}\right)}$$

The **quality of instruction** variable ranges from poor to optimal on a [0,1] interval scale; the **ability to understand instruction** variable is described on a standard score scale, that is, with a mean of zero and a standard deviation of unity. The **time actually spent** numerator is adjusted in terms of these two variables, and the **time needed** denominator is measured as the minimum of the following three quantities, namely, **opportunity, perseverance**, and **aptitude**. The detailed mathematics of Carroll's Model was developed in Carroll (1962).

The content demands of many mathematics curricula around the world, coupled with the inappropriate use of time means that for most students, the ratio of $\frac{time\ actually\ spent}{time\ needed}$ for mathematical learning is not sufficiently close to unity to realize a conceptual and creative understanding of mathematics (Bishop, 1989; Carroll, 1968; Kirby & Lawson, 2012b; Vinner, 2007; Stacey, 2010). With a focus on Caroll's Model therefore, the most important underlying variable likely to affect powerful mathematical learning is **time-on-task**, and this complex variable has historically been "the most strongly contested in curriculum planning" (Keeves, 2004, p. 287).

So if powerful mathematical learning is to occur in mass education there are at least three essentials. First, mathematics curricula should not be content-heavy (Kirby & Lawson, 2012b). As a case in point and from personal experience, the International Baccalaureate Diploma Programme (IBDP) is 'too busy' for the majority of mathematics students to develop deep learning that is intuitive and analytical. It is also noteworthy that in the *Review of the Australian Curriculum: Final Report,* concern is expressed that the Mathematics Curriculum might require students to cover too much content at the expense of understanding mathematics relationally and hierarchically. Moreover, it was argued that compared to the Japanese mathematics curriculum, the Australian Curriculum could be improved with respect to "sequencing, succinctness, timing of introduction of new concepts and clarity of identification of important mathematical concepts to be learned" (Donnelly & Wiltshire, 2014, p. 174). It is likely therefore that the structure of the Australian Curriculum would result in inefficient pedagogical practices in mathematics classrooms.

Second, mathematics teachers cannot be expected to create learning environments that motivate, and make it possible for students to learn to a high level efficiently without being trained in this regard. Third, even though the psychology of homework is "a complicated thing," (Corno, 1996, p. 27) both mathematics teachers and students need to view homework not as a burden, hindrance or punishment, but as a **regular** opportunity for learners to engage with carefully prepared, interesting, and challenging tasks (e.g., Olympiad-type or open-ended problems). Homework should not be overtaxing but scaffolded to mesh with, or reinforce classroom instruction, especially for the purpose of improving academic achievement (Cooper, Robinson, & Patall, 2006; Dettmers, Trautwein, Lüdtke, Kunter, & Baumert, 2010). That is through the intuitive compression of knowledge into intentional and emotional mental structures that can be interrogated for mathematical coherence and accuracy. However, all classroom and homework activity ought to be guided by the Pareto principle, because the "minority of causes, inputs, or effort usually lead to a majority of the results, outputs, or rewards" (Koch, 1998, p. 4).

Furthermore, the Carroll (1963, 1989) Model was informed by an **equality of opportunity** philosophy, rather than an **equality of attainment** ontic. It is noteworthy that although Bloom's (1968, 1981) concept of mastery learning was inspired by Carroll's Model (Block, 1971), as was a mathematical model for mastery learning (Aldridge, 1983), Bloom's mastery learning appeared to emphasize the uniform attainment of specific outcomes rather than the quality of the learning opportunity. As a consequence for many students in the United States — especially in the 1970s — time-on-task was reduced to copious amounts of rote learning which often resulted in an instrumental mastery of that which was practised (Bloom, 1968, 1981).

Prospectively, a philosophical understanding of Carroll's Model is for powerful mathematics teachers to focus on the development of each student's potentialities through appropriately differentiated instructional designs. If in contrast the teaching goal is for all students to master the same material equally, then student learning is likely to attain 'being mathematical but not Being-mathematical'. Therefore teaching for powerful mathematical learning is in harmony with the sentiment that

> educational programs should be devised and selected to permit students to travel as far as possible towards realizing their capabilities for learning. There should also be continual reassessment of potentialities with corresponding adjustments in educational programs. (Carroll, 1989, p. 30)

Concluding Remarks

The world and the people therein are becoming more interconnected (e.g., in 2013 the Eurozone and the United States launched free trade talks), because in part, "real strength lies in differences, not in similarities" (Krogerus & Tschäppeler, 2012, p. 134).

In this globalizing context powerful mathematical learning is unlikely to succeed in classrooms and schools unless complexity is presented simply and dialogically. However, the strength of the respective taxonomies taken as a whole; the structured (S−R−O−C) and unstructured (CAME) learning protocols, as well as Carroll's (1963) Model of School

Learning lies in their simplicity to inform the scope and sequence of teachers' pedagogies, despite the vibrantly different cultural histories, narratives, and values of Being-human. This also implies the possibility that powerful learning ecologies can be enriched and structured, at least in part, through dialogues that relate and interrelate the taxonomies, the protocols, and the model of school learning. If so, then a mind in society can develop an embodied society of mind, and vice versa, through an extended self that experiences dialogically both similar and dissimilar approaches to Being-mathematical. However, to learn mathematics in these terms requires Being-intelligent in ways that can progress and goal-orient human cognition and metacognition towards the advancement of different types of knowledge, positive affect, and the language of the Self-system which includes the aims of education, as well as quality teaching and epistemic instructional ideas.

End Note

1. Being-able implies that *I*-consciousness through an embodied mind in action 'can do', or achieve an outcome. There is a definiteness, or confidence associated with Being-able, because the 'can' in doing has a present history of demonstrated activity, or similar successful performance that serves as a historical record of being able to accomplish a particular task. In contrast however, the term 'Being-capable' emphasizes what is possible, or what is potentially likely to occur in Being-able, as a consequence of the present-past of Being-able, or the holistic capacity of the individual which is essentially the Being-there of Being-able.

Chapter Seven

Being-intelligent

Intelligence

Powerful mathematical learning requires a complex intelligence of Being-mathematical. If students are to develop intelligence in these terms, then mathematics teachers as Beings-intelligent need to role model mathematical intelligence through dialogues that are autotelic and can facilitate a flow of learning, or momentum in classroom situations which many students would otherwise find boring, distressing, or too difficult to access intellectually and emotionally. In other words to experience flow in the mathematics classroom, it behooves the intelligent teacher to moderate learning tasks so that they are neither ‘too easy’ nor ‘too difficult’ (Goldman, 2009). Therefore to teach autotelically for the Conceptual Age implies the emergence of an intelligence that is capable of adapting to, shaping, selecting, and even creating real or virtual environments so that students can respond meaningfully and innovatively to familiar and novel circumstances (Sternberg & O’Hara, 1999; Davis & Simmt, 2006).

In ‘Being-intelligent’ the mathematics teacher is able to create or reconfigure circumstances within, or alongside an evolving zone of proximal development (ZPD). Simple or advanced technologies can assist the creative teacher in this regard. For example, Adobe Creative Cloud supports an intuitive and highly connected approach to Being-creative, through Adobe’s substantial suite of design software that unites a virtual community of ‘creatives’ around the world, namely, **Behance** (Adobe Systems Incorporated, 2012). Moreover, the taxonomies and learning protocols dicussed in the previous chapter need to be grasped intuitively by the mathematics teacher to enrich Being-there, namely, through embodied cognition and in-relation to the aims and epistemic principles of mathematics education. It is through intuitive functioning alone that systems of Being can be comprehended by the individual teacher, student, or group so that in seeing learners are no longer 'blind' to possibilities of Being-mathematical.

Furthermore, Being-intelligent is rooted in both the physical and socio-cultural substantiality of being. Those who have not learned from, reflected upon, or retained the experience of the past cannot consciously experience Being-mathematical as an unfolding present history that is answerable, or accountable to the 'living space', or *Lebensraum* of the Self. Therefore an intelligent philosophy of powerful mathematical learning through *auto* and *telos*; *I* and *Other* is fundamentally a moral philosophy that inhibits the individual from naïvely repeating the past in perpetuity (Bakhtin, Liapunov, & Holquist, 1993; Santayana, 1953). Moreover, Being-mathematical is dependent upon a future oriented intelligence that is underpinned by a moral or ethical philosophy so that "voices are not silenced, denied or suppressed on the basis of race, gender, age or any other social or personal characteristic" (Hermans & Gieser, 2012).

In Being-mathematical therefore, mathematics teachers need to teach for the development, or acceleration of a holistic intelligence that subsumes the Piagetian and Vygotskyan epistemology espoused in the cognitive acceleration projects that were CAME and CASE. However, studies in intelligence have not been framed in terms of Piaget's (1970) genetic epistemology and Vygotsky's (1978, 1997) inter- and intrapersonal psychological planes of the self. Furthermore, theories of intelligence have historically not emphasized the wholeness of being in relation to the interpersonal and intrapersonal dimensions of Being. Einstein nevertheless made the point that "a human being is a part of the whole, called by us 'Universe', a part limited in time and space" (as cited in D. Siegel, 2010, p. 255).

Intelligent persona-abilities. Each *I*-position can be thought of as an 'intelligent persona-ability'. Spearman (1927) maintained that all intelligent functioning involved a general factor (g) and at least one specific factor(s) (Duncan et al., 2000; Kokot, 1992). Renzulli (1977, 1999) conceptualized student learning as an intelligence complexity that included a general factor, a volitional and affective factor that was task commitment, as well as creativity. Sternberg's (1985) Triarchic Theory of Intelligence was superordinate in three different abilities, namely, the consummate balancer who understood intelligence as a complex ability that was analytic, creative, and practical (Zhang & Sternberg, 2001).

Therefore all '*I*-position intelligences' overlap, or are underpinned by a general ability. If it were not for this common ability then an intentional, sustained, and emotive dialogue between the different *I*-positions could not result in a consummate balancer who was able to fuel Being-mathematical in the dialectics of Piagetian−Vygotskyan learning.

However, it was Carroll (1993) who developed a comprehensive understanding of intelligence through a meta-analytic structure of cognitive abilities. The structure was formulated by reanalysing more than 460 data sets collected over a 60 year period from 1927–1987. As indicated in **Figure 7·1**, the stratum model of intelligence involves three levels of generality, namely, general (Stratum III), broad (Stratum II), and narrow (Stratum I). Importantly for powerful mathematical learning, Carroll's (1993) factor analysis showed that a single general ability (G or 3G) applied to all cognitive, intellectual tasks, and correlated very strongly with fluid intelligence (2F). That is the mental processes of General Sequential Reasoning (RG), Induction (I), Quantitative Reasoning (RQ), Piagetian Reasoning (PR), and Speed of Reasoning (SR). Of particular interest, Carroll (1993) reported that the 2F processes were not strongly dependent on the socio-cultural dynamic of the learner, but it was the correlation between fluid reasoning and 3G that was approximately equal to unity (Gustafsson, 1997).

Therefore from a factor analytic perspective it is clear why the CAME and CASE projects were so successful in raising the mathematics and science achievement of many students (Adey, Csapó, Demetriou, Hautamäki, & Shayer, 2007). It is the single general ability 3G that empowers all the cognitive abilities, or *I*-positions. Thus in order to effect meaningful change

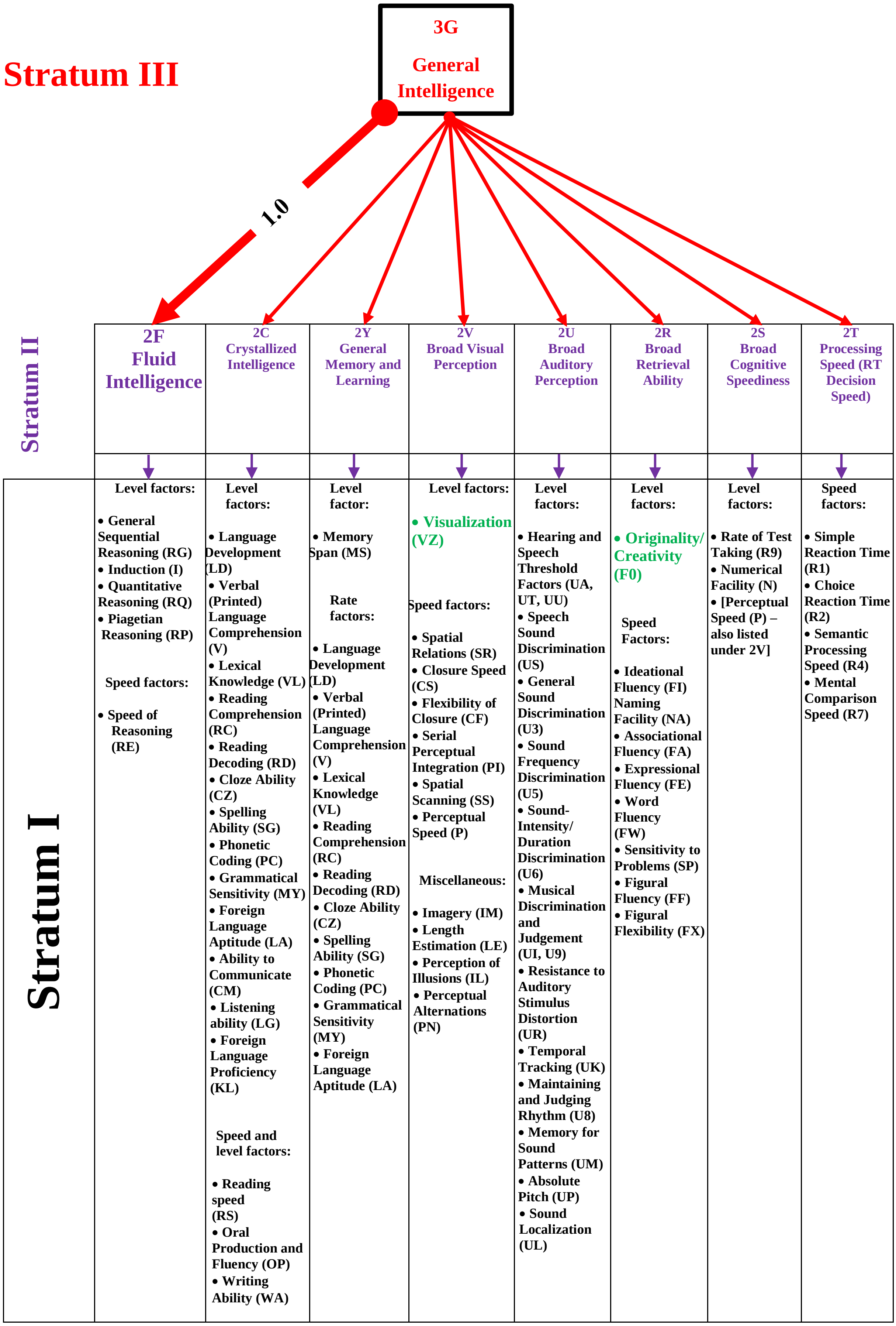


Figure 7·1. Carroll’s meta-analytic structure of cognitive abilities (adapted from Carroll, 1993, p. 626) in mathematics and science classrooms, a general intellectual factor that incorporates fluid reasoning is probably the most economical explanation of the many factor analyses, and a useful construct on which to base educational practice including cognitive acceleration. There is however, a major proviso. If a student is limited in Being-dialogical, then the intelligence potential of the *I*-positions will inevitably be constrained.

As was mentioned previously in this study, in certain Pakistani schools CAME and CASE outcomes were not very favourable, because culturally students and teachers were inhibited dialogically, and as a consequence dialogue did not have a multiplier effect on Beings-mathematical in relation to the foundational tenets of the ‘Teaching for Thinking’ intervention. Thus to learn mathematics as a dialogue, or through the medium of dialogue is likely to be a significant challenge for cultures that tend to be monological, or do not embrace individual differences and critical thinking (e.g., Saudi Arabian culture relative to Japanese culture relative to Australian culture).

Nevertheless, Carroll (1993), Gustafsson (1997), and the CAME and CASE interventions of the 1980s and the 1990s (Adey & Shayer, 1994; Iqbal & Shayer, 2000; Shayer & Adey, 2002) all suggest that intelligence is complex in the generality of specific abilities. If Being-human therefore is to think in generalities, but to live in details (Whitehead, 1943), what is the means, the mechanism, or the operating system whereby the general and the particular of intelligence can comport towards powerful mathematical learning? It is a fundamental question of Being-intelligent for Beings-mathematical (Dreyfus, 1991; Heidegger, 1927).

Schoolwide Enrichment Model. Over a period of more than 25 years, Renzulli (1999) modified his perspective on high level learning. The intelligence complexity that included a general factor, task commitment, and creativity was enlarged into a Schoolwide Enrichment Model (SEM) that was interrelational between the teacher, the student, and the curriculum (Renzulli & Reis, 1997, 2008). This model espouses the notion that Being is fundamentally relational in interpersonal, intrapersonal, and extrapersonal terms. Thus in grappling with the notion of giftedness, Renzulli (1977, 1988) was struggling with the complex problem of how to optimally educate Beings-in-the-world.

As indicated in **Figure 7·2**, the SEM was based on a premise that the structure and content of the curriculum would likely appeal to the imagination (creativity) of the learner, if the teacher was sufficiently knowledgeable and passionate in his or her subject, and had the necessary instructional skills to connect meaningfully with the abilities (cognitive and non-intellective), interests, and learning style(s) of the individual student (Renzulli & Reis, 2008). Learning style refers to the mind's manner of acting or Being-able. Therefore each learning style is holistic in relation to its Being, and can consequently complement or enhance the functionality of Gardner's (2006b) five systems of mind.

In a systemic sense therefore, different learning styles are required for powerful mathematical learning. That is at least three interacting continua, namely, the concrete experimenter (practitioner) and abstract theorizer; the logical analyzer (verbalizer) and intuitive synthesizer (visualizer); as well as the deliberate imitator and creative innovator (Delahoussaye, 2002; Leite, Svinicki, & Shi, 2010; Peters, 2007; Sadler−Smith, 1996). In Being-creative the emphasis is on the process of creativity, but in Being-innovative the emphasis is on the product of creativity for the purpose of addressing a real world problem practically. Some of the most successful entrepreneurs in history have been creative innovators (e.g., William Henry Gates III (1955−) of *Microsoft*, and Walter Elias Disney (1901−1966) who co-founded *The Walt Disney Company* with his brother Roy).

Rising IQs. It was not only Renzulli (1977) who modified and enlarged his understanding of human intelligence. Since at least 1930, the average IQ of individuals has increased over successive generations internationally (Sternberg & Kaufman, 1998). This global phenomenon has become known as the **Flynn effect** (Flynn, 1984, 1987, 1994). Not

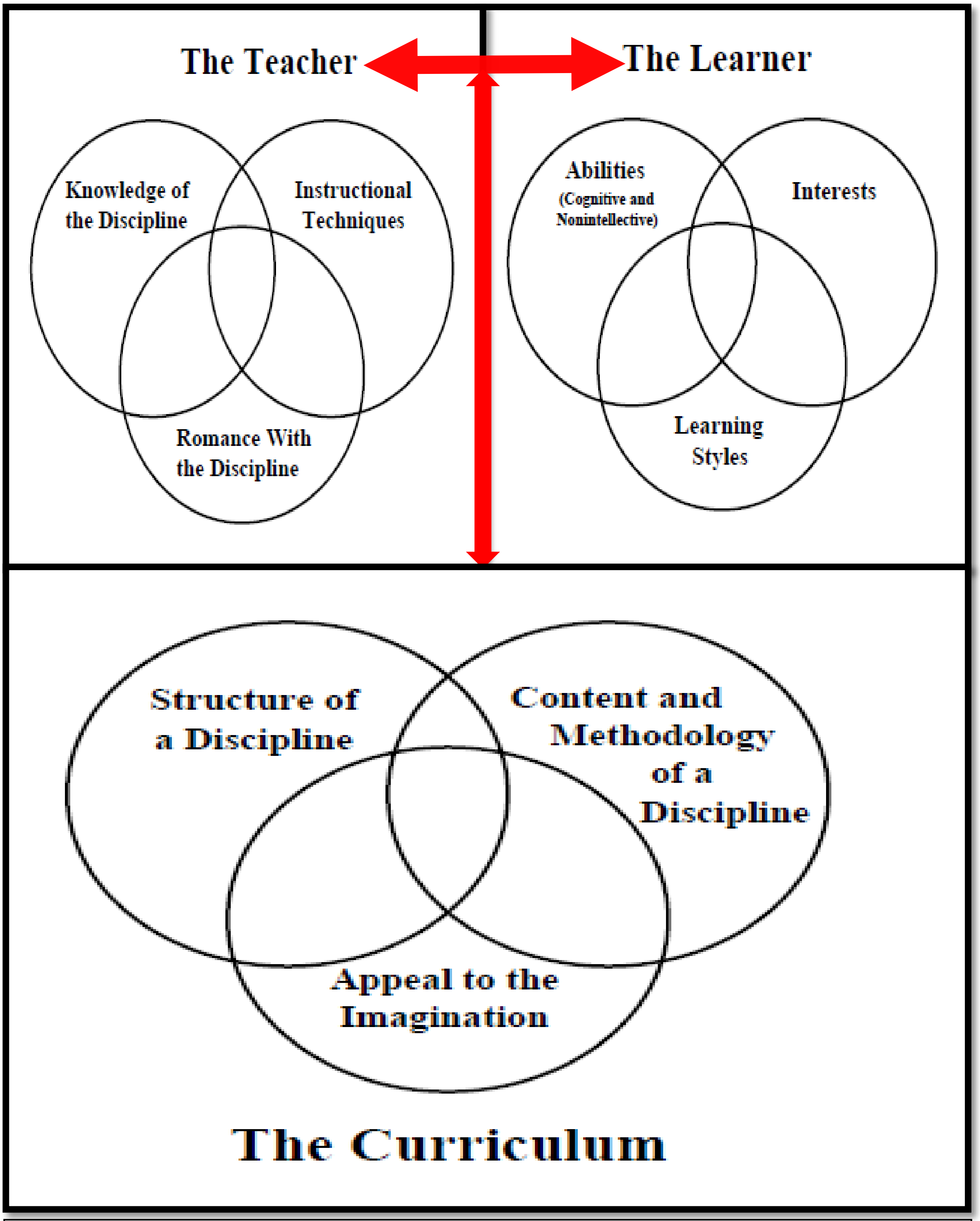


Figure 7·2. In a school, human intelligence involves a triadic and bi-directional interaction between a teacher, a learner, and a curriculum. In the figure, the curriculum is positioned off-centre to emphasize the authority and expertise of the teacher in Being-able to implement, guide, and scaffold the curriculum successfully. That is, Being-intelligent from the teacher's perspective means connecting holistically with the Self of the student, so that the curriculum appeals to the imagination of the powerful mathematical learner (adapted from Renzulli, 1999; Renzulli & Dai, 2001; Renzulli & Reis, 2008).

only has the effect been confirmed in OECD-type nations and the economically developing nations of Brazil, Dominica and the Sudan, but IQ correlated highly (Pearson's r = 0.917) with educational attainments (EAs) in 86 countries around the world (Lynn & Meisenberg, 2010).The effect was substantial, particularly for tests of fluid intelligence where a minimum increase of at least 15 IQ points per generation has been observed (Sternberg & Kaufman, 1998). But what does an IQ difference of 15 points actually mean? In the context of the United States for example,

> a person with an IQ of 100 might be expected to graduate from high school without much distinction and then attend a year or two of a community college, whereas a person with an IQ of 115 could expect to graduate from college and might go on to become a professional or fairly high-level business manager. In the other direction, someone with an IQ of 85, which is at the bottom of the normal range, is a candidate for being a high school dropout and could expect a career cap of skilled labor. (Nisbett, 2009, p. 5)

It is important to note however, that the average IQ is not uniform across nations. For example, a growing disparity between student IQs in Japan and the United States was noted before the release of *A Nation at Risk* in 1983. Moreover, in the case of East Asian students a relatively high IQ was sufficient to explain their superior achievements in international mathematics and science assessments, without controlling for the Confucian values of filial piety, humaneness, and ritual consciousness (Chia, 2011; Lynn, 1982, 2010; Oxnam & Bloom, 2013). This finding by Lynn (2010) is not all that surprising because the cognitive acceleration of students in algebra has been encouraged in countries like Japan, Singapore, and South Korea (Ginsburg et al., 2005; Sami, 2012; Yoong et al., 2009).

Overall though, the reasons behind a generational increase in IQ are complex, but the increase should encourage all teachers that powerful mathematical learning might be a possibility for the majority of learners in mass education, in spite of political, curricula, time, and cultural restrictions.

Wisdom. Although on average IQs have risen substantially post-1930 there appears to be little evidence, if any, that humanity is learning from history and becoming more intelligent in Being. This means becoming more wise, especially if wisdom is that ability which enables the individual to use knowledge for the greater benefit of humanity in complex situations (Sternberg, 2003a). In adapting the ethics of the French−Jewish philosopher Levinas (1905−1995), wisdom implies that the different *I*-positions learn intelligence not primarily from the *Other* but about, and through the *Other* for the purpose of empowering the being of the *Other* (Blades, 2006; Mautner, 2005; Merleau−Ponty, 1962).

Thus in empowering the *Other* the *I* increases its agency of Being-intelligent, that is, in comporting towards the *Other* the intelligence of *Da-Sein* is enhanced in terms of the *Other*. As a case in point, the Baptist minister and eminent leader in the African−American Civil Rights Movement, Martin Luther King (1929−1968) *vis-à-vis* the *Other*, namely his and other children, understood that

> I'm going to work and do everything that I can do to see that you get a good education. I don't ever want you to forget that there are millions of God's children who will not and cannot get a good education, and I don't want you feeling that you are better than they are. For you will never be what you ought to be until they are what they ought to be. (as cited in Darling−Hammond, 2010, p. 328)

In the sentiment of Levinas and King, it was Sternberg (1999, 2000, 2003a, 2003b) who realized that his Triarchic Theory of Intelligence (TTI) needed to reflect the notion of 'Successful Intelligence'. Consequently, the TTI was enlarged phenomenologically to include the purpose and intent of intelligence which was more than just embellishing the self with the individual, but rather intelligent

> citizens of the world need creativity to form a vision of where they want to go and to cope with change in the environment, analytical intelligence to ascertain whether their creative ideas are good ones, practical intelligence to implement their ideas and to persuade others of the value of those ideas, and wisdom in order to ensure that the ideas will help achieve some ethically based common good, over the long and short terms, rather than just what is good for them and their families and friends. (Sternberg, 2009, p. 10)

Therefore, Sternberg (2003b) began to develop a Balance Theory of Wisdom (BTW). The history of humankind suggests that the idea of balance has always been an inextricable part of human flourishing, wellbeing, or *eudaimonia* (Clark, 2008; Gibbons, 2004). For example, Aristotle articulated the Golden Mean not as an end in itself but rather as a tacit intent towards a virtuous life that was essentially an enactive and bodily balance between extremes (Aristotle, 2006; Lyon, 2009; Polanyi, 1966). From **Figure 7·3**, it is evident that Wisdom is a

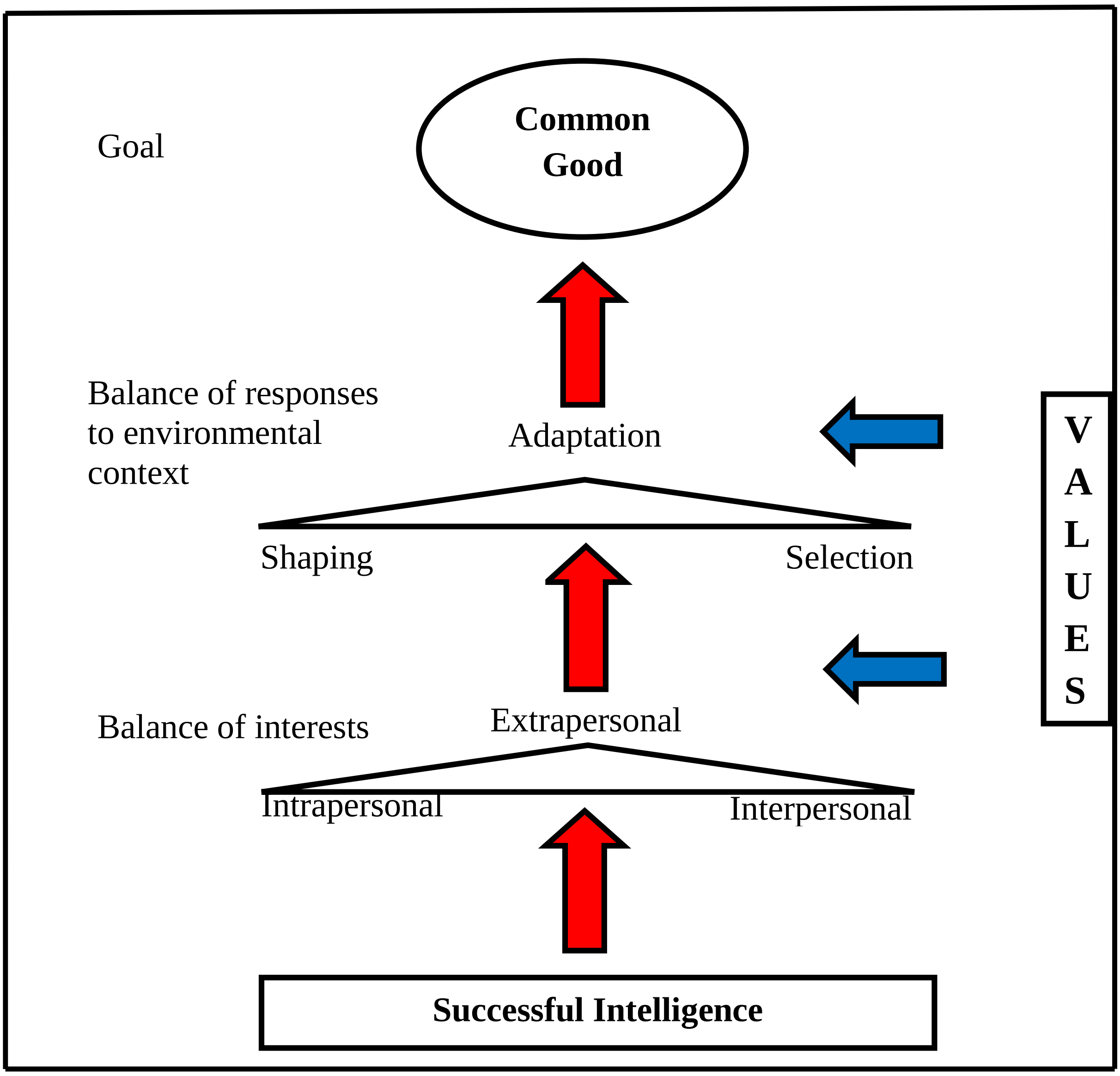


Figure 7·3. A Balance Theory of Wisdom (Sternberg, 2003a, p. 237)

'construct' that links Successful Intelligence with a virtuous and ethical outcome, or a Common Good that is infused with values. However, if the individual is to be successful in his or her environment, then Being-intelligent means that the *I* of the individual has to balance personal interests with his or her responses in the context of *I* and *Other*.

Since at least the times of Ancient Greece three kinds of goodness have been articulated, the morally right or good (L. *honestum* or *pulchrum*), the pleasurable (L. *jucundum*), and the advantageous (L. *utile*) for both the individual, the other person, and the environment (Mautner, 2005; Von Wright, 1963a, 1963b). But Successful Intelligence on the part of the individual implies an advantageous, pleasurable, and morally right outcome for both *I* and *Other*, with the proviso that if a Common Good is to be realized, then the *I* must have the liberty and capability to choose, shape, select, and adapt in-relation to the interpersonal or extrapersonal *Other* (Sternberg & O'Hara, 1999). Consequently, the dialogue associated with Being-intelligent needs to be creative, analytical, and practical, because as a dialectical unity of sense it is Wisdom, or *phronesis* that makes possible the potentiality of Being-successful in the Three Worlds. In this regard Being-intelligent for a Common Good means interrelating World 1 with the Intrapersonal of World 2, and the Extrapersonal of World 3 (e.g., the notions of equity, harmony, justice, and sexuality) so that the multi-faceted response of the individual enhances the value of the environmental context.

Emotional Intelligence

Successful Intelligence is complex in people relations. Wise people however, have learned intelligence by developing the self from a predominantly egocentric *I* (in Freud's structural theory, the ego was the *I* or executive agency of the self) towards a grasping of reality that includes "social and interpersonal knowledge, life knowledge, meta-knowledge, and postformal or dialectical thought" (Orwoll & Permutter, 1990, p. 174). However, although learning intelligence is epistemologically complex and diverse, learning is made possible through a physical body, because "the mind is embodied, in the full sense of the term, not just embrained" (Damasio, 2005, p. 118). It was the American social critic and satirist, Parker (1893−1967) who quipped insightfully that a particular actor's body had essentially 'gone to his head'. Therefore the Wisdom that is Being-intelligent is a spatio-temporal corporality event.

Consequently, a powerful mathematical learner is dependent upon having a body and knowing how to interact with that body for the purpose of actualizing an ethical, creative and dialogical self. The thinking body communicates in terms of emotions and feelings in order for the individual to know the current, or changing state of *Da-Sein* (Aldous, 2006; Kahneman, 2011; Kandel, 2006). The essence of a feeling however, is to know, experience, or be attentive to "what your body is doing *while* thoughts about specific contents roll by" (Damasio, 2005, p. 145). In particular it is the brain that has its own body map of somatosensory markers and it is through these markers that the body feels (Nelson, 2002).

The **emotion** of juxtaposed mental images or schema is communicated *via* the central nervous system, the peripheral nervous system, or the bloodstream to the markers (and viscera) which transform the brain communication into a **felt** state of Being (Barnacle, 2009; Diamond, Harris, & Peterson, 2002; Nelson, 2002; Damasio, 1999, 2005). In other words it is only through feeling that the body can respond cognitively or metacognitively, explicitly or implicitly to the emotion initiated by the mental images, because emotion tends to be a non-conscious electro-chemical event (Sylwester, 2000) . Essentially therefore, a brain that cognizes in-relation to a feeling body is what is meant by embodied cognition, and it is the notion of embodied cognition that suggests the possibility of an emotionally intelligent body, where emotional intelligence (EI) has been described as an **ability**

> to perceive accurately, appraise, and express emotion; the ability to access and/or generate feelings when they facilitate thought; the ability to understand emotion and emotional knowledge; and the ability to regulate emotions to promote emotional and intellectual growth. (Sternberg & Kaufman, 1998, p. 497)

Importantly for powerful mathematical learning therefore, why one student succeeds and another does not is almost always dependent upon the emotional intelligence (EI) of the individual (Bradberry & Greaves, 2009; Goleman, 1999, 2006), because without EI the person cannot establish an intellectual distance that simultaneously clarifies the proximal and the distal (Csikszentmihalyi & Rathunde, 1990; Polanyi, 1966). However, if a student in mass education is to become a powerful mathematical learner then that individual almost certainly will need to be part of a teacher's ZPD, or zone of collaborative development (ZCD), or better still a zone of dialogical development (ZDD) — all of which require the teacher to be emotionally intelligent in-relation to the social dynamics of 'serious play: where intuition and passion meet objectivity and logic' (adapted from Grinnell, 2009).

PME. Mathematics more than any other subject in schools has fomented anxiety, unfavourable attitudes, and negative experiences in both students and teachers (Hoffman, 2010; Putwain & Daniels, 2010). Consequently, and in response to the quietus of New Math, the International Group for the Psychology of Mathematics Education (**PME**) was established in 1976 in Karlsruhe, Germany at the third International Congress on Mathematical Education (Gutiérrez & Boero, 2006). The inaugural PME conference was held in Utrecht in the Netherlands in 1977. Since that event scholars from around the world have used PME as a forum for the specific purpose of addressing the many psychological challenges that are associated with the meaningful teaching and learning of mathematics. Much empirical knowledge has been advanced including neuroscientific studies which have demonstrated that

> high levels of anxiety reduce access to higher brain functions, interrupt the natural flow of information and processing between the hemispheres, and inhibit prefrontal cortex functions. Nervousness, fear, and tension block even learned knowledge. (Clark, 2008, p. 253; also see Goswami, 2004)

The emotionally intelligent teacher. It is the teacher's classroom behaviour, and instructional approaches that are primary causes of 'Being-inhibited' mathematically (Bekdemir, 2010). It is through a negative essentiality of Being-there that anxiety or boredom arises and inhibits classroom flow. The psychology of such embodied states is to avoid the pursuit of challenges in mathematics, as well as the quality and intensity of backwards and forwards motions between the question and the inquiry; the *I* and the *Me* in-relation to the

Other. Moreover, the student's zones of proximal development that are intended for collaboration and dialogue might devolve implicitly, or explicitly into self-handicapping, or self-limiting strategies that protect paradoxically the coherence of the self against negative social comparison or competition (Roeser, Peck, & Nasir, 2006). Therefore it is paramount that the mathematics teacher needs to be alert to the social and agentic psychology of embodied cognition, which through bodily feeling is a "gateway-triggering mechanism" for higher order cognition (Clark, 2008, p. 247).

Furthermore, the emotionally intelligent (EI) teacher needs to strike a balance between boredom, anxiety, and meaningful challenge for the purpose of promoting a psychology of ethics that includes gentleness and authority (Heller, 2002). A gentleness on the part of the teacher is desirable for the purpose of fostering a student's sense of belonging and value within a problem solving community, but a gentle disposition should not be allowed to erode the deontic leadership of the teacher, either towards authoritarianism, or an attitude that is *lassez-faire* (Mautner, 2005). Powerful mathematical learners require clear but flexible boundaries, or zones of promoted activity and freedom that require each student to take personal responsibility for his or her conduct and learning.

In broader terms, it is only through a deep ethical stance on the part of the mathematics teacher towards students, fellow teachers, administrators, and parents that the teacher can realistically connect with other persons in the diverse temporality of his or her situatedness (Diamond, 1999). This implies valuing the "contributions of others. You must listen with respect and humility, and when you have developed a voice, you contribute to the conversation, knowing that it is much greater than you" (Ernest, 2009, p. 37). The result over time is that the intelligence and character of the student are nurtured in-relationship with a benevolent authority who respects authority in the service of a dialogical teaching and learning community. Furthermore, the emotionally intelligent teacher has a responsibility towards the student to exact a form of corrective discipline and student accountability which is not only genteel in its tolerance, but also encourages a sense of humour through a manifest strength of Being-there that is tenacious and resilient towards becoming a powerful mathematical learner (Bahr, 2007; Curwin, 1995; Kotsopoulos & Cordy, 2009; Sunter, 1992; Wilson, 1998).

Therefore in-relation to Being-wise for a Common Good, the teacher with a high EI is hospitable towards the student by creating a dialogical space within which, and between which the student can become ‘better at who he or she already is’ (Curwin, Mendler, & Mendler, 2008; Esteva, 1987). In this regard all teachers have an ethical, or social justice informed duty of care to act against bullying in schools whether it be direct, indirect, or the cyber victimization of the individual or group (Campbell, 2005; Shelley & Craig, 2010). If need be, the caring teacher should provide his or her students with strategies that promote mental health and wellbeing, thereby facilitating improved social−emotional and academic outcomes and competencies (Dix, Slee, Lawson, & Keeves, 2011). However, EI in the fullness of Being-in-the-world does not mean insulating students from a globalizing world and mollycoddling them. The attitude that educators “should expect nothing of [students] but give everything to them; they should be cared for, counselled, and analyzed, and the whole school environment should be centred on their needs,” (Stout, 2000, p. 3) is out of step with the harsh realities of surviving, competing, or flourishing in the Conceptual Age.

Each student therefore needs to learn how to be autotelic, as should every teacher. Oftentimes teachers and students are embattled between indifference and hostility, and if not steadfast through an agency that ‘Being-gentle’ facilitates, both the teacher and the student are likely to capitulate into an attitude and a bodily sense that is akin to an unyielding and inflexible dogma (Hansen & Laverty, 2010). It is unfortunately a phenomenology of many societies that individuals tend to function in terms of an *I* that severely limits the Being of the other person in their lives (Schutz, 1970, 1972).

Summary insights: Emotional Intelligence. Success in the Conceptual Age is dependent upon emotional intelligence (Bradberry & Greaves, 2009; Goleman, 1999, 2006). Thus a fundamental goal of powerful mathematical learning means promoting the "best possible realization of humanity as humanity," (Dewey, 1916, p. 95) through the modelling of "humane behaviour for our students without sacrificing standards of learning or behaviour" (Heller, 2002, p. 77). Towards this goal, the wise teacher is emotionally intelligent as an educator and leader who is not only open-minded and creative, but is committed to the learning of each student through a value-based didactical contract.

If a student commits to such a teacher, which is a likely social event, given that Being-human is to respond to a deeply ethical posture that enables the humanness and the creativity of the *Other* person, then both the teacher and the student will likely learn intelligence as they become wise for a Common Good. Thus if intelligence unfolds and enfolds through activities which essentially mean 'Being-wise' (*Weise-Sein*), the self of the learner can progress from a relatively egocentric and narrow worldview towards a conscious reality that is complex, realistic, ethical, creative, and perhaps even cosmopolitan (Orwoll & Permutter, 1990). Nietzsche was one of the first philosophers to appreciate that consciousness "is really only a net of communication between human beings; it is only as such that it had to develop; a solitary human being who lived like a beast of prey would not have needed it" (as cited in Humphrey, 2006, p. 104).

A Triadic Model of Intelligence

Theoretical models are essential if teachers' actions are to be guided towards the goal of best practice in complex situations. Philosophically, a 'theory that is not practical' has not attained the status of a theory (Dewey, 1929a, 1929b; Kilpatrick, 2010). Moreover, simplicity as a doxastic virtue is desirable if the theory is to facilitate or signify a model towards more meaningful practice (Quine & Ullian, 1970). Teachers however, except for the outstanding few, are too busy to take up 'alluring' empirically-based research or philosophical ideas that pertain to the mind for example, unless concretized in simple-to-understand models that can be made feasible through a professional and sustained dialogue with empathetic scholars and colleagues (Black & Wiliam, 1998b).

Although the mind is highly complex in the sense of an embodied and extended self, teaching for powerful mathematical learning requires an understanding of mind. Its embodiment as a multidimensional edifice and hierarchy of general-purpose and specialized processes and abilities, can be viewed as a triadic interaction of frontal lobe executive functioning, working memory (WM), and domain specific thought (DST) systems (Demetriou, 2009). As indicated in **Figure 7·4**, intelligence is thought to involve bidirectional movements between WM and the complex DST system as mediated by the Directive-Executive Function (DEF). It is the DEF that is metacognitive in mediating the processing speed and signification of WM and DST. Importantly for powerful mathematical learning however, the New Taxonomy of Educational Objectives (see pp. 181−182) suggests that the DEF selection of domain specific thought systems — in relation with working memory — is influenced causally by the Self-system through affect and intentionality. Thus an autotelic, or emotionally intelligent individual rather than a 'bored or anxious self', is likely to experience enhanced working memory capability, especially if the mathematical process requires a kinetic, or visual−spatial interpretation of the problem solving event. Visual−Spatial intelligence is discussed later in this chapter.

A caveat however: Carroll's (1993) meta-analytic structure (see Figure 7·1) should be used to enhance Demetriou's (2009) triadic model of intelligence, because the latter appears not to emphasize, or to elaborate the strong link between General and Fluid Intelligence (especially Piagetian reasoning which is crucial for quality mathematical learning as was apparent in the

CASE and CAME studies), as well as the Visualization (VI) of Imagery (IM) which is dependent hierarchically on Broad Visual Perception (2V) and General Intelligence (3G).

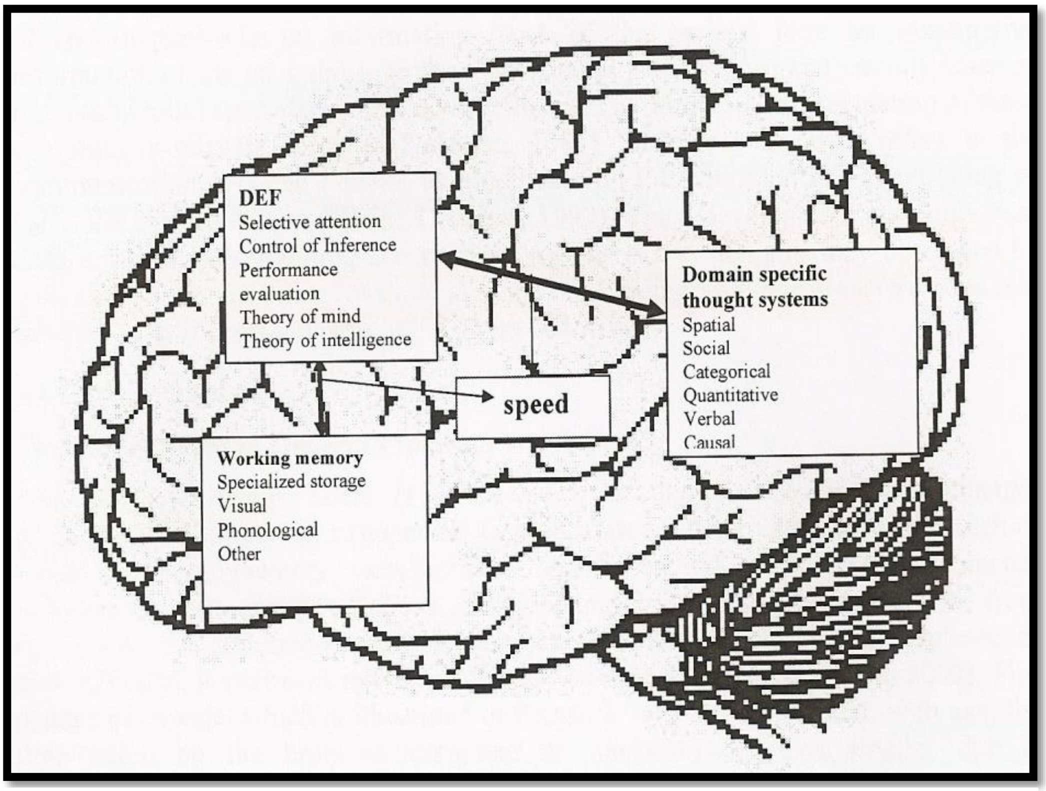


Figure 7·4. The general model of the domain-general and domain-specific systems of the human brain (Demetriou, 2009, p. 7)

Being-intelligent with working memory. As portrayed in **Figure 7·5**, working memory is a component of the super system that is human memory. WM is crucial for all learning. Although the embodied mind is capable of convoluted, highly abstract, and even postformal thought, a significant limitation on Being-human is the lack of capacity on the part of working memory. In neo-Darwinian terms it is ‘very difficult’ to justify the lack of evolutionary development in working memory, given the pivotal role that WM plays in the survival of Being-human. Nevertheless if students are to become powerful mathematical learners, then each student needs to learn how to optimize WM in relation to other memory systems and the human body as a whole, because Being-intelligent is essentially a growth in systemic consciousness that includes the Self-system, as well as the Metacognitive and Cognitive systems.

Working memory is largely underpinned by the prefrontal cortex and serves as a distinct type of short-term memory that not only integrates moment-to-moment perceptions, but also combines them with memories of prior learning or past experiences (Kandel, 2006). Therefore WM is essentially a temporary storage and activity system under the attentional control and agency of the *I*. In particular, WM facilitates the integration and orchestration of auditory and visual imagery in terms of a 'conscious mental space' that interrelates meaningfully with the DEF and Broca's area, which is crucially involved in the production of language and 'brain maps' of learning situations (Baddeley, 1983, 2007; McNeill, 2005).

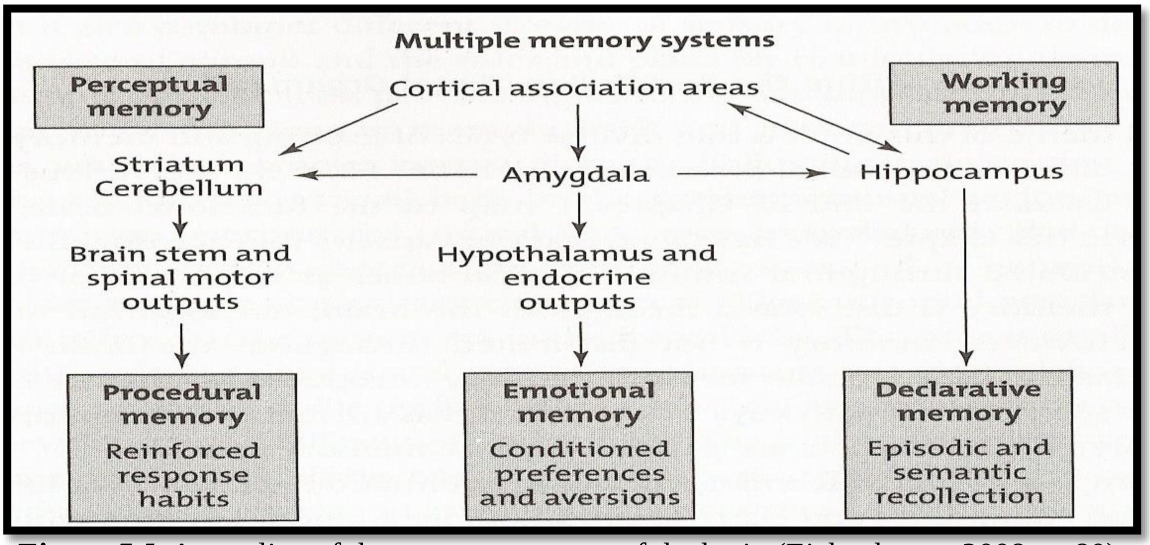


Figure 7·5. An outline of the memory systems of the brain (Eichenbaum, 2008, p. 80)

Furthermore, WM is characterized by "three dissociable components: A phonological loop for the maintenance of verbal information, a visuo-spatial sketchpad for the maintenance of visuo-spatial information, and a central executive for attentional control" (Nyberg & Cabeza, 2000, p. 506).Consequently, WM allows the individual to control, regulate, and actively maintain diverse information for the purpose of accomplishing a wide array of mathematical tasks (Raghubar, Barnes, & Hecht, 2010).

Working Memory Capacity. The capacity of working memory (WMC) is not only limited, but it also varies markedly across tasks and between individuals. In overarching terms however, the efficient use of working memory is imperative because WMC could perhaps explain “as much as 30% of the unique variance in mathematics performance above and beyond mathematics ability” (Hoffman, 2010, p. 277).

Cognitive Load Theory (CLT) is a theory that continues to flourish within the field of cognitive psychology. In June 2014 the *7th International Cognitive Load Theory Conference (ICLTC)* was held in Taipei. As a theory it has its roots in the seminal work of Miller (1956), and the empirical problem solving research of Sweller (1988, 1994), which interestingly coincided with the rise of the mathematics problem solving movement in the United States in the 1980s.

In particular however, CLT posits that the adept use of working memory means reducing irrelevant cognitive load, increasing relevant load, and managing intrinsic load when teaching and learning occurs. That is by excluding mental work which is **extraneous** to the learning goal, valuable mental resources can be conserved for **germane** activities that might enable the individual to better manage the complexity of the task at hand (Clark, Nguyen, & Sweller, 2006). Moreover, CLT research has indicated that learning efficiency can be improved significantly if: (a) a balance is struck between visual and auditory teaching modes, (b) the learner’s attention is supported, and (c) the amount of information to be processed in working memory is reduced.

Also, and as depicted in **Figure 7·6**, Efficiency (E) means maximizing Performance (P) and minimizing Mental Load, or Effort (ML) according to the equation E = P – ML. That is, CLT teaching and learning strategies should be designed so that P is approximately +1.0, and ML is approximately -1.0. But it is not being suggested that powerful mathematical learning can materialize in terms of an extended self without significant mental effort and perseverance on the part of both *I* and the interpersonal *Other*. Nevertheless, if instructional designs are to be consistent with the notion of Being-intelligent with working memory, then mathematics teachers need to aim for a point of balance on, or preferably above the line E = 0 so that an

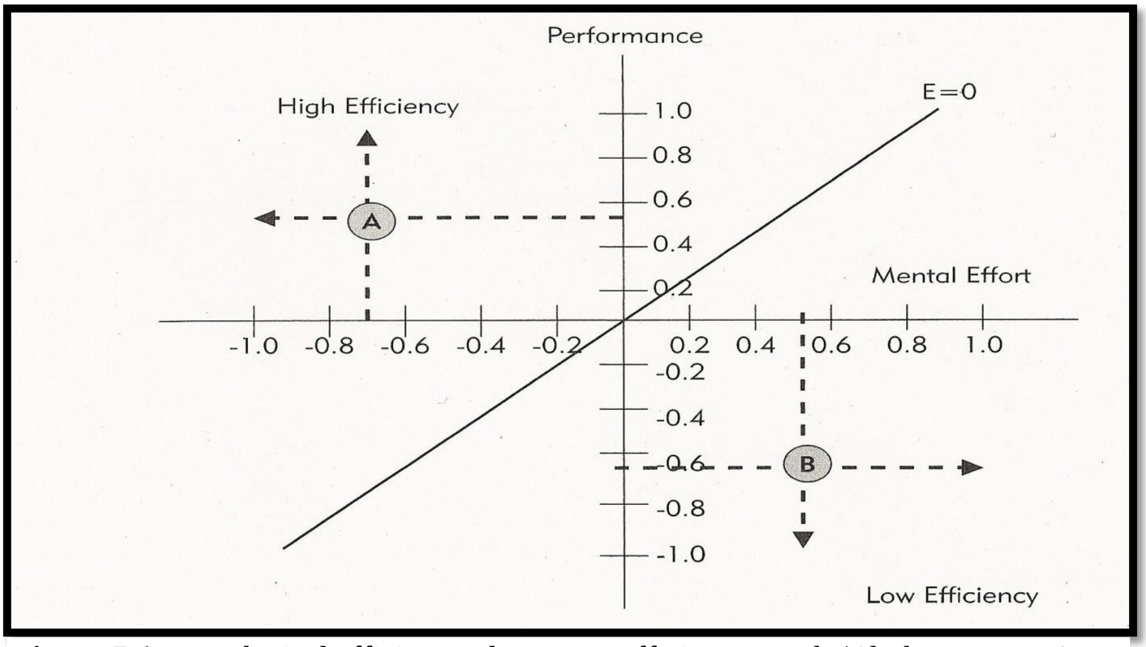


Figure 7·6. Hypothetical efficiency plots on an efficiency graph (Clark, Nguyen, & Sweller, 2006, p. 23)

increase in mental effort on the part of the student corresponds to a proportionally greater increase in student performance.

However, not all aspects of Being-intelligent are fluid and subject to enhancement through appropriate pedagogical strategies. For example, working memory is a short-term mental facility that appears to be invariant in its capability of holding more than a half dozen chunks of information at any single moment (Miller, 1956). Furthermore the mental act of "recognition takes nearly a second, and the simplest human reactions are measured in tens and hundreds of milliseconds, rather than microseconds, nanoseconds, or picoseconds" (Simon, 1990, p. 17).

Nonetheless, the overarching goal of CLT facilitates powerful mathematical learning, namely, to maximize the limited resources of working memory so that coherent and integrated schemas can be developed in long-term memory more efficiently. With this goal in mind, CLT is based on a psychological foundation of four tenets (Clark, Nguyen, & Sweller, 2006; Demetriou, 2009; Sweller, Ayres, & Kalyuga, 2011). First, experts in a domain of knowledge need a rich repository of well organized, and readily accessible schemas in long-term memory if the resources of working memory are to be optimized effectively (Anderson, 2010; Simon, 1990). Second, if students in mass education are to learn cogently then such individuals require scaffolded and instructional support to substitute for their lack of suitably assimilated and accommodated mental structures. Third, didactical events should focus student **attention** on relevant material through the **activation** of related prior learning. In addition, the **deliberate practice** of new content is necessary if carefully **encoded** schema are to be elaborated in long-term memory, especially for the accurate **retrieval** of knowledge structures that can facilitate problem solving capability. Fourth, all instructional and learning events place demands on WMC, and therefore need to be supported by didactical and pedagogical principles that can enable all students to manage the cognitive, affective, and volitional complexities of Being-intelligent more effectively, especially with respect to a working memory that has limited capacity.

S−R−O−C as an application of CLT. The origins of Cognitive Load Theory (CLT) can be traced back to one of the founders of modern cognitive psychology, namely, G. A. Miller (1920−2012). Over his career he worked primarily in the United States at Harvard University, MIT, and Princeton University. His most well known result was in demonstrating (partly in response to behaviourists who maintained that it was not feasible to study mental processes scientifically) that the maximum number of objects that could be held in working memory was 7 ± 2. Since Miller's finding however, the capabilities, limitations, and architecture of human cognition have been studied extensively, empirically, and particularly in relation to problem solving (Plass, Moreno, & Brünken, 2010; Sweller, Ayres, & Kalyuga, 2011). Consequently the cognitive sequence that is 'attention, activation, elaboration, encoding, and retrieval' has become the cornerstone of effective cognitive load management. In these terms the cognitive and metacognitive learning protocol that is S−R−O−C (**Select−Relate−Organize−Check**) can be thought of as a four stage application of CLT, because (a) S−R−O−C is also underpinned by ideas from cognitive psychology, and (b) the efficient and effective use of working memory in relation to other memory systems is at the core of the four stage learning process.

Stage 1. In the first five minutes of each lesson the teacher should **focus the attention** of each student by framing clearly the main goal of the lesson as an adventure in important ideas (Anderson, 2010; Butler & Winne, 1995; Whitehead, 1948). The beginning of the lesson is crucial for the purpose of stimulating student interest. Particularly during this initial period, the teacher ought to instruct deliberately because different students process different material in different ways and not necessarily at the same speed (Clark, Nguyen, & Sweller, 2006). Therefore, a multi-perspective or spiral approach to the **selection** or **identification of key ideas** is necessary to **activate**, or prime effectively and efficiently the prior learning of students with respect to the learning goals of the lesson (Askell−Williams & Lawson, 2009; Bruner, 1960; Bruner & Anglin, 1973).

Stage 2. In psychological and sociological terms, 'intelligent learning' has occurred when a cognitive, metacognitive, or director system influences a change of state in Being-intelligent (Berger & Luckmann, 1971; Skemp, 1979). In so Being the individual then has a stimulated capacity to acquire knowledge, reason abstractly, or to solve problems (Nisbett, 2009).

However, for the learner to enhance his or her state of Being-there, the student needs to **relate** or **elaborate** his or her prior knowledge, not only to, but in terms of the key ideas of the lesson or the task (Ausubel, 1968; Bruner, 1973). In this regard, Japanese mathematics teachers often required, or inspired their classes to grapple with problems that would lead to cognitive disequilibrium in the minds of their students (Furner & Robison, 2004; Stigler & Hiebert, 1999). It was Piaget who felt that if student learning was to be genuine, then the establishment of robust connections between the prior learning of the student and the new material was essential, and this robustness of relation was characterized by a new state of mind, or Being that had effectively equilibrated 'the old with the new' (Piaget, 1973, 1985). In Piagetian theory it is the central mechanism of **equilibration** that facilitates assimilation and accommodation[1] through a biological Being-there who interacts with the environment (Ernest, 2009; Piaget, 1977). In so Being a mental system was said to be in a state of equilibrium when the individual was able to resolve a problem situation by executing a cognitive structure of mental operations correctly and efficiently. This meant the "achievement of balance within the knower in response to perturbations," (Ernest, 2009, p. 42) especially through the cognitive processes that are inversion, negation, or reciprocity (Inhelder & Piaget, 1958).

Stage 3. Excellent teachers assist students to scope and sequence their learning in practical and efficient ways (Hattie, 2012). For example after grappling with an unfamiliar problem, students should not be left to their own devices and insights as teacher and student understandings of mathematics can be very different (Shimizu, 2006). Mathematics teachers need to address discrepancies in student learning through a direct intervention (perhaps on more than one occasion) that assists students to **organize**, or **encode** a quality of knowledge that is correct and useful. Encoding in this regard refers to the manner in which information is stored or represented in memory, and is facilitated by (a) appropriate note taking; (b) the writing down of meaningful images; (c) the sequential linking of ideas in a flow-type diagram; and (d) the drawing of a conceptual map (Askell−Williams & Lawson, 2009).

However, it is the use of spatial and relational imagery that is especially desirable when encoding cognitive memories because "such encoding is facilitated by deep processing of items' meanings, rather than superficial processing of items' physical characteristics" (Eichenbaum, 2008, p. 292). For example, the mathematics teacher can use the board as a historical record and visual feedback tool (Butler & Winne, 1995; Hoon, Kaur, & Kiam, 2006; Stigler & Hiebert, 1999). All humans relate well to spatial representations (Schwartz & Heiser, 2006), because Being-intelligent is empirically consistent with Aristotle's insight that thought without images is a human impossibility (Kosslyn, 1983).

Stage 4. Having organized their knowledge, the quality of students' work must be **checked** to ensure that each individual's understandings cohere logically with his or her prior knowledge; that of the learning community, and the overarching goal of the lesson (Anderson, 2010; Askell−Williams & Lawson, 2009; Hattie, 2009). Thus in-relation to a meaningful *Other*, each student ought to be encouraged to explain the key ideas of the lesson, or learning event through his or her conceptual map as mediated by the S−R−O−C protocol (Goldin−Meadow, 2003). In turn, the listening *Other* should try to make sense of the student's explanation, not only on the basis of the *I*'s conceptual map but also *via* the *Other's* conceptual understanding (Jensen, 2008; Kinchin & Hay, 2000). In analyzing and reflecting on the respective understandings, both individuals will likely improve the quality of their learning structures, and strengthen **retrieval** capability from long-term memory. As a consequence of interacting dialogically therefore, both individuals ought to enhance, or modify their note taking and spatial representations accordingly, and if necessary, in comparison to a template of what the teacher understands by the mathematics studied. Powerful mathematical learners are not only intuitive, but also self-directed, and in Being-intelligent are deliberate, thorough, and systematic.

Semantic and Episodic Memory

Powerful learners of mathematics use working memory to their advantage. However, this requires a holistic understanding of the different memory systems. In particular, and as indicated in Figure 7·5 (see p. 214), working memory (WM) and declarative memory (DM) are mediated by a limbic system structure, namely, the hippocampus. This structure lies deep within the temporal lobe of the cerebral hemispheres, and plays a crucial affective role in interrelating WM and DM (Eichenbaum, 2008; Gazzaniga, Ivry, Mangun, & Steven, 2009).

Declarative knowledge refers to semantic or episodic recollection. Semantic knowledge is a form of abstract knowledge that is not directly situated in the temporality and concrete specifics of World 1 or World 3. In contrast, episodic knowledge is information about particular episodes or events that can be retrieved from long-term memory as separate entries or entities (Schraw, 2006). Consequently, a genuine episodic memory is causally self-referential knowledge where the causal link is invariably experiential. Thus if *I* 'was told' rather than *I* 'sees', the memory is not episodic (Perner, 2000), because "at its best a learning episode reflects what has gone before it and permits one to generalize beyond it" (Bruner & Anglin, 1973, p. 422).

However, powerful learners of mathematics require a considerable amount of semantic knowledge to be at their disposal if they are to be cogent and efficient in Being-mathematical. This implies that facts, concepts, abstractions, and problem solving principles (e.g., heuristics) need to be embedded and well structured in a student's long-term memory, so that memory recall is not only fluid and accurate, but also requires less electro-chemical energy to maintain the retrieved memory in working memory. In other words students who enrich their semantic memories with positive affect, are more likely to remember those memories more easily and effectively, because the memories are infused with greater meaning, or Being-there than would be the case if the hippocampal formation (a brain triad that includes the hippocampus, the dentate gyrus, and the subiculum) was not invoked (Kandel, 2006).

Therefore to avoid 'rigid' semantic memories, mathematical learning should occur most often as an integrated or relational semantic−episodic event. The memory systems that are episodic and semantic complement each other by allowing the individual to interleave specific concrete experiences into the semantic network, thereby enabling learning transfer as an elaboration and expression of semantic or symbolic knowledge (Eichenbaum, 2008). It is however, the hippocampus that enables the powerful mathematical learner to integrate semantic knowledge in the relational context that is afforded by the episodic event. In so Being the individual encodes, and is able to retrieve 'relational' memories (Gazzaniga, Ivry, Mangun, & Steven, 2009). It is these relational memories that powerful mathematical learners can use to construct mental models for the purpose of understanding unfamiliar problem situations, because semantic‒episodic knowledge has structural capacity through concrete specifics and abstract relations.

However, mental models or representations of powerful mathematical learning are not haphazard in construction or development. The psychology and sociology of powerful mathematical learning entails a bidirectional epistemology and ontology, which through dialogue enables the growth of an autonoetic (self-knowing) consciousness that "mediates an individual's awareness of his or her existence and identity in subjective time extending from the personal past through the present to the personal future" (Tulving, 1985, p. 1). In these terms an example of Being-mathematical involved Japanese mathematics teachers who contemplated each lesson as an almost religious episodic event (Dubin, 2010; Furner & Robison, 2004; Stigler & Hiebert, 1999). As a result each mathematics lesson was crafted as an uninterrupted episodic whole in order to preserve the semantic integrity of the relational event in long-term memory.

In accord therefore, teaching for powerful mathematical learning is episodic intentionally because it fosters 'whole-brain learning'. However, the retrieval of complex semantic‒episodic declarative memories requires coordinated activity in numerous brain regions, and such "an extensive system seems quite vulnerable" (Nyberg & Cabeza, 2000, p. 506) if learning has not occurred in terms of well-crafted relational events. In order to illustrate the 'brain-spread' of verbal and non-verbal episodic learning, **Figure 7·7** is included as a visual summary of activations from brain imaging studies that comprised both verbal and non-verbal episodic memories. Therefore to emphasize, if the encoding process is **strong relationally** in multiple brain regions as a consequence of episodic learning, then the stored synaptic firing sequences in long-term memory are likely to facilitate the fluent and accurate recall, or future reconstruction of knowledge structures in working memory (Damasio, 2005).

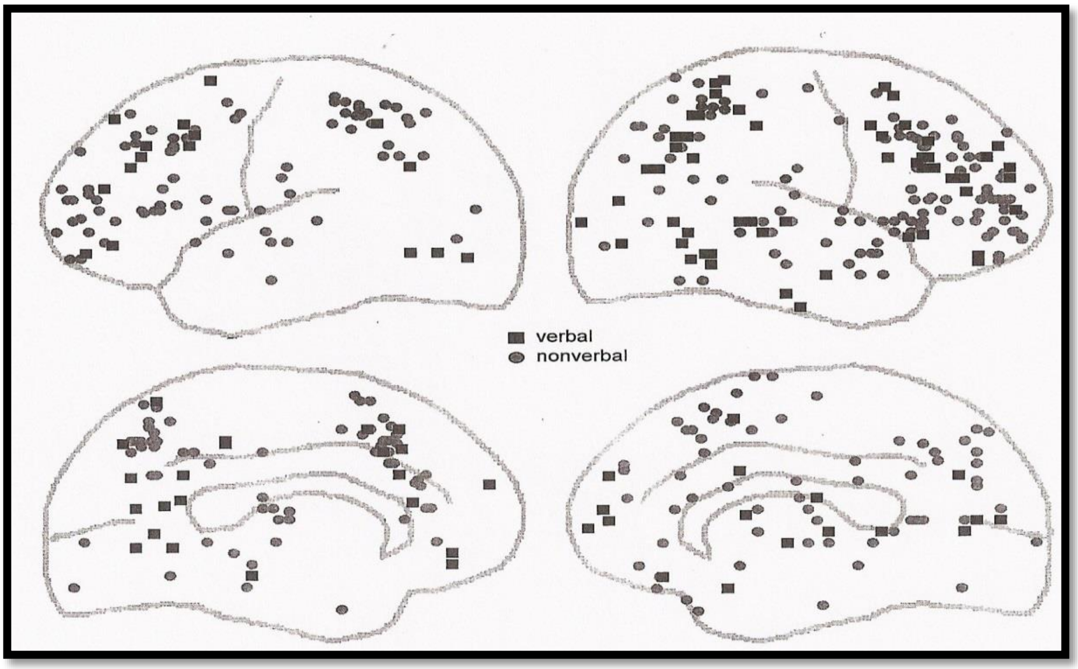


Figure 7·7. Summary of activations from brain imaging studies of episodic memory (Nyberg & Cabeza, 2000, p. 507)

The amygdala. As depicted in Figure 7·5 (see p. 214), the amygdala is a central component of the memory system and shares a ‘synergy of relationship’ with the hippocampus, the hypothalamus, and the striatum cerebellum for the purpose of encoding a memory with positive or negative affect. In particular however, following an emotional event the amygdala and the hippocampus consolidate the memory through a synaptic cross talk between axons and dendrites. The dual activation and resultant dynamics between the amygdala and the hippocampus is probably what makes emotionally based memories unique and long-lasting (Richter−Levin & Akirav, 2001). For example, the amygdala mediates the relationship between the striatum cerebellum and the hippocampus in the laying down of long-term procedural memories that necessarily involve reinforced response habits.

In general however, semantic‒episodic learning that encourages flow and Being-autotelic is likely to strengthen and enrich all cross talk between the amygdala and hippocampus, with the result that knowledge structures in long-term memory can be accessed more readily. As shown in **Figure 7·8A** and **Figure 7·8B**, the amygdala is located atop the hippocampus (Gk. *hippókampos* which translates as 'sea-horse') and acts as the sentry of the emotional brain by labelling each person's sensory, perceptual, and cognitive inputs as pleasurable, threatening, or non-threatening (Bath, 2005; Phelps, 2004).

The amygdala (L. *corpus amygdaloideum*) is a complexity of almond-shaped neurons that not only modulates reflexes and thought, but also influences the organization of human cognition and behaviour at all levels (Adolphs, 2004). In particular, the amygdala connects with the hippocampus for the purpose of modulating the affect of episodic memories of events, and mental maps of situations (O'Keefe & Nadel, 1978; Teasdale, 1999). As a result semantic‒episodic memories that are imbued with positive affect not only enhance memory retrieval but also facilitate the transfer of learning to new situations.

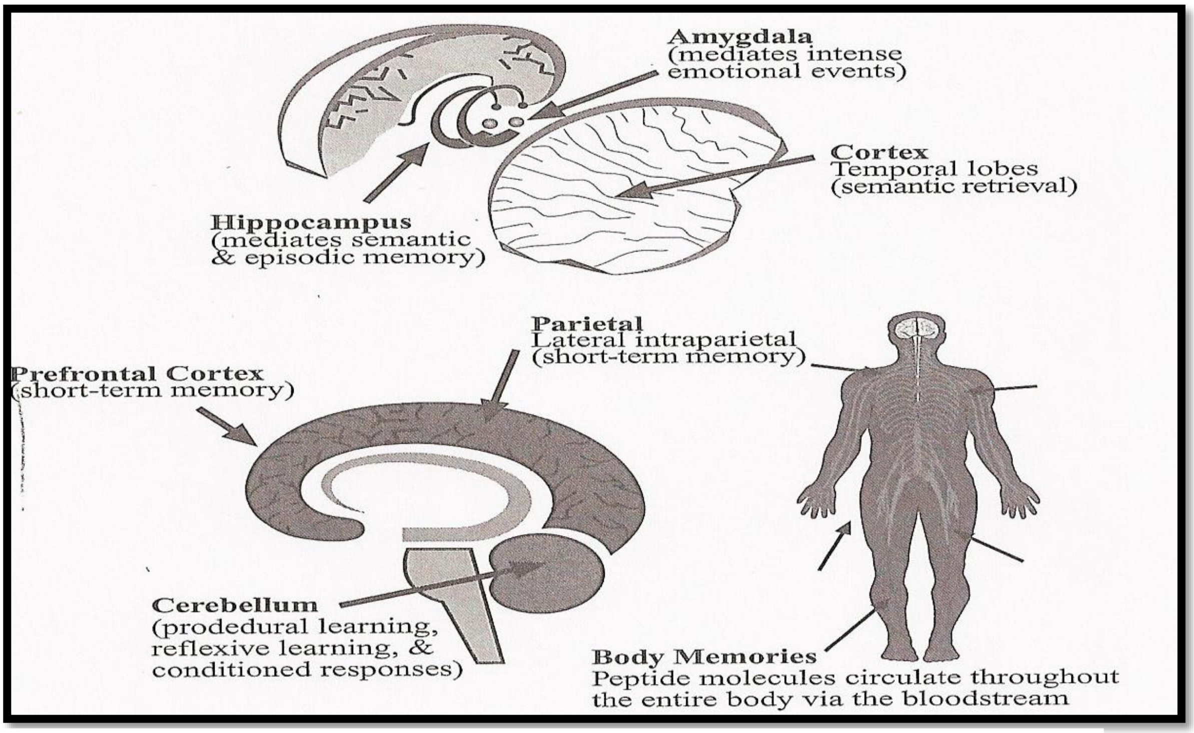


Figure 7·8A. Body structures that facilitate the dynamics of human memory (Jensen, 2000b, p. 23)

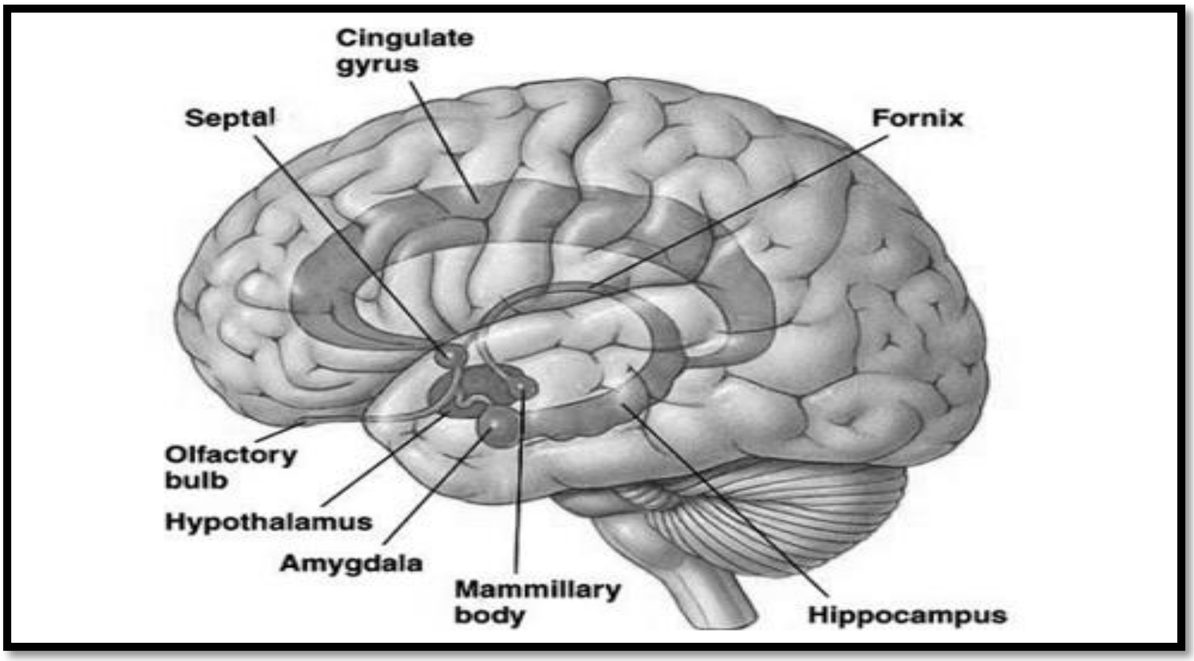


Figure 7·8B. The structures and nuclei of the limbic system began to evolve millions of years ago. Over the course of evolution, these emotional structures have expanded in size, some becoming increasingly cortical in response to increased environmental opportunities and demands. Until the neocortical forebrain expanded, the cerebrum of the ancestral line that would eventually give rise to humans, was dominated by the limbic system, but the latter nevertheless still remains highly influential in the reasoning ability of humans. (adapted from R. Joseph, 2012; also see LeDoux, 1989, 1995, 1996)

Summary insights: Semantic and Episodic Memory. The fundamental challenge of human memory is not encoding or storage but retrieval (Roediger, 2000). Semantic−episodic learning is a highly relational form of ‘cognitive−affective’ learning that affords the powerful mathematical learner with the schematic capacity to accurately and efficiently retrieve or transfer his or her learning. This is especially the case if the learner has been instructed didactically as an “active participant in the knowledge getting process, one who selects and transforms information, who constructs hypotheses and who alters those hypotheses in the face of inconsistent or discrepant evidence” (Bruner & Anglin, 1973, p. 397).

Visual−Spatial Intelligence

Powerful mathematical learning is a complexity of stimulus−response mediated interactions. Therefore powerful mathematical learning is dependent fundamentally on the signification that relates the stimulus with the response. In mathematics however, the "interplay between properties of visual images and abstract knowledge runs very deep in visual thinking" (Kosslyn, 1983, p. 190; also see Kidron, 2009). Consequently, the great value of visual–spatial self-awareness in mathematics is the heuristic benefit that it affords the mind of the mathematician in Being-able to interrelate the concrete with the abstract meaningfully, and especially the algebraic with the geometric (Giaquinto, 2007).

For example, a likely reason that Asian students have been superior in almost all aspects of mathematics compared to Western students, is not only because they have tended to practice more (Anderson, 2010), but because the orthography (visual text) of many Asian scripts (particularly Mandarin Chinese) is characterized by pictograms, symbolic words, associatives, and pictophonetic characters (Galligan, 2001). By implication therefore, students who study mathematics in a language like Chinese have a distinct advantage in Being-able to mediate successfully the question and the inquiry; the stimulus and the response, because a script based on pictures is more context dependent, less word dependent, and as a result understandings that are encoded in long-term memory are less prone to fragmented concept images and erroneous concept mapping (Bohm, 1980; Galligan, 2001; Hasemann & Mansfield, 1995).

The Irish philosopher Berkeley (1685–1753) was adamant that words often impeded the process and fluency of thought (Berkeley, 1950; Berkeley & Armstrong, 1965). Einstein concurred (Einstein, 1952; Gardner, 1993). The English Victorian polymath, Galton (1822–1911) went so far as to say that when thinking at a 'high and abstract level' he was not influenced by words at all (Galton, 1911, 1970).

However, the **Concrete–Pictorial–Abstract** (CPA) approach to the learning of mathematics is not widely used in many United States classrooms for example, perhaps because teachers feel that "concrete objects may be perceived by students as too elementary, or it may be that the content demands of the curriculum push teachers directly to the abstract level to save time" (Sousa, 2008, p. 187). Another possible reason that the CPA approach is infrequently used in the classroom learning of mathematics is that although young children readily visualize, as they grow older they are often encouraged away from imaging towards verbal expression (Kosslyn, 1983). To stress the active and epistemological possibilities of mental imagery, both teachers and students probably need to use the verb **imaging** rather than the noun **imagery** (Wheatley, 1997). This view is consistent with the Revised Taxonomy for Learning, Teaching, and Assessing which emphasized learning as an active process in the 'present progressive tense' (Anderson & Krathwohl, 2001). In addition, it was observed that although relatively few secondary school students used dynamic, or kinetic images when problem solving, those who did enhanced their chances of solving the problem (Presmeg, 1997a; Schwartz & Heiser, 2006)

The concept of visualization emphasizes imaging, because to visualize not only includes the generation and transformation of visual mental images, but also the construction and manipulation of physically drawn figures and diagrams on the page, the classroom board, or the computer screen (Presmeg, 1997a). In teaching the fundamental concept of the calculus to school students for example, the Cambridge Mathematics Project of Educational Services argued for a versatile learning sequence that enabled the student to visualize the concept of the limit through manipulatives, images and diagrams, and only then to engage more formally with abstract symbolic expressions (Bruner & Anglin, 1973). Moreover, a computer mediated approach that conceptualizes and embodies the limiting process graphically has been advocated by Tall (2000) for the purpose of developing a deep understanding of the calculus. Such an understanding would involve ideally the integration of maxima and minima problem solving, the dynamics of movement and forces, formal epsilon−delta analytic proofs, and the infinitesimal calculus of non-standard analysis.

However, mathematics is often characterized by the dialectical (e.g., $\sqrt{2}$ is a finite number that cannot be known, or represented exactly because its decimal is infinite and non-recurring; Euclidean Geometry involves an infinite plane). In other words visualization in World 2 enables the powerful mathematical learner to overcome the sensory limitations associated with World 1, particularly with respect to different kinds of infinity, irrational and imaginary numbers, and the algebra−geometry of more than three dimensions. Therefore, Being-able to image mentally that which cannot be experienced directly in the real world is crucial for the growth, development, and usefulness of mathematics in all Three Worlds.

From a phenomenological perspective, a deep understanding of a mathematical concept is limited to the degree that it has not been grasped epistemologically as a highly visual eidetic intuition (Merleau−Ponty & Lefort, 1974). It is through visual−spatial imaging that 'intuitive knowing' can signify an embodied conceptual understanding, or a feeling of 'bodily certainty'. It was the French mathematician, Hadamard (1865−1963) who reflected on the psychology of mathematical invention before declaring that

> any mathematical argument, however complicated, must appear to me as a unique thing. I do not feel that I have understood it as long as I do not succeed in grasping it in one global idea and, unhappily, as with Rodin [French sculptor, 1840−1917], this often requires a more or less painful exertion of thought. (Hadamard, 1945, pp. 65–66)

Therefore Being-intelligent to image effectively and creatively is crucial for powerful learners of mathematics in the Conceptual Age.

Gender differences. Male and female school students have compared favourably on measures of mathematics and science, intelligence, deductive reasoning, decision making, and working memory (Wigfield, Byrnes, & Eccles, 2006). However, girls have not always been as successful as boys when problem solving (Ackerman & Lohman, 2006; Benbow, 1988; Benbow & Stanley, 1980; Halpern, 2006; Jackson & Rushton, 2006; Leder, 1985; Wigfield, Byrnes, & Eccles, 2006). The reasons are complex, but mathematics teachers for powerful learning need to be aware that female students may require ongoing and deliberate support with respect to spatial perception, spatial visualization, mental rotation (the kinetic imaging of multi-dimensional structures in working memory), and spatio-temporal reasoning (Blakemore & Frith, 2005; Else−Quest, Hyde, & Linn, 2010; Fennema & Sherman, 1977, 1978; Halpern, 2006; Hyde, Fennema, & Lamon, 1990; Voyer & Doyle, 2010).

Visual−spatial cognition for powerful mathematical learning is embodied, and as such “the visual system is linked to the motor system, via the prefrontal cortex. Via this connection, motor schemas can be used to trace out image schemas with the hands and other parts of the body” (Lakoff & Núñez, 2000, p. 34). Therefore visualization is not a single ability, but a complexity of abilities that can be developed holistically through appropriate practice and a recognition that individuals exhibit differences when imaging (Kosslyn, 1983). For example, controllability differences might mean that an uncontrollable image could appear unbidden in a learner’s thought processes and persist in the face of contrary evidence including verbal learning (Mayer & Massa, 2003; Presmeg, 1997a; Zull, 2002).

In broad terms therefore, all powerful mathematical learners need to learn how to intentionally generate, maintain, transform, and scan images in working memory as an intelligent interplay involving words, spatial zones, and static or dynamic pictures and diagrams (Halpern, 2006). The following seven guidelines can aid students to grasp mathematics visually and haptically, and therefore intuitively, through the practised and imaginative use of four basic mental operations, namely, (a) **generating the image**, (b) **inspecting it**, (c) **maintaining it**, and (d) **manipulating it**:

- **(1)** Picture the situation as simply as possible;
- **(2)** picture the object or scene from multiple vantage points;
- **(3)** focus on the salient and idiosyncratic features of the object (e.g., colour, surface texture, shape, size, mass, smell, as well as static and dynamic elements);
- **(4)** make changes to the image and reflect if the changes lead to anything;
- **(5)** play with the material in an image, bending, folding, rotating, and moving parts around;
- **(6)** try to image an abstract model of the problem; and

(7) try to image a chart or graph that describes the problem situation holistically.
(adapted from Kosslyn, 1983, pp. 187, 190–192)

Perhaps the 'top-end' of such imaging is to conduct comprehensive thought (G. *gedanken*) experiments as was demonstrated by Einstein (1879−1955), the Serbian inventor Tesla (1856–1943), and the Austrian physicist Zeilinger (1945−) who investigated quantum oddities (Aldous, 2007; Brown, 2007; Gardner, 1993; Patrick, 2013; Zeilinger, 2010). Tesla in particular, used his vivid imagination to picture mentally and set in motion complex machinery, then after a few weeks of imaging he examined the machine parts for signs of wear and tear (Kosslyn, 1983)! This example is consistent with Carroll's (1993) meta-analytic structure of cognitive abilities: Originality and Creativity (FO) are dependent on General Intelligence (3G), which influences Broad Retrieval Ability (2R), which in turn influences Figural Fluency (FF) and Figural Flexibility (FX).

Progressive Insights: Being-intelligent

Human intelligence comprises multiple embodied systems. Therefore a systems metaphor of mind informed Gardner's Theory of Multiple Intelligences, Sternberg's Triarchic Theory of Intelligence, and Ceci's Bioecological Theory of Intelligence (Ceci, 1996; Gardner, 1983, 2009; Sternberg, 1985, 1990). Although all three theorists differed in their conceptualization of human intelligence and mind, none would have disagreed that working memory capability; the executive functioning of the pre-frontal lobes, and diverse brain systems require holistic development if the complexity that is Being-intelligent is to emerge and evolve optimally.

In a more recent model of the domain-general and domain-specific systems of the human brain, Demetriou (2009) posited that the architectural development of (mathematical) thought was crucially dependent upon the quality of working memory and ultimately that of a learned intelligence. In this regard it was the Directive-Executive Function (DEF) that capacitated and enhanced the processes and outcomes of working memory. Especially important for powerful learners of mathematics, the DEF actually mediates working memory — visually and phonologically — through the selective engagement, inferential control, and performance evaluation of visual−spatial, social, categorical, quantitative, verbal, and causal activities that involve the self of the individual.

However, students' problems with mathematics have often stemmed from their lack of correct images, or the use of inappropriate images, or the tendency to work instrumentally with algorithmic formalizations that were not understood relationally (Borgen & Manu, 2002). Oftentimes curriculum pressure has meant that mathematics teachers have moved onto new topics before students had imaged and laid down well-organized and accurate, but flexible and adaptable cognitive−affective intuitive structures in long-term memory. Notwithstanding, the trade-off between syllabus coverage and meaningful learning can be alleviated by embedding Cognitive Load Theory (CLT) protocols like Select−Relate−Organize−Check (S−R−O−C) into semantic−episodic lessons, particularly if the primary reason for implementing S−R−O−C is to enhance students' multiple intelligences for the purpose of Being-wise.

Being-intelligent for the Conceptual Age is inextricably linked to Being-wise as per Sternberg's (2003b) Balance Theory of Wisdom. As an example, the following metaphor between chemistry and powerful mathematical learning might assist mathematics teachers to grasp a '*phronesis* understanding' of what Being-intelligent could mean in their respective classrooms. As illustrated in **Figure 7·9**, the role of a catalyst in a chemical reaction has been to speed up the formation of a product by changing the mechanism of the reaction (Moore, Davies, & Collins, 1978). In effect the catalyst provides "an alternative path for the course of the reaction, in which the '*energy hill*' is lowered. It is possible that the catalyst forms an alternative activated complex requiring less activation energy" (Brink & Jones, 1979, pp. 131–132). Metaphorically therefore, the cornerstone principles of CLT (attention, activation, elaboration, encoding, and retrieval) might be able to facilitate 'catalytic dialogues' between *I* and *Other*, with the result that more feasible learning pathways can be established that require

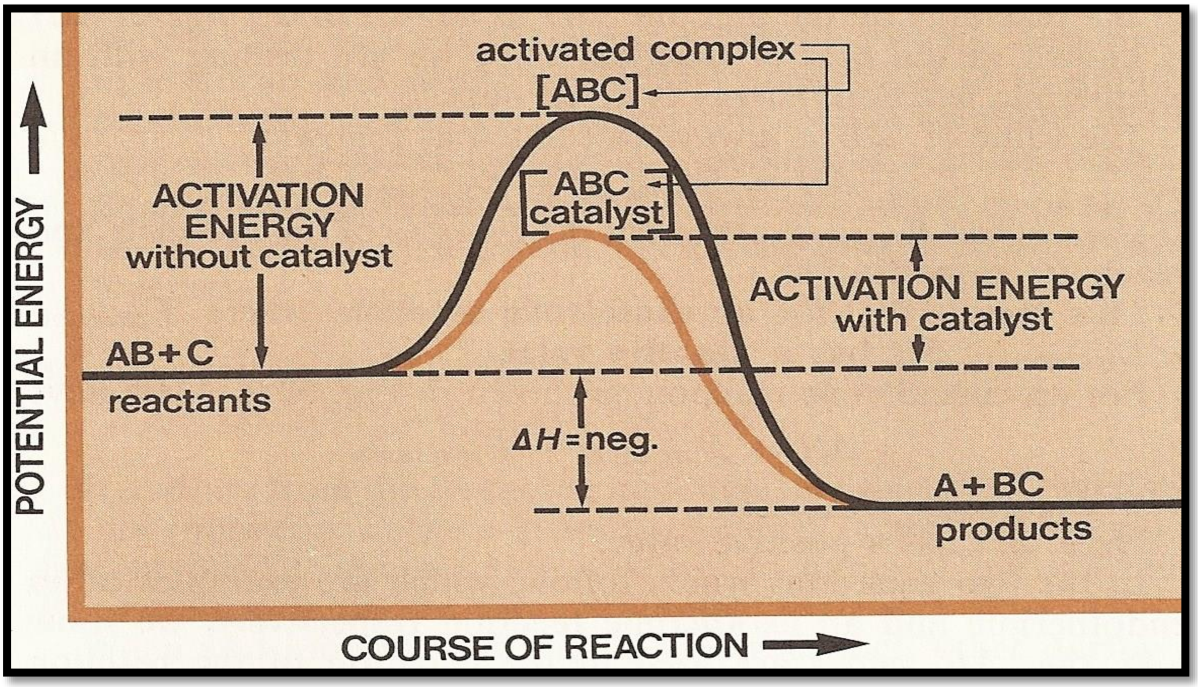


Figure 7·9. The effect of a catalyst on the activation energy of an exothermic reaction: $AB + C \rightarrow A + BC$, $\Delta H < 0$ (Brink & Jones, 1979, p. 132)

less mental and physical energy, or sustained activity than otherwise would be the case. In Being-dialogical therefore, it is hoped that powerful mathematical learning will be realizable for many students, at least in part because the activated complex between a challenging stimulus (reactants) and a successful response (product) will be more attainable.

Although the mind, or personalised brain is the most complex single object yet studied in our solar system,[2] it comprises just two per cent of the body's mass, but "has a ravenous appetite, consuming fully 20 per cent of our total energy (in newborns, the brain consumes an astonishing 65 per cent of the baby's energy), while fully 80 per cent of our genes are coded for the brain" (Kaku, 2014, p. 4). However, it is not only CLT principles that can make Being-intelligent and System II thinking (see p. 24) less cognitively demanding and more energy efficient. Nisbett (2009) has argued that the following social activities can also enhance, or accelerate intelligence cognitively from an early age:

(1) Breast-feeding up to nine months might increase IQ by approximately five points.
(2) Parents need to read to, and engage their children in 'adult-like' conversations where relatively high-level vocabulary is contextualized; curiosity is fostered; imagination is stimulated, and confidence is instilled.
(3) Children ought to be encouraged to play games of challenge that exercise working memory and kinetic imaging — especially through computer games that require attention and self-control, anticipation, stimulus discrimination, and conflict resolution.
(4) Instead of praising children for being intelligent, children should rather be praised for a sustained effort that is under their motivational control, and also facilitates flow. When children are praised for intelligence they tend to resist accepting a challenge that might be very beneficial for their learning, because it falls outside their emotionally-oriented 'praise zone'.

Point (1) reflects the rudimentary social nature of Being in the sense that the baby becomes part of the extended and 'languaging' self of its mother as it breastfeeds. Vygotsky (1978) noted that babies were born into, and developed psychologically through ongoing dialogues (e.g., between mother and child; father and child; siblings and child, etc.). Points (2) and (3) taken together reflect the Piagetian−Vygotskyan tenets of the CAME and CASE projects, and Point (4) is consistent with the research findings of Dweck (2000).

Therefore to a large degree Being-intelligent can be learned, and although "genes count, and given a constant environment they may have a big influence in determining talent," (Nisbett, 2009, p. 28) general intelligence is plastic (Adey, Csapó, Demetriou, Hautamäki, & Shayer, 2007), and therefore

> we can now shake off the yoke of hereditarianism in all of our thinking about intelligence. **Believing** [for emphasis] that our intelligence is substantially under our control won't make us smart by itself. But it's a good start. (Nisbett, 2009, p. 199)

End Notes

1. Assimilation and accommodation are complementary processes and the latter cannot proceed without the former (Anderson, Reder, & Simon, 1999). As Piaget (1977, 1985) argued, the progress of accommodation is dependent upon the schemata of assimilation, that is, it is not possible to build effectively upon, or modify cognitive structures which do not exist, or are largely incoherent. Notably however, assimilation is a relatively passive process compared to accommodation, because a student attempts to make sense of his or her experience by first strengthening his or her prior learning, or establishing new connections within the existing framework of his or her mental representations and operations.

2. The development of MRI machines and other advanced brain scans since the mid-1990s has revolutionized neuroscience. Consequently, institutes like the Max Planck Institute in Tübingen, Germany have prioritized brain research towards the goal of understanding the mind systemically in relation to an embodied brain that has approximately 100 billion neurons, with each neuron having up to 10,000 synaptic connections (Kaku, 2014).

Chapter Eight

Being-ethical

> We can only call a man's actions just or virtuous when the man who does them knows what he is doing; when he acts with deliberate choice, and his choice is based on the real nature of his action; and thirdly, when his tendency so to act is steady and not easily changed. (Aristotle as cited in Wallas, 1925, p. 77)

> The School is an entity surrounded by the rest of the world in which each individual struggles against that which restrains him — himself. (a graduating student as cited in Kohlberg, 1981, p. 48)

From a 'Deweyan progressive perspective' holistic human development should be the primary aim of education, and in this sense "the most important issue confronting educators and educational theorists is the choice of ends for the educational process" (Kohlberg & Mayer, 1981, p. 49). In other words what is the point of Being-intelligent? This question is underpinned by a moral philosophy that is ethics, where essentially, ethics is a substantive and analytic inquiry into how people should interrelate with one another as well as with themselves (Jewell et al., 2011; Mautner, 2005). The aim of such an inquiry, at least in Socratic terms, is to ascertain the most basic values or virtues (character traits) of a society, which are then "termed *moral*, and the major moral values in our society are the values of justice" (Kohlberg, 1981, p. 37). Succinctly put, especially in a Kantian sense, is to treat each individual in terms of a deontological reality, that is as an end in him or herself and not as a means to an end (Ruggiero, 2012). It is this ethical philosophical perspective that informs the teaching and learning of mathematics dialogically, because Being-dialogical as an end-in-itself is the exemplar modality of Being-ethical.

Humanity is globalizing towards a Type I civilization, which in effect means that within the next 100 to 200 years all power resources on the earth are likely to be exhausted. If humanity is to survive therefore, humankind needs to develop into a Type II civilization which implies being able to harness power increasingly from the solar system, and then as a Type III civilization would need to access large portions of the Milky Way galaxy (Kaku, 2006).[1, 2]

However, if earthlings are to realize such growth and development tremendous creativity is necessary. But as a myriad of globalizing‒localizing societies and cultures, the creative potential of our species is hindered severely by the enormity of inequity and injustice on the earth.

Ethics

Creativity is essential for our survival as a race of evolutionary hominids. In Being-creative the individual links essentially with the wholeness of its Being, and in Being-intelligent for a Common Good interrelates meaningfully and culturally with other minds in society. Thus learning to be creative involves the positive development of the whole Self, provided that development occurs in terms of a moral philosophy. If a Common Good is to be fostered however, which is tantamount to Being-wise for the culturally and environmentally situated individual, and potentially epistemic for Conceptual Age societies and future generations, then history suggests that human intelligence needs to include a mental disposition that is fundamentally ethical.

Moral Reasoning. The very core of **powerful** mathematical learning encompasses learning how to be ethical through moral reasoning, because Being-wise-in-the-world involves the resolution of disputes or points of difference in relation to individual differences. Essentially, the degree to which learning is not ethical through either cognition or affect is the degree to which the potential of the individual, the group, or the system is limited or undone. In accordance with the sentiments of Dewey and Pestalozzi for example, mathematics education needs to be underpinned by 'Being-caring — ideally the *Other* for the *I*; the *I* for society' (Ernest, 2009). History has indicated that 'exceptional notions of love' can unify different people towards a significant and Common Good (Buckley, 2012).

Therefore in order to facilitate Being-intelligent through an increasingly complex modality of Being-able to reason morally, Krathwohl and Anderson's (2001) Cognitive Processing

Objectives; Krathwohl, Bloom, & Masia's (1964) Affective Objectives, and Kohlberg's (1981) Six Stages of Moral Judgement have been sequenced together in **Table 8·1**. The developmental sequence culminates in universal ethical principles that can add or multiply value to Being-in-the-world, especially if the individual has learned how to be creative in relation to his or her glocalizing society.

However, because Being-ethical involves both Critical and Caring Thinking, a dialogical approach to teaching and learning suggests the use of the Socratic philosophic method. The Ancient Greek philosopher did not instruct directly, but posed insightful questions that were influenced by a fundamental belief that "morality is more than a matter of personal choice or convenience" (Ruggiero, 2012, p. 145). Socrates' morality was perhaps consistent with "a deontological ethic like Kant's, which says that rightness is only a matter of the universal form of the principle followed" (Kohlberg, 1984, p. 293). Nevertheless, Socrates examined the respondent's answers for vagueness and inconsistency with a view to further inquiry that would result ideally in the clarification of the problem or issue at hand. The first two columns of Table 8·1 list questions that can facilitate a Socratic dialogue between teachers or students for the purpose of developing a cognitive−affective 'ethical platform' from which, and about which powerful mathematical learning and Being-creative can be initiated.

***Streams of dialogue*.** The acronym **P−A−V−E (**Principles−Agreements−Values−End-consequences) has been developed as a tool to simplify the teaching of ethics in classrooms and schools, as well as to integrate the ideas articulated in Table 8·1 (Jewell et al., 2011). Epistemic **principles** are universally applicable standards that inform a 'morality that is reasonable'. Although not exhaustive, **Table 8·2** lists those moral ideals that are prized in many cultures around the world (Ruggiero, 2012). At the very least therefore, these ideals should influence all glocalizing contexts dialogically (cf., Franklin's list of values on p. 79).

Table 8·1. Learning ethics through six stages of moral development: Each stage involves both critical thinking and caring thinking which can lead to a mental disposition of Being-ethical (adapted from Jewell et al., 2011, pp. 25−27; Kohlberg, 1981, pp. 409−412).

Krathwohl & Anderson's (2001) Revised Taxonomy of Cognitive Objectives: Critical Thinking	**Krathwohl, Bloom, & Masia's (1964) Taxonomy of the Affective Domain: Caring Thinking**	**Kohlberg's (1981) Six Stages of Moral Judgement from a Social Progressive Perspective: Being-ethical**
1. Remember • What are the facts? • Who was involved? • What happened? • What proof is there?	**1.0 Receiving** • How did that make you feel? • How would you feel if ... ? • What did you notice?	**Stage 1:** Right is literal obedience to rules and authority, avoiding punishment, and not doing physical harm.
2. Understand • What is puzzling you? • Can you say that in a different way? • Can we clarify that point? • How/Why is that fair?	**2.0 Responding** • What was your initial reaction? • How do you feel now? • What should we think about first?	**Stage 2**: The person is aware that everybody has individual interests to pursue and these may conflict, so that right is relative in a concrete individualistic sense.
3. Apply • Are you aware of a similar problem? • How was that problem solved? • Could that resolution apply in this particular case?	**3.0 Valuing** • How could you defend his/her actions? • Whose idea do you identify with most closely? • What appeals to you most about his/her argument?	**Stage 3:** The person relates points of view through the 'concrete Golden Rule', putting oneself in the other person's shoes. He or she does not consider a generalized system perspective.
4. Analyze • Is that an assumption? • Is that a good reason? • If that is true, what else is true? • If that is true, what then must be false? • What positive outcomes happened as a result? • Imagine yourself as a bystander. How does your point of view change? • Did you act in a manner that was expected of you by your teachers/peers?	**4.1 Conceptualization** • Which of your values can be applied to structure this situation? • What do you think is the best choice, based on what/who you care about? • What do you feel was your duty to the other members of your group? • Why do you think that the leader of the group complained about your attitude?	**Stage 4A:** A person at this stage takes the viewpoint of the system, which defines roles and rules. **Stage 4 B/C:** An individual stands outside of his or her own society and considers him or herself as an individual making decisions without a generalized commitment or contract with society. **Stage 4C:** Moral decisions are generated from rights, values, or principles that are agreeable to all individuals composing or creating a society designed to have fair and beneficial practices.
5. Evaluate • Are your sources reliable? • Is this point relevant? • How do we know? • Are these ideas compatible?	**4.2 Organization** • What was the right thing to do? • How can the details be fitted together so that what happened makes sense?	**Stage 5:** The person considers the moral point of view and the legal point of view, recognizes that they conflict, but finds it challenging to integrate them.
6. Create • If you could, what would you change? How? Why? • Where do we go from here? • What if we do this ... ?	**5.0 Characterization by a Value Complex** • He/she changed his/her mind. Do you respect that? • How would you have acted?	**Stage 6:** The stage of universal ethical principles, especially the premise of respect for other persons as ends, not means.

Through the intentionality of consciousness however, all ethical principles must enhance the Beingness of the individual. Consequently, social contracts between people (including didactical contracts) need to be principle-based **agreements** that facilitate the internalisation, or development of **values** through dialogic activities that are associated with Being-there, or Being-in-the-world. Thus a powerful mathematical teaching and learning contract ought to specify that which is relationally important and ethical in Being-mathematical. In Being-ethical therefore, the **end-consequences** of principle-based teaching and learning are those agreements that are valued through "consent and consensus," (Partridge, 1971) and which means necessarily the transformation of multiple selves as a result of multiple streams of dialogue.

Table 8·2. Highly ethical people tend to view ideals as practical obligations that characterize their Being as a stable 'value complex' (Krathwohl, Bloom, & Masia, 1964; Ruggiero, 2012)

Moral Ideals	**Description** (adapted from Ruggiero, 2012, pp. 111−114)
1. Prudence	The exact opposite of rashness and impulsiveness. It can be thought of as practical wisdom or *phronesis*.
2. Justice/Fairness	The evaluation of situations according to their merits, without fear or prejudice.
3. Temperance	Socrates considered temperance almost equivalent to self-mastery.
4. Courage	A disposition of Being that steels the will and reinforces its resolutions in the face of significant challenge or harm.
5. *Agápe* love	An unconditional commitment to do right by one's neighbour, irrespective of the consequences to oneself or others.
6. Honesty	A refusal to mislead or deceive.
7. Compassion	An empathy for, and a willingness to help a person in need (even a bully or an enemy).
8. Forgiveness	Granting others absolution for their offences against us.
9. Repentance	An heartfelt apology for, and a turning away from wrongdoing.
10. Reparation	Undoing the harm, if possible, that the *I* inflicts on the *Other*, or vice versa.
11. Gratitude	A sense of appreciation and thanks for an act of kindness or generosity.
12. Beneficence	The performance of good acts for no other reason than that they are good.

In particular, transformative P−A−V−E **dialogues** have the potential to underpin and augment problem solving, Being-mathematical, and who human beings can become, because at its source (Gk. *dialectos*) the *raison d'être* of dialogue is to facilitate "opposing voices in search of truth" (Baker, Jensen, & Kolb, 2002, p. 11). Dialogue is fundamentally a 'unifying' conversational complexity. In this regard and as indicated in **Table 8·3**, five different dialogic streams or conversational flows have been considered, namely, Hearing *Others*, Heard by *Others*, Aware of *Others*, Differ with *Others*, and Compare with *Others*.

The conscious *I* of an embodied self hears the interpersonal *Other* in the sense that hearing is a 'reaching activity'. Hearing has been conceptualized as an embodied action that 'reaches for' mutual understanding whereas listening tends to be more passive (Davis, 1996). Thus the hearing *I* attempts to amplify the languaging body of the *Other* by resonating with, and reflecting on the vocal and gestural actions that are seen and heard. In these terms it is possible for different minds "to resonate, like tuning-forks, in harmony with one another" (Ryle, 1949, p. 57). Therefore the dialogic purpose of **Stream I** is to internalize the *Other* as an *I*-position, or as a particular *other-in-the-self*. In so Being the *I* is able to language with the interpersonal *Other*, because **languaging** can potentially take the form of **any** "symbolic display, action, or communication within human communities — verbal or nonverbal — intended to establish, question, or otherwise negotiate social and personal meanings and coordinate behaviour" (Neimeyer & Mahoney, 1995, p. 406).

Table 8·3. Five Streams of Conversational Learning (Jensen & Kolb, 2002, p. 127)

Stream I	Stream II	Stream III	Stream IV	Stream V
Resonating and Reflecting	*Expressing and Interacting*	*Attending and Appreciating*	*Interacting and Conceptualizing*	*Listening and Analyzing*
Gaining understanding of the meaning of one's own experience and/or others' experiences through resonating and reflecting in and through conversation.	Gaining understanding about one's own perspectives and feelings through expressing them, and feeling and hearing others resonate and respond during the course of conversation.	Gaining understanding of specific others and self through attending to and appreciating the interaction in the "here and now" of the conversation.	Gaining understanding of one's own and others' perspectives and feelings through interacting in conversation with others who hold and express different perspectives.	Gaining understanding of others' perspectives and feelings about the topic of conversation through listening and interpreting others' interaction in the conversation.
Hearing Others	*Heard by Others*	*Aware of Others*	*Differ with Others*	*Compare with Others*

As an outflow of Stream I, the dialogue that is **Stream II** emphasizes the hearing, seeing, and Being of the interpersonal *Other* who relates dialogically to the languaging of the *I*, whose communication resonates with the Being of the *Other*, and is heard consequently by the *Other*. In turn the interaction of *I−Other* is extended to include *Others* in the dialogic engagement. Then in Being-aware of *Others, I−Other* initiates **Stream III** by inviting multiple similar or contrasting points of view. Through critical self-reflection however, the *I* needs to develop intrapersonal meta-positions, or 'promoter positions' for the express purpose of meaningfully organizing, and giving order and direction to the *others-in-the-self* who corresond dialogically to the voices that are essentially the polyphonics of Stream III. Consequently, in the 'making of someone' through dialogue (Abbey & Valsiner, 2005; Valsiner, 2004), promoter positions are vital as innovators and creators of the self in the sense that they

> imply a considerable openness towards the future of the self and have the potential to produce and organize a diverse range of more specialized but qualitatively different positions in the service of the development of the self as a whole. Due to their openness and broad bandwidth, they have the potential to synthesize a variety of new and already existing positions in the self and reorganize the self towards a higher level of development. (Hermans & Gieser, 2012)

However, to compare and contrast is not only the basic method of science but also of dialogue. Therefore the conceptualizing and analyzing of dialogues that are **Stream IV and Stream V** respectively, in Differing with *Others* and Comparing with *Others*, are attempts on the part of the individual *I* to substantiate and test creative understandings for the purpose of preserving, or effecting the coherence of both the embodied and the extended self. The degree to which the *I* achieves such coherence, albeit dialectical, is dependent fundamentally upon the ethics of the dialogic community to which the *I* belongs, which means essentially 'Being-committed' to a Common Good for all participants, and perhaps even for society at large. The outflow of Streams IV and V is to eventuate a 'spiral dialogue' that commences with a more complex Stream I dialogue: Hearing *Others*.

A spiral dialogue. In order to develop further the notion of 'an ethical dialogue' the Wyss−Flamm (2002) Model of Conversational Learning is required and discussed. **Figure 8·1** is a diagrammatic representation of the model which has been characterized by four phases of conversation. Although the term 'conservational' was used to label the model, the spiral structure and associated terminology are useful for the purpose of understanding the spiral nature of an ethical dialogue. It is noteworthy that a conversation tends to be an informal interaction between individuals, whereas in this study, a dialogue is an intentional and deeply principled human engagement whereby 'Being-*I*' is transformed in-relation with an *Other*, and very often a 'knowledgeable and respected *Other*' (Ripley, 2013; Takahashi, 2014).

The Wyss−Flamm Model was developed, at least in part, to promote democracy and social justice in classrooms, schools, and society. The development of the model was influenced by Kolb's (1984) notion of experiential learning, where learning was defined as a process of knowledge discovery and invention, fuelled by the dialectic complementarity that is the grasping and transformation of experience. Moreover, the model is underpinned by three assumptions, namely, (a) that humans are intrinsically curious and want to learn; (b) the encounter of difference and redundancy in conversation is essential if meaningful learning is to occur, and (c) the self of the individual is unlikely to grow in a positive sense if conversation takes place in an environment which is not psychologically safe.

In **Phase 1** the *I* of the individual is exposed to, and experiences the difference of the interpersonal *Other*. Therefore Phase 1 is tantamount to a Stream I dialogue, where the *I* resonates with, and reflects on the Being of the *Other* who is in-relation to the *I*. Consequently, Phase I includes both the interpersonal and intrapersonal dimensions of 'Being-*I*', or otherwise stated the explicit and tacit dimensions of Being-*I*. At this stage of the conversation the Self-system of the *I* may choose to no longer participate.

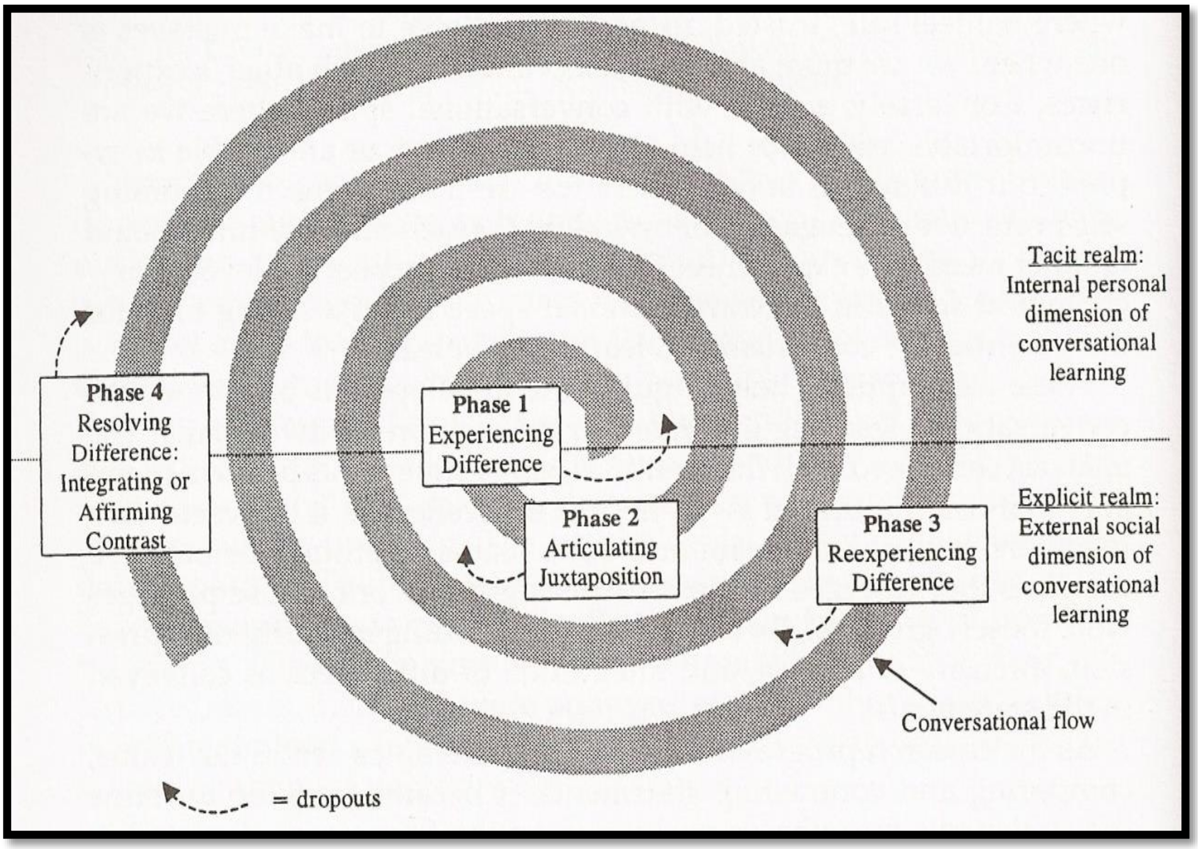


Figure 8·1. A Model of Conversational Learning where individuals may choose to dropout out of the conversational flow after any one of the respective phases (Wyss−Flamm, 2002, p. 152)

If the conversation proceeds however, it tends to do so explicitly through languaging bodies,which means that the conversation 'widens or deepens' psychologically through the juxtaposition of interacting ideas. **Phase 2** of the conversation is essentially a Stream II dialogue. If the conversation proceeds further, then it is likely to do so because the *I* attends to, and appreciates the contribution of the *Other*, and consequently chooses to re-experience the points of difference that were communicated by the *Other* as part of Phase I. **Phase 3** is essentially a Stream III dialogue. In Being-aware of the *Other*, this explicit social interaction is likely to enhance an existing *I*-position, or facilitate the development of a new *I*-position as an *other-in-the-self*, which corresponds to the interpersonal *Other* who is part of the extended self of the *I* through a substantive ethical conversation, or dialogue. In turn, **Phase 4** as a Stream IV and Stream V dialogue, is a transformative and ethical attempt on the part of the *I* to reconcile the 'experienced difference' of the *Other* with respect to existing *I*-positions.

This could involve a dialectical integration of thesis and antithesis, or an affirming of contrast. Depending on the outcome, the dialogue between *I* and *Other* may continue to spiral between the tacit and explicit realms of conscious awareness, and include not only a single *Other* but also multiple *Others*. In these terms it is noteworthy that the conscious awareness of the *I* is situated, or located explicitly and tacitly in a present history that includes a possible, or likely future in terms of both autonoetic (episodic) and noetic (semantic) consciousness respectively, namely, that

> autonoetic consciousness is intimately associated with our awareness of ourselves as persons with a history, and a future. It gives us the ability to mentally travel through time in a self-reflective way. Noetic consciousness is a more abstract sense of the past, and the future. It does not entail mental time travel but our awareness of knowledge that we have about the world we live in. (Gardiner, 2000, p. 160)

Progressive Insights: The Tacit and Explicit Realms of Being-ethical

In January 1877 a famous article, ‘The Ethics of Belief’ was first published. In the article the University of Cambridge mathematician and philosopher, Clifford (1845−1879) contended that it was wrong “always, everywhere, and for everyone, to believe anything on insufficient evidence” (as cited in Mautner, 2005, p. 202). However in 1896, the American psychologist and pragmatist philosopher James (1956) repudiated Clifford’s argument. In an essay entitled *The Will to Believe*, he asserted that what Clifford had written was ‘a prehistoric form of philosophic idolatry’. In relation to these ‘extreme views’, the highly influential Vienna Circle (also known as the Ernst Mach Society) developed and promoted the experiential ‘inductive and deductive’ philosophy that was logical positivism. From its ‘seminal beginnings’ in the first decade of the 1900s until the disbandment of the Vienna Circle in the 1930s when the Nazis came to power in Germany, the Circle’s unfolding discourse was to unify science and scientists through a common language that articulated a specific epistemology, namely, the scientific method. Consequently, the Austrian scientist Mach (1838−1916) in agreement with the pragmatism of James insisted that:

> According to our conception, natural laws are a product of our psychological need to feel at home with nature; all concepts transcending sensation are to be justified as helping us to understand, control and predict our environment, and different conceptual systems may be used to this end in different cultures and at different times with equal propriety. (as cited in Stokes, 2006, p. 125)

Essentially therefore, the philosophy of logical positivism was founded upon the **Verifiability Principle** “which accepted two kinds of statements as meaningful: the analytic ones (plus their negations), and those whose truth or falsity could be tested by perceptual and empirical experience. Other statements were rejected as unscientific and, indeed, as cognitively meaningless” (White, 2005b, p. 646). In particular however, although not always verifiable by experiment the Vienna Circle did accept logical mathematical statements as correct. This was the major difference between ‘their’ positivism and previous forms of positivism that could be traced back to the political and liberal ideas of the French philosophical sociologist Comte (1798−1857). Nevertheless, the only knowing that logical positivists would accept as ‘true knowledge’ was that which was explicit epistemologically through method and logic, and therefore clearly and independently verifiable by different people.

However, by the 1920s the debate surrounding the question of the ‘certainty of knowledge’ had become fundamentally an ethical debate, because that which was valued and ‘certified’ as knowledge would inevitably limit and delimit; empower and disempower human beings in relation to their Being. But by this stage, the German philosophers Husserl (1859−1938) and Heidegger (1889−1976) were engaged in ongoing interpersonal dialogues that would ultimately realize the foundations of phenomenology, ‘the method of reduction’, and a rejection of the ‘narrow’ tenets of logical positivism. Although it was left to Merleau−Ponty (1962, 1964) to articulate the phenomenology of languaging bodies or the intercorporeality of beings, Heidegger (1967, 1970) claimed that the phenomenological method was readily experienced in conversations that he had with his mentor, Husserl.

The phenomenological method of reduction. Neither Husserl nor Heidegger described their experience of the reductionist method in straightforward terms. This is not surprising, because ‘the method’ relates to a complex modality of Being that cannot be understood solely as a process or a mental representation (Dreyfus, 1991). The method necessarily involved both the interpersonal (explicit) and intrapersonal (tacit) dimensions of Being. Consequently, Husserl (1927) described ‘the method’ as the only method which resulted in ‘true inner experience’, because it was based inter-subjectively on a ‘society of persons’ who shared a conscious life through phenomenologically reduced and concretely apprehended *phainomena*. For example, Heidegger’s (1927) written words in *Being and Time* expose the reader to physical things or objects that are signified (*noumena*), but in a sense of the author’s ‘actual thoughts’ which can lead to an imagined ‘wordless meeting of minds’ between the reader and the philosopher (Heidegger, 1927; Merleau−Ponty & Lefort, 1974).

Moreover, Husserl (1927, 1970) in particular claimed that ‘the method’ could be practiced and there was no limit to its practice because the intentionality of human consciousness was essentially limitless. The purpose of the method was in accord with the notion that “men now demand that empirical psychology shall conform to the exactness required by modern natural science” (Husserl, 1927, p. 5). If however, ‘a thing’ was to be experienced in ‘pure psychological terms’, or independently of the person’s beliefs, values, and cultural sensitivities as a consequence of Being-in-the-world, then the individual had to practice ‘epoche’ which meant ‘bracketing the world’ from the eidetic intuition, or *phainomenon* that emerged into the conscious mind of the individual. The point of conversing therefore, or dialoguing at length with *Others* concerning ‘a thing’, was to pave the way for a ‘unity of sense’ to emerge in the mind of the individual, that is, what the *noumenon* actually was independent “of the world that was simply there” in the mind (Husserl, 1927, p. 4). It was for this reason that the adjective ‘reductionist’ was used to describe the phenomenological method and to stress the *phainomenon*, or outcome of the reductionist method as a thing literally but subjectively experienced.

Phenomenological method enhanced. Husserl's approach to Being-phenomenological was ethical, because he did not wish to 'contaminate' the eidetic constituents of his mind with 'who he was' by Being-in-the-world. Consequently, Husserl emphasized the importance of 'going back to the things themselves' (Husserl, 1927). Essentially therefore, he grappled with the possibility of establishing an isomorphic relationship between the explicit things of the real world and the tacit things of the mind. In so Being, the phenomenological method of reduction commenced with the *I* thoroughly exposing the mind to the thing being scrutinized. It was a mind−body literal event through the executive and dialogical functioning of the *I*. A foundational tenet of phenomenology is that consciousness is intentional — an idea that was espoused by the German philosopher Brentano (1838−1917) when he lectured at the Universities of Würzburg and Vienna. It was Dreyfus (1991) however, who noted that "*comportments* have the structure of directing-oneself-toward, of being-directed-toward. Annexing a term from Scholasticism, phenomenology calls this structure *intentionality*" (p. 51).

Consequently, the intentional consciousness that is *I*-consciousness 'refers' the scrutinized and analysed thing, or object to a self-organizing mind that operates largely at a non-conscious level of functioning. Cognitive Science has demonstrated empirically that the individual's mental functioning occurs largely outside the conscious awareness of the person (Lakoff & Johnson, 1999; Lakoff & Núñez, 2000). This concept of mind is compatible with a self-organizing, or 'ecological autopoietic complexity' that is perhaps similar to certain biological systems that have the capacity to maintain and reproduce themselves (Camazine et al., 2001; Maturana & Varela, 1980; Popper & Eccles, 1977; Varela, Thompson, & Rosch, 1991).

In other words the Directive-Executive Function (DEF) of the agentic *I* can direct the self-organizing mind to illuminate in consciousness a ‘pure’ eidetic intuition, which is essentially a detailed and accurate mental image of the object or thing in World 1 or World 3. There is necessarily a ‘time delay’ between the directive of the *I* and the emergence of the eidetic intuition in consciousness, because the self-organizing mind requires time to synthesize a System I narrative, and then for the *I* to become aware of that holistic narrative. Once the narrative is complete, or partially complete the *I* can engage with the image or intuition descriptively, analytically, or interpretively which means that System II thinking can occur in relation to relevant or disparate *I*-positions. The emergence of the intuition in consciousness is largely dependent on the ‘agentic expectation’, or self-belief of the *I* that this **will** occur. In other words, the intentionality of the *I* ‘wills it into conscious existence’.

However to enhance the possibility, efficiency, and quality of intuitive functioning, it is beneficial for the *I* to have a high level of expectation that the self-organizing mind **has** the ability to synthesize a meaningful narrative, or mental image in response to the intentionality of *I*-consciousness. Moreover, given that the extended and embodied self is social, it is probably advantageous to treat the self-organizing mind as an intrapersonal *Other* that is capable of profound intellectual synthesis through System I thinking and Being-intuitive. However, relying upon the unseen ability of an intrapersonal *Other* to act in the best interests of, and in agreement with the directive functioning of the *I* requires trust or faith on the part of the *I*.

Faith. Historically throughout the world and even in the twenty-first century, “every culture has maintained a belief in some form of a spiritual reality” (Alper, 2008, p. 3). Therefore human beings appear to have an innate ability, or need to believe in the intangible as if it were tangible, or in the possibility that the intangibility of the tangible can be made to materialize in the real world or in the mind. It is through a particular intentionality of *I*-consciousness – referred to as ‘faith’ — that ‘connects or links’ the needs and desires of the conscious *I* with an **unseen** *Other* who is capable of responding to those needs or desires, especially creatively. In the language of Dialogical Self Theory (Hermans, Rijks, & Kempen, 1993; Hermans & Gieser, 2012), the unseen intrapersonal *Other* can be understood as a ‘special kind’ of promoter position that is dependent upon the level of faith, or affective intentionality of *I*-consciousness as to the nature of response that the *I* might experience from the unseen intrapersonal *Other*.

It is through thought, speech, and gesture that the *I* actively stimulates or provokes the intrapersonal *Other* to respond intuitively, because cognition is essentially embodied and social. It requires faith on the part of the *I* to ‘approach’ or comport toward the unseen intrapersonal *Other* intentionally, because the self-organization or autopoiesis (Maturana & Varela, 1980, 1987) on the part of the intrapersonal *Other* is beyond the control, determination, and vision of the *I*. The resulting intuition might be brilliant or inept in relation to the real world, but the idea is that Being-human occurs to a large degree outside the conscious awareness of the subject that is *I*, and if the object that is *Me* is to unfold through a creative, coherent and unified narrative, or System I thinking, then such transformative change is dependent upon “a particular complexity of beliefs, or a faith, if you will, in *something*” (Holden, 2009, p. 576).

Therefore in terms of a dialogical self, ‘*Other*’ refers not only to another person, but also to an embodied and unseen (neither physically nor with the mind’s eye) entity that can operate quasi-independently of the human subject that is *I*. The unseen intrapersonal *Other* is not to be confused with the simplistic and mythical homunculus, namely, a dwarf-like creature who is supposed to abide within the embodied self for the purpose of organizing and empowering thought. The idea of an unseen tacit *Other* is to emphasize that the human mind is not only complex through *others-in-the self*, which are essentially the *I*-positions, but that each *I*-position has a non-conscious dimension, or complex intrapersonal *Other* in relation to the prior learning and Being-there of the individual. Notably, the Christian apologist Lewis (1898−1963) remarked that “a person cannot help thinking of himself as, and even feeling himself to be ... two people, **one** [for emphasis] of whom can act upon and observe the other” (as cited in Hermans & Hermans−Konopka, 2010, p. 120).

Moreover in Jungian psycho-analytic terms, the unseen tacit *Other* may include the ‘collective non-conscious’, or rich tapestry of universal archetypes, archaic patterns and images that represent the historical record of humanity and civilization in the socio-cultural and experiential heritage recorded by Being-there (Jones, 1999). In Being-human the unseen tacit *Other* may be this repository, or may draw on this repository in response to the agency of the *I*.

When teaching for powerful mathematical learning the teacher can perhaps use the notion of a ‘more’ powerful tacit *Other* to prime the expectation of the individual learner, as to what may be possible in relation to his or her learning, creativity, and achievement. In the development of humankind, it appears as if people haved needed an unseen reality as part of their psychology to promote a level of Being that otherwise would probably not have been attained. Thus phenomenologically and dialogically, ‘*I*-faith’ in an unseen tacit and beneficent *Other* is perhaps a prerequisite condition of possibility, if students in mass mathematics education are to experience a level of Being-mathematical that is powerful (Churchill & Richer, 2000). But not students only, “teaching is an act of faith, which requires, for many strong investment of the self” (Woods & Jeffrey, 1996, p. 7). In *An Essay on Man,* the English poet, Pope (1688−1744) linked faith to an unseen *Other,* which according to the poet led to an enduring hope, namely,

> *Hope springs eternal in the human breast;*
> *Man never is, but always to be blessed:*
> *The soul, uneasy and confined from home,*
> *Rests and expatiates in a life to come.* (Pope & Boynton, 1903)

However, in the hope that powerful mathematical learning might be didactically and dialogically possible between a mathematics teacher and his or her students, it is an ethical duty of the teacher to discuss the notion of an intrapersonal and unseen *Other*. That is not only in religious terms, but also as an autopoietic self-organizing reality which is part of the *Me*, and thus dependent on an organismic and social brain for its global−synthetic and intuitive functioning (Camazine et al., 2001; Popper & Eccles, 1977; Varela, Thompson, & Rosch, 1991; Wilde, 2010).

Husserl and Heidegger diverge. Although Husserl appears to have moderated his phenomenological intent towards the end of his life, for him, the method of reduction was essentially a descriptive sequence of subjective mental processes; the aim of which was to illuminate the real world into human consciousness in 'purely inter-subjective' terms. Through rich and detailed imagery, although in themselves abstractions of the real world, Husserl's intent was that each mental representation should accurately and holistically reflect a thing of the real world. This generation of 'subjective reality' was a direct consequence of an intentional act of consciousness. However, if the eidetic intuition as a subjective reality was to be objective in relation to the real world, then for Husserl the self of the individual was not permitted ethically to perpetrate the act of consciousness, but was

> the observing subject of the act. But this subject is never given in experience, is never, in Husserlian terms, the object of an intentional act. Accordingly, Husserl endorses a view akin to Kant, that the subject of experience is transcendental — outside the spatio-temporal causal order. (Stokes, 2006, p. 149)

Heidegger however, did not concur that Being involved a transcendental component because for him, Being-there "is a perspective, which, it turns out, is a locus of action extended through time. In sum, *Dasein* is a perspective from which action originates" (Stokes, 2006, p. 151). For Heidegger therefore, the illumination of an intuition in consciousness represented an opportunity for the agentic *I* to investigate, inquire, and interpret the imaging in relation to the real world, or the usefulness of the intuition in enabling the *I* to make sense of the real world.

Furthermore, Heidegger came to the conclusion that without a transcendental perspective, Being-there **and** knowing a thing absolutely was not only impossible for the human mind, but it was also not essential (Heidegger, 1927, 1967). The meaning of Being-there was not to know absolutely, or to comprehend the real world objectively, but rather the essentiality and purpose of Being was an ongoing interpretation and re-interpretation of essences-in-mind in relation to being in the real world. For Heidegger therefore, the growth and development of Being-there occurred as an ‘ever-deepening’ backwards and forwards interpretive motion on the part of the *I* between the question, or questions posed and corresponding inquiries. Without these bidirectional subjective motions, Being-in-the-world would have no essential reality and consequently there would be no possibility of Being-ethical. It is only as a developing essence of ‘Being-interpretive’ that the *I* comes to understand in relation to the three Worlds.

Progressive Insights: Being-ethical

The main point of Being-intelligent in the twenty-first century is to facilitate a creativity of Being-ethical whose outcome is a Common Good for a glocalizing world, which implies a Successful Intelligence that “if two responses produce good or two produce harm, choose the one that produces the greater good or the lesser harm” (Ruggiero, 2012, p. 155). This ethical view of Being-intelligent is underpinned by Sternberg’s (2003a, 2007) Balance Theory of Wisdom. That is in the sense that Successful Intelligence is dependent upon Values to balance Interpersonal, Intrapersonal, and Extrapersonal interests on the one hand, and the Selection, the Shaping, and the Adaptation of the individual’s environmental-related responses on the other hand, provided that the person’s intentionality is to realize a Common Good.

Therefore, Being-ethical is crucial to the type of intelligent and creative based outcomes that are desirable for the Conceptual Age. But Being-ethical is developmental (Kohlberg, 1981), and this development is closely associated with an individual's ability to cognize affectively and epistemologically so that the potential of *I-Other* is not unduly limited by Being-in-the-world. Balance is a key. Clifford emphasized that beliefs and values should be underpinned by sufficient evidence. James inferred that the human Will was sufficient to believe. The logical positivists contended that beliefs and values were meaningless if not substantiated rigorously through the evidence that was sense perception or logic.

Holistically, the philosophy of phenomenology has attempted to make sense of Being-human and is therefore a pursuit in ethics. In particular, Husserl's goal was to develop an ethical method that would validate an intrapersonal understanding of the real world that was consistent essentially with the different perspectives articulated by Clifford and James, as well as the knowledge claims of the logical positivists. But both Husserl and Heidegger realized that such a validation would have to commence in terms of a locus of interpersonal being where the human will, or consciousness played an intentional and mediating role.

However, it was primarily Husserl who made the ethical, aesthetic, and pragmatic decision that the phenonomenological method needed to be 'reductionist' fundamentally if things in the real world were to be pure and holistic manifestations, or representations in consciousness. In accord therefore with minimalist Gestalt psychology, if any attribute of the representative image in consciousness was to be removed, then the thing in consciousness would no longer be recognizable, or representative of the thing as it was in the real world (diSessa, 1983; Resnick & Ford, 1981). Moreover, Husserl and Heidegger appreciated that if a pure manifestation of a thing was to be represented ethically in terms of a conscious reality, then the intuitive functioning of the individual should not be influenced by the personal beliefs and values of the individual. Therefore the human will, or intentionality of consciousness could mandate the generation of a literal response in consciousness, that is in relation to the thing or *noumenon* in the real world, but the self was not permitted to participate in the act of generating the pure image in consciousness. Over time, Heidegger rejected the possibility of Being-human in these terms, but Husserl in accord with other dualist philosophers like Descartes and Kant, endorsed the view that the subject that was *I* must stand outside the ordinary causal order, if the things of the real world were to be grasped independently of the values espoused by the *I* (Stokes, 2006).

Yet if the self is fundamentally dialogical through a 'social intentionality' that is consciousness, then Husserl's and Heidegger's respective views are not necessarily incompatible. In terms of the emergent and developing theory that is Dialogical Self Theory, an unseen intrapersonal reality can be thought of as a transcendental *Other,* who although embodied, acts outside of *I*-consciousness in response to an agentic and sensory perceptive *I* that directs the unseen intrapersonal *Other* to illuminate in consciousness a literal and intuitive grasping of real world things. Thus the dialogical self through the intentionality of consciousness is able, at least to some degree, to separate agency from capability.

Consequently, a Heideggarian backwards and forwards interaction between an agentic, analytical, and questioning *I* is theorized in relation to an embodied transcendental *Other* who is capable of profound global−synthetic acts, at least in part, because the *Other* is not constrained by the limitations of working memory. Moreover, backwards and forwards question−inquiry movements between diverse *I*-positions are possible, especially if mediated by metacognitive and promoter *I*-positions.

So by bringing together the two phenomenologies of Husserl and Heidegger in the dialogical self, it is phenomenologically possible to develop a society of mind on the basis of a mind in society and a body in the real world. Although Husserl and Heidegger appreciated Being in interpersonal and intrapersonal terms, it was Merleau−Ponty (1962, 1964, 1974) who elucidated the notion of an 'intercorporeality of being', or languaging bodies as the vital mediator between the explicit interpersonal realm of being and the tacit intrapersonal realm of

Being. Currently however, Massive Open Online Courses (MOOC) characterize distance education (Darmawan & Keeves, 2013). The question is posed, "If the online learning does not involve languaging bodies in terms of a zone of proximal development, are the courses ethical?"

The word 'tacit' is chosen in this study with the intent to illuminate the intrapersonal dimension of Being. If a real world thing, or coherent entity is to be known tacitly, then it is the responsibility of *I*-consciousness, or as a mind in society to make the self aware of the particulars of the entity. However, if *I*-consciousness attends to the particulars of the entity then *I*-consciousness can lose sight of the entity as a whole. Consequently, a society of mind involves at least two levels of tacit reality, namely the proximal, which focuses the particulars of the entity, and the distal which is a global-synthetic comprehension of the entity. The respective levels of reality are necessarily

> controlled by distinctive principles. The upper one relies for its operations on the laws governing the elements of the lower one in themselves, but these operations of it are not explicable by the laws of the lower level. And we could say that between two such levels a logical relation holds, which corresponds to the fact that the two levels are the two terms of **an act of tacit knowing** [for emphasis] which jointly comprehends them. (Polanyi, 1966, pp. 34–35)

Therefore 'the distal' of Husserl's phenomenology was a single, holistic representation in the mind of the philosopher. It was not interpreted by Husserl, but was described as accurately as possible in terms of the proximal details, because the eidetic intuition was meant to reflect a pure manifestation of the 'real' in *I*-consciousness. But Heidegger's phenomenology was more complex. It probably involved whole sequences of modified and interrelated intuitions as the philosopher analysed the question, that is the particular, in relation to the intuitives which were the distal, or vice versa.

The testing of ideas. The integrity of Being-ethical requires that ideas, including those that have their origin in the liminal space, or the 'space between' the languaging bodies of Husserl and Heidegger, be tested for coherence in relation to World 1 and World 3. In

Popper's philosophy of Three Worlds, The Mind has no essence apart from World 1 and World 3. By implication therefore, the meaning making or phenomenology of World 2 cannot be coherent essentially unless related logically to an 'understanding' **between** people on the basis of The Natural−Physical World (World 1) and The Culture and Creativity of Diverse Human Societies (World 3). It was Merleau−Ponty and Lefort (1974) who emphasized that understanding was an **interpersonal** event. In terms of the dialogical self however, the tacit dimension of Being is not only embodied but includes the extended self. Therefore the distal that is understanding and the proximal that is meaning making, or vice versa means that the distal and the proximal can interrelate logically and interchangeably between the tacit (intrapersonal) dimension of Being-ethical, and the explicit (interpersonal) dimension of Being-ethical.

From the perspective of Clifford and the logical positivists who attended Schlick's (German physicist, 1882−1936) Saturday morning seminars in Vienna in the 1920s however, mathematical understanding and the certainty of knowledge was established solely through logical deduction on the premise of postulates or axioms that 'made sense' **to** the minds of the mathematicians. Therefore understanding and certainty is linked inescapably to the intuitive objectification of knowledge through the object that is the embodied *Me* of each mathematician. Consequently, elegant proofs were highly esteemed because (a) if the logical deduction was rigorous and efficient (reductionist) then the process was less likely to contain error, and (b) the proof constituted a demonstrated and 'shared understanding' of the 'aesthetically pleasing' insight that gave rise to the proof in the first place.

Moreover in relation to Applied Mathematics for example, experimentalist philosophers like Mayo (1996, 2010) of the Virginia Polytechnic Institute and the London School of Economics, have argued that if experiments are to facilitate understanding between embodied minds, and validate that understanding so that it is accepted as knowledge, then the 'nature of experiment' should not be confounded with high level theory. The reason being that high level theory is always underpinned by an intuitive sense that is likely to differ markedly between scientists. However, an experiment that is **basic and reductionist** in its implementation can be more easily and effectively used by scientists in different situations to validate, or falsify scientific claims (Chalmers, 1999; Mayo, 1996). In addition to Popper's (1965, 1979) idea of falsification, Mayo (1996) advocated that experiments conducted rigorously should advance knowledge by uncovering false assertions, as well as the **effect** that led to the falsification conclusion. It is noteworthy therefore, Fisher (1966) contended that the experimental design was inconceivable without the corresponding statistical procedure because the two were but different aspects of the same whole, and provided that the experimenter did not introduce error with his or her test treatments, "it may be said that the simple precaution of randomisation will suffice to guarantee the validity of the test of significance, by which the result of the experiment is to be judged" (p. 21).

Although theory and practice cannot be separated — at least from a critical realist perspective — experimental philosophy has nonetheless emphasized the 'physical act' in order to facilitate the quasi-objective testing of ideas. Dewey, Heidegger, and James all acknowledged the 'integrity' of the physical act if that which was claimed was to be accepted as correct, or as a shared understanding. Clearly, this approach to Applied Mathematics and Science continues to affirm the values of logic and sense perception as the key attributes to 'good science'. This affirmation is consistent with the reflection by Aldous Huxley (English writer, 1894−1963) that "the charm of history and its enigmatic lesson consist in the fact that, from age to age, nothing changes and yet everything is completely different" (as cited in Hermans & Hermans−Konopka, 2010, p. 82).

Concluding Remark

Learning to be ethical is complex, but necessary if students and teachers in mass education are to grow and develop in powerful mathematical learning. Following Comte, the French sociologist and philosopher Durkheim (1858−1917) argued passionately that the "fabric of all human societies is bound together by moral rules. These rules serve a central function in the organization of society. We must undertake a thorough investigation in order to understand them" (Stokes, 2006, p. 191). Therefore learning mathematics powerfully is fundamentally a dialogical and experimentalist inquiry, or event in the phenomenological ethics of Husserl, Heidegger, and Merleau−Ponty.

End Notes

1. The American theoretical physicist and futurist, Kaku (2006) wrote that a Type I civilization was a technologically advanced society that had harnessed all the solar energy striking its planet — in the order of 10^{16} watts. A Type II civilization had exhausted its planetary power and found ways to use the power of an entire star, or approximately 10^{26} watts which is 1,000 times larger than Avogadro's number (the number of constituent particles in a mole of substance). A Type III civilization had exhausted the power of its solar system and accessed large energy resources in its home galaxy — in the order of 10 billion stars, or approximately 10^{36} watts. The ranking scale was developed by the Russian astronomer Kardeshev in 1964 who wished to categorize radio signals from possible outer space civilizations on the basis of their technological development.

2. The three types of civilizations are not linked directly to the Type I and Type II processes of mind, but without the two systems of mind operating interactively, or as a System I – System II complementarity, it would not be possible for humanity to attain the level of even a Type I civilization.

Chapter Nine

A Phenomenology of Creativity

An indefinable something at the nexus between science and art that makes an answer soar, with the power, on occasion, to upend our understanding of the cosmos. (Lehmann, 2014)

Being-intelligent and Being-ethical are essential modalities of Being for the growth and development of the Conceptual Age. However, if this motion is to indeed benefit a glocalizing world, then learning to 'Be-creative' needs to be taken seriously in schools, tertiary institutions, and particularly in vocational education (Robinson, 2011; Robinson & Aronica, 2013). Consequently, learning creativity is advocated from the standpoint that it is a deeply ethical stance on the part of society towards the individual, because it is fundamentally about empowerment. Each person is given the opportunity to actualize their potential **holistically and participatively** for the greater good of the individual, that is in relationship with his or her society and embodied world, which means essentially having a world.

In the Revised Taxonomy for Learning, Teaching, and Assessing for example, the educational objective of 'Creating' subsumes all the other objectives, namely, 'Remembering, Understanding, Applying, Analyzing, and Evaluating', and these are underpinned by the four different knowledge types that are 'Factual, Conceptual, Procedural, and Metacognitive Knowledge' (Anderson & Krathwohl, 2001). Moreover, and with reference to **Figure 9·1**, Creativity mediates the Being of the **whole** person with respect to an embodied cognition that 'Learns for Understanding', and a curiosity through affect that fosters an 'Intrinsic Motivation' which is underpinned by moral ideals like those mentioned in Table 8·2 (see p. 238). Therefore creativity 'conciliates' cognition and affect in Being-creative.

The Process of Creativity

Being-creative includes a process that may lead to an outcome that is novel for the individual, the group, or the community who engages in the creative process. For example, in the case of "a mouse that finds an escape route when confronted with the household cat — and can do so even if the situation is somewhat different from what it has ever encountered before — is being creative" (Kurzweil, 2012, p. 116). creative" (Kurzweil, 2012, p. 116).

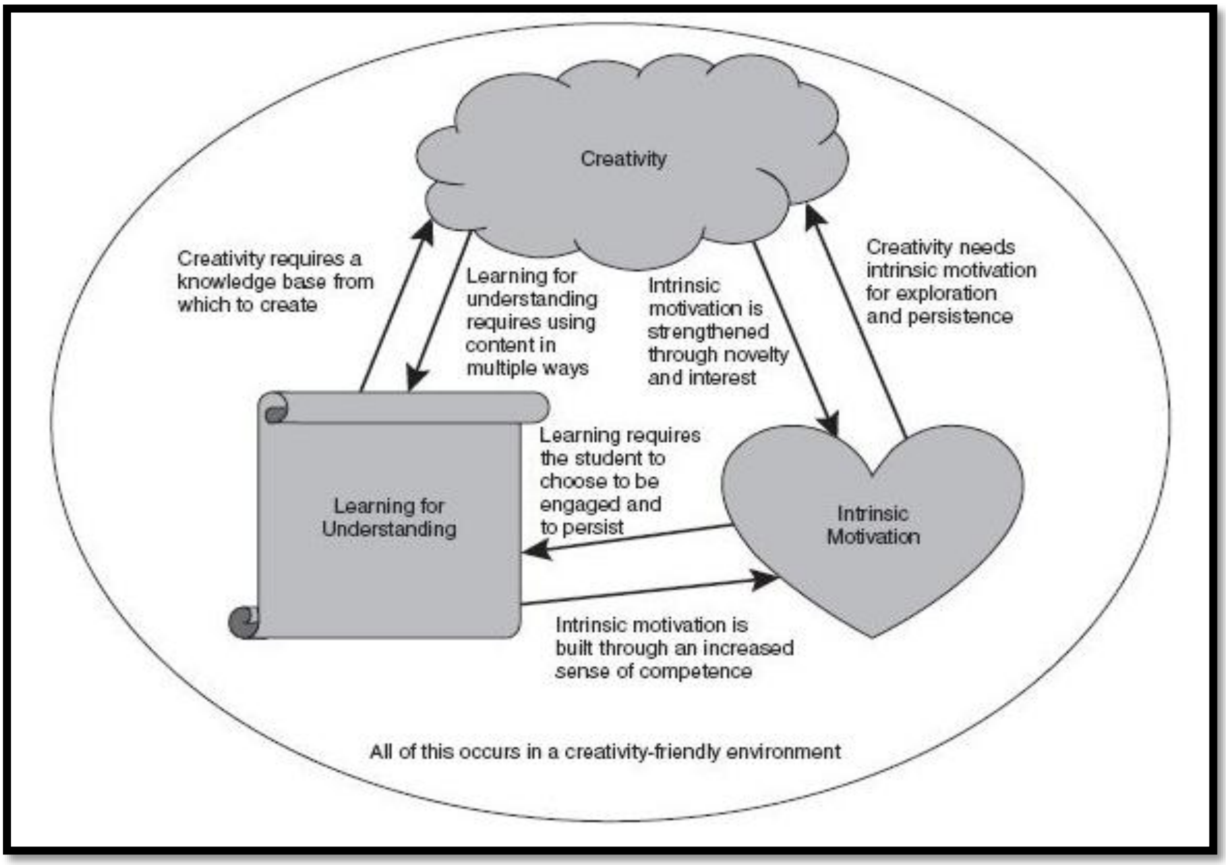


Figure 9·1. Creativity in the classroom: Schools of curious delight (Starko, 2014)

However, a creative outcome need not be immediately 'useful' for it to be considered 'creative': Applied Mathematics has often lagged behind Pure Mathematics by hundreds of years. Consider for example the creativity of the French mathematician, Galois (1811−1832) whose ideas in abstract algebra have provided impetus for the development of physics into the twenty-first century. Thus in accordance with Heidegger's understanding of Being-in-the-world, learning to Be-creative emphasizes that Being-creative involves epistemologically a developmental process that has no finality in relation to previous creative products.

Creativity as a Four Stage Process. Wallas was an English social psychologist and political analyst who lectured on both sides of the Atlantic.[1] Moreover as an educationist; a leader of the Fabian Society (1886−1904), and a co-founder of the London School of Economics, he was a futurist who espoused societal change through carefully articulated ideas.

In Being who he was and who he became, Wallas attempted to ‘create the idea’ of a society that was permanently more harmonious than his own (Wallas & Murray, 1940). In particular, he advocated social reform through educational processes that were marked by gradual rather than instantaneous change. In order to facilitate such change he grappled with the nature of the human condition, by focusing especially on the politics of power relations (Wallas & Rowse, 1948), as well as on a globalizing social milieu that had its ‘technological roots’ in the Industrial Revolution of the eighteenth and nineteenth centuries. In a socio-psychological analysis entitled *The Great Society,* Wallas (1925) wrote:

> Men find themselves working and thinking and feeling in relation to an environment, which, both in its world-wide extension and its intimate connection with all sides of human existence, is without precedent in the history of the world. (p. 3)

In the context of *The Great Society* and other socio-historical and eco-political writings, two key relationships were inferred by Wallas (1898, 1921). First, if *The Great Society* was to be harmonious it would need to acknowledge that it was fundamentally dialectical, and therefore was dependent upon the ‘interbreeding’ of individual differences for its success. Second, humankind was ‘completely’ unaware of the conditions that fostered a successful dialectical society. Therefore, although it was idealistic to suppose that all students everywhere could be educated as if they were ‘noblemen’, it was essential that all communities enabled sufficient individuals to perform the ‘process of thought with unusual efficiency’ (Wallas, 1926).

Thus the universal goal of the social libertarian was in a sense similar to the social philosophy espoused by the Frankfurt School[2] and the German critical theorist Horkheimer (1895−1973), namely, that humanity **needed** to be liberated from powers of authority that limited the Beingness of the individual and therefore of society (Abromeit, 2011; Higgs & Smith, 1997).

Stated differently, Wallas was sceptical of ‘all’ authority, in part because of the childhood experience that he had with his father who was described as the ‘puritanical’ Vicar of Barnstaple (Rogow, 1968; Wallas, 1940). He wished to ‘free’ himself and ‘anyone who would listen’ from the shackles of a ‘non-thinking’ society whose *Lebensraum* was arbitrary through tradition. Moreover he was an ‘existentialist’, who like Heidegger differentiated ‘Being-authentic’ from mere social existence, which tended to occur when the Being of the individual was submerged, or co-opted into a larger public group, which meant essentially that the uniqueness of the individual was ‘silenced’ (Solomon, 2005).

Unsurprisingly therefore, Wallas (1898, 1921, 1926) developed a perspective that ‘thought’ was more a ‘social form’ of art than a rigorous science. Dewey held a similar view (Bailey, Barrow, Carr, & McCarthy, 2010). At the time of Wallas’ death however, a social reformer and a fellow member of the Fabian Society, Webb (1859−1947) described Wallas’ true genius as the propensity to engage with “persons and their relations” (Rogow, 1968). In so Being, Wallas (1926) articulated four stages of ‘thought control’ as the essential features of ‘artful thinking’, which for him implied the development of more humane and harmonious societies that would eventually emerge into *The Great Society*. If however, powerful mathematical learning is to emerge in mass mathematics education then Wallas’ four stages for creativity need to be interpreted in a manner similar to the following process description:

> **Stage I (Preparation):** The mind is intentional in a willed belief that the (novel) problem can be solved. If the problem is to be solved successfully however, then it should be investigated in all directions. For the French mathematician, Poincaré (1914, 1952a) this was the stage of ‘mental toil’: a conscious effort to understand the problem analytically and systematically. If the problem is not solved at this stage then the problem solver needs to ‘hand the problem over’ to a dimension of mind that was more capable through different powers of mind.
>
> **Stage II (Incubation):** Essentially, through Stage I the problem is parametrized in terms of a zone of freedom of movement, or a zone of promoted activity within which the self-organizing mind can synthesize a possible solution, or ‘way forward’. This mental activity requires a sufficient amount of energy. Therefore, *I*-consciousness should not engage with the problem, otherwise mental resources and focus would be split between consciousness and non-consciousness. The likely result being that the

activated complex necessary to solve the problem is unlikely to be achieved. During the stage of Incubation therefore, bodily and mental relaxation, or physical exercise or sleep is deemed necessary to facilitate the non-conscious activity of mind. However, if efficiency or productivity are requirements for the problem solver, then other non-similar problems can be tackled.

Stage IIIA (Intimation): The thinking body, through feeling (an indistinct or strong impression of mental activity, or a sense of separation between the *I* and the rest of the self), prepares the conscious mind that 'a solution' is taking form on the fringes of consciousness. *I*-consciousness can then play a part in willing the synthesis of the non-consciousness mind into working memory. It was the educator McMurray (1909) who wrote, "Many of the best thoughts, probably most of them, do not come, like a flash, fully into being but find their beginnings in dim feelings, faint intuitions that need to be encouraged and coaxed before they can be surely felt and defined" (p. 278).

Stage IIIB (Illumination): The 'happy' moment occurs when the intuition flashes into consciousness and the problem solver sees a meaningful way forward, or if fortuitous, the problem is solved essentially.

Stage IV (Verification): At this stage the human Will[3] has once again comparatively full control over the problem solving process. Consequently, that which has been illuminated in consciousness is implemented to solve the problem. All workings should be verified or evaluated by the problem solver or other suitably qualified individuals.

A creative process narrative. Importantly for the powerful learning of mathematics in mass education, Sternberg's (1985) Triarchic Theory of Intelligence informs the process of creativity dialogically (Nisbett, 2009):

'Analytic Alice' primes and focuses the mathematical activity of the group by breaking the problem down into components that are relatively easy to understand (**Preparation**), but 'Creative Cathy' is more astute at using the ideas of others to spark novel avenues of thought and investigation (**Incubation and Illumination**). Although Alice is an excellent critic of the product of others (**Verification**), it is usually 'Practical Patty' that mediates the group discussion by helping to effect the most sensible and efficient, or elegant ways to solve the problem (**Consummate Balancer**). Cathy has mentioned that she enjoys working with Alice and Patty, not only because she gets to test her unusual ideas meaningfully, but also because her own ability to problem solve improved analytically and practically. Alice and Patty made similar comments.

This narrative represents a microcosm of society through different abilities, or varieties of intelligence. Put simply, the process of creativity at least potentially, is inclusive of all possible human modalities of Being-intelligent. Thus, if the creative process is embedded dialogically and ethically in the teaching and learning of mathematics curricula, it might be the preeminent mechanism to optimize Being-mathematical in classrooms and schools internationally. Ideally for the Conceptual Age however, learners should learn Successful Intelligence by Being-mathematical in the multiplied sense of developing superordinately in Sternberg's (1985) abilities. Consequently, if **all** students are allowed to engage with mathematics through a social and individual dynamic that is **analytical**, **creative**, and **practical**, then it is likely that the powerful learning of mathematics can be made accessible to many more students than otherwise would be the case, at least in part because

> Sternberg's measures of practical and creative intelligence show much less of a separation between minority and majority groups than do analytic tests, meaning that they become a way to bring more minorities into educational and occupational roles where their entrance might be blocked by tests of analytic intelligence. (Nisbett, 2009, pp. 13−14)

Incubation and its accompaniments. In his book *How to Create a Mind*, Kurzweil (2012) concluded that creativity was underpinned affectively by a courage of mind that believed doggedly in the power of the metaphor, or metaphors that linked the question and the inquiry to the novel outcome. Notably therefore, Wallas (1926) used a 'sexual or generative' metaphor to describe the non-conscious stage of the creative process, namely, Incubation. Wallas was aware of, and perhaps to some degree was influenced by the psycholanalytic ideas of the Austrian neurologist Freud (1856−1939), who espoused that human sexuality was the core attribute of Being-human in the sense that the *raison d'être* in Being-human was to pro**create**, not only through erotic and copulatory means, but also with respect to higher order functioning (Freud, 1936; Gardner, 1993). If Freud is correct then powerful mathematical learning is a misnomer without the process of creativity as a focal modality of Being-human.

Nonetheless, Wallas (1926) argued that 'co-consciousness' mediated "an unbroken series of grades from unconsciousness up to the highest level of consciousness" (p. 49). Significantly for powerful mathematical learners therefore, the idea of co-consciousness or intuition has been recognized as

> the most powerful area of brain function. It is probably the area that promises the most for the continuance and fulfilment of human kind. All other areas of the brain provide support for and are supported by this area of function. As each area evolves to higher levels, more of the intuitive and creative functions become available. (Clark, 2008, p. 256)

'Being-intuitive' is a complex modality of Being that links the first and last stages of Wallas'(1926) creative process. Therefore without intuitive functioning it would be impossible to create, because there would be no product to verify or validate. 'Being-intuitive' means that the self-organizing dimension of mind — outside the awareness of *I*-consciousness — responds to the intentionality, and 'leap of faith' of the Will by **germinating** a synthetic response in relation to the ideas that were 'flagged', or reflected upon by *I*-consciousness during the Preparation stage. However, it should be noted that the self-organizing mind is autopoietic, which means that it is not limited solely to those ideas that were primed during the Preparation stage unless directed to do so by the executive functioning of the *I*. It was conjectured that intuitive functioning had its essence in the Directive-Executive Function (DEF) of the pre-frontal cortex (perhaps the most recent evolutionary addition to the neo-cortex), because this brain region was thought to focus "on behaviours associated with planning, organizing, and **creating insight** [for emphasis], empathy, and introspection" (Cropley, 2001, p. 44).

It is during Wallas' Stage II, or Incubation that the germinated autopoietic synthesis develops into that product which ultimately flashes, or illuminates into working memory with a feeling of certitude. Although Stage II remains outside the conscious awareness of the *I* it is important that the *I* through the Will, or DEF maintains the intentionality and the faith that initiated Stage II in the first place. The word 'autopoietic' means that the network of interrelationships that germinated and subsequently developed the synthesis of mind — in relation to *I*-consciousness — are exactly those interrelationships that constitute the synthesis itself (Maturana & Varela, 1980). In terms of complexity however, the synthesis is likely to be more than the sum of its parts, which means that when it eventuates into illuminatory conscious awareness, System II thinking is necessary to evaluate carefully, or verify the whole analytically in relation to its parts.

Other than sustaining a consistency of intentionality, and an unswerving faith, or confidence in the capability of the self-organizing mind to synthesize a meaningful and illuminatory product, *I*-consciousness should not 'interfere' in the functioning of the autopoietic self, because the autopoietic self is very sensitive to the Will of the *I*, and its synthesis can be undone if there is negative affect, or if the intentionality of the *I* vacillates. As an 'act of faith' therefore, the *I* needs to 'rest' from its activities with respect to the novel problem that was 'grappled with' in Stage I of the creative process. In desisting therefore from attempting to solve the mathematics problem consciously, the DEF of the *I* maintains control of the overall creative process.

The self-organizing period, or Incubation Stage may be short or relatively long, but is dependent on at least four factors, (a) the complexity of the problem relative to the organization and quality of ideas, concepts, and experiential knowledge that already exists in the long term memory of the problem solver (Kahneman, 2011; Scott, 1999); (b) the deliberateness and thoroughness of Stage I processing (Anderson, 2010); (c) the stress level of the *I* in relation to the specific problem, or mathematics in general (Goswami, 2004, 2008); and (d) serendipity, or 'chance favours the prepared mind' (Fraleigh, 1989; Gallian, 1998; Peterson, 1954). It is however, possible to hasten the incubation period without interfering unduly with the cogent development of the autopoietic synthesis. First, if the problem solver learns mathematics in terms of interacting brain networks, or systems like the Self-system, the Metacognitive system, and the Cognitive system (see Figure 6·3 on p. 181), then it is likely that such learning would enhance the activity and efficiency of the multiple, interdependent processes that constitute intuitive functioning (Buckner & Schacter, 2004). Second, through deliberate practice and experience over time, the problem solver can become increasingly sensitive to the feelings and workings of his or her thinking body (Kahneman, 2011).

In other words the body mediates the relationship, or interaction between the conscious and the non-conscious dimensions, or aspects of mind, in the sense that the "body, with its urge to exist, procreate, and secure meaning for itself, is the receptacle of intuition" (Noddings and Shore, 1984, p. 204). There comes a time during incubation that the body feels the existence of a synthesis of mind in a dimension of Being that Wallas (1926) referred to as 'fringe consciousness', which in essence is a liminal space between full conscious awareness and non-conscious self-organizing activity. Through the feeling body therefore, the problem solver becomes aware of the synthesis of mind, and can consequently 'lock onto' the structure intentionally and in so Being, Will the intuition, or stimulate the intuition affectively so that there is a 'release' of electro-chemical energy and it illuminates in a moment of insight, or inspiration. In a Freudian sense, and therefore dependent on the individual's sexuality and desire for fulfilment, or gratification this illuminatory event might constitute a higher order orgasmic moment of Being as an expression of a thinking body.

However, if the problem solver does not appreciate the role that intuition needs to play in the creative process, because the individual does not see intuitive and analytical functioning "used effectively by his elders," (Bruner, 1960, p. 62) then the learner in mass mathematics education is unlikely to learn how to be bodily and cognitively minded, and as a result ***I*-**consciousness will ultimately be "captive to ideas that seem to implant themselves in [the] neo-cortex and take over" (Kurzweil, 2012, p. 240).

Nevertheless, by interrelating the stages of Preparation and Verification, "the concept of intuition expresses a fundamental, very consistent tendency of the human mind: the quest for certitude" (Fischbein, 1987, p. 14). Thus in the totality of Being-mathematical or creative, "intuition by itself yields a tentative ordering of a body of knowledge that, while it may generate a feeling that the ordering is self-evident, aids principally by giving us a basis for moving ahead in our testing of reality" (Bruner, 1960, p. 60).

Knowledge development. Being-human is complex and consequently the development of all knowledge, including mathematical knowledge requires many different intentionalities of consciousness. However, if Being-human is not only irreducibly social but also irreducibly creative, then it is plausible that Wallas' (1926) process of creativity can form, or inform a basis for all knowledge development, provided that the process steps characterize human creativity at a superordinate level of understanding. As delineated in **Table 9·1**, Cropley and Cropley (2008) have expanded, or amplified Wallas'(1926) four stage process to a seven stage process in response to an increased knowledge of best practice in teaching and learning. Consequently, in mathematics education the focus of the creative process has been novel problem solving, in part because of the 'authentic' problem solving emphasis in mathematics education, and education generally since the 1980s.

From this particular World-view therefore, creativity in all its forms tends to co-occur with novel problem solving, and is dependent upon a knowledge base that is sufficient to bridge the gap between the problem and a solution. It has been shown empirically that expert problem solvers are not necessarily more intelligent than non-expert problem solvers, but do have a well-organized, and readily accessible knowledge base that enables them to tackle problems in their fields of expertise from multiple and often diverse perspectives (Anderson, 2010). Consequently, their intuitive and metacognitive functioning is greatly augmented (Atkinson & Claxton, 2000).

Therefore Cropley and Cropley (2008) extended Wallas' (1926) Preparation stage to include not only the idea of Preparation but also **Activation**, which is fundamentally metacognitive through problem identification, as well as the clear setting of goals and associated solution criteria. However, Cropley and Cropley (2008) did not embrace Wallas' (1926) idea of

Table 9·1. Different approaches to knowledge development (adapted from Cropley & Cropley, 2008, pp. 366–367; Higgs & Smith, 1997; Husserl, 1927; Wallas, 1926)

Wallas' (1926) Four Stage Creative Process	**Logical Positivists: The Scientific Method (Higgs & Smith, 1997)**	**Phenomenological Method of Reduction (Husserl, 1927, 2002)**	**Cropley & Cropley's (2008) Seven Stage Process of Creativity**
1. Preparation: Investigate the problem in all its aspects	**1. Problem Identification**:Through the five senses, a phenomenon, or a group of related phenomena are carefully observed and thoroughly described (qualitatively). **2. Formulation of the Problem**: One or more hypotheses are specified algebraically (including possible relationships between variables) to explain the phenomenon, or phenomena. **3. Exhaustive Experimentation**: Copious amounts of data are collected under 'like' conditions using instruments that facilitate precise and accurate measurements.	**1.** A 'thing' is observed and discussed at length between multiple participants. **2.** Intentionally, the world of the person is 'bracketed' from the thing. **3.** The *I* refers the discussion to the intrapersonal unseen *Other* for the purpose of intuiting 'the thing' as a pure 'unity of sense' in consciousness.	**1. Preparation**: A knowledge base is established. **2. Activation**: Probems are identified; goals are defined, and solution criteria are established. **3A. Cogitation**: At least one candidate solution is initiated.
2A. Incubation: A form of non-conscious meditation or reflection **2B. Advanced Incubation, or Intimation**: A bodily sense that a eureka moment, or a possible way forward is pending.	**4. Reflection**: The data are studied carefully and reflected upon.	**4. Incubation**: The individual does not reflect on 'the thing', but allows the mind to self-organize a 'unity of sense' that corresponds to the *noumenon*.	**3B. Cogitation**: One or more candidate solutions are allowed to develop, especially non-cognitively.
3. Illumination: A sudden flash of insight emerges in conscious awareness. The insight influences the development and direction of immediate, or future thinking.	**5. Induction**: At best, a eureka moment occurs. A 'data-unifying' pattern is generated that supports the hypothesis (or not) and leads to the formulation of a scientific law.	**5. Illumination**: The *phainomenon*, or 'unity of sense' is described as accurately and literally as possible.	**4. Illumination**: A solution that the person recognizes as promising emerges.
4. Verification: The intuition as a synthesis of mind, or mental structure is analysed and structured, or re-structured towards a correct solution.	**6. Verification by Experiment**: Multiple scientists in diverse situations attempt to reproduce the experimental results. **7. Verification by Logical Deduction:** The scientific law is used to deduce other results. As long as there are no contradictions the result is considered correct.	**7. Verification**: The *phainomena* of the different participants are compared and contrasted at length towards a common 'unity of sense' understanding of the *noumenon*.	**5. Verification**: The solution above is explored by the individual and judged to be appropriate, or not. **6. Communication:** Appropriate judges become aware of the proposed solution. **7. Validation:** A novel product that appropriate judges accept (or reject) on the basis of objective empirical evidence.

Incubation as an intermediary non-conscious stage between what is essentially Preparation and Illumination. The term **Cogitation** was introduced to mediate Activation and Illumination. This implies that Cogitation includes the conscious mind, but does not exclude the possibility of non-conscious human functioning. Although Wallas (1926) did not grasp Being-human in terms of a dialogical self, where it is possible to distinguish between *I*-consciousness and an intentional belief in the capability of an unseen and intrapersonal self-organizing reality that can be personalized, he did advocate that consciousness varied from 'full' consciousness to "unconsciousness, and from comparatively unified consciousness to 'co-consciousness'; and the thinker must train himself to observe his less conscious as well as his more conscious psychological experiences" (p. 8).

However, although Cropley and Cropley (2008) did not elaborate on the tacit, intuitive, or non-consciousness dimensions of Being-creative, Cogitation was described as "information processed in the person's head," (p. 364) that is, as a precursor to the Illumination stage when the 'eye of the mind' sees a possible solution (Noddings & Shore, 1984). The problem solver then attempts to **Verify** the solution analytically. If successful, the solution is **Communicated** to knowledgeable others who provide feedback on the proposed solution, and if possible the solution is **Validated** empirically. Dasgupta (2004) stressed the importance of Communication in order to propel the aesthetic sense of the problem solver towards a complete or mature product that was accepted by a (sympathetic) community.

Therefore creativity includes a social and cultural process that not only involves World 1 (The Natural−Physical World) and World 2 (The Mind), but also World 3 (The Culture and Creativity of Diverse Human Groups or Societies), which means that the creative process enhances the dimensions of Being that are interphysical (World 1), intrapersonal (World 2), and extrapersonal (World 3). However, as exemplified in Table 9·1 the nature of the enhancement of Being through knowledge development ultimately depends on the particularities of the creative process.

For example the logical positivists attempted to facilitate an objective understanding of objects, or *noumenon* external to the mind of the individual. Influenced by *Euclid's Elements* (Euclid of Alexandria, fl. 300 BC), as well as by *The Principia* which was Newton's (1726) thesis of the mathematical principles of natural philosophy (Newton & Cohen, 1999) — a scientific exemplar — the logical positivists limited 'Being-scientific' to sense perception, measurement and inductive and deductive logic. The result was the 'scientific method', which was articulated for the purpose of generating and validating empirically-based knowledge. Consequently the beliefs, the values, the history, and the emotion of the problem solver were subordinated to a method that did not include probabilistic or stochastic modelling, and was designed to 'bring into existence' knowledge that was precise and accurate in relation to World 1. From the perspective of the Vienna Circle therefore, unless the scientific method was 'embedded' in that which was expressed or generated, it was a human impossibility to validate it as 'true knowledge'.

Influenced by Brentano however, both Husserl and Heidegger realized that the agency of Being-human was fundamentally subjective through a conscious Will that was intentional. Therefore, both men set themselves the task of grasping real world objects, or things in subjective but literal terms, namely, through a reductionist method that excluded the possibility of an eidetic intuition also including 'contaminants of mind' as a result of Being-in-the-world. Although neither Husserl nor Heidegger were able to verify the reductionist method, Husserl (1927) did claim that

> my psychological experiences, perceptions, imaginations and the like remain in form and content what they were, but I see them as 'structures' now, for I am face to face at last with the ultimate structure of consciousness [the eidetic intuition]. (p. 8)

The point is that both the scientific method and the reductionist method are but two examples of Being-creative. Essentially, both methods include complex modalities of Being that are invoked through a consciousness that is intentional, and in terms of Being-mathematical both are necessary, because the one approach advocates an objective understanding of mathematics and the other corresponds to a unity of sense that is subjective and meaningful.

Cropley and Cropley (2008) in **Table 9·2** contrast the creative learning of mathematics by Japanese students on the one hand, and United States and German students on the other. The 'Japanese' approach to learning and knowledge creation is more phenomenological than scientific, and the 'United States and German' approach to Being-creative is consistent with logical positivism. However, it is argued that the one learning approach is not more desirable or necessary than the other. The two approaches are but different intentionalities, or polarities of Being-mathematical, and powerful mathematical learners need to draw dialectically on both 'flows' if powerful mathematical learning is to be realized on the part of the learner. The one polarity of Being-creative is represented by the Japanese students, while the other polarity of Being-creative is represented by students from the United States and Germany. It was McMullan (1978) who identified seven bi-polar characteristics of Being-creative, and it is possible that a continuum links the two polarities in each case:

- **(1)** Openness to new ideas : Closure of incomplete Gestalts;
- **(2)** Acceptance of intuitions into consciousness : Objective reality;
- **(3)** Deconstructionist activity: Constructive problem solving;
- **(4)** Impassive neutrality : Passionate engagement;
- **(5)** Self-centredness: Altruism;
- **(6)** Self-criticism : Self-confidence; and
- **(7)** Tension and concentration : Relaxedness.

In other words powerful learners espouse a dialogical self that is Being-able to move between different 'psycho-behavioural waves', or *I*-positions (Koberg & Bagnall, 1991). However, the seven polarities which characterize *I*-position functioning are not limited to any particular stage of the creative process, but Being-creative means having the flexibility and confidence to alternate within and between the different polarities, or psycho-behavioural waves (Koberg & Bagnall, 1991). In addition therefore, powerful mathematical learners should be willing and capable of 'shifting back and forth' between the scientific and phenomenological methods, because both methods are essentially different expressions of the creative process.

Table 9·2. Learning 'creatively': A 'TIMSS' comparison between the cognition and affect of Japanese students, and United States and German students (adapted from Cropley & Cropley, 2008, pp. 368−369).

Learning Phases	**Cognition: Central Processes**		**Affect: Favourable Motivation**	
	Japanese	U.S./German	Japanese	U.S/German
Preparation	Acquiring broad knowledge	Acquiring specific facts	Curiosity	Desire to please the teacher
Activation	• Problem finding • Goal setting • Specifying solution criteria	Perfecting factual knowledge	Dissatisfied with the status quo	Desire to preserve the status quo
Cogitation	• Making associations • Building networks • Seeing implications	Acquiring the solution	Drive for complexity	Drive for simplicity
Illumination	Recognizing possible solutions	Grasp a meaningful solution that is right	Drive to find an elegant solution	Desire to perfect mastery of the solution
Verification	Checking a potential solution	Confirming that one has the right solution	Self-evaluation	Desire for rapid closure
Communication	Revealing one's personal position	Displaying one's mastery of existing facts	The urge to face a challenge	Desire for approval
Validation	**Being** encouraged to continue generating ideas	**Being** reassured that one has got it just right (grades)	Drive for perfection	Drive for perfection

A cautionary note though: powerful mathematical learning is not haphazard and arbitrary but systematic (Cropley & Cropley, 2008); it is structured by the different learning phases that constitute the creative process. Nonetheless, the different phases or stages of creativity are quintessentially a learning protocol for the teaching of mathematics in the Conceptual Age, but as informed by the seven polarities of Being-creative (McMullan, 1978), as well as the everyday practice of mathematics which is an emotive and logical interaction between intuition and objectivity (Grinell, 2009). It was Wallas (1926) who contended that "the daily conflict between the stimulus of habit-keeping and that of habit-breaking, is only part of the larger problem of regularity and adventure in the life of a creative thinker" (p. 13).

A System of Creativity

Creativity is more than a process, because Being-creative is a complex interaction within and between Three Worlds. It is possible therefore that creativity is a system, or at the very least, needs to be grappled with as a social system if Being-creative is to be facilitated in mass education (Amabile, 1983; Feldman, Csikszentmihalyi, & Gardner, 1994; Hennessey & Amabile, 2010; Sriraman & Lee, 2011; Wallace & Gruber, 1992). For example, **Figure 9·2** represents an empirically substantiated systemization of novel problem solving in mathematics (Aldous, 2005, 2006, 2007). The conceptual framework was underpinned by the notion that creativity is the production of effective novelty (Cropley, 1999; Lubart, 2001; Mumford, 2003), especially through

> the art and the science of thinking and behaving with both subjectivity and objectivity. It is a combination of feeling and knowing: of **alternating back and forth** [for emphasis] between what we sense and what we already know. Becoming more creative involves becoming awake to both; discovering a state of wholeness which differs from the primarily objective or subjective person which typifies our society. (Koberg & Bagnall, 1976, p.8)

In particular, the different stages of creativity were theorized to function psychologically through four feedback loops (Shaw, 1989; Cropley, 2001). First, the **Areti loop** is essentially an executive function of the frontal lobes that influences the Incubation stage of the creative process as a consequence of a thorough, earnest and playful preparation of ideas in *I*-consciousness (Grinnell, 2009). However, given the interconnectivity of Being- mathematical it is likely that the Incubation stage commences before the Preparation stage is complete. This implies mental activity that *I*-consciousness is not aware of directly, but might contribute to the student's preparatory problem solving especially as 'feedback' which does not illuminate an actual solution.

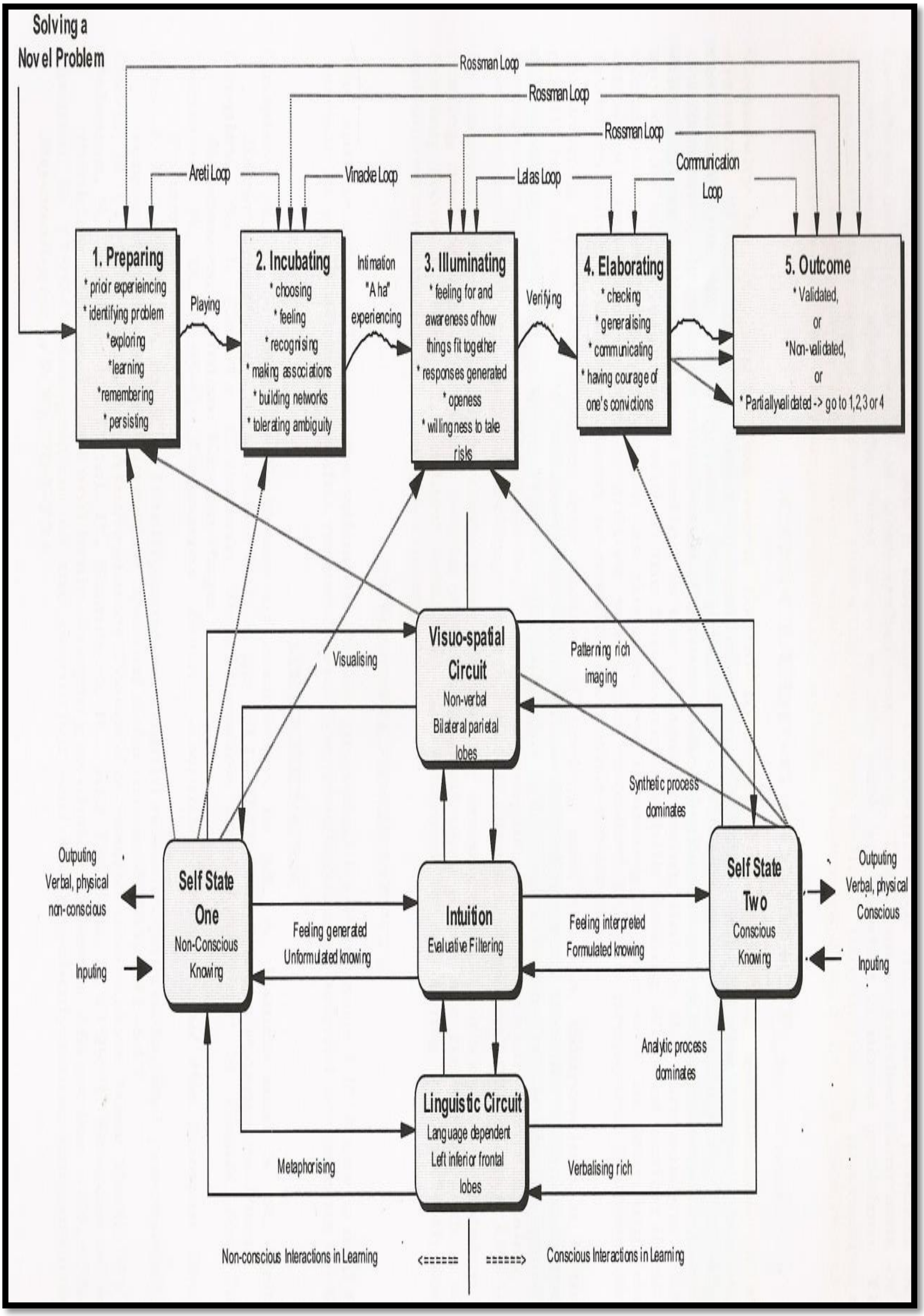


Figure 9·2. A conceptual framework of listening to the Self when solving novel mathematics problems (Aldous, 2005, p. 54).

In other words the Areti loop relates to Cropley and Cropley's (2008) phase of Cogitation, which almost definitely involves mental activity in both the Preparation stage and the Incubation stage. Second, the **Vinacke loop** interrelates Incubation and Illumination as a consequence of one or more Aha moments. This loop is also likely to involve both conscious and non-conscious mental activity, because the two stages are mediated by fringe consciousness, which is essentially a liminal, or 'linking space' between the emergence of an intuition in consciousness and the development of that intuition prior to the illuminatory moment.

The third feedback loop is the **Lalas loop**. It occurs predominantly in *I*-consciousness; the goal of which is to increasingly enhance and substantiate the plausibility (or not) of the intuition. At this stage of its progression towards becoming a deep and compressed mental attitude with structural capacity in long-term memory (Fischbein, 1987, 1999; Semadeni, 2008), the synthesis of mind that is the intuition is both a process and a product. Its development began in the Preparatory phase; was further enhanced during the self-organizing Incubation and Illumination phases, and was scrutinized in the Verification or Elaboration phase. The Lalas loop is 'complete' when the intuition, or successive intuitions have been elaborated as a logically deduced solution, or outcome by the student.

The fourth loop is the **Communication loop**. This loop mediates the outcome of the creative process and necessarily involves an interpersonal dynamic that might result in the problem solver 're-enacting' the creativity process, wholly or in part. As the student reflects on the validated, or non-validated solution, all prior stages of the creative and problem solving event can be influenced or modified, namely, through the bidirectional loops that are the Communication loop and the three **Rossman loops** respectively (Shaw, 1989; Cropley, 2001).

The languaging embodied mind. Although the different loops have not been validated holistically and empirically, the theoretical importance of the loops has been described at length by Aldous (2009, 2012, 2013), and in addition, initial quantitative modelling using Mplus has provided positive evidence for the existence of the Vinacke and Lalas loops in particular. Consequently, and as depicted in Figure 9·2, intuitive functioning is central to the entire process of creativity, because intuition is a bodily function that is "always associated with the sense of beauty, because intuition is always associated with unity in a multiplicity of appearances" (Fischbein, 1987, p. 4).

Essentially therefore, intuition as a 'unity of sense' mediates two modalities of Being-mathematical: a Visual−Spatial modality and a Language modality — both of which interrelate to enable Being-mathematical through a creative Self[4] that is aware consciously and non-consciously. The leitmotiv of the German–Jewish educator, Hahn (1886–1974) was that there was 'more in students than they thought' (Van Oord, 2010). However, it was Husserl (1927) who distinguished between a 'psychological *I*', and a 'transcendental *I*' that should be

> comprehended in the concreteness of transcendental consciousness. But though the transcendental "I" is to my psychological "I," it must not be considered as if it were a second "I", for it is no more separated from my psychological "I" in the conventional sense of separation, than it is joined to it in the conventional sense of being joined. (p. 8)

In effect therefore, the transcendental *I* functions in terms of a self-organizing brain that relates to the intent of the psychological *I*, but independently of the conscious awareness of the psychological *I*, or *I*-consciousness. Therefore Being-mathematical involves a complex intuitive interaction between two states of Being that are essentially a Self-system. Self State One (Non-Conscious Knowing) includes the agentic transcendental *I*, but Self State Two (Conscious Knowing) is agentic through the psychological *I*. Importantly for powerful mathematical learners however, the psychological *I* relates primarily to the mind, but the transcendental *I* to the body through feeling and intuition. This mind−body interaction is made possible by a Two State Self−system that requires the operation of both the psychological *I* and the transcendental *I* to facilitate the creative process, which involves dialogically the linguistic and visual−spatial synaptic organization of the brain (Ingram, 2007; Shepherd, 1998).

Growth of the Self. Dialogue has been made possible by a neurophysiological language system that links thought, speech, and gesture (McNeill, 1992, 2005; Sfard, 2009). In particular, it has been argued that gestural hand movements that accompany speech facilitate the emergence of complex intuitions in *I*-consciousness (Wilson, 1998). Although the notion of emergence has “considerable currency in mathematics education” it is not well understood theoretically and therefore needs to be clarified (Roth & Maheux, 2014, p. 32). It is however, always characterized by convoluted key features (Sheets−Johnstone, 2011).

In particular, the Aha moment is an ‘emergence’ in consciousness and belongs to two very different states of Being. The *I*-conscious world of Preparation and Illumination, and the non-conscious world of Incubation, or Self State Two and Self State One respectively. ‘Being-an-intuition’, the emergence encompasses the **whole** transition between the two States, and the new State of Being that is Illumination is not derived directly from Preparation and Incubation. Consequently the substantiality, or complexity of the Aha moment cannot be predicted by *I*-consciousness before the Illumination takes place. Moreover, since the sociality of the emergent “is the very structure of our minds,” (Mead, 1932, p. 90) namely, a structured mental disposition with a mathematical capacity that **originated** through different *I*-positions as a consequence of Being-in-the-world (Dreyfus, 1991; Fischbein, 1987; Heidegger, 1927), the emergent is a heterogeneous thing, or a *phainomenon*.

Therefore the Aha moment **is** the emergence of two very different states of Being, possibly resulting in the growth of the Self (Varela, 1995). Essentially, an intelligent and embodied mind is capable of expressing or communicating a thought, or idea as part of a ‘speech act’ in basically two different ways (Goldin−Meadow, 2003; McNeill, 2005). Through oral communication an idea is communicated, or languaged in a form that is linear−analytic and combinatoric. While speaking however, the same idea can be expressed as a global−synthetic gesture, or literally as an embodied image in the hands.

In other words if the communicator gestures while speaking, then it is possible for an idea, or aspects of the idea to emerge holistically 'in the hands' before the emergent is expressed verbally (Broaders, Cook, Mitchell, & Goldin−Meadow, 2007; Cook, Mitchell, & Goldin−Meadow, 2008; Ehrlich, Levine, & Goldin−Meadow, 2006; Goldin−Meadow, 2003). It is important for mathematics teachers to be aware that communication in the hands is always explicit to *I*-transcendental, but might only be implicit to the visual−spatial awareness of *I*-psychological.

In terms of powerful mathematical learning however, if hand movements are encouraged as part of a bi-directional thought–language−hand link, then this enaction might facilitate the emergence and bodily communication of intuitions through gesture (or perhaps even by drawing a physical diagram), which might mean that the intuitive is more likely to illuminate in the student's *I*-consciousness than otherwise would be the case (Davis, 1973; Goldin−Meadow, 2003; Hendrix, 1973; Wilson, 1998). Furthermore by attending not only to feeling but also to his or her 'languaging hands' (as well as the gestures of *Others*), the problem solver might gain insight into the intrapersonal world of *I*-transcendental; perhaps discovering or gaining access to the necessary affordance, or visual clue that unlocks the solution to the mathematics problem (Broaders, Cook, Mitchell, & Goldin−Meadow, 2007; Lowry, 1967; McNeill, 2005).

Nevertheless, the growth of the Self through the development and expression of embodied imagery and intuitions emphasizes the 'primacy of movement' for the meaningful linking, or integration of Self State One and Self State Two. Even antagonistic movements however, are required in this regard (McNeill, 2012; Sheets−Johnstone, 2011; Sylwester, 2000), because fundamentally, Being-human or Being-mathematical is a 'social co-expression' in the dialectics of oral and gestural speech acts. A gesture embodies a material and fixed basis, or synthesis in connection with *I*-transcendental, but simultaneously in relation to an unfolding oral communication, or verbal reality that engages *I*-consciousness in the speech act — syllable by syllable, word by word, symbol by symbol.

The socio-cultural psychologist Vygotsky proposed the idea of a dialectic for the purpose of understanding humanness, especially through the cultural signification of language whereby change, or difference is combined with that which remains unaltered. The genius of Vygotsky (1986, 1991, 1997) was to realize that human potential **cannot** be actualized in its possibilities of Being-human, unless the individual can develop his or her uniqueness through an embodiment of Being that is dialectical. In particular, if the embodiment of culture through language was not dialectical then human traditions, institutions and formations, hegemony, signs and notations, conventions (established relationships), genres and forms, and creative practice would all be reduced to a state of humanness that was pre-language — in the sense that communication would almost definitely be constrained to demonstrable actions between human bodies (Merleau−Ponty & Lefort, 1974; Otte, 2006; Shank, 2006; Williams, 1977; Wilson, 1998).

Therefore, the growth of the Self is dependent upon the 'intuitive' being expressed relationally and differentially through a languaging body that is complex in thought, word and action. Based on Vygotsky's (1978, 1986, 1991) concept of a psychological predicate, or a unit of inner speech that is dialectical, or a 'zone of proximal contrast' that is founded on the notion of 'antireductionist holism' (Higgins, 1999; Williams, 1977), the idea of a "growth point" or GP was introduced by McNeill (2005), namely,

> a minimal unit of dialectic in which **imagery and linguistic content are combined** [for emphasis]. A GP contains opposite semiotic modes of meaning capture —

> instantaneous, global, non-hierarchical imagery with temporally sequential, segmented, and hierarchical language. The GP is a unit with demonstrable self-binding power, and the **opposition** [for emphasis] of semiotic modes within it fuels the dialectic. (p. 18)

The essential characteristic of the dialectic is that the two semiotic modes operate concurrently and co-dependently in relation to the mental and bodily languaging of the speaker. Consequently, when the same idea is represented simultaneously in opposite modes a 'benevolent instability' is created on the dynamic dimension of language, namely, imagery and linguistic content. However, the instability is often resolved by accessing languaging forms on the static or stable dimension of language, especially through suitable language constructions and lexical choices, that is, appropriate words, symbols, and sentence structures are chosen and strung together in a manner that is coherent ― the outcome being that *I*-consciousness has a temporary feeling of relative certitude and stability in relation to its Being. Essentially the dialectic is resolved intuitively when it is transformed through the use of language into a unity of sense, and it is this sense of 'completeness' that manifests a psychoanalytic feeling of 'Being-whole', which in turn results in states of temporary inactivity, or repose on the part of the languaging individual (Cropley, 2001; Noddings & Shore, 1984).

Therefore the growth of the Self occurs when the two dimensions of language, namely, the static dimension and the dynamic dimension combine to unpack the unstable GP into language constructions and vocabulary items that are meaningful to the individual. When the imagery and linguistic content of the GP emerge increasingly as 'equilibrated' (in a Piagetian sense), or stable language structures on the static dimension, then this is an indication that the Self is more complex because learning has occurred. This however, is unlikely to be the case if the individual who is languaging does not have a sufficiently sophisticated repertoire of language to articulate the visuo‒spatial imagery. An essential part of the CAME project for example, was to ensure that the learners had adequate language expression before commencing with unfamiliar mathematics problems (Adey, 2006; Shayer & Adey, 2002).

Growth point model. Cognitivism that is asocial is antithetic to the idea of a languaging body. However, a critical realist or process−relationist philosophy is consistent with a psycholinguistic and neurophysiological understanding of language, namely that thought, speech and gesture are inextricably linked. In particular, the ‘social’ brain coordinates both manual and vocal (oral) motor systems through a directive-executive meaning-controlled system that involves understanding (Kelso, 1995; Demetriou, 2009).

Although Johnson (1987) argued that meaning and understanding were equivalent because cognition was embodied, in a radical constructivist sense meaning does not imply the discovery of a pre-existing reality, or understanding external to the mind−body of the knower. However, the nature of the dialogical self is such that meaning is embodied but understanding occurs between people (Hermans & Hermans−Konopka, 2010; Merleau−Ponty & Lefort, 1974). Therefore a goal of powerful mathematical learning is to develop a ‘meaningful understanding’ of mathematics through a dialogue that essentially ‘bridges’ the gap between the interpersonal and intrapersonal psychological planes of the self. The dialogical self is a ‘heterogeneous unity’, or an existential psychological plane in the dimensions that are ‘meaning’ and ‘understanding’.

McNeill (2005) has theorized that an individual is capable of developing a meaningful understanding through manual and vocal motor systems that support the convergence of two cognitive modes or circuits, namely, a visuo-spatial mode and a linguistic mode that interact through a specific motor sequence. With reference to **Figure 9·3**, the locus of convergence is likely to be Broca’s area, because this area of the brain orchestrates manual and oral−vocal tract actions (MacNeilage, 1998). However, brains have evolved to facilitate increasingly complex bodily movements, and consequently have become more complex in themselves (Blakemore & Frith, 2005). For example, Wilson (1998) argued persuasively that the evolutionary development of the thumb ‘in opposition’ to the other fingers on the hand, expedited or advanced the executive functioning of the pre-frontal lobes, and as a result enhanced the metacognitive and creative possibilities for humankind.

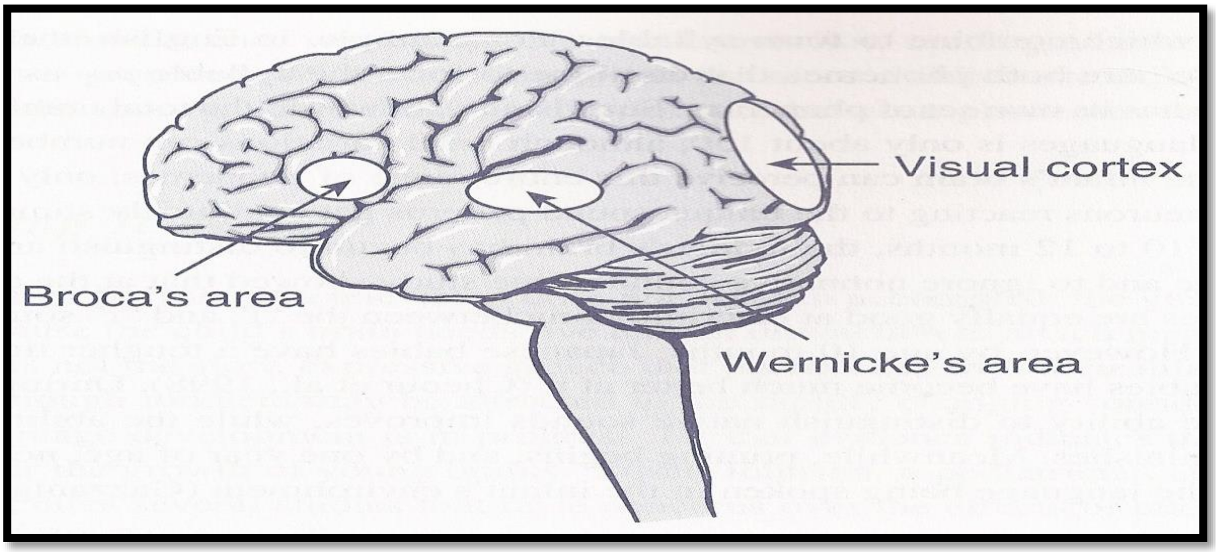


Figure 9·3. Broca’s area and Wernicke’s area, located in the left hemisphere in most individuals, are the two major language processing centres of the brain. The visual cortex, across the back of both hemispheres, processes visual stimuli (Sousa, 2007, p. 67).

However, it is not only Broca’s area that underpins language as a static−dynamic event. Wernicke’s area; the left hemisphere (sequential, literal, functional, textual, and analytic); the right hemisphere (simultaneous, metaphorical, aesthetic, contextual, and synthetic); and the frontal lobes all appear to be part of a circuit, or interrelating circuits that enable the individual to language the bodily movements of a visuo-spatial and linguistic self (Goldin−Meadow, 2003; McNeill, 1992, 2005; Pink, 2005). In particular, it is probable that Wernicke’s area facilitates language comprehension through the categorization of imagery. Although imagery emerges predominantly in the right hemisphere, the imagery requires Wernicke-originated categorizations to initiate, form, or give shape to the corresponding GPs, which are unpacked in Broca’s area. The central claim of McNeill’s (2005) growth point, or brain model is that Broca’s area “is the unique point of (a) convergence and (b) orchestration of manual and vocal actions guided by GPs and semantically framed language forms” (p.

232).

Therefore although GPs arise in the right hemisphere, multiple brain areas including the frontal lobes, the posterior left, and the right and anterior left are involved in their formation (Spencer, Zelaznik, Diedrichsen, & Ivry, 2003). Consequently, whole brain learning is necessary if GPs are to be contextualized and meaningfully understood through dialogue. In a Saussurian structured and semiotic language sense therefore (Culler, 1976; J. E. Joseph, 2012),[5] powerful mathematical learners language a system of creativity that depends on intuition to form and sustain the development of growth points; the coherent resolution of which ultimately constitutes powerful mathematical learning. In other words Being-mathematical is a social phenomenon that involves a structured system of languaging that is simultaneously synchronic and diachronic. In short, powerful mathematical learners are able to handle the stasis and flux of Being-mathematical.

Summary insights: The languaging embodied mind. As a consequence of Preparation and Incubation an eidetic intuition might emerge in consciousness. If this occurs, the content of the intuition is global and detailed imagery. However, as the individual reflects on and begins to describe the vivid imagery, the intuition is increasingly transformed through a sequence of growth points, where each growth point is essentially a minimalist, or primitive dialectical 'unit of thought'. The term 'minimalist' is used in the Gestalt sense that if any aspect of the GP is removed then the GP has no meaning.

However, as long as the holistic imagery is in relation to a linguistic content that is incomplete, the GP is dialectical because it lacks meaning in *I*-consciousness. But, Being-intuitive implies an affective drive for closure through meaning making or interpretation. Thus the movement of language between the dynamic, or dialectical dimension and the static dimension is likely to continue until it is halted by a 'stop order' which is signified by an intuitively complete or well-formed mental–language structure. That is when the imagery of the eidetic intuition is equilibrated on the static dimension of language, the languager feels a sense of completeness (even relief or satisfaction), especially if the embodied and ideated imagery is expressed as a holistic and meaningful speech act.

It is likely however, that if a learner is grappling with an unfamiliar learning situation then he or she will experience numerous cycles of overlapping GPs that are not adequately resolved into meaningful or useful sentences. It is only when the learner resolves the GP sequence adequately that learning takes place. The complexity of the semiotic-imagery, the prior learning of the individual, and the linguistic style of the individual can all influence how many GPs need to be unpacked before a well-formed and meaningful sentence, or sentences are expressed logically on the static dimension of language.

Nevertheless since the times of Aristotle, Socrates, Plato, and Archimedes it appears that dialogue is the preeminent linguistic form, or mode of Being to facilitate the logical development of an eidetic intuition — through a sequence of dialectical thought units, or growth points that culminate in a meaningful understanding duly expressed on the static dimension of language. Importantly for powerful mathematical learning it is necessary however, for teachers and students to realize that gesture embodies mental imagery (implicitly or explicitly) as part of the speech act, and therefore the hands can support the resolution of GPs into mathematical sentences that are deduced logically and elegant. Therefore dialogue that includes active gesturing is especially useful if the context of learning is framed in terms of novel problem solving, or a challenging "field of oppositions" that is inherently dialectical (McNeill, 2005, p. 19). The more difficult the field of oppositions, the more useful a dialogical and gesticulatory approach, because the brain uses the hands as a second 'working memory', or a source of embodied visualization to mediate growth point imagery that can be verbalized and ultimately written down in a rigorous manner.

An Ecology of Creativity

Aldous' (2005) conceptualization of creativity focused on the intrapersonal and interpersonal dimensions of creativity as well as the environment. However, Sternberg's (2003a) Balance Theory of Wisdom (see p. 205) emphasizes that Successful Intelligence for a Common Good is dependent upon a balanced interaction, or a differentiation of the Self that involves the intrapersonal, interpersonal, and extrapersonal dimensions of Being. The extrapersonal dimension relates strongly to Being-ethical. Nevertheless, if the complex interaction is to result in a Common Good that is both the sum and the multiplication of the parts, then a principle of complexity suggests that the nature and the quality, or substantiality of the 'neighbour interactions' is crucial (Davis & Simmt, 2003; Hurford, 2010).

In Dialogical Self Theory the 'neighbour' of the *I* is essentially the *Other*. The subject that is *I* is always in vital relationship to the object that is *Me*, because the intentionality of consciousness objectifies the *Me* in terms of a social and agentic *I* that is fundamentally essential and intuitive. In this sense therefore, the *Me* is the 'closest' *Other* in relation to the *I*. However, the *Other* may refer conceptually to anyone or anything that the *I* chooses to relate to. Therefore the *Other* may have its form in The Natural−Physical World (World 1), or in The Mind (World 2), or in The Culture and Creativity of Diverse Human Groups or Societies (World 3). Thus in primitive terms the *Other* is essentially a thing (see p. 4).

It is a basic tenet of this study that the *I*, or the subject of 'Being-conscious' needs to embrace, or grapple with the individuality or uniqueness of the *Other* if the self is to comprehend, construct, or realize its potential through the *Other*, because the *Other* mediates the potential of the *I* to Be what otherwise cannot Be. Hence, the future of the *I* is not fixed but dependent upon the *Other* in the multitude of its manifestations and complexities. The powerful mathematical learner is wise in the intention to learn from every person, situation, or thing that he or she encounters. Thus if the *I* is not exposed to the *Other*, or chooses not to develop an *I−Other* action or interaction, then the consequence for the *I* is that the self of the *Me* cannot expand in relation to the *Other*. Consequently, powerful mathematical learners seek opportunities to expand the Self by not only engaging with intrapersonal (e.g., *I-*positions or *others-in-the-self*) and interpersonal *Others*, but also with extrapersonal *Others* (e.g. social justice, freedom of religion, and animal rights), because the latter often mediate *I−Other* interpersonal and intrapersonal relations.

The power of words. Without exception, the words of the wise mathematics teacher are characterized by positive affect as an expression of his or her Being-mathematical (Csikszentmihalyi, 1990, 1994; Darling−Hammond, 2002). Consequently, students who are valued affectively through words are highly likely to experience positive feelings towards the *Other* (mathematics teacher), or *Da-Sein* who is a locus of Being-mathematical in that particular teaching and learning situation (Denton, 2008; Jensen, 2008). In turn these interpersonal feelings facilitate or enhance the students' aesthetic sense of Being-mathematical. It is essentially a mirror-neuron based response, or bodily interaction on the part of the students "who are indeed involuntarily caught in Nietzsche's net of shared experience — the net that is a crucial feature, perhaps *the* crucial feature, of human social life" (Humphrey, 2006, p. 109). In so Being the mathematics teacher awakens or primes the body of the student to engage in the creative process.

For example, consider Kaku's (2006) words on the influence that 'hyperspace' has had on theoretical physics since the late 1960s when two early career physicists (Veneziano and Suzuki at CERN, Geneva) connected, independently and serendipitously, the subatomic world with the (obscure) eighteenth century Euler Beta function:

> Higher dimensions are now in the center of a profound revolution in physics because physicists are forced to confront the greatest problem facing physics today: the chasm between general relativity and the quantum theory. Remarkably, these two theories comprise the sum total of all physical knowledge about the universe at the fundamental level. At present, only M-theory has the ability to unify these two great, seemingly contradictory theories of the universe into a coherent whole, to create a "theory of

> everything." Of all the theories proposed in the past century, the only candidate that can potentially "read the mind of God," as Einstein put it, is M-theory.
>
> Only in ten- or eleven-dimensional hyperspace do we have "enough room" to unify all the forces of nature in a single elegant theory. (p. 185)

Therefore, the student who listens or reads intently 'reaches towards' words in an enactivist or bodily sense (Davis, 1996). Thus, Being-mathematical is to engage essentially with words (and symbols) through a locus of mathematical activity, or a zone of proximal development. After Preparation and Incubation the 'reaching body' transcends the constraints of the situated and limited interpersonal words because the words that were spoken, or written 'take on a life of their own' in relation to a feeling body that connects uniquely with a self that knows epistemologically through the creative process. In this regard it is likely that the "cognitive processes and interpersonal communication processes are but different manifestations of basically the same phenomenon" (Sfard, 2008, p. 83). It was Vygotsky (1986) who argued that 'words' are essentially a microcosm of

> consciousness-for-myself, then not only one particular thought but all consciousness is connected with the development of the word. The word is a thing in our consciousness … that is absolutely impossible for one person, but that becomes a reality for two. The word is a direct expression of the historical nature of human consciousness. (p. 256)

Therefore as part of a dialogic system of conscious and non-conscious knowing, the creative process can be precipitated by words that mediate or signify the actions of Beings-mathematical. Then in response to a deeply felt intuitive need, or desire to 'complete the incomplete', a narrative solution to the novel problem can unfold through a dialectical sequence of developmental growth points. A successful outcome is the transformation of intuited imagery into increasingly stable language forms, or cognitive models. Once the process solution has been verified by knowledgeable *Others*, the words are no longer linked directly to the problem solver but to an organismic community or society. At this stage the solution is a thing with an 'independent' existence, which implies that 'the words' are extrapersonal and a part of World 3.

Personality of place. Being-creative is influenced relationally by things in Three Worlds because powerful mathematical learners are an integral part of their environment, and vice versa. Consequently, if students in mass education are to be creative in mathematics then their surroundings need to be used to inspire creativity, which implies that new ways of teaching and Being require a different understanding, or an enhanced perspective of space and objects (Taylor, 2009). Essentially therefore, powerful mathematical learners are an intrinsic part of a learning ecology that is animated relationally by Being-creative. For example, the Indian mathematician Ramanujan (1887−1920) developed a 'romance' with the discipline. In particular, he became so familiar with the first ten thousand natural numbers that he considered each number as an 'extrapersonal friend'. When the English mathematician Hardy (1877−1947) commented that the number 1729 was not a very interesting number, Ramanujan immediately replied 'protectively' that it was a most interesting number, because it was the smallest number that could be represented as the sum of two cubes in two different ways ($1729 = 1^3 + 12^3 = 9^3 + 10^3$).

However, it is not only numbers and symbols that can be animated but buildings as well. Schools for the future need to be constructed in ways that foster an essential quality of delight, or 'personality of place' for the purpose of stirring both teachers and students towards levels of increasing insight and creativity (Hansen & Childs, 1998; Tanner, 2000). It was Bronowski (1973) who said that "in a sense all science, all human thought, is a form of play" (p. 432). For example, the campus of the Mathematics and Science High School of the National University of Singapore (NUS) was designed intentionally to be a three-dimensional learning tool that would appeal to students' aesthetic sensibilities and inspire achievement. Most notably, the double helix structure of DNA was used to model the nanotube stairway at the entrance of the school lobby; the main façade of the auditorium, in elevation, reflects the different groups of the periodic table, and the'Pi-Wall' (adjacent to the track and field) is a colour coded and sequenced mosaic construction of rectangular perforated aluminium panels that represent the first few digits of the number π (OECD, 2006b). Moreover and as depicted in **Figure 9·4**, the school logo of NUS is a test tube, the irrational number π, and a sparkle. By implication therefore, the learning ecology that is NUS High School exists so that learners can ‘light up’, or enlighten their world through mathematics and science.

Figure 9·4. The Mathematics and Science logo of NUS High School reflects the school’s approach to learning: *Experiment–Explore–Excel*.

Although most schools in the world do not have the funding to develop a learning ecology that NUS High School currently enjoys, every effort should be made to create a learning environment that is an expression of Being-creative through the innovative use of diverse materials, space, and the play of light. For example, the Victoria School in Singapore has geometrically shaped floor patterns and ceilings that depict stars and constellations (OECD, 2006b). Nevertheless, through Being-autotelic a teaching and learning community can establish teaching and learning environments that are resource-full, even though resources are limited. Consider for example the Sidi Bouskri School in the rural town of Smimou, Morocco. The primary school has a program called ‘Maths Grows on Trees’ where students apply their learning of numeracy to the process and production of olive oil.

However, although it is the people who influence the learning environment primarily by Being-there — ‘the walls speak and still speak’ (Uline & Tschannen−Moran, 2008; Uline, Tschannen−Moran, & Wolsey, 2009) — which means that the affect and socio-cultural interplay of teachers and students over time (Huizinga, 1950) can be enhanced through

> the relationship of the buildings to the surrounding environment, from the choice of materials, from form and proportion, and from the subtle modulation of colour, lighting and acoustics. Delight that lifts the spirit and affirms to both students and staff that there is more to education than simply acquiring the skills and knowledge to survive in an increasingly competitive world. (OECD, 2001, p. vii)

***Embodied technology*.** Although mathematics education technologies abound, the use of technology does not necessarily facilitate Being-creative. However, the emergence of the ‘fyborg’ is a distinct possibility within the next few decades. A fyborg is an individual who has inbuilt technology for the purpose of boosting, or augmenting his or her genetic potential to learn, to innovate, and to perform beyond what would be naturally possible for that particular individual (D’Ambrosio, 2007a). This development — in part through biomechatronics — is potentially very exciting for humankind and education as well, because the embodiment of *homo intelligens* is likely to be a reality sooner than was imagined (Brooker, 2012; Masuda, 1985).

However, there is an ethical dilemma. Not all students may choose to become fyborgs. Many individuals may not be able to afford fyborg technology. Therefore in institutions of learning as well as the job market for example, non-fyborgs may face significant disadvantage in the twenty-first century, and the gap between the ‘Haves and the ‘Have Nots’ might increase substantially. Thus the idea of fyborg technology, although compelling, needs to be guided by wisdom for a Common Good. Nonetheless if embodied technology can amplify working memory capacity; improve long-term memory organization and retrieval, and enhance the creativity stages of Incubation and Illumination, then powerful mathematical learning will almost certainly be attainable for the majority of students in mass mathematics education, including many with disabilities like dyslexia, or perhaps even dyscalculia.

Therefore the fyborg embodies technology, and as a ‘metaphor of embodiment’ the individual is empowered to change the personality of place by interlinking with Three Worlds creatively, because essentially

> technologies transform our experience of the world and our perceptions and interpretations of our world, and we in turn become transformed in this process. Transformations are non-neutral. And it is here that histories and any empirical turn may become *ontologically* important, which will lead us to the pragmatist insight that histories also are important in any philosophical analysis as such. (Ihde, 2009, p. 44)

Summary insights: Being-creative

Wallas realized that a core feature of society was politics, and as a consequence of the human condition many people were disadvantaged and their potential constrained, or usurped for selfish purposes. However, if each individual was educated in the basic creative process of Preparation, Incubation, Illumination, and Verification, a more ethical and capable society would likely result. In other words Being-creative for a Common Good has positive ramifications interpersonally, intrapersonally, and extrapersonally for the individual and society at large.

However, Being-creative is a complexity. Aldous (2005, 2006, 2007) systematized the creative process by relating two states of Being through intuitive functioning, namely, Self State One (Non-Conscious Knowing) and Self State Two (Conscious Knowing). Essentially, in response to a novel mathematics problem, Self State Two initiated an interaction between the Visuo-spatial Circuit and the Linguistic Circuit in Self State One. The goal of which was to illuminate imagery and semiotic−linguistic content in consciousness so that the mathematics problem could be unravelled analytically and deductively. This meant transforming the essence from the dynamic, or dialectical dimension of language to the static, or stable dimension of language through successive growth points. In this regard, McNeill (2005) hypothesized that Broca's area in the brain was the most likely point of convergence between the Visuo-spatial Circuit and the Linguistic Circuit. The manual and vocal actions of a languaging body are then orchestrated from Broca's area under the guidance of GPs towards an elegant and semantic understanding of the illuminated essence.

However, from a Husserlian perspective the process of creativity is dependent upon an intentionality of consciousness that facilitates the Being of the individual towards a literal but subjective understanding of things. This can be achieved through a phenomenological method of reduction that 'brackets the world' from the intentional content that is illuminated in consciousness after Preparation and Incubation (Dreyfus, 1991). But intentionality is complex, and Heidegger (1927) emphasized a different dimension of intentionality. He argued that intentionality was not primarily psychological or mental, but rather a locus of directed activity as a consequence of *Da-Sein*, or Being-there. In other words the individual comported (*Verhalten*) towards the thing in order to know the thing for what it was, or what it could be in relation to the activity of the individual. Thus essentially, Husserl and Heidegger focused on different aspects of intuitive functioning. For Husserl, intuition was a Willed intent through a transcendental psychology of Being that eventuated in an essence, or eidetic intuition that was a conscious mental state. For Heidegger however, intuition was a consequence of an agentic body 'grasping' a thing, or an insight to a novel mathematics problem for example.

Logically therefore and from a cognitive psychology perspective, Aldous (2005, 2006, 2007) understood intuition as an 'enactivistic or mental−body' mediator of conscious and non-conscious knowing that was dependent fundamentally upon a **feeling** body. Contrary to 'traditional viewpoints', Aldous (2009) found through empirical research that

> attending to feeling is an essential ingredient of successfully solving novel problems and, moreover, that in the absence of this feeling individuals are unlikely to solve novel problems at all. Indeed, no model of solutions could be found that was without the involvement of feeling. (p. 339)

In contrast however, the logical positivists and the Vienna Circle developed the scientific method to understand the things of World 1 and World 3 'objectively'. This meant that emotions, values, and cultural attributes of Being-in-the-world were inhibited through a rigorous application of the scientific method. The method emphasized sensorial experience through the bodily and mental activity of repeated and thorough observation, experiment and measurement, so that scientists could use the data to **reason** inductively and deductively. Although the logical positivists did not value Being-intuitive in relation to a feeling body, Heidegger (1927, 1967, 1970) implied that a languaging body implied intuitive functioning. Therefore the scientific method cannot exclude Being-intuitive because its application always involves a human body.

Nonetheless, a system of creativity allows powerful mathematical learners to engage with Three Worlds differently through different intentionalities of Being-creative. This is especially the case if the teaching and learning environment is 'objectified' to reflect and inspire an interrelated ecology of things, or a super-system of creativity that is bi-directional in terms of Three Worlds. Phenomenologically, philosophers have argued that to Be-human is essentially an embodied mind with an agency to facilitate a permanency of change in both the natural and the socio-cultural world (Hanna & Maiese, 2009; Ricoeur, 2002). However, the complexity of perspective that is informed by Heidegger (1977), Whitehead (1963), and the experimentalists is the need for modern physics and science to be technologically and experimentally embodied for the purpose of fostering a 'certainty of knowledge' that then constitutes mutual understanding. Consequently 'all' science should be embodied technologically, and the study of essences must therefore integrate technology towards human stability and change over time (Ihde, 1979, 1990, 2009).

End Notes

1. Wallas (1858−1932) was a Lowell Lecturer at Harvard University in 1910; a Dodge Lecturer at Yale University in 1919, and a Professor of Political Science at London University from 1914−1923 and at Oxford University in 1931.

2. The Frankfurt School was established in Frankfurt am Main in 1923 through a private endowment bequeathed by Felix Weil. Although the group of intellectuals espoused a Marxist ideology that was not connected to any communist party, their Critical Theory was nonetheless a critique of modern society. As a consequence of their 'activism' it was hoped to inspire an emancipated society of individuals, where each individual had the autonomy to cooperate with any other individual in that society (Mautner, 2005).

3. The **'**Will' is essentially Being-intentional.

4. The 'self' is embodied, but the 'Self' that is dialogical includes not only the physical body but also the relational realities that are the individual's World 2 and World 3 experiences and events.

5. Ferdinand de Saussure (1857−1913) was a Swiss linguist who developed a philosophy of language. Like Vygotsky (1978, 1986, 1997) he understood language as a core feature of Being-human, especially that language was akin to a system of semiotics, that is, a 'sign of signs' that not only pointed to, but through the signs the essence of the language was 'grasped' as a socio-cultural and historical event. That is the Beingness of each individual developed in terms of his or her language, and in this sense Beingness meant 'living the language' which resulted in both stability and change over time for both the language and the person (Gasparov, 2012).

Chapter Ten

Being-mathematical

Australia's future prosperity, and that of every globalizing country depends on pure mathematics being understood and applied through a process of creativity, because new technological capability originates in the ideas of pure and applied mathematicians who wish to make sense of their world (Wormald as cited in Collis, 2014). Correspondingly therefore, Pradier (2014, p. 3) stated:

> It is the mathematically literate people who, figuratively speaking, make the world turn. Maths is pretty much where all technology and science starts.
>
> …We have to put STEM [Science, Technology, Engineering, Mathematics] back on the radar of parents, students and career advisers because out in the real world the demand for graduates in these fields is massive and accelerating.

It is however, the process of creativity that is **crucial** if STEM learning is to be powerful in the twenty-first century. It is through the creative process that powerful mathematical learners integrate dialogically the aspects of science, technology, engineering, and mathematics in Being-mathematical. In connection with these ideas, **Table 10·1** provides an insight into the creative activity of six Australian STEM learners. Of particular relevance to Being-mathematical are their problem solving attitudes, or the intent that focuses or assists their multiple intelligences, prior learning, and experiences to develop a new product or solution for a Common Good. For example, there is a need for Being-creative to facilitate communication between a Self that is intrigued with the real, the abstract, and the disentanglement of convoluted problems with the aid of computer technology; as well as the need for patience if aesthetically good ideas are to be analysed and validated logically and practically by STEM communities-of-practice.

Notably the first female, and 2014 recipient of the Fields Medal (officially known as the International Medal for Outstanding Discoveries in Mathematics), namely, Mirzakhani of Iran and Stanford University, remarked that it was through patience that the beauty of mathematics was revealed in the geometries of dynamical systems (Lehmann, 2014; http://www.mathunion.org/general/prizes/2014). However, STEM learners constitute a small minority in mass education. The question is posed, What will be the percentage increase in high level STEM problem solvers if powerful mathematical learning is introduced in schools across Australia and internationally?

Table 10·1. Being-creative through the use of mathematics: The attitudes and perspectives of six STEM-educated Australians (adapted from Dobos, 2014)

Beings-mathematical	Level of Education	Attitude/Perspective	Being-creative
Yaya Chenue Lu	Year 12 student in Hobart, Tasmania	The only way we can truly understand and appreciate technology, and how we interact with it, is if we focus on the communication between **the computer and the Self**.	Yaya has protyped a low-cost voice-controller that allows quadriplegics to guide their wheelchairs more easily.
Terence Tao	Professor of Mathematics at the University of Los Angeles, California. Terence grew up in Adelaide and attended Flinders University.	In my job it is not about how quick or how smart you are, but more about **patience**.	Terence's algorithm allows equations to be solved with very limited data, and this has improved the speed and accuracy of diagnosing tumours and spinal injuries through MRI technology.
Andrea Morello	Associate Professor at the University of New South Wales, and Winner of the 2013 Prime Minister's Prize for Physical Scientist of the Year	Science is about the **triumph of good ideas**.	Students will pioneer machines that exploit the laws of quantum mechanics to solve problems that are otherwise intractable on classical computers.
Joanna Masel	Associate Professor at the University of Arizona	I dislike competitiveness. I prefer cooperation — **working to solve problems together**.	Joanna and her team analyse other people's data to test theories of how evolution works.
Nick Beeton	Post-doctoral researcher at the University of Tasmania's School of Zoology	I have always enjoyed problem-solving and I wanted to know more about **how things work at an abstract level**.	Ecological modelling to benefit native species
Benjamin Burton	Computational topologist and 'knot theorist' at the University of Queensland	Mathematical **intrigue**	He 'teaches' computers how to solve problems that humans find easy, but that are unexpectedly difficult for a machine.

An International Perspective on Problem Solving in Mathematics

Problem solving is a core feature of Being-human, and thus without problem solving Being-mathematical would not have a present history or a viable future. There are many different problem solving experiences and perspectives in mathematics education (e.g., Dominowski & Bourne, 1994; English & Sriraman, 2010; Hiebert et. al, 1996; Schoenfeld, 1985). However, the international problem solving perspective that follows, and which is described more fully in **Appendix A**, is based on two special issues of the *International Journal on Mathematics Education* (i.e., the *ZDM* journal, or the *Zentralblatt für Didaktik der Mathematik*) that were published in October 2007. The motivation behind the publication was to sum up the "state of the art" of problem solving around the world, because although mathematics and human cognition are to a large degree universal, "mathematics teaching and the conduct of research into mathematical thinking, teaching, and learning are very much cultural matters" (Törner, Schoenfeld, & Reiss, 2007, p. 353). By comparing and contrasting diverse teaching and learning contexts and situations in mathematics education, educators from different countries can enrich their educational and problem solving strategies, competences, attitudes and views, which in turn can be used dialogically to inform the mathematical creativity and classroom pedagogy of educators for a globalising world.

Internationally however, mathematical problem solving in schools, as is considered in Appendix A, has an underlying theme: meaningful and authentic problem solving in classrooms is dependent fundamentally on the Beingness of the mathematics teacher in relation to his or her mathematical, pedagogical, technological, and social affordances. Teachers in **China** and **Israel** for example, benefit from a rich knowledge-base of mathematical problem solving acumen in schools. The 'problem solving tradition' is maintained and enhanced as novice mathematics teachers are mentored didactically in the static−dynamics of mathematical problem solving. In this regard, with reference to **Germany** in particular, it appears essential that the basic principles of mathematics education and problem solving are expressed in a formal document, and are then role modelled in primary schools, before being expanded upon in high schools and tertiary institutions. The simple reason is that deep learning takes time, especially when pursued through spiral STEAM (Science, Technology, Engineering, Arts, Mathematics) curricula.

However, if the curricula place an overemphasis on testing, as was the case in **Brazil**, then it becomes almost impossible for the mathematics teacher to implement effectively the 'new problem solving paradigm' as is delineated in **Table 10·2**. Moreover, if mathematics teachers do not have sufficient knowledge or curricula support in implementing the 'new', then the intended curriculum and that which is realized in classrooms can be very different. For example in **The Netherlands**, mathematics textbooks did not reflect 'realistic mathematics education' and many mathematics teachers found it less than straighforward to develop their own materials. In **Italy**, many mathematics teachers were not sufficiently qualified to implement the intended curriculum, and specifically, the in-service training of novice teachers was often inadequate, especially in relation to the timing and nature of the feedback that was necessary if students were to progress mathematically at a reasonable rate. And in the **United Kingdom** many 'well-intentioned' systemic changes failed to materialize, often because different professional bodies did not communicate with government that the proposed curriculum changes were out of step with the training, experience, and pedagogical or mathematical 'beliefs' of many teachers.

Table 10·2. A comparison between the essentials of 'New thinking' and 'Old thinking' problem solving paradigms (adapted from D'Ambrosio, 2007b, p. 517)

'Old' Problem Solving Paradigm	'New' Problem Solving Paradigm
Given problems to solve	Identify problems; Problem posing
Individual work	Cooperative work; Teams
One solution problems	Open ended problems
Exact solutions	Approximate solutions

In **Singapore** however, overall 'best practice' was improved through a centralized and future-oriented educational system that not only promoted a focused problem solving curriculum, but also advocated that both teachers and students needed to engage with the mathematics syllabus more meaningfully, and usefully by learning cognitively and metacognitively (see p. 112). Furthermore, although the 'Singaporean journey' in mathematics education indicates that teachers should be treated as a valuable national resource, 'too much' democracy in the educational system is unlikely to optimise teaching and learning outcomes (Yoong et al., 2009). Nevertheless, if a problem solving focus in mathematics classrooms is to succeed then a 'socio-political lesson' from mathematics education in the **United States**, is that a problem solving basis cannot be at the expense of students being able to perform 'the basics' fluently (e.g., each student knowing his or her times tables off by heart). In other words if a student understands 'the basics' then he or she must have the demonstrated skills to address familiar real-world problems conceptually and correctly (Stacey, 2010). As inferred from mathematics education research in **France** however, it is the wise teacher who establishes a didactical contract between him or herself, the students and parents, making clear the basic educational processes and responsibilities of the different role players with respect to the curriculum. In particular, and informed by the experience of **Japanese and Australian** mathematics educators, a meaningful didactical contract for the Conceptual Age can aid the teacher in aligning the curriculum and assessment through a balance of direct instruction and guided discovery that both involve the creative process.

Problem solving research. Relatively little progress has been made in 'problem solving research' since the 1990s, and problem solving *per se* "is fading away as a specific subject in curricula and as themes of conferences and symposia" (D'Ambrosio, 2007b, p. 515). This is not surprising because non-routine problem solving in classrooms has by and large not been studied as part of the creative process, and novel problem solving apart from the creative process, or Being-creative is a misnomer.

A fundamental shift therefore in mathematics curricula, and problem solving research is necessary if powerful problem solving (in a Gestalt sense) is to be realized in mathematics classrooms internationally. Visionary, authoritative, and centralized–decentralized leadership is essential to direct and integrate curricula, instruction and assessment towards a creative and systemic focus that acknowledges both the global and the local in mathematics education. In other words if learners who are situated in local contexts are to experience mathematics in terms of a holism that involves both conscious and non-conscious knowing, then teachers need to be taught and mentored implicitly and explicitly in this regard.

Problem 1: Determine 9+5 in a unique way.

- As part of the creative process, the *I* intentionally referred the problem to the intrapersonal and unseen *Other*.
- The *I* relaxed. While thinking about something else, a visual came to mind of the '9' crossing over the plus sign and positioning itself next to the '5'.
- The *I* saw the number as 95 and immediately prime factorized the number as 19*5. This was an automatic and trained response on the part of the *I*.
- The *I* reflected on what to do with 19*5, but did not know how to proceed.
- The *I* referred the problem to the unseen *Other* expecting a meaningful response.
- A short time later, while doing something else the problem solver experienced the A-ha moment: 19 – 5 =14.

Problem 2: Solve for *all unknowns* if $(x-a)(x-b)(x-c)(x-d) = 49$ .... A
$a+b+c+d = 4$ …………………….... B

- Five unknowns, but only two equations.
- A **literal** phenomenological and dialogical approach was adopted.
- What is equation A? Make the L.H.S. **exactly equivalent** to the R.H.S.
- What is the L.H.S.? Four linear and different **factors**
- 'Prime' factorize the R.H.S. into four different linear factors: 49 = (7)(-7)(1)(-1)
- $\therefore x-a=7;\ x-b=-7;\ x-c=1;\ x-d=-1$
- Solving simultaneously, $4x - (a+b+c+d) = 0$ ……….. (LHS and RHS addition approach)
- $\therefore x=1$
- By substituting $x = 1$, a = −6; b = 8; c = 0; d = 2
- Check solution

Figure 10·1. Two problems that were solved intuitively and analytically by Being-mathematical through Wallas' (1926) four stages of creativity.

With reference to **Figure 10·1** for example, creative problem solving in mathematics can be different from teachers' prior learning, especially in relation to Being-intuitive. It is noteworthy that problems in mathematics need not be real world problems in order to motivate students, because it is fundamentally Being-ethical, Being-creative, and Being-mathematical through eureka moments and intuitive‒analytical functioning that sustains students' positive affect, and problem solving flow inside and outside of classrooms.

However, although mathematical problem solving is a primary, or even a systemic curricular goal in many countries, in nations like Cyprus and England for example, "relatively little is known about how teachers construe problems and problem solving in relation to curricular intentions, not least because they interpret and adapt curricula according to their experiences, capabilities and beliefs" (Xenofontos & Andrews, 2014, p. 279). Importantly therefore**,** **Appendix B** depicts a comprehensive protocol structure that can be used by mathematics educators and researchers as a dialogic or instructional tool with which to reflect upon, or to discuss and articulate various mathematical and problem solving pedagogies, beliefs and awarenesses.

Universals of Being-mathematical

Being-mathematical in classrooms involves mathematical problem solving, but in agreement with Hino (2007),

> by recognizing the problem solving approach as a powerful way of learning mathematics, we continue to investigate conditions for, and roles of, the teacher in realizing a classroom in which the students are actively engaged in the activity of solving problems and **developing** [for emphasis] mathematics. (p. 513)

The world is globalizing in terms of Three Worlds and therefore mathematics teachers, as lifelong learners, need to advance pedagogically through different types of inquiry including action research. However, in order to facilitate a commonality of Being-mathematical internationally, it is essential that teachers of powerful mathematical learning be made aware of universal problem solving competences that are applicable to all socio-cultural situations.

General Problem Solving Competences. From an educational psychology perspective,

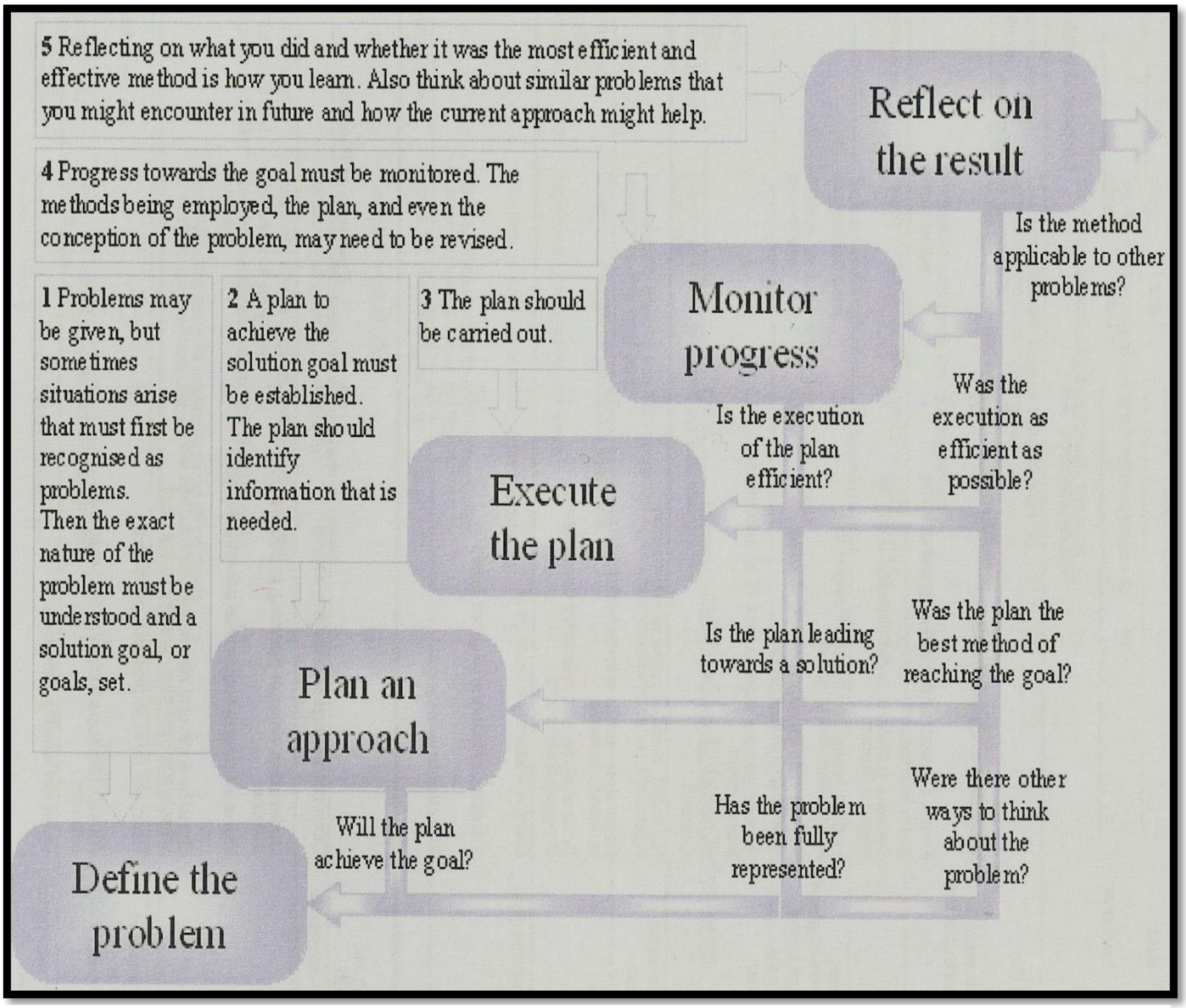


Figure 10·2. The five steps: Generic components of the problem solving process (adapted from Curtis, 2010, p. 413)

problem solving involves processes that appear to be underpinned by generic competences. The five steps, or competences of problem solving are described in **Figure 10·2**. The steps unfold epistemologically ('how' the *I* comes to know), and enfold ontologically (what the *I* 'sees' as a result of his or her knowing). Therefore problem solving is a bi-directional activity between the different steps involved in the Beingness of problem solving, namely, **Defining** the problem, **Planning** an approach, **Executing** the plan, **Monitoring** progress, and **Reflecting** on the result. Even though in a language sense all five stages are a present history of 'doing words', the first three steps or stages are primarily cognitive activities, whereas the last two are largely metacognitive. Using these ideas and terms, Curtis (2010) modelled, or synthesized diverse conceptions of problem solving that are not only applicable to academic learning but also to lifelong learning. At least to some degree this model has been substantiated quantitatively through empirical research.

In a Kantian sense however, the word 'abstract' means to 'draw away from that which already exists' (Gray & Tall, 2002; Kant, 1950; Mautner, 2005). Therefore when solving a mathematics problem there is a drawing away from the concrete, which might be described as "that property which measures the degree of our relatedness to the object (the richness of our representations, interactions, connections with the object), how close we are to it, or if you will the quality of our **relationship** [for emphasis] with the object" (Wilenski, 1991, p. 198). Consequently, the abstract only has meaning in connection with the concrete. Thus to understand the underlying generic competences of problem solving it is crucial to appreciate the generic nature of the transformative movement between the concrete and the abstract. As is depicted in Figure 10·2, Curtis' (2010) model includes epistemic or ontic questions that are likely to facilitate the psychological or intellectual movements between the different competences. For example, consider the bidirectional questioning between Step 1 and Step 2 — How can a plan of action be derived from the problem definition? Will the plan of action achieve the goal as described in the problem definition? The first question relates to epistemology and the second question to ontology.

Piagetian developmental psychology suggested that learning occurs from the concrete to the abstract. In the case of Curtis' (2010) model, moving from Step 1 through to Step 5 would involve moving increasingly from the concrete to the abstract. However, dialectical philosophy has indicated that the opposite might also be true (Mepham & Reuben, 1979; Merleau−Ponty, 1974). Crucially for novel problem solving, it is probable that the **development** of the 'concrete−abstract' solution as a single complex process or object “occurs in a double ascension that simultaneously moves in both directions: it is a passage of one in the other” (Roth & Hwang, 2006b, p. 334). Consequently, if an individual is to abstract a non-trivial solution it is not possible without referencing continually concrete detail and lived experience (Roth & Hwang, 2006a, 2006b), because embedded in the concrete–abstract relation is a decontextualised ‘ascent’ from the concrete to the abstract and a contextualized ‘ascent’ from the abstract to the concrete (Van Oers, 2001). Thus in moving from the concrete to the abstract, the abstraction that is the concrete–abstract is co-emergent with the abstract−concrete.

Therefore in a dialectical materialist sense, the development of the concrete to the abstract and the abstract to the concrete is not two separate movements but a single complexity of movements. Consequently it is germane that Curtis (2010) interrelated all five component processes of problem solving into a single bi-directional whole, so that when completed, the concrete−abstract constitutes a psychological or reified basis for the meaningful understanding of the problem in the mind of the problem solver.

Epistemic Action Model. The Curtis (2010) problem solving model was based almost entirely on the four process-based models of problem solving described in **Table 10·3**. Each of the four models are general problem solving models. However, in mathematics education an empirically and process-based epistemic action model was developed through extensive qualitative research (Hershkowitz, Hadas, Dreyfus, & Schwarz, 2007; Hershkowitz, Schwarz, & Dreyfus, 2001; Schwarz, Dreyfus, & Hershkowitz, 2009; Schwarz, Hershkowitz, & Dreyfus, 2002). An epistemic action is an embodied cognition that is apparent outwardly to a a thinking body (Todd, 1937). Therefore mental problem solving activity is made visible to sense perception by means of epistemic actions.

The four epistemic or mental actions are underpinned by a **building** metaphor when abstracting a new piece of mathematics, or solving a novel problem in a particular context.

Table 10·3. Process-based models of problem solving (Curtis, 2010, p. 118)

Cognitive and metacognitive processes	**Sternberg's (2000b) meta-components**	**Pólya (1957, pp. 5−19)**	**Bransford and Stein (1984, p. 12)**	**Hayes (1989, p. 3)**
Apprehend	Recognise the problem	Understand the problem	Identify the problem	Finding the problem
Represent	Decide nature of problem Select a representation		Define and represent the problem with precision	Representing the problem
Plan	Select problem solving processes	Devise a plan	Explore possible strategies	Planning the solution
Act	Allocate resources; Encode; Infer; Compare, and Respond	Carry out the plan	Act on those strategies	Carrying out the plan
Reflect Evaluate	Monitor progress Evaluate effectiveness	Look back	Look back and evaluate the effects of activities	Evaluating the solution Consolidating gains

The first action was labelled **Recognizing**. The degree to which the student can grapple meaningfully with the novel problem is dependent on the prior learning and affective intent of the student. In other words if the Beingness of the student is insufficient to make sense of the mathematical problem, then the problem is inappropriate for the individual. Therefore, the mental action of Recognizing is always subjective because Being-there is a unique experience for the individual.

The second epistemic action means **Building-with** the components of the problem and the prior learning of the student towards an elegant solution. The third epistemic action is the present continuous verb **Constructing**. This action is central to the process of developing a concrete−abstract and refers to the first time that the new construct is articulated or expressed by the learner. The fourth epistemic action was termed **Consolidation** (Dreyfus & Tsamir, 2004; Hershkowitz, Hadas, Dreyfus, & Schwarz, 2007; Kidron & Dreyfus, 2010; Schwarz, Hershkowitz, & Dreyfus, 2002; Tsamir & Dreyfus, 2002). Consequently, the epistemic action model was labelled the **RBC−C** nested model, because the four epistemic actions, if completed successfully, were viewed psychologically as a hierarchical mental structure that enabled the learner to understand the problem through embodied cognition and metacognition (Hershkowitz, Schwarz, & Dreyfus, 2001; Schwarz, Hershkowitz, & Dreyfus, 2002).

Moreover, the consolidation of a new mental structure has at least five different emergent characteristics and therefore can be understood as a complex form of self-regulated learning (Schwarz, Dreyfus, & Hershkowitz, 2009; Schwarz, Hershkowitz, & Dreyfus, 2002). The first characteristic of consolidation is that of **Immediacy**. It refers to the speed and goal-directedness with which a mathematical structure is recognized, or built-with in relation to the student's prior learning. The second self-regulating characteristic is **Self-evidence**, or the student **feels** that his or her building-with is 'obviously correct'. The third characteristic is the **Confidence** or certainty with which a mathematical structure, or structures are applied in attempting to solve the problem. Consequently, the second and third consolidatory characteristics involve intuitive functioning on the part of the individual. Fourth, the regular use of a structure is likely to facilitate the establishment of new connections and thus promote the **Flexibility** of its use. The fifth consolidatory action is metacognitive in the sense that the student reflects on the structure for the purpose of increasing mathematical **Awareness** and depth of knowledge.

Therefore a new mental structure that is consolidated by being exposed to unfamiliar problem solving situations is likely to be more robust cognitively and affectively. Moreover, if 'abstraction in context' culminates deliberately in acts of consolidation that are verified by more knowledgeable interpersonal *Others*, then the student will be more confident in transferring his or her learning to different situations than otherwise would be the case **—** with the proviso that the new situations are not too dissimilar from that which the individual has already experienced (Anderson, Reder, & Simon, 1996, 1997; Schwarz, Dreyfus, & Hershkowitz, 2009).

Because both the RBC−C model and S−R−O−C learning protocol are informed theoretically by educational psychology, and the goal of both is ultimately to establish coherence between the student's **concept image** (CI), and the **concept definition** (CD) which is external to the mind−body of the learner, that is a *phainomenon* and *noumenon* respectively, it is not surprising that there is a close overlap in sequential learning actions between the model and the protocol. In this regard consider **Table 10·4**.

Table 10·4. The RBC−C epistemic action model is compared with the S−R−O−C teaching and learning protocol.

RBC−C	**S−R−O−C**
Recognizing	Selecting
Building-with	Relating
Constructing	Organizing
Consolidating	Checking

The idea of a CI and a CD was first introduced to the Mathematics Education community more than three decades ago (Vinner & Hershkowitz, 1980; Tall & Vinner, 1981). Importantly, research has shown that the mental image 'wins' over the definition when students engage with non-trivial mathematics (Niss, 2006). However, a powerful learner of mathematics is able to establish coherence between the intuitively infused CI and the formulaic or logically deduced CD, because the individual has learned that the

> mastery of symbolism and formalism requires students to develop a kind of 'controlled schizophrenia' between intuition (and sense-making) and formalism that allows them to switch between the two so as to distinguish between interpretation and meaning, on the one hand, and notation and rules, on the other hand. (Niss, 2006, p. 61)

The Pirie−Kieren Model for the Growth of Mathematical Understanding. Being-mathematical is a complex interaction between concept images and concept definitions requiring both intuitive and analytical thinking. Consequently, a student's mathematical understanding is dependent upon the nature and the quality of his or her concept images and associated definitions. Over the past 25 years the Pirie−Kieren model (PKM) has been used and enhanced to study the development of students' mental images and mathematical understandings (Martin, 2008; Pirie & Schwarzenberger, 1988; Pirie & Kieren, 1989, 1992a, 1992b, 1994; Pirie & Martin, 1997; Wright, 2014). **Figure 10·3** depicts the PKM's hierarchical or nested levels of mathematical growth.

The PKM is based on the notion that more formal, abstract mathematical knowledge is grounded in informal or previous learning (Gravemeijer, 2002). Consequently, the first level was termed **Primitive Knowing**. This level refers to that which a learner has 'in his or her head' when approaching an unfamiliar mathematics problem. It is the responsibility of the teacher to be aware of his or her student's prior learning, or 'set-befores' and 'met-befores'. A 'set-before' is defined as an embodied and stable mental structure that shapes long-term learning and enables, or inhibits the individual to function mathematically in specific ways (McGowan & Tall, 2010). Consequently, set-befores need to be checked for coherence and consistency. There are at least three different kinds of set-befores. First, those that assist the student to recognize different mathematical patterns. Second, there are those that facilitate the automatic repetition of action sequences like procedures. The third type refers to the manner in which the individual has been enculturated mathematically over time. For example, whether the learner is an instrumental or a relational learner of mathematics?

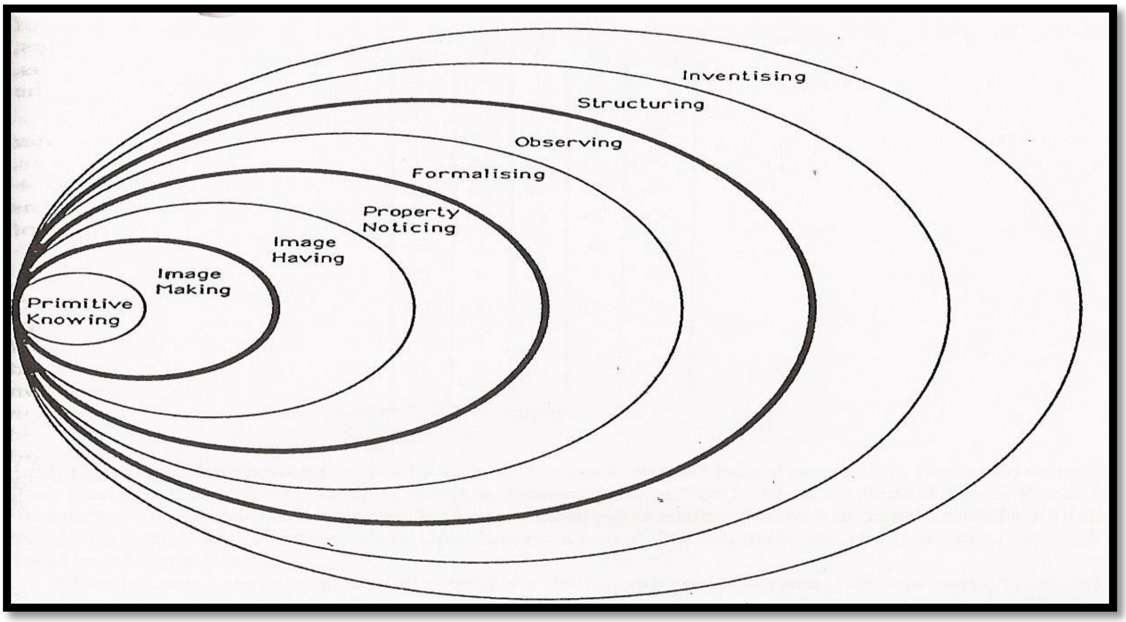


Figure 10·3. The Pirie–Kieren model: A dynamic process of mathematical understanding (Pirie & Kieren, 1994, p. 167)

However, not all mental facilities have the same degree of influence on long-term learning. 'Met-befores' have been described as current mental facilities that might include set-befores, but in many instances the internalization of specific prior experiences of the learner was tenous, incomplete, or lacked meaning (Tall, 2008). Therefore it was incumbent upon mathematics teachers to 'work with' students to identify any inadequate set-befores or met-befores. In particular phenomenological primitives (basic intuitions), because these constitute the psychological foundation upon which new understandings can be developed (diSessa, 1983). Consequently teachers need to "appreciate their power, and confront them directly and repeatedly. Only then is it possible to construct more adequate mental representations that become robust and enduring" (Veenema & Gardner, 1996, p. 71).

Therefore in order to assist a student to develop an understanding of a new mathematical situation, the individual can be requested to complete certain tasks for the purpose of 'opening up' the mathematics problem 'for what it is', or 'could be' in the mind of the student. This is the second stage of the student's mathematical development and was referred to as **Image Making** in the Pirie−Kieren model, because the learner attempted to 'picture' the problem situation as a whole, either physically or mentally using his or her primitive knowing. Once the student had developed his or her mathematical understanding to the point of being able to visualize the problem situation, then the learner had attained the third level of mathematical understanding which was **Image Having**. This implied that the learner had a 'mental visual' available that could be used to tackle this exact type of problem without first reconstructing the mental representation.

At the fourth level the student could examine his or her image of the problem situation for pertinent properties, or **Property Noticing** that would position the problem solver to sequence a logically deduced solution. The fifth level of mathematical understanding was labelled **Formalizing**. The learner symbolized, or developed the concrete−abstract relationship formally and analytically. The sixth level was the student **Observing**, or standing back from the complex activity of formalizing the Property Noticing for the purpose of generalizing the problem situation and solution to include multiple similar problem situations. In more advanced mathematics the activity of Observing could result in an elegantly stated mathematical theorem with proof.

The seventh level indicated that the student understood the assumptions that gave rise to a substantial mathematical structure, or theory of which the theorem was but one of many theorems and lemmas (e.g., Euclidean geometry). Consequently, the level of **Structuring** implied thinking axiomatically, proving rigorously, and reflecting on meta-mathematical argument. The eighth and outermost ring of the Pirie−Kieren model was called **Inventising**. The student used his or her imagination to enlarge the bounds of a comprehensive or mathematical theory (e.g., Group, Ring, or Graph Theory) by asking new or different questions. In Being-mathematical the student ‘moved away’ from existing ideas or preconceptions in an attempt to concretize novel or more complex concrete−abstracts.

Learning by folding back. The Pirie–Kieren model has been considered to be hierarchical because all previous forms of mathematical understanding are embedded in the outermost level of knowing that the student had attained (Martin & Pirie, 2003). However, to develop within and beyond a particular level of understanding a student did not need to be overtly aware of all earlier forms of understanding. The notion of ‘do not need’ or threshold boundaries is represented by the **bold rings** in Figure 10·3. These rings denote points of abstraction in the Pirie–Kieren model (Kieren, Pirie, & Calvert, 1999). For example, the bold ring above Image Making indicates that a student does not need to make direct use of his or her earlier understandings (Primitive Knowing and Image Making) for the purpose of progressing from Image Having to Property Noticing. However, in order to abstract meaningfully student learning does not only occur from the concrete to the abstract.

Therefore a very important feature of the Pirie–Kieren model is the notion of folding back, because

> understanding in action continually entails *folding back* (at least for students from ages seven to university level whom we have studied). No matter what level or how sophisticated the understanding of a person, whenever they find their mental and physical actions and their situation incoherent or incomprehensible, they are prompted to *fold back* to an inner level of activity in order to *extend* their current action capabilities and action spaces. (Kieren, Pirie, & Calvert, 1999, p. 218)

Thus when students construct their own (conceptual) knowledge with respect to an evolving or emergent situation, it is a case of structuring and re-structuring their explanations and re-examinations (Borgen & Manu, 2002; Schoenfeld, 1992). For example, **Figure 10·4** represents a mapping of a high school student's mathematical actions, or path of thinking as she engaged with an unfamiliar mathematics problem. The solid lines indicate Jane's mathematical connections that were clear and correct, and the broken lines represent those connections that were incoherent or incomplete.

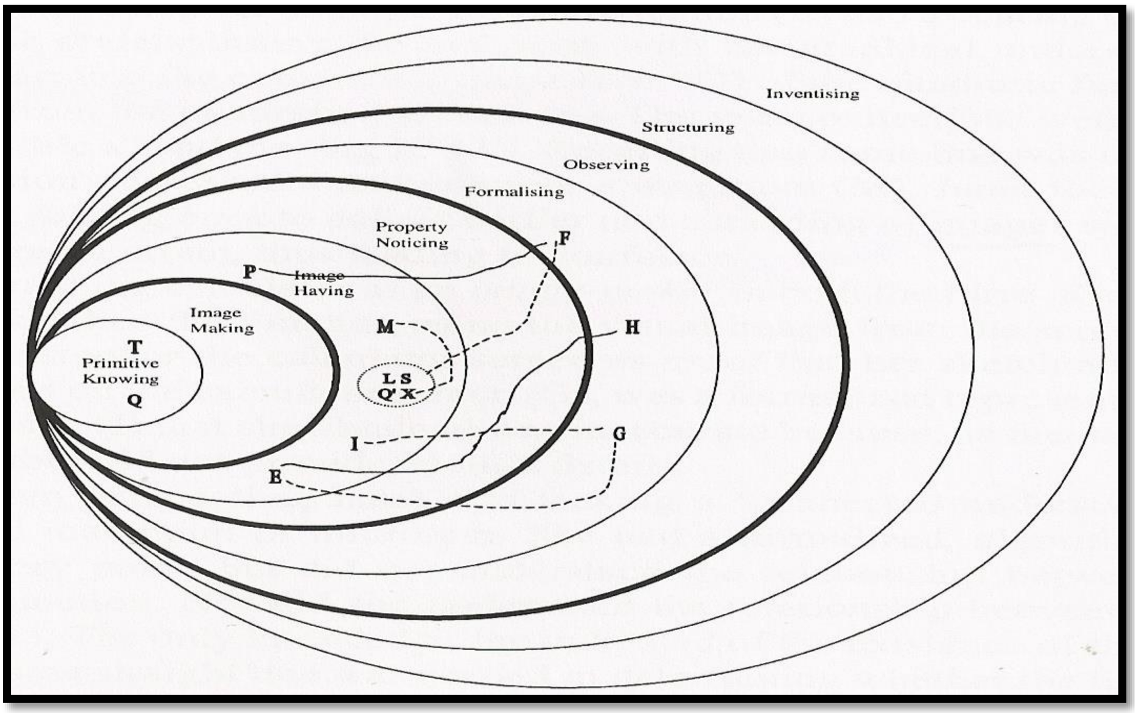


Figure 10·4. A mapping of Janet's (a high school student) mathematical actions on the Pirie–Kieren model for the growth of mathematical understanding. Janet commenced her work at the formalizing level, F (adapted from Borgen & Manu, 2002, p. 161).

Thus the ebb and flow of a student's mathematical actions do not necessarily constitute effective or correct learning (Borgen & Manu, 2002; Martin, 2008; Warner, 2008). However, although "explaining and accounting for mathematical understanding is a complex and challenging problem," (Martin, 2008, p. 83) the Pirie−Kieren model is a visual tool that can enable the teacher and the student to identify mathematical understandings, or misunderstandings by carefully tracing out a sequence of sense-making actions within and between the different levels of abstraction (Pirie & Kieren, 1992a).

Importantly therefore, the PKM was developed to promote mathematical understanding through **acts of communication**. As depicted in **Figure 10·5**, "each level beyond primitive knowing is composed of a complementarity of *acting* and *expressing*" (Pirie & Kieren, 1999, p. 175). At any level, acting was observed to be based on all previous understanding and therefore provided continuity and coherence with inner levels. In addition, students who expressed themselves verbally and gesturally within a specific level substantiated their acts of learning within that level of abstraction. For example, Image Having was understood to be a

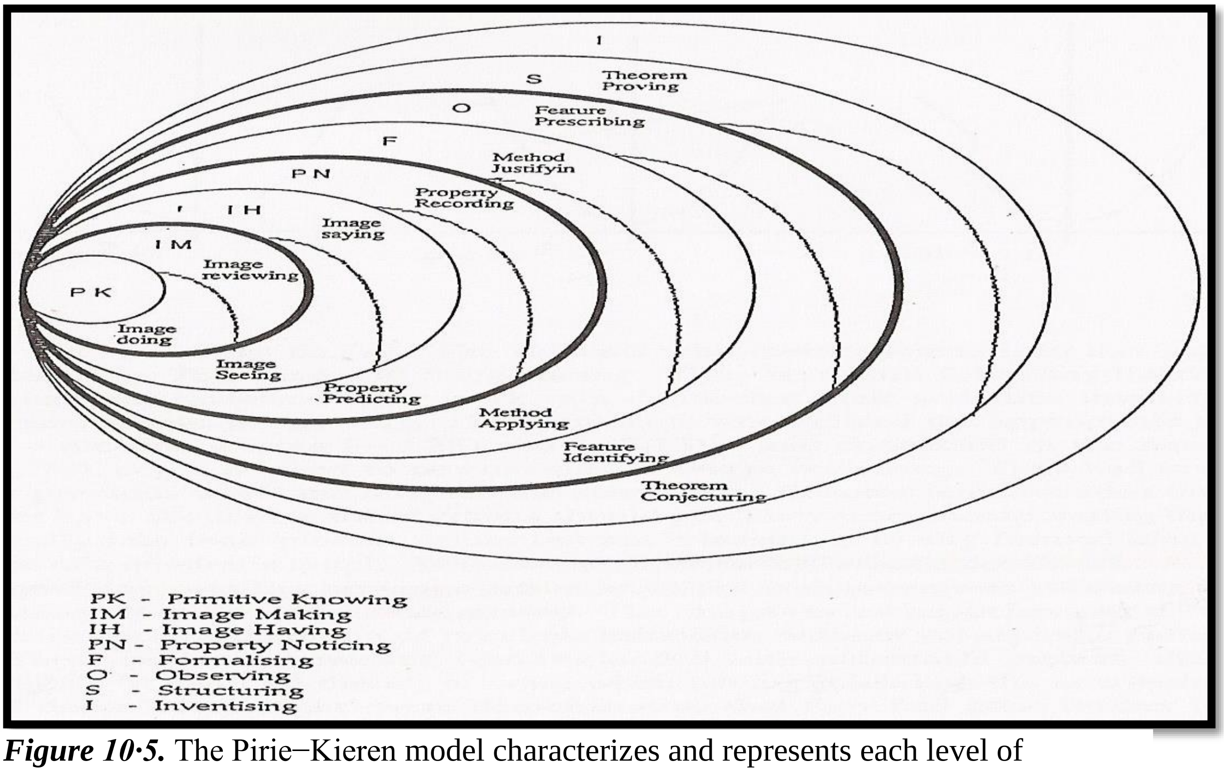


Figure 10·5. The Pirie−Kieren model characterizes and represents each level of mathematical understanding as a complementarity of acting and expressing (adapted from Pirie & Kieren, 1994, p. 176).

complementarity of **Image Seeing** and **Image Saying**. In other words a student who attained the level of Image Having could not only act correctly on the basis of his or her mental representation of the problem situation, but was also able to explain his or her mathematical actions, either to him or herself, or an interpersonal *Other*.

Therefore the Pirie−Kieren model would probably be a useful tool to facilitate a language of mathematical dialogue in classrooms, because at least in part it offers a common vocabulary for teachers, students, and researchers to describe and act mathematically through different layers of understanding. Moreover, 'whether a student understands or not' cannot be ascertained from his or her written work alone (Borgen & Manu, 2002). Thus in particular, dialogic student−teacher acts that involve folding back are likely to be advantageous in assisting students to overcome epistemological obstacles that might exist between formal mathematical knowledge (e.g., the concept of a limit), and non-formal or unstructured intuitive mathematical knowing (Kidron, 2008, 2009; Kieren, Pirie, & Calvert, 1999; McGowen & Tall, 2010; Sierpińska, 1987; Tall & Vinner, 1981).

The Conceptualization of Symbol Processing. The Pirie−Kieren model emphasizes that the growth of mathematical understanding is grounded in situated action which means developing a mental image that describes the problem situation in such a way that it can be formalized, or deduced logically. However, students have also benefited by learning mathematics through efficient structures (Bruner & Anglin, 1973; Inhelder & Piaget, 1958). In particular, being trained in mathematical procedures has been a central theme in the history of mathematics education (Kilpatrick, 1988), where a procedure is a visually mediated sequence of coherent step-by-step actions that lead to a logically correct outcome (Davis, 1984).

Thus procedural learning is a form of instrumental or relational embodied cognition, where sense perception is paramount in Being-able to prosecute correctly and repeatedly a chain-like action sequence. As indicated in **Figure 10·6**, progress in mathematical learning can commence with the deliberate practice of routines (procedures) towards the goal of mastery. This constitutes the first stage in learning mathematics through symbol processing. As a result of sufficient practice that includes procedural variations (especially with respect to the real number axioms) and different straightforward applications or proofs, the procedure is likely to emerge in the mind of the individual as a process (Skemp, 1979; Sun, 2011; Watson & Mason, 2006). For example, consider the application of the commutative and distributive axioms to show formally that $a(b+c) \equiv a(c+b) \equiv (b+c)a \equiv (c+b)a.$ Thus from the perspective of the Pirie−Kieren model, it is important that students appreciate the foundations upon which their respective understandings of mathematics are based, and this in part can be achieved through a reflective practice that involves the dynamics of procedural variations; words, symbols and spatial relations.

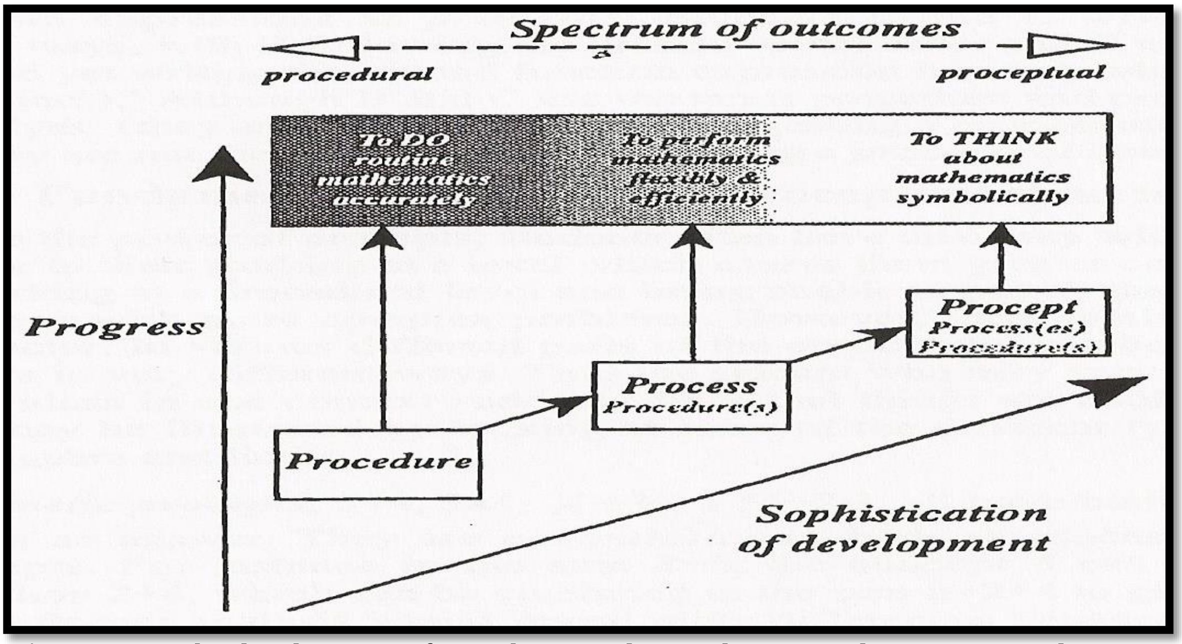


Figure 10·6. The development of a mathematical procedure towards an increasingly advanced, or effective cognitive−affective state in the mind of a learner (adapted from Tall, 2000, p. 37)

Nonetheless a process is essentially an effect that involves a cause, because the beginning (the cause) and the end (the effect) of the procedure become the foreground of interest, while the intermediary actions (operations) are but necessary, or background interventions on the path that leads to the logically deduced and 'felt' outcome. Consequently at the level of a process, the procedure has begun to objectify as a mental representation in the mind of the individual.

This implies a psychological compression, or differentiation of knowledge between the cause and the effect, and the mathematics that links the cause and the effect. Therefore a process-in-mind is more sophisticated and abstract, or less concrete and situated than its intrinsic procedure, which as a result allows the learner greater flexibility of thought in application as well as reduced cognitive load. At the level of a process the individual has begun to grasp the procedure intuitively and therefore holistically.The next stage of mathematical development is to reflect intentionally on the process symbolically, and then to relate the process to other mathematical ideas and processes. In so Being the learner can conceptualize a process not only as a bodily activity but also as a concept to ponder. Thus the main goal of procedural learning is to understand mathematics **proceptually,** or in other words as "a process to do and a concept to think about" (Gray & Tall, 2002, 2007), in the affective volition that "the common tendency in the mathematical education community today is to move from meaningless procedures (rituals) to meaningful actions" (Vinner, 2007, p. 10).

Learning mathematics structurally and developmentally. The **SOLO taxonomy** was developed to quantify the **S**tructure of the **O**bserved **L**earning **O**utcome. The student was provided with a mathematical stimulus and then requested to respond. The quality of the response was graded by counting the **number** of correct and relevant connections (Biggs & Collis, 1982). The respective gradings increased in number and were labelled Pre-structural, Unistructural (U), Multistructural (M), Relational (R), and Extended Abstract (EA). Consider, **Table 10·5** which compares and aligns the different stages of proceptual learning with the RBC−C nested model of epistemic actions, and also with the SOLO taxonomy which is underpinned by Piaget's Stage Theory of Cognitive Development (Inhelder & Piaget, 1958;

Piaget, 1985).

The great value of Piagetian theory, lies not in limiting the learning possibilities and potential of students based on age, but rather in the sequencing and description of the necessary learning stages towards the goal of formal operational reasoning (Biggs & Collis, 1982; Brown & Desforges, 1979; Shulman, Restaino−Baumann, & Butler, 1985). In embracing the ideas of both Piaget and Vygotsky however, the CAME and CASE projects demonstrated the educational value of cognitive acceleration in mass mathematics and science education respectively (Adey & Shayer, 1990; Shayer & Adey, 2002).

Table 10·5. The development of a mathematical concept: Three stage-wise models compared (adapted from Biggs & Collis, 1982; Hershkowitz, Schwarz, & Dreyfus, 2001; Gray & Tall, 2001)

Proceptual learning (Gray & Tall, 2001)	**RBC−C model** (Hershkowitz, Schwarz, & Dreyfus, 2001)	**SOLO taxonomy** (Biggs & Collis, 1982)
Procedure	**Recognizing**	**Pre-structural** (Pre-operational) **Unistructural** (Early Concrete)
Process	**Building-with**	**Multistructural** (Middle Concrete)
Procept (process to do)	**Construct**	**Relational** (Concrete Generalization): Early Formal contains the elements of abstract thinking, but the student can only generalize from within the context of his or her own experiential learning.
Procept (concept to think about in relation to other concepts and ideas)	**Consolidate**	**Extended Abstract** (Formal Operations): The learner can hypothesize about 'possible' concepts. Advanced, or imaginative combinatorial thinking enables the individual to develop 'novel' results beyond his or her own (empirical) experience.

Since the 1980s however, the SOLO taxonomy has been developed from a classification and assessment tool into the **SOLO model** for the construction and development of mathematical concepts (Pegg & Tall, 2010). The model proposes three main levels of increasing abstraction and complexity — the **Ikonic Mode** (IM), the **Concrete Symbolic Mode** (CSM), and the **Formal Mode** (FM). The Ikonic Mode refers to the internalization of bodily actions as mental images, and therefore is consistent with Bruner's (1960, 1966) conceptualization of abstract learning, because the Ikonic Mode includes his Enactive and Iconic Modes. Notably, IM relates to Image Making and Image Having in the Pirie−Kieren model. CSM is essentially symbol processing within an abstract mathematical (number or spatial) system. The Formal Mode implies that the learner is no longer dependent upon a concrete referent and works logically with principles, theorems, and theories.

Figure 10·7 is an example of SOLO model learning. The core feature of the SOLO model is the UMR (Unistructural, Multistructural, and Relational) learning cycle. In this particular example, learning occurs within the Concrete Symbolic Mode through two successive UMR cycles. The impetus for the first UMR cycle is relational imagery developed in the Ikonic Mode. Similarly, $U_2M_2R_2$ is the second relational response in the CSM that facilitates the first Unistructural response in the Formal Mode. Therefore, UMR cycling is grounded in the Ikonic Mode

> with actions on known objects (which may be physical or mental) which are practised to become routinized step-by-step procedures, seen as whole processes [in the CSM], then conceived as entities in themselves that can be operated on at a higher level [FM] to give a further cycle of construction. (Pegg & Tall, 2010, p. 180)

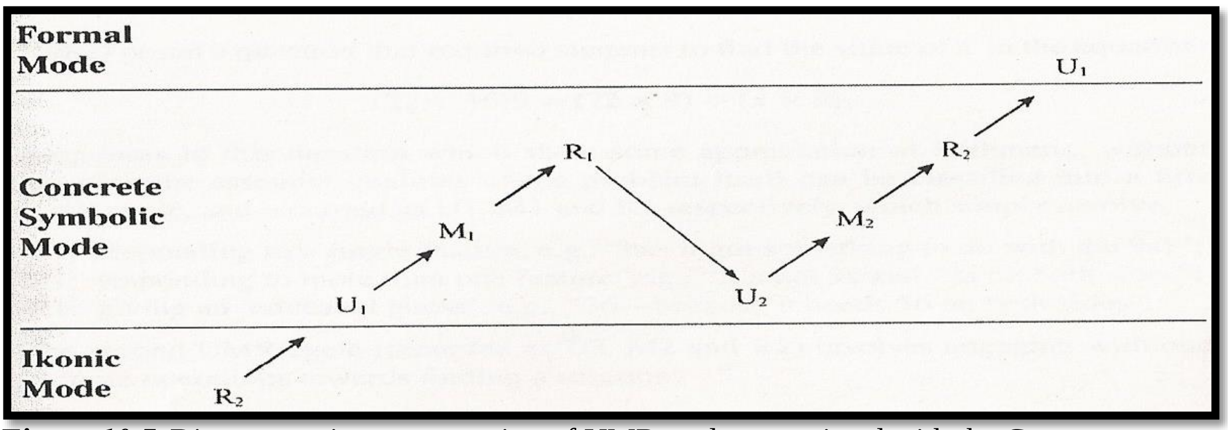


Figure 10·7. Diagrammatic representation of UMR cycles associated with the Concrete Symbolic Mode (Pegg & Tall, 2010, p. 177)

However, the notion of UMR cycling through three different modes of Being-mathematical are but a single, albeit substantial perspective, of how students can learn mathematics conceptually. **Table 10·6** compares and contrasts UMR developmental cycling to the ideas of Davis (1991), Sfard (1991), Dubinsky (1991), as well as Gray and Tall (2002, 2007). Although there are differences between the perspectives, the pattern of ideas contained in Table 10·6 concurs with the essential ideas of Bruner (1960, 1966, 1973):

(1) Students learn first through actions. This view is consistent with Vygotsky (1978, 1986) who maintained that bodily actions were necessary to develop words and sentences.
(2) Mental imagery summarizes actions and influences further actions.
(3) Because symbols are arbitrary and remote in reference, they enable the individual to transform his or her reality beyond that which is possible through either actions or mental imagery.

But especially important for powerful mathematical learners is Dubinsky's (1991) **Action→Process→Object→Schema** (APOS) view of advanced mathematical thinking. In APOS theory a stage of mathematical learning was described that went beyond objectification or reification, namely, the development of a schema which comprised interrelated mental objects and processes as a **circular feedback system**.

The research scholars mentioned in Table 10·6 all contend that students learn mathematics by developing mathematically through different bands or levels of abstraction. The Pirie−Kieren model is also a case in point. In this vein however, but not mentioned thus far, is a developmental model for the rigorous learning of geometry. It is important that powerful mathematical learners have both an algebraic and geometric understanding of mathematics, because essentially, the algebra of geometry and the geometry of algebra in all possible forms is Mathematics. As described in **Appendix C**, Van Hiele's Five Levels of Geometric Thought can be related to the **extended** epistemic action model of mathematical learning that is RBC−CE. Given that Being-there is fundamentally a locus of physical actions that are communicative in time and space, it seems appropriate to juxtapose sequentially the

Table 10·6. A comparison of five basic learning sequences; each of which can lead to the objectification of a mathematical concept through symbol processing.

Biggs & Collis, (1982, 1991); Pegg & Tall (2010): SOLO model	**Davis (1984): A cognitive science approach to mathematics education**	**Sfard (1991): Processes and objects as different sides of the same coin**	**APOS of Dubinsky (1991)**	**Gray and Tall (1994, 2001, 2002, 2007); Tall (2000, 2008): Learning mathematics proceptually**
			Objects: The term object refers to a mental or physical object that includes mathematical objects (e.g., functions)	**Base objects** (known objects)
Unistructural **Multistructural**	**Visually Moderated Sequence** of physical actions: A visual cue V_1 prompts, or facilitates a procedure P_1; the execution of which produces a new visual cue V_2, which prompts a procedure P_2, *et cetera.*		**Action:** Perform physical actions on objects like moving 'up and down' a number line.	**Procedure**: DO routine mathematics accurately.
Relational	**Process**: An integrated sequence so that every V_i and corresponding P_i are well represented in memory, and can be retrieved without requiring external visual inputs.	**Interiorization**: A process performed on already familiar objects.	**Process**: An interiorized action is a process. Interiorization permits the student to be conscious of an action; to reflect on it, and to combine it with other actions. Interiorizing actions is one way of constructing processes. Another way is to work with existing processes to form new ones.	**Process**: The compression of the procedure into a meaningful process so that the student's focus shifts from the individual steps of the procedure to the cause and effect of the procedure.
Unistructural (old cycle→start of a new cycle)	**Process→Entity:** The sequence $V_1 \leftrightarrow P_1 \leftrightarrow V_2 \leftrightarrow P_2$... is 'welded' together. Consequently, $V_1 \leftrightarrow P_1 \leftrightarrow V_2 \leftrightarrow P_2$... can be retrieved from memory as a single entity.	**Condensation (Reification)**: The idea of turning this process into an autonomous entity should emerge. The ability to see this new entity as an integrated, object-like whole is acquired.	**Process→Object (Encapsulation)**: In addition to using processes to construct new processes, it is possible to reflect on a process and convert it into a mental object. For example when composing functions, the learner needs to alternate between thinking about the same mathematical entity as a process, and as an object.	**Process→Procept**: THINK about mathematics symbolically, especially with respect to elementary procepts (an elementary procept has a single symbol, say 3+4).

mathematical learning of time and space to the embodied, or enacted activities that are Recognizing, Building-with, Constructing, Consolidating, and **Extending**.is Mathematics. As described in **Appendix C**, Van Hiele's Five Levels of Geometric Thought can be related to the **extended** epistemic action model of mathematical learning that is RBC−CE. Given that Being-there is fundamentally a locus of physical actions that are communicative in time and space, it seems appropriate to juxtapose sequentially the mathematical learning of time and space to the embodied, or enacted activities that are Recognizing, Building-with, Constructing, Consolidating, and **Extending**.

Model Limitations

Although mathematical problem solving is a core feature of Being-mathematical, Being-mathematical is not reducible to problem solving. For example sense perception, primitive knowing that involves phenomenological primitives, rote learning and memorization, embodied cognition, as well as drill and intelligent practice *via* procedural variation do not necessarily involve problem solving directly. Nevertheless, powerful mathematical learning is culturally and historically situated in the implementation of universal principles that inform Being-mathematical, where Being-mathematical involves backwards and forwards movements between different steps or stages of mathematical development and problem solving.

In this regard the problem solving and mathematical learning models that are discussed in this chapter are a rich, and empirically-based source for understanding the educational processes that are pertinent to the meaningful teaching and learning of mathematics globally. However, the models all share two fundamental weaknesses. First, none of the models represent the learning of mathematics as a system of conscious and non-conscious knowing; nor do any of the models embrace the creative process (in its varying forms) that was espoused by Wallas (1926), Mayer (1992), Cropley (2001), Cropley and Cropley (2008), and Aldous (2012, 2013). Although Curtis (2010) noted that the Gestalt model of problem solving proceeded through the four phases of Preparation, Incubation, Illumination, and Verification, he considered the Illumination phase to be

> problematic in attempting to build an instructional program or an assessment tool on the Gestalt model. Illumination is regarded as being internal to the problem solver and may be considered to occur at a subconscious level and to be unavailable to intervention or assessment, and that limits the value of the Gestalt model. (p. 109)

Consequently, but not justifiably, Curtis (2010) chose to circumvent the creative process, as well as intuition and aesthetic influences — both of which are characteristics of expert (mathematical) problem solving (Atkinson & Claxton, 2000; Silver & Metzger, 1989; Sinclair, 2004)). However, if the teaching of mathematical problem solving does not engage with intuition and the non-conscious dimension of Being-mathematical, then the powerful learning of mathematics in mass education is unrealistic, because the **power** to understand mathematics conceptually and creatively is dependent upon the incubation or self-organization of ideas; their intuitive illumination in consciousness, and subsequent compression in long-term memory as deep or hierarchical intuitions that are in effect well-organized concept images (Bingolbali & Monaghan, 2008; Semadeni, 2008).

The second limitation of the models discussed in this chapter is the lack of explanation on **how** to interrelate the different model steps or stages. This is an area that requires substantial research in mathematics education. But dialogue that engages the sensory modalities (visual, spatial, auditory, olfactory, gustatory, tactile, and motoric) through the figures of speech that are metaphor, analogy, and metonymy is crucial for the growth and development of abstract mathematical thought because of the situatedness, or 'concreteness' of Being-there (Lakoff & Núñez, 2000; McNamara, 1994; Schwartz & Heiser, 2006). Put succinctly, the aspects of dialogue that involve figures of speech warrant consideration.

Figures of speech. A metaphor was understood by Johnson (1987) to be a one-to-one and onto mapping from a source domain to a target domain. In 'Being-metaphorical' therefore, a powerful mathematical learner can use isomorphic language structures to expand an already existing source domain into an emergent target domain, where the source domain and the target domain are represented psychologically by different *I*-positions. Thus metaphors can be applied creatively to develop an abstract or disparate understanding from a concrete situation, and vice versa. In analogical reasoning however, only structural relations that operate in the source domain are mapped to corresponding objects in the target domain. Consequently, in the case of analogy (in contrast to metaphor) the particular attributes of the objects in the source domain are not mapped to those objects in the target domain. This means that the different growth steps in mathematical understanding — through the use of analogy — can be interrelated with a more flexible narrative structure than is the case with metaphor.

Nevertheless, both metaphor and analogy can be used as rich languaging instruments "for expressing shortly, perspicuously, and suggestively, the exceedingly complicated relations in which abstract things stand to one another" (Jourdain, 1956, p. 31). However, metaphors and analogies are themselves relationships. In order to simplify 'complicated relations', the figure of speech that is metonymy may be employed to name the target domain of the metaphor (or analogy) with the name of the source domain, or vice versa (Berger, 1995; Presmeg, 1992, 1997b). By naming in this manner, the metonymical outcome of the metaphor or the analogy might enable the learner to structure or understand the concrete−abstract (source domain ↔ target domain) as a complex yet simplified singularity. This particular form of metonymy is synecdoche because the whole is grasped constituently (English, 1997, 2004). In mathematics synecdoche underpins all mathematical symbolism (Presmeg, 1997b). For example, the variable x represents each number on the real number line but is also mapped to the number line as a whole. In this case the mapping between any number and the number line is the 'mathematical metaphor', and the synecdoche is x because any number and the number line share a common label.

Logical thinking. In addition to the creative use of figures of speech, the various stages of mathematical understanding and problem solving need to be developed and linked analytically through different kinds of deductive thinking. Fiske (1893) maintained that “the ability to imagine relations is one of the most indispensable conditions of all precise thinking” (as cited in Moritz, 1914, p. 31).

In particular, Piagetian reasoning is necessary to facilitate the interplay between reality and possibility in a hypothetico–deductive manner (Flavell, 1963). However, even though deductive thinking is crucial for the verification and expansion of intuitive insights, it is largely absent from (high school) mathematics classrooms, especially those that are limited to the simplistic theory of constructivism (Keeves, 2002). Nevertheless, although logical “reasoning can only give us immediately evident truths borrowed from direct intuition,” (Poincaré, 1952b, p. 2) the ‘if…then…style’ of deduction is perhaps the single most important long term goal in the teaching and learning of mathematics (Peard, 1984; Piaget, 1973). It is therefore imperative that all mathematics students be trained to argue logically on a basis of propositional arguments that support a conclusion (Pendlebury, 1997; Selden & Selden, 1995). Deductive reasoning is a unique form of thoughtful knowing, because its correctness does not depend on experiment but the rigour of the deductive process as derived from the given information (Ayalon & Even, 2008). Consequently, it is apparent intuitively that “only mathematical deduction allows us to leave a proof completely aside and replace it by its conclusion” (Hadamard, 1945, p. 99).

The building blocks of deductive logic according to Serra (1997) include *modus ponens* (if P implies Q and P is correct then logically Q is correct), *modus tollens* (if P implies Q and Q is not correct then logically P is not correct), the law of the contrapostive (if P implies Q is a correct statement then not Q implies not P is also a correct statement), and the law of syllogism (if P implies Q and Q implies R then P implies R). The rules of logic exist to unfold correctly, or verify the concrete−abstract, or to answer the essential question of whether the conclusion logically coheres with the initial ideas or premises (Dominowski & Bourne, 1994).

In other words powerful mathematical learners are able to interrelate System I and System II thinking. It is through System I that a 'feel-good' or intuitive narrative is developed, but System II thinking is needed to analyse and enhance the narrative logically. High school students in particular can readily learn to think logically by proving simple logic proofs, or by writing computer programs in C++ that generate numerical sequences based on different mathematical patterns.

Moreover, students can learn to think in a hypothetico–deductive manner through individual or group conjecture, and then attempt to prove or negate the statement of conjecture by providing a proof of existence, counter example, or the development of a logical process that leads to a direct, indirect (*reductio ad absurdum*), or a conditional proof (Mercer, 1972; Peard, 1984; Serra, 1997; Zhou & Bao, 2009). Therefore the powerful learning of mathematics means being able to use symbolic logic rigorously to identify those mathematical inferences which are sensible with the laws of correct reasoning (Voyat, 1982, p. 149; also see Wertheimer, 1961). Consequently, logic and Euclidean geometry proofs can form the basis for classroom dialogues and zones of promoted activity that start

> with an empirically based conjecture [that leads to an] attempt to construct a proof, discover cases that are inconsistent with the conjecture, modify the conjecture to exclude the anomalous cases, and proceed through further cycles of proof, refutation, and theory revision until [the students] arrive finally at a provable theorem that withstands criticism. (Bereiter & Scardamalia, 2006, p. 701; also see Lakatos, 1976)

However, deduction is not the only form of logical thinking that is necessary for a classroom community of inquirers to engage in meaningful problem solving and proof construction. The processes of **deduction**, **induction**, and **abduction** are complementary learning strategies that can fuel student thought and the logical construction of ideas. Abduction might initiate the processes of induction and deduction toward the discovery or invention of mathematical coherences (Meyer, 2010). By definition, abduction proceeds concretely and abstractly from the basis that

> the surprising fact, C, is observed;
> But if A were true, C would be a matter of course,
> Hence, there is reason to suspect that A is true.
> (C. S. Peirce as cited in Meyer, 2010, p. 189)

The individual student or group mind then attempts to construct or conceptualize a generalized rule or claim by considering numerous particular cases. If successful, the conjecture that is A is said to be evidenced by induction, where conjecture is defined as a triplet comprising "a statement, an argumentation and a system of conceptions" (Pedemonte, 2007, p. 28). Consequently, the generalization that is A can be used to deduce or infer mathematical facts other than C. However, in mathematics the only 'certain' knowledge is that which has been deduced logically (with the aid of axioms and postulates). Abduction and induction cannot assure mathematical certainty. And as a result conjectures take on the status of a theorem only if they constitute the product of an ending chain of deductions (Meyer, 2010).

Inductive and deductive learning pathways. Epistemologically, inductive reasoning has equal status to deductive reasoning in the sense that the certainty of logical and mathematical 'truths' is dependent upon "our experiential knowledge of the veracity of our understanding" (Couvalis, 2004, p. 32; also see Dewey, 1929b). Nonetheless for practical purposes, **Figure 10·8** depicts learning pathways that involve inductive and deductive processes. It is noteworthy that inductive reasoning might be challenging for some students because of the demanding nature of selective coding and comparison processes, both of which involve distinguishing between relevant and irrelevant information (Bisanz, Bisanz, & Korpan, 1994).

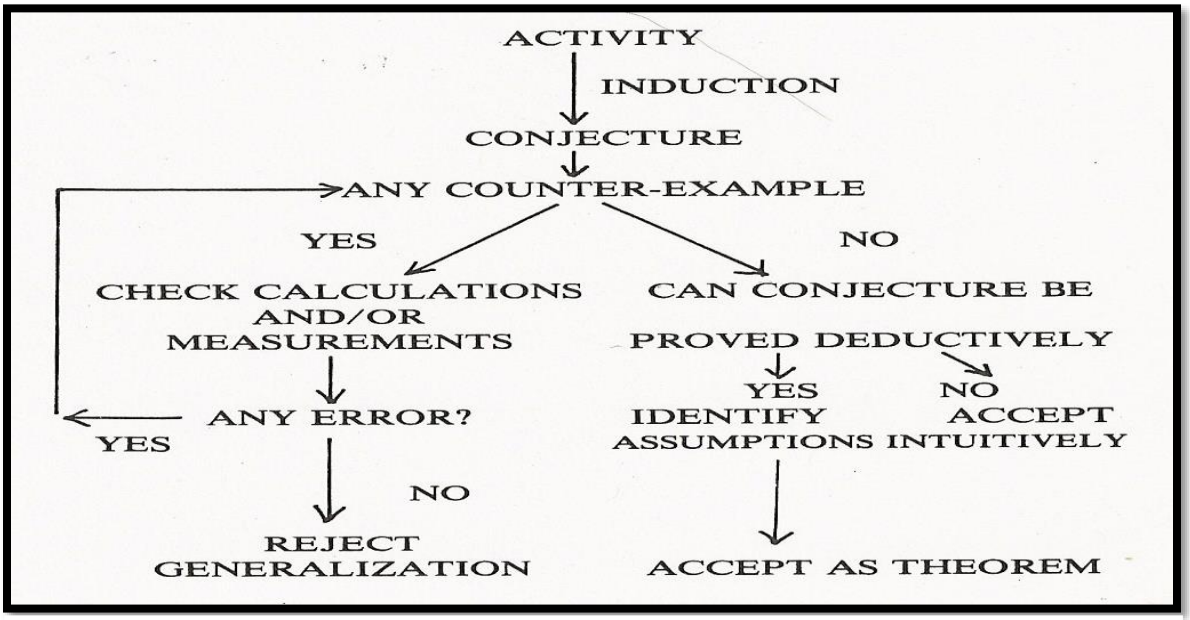


Figure 10·8. The flow chart describes possible learning pathways that include the processes of induction and deduction (Peard, 1984, p. 13).

Summary insights: Being-mathematical

Powerful mathematical learners in Being-mathematical move backwards and forwards between the different stages of mathematical understanding and problem solving in order to engage in loci of situated and abstract mathematical activity. These embodied mathematical movements form part of the creative process and are enhanced and facilitated through language structures and logic. The ultimate purpose of such activity is the hierarchical development of mental attitudes with structural capacities, or intuitive objectified processes that have been compressed into complex schemata feedback systems by Being-dialogical. In Being-mathematical the dialogical learning of mathematics is likely to be

> amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurston, 1990, p. 847)

In this regard the different models described in this chapter are likely to be very useful towards the development of powerful mathematical learning in mass education. In terms of Being-ethical however, none of the models have been validated stochastically in multiple and diverse teaching and learning situations. The following example demonstrates why it is essential for the structure of **every** educational model to be tested in a quasi-objective and statistically significant manner.

For example, Bloom's (1956) Taxonomy of Cognitive Objectives was underpinned by the Eight-Year Study (Aikin, 1942), as well as substantial discussions that took place between dozens of educators during the period 1949−1953. Consequently, a linear and hierarchical structure of cognitive objectives or categories was proposed, namely, Knowledge, Comprehension, Application, Analysis, Synthesis, and Evaluation. Since its publication however, a few attempts have been made to validate the structure of Bloom's (1956) Taxonomy with mixed results. Madaus, Woods, and Nuttal (1973) in particular, hypothesized a causal model using principal components to identify a factor of general ability, and the use of regression analysis to examine the links between the taxonomic levels. The three lowest levels were shown to be linear and hierarchical, but the remainder of the hierarchy involved a bifurcation into two groups, namely, Analysis in the one group, and Synthesis and Evaluation in the other group.

Therefore the finding by Madaus, Woods, and Nuttal (1973) serves as a poignant example as to why any 'educational structure', or model that pertains to powerful mathematical learning needs to be tested empirically and thoroughly before it can be accepted with confidence as a part of World 3. In other words a validated 'knowledge structure' for the powerful learning of mathematics implies that the hierarchical structure of the grouping, or the hypothetical framework of categories in relation to World 1 has been examined rigorously. In this regard there appears to be two main strategies of analysis that can be employed to test the operation of such ideas and relationships, namely, (a) Rasch measurement using the Saltus model (M. Wilson, 1998, 2004), and (b) structural equation modelling using Mplus (Hox & Roberts, 2011; Muthén & Muthén, 2012).

Chapter Eleven

Epilogue

Research Question One: Who are powerful learners of mathematics?

Research Question Two: How can powerful mathematical learning be realized through an ontology of Being that is dialogical?

A present history of Mathematics stretches back thousands of years in time and space. Before Stonehenge for example, Smith (2014) has argued that the architects, engineers and labourers of the Orkney Islands (c. 3200 BC) developed vision and technology millennia ahead of their time. Traders and pilgrims from far afield visited the large and sophisticated stone structures that were the cultural and spiritual Ring of Brodgar, the Ness of Brodgar, the Stones of Stenness, and Maes Howe. In so being the complexity of the religious sites likely inspired different ideas and interrelationships for the growth and development of Being, and Being-mathematical in other parts of the world as well as for future generations. However, or consequently, since antiquity mathematicians have diverged

> on how best to do mathematics, on what methods to use for attacking problems and establishing results. Some advocate formal, rigorous proofs, others intuitive, heuristic ones (and some do not see the difference between the two). Adherents of the synthetic method battle supporters of the analytic method. Rationalists confront empiricists and formalists oppose intuitionists (to use current terms rather loosely). Of course the tensions between these groups have, in general, been healthy for mathematics (though, perhaps, less so for the protagonists). (Kleiner & Movshovitz−Hadar, 1990, p. 28)

In contrast, powerful mathematical learners choose to develop an intentionality of consciousness that is willing to embrace all possible modalities of Being-mathematical. Powerful mathematical learners are not willing to be limited, or restricted to a narrow or embodied self, but rather recognize that Being-mathematical for the Conceptual Age requires patient engagement with the real world and the human imagination (Stewart, 2012). That is in terms of multiple psychological; scientific and technological, as well as social systems that interrelate ecologically, which implies

> an ecological metaphor for the study of mathematical activity. In ecological terms, the biosphere is comprised of interconnected and interrelated ecosystems. I argue that, analogously, there are nested and interrelated mathematical activity systems and structures in which “mathematical sense-making” plays the role of “health” in ecosystems. (Schoenfeld, 2013)

However, Being-mathematical might not be a system in itself because it is so complex in the dimensions of cognitive and non-cognitive knowing and Being-able to say and empathize more than you think you know (Aldous, 2014; Wertsch & Kazak, 2011). Notably, as was demonstrated by Russell, the set of all sets in mathematics does not exist because it is too large to be underpinned by a finite number of set theoretic axioms (Enderton, 1977). In metaphorical terms therefore, it is probable that Being-mathematical is not reducible to the systems view of Meadows and Wright (2008) as described on pp. 179–180 of this study, but to Be-mathematical in the sense of powerful learning is more comparable to ecosystems that do not “settle down to some kind of static balance of nature: instead they wander around on strange attractors, usually looking fairly similar, but always changing” (Stewart, 2012, p. 307).

From a Husserlian phenomenological perspective however, it is not paramount to understand powerful mathematical learners ‘as a holistic system’, but what is crucial is to grasp the whole that is Being-mathematical in terms of an essential basis that is a system, and which can be used to generate, and increasingly so, powerful mathematical learning. The phenomenological principle is that ‘Being-historical’ through the activity of *Da-Sein*, or Being-there grasps the whole of Being-mathematical in terms of the essences of what it means, or can mean potentially to Be-mathematical (Heidegger, 1967; Husserl, 1927, 1970, 2002). It was Nietzsche who wrote, “Enormous *self-reflection!* To become conscious not as an individual but as mankind. Let us reflect, let us think back: let us go all the small and the great ways!” (as cited in Heidegger, 1967, p. 43). A heightened conscious awareness of Being-mathematical however, is unlikely if cognitive reasoning does not involve affective and environmental relationships (Aldous, 2005, 2006, 2007, 2014).

Powerful Mathematical Learning

The word **powerful** in ‘powerful mathematical learning’ is linked to the notion of ‘power’ in physics which is defined as the total expenditure of energy or work per unit time. In this study **powerful** refers to a particular quality and kind of learning over time that is characterized by the extent, the well-foundedness (integrity), the structure, and the complexity of the learning (Gibbons, 2012; Lawson & Askell−Williams, 2012). However, if the process of learning is not ethical, creative, and dialogical then it is unlikely to result in a generativity and representational format of Being, and Being-mathematical that will help meet the needs of a globalizing world through mass education. This includes Massive Open Online Courses like those offered in Australia by Melbourne University, Graduate School of Education. For example, the **Assessment and Teaching of 21st Century Skills** is available through the international online platform Coursera (McFarlane, 2014).

Nevertheless, an “elite standard for everyone” (Resnick, 2010, p. 184) is highly improbable in (mathematics) education unless change is implemented systemically. **Figure 11·1** depicts a triadic interaction of Human Capital (HC), Social Capital (SC), and Instructional Tools and Routines (ITR) that is considered fundamental for the purpose of fostering mathematics education reform. In response to Research Question Two, the major purpose of the HC−SC−ITR triangle is to understand and implement system reform, or the “growing professionalisation of reform — self-conscious, deliberate attempts to use the growing body of change knowledge to continuously improve whole systems” (Fullan, 2009, p. 112; also see Mourshed, Chijioke, & Barber, 2010).

HC refers in particular to the beliefs, attitudes and values (McLeod, 1992; Bishop, Seah, & Chin, 2003) held by the teacher, especially with respect to his or her conceptual understanding of mathematics, pedagogical knowledge, and instructional leadership. The interaction between HC and SC is mediated by ITR for the purpose of reducing failure rates, and enhancing the quality of the learning community, which includes the idea of 'systems engineers' of mathematics education (Lesh, 2006; O'Shea, 2006; Resnick, 2010).

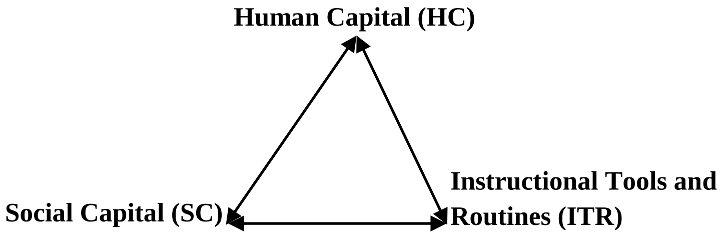


Figure 11·1. Policy triangle to influence policy design in diverse educational settings (adapted from Resnick, 2010, p. 190)

However, the quality of the learning community is also dependent on the socio-cultural static−dynamic of the school or organization, because an institution "vastly affects how a teacher can teach and how a student can learn" (Sousa & Pilecki, 2013, p. 92). Essentially therefore, powerful mathematical learning is a 'philosophical strategy' designed to facilitate systemic change in communities-of-practice (e.g., Star, 1996). In Being-intelligent for a Common Good, Sternberg's (2003a, 2003b, 2009a, 2009b) Balance Theory of Wisdom suggests that mathematics education reform in teaching and learning communities needs to occur in terms of holistic change (Westoby & Dowling, 2013), that is, STEM change which includes the arts, or a social system that is **Science−Technology−Engineering−Arts−Mathematics** (STEAM), which for many educators

> is a 'call to arms' for an overhaul of how we train teachers and administrators, how we inform our politicians, and — the biggest challenge of all — how we significantly increase parental involvement. In many ways, realigning the arts with the sciences puts trust back in the teachers and their capabilities and instincts and makes for a more exciting, creative, and successful classroom. (Sousa & Pilecki, 2013, p. 242)

In particular, STEAM change means that powerful mathematical learners need to keep pace with the dynamism that is 'human mathematics', especially in relation to the basic ideas of 'chaos and complexity', which means developing an essential understanding of infinity. In this regard, although long and convoluted, the hi**story** of mathematics is ongoing (Stewart, 2012).

Therefore mathematics curricula that are relevant to a globalizing world must include content that relates to the notion of a complex system, whose parts interconnect in ways that are linear and nonlinear — resulting in a 'collectivity' of self-organizing behaviour that implies the spontaneous formation or emergence of temporal, spatial, and functional mathematical structures (Meyers, 2011). In the modelling of an ergodic system for example, the initial state is independent of the limiting distribution of the various states of the system, and thus in order to model such a system in mass mathematics education, 'visual and instrumental' technologies like MATLAB would be a very useful tool for the powerful mathematics teacher and his or her students (Buzzi, 2011; Lynch, 2014).

Teacher Training for a Major Reform in Mathematics Education

Ideally, this epilogue heralds a major reform of mathematics education in Australian schools and futher afield. History has demonstrated however, that it should not be assumed that teachers will be motivated or capable of engaging with a new reform in mathematics education. For example with respect to the implementation of New Math, Keeves (1965) reported that in the United States professional mathematicians initiated radical changes in the content of both primary and high school courses without careful attention being paid to the research findings of Brownell (1944, 1945, 1964), Hartung (1964), and others, whose ideas had developed steadily over time and with specific reference to the classroom teaching and learning of mathematics. As a result, New Math in United States classrooms was often out of step with the prior learning and mathematics education psychology of teachers, students, and parents.

Across the Atlantic in England, it was recognized that a predominantly student-centred, or discovery approach to the teaching and learning of mathematics was not feasible for most students (Keeves, 1965; Thwaites, 1961). Nevertheless, important topics should be covered by all learners so that each individual was prepared to play a meaningful role in the life of his or her society, and consequently, the treatment of mathematical concepts need not be too abstract, or 'overly symbolic' for the purpose of satisfying high standards of mathematical rigour (Cockcroft, 1982; Shayer & Adey, 2002; Skemp, 1964, 1972, 1979). A lesson for Australia was that we should "never forget that it is the classroom teacher who is ultimately responsible for the effectiveness of the projected change" (Keeves, 1965, p. 16).

In terms of powerful mathematical learning therefore, it is crucial that teachers are trained in the principles of dialogue (Hermans & Hermans−Konopka, 2010) to implement a spiral curriculum through an educational process (Bruner, 1960, 1966) that is informed by Dewey's (1897, 1916) progressive ideas on a social pedagogy for democratic change; Whitehead's (1911; 1932) process−relational aims of education; Conant's (1947) ideas on the static−dynamic historicity of science; Tyler's (1949) basic principles of curriculum and instruction; Bloom's (1956) taxonomy of cognitive objectives; Ulich's (1961) wisdom for education; Carroll's (1962, 1963, 1989) model of school learning; Krathwohl, Bloom, and Masia's (1964) taxonomy of the affective domain; Carroll's (1993) meta-analytic structure of cognitive abilities; Krathwohl and Anderson's (2001) taxonomy for learning, teaching, and assessing — which is a revision of Bloom's (1956) taxonomy of educational objectives; Tomei's (2005) taxonomy for the technology domain; Gardner's (2006b) "five minds for the future," and Marzano and Kendall's (2007) systemic approach towards understanding embodied cognition and metacognition in relation to the executive functioning of an intentional and holistic Self.

Moreover, mathematics teachers need to be exposed to the principles of complexity science and Being-dialogical, because each teacher is influential as to whether a community-in-practice will be able to implement powerfully the creative and collaborative learning of mathematics in mass education (Thousand, Villa, & Nain, 1994). In this regard the powerful mathematical teacher or learner is especially dependent upon the **internal diversity** of the interpersonal *Other* for the purpose of enabling or enriching his or her uniqueness. It is the individuality of the person in relation to the interpersonal *Other* that is a key attribute of being successful in the Conceptual Age. Therefore the mathematics teacher who learns and teaches in-relation to individual differences is preparing students to be productive in a globalizing world that not only involves Three distinct and inseparable Worlds (Popper, 1978), but also

> involves more than an international and global approach to economic and political issues. Today, more is at risk than financial transactions and only the process of education, considered as a world-wide whole, can resolve the complex issues faced. (Keeves & Darmawan, 2010, p. 5)

Importantly therefore, not only ‘differences’ should be emphasized. If a learning dynamic, or ‘group mind’ is to emerge uniquely in the mind of the student then there has to be substantial overlap between the learners in terms of ideas and understandings. Fundamentally, dialogue is not possible without **commonality** or sameness between agentic bodies if a collection of ‘me’s is to transition to a collective of ‘us’ in both an interpersonal and an intrapersonal sense (Davis & Simmt, 2003). Thus the languaging of comparable experiences is necessary to facilitate a group mind through a locus of learning, or zone of proximal development that requires the teacher to maintain overall control of activity, and correctness by **decentralizing control** to the individual students and allowing them to take responsibility for their own actions. In Being-dialogical therefore, the teacher promotes a zone of promoted freedom and action that permits all students to participate relationally in Being-mathematical. Consequently, the teaching and learning situation is a rule-bound complexity that is characterized or structured by **organized randomness**, which implies **neighbour interactions** so that learners not only influence one another’s activities directly, but also by ‘stumbling across’ different perspectives, persuasions, and intiatives (Davis & Simmt, 2003).

Therefore powerful mathematical learners require not only direct and traditional instruction that involves deliberate practice (e.g., Thorndike, 1903, 1906, 1924, 1931), but also more adventurous and unstructured acceleration programs (e.g., Adey & Shayer, 1994; Shayer & Adey, 2002; Shayer & Adhami, 2007). Cognitive acceleration, or 'teaching for thinking' programs (Adey, 2006) need to involve an individual−group dialectic, which implies a relational 'inside−outside' ontology of Being, namely, the developmental psychology and genetic epistemology of Piaget (1932, 1970, 1973), and the socio-cultural thought−language dynamics of higher order functioning as proposed by Vygotksy (1978, 1986, 1991).

However, since Being is dialectical so too is Being-mathematical, but also because Being-mathematical embodies a brain which is essentially a physical symbol system (Simon, 1990) that articulates relationally with a Being-in-the-world who has a historically situated mind in society (Valsiner, 1997, 2004; Vygotsky 1978, 1997). This enables Being-mathematical to function dialectically in terms of a material or concrete basis of being situated, but the basis is subject to change through an intuitive and abstract reality that is mediated by symbols. The Austrian-born and Cambridge University philosopher Wittgenstein (1889−1951) argued that mathematics was essentially an activity with symbols, where each symbol represented a process, or was 'in itself' a process (Frascolla, 1994; Wittgenstein & Diamond, 1989). The primary purpose of such mathematical functioning, or problem solving for powerful learners is to interrelate the Three Worlds (Popper, 1978) logically, or in ways that are new to them for a Common Good (Sternberg, 2003a, 2009a).

Towards this goal the powerful learning of mathematics requires teachers to learn **how** to role model the development, construction, and emergence of what is probably the most influential dialectic in Being-able to learn mathematics powerfully, namely, the objectified entity that is the eidetic intuition of the concrete−abstract (Roth & Hwang, 2006a, 2006b). Through a complementarity of symbol processing and situated action, the mathematical entity emerges in 'opposing' directions and with different actions and operations. Situated action implies making sense of a particular concrete situation *via* the development of an abstraction that enables the learner to solve the problem logically and procedurally. The procedure is thus specific to the particular problem.

However, symbol processing means essentially modifying or adapting an already generalized pattern, or procedure to the problem situation in a meaningful way, but nonetheless always involves backwards and forwards movements between the concrete in the abstract, and the abstract in the concrete. In these terms when the problem solver grasps the mathematical entity, or concrete−abstract as an eidetic intuition creatively, the learner has necessarily understood his or her (novel) problem at the level of Wallas' (1926) creative process. As proposed by Aldous (2005, 2006, 2007) in her model of creative problem solving, there are bi-directional movements in the 'concrete and abstract' of Conscious and Non-conscious Knowing. These movements are facilitated by Verbal−Spatial imagery and Linguistic static−dynamic forms. The desirable outcome being the illumination and elaboration in *I*-consciousness of the intuitive concrete−abstract that enables the student to address the novel problem meaningfully.

Therefore situated action facilitates the emergence of the general (abstract) in the specific (concrete), whereas symbol processing implies the emergence of the specific (abstract) in the general (concrete). Nonetheless, both mathematical flows occur between the concrete and the abstract, and this betweenness of Being is evidence of the powerful learner Being-mathematical (Heidegger, 1967; Kant, 1934) in relation to the process and emergence of Being-mathematical through Wallas' (1926) creative process, which is a core feature of powerful mathematical learning (cf., Oakley, 2014). That is by Being-creative all three strata of Carroll's (1993) meta-analytic structure of cognitive abilities are engaged, namely, General Intelligence (3G) influences Broad Retrieval Ability (2R), which in turn influences the Stratum I factor, that is Originality or Creativity (FO), through Associational Fluency, Sensitivity to Problems (SP), Figural Fluency (FF) and Figural Flexibility (FX).

Bringing together the phenomenological perspectives of Husserl (1927, 1970, 2002), Heidegger (1927, 1970) and Merleau−Ponty (1962), it is the interpersonal and ethical (caring) conversation between human beings that facilitates the subjective embodiment of a literal, or interpretive intercorporeality of betweenness in the mind of the individual. And it is this essentiality of betweenness that provides the platform for Dialogical Self Theory (Hermans & Gieser, 2012) to be a bridging theory for Being-mathematical, which is constituted fundamentally in terms of the intrapersonal concrete−abstract, namely, the emergent *phainomenon* in relation to a concrete or abstract *noumenon*, depending on the direction of mathematical flow that is situated action or symbol processing respectively (Heidegger, 1967; Kant, 1934).

Therefore teaching for powerful mathematical learning is both a structured and unstructured ethical event in learning how to dialogue mathematically and creatively with *Others* and with oneself. Teaching in these terms is to understand Being-mathematical as a dialogical self that is not only embodied, but extends organismically and ecologically beyond the physical body to include essentially all relationships that are largely the choice of the learners. In so Being powerful mathematical learners develop a social formation of mind, especially a mathematical society of mind in-relation to *I−Other* mathematical minds in society (Chambers, 2014; Minsky, 1985; Hermans & Gieser, 2012).

The Testing of Ideas and Relationships: Design of a Research Study

From the perspective of a process−relationist (Whitehead, 1943, 1953) and basic experimentalist philosophy (Mayo, 1996, 2010), or the pragmatism of Dewey (1929b) and Peirce (1940), relational ideas need to be tested or applied empirically if they are to have validity across the Three Worlds. From an epistemic point of view, which includes those virtues that underpin empirical science (Lycan, 1988; Quine & Ullian, 1970), if a new philosophy for mathematics education is to be presented with confidence and integrity to teachers and educators, then it needs to be supported with substantial coherence that has been analysed and systematised in relation to diverse teachers and students; classrooms and schools.

However, there is a significant difference between ideas being **verified** and ideas being **validated**. Cropley and Cropley (2008) expanded Wallas'(1926) creativity process not only to include verification but also validation. Verification implies that a knowledgeable and relevant authority has placed his or her 'stamp of approval' on the quality of the creative product. Thus verification is essentially a scholarly judgement that is based on empirical evidence and is therefore a necessary part of the creative process. Validation on the other hand, requires that ideas are examined in a rigorous manner where the emphasis is not on the person who is examining, but on the method of examination so that which is examined can be adopted at the level of general ideas and relationships, or principles. This view is in agreement with 'modern science' which is underpinned by Kant's epistemology "that in any particular doctrine of nature only so much *genuine* science can be found as there is mathematics to be found in it," (as cited in Heidegger, 1967, p. 68) or as Whitehead (1962) stated, "through and through the world is infected with quantity. To talk sense is to talk in quantities" (p. 11). In the same vein, influenced by the pragmatic thinking of Peirce, James, and Dewey, the philosopher and sociologist Kaplan (1964) contended that

> mathematical advances seem to me to hold our great promise for behavioral science, especially in making possible exact treatment of matters so long thought to be 'intrinsically' incapable of it. I have no sympathy with principled, purposeful vagueness, even where it is not a cover for loose thinking. (p. 409)

However, although mathematics continues to be the foundation of science and quantitative educational inquiry, it was Popper who extended the logical positivism of the Vienna Circle (c. 1908−1933) when he argued that "no matter how often a theory is tested we can never say 'this theory is right' because all theories are based on induction" (Higgs & Smith, 1997, p.

115). Consequently, the stochastic modelling and testing of complex ideas through advanced statistics has become standard practice for scientists the world over (Lindsay, 1995; Oliveira, Temido, Henriques, & Vichi, 2012). In particular however, it was the development of matrix algebra (e.g., the mathematician Taussky–Todd used matrices during World War II to analyse vibrations on aircrafts) that made statistical methods and computing tractable mathematically and applicable for research workers (Bradley & Meek, 1986; Fisher, 1958, 1966; Meyer, 1990).

The systemic modelling basis, or philosophical structure that is thought to generate powerful mathematical learning is depicted in **Figure 11·2**. The basis reflects the philosophy of powerful mathematical learning, and essentially 'answers' Research Question One: Powerful mathematical learners are influenced by ethical values and the principles of dialogue to create complementarities of symbol processing and situated action for the purpose of successful mathematical performance that includes novel problem solving. Therefore each learner not only 'works mathematically' (epistemologically), but also learns to 'see mathematically' (ontologically) through an increasing and ethical sense of his or her *Da-Sein* that is Being-mathematical.

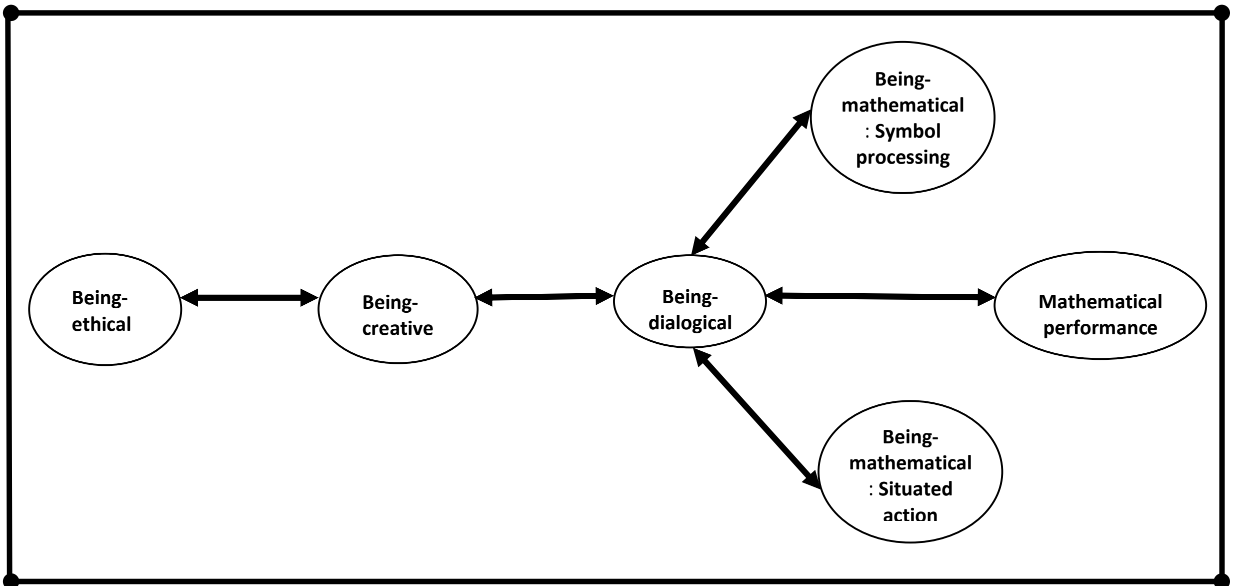


Figure 11·2. A structural basis for the powerful learning of mathematics.

However, through an appropriately designed research study the structural relationships of the basis need to be examined systematically and empirically for statistical significance towards the goal of verifying and validating the model globally. In this regard the integrity of the modelling is to be infused with the principles of Rasch measurement and structural equation modelling.

Data collection. The world is globalizing in often very different local situations. Therefore if the study is to be authentic and generalizable it is pertinent to collect data from diverse sources with representation from appropriate, or 'forward-thinking' learning environments. In addition to research possibilities in Penang, Malaysia as well as Indonesia (Ministry of Education and Culture), the following institutions were approached and have indicated a willingness to participate in a secondary school mathematics education research project:

(1) Government and independent schools in South Australia;
(2) an international school in Hong Kong;
(3) independent schools in Mumbai, India and Johannesburg, South Africa; and
(4) the Ministry of Education in Singapore has agreed in principle to support the research effort provided that at each participant school a maximum of 10 per cent of the target population are involved.

The teaching and learning of mathematics is complex and highly interconnected. In particular therefore, because students learn in classrooms and classrooms are located in schools, it is necessary to model and examine for statistical significance, the systemic basis for the powerful learning of mathematics in terms of a multilevel and multivariable structure (Heck & Thomas, 2009; Rabe‒Hesketh & Skrondal, 2008). The diagram in **Figure 11·3** is a graphic representation of a multi-level modelling framework that includes mediating and bi-directional effects as well as cross-level moderation.

However, three level **structural equation** modelling requires a large number of degrees of freedom to ensure the statistical significance of the effects within and between the three

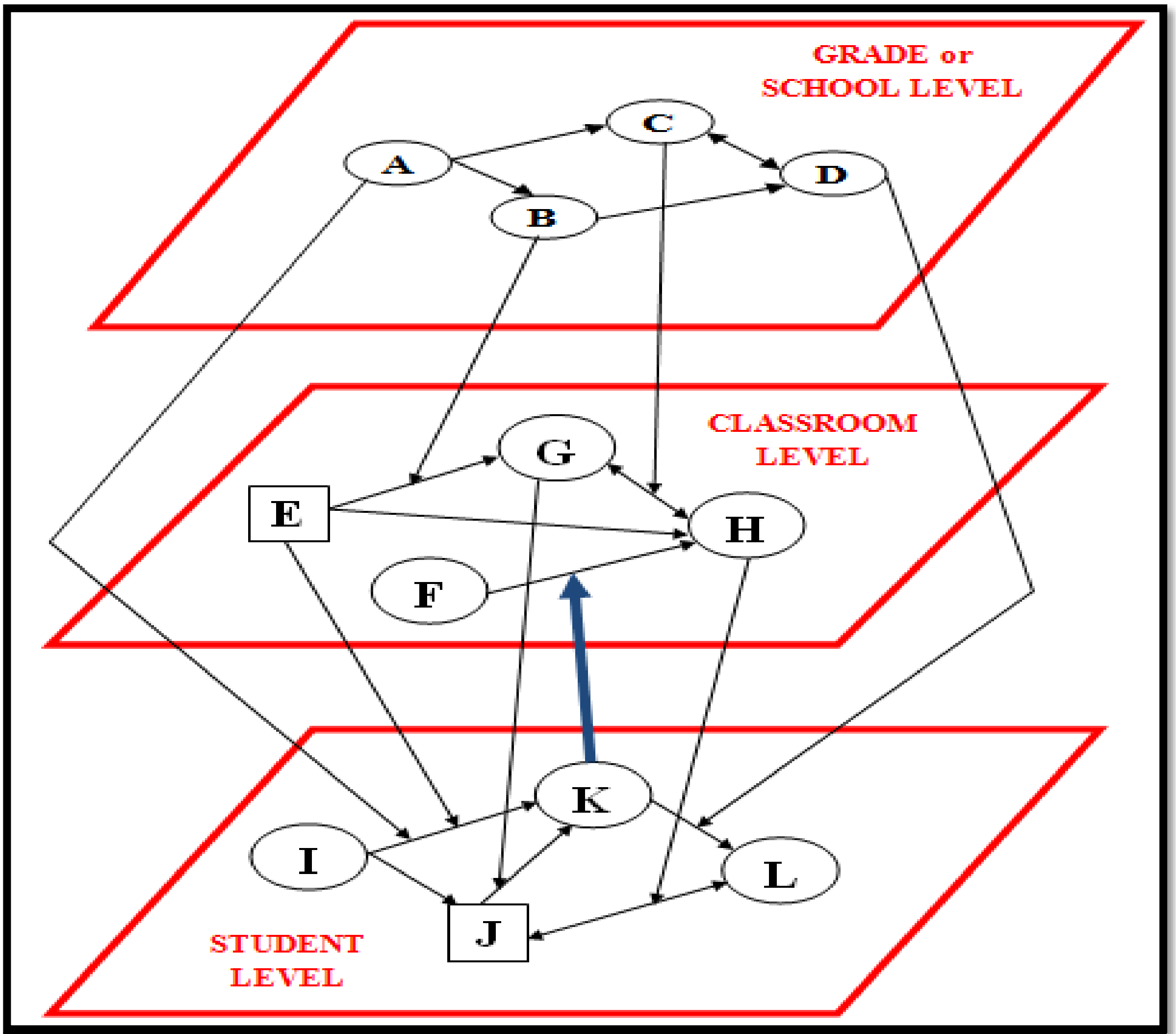


Figure 11·3 represents a three level modelling structure for the powerful learning of mathematics. The **latent variables** are depicted by oval shapes, and the **manifest variates**, which are observable or scaled indicators of the raw data, are depicted by square or rectangular shapes. A latent variable cannot be measured directly but is constructed essentially through the factor analysis of preferably three or more Rasch scaled item response groupings, each of which is characterized by a different unidimensional latent trait. The **blue arrow** emphasizes an influential effect (particularly in relation to individual differences) that is cross-level in an upward direction. However, Mplus (Version 7.3) does not appear to have the capability to model bi-directional cross-level interactions.

levels. As a guideline approximately 27,000 students need to participate in the study, or 30 students from each of 30 classes in each of 30 schools (Hox, 2010; Hox & Roberts, 2011). This is a large scale study. If necessary, the size of the study can possibly be reduced without loss of statistical significance by using bootstrapping techniques innovatively (Chernick, 1999; Davison & Hinkley, 1997; Osborne, 2008). As a general principle however, bootstrapping works well provided that the sampling distribution is robust when small changes are made to the process that generates the resampled data. In particular for non-parametric bootstrapping, this means that the removal of a few observed data points should have a negligible effect on the overall data structure of the manifest variate, and consequently, also on the construction of the latent variable with which the manifest variate is associated. In this regard it is important to be aware of outliers.

Data collection instruments. Surveys allow a researcher to collect a large amount of data efficiently, especially if the data are collected online through an advanced software tool like SurveyGizmo (Source: http://www.surveygizmo.com). However, survey questions need to be constructed carefully because "cognitive research into survey methodology starts from the premise that responding to survey questions involves many, frequently iterative, steps of complex information processing" (Lietz, 2010, p. 249). Consequently, in working with the Australian Council for Educational Research and PISA, Lietz (2010) summarized the literature on questionnaire design as follows:

1. Questions should be clear and concise. For the most part, survey items should be a maximum of 14 to 16 words in length.
2. It is preferable that the focus of the questionnaire should be on current attitudes and recent behaviours and learning experiences.
3. General questions should precede specific questions.
4. Vague quantifiers like 'usually' are to be avoided.
5. The most appropriate Likert-type response scale length lies between five and eight categories.
6. If a middle option is included there is likely to be a slight improvement in the reliability and the validity of a response scale.
7. A unipolar numerical scale with matching verbal labels as anchors is preferable.
8. 'Extremely' and 'not at all' are probably the most well understood verbal intensifiers.
9. All numeric labels ought to be displayed on each survey item.
10. 'Negative' statements and 'positive' statements involve different psychological processes, and therefore grouping the two kinds of statement is likely to confound student responses with respect to a particular view or attitude scale.

Table 11·1 refers to, and **Appendix D** specifies survey items and assessment problems that relate specifically to the latent variables depicted in Figure 11·2, namely, Being-ethical, Being-creative, Being-dialogical, Being-mathematical, and Mathematical performance or 'being-mathematical'. It is noted that not all items are applicable to all students, because different students are at different levels of cognitive, metacognitive, and non-cognitive development. Therefore it is important to pilot each item diligently.

In particular, **Appendix D (Part 5)** gives examples of mathematics assessment questions that can be used to reflect the latent variable 'Mathematical performance', but whether a question is meaningful or not is dependent on the prior learning and mathematical experiences of the individual who is attempting to answer the question. In this regard therefore, any mathematics assessment should offer students a range of questions to choose from, and moreover, to facilitate insightful and reflective responses individual respondents need to be given sufficient time to respond adequately. The questions included in Appendix D (Part 5) involve numeracy; algebra and geometry; trigonometry; the infinite and the undefined; 'real-world'story-type problems; proof; modelling, and problem construction and solution.

However, the teaching and learning of mathematics is complex as a result of many different interrelating factors. Therefore the structure of the systemic basis for powerful mathematical learning needs to be examined in relation with other factors if significant relationships between the basis factors are to be validated as non-spurious. **Table 11·2** lists, and **Appendix E** delineates item response variates that are considered influential in the teaching and learning of mathematics in classrooms and schools. Feedback on many of the survey items has been received from secondary school students in Adelaide with Australian and international backgrounds in mathematics education. Moreover, the attitude and view scales of Husén (1967a, 1967b), as well as Carlgren's (2013) communication, critical thinking, and problem solving course for high school students in the twenty-first century can be used to augment, modify, or enrich the various item response variates.

Data analysis. Item Response Theory articulates that each latent variable, or educational factor be developed in terms of (highly) correlated items, because this implies that the respective items are measuring the same dimension or trait of the latent variable or factor. Principal component analysis (PCA) is a simple but effective method of identifying and combining linked, or correlated items. Essentially therefore, PCA tests for the existence of an implicit latent variable by establishing linear combinations of the student response items (raw data), namely, the principal components. Those items with relatively high weightings in the same principal component are linked, or correlated. And although *p* orthogonal (independent) components can be extracted from a group of *p* items, it is most often the case that k < p components explain most of the data variability.

Table 11·1. Basis factors for the powerful learning of mathematics

Basis Factors	**Number of Available Items**
1. Being-ethical: A First Philosophy of Learning Mathematics	17 items
2A. Creativity: Generic Problem Solving Competences	6 ordered response items
2B. The Process of Creativity	8 ordered response items
3. Dialogue	21 items (inludes 4 ordered response items)
4A. Symbol Processing	7 ordered response items
4B. Situated Learning	9 ordered response items
5. Mathematics Assessment Questions	18 'short' and 'long' questions

Table 11·2. Influential factors for powerful mathematical learning

Mathematics Education Factors	**Number of Available Items**
1. Being-human	12 items
2. Gender	1 item
3. Age (date of birth)	1 item
4. Grade Level	1 item
5. Family (informed by OECD, 2003)	6 items
6. Socio-economic status (informed by OECD, 2003)	5 items
7. What is Mathematics? (informed by OECD, 2003)	16 items
8. Intentionality/Autotelic Personality	13 items
9. Learning/Instructional Principles	25 items
10. Teaching Behaviour	14 items
11. Social-emotional Intelligence/Didactical Contract	21 items
12. Time-on-task (informed by OECD, 2003)	11 items
13. Personality of Place	10 items
14. Individual Differences	12 items
15. Deliberate/Intelligent Practice	13 items
16. Embodied Cognition	13 items
17. RBC−C Epistemic Model of Learning	5 ordered response items
18. S−R−O−C Learning Model	7 ordered response items
19. Literacy	12 items
20. Technology	15 items
21. Globalization	13 items
22. Problem Solving Heuristics	14 items
23. Higher Order Thinking	13 items
24. Formal Operational Thinking	21 items
25. Visual-spatial reasoning	9 items

In particular the first component explains the majority of the variability in the data, and when a PCA is employed on data that has been standardized with respect to variance, the first principal component weightings for each item provide a measure of the contribution of each item to the overall value of the first component. Consequently, the items that have the highest (positive or negative) weightings within the first principal component are those that are scrutinized as part of the next phase of data analysis, namely, Rasch measurement. Similarly for all other principal components, but especially those whose associated eigenvalue is greater than unity, because this indicates that the principal component is explaining a significant proportion of data variability. It is nonetheless important to note that a PCA based on the sample covariance matrix and a PCA based on the sample correlation matrix differ (Meyer, 1990). If a covariance matrix is used then the items with the **largest variability** are most prominent in the first principal component and are not necessarily highly correlated.

However, in the proposed research study it is envisaged that PCA will be employed solely as an exploratory and descriptive data analytic tool. Therefore there will be no need to assume an underlying population distribution for the purpose of making inferences about the parent population. Furthermore, concerning an appropriate sample size there appears to be little consistency in the literature, but for exploratory factor analysis and principal component analysis it has been suggested that "the adequacy of the sample size might be evaluated very roughly on the following scale: 50 – very poor; 100 – poor; 200 – fair; 300 – good; 500 – very good; 1000 or more – excellent" (Comfrey & Lee, 1992, p. 217).

Nevertheless before a large scale international study is carried out, a pilot study involving at least 400 students is considered essential for the purpose of establishing the statistical strength of the survey construct. Notably, the number '400' has particular significance in sampling theory. Consider for example a random sample of n students who respond to a survey item. If the number of responses n is sufficiently large then from the Central Limit Theorem, the distribution of $\overline{X_n}$ for the item is asymptotically normal with population mean μ and variance σ^2/n (Steen, 1982). However, the population parameters are likely to be unknown and if estimates are required then a standard error of five per cent is obtained when n = 400 (i.e., $1/\sqrt{400} = 1/20 = 0.05$).

Moreover, the pilot study cohort should match the characteristics of the international cohort as closely as possible, which is not unrealistic in Australia because of the diverse ethnic and mathematical backgrounds of the student population. In 2009 for example, Australia had a "higher proportion of international students relative to the total population than any other country in the world with almost 20% of all enrolments being international students" (Kell & Vogl, 2012, p. 1).

Rasch measurement. In terms of rigour, educational measurement requires that if a manifest variate is to reflect, or inform a latent variable then the items which constitute the manifest variate need to be scaled meaningfully and probabilistically. There are numerous probability scales of measurement including the normal, logarithmic, exponential, and Poisson distribution scales. However, it is the strong measurement and mathematically tractable properties of the

> logistic model proposed by Rasch [that] warrants the use of this model, and the rejection of items or tasks that do not conform to the model in order to develop a unidimensional scale that measures an identifiable latent trait with a pattern of responses that approximates the pattern described by a Guttman scale. Moreover, the Guttman response pattern is consistent with the use of the logistic function as a model of response distribution. (Keeves & Alagumalai, 1999, p. 24).

Moreover, education is concerned with learning and development, and in these terms, change over time can only be measured on a **unidimensional** interval scale.The formula in **Figure 11·4** denotes the probability density function of the measurement principle that is the Rasch scale. It is relevant to note that the ability of the person and the item difficulty are regarded as conjoint in all analyses of responses. Consequently, the task of measurement is implied by a principle of relativity involving the learner with respect to the item. Essentially, it is this principle that has made measurement in the social and behavioural sciences possible, because it is not β_n or ∂_i that is the unit of measurement but $\beta_n - \partial_i$, which is the difference between the ability of the individual relative to the difficulty of the item.

Importantly for the empirical research study however, the simple Rasch model can be extended to include more than one parameter. In particular if items are ordered (e.g., in the creative process), then an interaction parameter τ can be incorporated into the standard Rasch model equation for the purpose of linking the different levels developmentally, where "a *developmental level* is a step in a sequence postulated as part of a theory of the progression of an individual toward maturity, broadly or narrowly defined" (M. Wilson, 1998). The probability equation in **Appendix F** specifies the Saltus model mathematically (M. Wilson, 2004). Moreover, various Rasch models for ordered response categories have been described by Andrich (2005a).

$$P_{ni} = \frac{\exp(\beta_n - \partial_i)}{1 + \exp(\beta_n - \partial_i)}$$

- β_n is an index for the underlying ability of person *n* on the attribute or trait that is being measured.
- ∂_i is an index for the underlying difficulty or facility level of the item or task *i*.
- P_{ni} is the probability of a correct, or particular response by a person *n* on a task *i*.

Figure 11·4. The single parameter Rasch model (adapted from Keeves & Alagumalai, 1999)

Notwithstanding, the basic details of how to scale items through Rasch modelling is made clear in Masters and Keeves (1999), and through numerous exemplars in Alagumalai, Curtis, and Hungi (2005). It is especially noteworthy that different sets of scale scores relating to response items can be equated, with the guideline that at least 50 per cent of items are common to both scales (Alagumalai, Curtis, & Hungi, 2005; Andrich, 2013; Christensen, Kreiner, & Mesbah, 2013). This implies that not all students need to respond to all items as part of the proposed international study, but nevertheless all students can be located on each conjoint scale without loss of statistical significance. However, the calibration of a conjoint or equated scale may require the rejection of certain items and respondents. This is a "small price to pay for strong measurement" (Keeves & Alagumalai, 1999, p. 28) which means that the scale's Item and Person characteristic curves are logistic functions that mirror one another about the (0,1) probability interval.

The structural design of data to be collected in the proposed empirical study is presented in **Figure 11·5**. The design consists of five item booklets. Each booklet comprises 180 items, 90 of which overlap with another booklet. Therefore a total of 450 items would constitute the data collection. Thus theoretically, five different interval scales relating to the same mathematics education factor can be equated sequentially in a pairwise manner across the

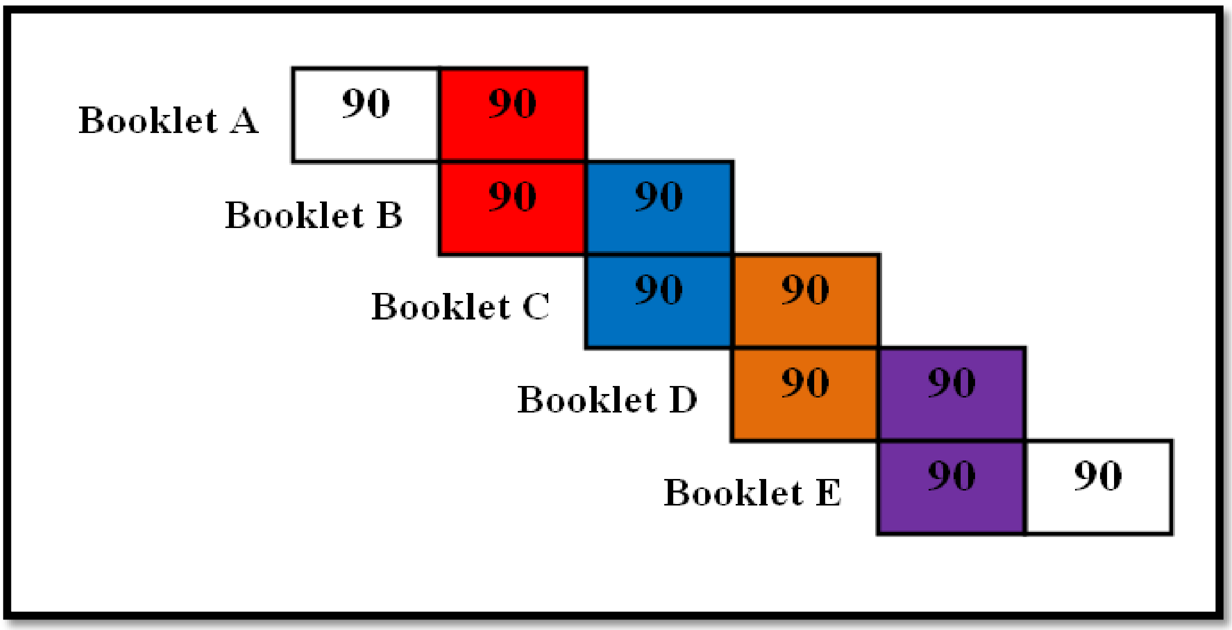


Figure 11·5. Five booklet survey design structure

successive booklets. An initial pilot study in Adelaide has indicated that students at the upper secondary school level require a maximum of 45 minutes to complete a booklet of items.

Moreover, the generalized item response software RUMM (Rasch unidimensional measurement models) engages a variation of the PAIR Method to obtain stable estimates with fewer items and students than may be necessary if other available software is used like Mplus for example (Keeves & Alagumalai, 1999; Andrich, Lyne, Sheridan, & Luo, 1997, 2003, 2010; Mesbah, 2013; Muthén & Muthén, 2012). Therefore the use of RUMM 2030 software is likely to enhance the efficiency and cost effectiveness of the proposed international research study (Sources: http://www.rasch-analysis.com/recommended-rasch-software.htm; http://www.rummlab.com.au). The main characteristics of the PAIR Method are as follows:

1. Create a paired comparison matrix (item by item comparisons).
2. Entries above the diagonal indicate count of persons who succeed on the first item and fail on the second item in pairs.
3. Fit a linear model to logarithms of cell entries using least squares or maximum likelihood estimation. (adapted from Engelhard, 2013, p. 145)

From a philosophical perspective, Rasch (1977) held the view that if a generalization (the ultimate goal of educational measurement) was to have real-world value then it had to be founded on systematic comparisons, either experimental or observational (Andrich, 2005b).

Factor analysis. Each student's fitted items on a particular Rasch scale are averaged, which is appropriate because the fitted items are all located on the same unidimensional interval scale. The averaged scaled scores for all students are then combined to form a single manifest variate. The process is repeated for at least another two item response groupings. Then the averaged scaled scores for each manifest variate are used as part of a factor analysis (FA), namely, to analyse the shared variance between the independent manifest variates for the purpose of constructing a latent variable. Ideally the overlap in variance should be close to 100 per cent, and the specific or unique variances associated with each of the manifest variates respectively, should be as small as possible in accordance with the Factor model described in **Appendix G**. It is preferable that at least three manifest variates constitute the latent variable in order to minimize the unexplained variance associated with the latent construct (Keith, 2006).

The diagrams in **Figure 11·6** indicate the essential difference between factor analysis and principal component analysis in the construction of a latent variable. If a group of manifest variates (MV1, MV2, MV3) — consisting of average scaled scores — are all linearly independent with no excessive intercorrelations, then factor analysis can be used to construct a latent variable (FA). The different traits of the latent variable are **reflected** by the predictor, or 'outward mode' variates MV1, MV2, and MV3 respectively (Pedhazur, 1997). Alternatively if a grouping of student response items (raw data) are correlated strongly, then the items that fit a Rasch interval scale can be combined using principal component analysis (PCA) to generate an 'inward mode' linear combination, which **informs** the constructed latent variable, or educational factor. However, Mplus (Version 7.3) appears only to make use of the outward mode in the construction of latent variables (Muthén & Muthén, 2012). Nevertheless, Mplus does take into account non-normality of outcomes and affords maximum likelihood estimation for all models.

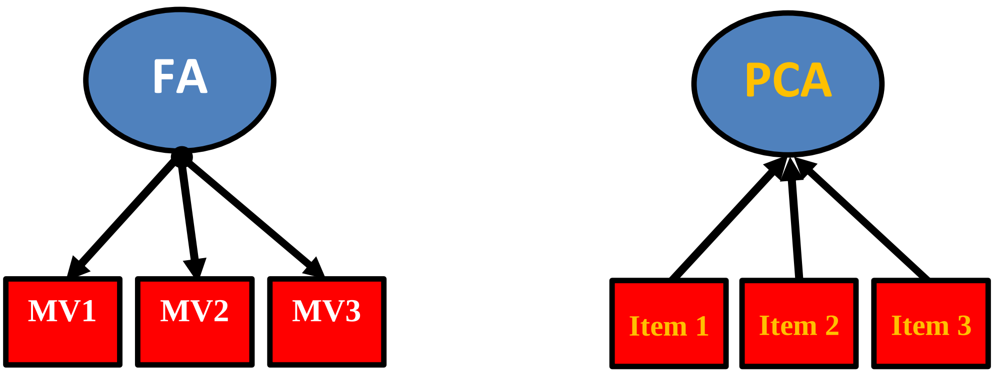


Figure 11·6. Factor analysis (FA) compared to principal component analysis (PCA) in the construction of a latent variable.

Structural equation modelling. Wold (1964, 1982), Jöreskog and Sörbom (1978), and Jöreskog and Wold (1982) pioneered structural equation modelling (SEM). It is highly probable that the development of SEM was influenced by the ideas of Nelder and Wedderburn (1972), who 'combined' statistical regression models (e.g., linear, logistic, and

Poisson) for the purpose of constructing the generalized linear model (GLM) in terms of an iteratively reweighted least squares method. In the case of SEM, multiple regression and factor analysis were incorporated into a single path-wise modelling system that minimized standard errors between the observed data and the latent variables through the partial least squares analysis of the latent variable covariance matrix. In general terms latent variables are theoretical constructs that cannot be observed or measured directly but are informed, or reflected by multiple observed or scaled indicators called manifest variates (adapted from Byrne, 2001; Falk & Miller, 1992).

A major strength of latent variable partial least squares (LVPLS) analysis is that no underlying probability distribution needs to be assumed (Sellin, 1990; Sellin & Keeves, 1997). This is advantageous for the purposes of measurement in the social sciences because data are often sampled from a population whose probability distribution is not normal or is unknown. In addition, as an exploratory data analytic tool LVPLS modelling can be constructed effectively with as few as a 100 respondents. However, LVPLS models are limited by the fact that they involve only direct or mediating variables within a single level.

Consequently, for the purpose of including cross-level effects in educational research multilevel modelling was initiated. In particular, hierarchical linear modelling (HLM) was developed by Bryk and Raudenbush (1992). HLM assumed multivariate normality and used (Bayesian) maximum likelihood estimation (MLE) for the purpose of estimating effects. In broader terms, the assumption of normality has been widespread in multivariate analysis because as described by Meyer (1990):

- **(a)** The multivariate normal distribution is easily derived from its univariate counterpart;
- **(b)** the multivariate normal distribution is completely defined by its first and second moments which implies that only the mean vector and the covariance matrix are estimated for the data analysis. In particular, the covariance matrix Σ_{pxp} is symmetric and therefore only ½p(p+3) parameters in total have to be estimated;
- **(c)** with respect to normal variables a zero correlation indicates independence; and
- **(d)** linear functions of a multivariate normal random vector are univariate normal.

Nevertheless, whether a system is modelled in terms of normality or not, there are system characteristics that are not addressed meaningfully in either LVPLS or HLM. For example, bidirectional as well as mediating and moderating effects cannot be measured within the same modelling process and structure. The latent variable modelling program Mplus (Version 7.3) has addressed these shortcomings (Muthén & Muthén, 2012). Consequently, three level within and between structural equation modelling is currently available in the social sciences. In other words Mplus offers a wide array of models, estimators (e.g., maximum likelihood estimation, weighted leasted squares, and Bayesian analysis), and algorithms that can be used as building blocks towards the creation of unique bidirectional, linear and quadratic model structures that involve direct, mediating and moderating effects (Hox, 2010; Möller, Retelsdorf, Köller, & Marsh, 2011; Muthén & Muthén, 2010, 2012). This statistical capability within a single software program is especially convenient for the purpose of verifying and examining stochastically the systemic and philosophical basis of powerful mathematical learning.

Concluding Remarks: The Past and the Future in the Present

There is concern internationally that so many students do not know how to engage with mathematics at a level that constitutes deep mathematical learning (Muir, 2014). Specifically, multitudes of teachers and their students do not know how to learn mathematics in terms of a thinking−feeling body. It is precisely this aspect of Being-mathematical that constructivism has struggled to make sense of, largely because of a constructivist metaphor which limits Being to a psychology of *I*-conscious learning and thinking. But the route to deep mathematical learning is through the body, because the body is uniquely positioned to interrelate the Three Worlds meaningfully and powerfully. In World 1 ‘the body’ relates to its physical substantiality, objects and things through bodily movement, comportments of Being, and sense perception. In World 2 an embodied mind in society can facilitate a social transformation of mind in the powerful learning sense that the embodied mind develops intentionally a dialogical society of mind. This occurs as a result of an intercorporeality of being, namely, a languaging body that bridges, or integrates the Vygoskyan gap between the interpersonal and intrapersonal psychological coordinates of the Self. That is in part by Being-creative in-relation to World 3 structures and entities that are incorporated into the socio-cultural network of relationships that characterizes the embodied and extended Self.

From a Gestalt phenomenological point of view however, powerful mathematical learning can only be experienced through a languaging body that facilitates the emergence of a dialogic centrality, namely, the eidetic intuition. It is through the eidetic intuition that Being-able is capable of mediating the 'passibility' of knowing mathematics between Self State One and Self State Two, or non-conscious and conscious knowing respectively (Aldous, 2005; Roth, 2011). It is fundamentally the emergence, the analysis, and the use of eidetic intuitions over time and in different problem solving situations, that the creative learning process is objectified as a complexity of cognition and affect with well structured mathematical capacity (Fischbein, 1987, 1999). If however, a student is not patient in the consistency of Being-mathematical through the embodiment of dialogue as a value in the Self, powerful mathematical learning is highly unlikely to become a 'settled' modality of Being in-relation to his or her network of Being-there (De Leo, 2012b; Wegerif, Boero, Andriessen, & Forman, 2009).

But in contrast, and with reference to Figure 0·1 (see p. xiii), the mind of the powerful mathematical learner empowers 'The Mind'(which is a self-actualizing and self-organizing complexity), as he or she interacts mathematically and technologically with World 1 and World 3 through the intuitive and analytical modalities of Being that are Being-ethical, Being-wise, and Being-creative. Therefore if mathematical potential is to be optimized, then Being-mathematical needs to become a literal↔interpretive and subjective↔objective entity, namely, through the static↔dynamic capabilities and potentialities of an embodied and extended self that grasps situated action and symbol processing as an organismic problem solving complementarity. A major goal of which is to expand and develop Being-there towards the rich possibilities associated with Type I, Type II, and Type III civilizations.

From an ethical point of view therefore, the purpose of validation in the proposed modelling study is not only to establish quasi-objective support for a particular interpretation of Being-mathematical, namely, powerful mathematical learning but "to find out what might be wrong with it. A proposition deserves some degree of trust only when it has survived serious attempts to falsify it" (Cronbach, 1980 as cited in Lather, 1986, p. 67; also see Popper, 1965). In this regard modelling strategies and causal structures are to be implemented for the purpose of understanding powerful mathematical learning for what it is essentially, and perhaps more importantly, what it is not in a global sense. In this regard Rasch modelling is likely to be particularly useful, because a major focus of the Rasch paradigm is to identify and examine anomalies that might be in the sample data (Andrich, Lyne, Sheridan, & Luo, 2010).

However, objectivity in the social sciences always involves subjective experience (Schutz, 1970, 1972; Wittgenstein, Anscombe, & Von Wright, 1979), at least in part because "there is no such thing as an immaculate perception" (Kosslyn & Sussman, 1995, p. 1035). Therefore no single research method can suffice in illuminating the "validity of knowledge in process" (Reason & Rowan, 1981, p. 250). Therefore it is recommended that for future research, since Being-mathematical is fundamentally dialogical and creative, "one must enlarge the conception of what the process is, moving from an exclusive focus on the individual to a systemic perspective that includes the social and cultural context in which the 'creative' person operates" (Csikszentmihalyi, 1994, p. 135). In these terms the Beingness of different communities-of-practice needs to be observed directly and repeatedly, namely, in their respective situated learning environments for the purpose of stripping "the system's complexity down to its bare essentials" (McDonald, Suleman, Williams, Howison, & Johnson, 2011, p. 2). In particular, the activity systems in mathematics classrooms, especially those that involve 'new actions on old objects' can be made visible (Gresalfi, Martin, Hand, & Greeno, 2009; Schwarz & Dreyfus, 1995) through well planned and carefully coordinated video strategies (Janik & Seidel, 2009), as well as the use of advanced network visualization, manipulation, and (non-linear) analysis software like Gephi or Pajek (Bastian, Heymann, & Jacomy, 2009; Batagelj & Mrvar, 2010; De Nooy, Mrvar, & Batagelj, 2005). Importantly, the network software is underpinned mathematically by structural graphing theory, which provides the tools (e.g., spanning trees) to 'abstract' the system, or various patterns of communication from the social complexity that is Being-there (Chartrand & Lesniak, 1996; Wilson & Beineke, 2013).

Nevertheless, all research methods that relate to powerful mathematical learning should be consistent with the ideas of Dewey, Russell, and Whitehead, in the sense that these progressive and reflective thinkers were concerned primarily with the challenge of encouraging educators and society with fresh ideas and possibilities of being — for the greater good of current and future generations (Winchester, 1985). It is in these terms that the philosophy for the powerful learning of mathematics has been expressed from a deep conviction, or sense of ethical and intuitive–analytical intent that

> the intellectual obligation upon all thoughtful persons is to seek a balance. He or she may not achieve it, for we are human and fallible. In seeking that balance, however, they are seeking the best possible decision at a particular time in particular circumstances. Such was the wisdom ascribed to Solomon. (Gibbons, 2012, p. 12)

Informed and guided by integrity therefore, it is hoped that powerful mathematical learning will herald a quiet revolution in mass mathematics education. It is approximately 50 years since the major reform that was New Math. Since that time mathematics education research has developed increasingly towards an aggregate of factors that are influential, or causal in enabling the 'whole person' to make sense of mathematics for him or herself (e.g., Begle, 1979; Ray, 2013; Schoenfeld, 2008a). During this time period humankind has emerged into an exciting but challenging era of Being — the glocalizing Conceptual Age. From an ethical and developmental perspective, this epoch of present history requires creators and empathizers to combine (disparate) knowledge from the Information Age to improve the wellbeing of societies and individuals across the globe, especially through advancing technologies.

If however, mathematics teachers are to facilitate Being-mathematical for this stage of human growth and development, which includes the extremes of mass open online courses and face-to-face tutorials, then a new psychology of *I-Other*, or Being-dialogical is essential. In large part for the purpose of combining *epistémé* and *techné* successfully through a situated *phronesis* that enables Being-wise, for the teaching and learning of mathematical complexity and dynamical systems in-relation to increasingly complex ethical, ambitious, and marginalized Selves. Instead if any reform in mathematics education is implemented in a manner which is not systemic and dialogical in the factors that are Human Capital, Social Capital, and Instructional Tools and Routines (Resnick, 2010), then it is likely that many students will continue to work on their mathematics with little hope of substantial improvement, which in the majority of instances may be a present history metaphor, or a predictor of their actual futures. In the sentiments of the English poet Coleridge:

And would you learn the spells that drowse my soul?
Work without Hope draws nectar in a sieve,
And Hope without an object cannot live.
(as cited in Wordsworth & Wordsworth, 2001, p. 809)

Nonetheless, it is hoped that a philosophy for the powerful learning of mathematics will be an ‘object’ in mass education that facilitates Being-dialogical and Being-creative, because in so Being students and teachers can develop the whole Self — not only epistemologically but also ontologically — with the result that Beings-mathematical can **see** symbol processing and situated action as a complementarity that is influenced by an ethical intent for a Common Good.

Appendices

Appendix A: A Narrative Discourse on Mathematical Problem Solving

The Middle East has a long history of mathematical problem solving. For example, consider the Rhind Papyrus (c. 1850 BC), the Moscow Papyrus (c. 1850 BC), and the Cairo Mathematical Papyrus (c. 300 BC) of Ancient Egypt. The Rhind Papyrus was a manuscript of 84 administrative and building-works problems that involved numerical operations as well as geometric series and shapes. The Moscow Papyrus included Ancient Egypt's greatest geometrical achievement, namely, the exact formula for the volume of a truncated pyramid.[1] However, it was the Cairo Mathematical Papyrus that indicated knowledge of Pythagoras' Theorem. Nonetheless, the Babylonian clay tablets (c. 3500 BC – 300 BC) demonstrated that the Mesopotamians had a superior knowledge of fractions, algorithms, and quadratic and cubic equations as compared to the Ancient Egyptians (Netz & Noel, 2007).

From a regional and historical perspective therefore, it is not surprising that Jews have a proud tradition in mathematics. In fact, approximately one-third of Fields Medals in Mathematics for Americans have been awarded to Jews, and moreover, "most reports place the average Ashkenazi Jewish IQ at two-thirds to one standard deviation above the white average. That is equal to an IQ of 110 to 115" (Nisbett, 2009, p. 172). Thus in a correlative and cultural sense, mathematical problem solving in **Israeli** elementary school projects had the following general characteristics (Arcavi & Friedlander, 2007):

1. Drill and practice was not considered problem solving.
2. Problem solving was a central component in the teaching and learning of mathematics even at the very earliest stages of elementary school.
3. Problems were diverse in nature.
4. Complex problem solving was an activity for all students.

Although **French** schools have a rich tradition in mathematical problem solving, teachers and students in classrooms have been exposed to different theorists compared to Israel and the wider Middle East. This is an example of the situatedness of Being-human especially in relation to philosophy and national pride. Since the implementation of New Math syllabi however, when at least some students were encouraged to express "the creative power of one's self," (Artigue & Houdement, 2007, p. 370) the teaching and learning of mathematics in France has been francocentric through Brousseau's (1997) Theory of Didactic Situations (TDS); Chevallard's (2006) Anthropological Theory of Didactics (ATD), and Vergnaud's (1994, 1996) Theory of Conceptual Fields (TCF). Consequently, French mathematics curricula emphasized diverse problem solving situations so that each student was given the opportunity to:

(1) Formulate a procedure; execute that procedure, and communicate his or her results in a meaningful way;
(2) defend the validity of his or her solution;
(3) grapple with an authentic research problem that was a novel event for the individual; and
(4) to generate interesting questions and solutions from a set of data (Artigue & Houdement, 2007).

In another Continental country over the past 30 to 40 years however, namely, **Italy** there was a vast difference between the documented, or intended mathematics curricula and that which was implemented in classrooms. As a result, Boero and Dapueto (2007) contended that the "challenging task for us as researchers is to understand the reasons for the failure of the effort to improve teaching and learning of problem solving in Italian classrooms, and to try to make realistic hypotheses about how to overcome the situation" (p. 384). In this regard recommendations were made along the following lines:

1. All mathematics teachers at all levels of schooling **needed** to have a minimum of four years of professional training at the university level.
2. Prospective teachers **needed** highly competent educators to effect good models of teaching mathematics, so that students could build concepts by developing mathematical arguments whilst solving real world problems.
3. Novice teachers **needed** to be critiqued on the nature and timing of the feedback that they gave students during classroom problem solving activities.

In **Brazil** and other **Latin American** countries creative problem solving in mathematics was stifled at the system level. Notably therefore, "most Brazilian mathematics educators feel that the separation of research into trends is a theoretical idealization that does not respond to the dynamics of the problems we face" (D'Ambrosio & Borba, 2010, p. 271). Nevertheless, by and large students learned how to pass mathematics tests rather than how to reason with understanding and creatively. Consequently, D'Ambrosio (2007b) developed the perspective that

> an innovative approach of Problem Solving, and in Mathematics Education in general, depends, paraphrasing Hassler Whitney [1976], on the courage to present hard, and even unsolvable problems, to children, and to listen to their proposals. But testing precludes this. (p. 520)

Nevertheless, innovative problem solving ought to be a priority in schools because a conceptual transition has taken place in the 'world of work'. **Table A·1** summarises the main characteristics, at a fundamental level, of how real world problem solving has changed, and not only in mathematics.

Table A·1. A comparison between the essentials of 'New thinking' and 'Old thinking' problem solving paradigms (adapted from D'Ambrosio, 2007b, p. 517)

'Old' Problem Solving Paradigm	'New' Problem Solving Paradigm
Given problems to solve	Identify problems; Problem posing
Individual work	Cooperative work; Teams
One solution problems	Open ended problems
Exact solutions	Approximate solutions

North of Brazil and Mexico lies the **United States**. In this nation mathematics education has been influenced strongly by a back to basics mentality that developed in the late 1960s and early 1970s. Many educators, administrators, and politicians did not learn from history and therefore were destined to repeat the mistakes of the past (cf., Santayana, 1953). For example, the back to basics mathematics curricula of the 1970s resulted in most students being neither adept at the basics nor at problem solving (Schoenfeld, 2004). Moreover, since the 1980s mathematics education in the United States has experienced ongoing resistance to a problem solving basis for the teaching and learning of mathematics. In spite of this however, many mathematics educators have been made aware of at least some of the vital elements that constitute highly productive ‘sense-making’ learning environments. **Table A·2** lists and describes four of these vital elements. Notably each of these elements is consistent with D’Ambrosio’s (2007a, 2007b) ‘new thinking’ problem solving paradigm.

Table A·2. Common elements of productive classroom cultures (adapted from Engle & Conant, 2002, pp. 400−401)

Common elements	Description
Problematizing	Students are encouraged to engage with problems that challenge their intellect.
Authority	Students are given the liberty to tackle such problems meaningfully.
Accountability	As part of a community of problem solvers, each student takes responsibility for his or her problem solving activities and outcomes.
Resources	Necessary resources and technologies are made available to students.

Although **Australia** has maintained close links with the United States in the field of Mathematics Education since at least the beginning of the 1960s, the back to basics debate has not been as vociferous as in the United States. This allowed mathematics education research to evolve rapidly beyond a relatively narrow problem solving focus. As inferred from **Table A·3**, if ‘a tangible difference’ in the teaching and learning of mathematics was to be realized in classrooms, then problem solving research needed to be ‘applied and generalised’ (Clarke, Goos, & Morony, 2007).

Particularly relevant for the powerful learning of mathematics was for teachers to help students develop a culture of inquiry (Goos, 2004; Groves, Doig, & Splitter, 2000) that involved mathematical sense making in out-of-school environments which were technology rich, and in particular, afforded students the opportunity to pose problems and interpret simulated three dimensional worlds (Lowrie, 2002a, 2002b, 2005); coupled with the scaffolding of quality learning situations that required small-group student collaboration (Barnes, 2001, 2003), for the purpose of ‘working mathematically’ by thinking spontaneously (intuitively) and creatively (Williams, 2002, 2004), as well as metacognitively (Goos & Galbraith, 1996).

Table A·3. Themes of problem solving research in Australia related to **Students'** problem solving performance and **Teachers'** instructional practices (adapted from Clarke, Goos, & Morony, 2007, p. 477)

Theme	Research domain	
	Students' problem solving performance	Teachers' instructional practices
1. Move towards "applied" problem solving research	1.1 How can students be assisted to form appropriate **visual representations** of problems?	1.2 What type of problem solving **tasks** should teachers choose for use in the classroom?
(Maturation)	Technology (Lowrie)	Problem posing (Lowrie)
	Diagrams (Diezmann)	Cognitive engagement (Helme & Clarke; Williams)
		Context (Clarke & Helme)
		1.3 How do teachers' **beliefs** about problem solving influence their classroom **practice**?
		Teachers' problem solving beliefs (Anderson et al.)
2. Broadening of the field to explore more general theoretical concepts and perspectives	2.1 How can a problem solving approach promote **mathematical thinking**?	2.2 What classroom processes promote a **culture of inquiry** to support problem solving?
(Generalisation)	Using modelling to connect mathematics with real world contexts (English; Galbraith & Stillman)	Communities of inquiry (Goos; Groves et al.)
		Collaborative learning (Barnes)
	Using investigations to develop mathematical reasoning (Diezmann et al.)	
	Developing creativity in mathematical thinking (Williams)	
	Teaching for abstraction (Mitchelmore & White)	

On the whole however, classroom practice in Australia did not keep pace with mathematics education research. Based on the TIMSS 1999 Video Study of Australian mathematics classrooms, Stacey (1999) declared that "the average lesson in Australia reveals a cluster of features that together constitute a syndrome of shallow teaching, where students are asked to follow procedures without reasons" (p. 487). This view was not contradicted by Clarke, Goos, and Morony (2007), but it was emphasized that if a problem solving culture was to take root in Australian mathematics classrooms then research showed that curriculum, instruction, and assessment **needed** to be aligned. This research outcome was consistent with the experience of Boero and Dapueto (2007) in Italy, and D'Ambrosio (2007b) in Brazil and Latin America respectively. Another case in point was **Mexico**. Curricula proposals failed to incorporate problem solving approaches in the organization and presentation of the mathematical contents to be taught in elementary schools. As an overall result, the main mathematical ideas of quantitative and proportional reasoning were not learned and assessed in classrooms in ways that fostered **both** mathematical habits and problem solving processes (Santos−Trigo, 2007).

Singapore however, like Australia adopted a problem solving focus as a consequence of the emergence of the problem solving movement in the United States and other regions of the world in the 1980s. But unlike Australia and the United States the education system in Singapore is highly centralized through the Ministry of Education. Therefore it has been much easier for the educational authorities to align the mathematics curriculum, classroom instruction, and the nature of assessment towards the goal of developing a best practice for Singapore.

In particular, Singapore's leaders appreciated the importance of creative problem solving in a globalizing world, because the island nation had to create literally a 'place for itself' in the world because of its small population and limited resources. But in order to facilitate a creative and innovative people, the Ministry of Education realized that it would need to influence the assessment practices of teachers at a fundamental level. Simply stated, "there is enough evidence that assessment affects teachers and their practice," (D'Ambrosio, 2007b, p. 520) and in the context of Singapore mathematics,

> although traditional assessment is powerful in assessing students' factual knowledge, it often receives criticism for being less effective in assessing students' conceptual understanding, higher order thinking skills, problem solving abilities, as well as communication skills, which are recognized to be more and more important nowadays. (Fan & Zhu, 2007, p. 499)

Consequently, mathematics education in Singapore has been characterized by both traditional (e.g., paper-and-pencil tests) and non-traditional assessments. The 'new' assessment strategies have included project and portfolio assessment, oral presentations, journal writing, and student self-assessment. These learning strategies have begun to be implemented and also evaluated (Fan & Quek, 2005; Yazilah & Fan, 2002). For example, commencing in December 2003 a two year pre- and post-test intervention study was undertaken. The study involved around 2, 400 students from eight primary and eight secondary schools. Approximately half the students constituted the ‘experimental’ assessment group, and the other half were an ‘intact’ comparison group. Initial analysis indicated that students benefited academically and affectively from being assessed in different ways. There were multiple reasons for the positive finding, but a variety of assessment learning strategies can be effective pedagogically, especially if the assessment is “self-directed learning oriented assessment,” namely, assessment **for**, **of**, and **as** goal-oriented learning in mathematics (Mok, 2011), but is still linked consistently to **teaching for** problem solving; **teaching about** problem solving, and **teaching through** problem solving (Ho & Hedberg, 2005; Stacey, 2005; Lam et al., 2013).

Interestingly however, students from the reform classrooms tended not to outperform the comparison group when solving novel or unfamiliar problems. Therefore, Fan and Zhu (2007) concluded that in Singapore the development of higher-order cognitive and affective functioning in mass mathematics education would take longer than was anticipated, because as was pointed out by Santos−Trigo (2007), albeit in a Mexican context,

> problem-solving performance seems to be a function of several interdependent categories of factors including: Knowledge acquisition and utilization, control, beliefs, affects, socio-cultural contexts, and facility with various representational modes (i.e., symbolic, visual, oral, and kinesthetic). (Lester & Kehle, 2003, p. 508)

Notably however, *Assessment in Mathematics* was the theme of the AME−SMS (Association of Mathematics Educators and Singapore Mathematical Society) Conference held at NUS High School in June 2014.

In **China** (Mainland China, Hong Kong, and Taiwan) many mathematics teachers used at least three different instructional or assessment strategies to contextualize the teaching and learning of mathematics, and as a result students were able to ‘make connections’ (Cai & Nie, 2007). These three strategies were (1) **one problem, multiple solutions**; (2) **multiple problems, one solution**, and (3) **one problem, multiple variations**. A focus on mathematical problem solving in Chinese education can be traced back to at least the time of the Han Dynasty (c. 206 BC – 220 AD), and consequently, many mathematics teachers were not only adept at solving a vast array of mathematics problems in different ways, but were also skilled at communicating this knowledge in classrooms. But in part because of the Cultural Revolution, teachers' problem solving and didactical acumen were often not recorded in textbooks or teaching manuals (Cai & Nie, 2007). Interestingly however, Cai and Cifarelli (2004) identified the following six characteristics of Chinese students' mathematical problem solving:

1. Computational skills and basic knowledge were more impressive than open-ended complex problem solving.
2. Symbolic representations and generalized strategies characterized problem solving attempts.
3. Student thinking was convergent rather than divergent or creative.
4. If requested, many students could generate a second solution to a given problem; probably not through a creative process, but because "the working of a problem is selected from various methods, and the method should suit the problem (Song mathematician Yang Hui in 1274, as cited in Siu, 2004, p. 164).
5. Unnecessary symbol manipulations were often carried out incorrectly.
6. Students did not like to take risks when problem solving.

Nonetheless, China has an 'examination culture' that is highly competitive. Therefore it is not odd that "a general finding from almost all existing international studies in mathematics was that Chinese students consistently outperformed US students across grade levels and mathematical topics" (Cai & Nie, 2007, p. 460). Moreover in the 16 years from 1999−2014, Mainland China placed first in the International Mathematical Olympiad on no less than 13 occasions.

In **Japan**, mathematics education has been linked cross-culturally to the United States, France, and China. In particular however, Japanese mathematics educators investigated, and continue to investigate the role of the teacher in 'facilitating guidance' and providing 'direct guidance' to classes of students who attempted to solve novel or open-ended problems (Ding & Li, 2014; Hino, 2007). In this regard an important part of mathematics teaching and learning in Japan was the notion of a carefully planned, or 'crafted' lesson (Furner & Robison, 2004). A typical lesson unfolded in six stages (Hino, 2007):

1. The previous lesson, or lessons were reviewed and a 'preliminary' problem for the day was presented by the teacher to the class.
2. The problem was worked on individually and then different students' presented their ideas to the class as a whole.
3. The problem for the day was refined and made clear by the teacher.
4. Each individual student worked on the problem by him or herself, and then the different solutions were presented to the class.
5. Solutions were compared, and when possible, an elegant method was detailed.
6. The teacher summarized the main ideas of the lesson, and each individual student proceeded to write comments on their own learning, insights, and errors.

As depicted in **Table A·4**, the crafted lesson is consistent with S−R−O−C, the Select−Relate−Organize−Check learning protocol.

Table A·4. Two representations of integrated learning compared

S−R−O−C Learning Protocol	**Crafted Lesson Sequence**
Select	Stage 1 – Stage 3
Relate	Stage 4
Organize	Stage 5
Check	Stage 6

Moreover, both during and after the Decade of the Brain (1990−1999) affect became increasingly important to Japanese mathematics educators. In particular the purpose of the crafted lesson was to integrate learning together with a 'zest for living' (Hino, 2007). Consequently, classroom management strategies were discussed and questions such as the following were posed by teachers (Hino, 2007; Lester, 1994):

a. Is the problem situation sufficiently meaningful and challenging so that each student can pursue the task vigorously?
b. Does each student feel that he or she can eventually solve the problem?
c. In what ways should the students' solutions be managed so that the class can be led to the mathematical understandings that are the goals of the lesson?
d. To what extent should teachers scaffold, or differentiate learning opportunities so that all students are cognitively and metacognitively active over the course of the lesson?

In the **United Kingdom** by the beginning of the twenty-first century there was sufficient knowledge to implement ‘non-routine’ problem solving in schools. Especially since World War II, a holistic body of knowledge was developed that specified the type and sequence of learning that students needed to experience if they were to make sense of mathematics for themselves (Burkhardt & Bell, 2007). In particular rote learning, memorization, and skill practice was a precursor to the meaningful learning of concepts and skills. This mathematical foundation paved the way for students to learn how to solve increasingly challenging problems. After learning how to solve given problems, the next stage of student development was to pose and investigate their own problems. The goal of all mathematical problem solving was to describe mathematical structures that reflected practical situations (applied mathematics), and then to intra- and interrelate these structures (pure mathematics).

Unlike Singapore however, the politics and the intricacies of the education system in the United Kingdom was such that different levels of government tended “to assume that policy decisions will be implemented, and on time, independent of the level of support they provide” (Burkhardt & Bell, 2007, p. 403). Consequently many ‘well-intentioned’ systemic changes failed to materialize, especially because the different education professions did not articulate to government that the proposed changes were often not realistic given the nature of teacher training and experience. Mathematics teachers on the whole did not have the problem solving and pedagogical acumen that was a part of the Chinese and Japanese teaching and learning cultures. However, although “the challenge of modifying the system dynamics so as to yield large-scale improvements remains an unsolved problem in the UK, as elsewhere; at least, it is now recognized and being worked on” (Burkhardt & Bell, 2007, p. 395).

In **The Netherlands**, implementing systemic change in mathematics education had limited success, but the outlook in this regard appeared to be more positive than was the case in the United Kingdom. In particular, many Dutch mathematics educators were influenced by 'realistic mathematics education' that was informed ontologically by Pólya (1962), namely, that

> solving a problem means finding a way out of a difficulty, a way round an obstacle, attaining an aim which was not immediately attainable. Solving problems is the specific achievement of intelligence, and intelligence is the specific gift of mankind: solving problems can be regarded as the most characteristically human activity. (p. v)

Overall however, the focus of textbook series and assessment in The Netherlands was not novel or non-routine problem solving that required the integrated use of technology and modelling (Doorman et al., 2007). Therefore mathematics teachers had to develop their own resources in this regard which was no trivial process. Nevertheless, teachers were encouraged to network with other 'problem solving' teachers; participate in virtual communities, commence problem solving activities with students from an early age, and use school-based examinations to foster types of assessment that were conducive to problem solving that necessitated reasoning and sense making. Thus grassroots change was promoted as a realistic option in an attempt to alter the textbook culture in the country.

The experience of mathematics educators in **Germany** was that if change could be brought about in elementary school classrooms then it was likely to provide an impetus for similar change at the secondary school level. In these terms there was "a shift in mathematics textbooks for all grades from rather algorithmically oriented tasks to more demanding problems" (Reiss & Törner, 2007, p. 440). This shift was influenced in part by the introduction of the German *Kultusministerkonferenz* (KMK) objectives for education in 2003 and 2004. These educational standards were similar to the NCTM's *Principles and Standards for School Mathematics* (Standards 2000). Consequently, it was no longer feasible for problem solving in mathematics classrooms to be but an isolated activity, but rather needed to become a 'habit of mind', which meant grappling with both well-defined and ill-defined problems on an ongoing basis.

Interestingly therefore, although there was acceptance of Pólya's (1954, 1957) ideas, there was no evidence to suggest that his approach to problem solving and reasoning had influenced classroom practice in Germany. But rooted historically in Gestalt psychology and the intuitive thinking of Gauss and Goethe, there was a view in mathematics education that problem solving was an exercise in structuring, or re-structuring the problem in a manner that gave shape to a solution that was a thing other than just the sum of the parts (Bortoft, 1996; Duncker, 1945; Koffka, 1936; Schaaf, 1964; Wertheimer, 1938). If this was not the case then the problem solver had not understood the problem meaningfully (Wertheimer, 1961).

Therefore it was considered essential that mathematical problem solving be a central feature of all pre-service teacher education. In particular pre-service teachers needed to develop a 'problem solving language' that would empower all students, not just high achieving students to engage purposefully with novel problems, because

> a 'new culture of problems' is emerging and influences the school curricula. In a way the situation in Germany now parallels that of the United States some years ago. Stanic and Kilpatrick (1989, p. 1) get to the point when stating: 'Problems have occupied a central place in the school mathematics curriculum since antiquity, but problem solving has not. Only recently have mathematics educators accepted the idea that the development of problem-solving ability deserves special attention'. It is probably a relevant coincidence that these changes emerged in both countries when standards for school mathematics were introduced. (Reiss & Törner, 2007, pp. 439−440)

End Note

1. If the base of a truncated pyramid of height **h** has area **ab**, then the exact volume of the pyramid is given by the formula, $V = \frac{h}{3}(a^2 + ab + b^2)$.

Appendix B: A Dialogical Protocol for the Development of Teachers' Problem Solving Acumen (adapted from Xenofontos & Andrews, 2014, p. 295)

Belief dimension	Problem-solving belief dimension	Examples of questions
Nature of mathematics	Nature of mathematical problems and problem solving	1. What is a mathematical problem to you? 2. What characteristics should a good mathematical problem have? 3. Can you give me examples of mathematical problems from your personal experiences? 4. What does mathematical problem solving mean to you? Can you define it?
Mathematics teaching in general	Teaching in problem-based activities.	1. What should teachers be doing during problem-based activities? 2. In what ways, and for what purposes, can problem-based activities be used in mathematics lessons? Are there different ways? 3. How can teachers organise the classroom for problem-based activities? Should they ask the pupils to work individually, in pairs or groups? Why?
Mathematics learning in general	Learning in problem-based activities	1. What benefits do learners gain from their interaction with mathematical problems, if any? 2. What should someone do, in your opinion, in order to improve her/his problem solving skills? 3. What difficulties can problem-based activities create for students as learners of mathematics?
Self-efficacy about mathematics	Self-efficacy in solving mathematical problems	1. How would you describe yourself as a problem solver? Why? 2. How do you feel when you face a mathematical problem? 3. What do you do when you face difficulties solving a problem? How do you overcome these difficulties?
Self-efficacy about mathematics teaching	Self-efficacy in using problem-based activities in future teaching	1. How do you feel about the idea of using problem solving activities in your teaching? 2. Suppose you were teaching a class of pupils. Would you describe a mathematical problem-based activity on a topic of your choice? What would you, as the teacher be doing? What would pupils be doing? 3. Suppose some children face difficulties during the problem solving activity. What would you do in order to help them?

Appendix C: Description of Van Hiele's Levels of Geometric Thought compared with the mathematical actions of the RBC−CE model (adapted from Fuys, Geddes, & Tischler, 1988; Hoffer, 1983; Schwarz, Dreyfus, & Hershkowitz, 2009)

Levels of Geometric Thought	Description of Thought Levels	RBC−C Epistemic Actions
Recognition	Although the student cannot yet identify specific features or properties of figures, the learner is able to differentiate figures on the basis of their overall appearance. This empirical observation is in agreement with Gestalt psychologists who argue that it is more natural for humans to 'see in wholes than in parts'.	**Recognizing**
Analysis	Properties of figures are analysed correctly, but different figures and properties are not explicitly interrelated.	**Recognizing/Building-with**
Ordering	Different figures and their properties are linked correctly, but justification of decision does not take the form of organized sequences.	**Building-with/Constructing**
Deduction	Formal operations: Sequences of statements are deduced logically within a single geometric system.	**Constructing/Consolidating**
Rigour	Post-formal operations: Various deductive systems are analysed rigorously in a manner that is comparable to Hilbert's foundational approach to mathematics.	**Consolidating/Extending**

Appendix D: Basis Factors and Survey Items for Powerful Mathematical Learning

1. Being-ethical: A First Philosophy of Learning Mathematics (17 items)

Strongly disagree	Disagree	Slightly disagree	Neutral	Slightly agree	Agree	Strongly agree
⬭	⬭	⬭	⬭	⬭	⬭	⬭

1. I am willing to go out of my way to help others learn mathematics.

2. In my mathematics class I make sure that my behaviour benefits the learning of others.

3. I am tolerant of mathematical ideas that are different from mine.

4. I am optimistic about my mathematical future.

5. I choose to have fun when learning mathematics.

6. I have faith in my teacher's ability to help me learn mathematics.

7. When tackling a difficult problem I trust my ability to solve the problem.

8. I exercise faith to bridge the gap between that which I see, and that which I need to see, to solve challenging mathematical problems.

9. I persist in mathematics and belief comes that I can actually do it.

10. I believe that there is more in me mathematically than I experience currently.

11. To be successful in mathematics I must have faith, which is the assurance (steadfast belief) that what I hope for will actually happen.

12. The inspiration to succeed in mathematics comes from what my teacher says, and how he/she says it.

13. I like learning mathematics with others because there is 'strength through unity'.

14. My mathematics teacher has high expectations for the class.

15. In mathematics class there is a lot of goodwill between the students.

16. The students in my mathematics class are friendly towards me.

17. I cooperate with other students in mathematics so that they can learn well.

2A. Generic Competences Relevant to the Creative Process (6 ordered response items)

Consider each of the following **six** steps. Do they reflect the way in which you go about solving challenging mathematics problems?

Almost never	Rarely	Sometimes	Often	Almost always
⬭ **1**	⬭ **2**	⬭ **3**	⬭ **4**	⬭ **5**

1. **Stage I**: I define the problem.

2. **Stage II**: I plan an approach.

3. **Stage III**: I carry out the plan.

4. **Stage IV**: I monitor my progress with a particular goal in mind.

5. **Stage V**: I reflect on the result (Was the outcome of my method effective/correct?).

6. **Stage VI**: I reflect on the efficiency of the procedure/method (Was my thinking sharp and to the point?).

2B. The Process of Creativity (8 ordered response items)

Almost never	Rarely	Sometimes	Often	Almost always
⬭ **1**	⬭ **2**	⬭ **3**	⬭ **4**	⬭ **5**

1. **Stage I**: *Encounter* (I identify the problem or challenge as clearly as I can.)

2. **Stage II**: *Preparation* (I try to gather all relevant information.)

3. **Stage III**: *Concentration* (I make a strong effort to solve the problem.)

4. **Stage IV**: *Incubation* (If necessary, I allow the non-conscious dimension of my mind to work on the problem over time—perhaps even a day or two.)

5. **Stage V**: *Illumination* (I suddenly get it! Often when I am doing something else)

6. **Stage VI**: *Verification* (I analyse my idea formally in a step-by-step logical manner.)

7. **Stage VII**: *Persuasion* (If I think that I am correct, I attempt to convince others that my idea or solution really does work.)

8. **Stage VIII**: *Elaboration* (I try to broaden my ideas through the feedback that I receive.)

3. Dialogue (17 non-ordered response items; 4 ordered response items)

Almost never	Rarely	Sometimes	Often	Almost always
⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**

As a member of my mathematics class ____________________.

1. I get the opportunity to hear the mathematical ideas of others
2. I communicate my mathematical ideas to others
3. I am aware of mathematical conversations that are happening around me
4. I get the chance to compare my ideas with the ideas of my classmates
5. the goal of group learning is for each student to explain the problem and solution clearly
6. I work most productively on my own, but with others around me who can give useful feedback
7. we discuss artwork that reflects, or has been inspired by mathematics
8. I use Facebook or Twitter to have mathematical discussions with my friends
9. I dialogue with another person to develop my initial mathematical idea

10. I make a note of my mathematical dialogues in a learning journal.
11. I dialogue mathematically with God.
12. I have mathematical conversations with myself based on conversations that I have had with others.
13. I solve novel problems by engaging in an imaginary dialogue with myself and another Person.
14. I talk aloud when solving mathematics problems on my own.
15. I feel that my mathematics teacher is a part of my thinking.
16. I dialogue with myself mathematically in order to learn from myself.
17. I work with the ideas of others in order to find out something new.

Do the following steps indicate how you resolve differences when learning mathematics?

18. **STEP I:** My friend and I each attempt a mathematics problem on our own.
19. **STEP II:** If there are differences in approach we express our differences to one another.
20. **STEP III:** We re-experience our differences together.
21. **STEP IV:** We resolve our differences by

A. determining whether our differences are actually differences.
B. determining how our differences are different.
C. bringing them together in a way that makes sense to both of us.
D. bringing them together in a way that emphasizes what they have in common.

4A. Symbol Processing/Proceptual Learning (7 ordered response items)

Almost never	Rarely	Sometimes	Often	Almost always
⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**

In my mathematics class learning proceeds as follows:

Stage I: **1.** The teacher gives step-by-step instructions on how to perform a particular method or procedure.
2. The meaning of the different mathematical symbols are made clear.
3. The teacher explains "why" at each step of the procedure.
4. The teacher answers any questions that the class may have.

Stage II: **5.** I practise the procedure by processing symbols (drill work) until I can do it accurately and quickly.

Stage III: **6.** The teacher applies the procedure to problems on the board.
7. The teacher assigns a set of problems to the class.

Stage IV: **8.** I attempt the problems.
9. I discuss any difficulties with my teacher (or friends).
10. I solve problems until I feel that I can work with the procedure confidently.

Stage V: **11.** I reflect on the work that I have done by comparing (and contrasting) *how* the procedure has been used in the different problems.
12. I try to grasp the procedure as a whole (e.g., by drawing a graph, a diagram, or a concept map).
13. I construct and then solve *my own* problems using the procedure.

Stage VI: **14.** I use the procedure flexibly when faced with more challenging problems.

Stage VII: **15.** I relate this procedure to other procedures (or methods) that I have learned by asking myself the following questions:

A. What do the procedures have in common?
B. How are the procedures different?
C. Can I develop a new procedure from the learned procedures?

4B. Situated Learning (9 ordered response items)

Almost never	Rarely	Sometimes	Often	Almost always
⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**

1. **Stage I: A.** My mathematics teacher gives me a mathematical task to complete.

2. **Stage II: A.** I engage with the task/problem and act on what comes to mind.
   B. My mathematical actions lead to the formation of a mental image (mental picture) of what works.
   C. I review my actions to sharpen my mental image of what works.
   D. I repeat A→C until the task/problem is complete.

3. **Stage III: A.** My teacher gives me more mathematical tasks to complete.
   B. I use my mental image to do the tasks.
   C. I describe my mental image (or understanding) in words.

4. **Stage IV: A.** I predict which properties of my mental image will help me solve the problem directly.
   B. I write down the properties that I notice.

5. **Stage V: A.** I apply the properties as a method (that is as a sequence of actions) to solve the given problems
   B. I say *why* the method works.

6. **Stage VI: A.** I adapt my method to *similar* tasks or situations.
   B. I identify the features of my method that are applicable to *all* the tasks.
   C. I generalize my method across the many similar situations.

7. **Stage VII: A.** I formally state my generalized method as a rule or theorem.
   B. I prove the rule or theorem.

8. **Stage VIII: A.** I apply my proved result in a *new* problem situation.
   B. I play or practise in the new situation.
   C. Consequently I deepen my understanding in the new situation.

9. **Stage IX**: **A.** I invent or construct a piece of mathematics that is new to me.
   B. I attempt to convince others that my idea is sound.

5. Mathematics Assessment (Possible assessment structure and questions)

Guidelines and Instructions

- Attempt all **eight** questions in Part A. Choose at least **three** questions from Part B.

- Make *your* thinking and feelings as clear as possible as you work through each question. Please express yourself mathematically in your own way. However, clear communication on your part is vital if the quality of your work is to be assessed accurately. Do not assume that I will 'know what you mean'—write down your problem solving attempts as they unfold.

- Marks will be awarded for
  - (i) knowing **what** to do,
  - (ii) being able to communicate **why**, or being able to **demonstrate a broader understanding of the situation**, and
  - (iii) for the **creative** use of mathematics.

- There is no maximum score for each question. However, there is a suggested **minimum amount of time** that should be spent on each question.

- The maximum time allowed is **4 hours**. No student may leave during the **first hour** or the **last 15 minutes** of the assessment.

- **Five minutes** reading time over and above the 4 hours.

- **No calculator** or electronic device may be used.

- The assessment is not so much a test, but an opportunity for you to express yourself mathematically. There is no single 'right' method to any question. **Enjoy** the experience!

Part A (Shorter questions)

Question 1 (5 minutes)

An army bus holds 36 soldiers. If 1,128 soldiers are to be bused to their training site, how many buses are needed? (Carpenter, Lindquist, Matthews, & Silver, 1983)

Question 2 (5 minutes)

Estimate $56+23\times9246\div125$

Question 3 (8−10 minutes)

Why is 12×8 four less than a hundred?

<u>Question 4 (5 minutes)</u>

Express $\dfrac{16(a^2b^4)^{-\frac{1}{2}}}{b^{-3}}$ as a simple fraction involving no negative exponents (indices).

<u>Question 5 (8−10 minutes)</u>

The floor of a room is covered with wooden rectangular blocks. When blocks measuring $\boldsymbol{a}$ cm by $\boldsymbol{b}$ cm are used, $\boldsymbol{M}$ blocks are needed. If blocks fit exactly, how many blocks are needed if each block measures $\boldsymbol{x}$ cm by $\boldsymbol{y}$ cm? (Husén, 1967a, 1967b)

<u>Question 6 (8−10 minutes)</u>

A wooden pole is stuck in the mud at the bottom of a pond. There is some water above the mud and part of the pole sticks up into the air. One-half of the pole is in the mud; $\frac{2}{3}$ of the rest is in the water, and 1 metre is sticking out into the air.

(a) How long is the pole?

(b) How do you know that your answer is correct? (adapted from Butterworth, 2002)

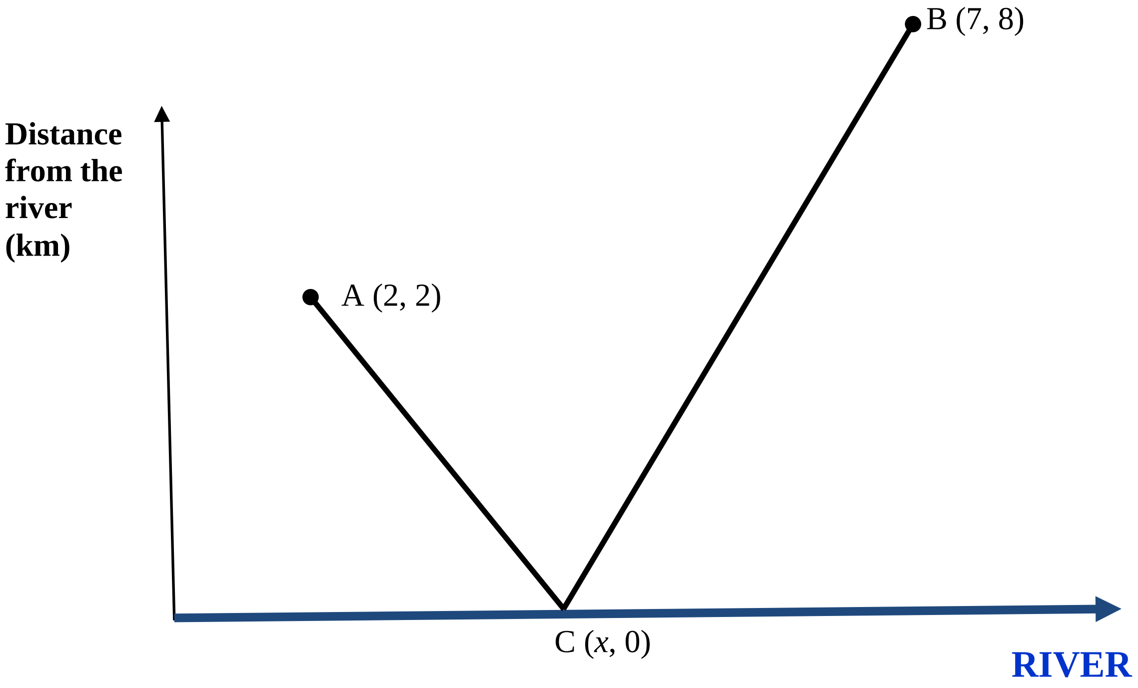
<u>Question 7 (8−10 minutes)</u>

(a) Factorise x^2 in y different ways, where y is the smallest whole number necessary to generalize your factorization.

(b) Generalize your factorization.

Question 8 (10 minutes)

Two towns A and B draw their water supply from the same river. Determine the value of x (distance from the source of the river) if the amount of piping, AC + BC is to be minimized to reduce the cost.



(adapted from Bossé, 2007)

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Part B (Longer questions)

Instruction

- Attempt a minimum of **THREE** questions.

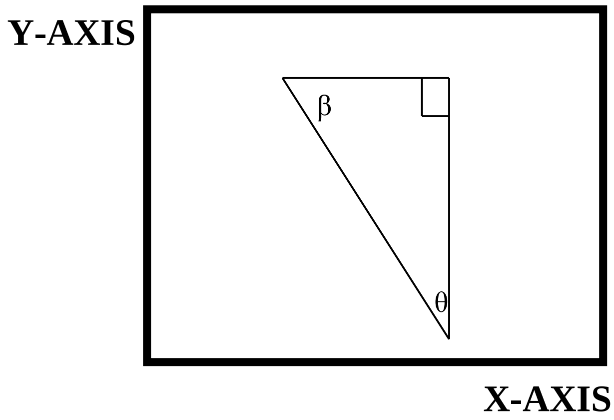
Question 1 (20 minutes)

(a) A triangle is drawn on a flat surface. In your mind, why does it make sense that the angle sum of the triangle is 180°.

(b) Consider the figure below. Imagine that the right triangle has been drawn on a rubber sheet. The sides of the triangle are assumed to have no width. The rubber sheet is stretched parallel to the X-axis while leaving all the distances parallel to the Y-axis unchanged. The stretching is uniform, that is, the same for every part of the sheet.

(i) What will happen to the angle sum, that is, $\theta + \beta$? Why?

(ii) What will happen to the angle sum of the right triangle? Why?



Question 2 (20 minutes)

Which is smaller $\frac{3}{4}$ *or* $\frac{301}{401}$?

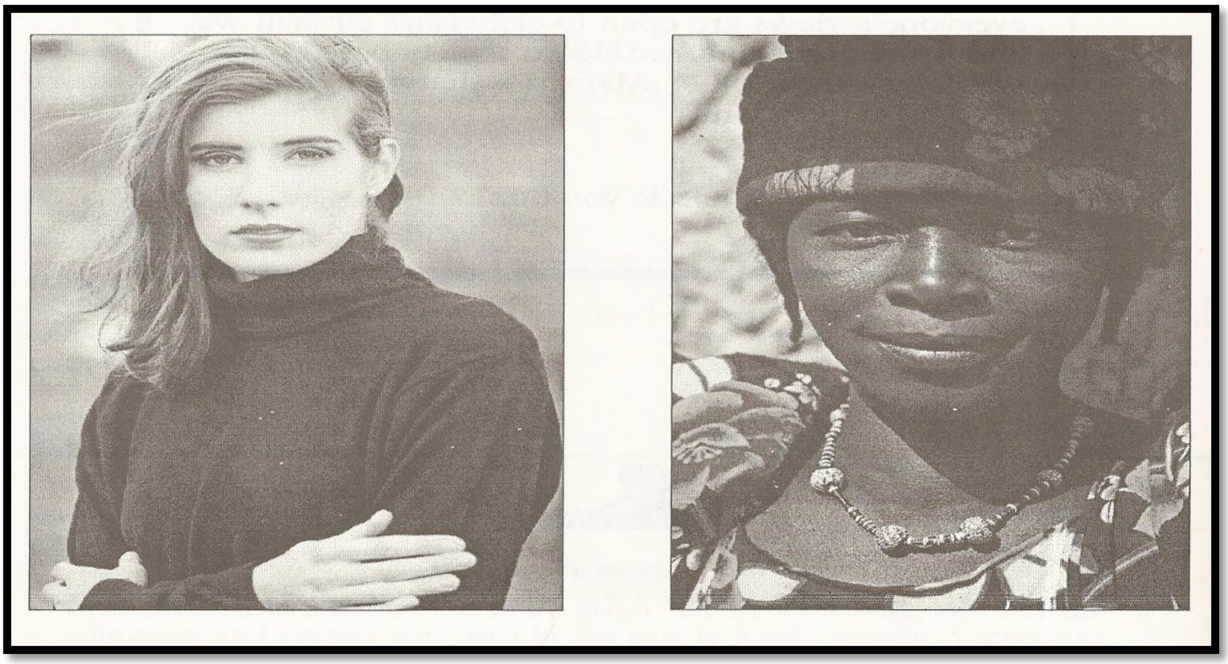
(a) Solve the above problem in three different ways.

(b) Relate, or connect meaningfully your three solutions. In other words, show (demonstrate) how <u>each</u> of your solutions helps you to understand the other solutions. Draw diagrams if necessary.

(adapted from Sullivan, 2008)

Question 3a (10 minutes)

Consider the following photographs. Model each face mathematically. Compare and contrast your models.



(adapted from Higgs & Smith, 1997)

Question 3b (10 minutes)

Consider the following equation: $x = x + 1$.

i. In the above equation what does x mean to you?
ii. When you focus on the equation what do you see, or what comes to mind?
iii. Solve for x in a way that makes sense to you. Motivate your reasoning.

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Question 4 (20 minutes)

Read the following theorem and its proof (adapted from Conradie & Frith, 2000).

R. T. P. (Required to Prove): $\sqrt{20}$ is an irrational number

Proof

Suppose there are integers m and n such that $\sqrt{20} = \frac{m}{n}$.
Without loss of generality we may assume that m and n have no factors in common.

Now $m^2=20n^2$.

Hence 5 is a factor of m^2, and so 5 must be a factor of m.

We can therefore write $m = 5k$, for some integer k.

Then $25k^2 = m^2 = 20n^2$, or $5k^2 = 4n^2$.

Hence 5 is a factor of n^2, and hence of n.

But then 5 is a factor of m and n, contradicting our assumption.

Now answer the following questions:

(a) What method of proof is used here?

(b) How is $\sqrt{20}$ defined?

(c) When is a real number irrational?

(d) Why may we assume that m and n have no factors in common?

(e) Given that 5 is a factor of m^2 how does it follow that 5 is a factor of m?

(f) Why does $5k^2 = 4n^2$ imply that 5 is a factor of n^2?

(g) Which assumption is contradicted, and how does the theorem follow from this?

(h) The equation $m^2 = 20n^2$ also implies that 2 is a factor of m^2. Could we have used this to prove the theorem? Motivate your answer.

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<u>Question 5 (20 minutes)</u>

Consider the following sets of numbers:

A = {1, 2, 3, 4, 5,...}

B = {2, 4, 6, 8, 10,...}

C = {3, 6, 9, 12, 15,...}

D = {2, 3, 5, 7, 11,...}

Two students, Peter and Sally are heard to make the following statements.

Peter: *Clearly, A has more elements than B, B has more elements than C and so on.*
Sally: *But they are all infinite sets!*

(a) Why do you think Peter made the statement he did?

(b) Why do you think Sally made the statement she did?

(c) Resolve Peter and Sally's dilemma. Is it possible that both students are correct? Develop, or construct a piece of mathematics that would solve the problem as to which of the four sets has the most elements.

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Alternative Assessment Questions

Question 1 (5 minutes)

Assume that you have 24 iPhones and wish to give each of your 12 friends the same number of iPhones. How many iPhones will you give to each of your friends? Why?

Question 2 (15 minutes)

Complete each of the following statements about a square.

(a) A square has ZERO ____

(b) A square has ONE ____

(c) A square has TWO ______________________and TWO ___________________

(d) A square has THREE ____

(e) A square has FOUR ______________________and FOUR__________________

(f) A square has FIVE ____

(g) A square has SIX ____

(h) A square has SEVEN ____

(i) A square has EIGHT____

Question 3 (15 minutes)

Construct an interesting mathematical problem that involves a right triangle, a circle, and the trinomial $X^2 - 3X + 4$. Solve the problem.

Question 4 (20 minutes)

Imagine a train passing through a tunnel.

(a) What do you **see** mathematically?

(b) What do you **hear** mathematically?

(c) What you **know** mathematically?

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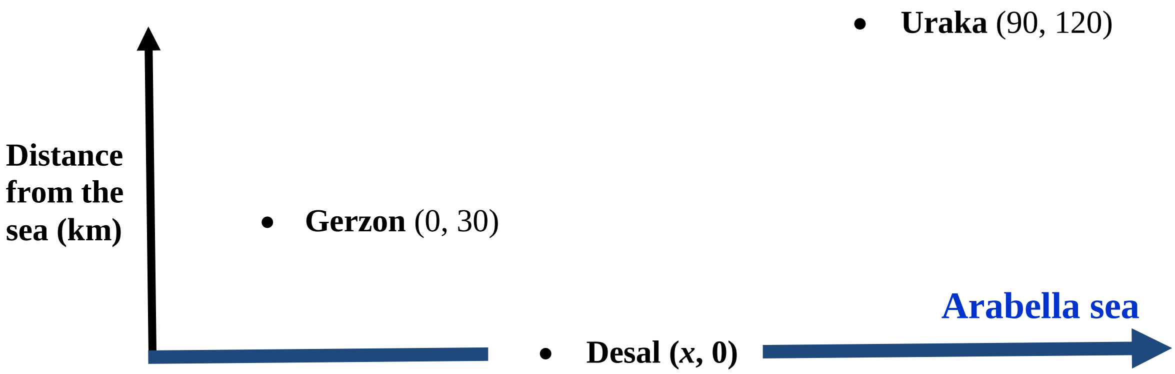
Question 5: Mathematics in Context (‘analytical, practical, creative, authentic, and ambiguous’)

Two cities, Gerzon and Uraka have experienced water shortages over the past decade because of decreased annual rainfall and a growing population. The decrease in rainfall is thought to be due to climate change and mathematical modelling has suggested that the situation is not going to improve over the next 25 years. In addition, the growth in population is due largely to improved medical facilities and the increased movement of people away from the rural or farming areas to the cities. The population of Gerzon is currently twice that of Uraka, but its growth rate is only half that of its trading partner and economic competitor.

In order to address the problem of an increasing population and diminishing water resources, the cities agree to embark on a joint venture through the construction of a single desalination plant. Ideally, both cities would like their own plant thereby not conceding any ground economically. However, due to a lack of funds (both cities are wary of over-borrowing in the light of the Global Financial Crisis of 2008 and the more recent Eurozone monetary difficulties), and in the national interest of protecting the environment, the two cities agree to a compromise solution whereby the plant will be built at a location ‘somewhere’ between the cities on the coast. As per the diagram, Gerzon is located 30 km inland from the ocean; Uraka is 90 km to the East and 90 km further inland. The area between Gerzon, Uraka and the sea is flat and uninhabited. The desalination plant is to be located at Desal (x, 0).



In the design of the desalination plant many mathematical problems have to be solved. Some of the problems require advanced engineering mathematics, but the following problems can be solved by secondary school students.

Make your reasoning clear and mention any underlying assumptions.

- **a)** If **Desal (*x*, 0)** is located to minimize the cost of piping that will deliver water to Gerzon and Uraka respectively, determine the value of ***x***.
- **b)** However, Gerzon does not wish to pay more than is ‘fair’ with respect to the cost of piping, because Gerzon is closer to the sea than is Uraka. Consequently, Uraka agrees to compensate Gerzon appropriately, or vice versa as the case may be. For each $100 spent on piping, how much should the ‘offending’ city compensate the other?

Gerzon requires more water than Uraka. In the desalination plant tanks will be set up to serve each city. Each Gerzon tank is large enough to supply water for six hours if the sluice gates are opened fully; each Uraka tank for four hours. For ease of operation, each Gerzon tank will be coupled to a Uraka tank and both will be filled simultaneously with fresh water. Moreover, the tanks will be synchronised electronically in order to dispense water at the same time. However, when the amount of water is twice the amount in one tank compared to the other tank, both tanks will be refilled. For how long will the tanks dispense water before being refilled?

Appendix E: Influential Factors and Items for Powerful Mathematical Learning

1. Being-human (12 items)

Strongly disagree	Disagree	Slightly disagree	Neutral	Slightly agree	Agree	Strongly agree
⬭	⬭	⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**	**6**	**7**

1. I am taught *how* to learn mathematics for myself. (autonomy)

2. I imitate the problem solving approaches of my mathematics teacher. (imitation)

3. In my school we care about each other because it is the right thing to do. (intrinsic moral value)

4. I am taking advantage of the opportunity to learn mathematics. (intrinsic moral value)

5. I take personal responsibility for my learning in mathematics. (moral accountability)

6. At home, or outside my mathematics classroom I have a quiet place to study mathematics. (privacy)

7. I learn mathematics by reflecting on what I am taught. (reciprocity)

8. By dialoguing mathematically with others, I am learning how to dialogue mathematically with myself. (reciprocity)

9. Mathematics has conventions (widely accepted ways of doing things). I practise these conventions or procedures. (conventionality)

10. I solve mathematics problems in ways that are different from my teacher. (creativity)

11. I feel that mathematics is becoming a part of me. (authenticity of relation)

12. My mathematics teacher gives me regular feedback on my problem solving attempts. (authenticity of relation)

2. Gender (1 item)

1. Please check one of the following:

I am   Male ☐   Female ☐

3. Age (1 item)

1. My date of birth is

   Day……………………..   Month ………………………….. Year …………………

4. Grade Level (1 item)

1. In which grade do you currently learn mathematics?

Grade 8 ☐   Grade 9 ☐   Grade 10 ☐   Grade 11 ☐   Grade 12 ☐   Grade 13 ☐

5. Family (6 items)

Strongly disagree	Disagree	Slightly disagree	Neutral	Slightly agree	Agree	Strongly agree
⬭	⬭	⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**	**6**	**7**

1. My father thinks that the learning of mathematics is important.

2. My mother uses mathematics in her work.

3. A family member is available to help me in mathematics if need be.

4. At least one of my parents is strict when it comes to doing mathematics homework.

5. My family expects me to do well mathematically.

6. I enjoy solving mathematics problems (puzzles) with my family.

6. Socio-Economic Status (5 items)

1. What is your father's main job? ___________________________

2. What is your mother's main job? ____________________________

3. How many of these items are at your home?

(a) Smartphone None ☐ One ☐ Two ☐ Three ☐ Four or more ☐

(b) Laptop computer None ☐ One ☐ Two ☐ Three ☐ Four or more ☐

(c) Motor vehicle None ☐ One ☐ Two ☐ Three ☐ Four or more ☐

(d) Bathroom None ☐ One ☐ Two ☐ Three ☐ Four or more ☐

(e) Tablet computer (e.g., iPad) None ☐ One ☐ Two ☐ Three ☐ Four or more ☐

4. I usually travel on holiday at least once per year Yes ☐ No ☐

5. How many times have you travelled internationally?

Never ☐ Once ☐ Twice ☐ Three times ☐ Four or more times ☐

7. What is Mathematics? (16 items)

Strongly disagree	Disagree	Slightly disagree	Neutral	Slightly agree	Agree	Strongly agree
⬭	⬭	⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**	**6**	**7**

1. Mathematics comprises rules and procedures to be memorized or practised so that problems and equations can be solved.

2. Mathematics is a ‘science of patterns’: An exploration of number and shape to find or construct new patterns.

3. Mathematics is ‘the language of science’: A form of abstract communication; the goal of which is to tackle meaningful problems.

4. Proof gives Mathematics credibility.

5. Mathematics involves movements between the question asked and a search for the answer.

6. There is always a rule to follow when solving a mathematics problem.

7. The field of Mathematics requires people with creative minds.

8. Learning Mathematics means coming to understand, at least in part, the mind of God.

9. Mathematics is like a friend to me.

10. Mathematics is a form of abstract beauty.

11. Mathematics means processing symbols in a logical way.

12. Mathematics is always involves a particular social and historical context.

13. The learning of mathematics means discovering an independent, pre-existing world outside my mind.

14. Within my mind I have an imagined World of lines, numbers, and symbols.

15. New mathematics is being developed all the time.

16. Mathematics means empowerment for my future.

8. Intentionality / Autotelic Personality (13 items)

Strongly disagree	Disagree	Slightly disagree	Neutral	Slightly agree	Agree	Strongly agree
◯	◯	◯	◯	◯	◯	◯
1	**2**	**3**	**4**	**5**	**6**	**7**

1. My attitude towards the learning of mathematics is "Yes, I can!"

2. If the mathematics is difficult I rise to the challenge.

3. I choose to enjoy my classroom experience of mathematics.

4. I study mathematics for the sake of mathematics rather than for some future goal.

5. When learning mathematics I live in the moment.

6. I have devised ways to enjoy mathematics.

7. I refuse to be weak at mathematics.

8. The joy of solving a challenging problem is its own reward in mathematics.

9. If the mathematics lesson is boring I use my imagination to make the work interesting.

10. When it comes to the learning of mathematics I find a way to win/succeed.

11. I ask as many questions as I need to in order to understand mathematics.

12. When attempting a difficult problem, I say that I can, and eventually I get the better of the problem.

13. I speak positively about my mathematics teacher or teachers.

9. Learning/Instructional Principles (25 items)

Almost never	Rarely	Sometimes	Often	Almost always
⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**

1. My mathematics lessons are well structured. (Contiguity effect)

2. I test out my mathematical ideas in a practical way. (Perceptual-motor grounding)

3.

- **A.** When explaining a concept my mathematics teacher does so in more than one way.
- **B.** My mathematics teacher uses multimedia in the classroom. (Dual coding and Multimedia effects)

4. I receive helpful feedback from my teacher after a mathematics test. (Testing effect)

5. I enjoy the pace of instruction in my mathematics class. (Spacing effect)

6. I do many past exam questions in the lead up to a mathematics examination. (Examination expectations)

7. My mathematics teacher requires me to think through problems for myself. (Generation effects)

8. For each mathematics lesson I organize the main points in a way that makes sense to me. (Organization effects)

9. My understanding improves when my mathematics teacher uses physical props to demonstrate an idea. (Coherence effect)

10. I make up stories to understand mathematics. (Stories and example cases)

11. My mathematics teacher explains a mathematics concept by using different kinds of example. (Multiple examples)

12. My mathematics teacher gives me prompt verbal or written feedback on my ideas. (Feedback effects)

13. If I make a mistake in class my mathematics teacher corrects me immediately. (Negative suggestion)

14. In my mathematics class I can solve almost any problem in 10 minutes or less! (Desirable difficulties)

15. When instructing, my mathematics teacher focuses on the main points. (Manageable cognitive load)

16. My mathematics teacher breaks the lesson down into bite-size chunks. (Segmentation principle)

17. I construct my own explanations of the mathematics discussed in class. (Explanation effects)

18. My mathematics teacher asks me ___________ questions. (Deep questions)

- **A.** "Why?"
- **B.** "Why not?"
- **C.** "How?"
- **D.** "What-if?"

19. I am given mathematics problems that involve paradoxes, or contradictions. (Cognitive disequilibrium)

20. If I can't solve a problem one way, I will try an alternative approach. (Cognitive flexibility)

21. Mathematics for me is neither too easy nor too difficult. (Goldilocks principle)

22. I think about my mathematical thinking to improve my learning. (Imperfect metacognition)

23. I often need guidance from my teacher to discover new things in mathematics. (Discovery learning)

24. **A.** I have mathematical goals.
B. I write down my mathematical goals.
C. I read my mathematical goals.
D. I say, "What can I do *today* that can help me achieve my goals?"
E. I follow through on what I have said in D. (Self-regulated learning)

25. Using mathematics I solve interesting real-world problems. (Anchored learning)

10. Teaching Behaviour (14 items)

Almost never	Rarely	Sometimes	Often	Almost always
⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**

My mathematics teacher

1. helps me fix my errors and misconceptions.

2. encourages me to use mathematics to make sense of **my** world.

3. allows me to construct mathematics that may be different from his/her mathematics.

4. has emphasized that mathematical knowledge may be fallible (not perfectly correct).

5. is like a fountain of knowledge, “Here is what is known, and here is how to use it.”

6. makes false starts or mistakes when working on the board.

7. is animated/lively when instructing the class.

8. makes use of metaphor to help me grasp mathematics.

9. negotiates with me **what** mathematics I will study.

10. is a show and tell teacher.

11. explains everything carefully.

12. orchestrates/manages classroom activities so that they run more or less smoothly most of the time.

13. conducts interesting lessons.

14. uses **my** understanding of a problem as a springboard for classroom discussion.

11. Social−Emotional Intelligence /Didactical Contract (21 items)

Strongly disagree	Disagree	Slightly disagree	Neutral	Slightly agree	Agree	Strongly agree
⬭	⬭	⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**	**6**	**7**

1. In my mathematics class/classes there is a genuine acceptance of individual differences.

2. In my mathematics class/classes there is a democracy of the emotions, because I am allowed to answer back in a polite manner.

3. I have a signed contract (agreement) with my mathematics teacher making clear what **he/she expects** from me as a student.

4. I have a signed contract (agreement) with my mathematics teacher making clear what **I can expect** from him/her as a teacher.

Almost never	Rarely	Sometimes	Often	Almost always
⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**

My Mathematics teacher

5. is patient.
6. gets angry.
7. is willing to learn from his/her students.
8. perseveres with me until I understand.
9. is passionate about teaching mathematics.
10. is a person of authority.
11. comes to class on time.
12. smiles.
13. is self-controlled.
14. prevents bullying in the classroom.
15. prevents ridicule/sarcasm in the classroom.
16. makes me feel part of the class.
17. takes my ideas seriously.
18. disciplines me in a manner that benefits my learning.
19. helps me to catch up work that I miss.
20. is a gentle person.
21. praises me publicly when I do well.

12. Time-on-task in Mathematics (11 items)

During the school term (semester), on average how much time do you spend each week on the following? (When answering please include time at the weekend as well)

1. Homework, or other study set by your school mathematics teachers: .... hours per week.

2. Extra lesson or remedial classes in mathematics at school: .... hours per week.

3. Enrichment/extension classes in mathematics at school: ....hours per week.

4. Working with an extra lessons mathematics teacher/tutor: .... hours per week.

5. Extra help in mathematics from your family: ....hours per week.

6. Other mathematical activities (e.g., mathematics competitions; mathematics club): .... hours per week.

7. During the school holidays **(exclude the end-of-school-year holidays)**, on average how much time do you spend doing mathematics: .... hours per week.

Almost never	Rarely	Sometimes	Often	Almost always
⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**

8. If you have double lessons (or lessons that exceed an hour) in mathematics do you find them too long?

Strongly disagree	Disagree	Slightly disagree	Neutral	Slightly agree	Agree	Strongly agree
⬭	⬭	⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**	**6**	**7**

9. My mathematics classes at school are usually productive.

10. If school started later in the day my mathematics learning would benefit.

11. On average how many mathematics lessons do you miss in a school week?

0 ☐ 1 ☐ 2 ☐ 3 ☐ More than 3 ☐

13. Personality of Place (10 items)

Strongly disagree	Disagree	Slightly disagree	Neutral	Slightly agree	Agree	Strongly agree
⬭	⬭	⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**	**6**	**7**

1. The walls of my mathematics classroom are decorated with useful mathematical items.

2. My mathematics classroom is easily recognizable as a mathematics classroom.

3. In my school we have a ‘Math Wall’ or a ‘Math Walk’ — a section (not in a classroom) that is dedicated to Mathematics.

4. The architecture of my school is ‘mathematical’.

5. My mathematics classroom is a physically comfortable place to learn.

6. In my mathematics classroom I can work well as an individual.

7. The design of my mathematics classroom means that the furniture can be easily rearranged to allow for different learning activities.

My mathematics classroom has

8A. audio-visual equipment.
8B. a Smart TV.
8C. a Smart Board (interactive whiteboard).
8D. internet connections for the students.

9. As a school we participate in an international event that celebrates the learning of Mathematics (e.g., World Maths Day).

10. My school environment inspires me to learn mathematics.

14. Individual Differences (12 items)

Almost never	Rarely	Sometimes	Often	Almost always
⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**

My mathematics teacher

1. adopts a one-size-fits-all approach to teaching mathematics.

2. provides the class with supplementary (additional) learning materials.

3. provides materials to encourage further exploration of topics of interest.

4. allows time for me to reflect on my own ideas during the lesson.

5. allows me to work as an individual or in a group.

6. gives me problems that are specific to my interests.

7. values different mathematical perspectives in the class.

8. emphasizes the mathematical culture and history of my ethnic group (e.g., Chinese mathematics, Greek mathematics, or Indian mathematics).

9. helps me learn mathematics kinaesthetically, that is, through bodily movement (e.g., by making use of dance).

10. adopts an algebraic approach to learning mathematics.

11. adopts a geometric approach to learning mathematics.

12. encourages me to understand mathematics by categorizing concepts (e.g., a square is a special case of a rectangle)

15. Deliberate/Intelligent Practice (13 items)

Strongly disagree	Disagree	Slightly disagree	Neutral	Slightly agree	Agree	Strongly agree
○	○	○	○	○	○	○
1	**2**	**3**	**4**	**5**	**6**	**7**

1. When practising mathematics I am careful to follow the method of my teacher.

2. When practising a procedure I reflect on what I am actually doing.

3. I practise mathematics with a specific learning goal in mind.

4A. I rote learn certain mathematical proofs.
4B. I remember my times tables well, because I learned them off by heart.

5. To learn mathematics I try to remember every step in a procedure.

6. My teacher has given me the tools to practise mathematics in different ways.

7. To remember the method for solving a mathematics problem, I go through examples again and again.

8. I grasp a mathematical concept through repeated use of the concept.

9. I devise new strategies to practise mathematics.

10. I practise until I master the work.

11. I tackle as many mathematics problems as possible to increase my expertise.

12. My mathematical practice involves mainly algebraic routines.

13. I use the axioms or rules (e.g., the commutative rule: a+b = b+a) of the real numbers to vary my practise of mathematics.

16. Embodied Cognition (13 items)

Almost never	Rarely	Sometimes	Often	Almost always
⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**

1. I use bodily actions to help me learn mathematics.

2. I use my hands to shape my mathematical thinking.

3. In my mathematics class I get up and move around.

4. My mathematical mind is like an embodied mind in action.

5. I learn mathematics using a Touch Screen.

6. I feel that I have the language to describe my mathematical thinking.

7. I know how to solve a problem but cannot explain what I am doing.

8. I feel limited when doing mathematics because I have to follow a strict set of rules.

9. To my mind the learning of mathematics has a spiritual (transcendental) dimension.

10. I **intentionally** look up and to the right in order to create a new mental image.

11. I **intentionally** look up to the left when retrieving stored pictures from my (long term) memory.

12. I **intentionally** move my eyes down to the left when I wish to engage in a mathematical dialogue with myself.

13. When the teacher instructs my mathematics class I pay special attention to his or her hand movements.

17. RBC−C Learning Model (5 ordered response items)

How do you proceed when solving a new mathematics problem?

1. **Stage I**: I *recognize* or *look for* familiar building blocks (e.g., a particular mathematical pattern).

2. **Stage II**: I *play*, or *build with* the building blocks.

3. **Stage III**: Using the building blocks I *construc*t a solution or an understanding.

4. **Stage IV**: Having constructed a solution, I *consolidate* my learning by checking whether my solution works for *similar* problems.

5. **Stage V**: Having understood the problem, I *use* my new knowledge to tackle *different* problems.

18. Learning Protocol: S-R-O-C Model (7 ordered response items)

Almost never	Rarely	Sometimes	Often	Almost always
◯	◯	◯	◯	◯
1	**2**	**3**	**4**	**5**

My mathematics teacher

1. **Stage I**: makes the main goal of the lesson *clear* in the first five minutes.

2. **Stage II**: *links* the main goal of the lesson to work that the class has seen before.

3. **Stage III**: *focuses attention* on the mathematics that is different in the lesson.

4. **Stage IV**: helps the class *select* or identify the key ideas.

5. **Stage V**: helps the class *relate* the key ideas.

6. **Stage VI**: helps me *organize* my new knowledge meaningfully.

7. **Stage VII**: helps me *check* whether I achieve the main goal of the lesson.

19. Literacy (12 items)

1. What language do you speak at home most of the time? ___________________________

2. What language to you speak with your friends most of the time?_____________________

3. How many English books are there in your home?

0−25 ☐ **26−100** ☐ **101−200** ☐ **201−500** ☐ **More than 500 books** ☐

4. I choose to read an English newspaper

almost every day ☐

2-3 times per week ☐

2-3 times per month ☐

less than once per month ☐

almost never ☐

Almost never	Rarely	Sometimes	Often	Almost always
⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**

5. My mathematics lessons are in English.

6. I write English poetry.

7. I read novels or magazines in English.

8. I debate or public speak in English.

9. I make up new words in English.

10. I like to read Shakespeare.

11. I find mathematical terms meaningful when written in English.

12. I am bilingual in English and Mathematics.

20. Technology (15 items)

Almost never	Rarely	Sometimes	Often	Almost always
◯ **1**	◯ **2**	◯ **3**	◯ **4**	◯ **5**

I use the following technologies to help me learn mathematics.

1. An eraser

2. A laptop computer

3. A graphing/graphics calculator

4. A mobile phone

5. Online chat rooms where mathematics is discussed

6. YouTube mathematics videos

7. An interactive computer program that allows me to play mathematically and get quick feedback (e.g., Geometer's Sketchpad or Autograph)

8. A Thinkpad Tablet

9. Excel (Microsoft Office)

10. PowerPoint (Microsoft Office)

11. *Interactive software* that enables me to think through mathematical concepts for myself.

Almost never	Rarely	Sometimes	Often	Almost always
◯ **1**	◯ **2**	◯ **3**	◯ **4**	◯ **5**

12. When I use mathematical software (e.g., graphing software) my learning of mathematics speeds up.

13. I use interactive computer-based programs to help me concretize/visualize complicated mathematics.

14. I use a computer to practise mathematical routines (procedures).

15. I model my mathematical ideas using computer-based technologies.

21. Globalization (13 items)

Strongly disagree	Disagree	Slightly disagree	Neutral	Slightly agree	Agree	Strongly agree
○ **1**	○ **2**	○ **3**	○ **4**	○ **5**	○ **6**	○ **7**

By studying mathematics I am learning ____.

1. to deal with uncertainties effectively

2. to be an active risk taker

3. to cope with a diversity of new situations

4. to evaluate solutions in relation to future possibilities.

5. to accept risk as a condition of excitement and adventure

6. how to justify my reasoning

7. to question what I am taught

8. how to engage meaningfully with others who think differently from me

9. for a society where the meaning of sexuality might be changing

10. that sexual equality is a core principle of democracy

11. from a teacher who is an excellent example of a lifelong learner

12. to become a flexible problem solver

13. to function in an open framework of global communications (e.g., I use SKYPE to discuss mathematics with students overseas)

22. Problem Solving Heuristics (14 items)

Almost never	Rarely	Sometimes	Often	Almost always
◯	◯	◯	◯	◯
1	**2**	**3**	**4**	**5**

1. If there is a problem I cannot solve, I construct an easier one which I can solve (in order to help me solve the initial problem).

2. I work backwards when I cannot work forwards.

3. I try to think ahead to see whether a particular technique will be useful.

4. I guess an answer that might work. I then adjust it to fit the problem.

5. Ugly or messy methods work well.

6. The surprising fact, C, is observed. However, if A were true, C would be clearly correct. Hence, there is reason to suspect that A is true. I try to show that A is correct.

7. The answer is in the question.

8. I take the given information quite literally — just the way things are written — little to no interpretation on my part.

9. If I reach a sticking point I work *around* the difficulty.

10. I sleep on the problem.

11. I imagine myself in the problem situation.

12. I tackle the problem by means of a thought experiment (an experiment that I do in my mind)

13. I reframe, or restructure the problem in a way that makes sense to me.

14. I draw a diagram.

23. Higher Order Thinking (13 items)

Almost never	Rarely	Sometimes	Often	Almost always
○ **1**	○ **2**	○ **3**	○ **4**	○ **5**

In order to learn mathematics

1. I compare and contrast. (analytical thinking)
2. I critique and judge. (analytical thinking)

3. I use mathematics to help address real needs in my community or city. (practical thinking)
4. In my everyday life I use the logic of mathematics to persuade others that my ideas have value. (practical thinking)

5. I think of mathematics as an art form. (creative thinking)
6. When learning mathematics I like to explore and imagine. (creative thinking)

The learning of mathematics helps me to

7. balance my own interests with the interests of others. (wise thinking)
8. understand how what is true can change over time and between different places. (wise thinking)

9. I question the mathematical understandings of those around me. (critical thinking)

10. If engaged in a mathematical argument, I try to find a 'middle way' between the opposing ideas. (holistic thinking)
11. When learning mathematics I integrate, or combine the different points of view in ways that make sense to me. (holistic thinking)

12. I use intuition (a bodily feeling) to point me in the right direction when grappling with a difficult problem. (intuitive thinking)
13. I 'think−feel' my way towards a possible solution. (intuitive thinking)

24. Formal Operational Thinking (21 items)

Almost never	Rarely	Sometimes	Often	Almost always
○	○	○	○	○
1	**2**	**3**	**4**	**5**

In my learning of mathematics I

1. think about the possible as well as the actual.

2. make predictions.

3. test my predictions/hypotheses.

4. examine what I observe in a careful and thorough way.

5. make many observations in order to find (or abstract) a pattern.

6. use existing patterns to construct new patterns.

7. I deduce results logically.

I solve mathematics problems of the form:

1. "If I make this change what *effect* does this have on...?"

2. "What is the *relationship or connection* between...?"

3. "How *strong* is the relationship or connection between...?"

4. "What is the *possibility* that...?" or "What is the *likelihood* that...?"

5. "What are all the possible *combinations* or *arrangements* of...?"

6. "$y \propto x$" (***y*** is *directly* proportional to ***x***).

7. "$y \propto \frac{1}{x}$" (***y*** is *indirectly* or *inversely* proportional to ***x***).

8. "Find different values for the product $Z = XY$ so that the quantity Z is conserved/does not change."

9. "What is the trade-off between X, Y and Z so that the system does not change, that is, maintains its equilibrium position?"

10. "What is the trade-off between X, Y and Z so that the system attains a new equilibrium, or position of balance?"

11. "What is the trade-off between X, Y and Z so that the system is optimised in relation to a particular goal?"

12. "How does this additional piece of information change your understanding of the answer?

13. "Explain your answer in relation to a different frame or point of reference."

14. What does the answer mean if we place the problem in a different context?"

25. Visual−Spatial Reasoning (9 items)

Consider the following **two** steps. Do they reflect the way in which you go about solving mathematics problems?

Almost never	Rarely	Sometimes	Often	Almost always
⬭	⬭	⬭	⬭	⬭
1	**2**	**3**	**4**	**5**

1A. Stage I: I visualize the given information (words) as a picture or diagram.

1B. Stage II: I translate the picture or diagram into mathematics (e.g., an equation to be solved).

Which of the following activities help you visualize mathematics in your mind's eye?

1. I picture the mathematical situation as simply as possible.

2. I make changes to my mental image in order to see if the changes lead to anything interesting or useful.

3. I play with my mental image **—** bending, folding, rotating, or moving parts around.

4. I construct a physical model of the abstract relationship that I am trying to understand.

5. I image mentally what I feel.

6. I say what I see mathematically.

7. I visualize what I say mathematically.

8. I act out physically what I see mentally.

Appendix F: Saltus Response Model for Hierarchical or Ordered Items (adapted from M. Wilson, 1998, 2004)

$$\Pr\,(y_{ni} = 1 | \varphi_{hn}) = \frac{\exp(\beta_n - \delta_i + \tau_{hk})}{1 + \exp(\beta_n - \delta_i + \tau_{hk})}$$

- y_{ni} is the response (1 represents success, 0 represents failure) of person *n* in level *h* to item *i* in level *k*.
- β_n is an index for the underlying ability of person *n* on the attribute or trait that is being measured.
- δ_i is an index for the underlying difficulty or facility level of the item or task *i*.
- $\varphi_{hn} = 1$ indicates that person *n* is in level *h*. Note that the interaction is a group-level interaction between all persons at a certain level and all items at a certain level.

Appendix G: The Factor Analysis Model (adapted from Meyer, 1990, p. 70)

Let X be a random p-vector with mean μ and covariance matrix Σ. The k-factor model holds (k < p) for X, if X can be expressed by the vector equation: X = δf + u + μ.

The pxk matrix δ is a matrix of constants, namely, the factor loading matrix. The k-vector f is a random vector indicating a (relatively small) number k of underlying factors. The p-vector u is a random vector which describes the variability that is specific to X. Importantly, the random vectors f and u satisfy the following **stringent** conditions:

- E (f) = 0
- E (u) = 0
- Covariance (f) = I, the kxk identity matrix
- Cov (u) = ψ = $\textit{Diagonal}\ (\psi_1, \psi_2, \ldots, \psi_p)$
- Cov (f, u) = 0.

Therefore the k factors are assumed independent of one another and of the specific random variables. In addition the factors are standardized with respect to mean and variance, while all the independent specific random variables are standardized only with respect to the mean.

If the factor model holds then **(1)** Cov (X) = Σ = $\delta\delta' + \psi$, and

$$\textbf{(2)}\ \text{Variance}\ (X_{(i)}) = \sum_{j=1}^{j=k} \partial_{ij}^2 + \psi_{ii}\,.$$

Moreover, the communality of $X_{(i)}$ represents the variance of $X_{(i)}$ which is shared with the other $X_{(j)}$ variables through the k common factors. A high commonality for $X_{(i)}$ means that this variable is well explained by the k factors, but a low commonality for $X_{(i)}$ means that this variable is poorly explained by the k factors.

The commonality of $X_{(i)}$ is defined by: $h_{ii} = \sum_{j=1}^{j=k} \partial_{ij}^2$. The remaining variance in $X_{(i)} = \psi_{ii}$.
This quantity is called the unique or specific variance for $X_{(i)}$ and it represents the variability in $X_{(i)}$ that is unique or specific to $X_{(i)}$ alone.

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