Assessing Learning Progression in the Domain of Fractions

by

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Thesis
Submitted to Flinders University
for the degree of

Doctor of Philosophy

College of Education, Psychology and Social Work

November 2019

ABSTRACT

Rational numbers are an important domain of mathematics learning and one which many students find difficult to learn. The present research has developed and validated an assessment instrument to assess the progression of student learning in the domain of rational numbers, with an emphasis on fractions. The assessment was based on a cognitive model of fraction learning and was innovative in that it distinguished two essential dimensions of fraction knowledge, namely conceptual and procedural.

The research has developed a hypothetical model of the two-dimensional fraction learning progression based on existing research. The hypothetical learning progression was first validated in a qualitative study, carried out through cognitive interviews. The results from the interviews were used to evaluate and revise the learning progression, which was subsequently tested in a second study with 516 students from grades 7 to 9 in a junior high school in Bogor, Indonesia. The fraction learning progression was validated using Bayesian Network Analysis. Two Bayesian Networks models were developed. Model 1 was a single latent variable model, while Model 2 was a multiple hierarchical latent variables model. Model 2 was found to have a better fit with the students' responses than Model 1 and had a number of innovative characteristics, such as incorporating the assumption of the hierarchical dependencies between the levels in the learning progression into a formal statistical model, measuring students' competency for each level and performing pseudo-guessing item analysis.

A confirmatory analysis was developed through Bayesian Network item level analysis and student level analysis to validate the hypothesized fraction learning progression empirically. The analysis has resulted in a learning progression with 7 validated levels of conceptual and 7 validated levels of procedural knowledge. About 48% of the students were grouped at very low levels of conceptual knowledge, indicating that the Indonesian curriculum is ineffective in developing a conceptual understanding of fractions beyond part-whole. In the procedural knowledge dimension, about 50% of the students reached the goals of the Indonesian curriculum at grade 7. However, the remaining students had difficulties with both additive and multiplicative fraction operations and often misapplied the algorithms for addition to multiplication.

There were substantial individual differences in the relationship between students' conceptual and procedural knowledge but some important dependencies between conceptual and procedural knowledge were also identified.

The present research is innovative in the area of fraction assessment research, because it has developed the first two-dimensional fraction learning progression based on conceptual and procedural knowledge and also because it has included aspects of fraction knowledge that were missing from previous assessments. The two-dimensional learning progression provided more accurate profiles of students' progression levels compared with previous research, thus making a significant contribution to research into fraction education. Finally, the development of Bayesian Networks Models has made a contribution to educational measurement research both in that it has validated a learning progression using item and student level analysis, and in that it has developed Bayesian Networks analyses of item difficulty, item discrimination, and pseudo-guessing.

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DECLARATION

I certify that this thesis does not incorporate without acknowledgment any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge and belief it does not contain any material previously published or written by another person except where due reference is made in the text.

Bakir Haryanto

ACKNOWLEDGEMENTS

I would like to acknowledge that this research was funded by Flinders University through a Flinders International Postgraduate Research Scholarship (FIPRS). I would like to express my deepest gratitude to my principal supervisor, Professor Stella Vosniadou, for her academic support, patience and availability whenever I needed it. Professor Vosniadou's supervision and guidance motivated me to keep going with this research. I would like also to express my gratitude to my co-supervisor, Dr. Darfiana Nur, for her guidance, especially with Bayesian modelling and analysis.

I would particularly like to thank Associate Professor David Curtis who supervised me in developing my research proposal and supported me throughout the first year of my candidature. I am also thankful to my research participants and everyone involved in conducting this study at the research site.

I would like to express my special thanks to Associate Professor Kelvin Gregory, supervisor of my Masters Degree in Educational Research, Assessment and Evaluation between 2008 and 2010 at Flinders University. Associate Professor Gregory inspired my interest in educational assessment and measurement and encouraged me to continue to study in this field at Flinders.

Finally, I would like to express my sincerest gratitude to my beloved family, my wife Ria, son Zaidan and daughter Hafiza for accompanying me during my study. All of you motivated me to keep going with my research and to finish my writing. To my parents, thank you for your prayers and encouragement.

CHAPTER 1 : INTRODUCTION

1.1 Aims of the research

The present research developed and validated an assessment instrument for students' progression in learning fractions. More specifically, the research (a) developed a two-dimensional learning progression for fractions based on two hypothesized knowledge dimensions: conceptual and procedural; and (b) validated the hypothesized model of the two-dimensional learning progression of fractions using Bayesian Network analysis.

1.2 Why fractions?

In mathematics learning, fractions, along with algebra and geometry, are an important domain of knowledge in secondary school mathematics (Wu, 2005). According to Torbeyns, Schneider, Xin, and Siegler (2015), it is a key factor underlying students' general mathematics achievement. Moreover, fractions, together with decimals, are commonly found outside mathematics, in fields such as economics, science, and psychology. Fraction knowledge can influence success in many professions (Lortie-Forgues, Tian, & Siegler, 2015). Despite its importance, research highlights that many students at secondary schools find it difficult to grasp rational number concepts (Moss and Case (1999). The difficulty in understanding fractions and their operations also goes beyond students in secondary school; it is, for example, identified in pre-service teachers and University students (Chinnappan & Forrester, 2014; Hanson & Hogan, 2000).

Much research has investigated students' difficulties with fractions (e.g. Durkin & Rittle-Johnson, 2015; Ni & Zhou, 2005; Robert S. Siegler, Thompson, & Schneider, 2011; Stafylidou & Vosniadou, 2004; Vamvakoussi, 2015; Vamvakoussi & Vosniadou, 2010; Van Dooren, Lehtinen, & Verschaffel, 2015). However, the research findings have not been translated into adequate assessments, especially formative assessments that can provide diagnostic information and improve teachers' understanding of the learning challenges students face in the development of fraction knowledge.

1.3. Why a two-dimensional learning progression?

One of the most important purposes of assessment is to provide diagnostic information about students' learning (Black & Wiliam, 1998; Pellegrino et al., 2001). Such information is crucial to generate effective feedback on learning. However, most current assessments provide limited diagnostic information about students' strengths and weaknesses in the domain of knowledge that is being assessed (Huff & Goodman, 2007). The failure of current assessments to provide adequate diagnostic information about learning has been referred to as the 'assessment crisis' by Richard J. Stiggins (2002).

Current assessments are typically developed based on Classical Test Theory (CTT) and Item Response Theory (IRT). These assessments focus on estimating students' general proficiency (Nichols, 1994). Based on such assessments, students can be ordered according to their levels of ability, usually against some curriculum standards. This ordering can inform summative assessment requirements and determine students' grades, or which students pass and which fail a course (de la Torre & Minchen, 2014). However, these assessments have limitations in generating diagnostic information about individual students' learning (de la Torre & Karelitz, 2009). They do not tell us why some students are failing and what cognitive challenges they face in their learning progress that influence their performance (Nichols, 1994).

Pellegrino et al. (2001) proposed that assessments can become more diagnostic if they are constructed on a cognitive model of learning, which describes students' representations of knowledge and the development of their competencies based on available theoretical frameworks and empirical research on students' learning. In such assessments, the cognitive model drives the assessment tasks and guides the interpretation of the students' responses. The results of assessments based on cognitive models can be diagnostic because they provide empirical evidence about the students' knowledge/skills vis-à-vis the cognitive model.

A learning progression[1] is a cognitive model that gives information about the level of students' learning against the specified progression (Pellegrino, 2014). This

[1] The term learning progression is similar to the term learning trajectory in terms of modelling the progression of student learning. However, the scope of a learning progression is wider than the learning

information reveals students' successes and difficulties in their learning journey (Berland & McNeill, 2010) and provides "intermediate goals, or stages" that can be used by teachers as a strategy to improve their instruction (Kane & Bejar, 2014, p. 118). By specifying the level of a student in the learning progression, the assessment provides meaningful information to teachers that can be used to guide classroom learning. In addition, the levels of learning can be used by students to guide their independent study. On a larger scale, the education system can use learning progressions to improve the design of the curriculum. The present research employs a learning progression as a cognitive model in developing an assessment framework for the learning of fractions.

Learning progressions usually provide unidimensional hierarchical-levels of students' learning (e.g. Briggs & Alonzo, 2009; Draney, 2009; West et al., 2012; Wilson, 2009b). Within these learning progression models, there has been little discussion so far about the need to introduce multiple hierarchical levels of progression in terms of the kinds of knowledge and skills that might underlie a given subject domain. For example, in mathematics, researchers have commonly argued that conceptual knowledge and procedural knowledge are two distinct key cognitive dimensions for mathematics proficiency (Crooks & Alibali, 2014; NCTM, 2000; Rittle-Johnson & Schneider, 2014). However, there has been no attempt to measure students' mathematical knowledge in these distinct dimensions and to understand how they interact with and influence each other.

The present research developed and validated a cognitive model of fraction learning progression based on the above-mentioned key knowledge dimensions of mathematical knowledge, namely conceptual and procedural knowledge of fractions. The development of a learning progression based on these two key dimensions of mathematical knowledge is a significant new step in assessment research and

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trajectory (Stevens, Shin, and Krajcik (2009). Indeed, the learning trajectory is a subset of a learning progression. A learning progression can be based on several learning trajectories (Rutstein, 2012). Learning progressions describe a common path in students' learning, while learning trajectories show the differences in students' paths in learning. The present resesarch focuses on learning progressions because they "provide the big picture of what is to be learned, support instructional planning, and act as a touchstone for formative assessment" (Heritage, 2008, p. 1).

represents an innovation in the development of learning progressions in general and of assessments of fraction knowledge in particular.

1.4. Why Bayesian Networks?

Another innovation of the present research is that it developed a measurement model based on Bayesian Networks to validate the two-dimensional learning progression. Two well-known measurement models exist in the literature to validate students' learning progressions, the Rasch Model (Draney, 2009; Wilmot, Schoenfeld, Wilson, Champney, & Zahner, 2011; Wilson, 2009b; Wilson & Cartensen, 2007) and Bayesian Networks (Rutstein, 2012; Jefrey Thomas Steedle, 2008; Jeffrey T Steedle & Shavelson, 2009; West et al., 2012). Bayesian Networks are preferable in this research because they model the conditional dependency between the various knowlegde within a learning progression. Moreover, Bayesian Networks approaches provide information about the uncertainties of the parameters being considered in terms of their (marginal) posterior distributions. Particularly in this research, the Bayesian Networks are used to identify the uncertainties of the parameters of items' levels and students' levels through their marginal distribution.

Bayesian Networks, which have been developed and implemented widely in educational measurement, provide a cognitive diagnostic measurement model (Mislevy, 1995; Mislevy et al., 2002; Mislevy & Almond, 1997; Mislevy, Almond, Yan, & Steinberg, 2000; Mislevy & Gitomer, 1996). They provide graphical probabilistic networks of the competencies underpinning students' performance (Almond, Mislevy, Steinberg, Yan, & Williamson, 2015). Using this approach in the learning progression model, the mastery and non-mastery of competencies at the proposed levels can be estimated, and the interconnection (inter-dependency) between competency components can be demonstrated.

Typically, a simple Bayesian Network with a single parameter (known as Bayesian Latent Class Analysis) is used to assess and validate learning progression models (Jefrey Thomas Steedle, 2008; Jeffrey T Steedle & Shavelson, 2009; West et al., 2010). In Bayesian Latent Class Analysis, the hierarchical levels of the learning progression model are captured by one latent variable, which has several categories referring to the levels.

Little attention has been given to modelling learning progressions with multiple latent variables, in which each level is represented by a single latent variable. The present research has expanded the Bayesian Networks model with a single latent variable into multiple latent variables based on the previous work by Rutstein (2012) and Jefrey Thomas Steedle (2008). Multiple latent variables are required to represent the hierarchical dependency between the levels. As a result, the measurement model developed from the Bayesian Networks model reflects the hierarchical assumption of the learning progression, which can lead to a more valid interpretation of the students' responses.

1.5 Participants and Design of the Research

The present research constructed a hypothetical learning progression of fractions and validated it empirically using Indonesian Junior High School students. The assessment triangle (the interconnection between cognition, observation, and interpretation in assessment) was used as the framework for the assessment design.

The research utilized a mixed methods design, consisting of two sequential studies, Study I (a qualitative study) and Study II (a quantitative study). Study I was carried out through cognitive interviews. The purpose of the interviews was to collect data to evaluate and revise the hypothetical model of the two-dimensional fraction learning progressions and corresponding items. Study II was a test of the revised fraction learning progression. Students' responses were analyzed using Bayesian Networks modelling to validate the learning progression. Figure 1.1 shows the mixed methods design used in this research.

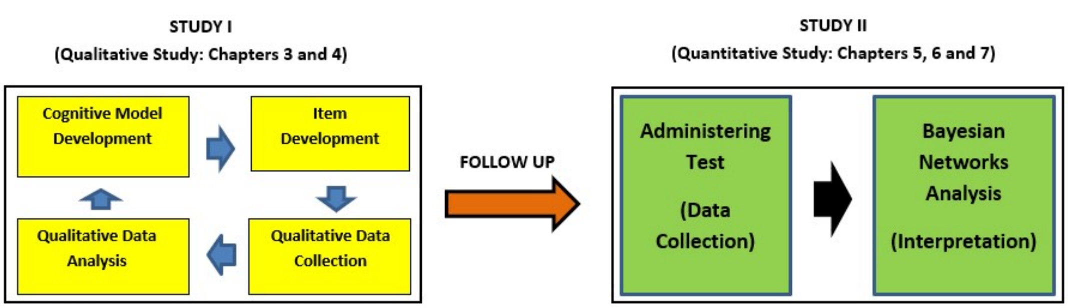


Figure 1.1 The mixed methods design of developing and validating the hypothetical learning progression model implemented in this research

1.6 Significance

The research produced a validated cognitive model of a two-dimensional fraction learning progression. This model captured the development of students' conceptual and procedural knowledge in learning fractions. The validated conceptual knowledge provided empirical evidence about the progression of students' understanding of the symbolic notation of fractions, while the validated procedural knowledge dimension showed the progression of students' knowledge of fraction operations. This is discussed further in Chapters 3, 4 and 6.

Another important result of the research was the development of two different measurement models based on Bayesian Networks. These measurement models provided information about uncertainties at both item and student levels. Model 1, which is known as Bayesian Latent Class Analysis, was developed based on a single latent variable. In this model, there was a latent variable with six categories corresponding to the six levels of the learning progression. Model 2 was developed based on a multiple latent variable measurement model. In Model 2, there were six latent variables corresponding to the six levels of the learning progression. These multiple latent variables were interrelated hierarchically to reflect the dependency between the competencies in each level in the learning progression model. The results showed that Model 2 had a better fit for measuring students' learning progressions than Model 1. Moreover, Model 2 had several properties which were superior to Model 1, such as diagnostic analytics, guessing analysis, and detection of the deviation of students' responses from the learning progression models. This is detailed further in Chapters 5 and 6.

The present research makes significant contributions to assessment, instruction and curriculum development. In relation to assessment, the research is significant because it further developed the cognitive element of the assessment triangle (Pellegrino et al., 2001). This cognitive model of two-dimensional learning progression can be used as a foundation to develop diagnostic assessment of conceptual and procedural knowledge, particularly in mathematics. Furthermore, the Bayesian Networks models developed in this study contribute to the development of models for measuring learning progressions, particularly in expanding the Bayesian Networks

Model from a single latent variable into multiple latent variables, and in developing item analysis using Bayesian Networks to validate learning progressions.

For instruction, the learning progression model is diagnostic because it provides a conceptual and procedural road map for teachers that can inform them of the learning difficulties students face at different levels of fraction learning. The learning challenges are investigated further in Chapter 7.

For curriculum development, the learning progression model developed in this research provides suggestions about how to develop and structure materials that can facilitate fraction learning across grades. Moreover, the learning progression model contributes to the potential development of curriculum materials that have a balance of conceptual and procedural elements. The current curriculum in Indonesia favours the procedural components of learning and teaching fractions. The implications of this research for curriculum development are discussed further in Chapter 7.

1.7 Structure of the Thesis

This thesis is divided into eight chapters. In Chapter 1, the aims and the background of the study are presented. The importance of proposing a two-dimensional knowledge of fraction learning progression to generate effective feedback on learning is discussed. Subsequently, the interconnected relationship between learning progression and assessment is described. The qualitative and quantitative approaches for validating the learning progression model within the framework of a mixed methods design are also reviewed. The chapter concludes with a definition of terms and an overview of the thesis.

In Chapter 2, through a review of the literature, the foundation and frameworks of assessment are examined. The role of cognitive models of learning progression in assessment is discussed. The fraction sub-constructs and the conceptual and procedural knowledge in learning fractions are then described. This discussion is extended in Chapter 3 by proposing the hypothetical model of fraction learning progression. Finally, the measurement model using Bayesian Networks is described. This explains the inference in Bayesian analysis related to the assessment framework, Bayes' theorem, and Bayesian Networks as a probabilistic network.

In Chapter 3, the theoretical foundation underlying the hypothetical learning progression is developed. The conceptual and procedural knowledge progressions of students in learning fractions are discussed. Next, the assessment tasks corresponding to the competencies of the learning progression model are detailed. Finally, the proposed model and the previous models of learning progressions are compared.

In Chapter 4, the results of the cognitive interviews are discussed. The findings are used to revise the levels of the learning progression model and the corresponding items. A deterministic approach is used to revise or validate the learning progression model, based on the results of the cognitive interviews.

In Chapter 5, the Bayesian Networks modelling that is used to measure and validate the learning progression model is presented. The specifications of Model 1 and Model 2 of Bayesian Networks are provided.

In Chapter 6, the validation of the progression levels using Bayesian Networks is analyzed. The item analysis and the student level analysis are used to validate the learning progression model.

In Chapter 7, the results from the Bayesian Networks analysis are discussed, in terms of conceptual and procedural dimensions and the relationship between them. The final discussion, implications for assessment, instruction and curriculum, and recommendations for future research are provided in Chapter 8.

1.8 Definition of Terms

The definition of terms, drawn from the literature review (Chapter 2) and applied in this research, is presented in Table 1.1.

Table 1.1 The definition of the terms used throughout the study

Term	Definition
Assessment triangle	The three interconnected elements underpinning effective assessment; cognition, observation, and interpretation
Bayesian Networks	A statistical model which generates a graphical probabilistic network of attributes underpinning students’ performance
Cognition	A model of learning which describes how students’ knowledge and skills are developed in the domain of interest

Competency	The combination of skills, abilities and knowledge which are required to to solve item tasks
Conceptual knowledge	The interconnected pieces of information about ideas and principles
Density	The property of fractions that there are unlimited numbers between two fractions
Fractions	A symbolic notation which is denoted as two numbers separated by the dash
Formative assessment	A type of assessment which is developed to diagnose students' successes and difficulties in the process of their learning
Interpretation	Inferences about students' cognition from the observational data
Item Level Inference	Items which were assigned at the given levels would be answered correctly by those students who were found to belong to this level or an upper level, but not by the students at lower levels
Learning Progression	A model of learning which describes developmental student understanding in a particular domain over time. It consists of hierarchical building blocks (levels) containing knowledge and skills that should be mastered sequentially in order to master more advanced concepts
Levels of Achievement	A hierarchical level of students' learning from naïve understanding to more sophisticated levels of learning
Measure	Understanding of the symbolic notation of fractions as a scale in the number line
Observation	Tasks which allow students to demonstrate their proficiency regarding the knowledge and skills defined in the cognition element
Part-whole relation	Understanding of the symbolic notation of fractions as representation of part and whole
Procedural Knowledge	Knowledge of the sequential steps or the algorithm to solve mathematical tasks
Progress variables	Essential concepts of the learning domain, as they are monitored across levels of progressions
Unbounded Infinity	Fractions are infinite numbers
Students' Level of Inference	The students who were assigned at a certain level in the learning progression would have sufficient competencies at that level and below, but not at the upper level(s)
Summative Assessment	A type of assessment which is developed to assess students' achievements after the process of instruction is completed

CHAPTER 2 : LITERATURE REVIEW

2.1 Introduction

The purpose of this chapter is to review the literature used to develop the rationale and framework of the present research. It is organized into two main sections: 1) Review of the literature on assessment and 2) Review of the literature on mathematics assessment of fraction learning.

In the first section, traditional assessments and assessments based on student cognitive models were compared. Subsequently, assessments based on cognitive models, including learning progressions, was discussed in greater detail. This sub-section also discussed the Bayesian Networks approach, which was used for modelling student learning progressions in the present research.

In the second section, the literature on fraction learning progressions was described, followed by a discussion of two dimensions of mathematical knowledge – conceptual and procedural – on the basis of which the learning progressions were developed in the present research.

2.2. Literature Review on Assessment

2.2.1 Comparison of Traditional Assessments and Assessments Based on Cognitive Models

According to Pellegrino et al. (2001), traditional assessments could be improved if they took into consideration cognitive models of student learning. Incorporating a cognitive model of learning in an assessment is also recommended by many other assessment experts (Embretson & Gorin, 2001; Mislevy, 1994c; Pellegrino et al., 2001; Pellegrino, Wilson, Koenig, & Beatty, 2014).

Traditional assessments have several differences from assessments based on cognitive models. First, traditional assessments are usually developed based on “logical taxonomies and content specifications” (Nichols, 1994, p. 577). The blueprint of such an assessment type is derived from a sample of the content and skill areas, which is typically drawn from the standards stated in the curriculum. As a result, the findings obtained from this type of assessment are limited to what “students know and can do” (Kane & Bejar, 2014, p. 119), by addressing only the list of competencies stated in the curriculum.

These assessments do not provide information about why students are succeeding or failing, or where their difficulties lie.

The second difference is in the use of formal statistical/psychometric models to draw inferences about students' proficiency. Such assessments are known as measurement models (Pellegrino et al., 2001). The measurement models in traditional types of assessment are used to produce general scores, which lie on the continuum scale of students' proficiency (de la Torre & Minchen, 2014). These scores are used for many different purposes such as "identifying a student's level of proficiency, differentiating passing from non-passing students, selecting candidates for a program, admitting students to a college, or determining the recipients of scholarships" (de la Torre & Minchen, 2014, p. 89). Although these general scores represent the general proficiency of the students, they do not provide information about students' strengths and weaknesses in their learning, which is important for diagnostic purposes (Kane & Bejar, 2014). Diagnostic information is important to improve teaching and learning, which is the main objective of the education reform (Pellegrino et al., 2001).

In contrast, assessments based on a cognitive model of learning utilize information about how students "represent the knowledge and develop competence in the domain" (Pellegrino et al., 2001, p. 178). The cognitive model is formulated from the results of empirical research on learning in specific areas of expertise. This cognitive model guides the selection of the assessment tasks and specifies the way inferences can be drawn from students' responses. Proposing a cognitive model of learning as a basis for developing an assessment is the critical difference between the new type of assessments and the traditional type of assessments discussed above (Pellegrino, 2014; Pellegrino et al., 2001).

Adopting a cognitive model of learning into assessment design enables the assessment to produce diagnostic information about the students' progression and their learning difficulties, which is consistent with the notion of formative assessment. In contrast, traditional types of assessment are used mainly for summative assessment and accountability purposes (de la Torre & Minchen, 2014). Formative assessment is a type of assessment that has been developed to diagnose students' success and difficulties in the process of their learning, while summative assessment was developed to assess students' achievement after the process of instruction is completed (Nitko & Brookhart, 2007).

Assessments based on cognitive models can provide important information for instruction for several reasons. First, the design of the assessment is based on empirical research of how students develop the learning encapsulated in the cognitive model. This cognitive model provides information about the levels of progression from novice to expert learning. The assessment can then be used to draw inferences about the students' levels on such a progression of learning. The information about the levels of students' learning can be used by teachers to create more effective instruction.

Furthermore, the development of the measurement models in the new foundation of assessment aim to measure specific competencies or skills in the learning domain (de la Torre & Minchen, 2014). Information about students' domain-specific skills and competencies (depending on the purpose of assessment) can inform educators' decisions so they can improve their teaching practices and hence student learning. This is an important difference from traditional assessments.

In summary, assessments based on cognitive models are grounded in empirical evidence about how students learn. These assessments have certain advantages over traditional types of assessment because they can provide information about where students are in their learning progression in specific subject matter areas and provide important diagnostic information to educators, which is necessary in order to produce effective feedback (Black and Wiliam (1998). The present research adopts the framework of cognitive assessments, to be discussed in more detail in the sections that follow.

2.2.2. Assessments based on cognitive models

The assessment triangle describes the interconnected elements of cognition, observation, and interpretation for crafting an effective assessment based on a cognitive model (Figure 2.1; Pellegrino et al. (2001). The cognitive model is the cornerstone of the assessment triangle, on the basis of which the observation and interpretation components are developed. Observation refers to the tasks or situations which allow students to demonstrate their proficiency in the knowledge and skills defined in the cognition element. The tasks that are created in the assessment provide information about what students know and can do. These tasks should be created to reflect the knowledge and skills specified in the cognition element in order to obtain valid assessment results.

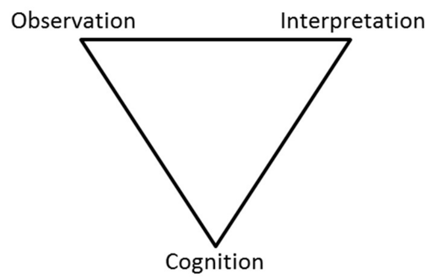


Figure 2.1 Assessment Triangle (Pellegrino, Chudowsky, & Glaser, 2001, p. 44)

Interpretation consists of technical tools and methods that make it possible to draw inferences about students' cognition from the observational data (student responses to the observation tasks). The interpretation component links the students' responses produced from observation to the knowledge formulated in the cognition component. In many cases, measurement models are used to draw inferences about students' knowledge (specified by the cognition component), based on their responses to the tasks (specified by the observation component).

According to Pellegrino et al. (2001) assessment is "a process of drawing reasonable inferences about what students know on the basis of evidence derived from observations"(Pellegrino et al., 2001, p. 112). What students know is organized in the cognition component; the evidence is obtained from the tasks specified by the observation component; drawing inferences from the evidence is represented by the interpretation component.

The following sections discuss the three elements of the assessment triangle as a framework for the present study in more detail.

2.2.2.1 Cognitive Models

Two broad categories of cognitive models have been discussed in the literature. In the first category, a cognitive model is thought to represent "the knowledge, processes and strategies" required to solve the given assessment item tasks (Gierl, Wang, & Zhou, 2008, p. 5). This type of cognitive model is expressed in lists of "attributes" which show the content knowledge, the process, and the skills underlying students' performance in an area of knowledge (Tatsuoka, Corter, & Tatsuoka, 2004, p. 901).

Table 2.1, presents a list of the content, process and skill attributes that are thought to be associated with the mathematics items of the TIMMS-R assessment (Tatsuoka et al. (2004). Figure 2.2 shows an example from a geometry item in this task. The attributes required to answer this item (from Table 2.1) are the following: C4 (basic concepts and operations in two-dimensional geometry), S3 (using figures, tables, charts, and graphs), S5 (evaluate/verify/check options, P3 Judgmental applications of knowledge in arithmetic and geometry), P5 (Logical reasoning—including case reasoning, deductive thinking skills, if-then, necessary and sufficient, generalization skills), P7 (generating, visualizing, and reading figures and graphs), and P9 (management of data and procedures).

Table 2.1 The content, processes and the skills underlying the mathematics items of TIMMS-R 1999 (adopted from Tatsuoka et al., 2004, p. 907)

Content attributes	
C1	Basic concepts and operations in whole numbers and integers
C2	Basic concepts and operations in fractions and decimals
C3	Basic concepts and operations in elementary algebra
C4	Basic concepts and operations in two-dimensional geometry
C5	Data, probability, and basic statistics
C6	Measuring or estimating: length, time, angle, temperature, etc.
Process attributes	
P1	Translate/formulate equations and expressions to solve a problem
P2	Computational applications of knowledge in arithmetic and geometry
P3	Judgmental applications of knowledge in arithmetic and geometry
P4	Applying rules in algebra
P5	Logical reasoning—includes case reasoning, deductive thinking skills, if-then, necessary and sufficient, generalization skills
P6	Problem search; analytic thinking, problem restructuring; inductive thinking
P7	Generating, visualizing, and reading figures and graphs
P8	Applying and evaluating mathematical correctness
P9	Management of data and procedures
P10	Quantitative and logical reading
Skill (item type) attributes	
S1	Unit conversion
S2	Apply number properties and relationships; number sense/number line

S3	Using figures, tables, charts, and graphs
S4	Approximation/estimation
S5	Evaluate/verify/check options
S6	Patterns and relationships (inductive thinking skills)
S7	Using proportional reasoning
S8	Solving novel or unfamiliar problems
S9	Comparison of two/or more entities
S10	Open-ended items, in which an answer is not given
S11	Understanding verbally posed questions

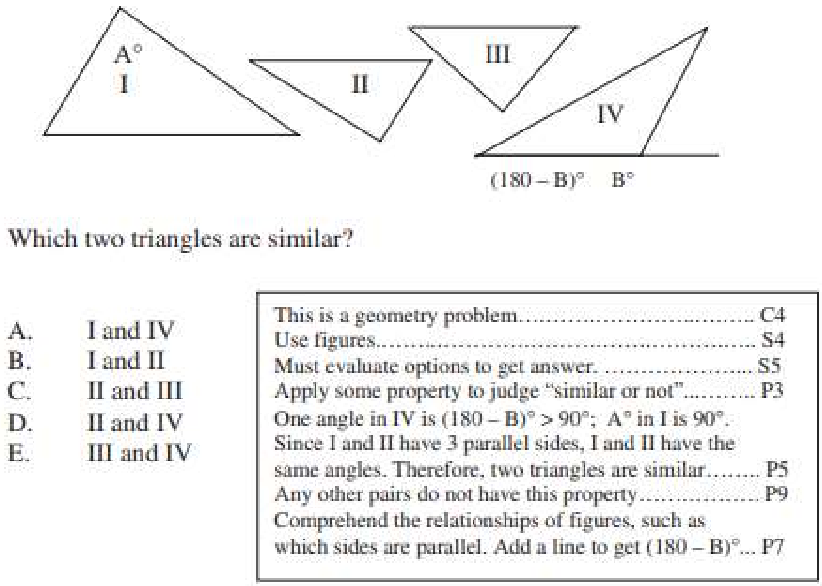


Figure 2.2 The sample of an item taken from TIMMS-R 1999 (adopted from Tatsuoka et al., 2004, p. 909)

The category of cognitive models discussed by Tatsuoka et al. (2004) does not assume that knowledge and skills are organised in hierarchical levels. This implies that the development of those attributes can be independent of each other, meaning that a particular knowledge or skill does not depend on the presence of other forms of knowledge or skills. Contrary to this, but still in the first category of cognitive models, Leighton, Gierl, and Hunka (2002) proposed an attribute hierarchy to describe the students' knowledge and skills. This hierarchy shows "the psychological ordering among the attributes required to solve a test problem" (Leighton et al., 2002, p. 5). Figure 2.3 shows four possible attribute hierarchies: linear, convergent, divergent, and unstructured. As can be seen, in the linear hierarchy presented in Figure 2.3, Column A, attributes 1, 2 and 3, are prerequisites for attribute 4, and attributes 1, 2, 3 and 4 are prerequisites to attribute 5 (Leighton et al., 2002).

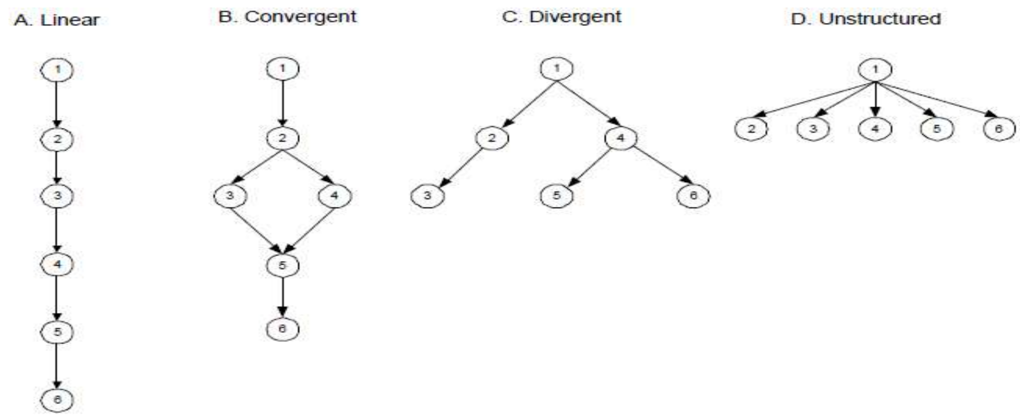


Figure 2.3 The attribute hierarchy in the cognitive model proposed by Leighton et al. (2002, p. 45)

The second category of cognitive models provides a more detailed learning progression of students' knowledge when compared with the first category. The term "learning progression" refers to a cognitive model that describes the development of student understanding in a particular subject matter area over time, based on the available empirical evidence (NRC, 2007). It consists of hierarchical building blocks (levels) that describe the knowledge and skills that are mastered sequentially as the students progress in their learning (Popham, 2007). Furtak, Roberts, Morrison, Henson, and Malone (2010) pointed out that learning progressions are hypothetical pathways for learning and provide "a road map" for instruction. Consequently, teachers can identify student successes and difficulties over time (Berland & McNeill, 2010), and they can create effective strategies to improve teaching and learning (Kane & Bejar, 2014).

According to Pellegrino (2014), a learning progression should consist of five key elements:

1) "target performances" or "learning goals" which are defined in the highest level of the learning progression model;

2) "progress variables" which refer to indicators of the essential concepts in the learning domain, monitored across levels of progressions;

3) "levels of achievement" which describe students' learning from naïve understanding to a sophisticated level of learning;

4) "learning performance" which describes what students can do at every level of the progression; and

5) “assessment” which is required to monitor the progression of students’ learning, based on their performance in the item tasks.

The hypothetical sequence of the progression of student learning is developed based on empirical research evidence. This hypothetical sequence needs to be further tested and validated empirically to generate a learning progression model which captures accurately the development of students’ understanding. Hence, a validated learning progression can provide a common road map of how students achieve the learning goals in a given subject matter area, which is supported by both theoretical and empirical research on learning and assessment (Popham, 2007).

An extensive discussion of learning progression as a cognitive model of learning in assessment can be found in the Berkeley Evaluation and Assessment Research (BEAR) Assessment System (Wilson & Cartensen, 2007; Wilson & Scalise, 2006). The concept of construct maps is a key concept in the development of assessments based on learning progressions. Construct maps show “…qualitatively different levels of performance focusing on one characteristic” (Wilson, 2009b, p. 3), which define what students’ can do and know at each level (Draney, 2009). Several construct maps can be joined together to form a learning progression, which specifies the competencies being measured in great detail. Table 2.2 presents an example of two construct maps, parts of the learning progression in the domain of the Carbon Cycle; learning developed under the BEAR Assessment system. The construct maps in Table 2.2 show the increase in the sophistication of learning across the levels in the two topics of interest, which are: 1) the Hierarchy of the Carbon Cycle System, and 2) the Material Kind and Properties of Matter.

Table 2.2 Constructs maps of the carbon cycle in the topics of hierarchy system and material kind and properties of matter

Level	Hierarchy of Carbon Cycle System	Material Kind & Properties of matter
7	Describes movements of matter through multiple processes at multiple scales	Correctly characterizes products of processes in terms of how they affect organic carbon compounds
6	Traces elements or atoms through a single life process, relating multiple scales	Correctly identifies reactants and products of a single life process
5	Describes movements of matter in simple chemical changes at atomic molecular scale. (not just events)	Correctly identifies reactants and products in simple chemical changes
4	Describes matter movement at macroscopic scale. (not just events)	Correctly identifies reactants and products in simple chemical changes. Identifies solids, liquids, but not gases involved in chemical or physical changes
3	Attention to hidden mechanism. Describes events as changes in materials.	Pays attention to hidden mechanisms but cannot identify any material kinds.
2	Describes changes as events (at macroscopic scale)	Identifies changes by using common sense of natural phenomena, but not as changes in materials
1	Egocentric/naturalistic reasoning: respondents use human analogy to explain the changes in materials	Egocentric/naturalistic reasoning: respondents use human analogy to explain the changes in materials

From the discussion of the two categories of cognitive models discussed above, it can be seen that the first category developed by Tatsuoka et al. (2004) and Leighton et al. (2002) provides “a fine grain size”, which shows in the details the kinds of attributes assumed to underlie the students’ performance when they solve a given test item. The diagnostics analysis can be performed by identifying students’ strengths and weaknesses in terms of the attributes involved. This information can inform teachers, who can then focus their instructions to specific important attributes which might be difficult for students to grasp. The second category of cognitive models provides less information about the assumed knowledge, processes and skills underlying students’ performance in solving a given test item. However, because they are expressed as learning progressions, they provide rich information about the level of development of students’ learning in a given subject matter area. Such information can be used as a framework to assess students’ progression in learning. As a result, the feedback and reports generated from this second category of assessment can inform the interested parties "where learners are in their learning at the time assessment and, ideally, what progress they have made over time", which is the fundamental goal of assessment (Masters, 2013, p. 8). This feedback and associated reports can be used by teachers to improve their instruction or by other agencies to improve the curriculum.

Pellegrino et al. (2001) highlighted that assessment, curriculum, and instruction should be tightly connected, and a cognitive model of learning progression can facilitate the interconnection between them (Wilson, 2009a). Ideally, assessment should assess the materials that are taught to the students through programs of instruction; and that instruction should teach the content knowledge specified in the curriculum. However, many factors can interfere with this ideal interconnection. For example, high-stake assessments can influence the teachers to focus on teaching the items in the test, not the materials in the curriculum. In such a situation, Pellegrino et al. (2001) suggests that teachers use a cognitive model of learning which provides a, "shared knowledge base about cognition and learning in the subject domain" (p. 53) that can be used to develop curriculum, assessment, and instruction. As a result, the curriculum, instruction, and assessment approach the same learning goals and can reinforce each other.

2.2.2.2 Observation

The observation component of the assessment triangle "represents a description or set of specifications for assessment tasks that will elicit illuminating responses from students" (Pellegrino et al., 2001). It includes the activities of constructing the tasks and collecting and summarizing students' responses (Shavelson, Ruiz-Primo, Li, & Ayala, 2003). These responses are the source of evidence about the students' knowledge that is being assessed. The observation component provides an indirect way to measure students' knowledge, which is not directly observable from the students' brain (Pellegrino et al., 2001). In order to draw valid inferences about the students' knowledge from the observation, the assessment tasks should be carefully designed in order to represent the competencies specified in the cognition model.

Nitko and Brookhart (2007) highlighted three fundamental principles in creating assessment tasks. First, the assessment tasks should focus on important learning targets. In the context of assessments based on learning progressions, learning targets are the competencies specified in each level of the learning progression. This implies that assessment should avoid “trivial performances and/or minor points of content” (Nitko & Brookhart, 2007, p. 133). Second, the assessment tasks should be created to obtain the competencies related to the learning targets only, meaning that if the students have already achieved the learning targets, then they should be able to perform the assessment tasks correctly. Conversely, if they have not mastered the learning targets, then their response errors can be used to examine their weaknesses in learning. Finally, the assessment tasks should be crafted by avoiding situations that can prevent students from demonstrating their abilities. For example, poor wording of questions or unclear diagrams can lead students who have the required competencies to answer the questions incorrectly.

In crafting assessment tasks based on a learning progression, the items in the tasks should have a good discriminant power among the discrete-levels. This means that the items in the tasks at a particular level should be designed to be answered correctly only by the students at that level or higher levels (West et al., 2010). The students at lower levels should not be able to solve these items successfully. The items which can be used to differentiate students across the different levels are essential to support the empirical validation of the progression model.

Another issue in designing assessment tasks in a learning progression is the dependency among the items. As discussed, the learning progression model describes the development of students’ competencies through the ordered levels. It indicates that the competencies at a particular level are developed based on competencies at the lower levels. To enhance the validity of this construct, the items in the tasks should be created to reflect the dependency between the levels. This means that the items in the tasks should be interrelated because in order to solve a task item correctly at a particular level, the students are expected to solve the items at the lower level.

Different types of task are required to assess students’ conceptual and procedural knowledge of mathematics. For assessing conceptual knowledge, mathematical concepts can be expressed using pictorial representations (Bayazit & Aksoy, 2010). For example, a shaded circle (a pie diagram) can be used as a cognitive tool to represent fractions. Bayazit and Aksoy (2010, p. 94) highlighted that, “an image of a mathematical idea cannot be separated from the concept itself; and it should be regarded as an essential part of thinking" (Bayazit & Aksoy, 2010, p. 94). The symbolic notations in mathematics (i.e., a fraction symbol) are another form of representation that can represent mathematical ideas (Hiebert, 1988). For example, a symbolic notation is used in mathematics to represent fractions as two numbers (the numerator and the denominator), separated by the dash. Kieren (1980) highlighted that the symbolic notation of fractions can have different meanings, for example, fractions as part-whole, measure, operator, quotient, and ratio.

Consistent with Bayazit and Aksoy (2010) and Hiebert (1988), Byrnes and Wasik (1991) identified three types of tasks that can be used to assess students' conceptual knowledge. The first type of task uses pictorial representations. In this task, students are given a diagramatic representation of a fraction (e.g., a pie diagram) and are asked to select the appropriate numerical, symbolic representation for this diagram. The second task is simple isomorphic items. In this task, students are given a diagram which represents a fraction and then they are asked to match this diagram to another diagram which has the same value of the fraction. The third task is fraction ordering (Byrnes and Wasik (1991). In this task, students' understanding of the value of the fraction symbol is tested by giving them two or more fractions (using numerical, symbolic representation) and asking them to select which fraction is the greater.

In order to assess students' procedural knowledge, tasks involving arithmetic computation including addition, subtraction, multiplication and division can be used (Rittle-Johnson, Siegler, & Alibali, 2001). The purpose of these tasks is to assess students' knowledge of the accuracy of the algorithms required to perform the computations (Rittle-Johnson & Schneider, 2014). Rittle-Johnson et al. (2001) also argued that a conceptual task becomes a procedural task if the students have repeated experience of solving this task that has resulted in making this a "routine" task (p. 349).

2.2.2.3 Interpretation

The interpretation component of the assessment triangle allows inferences about the students' knowledge to be drawn from their responses. In psychometric terms, the interpretation element of the assessment triangle refers to measurement models (Pellegrino et al., 2001). A Bayesian Inference Network (commonly termed a Bayesian

Network) is a prominent measurement model that has been developed and implemented widely for educational measurements (see Mislevy, 1995; Mislevy et al., 2002; Mislevy & Almond, 1997; Mislevy et al., 2000; Mislevy & Gitomer, 1996). Bayesian Networks are capable of handling cognitive models with complex dependencies among the competencies (Almond et al., 2015). This feature is relevant to the present research, which deals with fraction learning.

The next sections discuss Bayesian Networks in detail, including Bayesian inference, Bayesian Measurement Models, and Bayesian Network Models.

2.2.2.3.1 Bayesian Inference in Educational Assessment

According to Mislevy (1994b), inference is "reasoning from what we know and what we observe to explanations, conclusions, or predictions" (p.2). In practice, the information that is received is "typically incomplete, inconclusive, amenable to more than one explanation" (Mislevy, 1994b, p. 2). Particularly, in educational testing, students' responses can be considered as incomplete evidence and contain some degree of uncertainty.

The source of uncertainty in educational testing comes from at least two sources. First, uncertainty exists about measurement errors. For example, students have a competency in a particular item but they can slip in answering the item, or students do not have competency in a particular item but can correctly guess the answer (Almond et al., 2015; Nichols, Chipman, & Brennan, 1995). Second, there may be uncertainty because of the limited number of items given to students. The items given to students are a representation from a hypothetical item pool (Lord, 1965; Osburn, 1968). Assessing students' competency based on a limited selection of items causes some degree of uncertainty in the inference regarding their competencies.

Bayesian inference refers to inference based on Bayes' theorem, which takes into account uncertainty and incomplete evidence using a probabilistic approach. Bayes' theorem describes how to update the probabilities of parameters or hypotheses conditional on data or evidence. Bayesian inference provides a range of possible estimates about the event/parameter being investigated. It is a powerful method that gives "a guiding principle for building and reasoning about complex models, and provides correct solutions to problems that were not tractable under the classical approach" (Almond et al., 2015, p. 62).

The fundamental idea of Bayes theorem is based on a conditional probability of events. The notion of a conditional probability is about changing or updating beliefs. A belief is updated if there is new information related to the event. For example, if there are two related events A and B, and information about B is received, the theorem is concerned with how beliefs change about A after receiving this information. This change is denoted as $P(A|B)$.

In educational settings, the information about students' competency is assessed based on available information from the students' responses to a particular test. In this research, competency is defined as "the combination of skills, abilities, and knowledge needed to perform a specific task" (Jones & Voorhees, 2002, p. 7). Through a conditional probability, the relationship between students' competency (θ) and their responses (X) can be denoted as $P(X|\theta)$, which means that the students' correct/incorrect answers (X) are conditional or dependent on their competency (θ). The conditional probability of $P(X|\theta)$ reflects deductive reasoning because it highlights that the students' competency (cause) influences their correct/incorrect response (effect) (the reasoning goes from cause to effect). In contrast, the inference of students' competency based on their correct/incorrect response reflects an inductive reasoning (the reasoning goes from effect to cause). These deductive and inductive reasonings are illustrated using a graphical representation in Figure 2.4

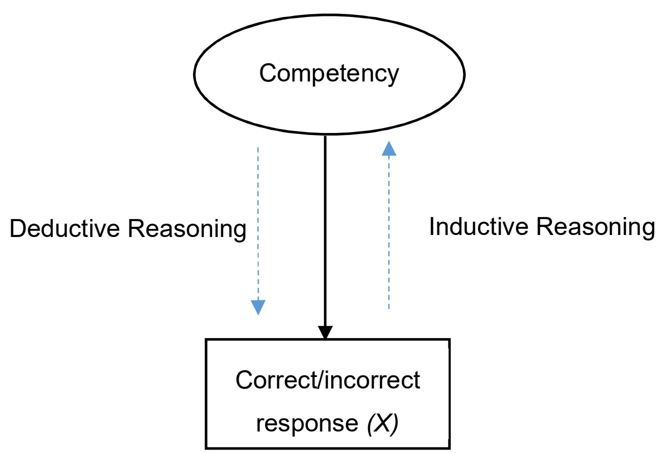


Figure 2.4 The relationship between students' knowledge/skills and their response to a particular item

The deductive reasoning is essential to structure the conditional probabilities in Bayesian inference. The conditional probabilities $P(X|\theta)$ reflect the cause – effect relationship among a set of variables that should be built, based on an available theory in the field or based on the findings of previous research. In other words, building Bayesian models should be driven theoretically (Mislevy, 1994a).

The Bayes theorem provides a reverse direction of the deductive reasoning discussed above. Bayes' theorem facilitates inference about students' competency given their responses on the test $P(\theta|X)$. Thus, the Bayes theorem facilitates inductive reasoning, in which the direction of the reasoning flows from effects to plausible causes (Mislevy, 1994a).

Within the Bayesian framework, deductive and inductive reasoning are incorporated in the assessment process, which is consistent with the assessment triangle proposed by Pellegrino et al. (2001). In the assessment triangle, the observation component should be developed based on the competency specified in the cognitive model. This shows the deductive reasoning because the direction of inference is developed from cause to effect, i.e., from the competency to the students' responses. Students who have the required competencies should be able to respond correctly to the assessment tasks, while students who do not have them should not be able to respond correctly to those tasks.

However, the interpretation component, which shows that the inference process is from the students' responses to the competency, demonstrates an inductive inference. This is consistent with the fundamental idea of assessment as "reasoning from evidence" stated by Pellegrino et al. (2001). This idea suggests that the inference about students' knowledge should be drawn from the data generated from their responses. The direction of the reasoning from the students' responses to what students know is an inductive inference.

Because assessment as reasoning from evidence is an inductive inference, it therefore contains uncertainties. Eysenck and Keane (2010) highlighted that, "a key feature of inductive reasoning is that the conclusions of inductively valid arguments are probably (but not necessarily) true", (p.533). Bayesian inference facilitates this inductive inference using probabilistic models to take into account the uncertainties in the inductive reasoning process. The inference offers the right direction on the reasoning process of assessment in terms of assessment as reasoning from evidence.

In summary, Bayesian Inference provides a framework of reasoning for building a complex model in assessment. Deductive reasoning guides the researcher to develop a hypothetical model and item tasks, which are theoretically driven. Meanwhile, inductive reasoning guides the researcher to draw inferences probabilistically about the students' knowledge based on the observed data. This framework is consistent with the assessment triangle proposed by Pellegrino et al. (2001), which enhances the effectiveness of the assessment design in this study.

The next section provides more technical detail about Bayesian modelling in the context of educational measurement.

2.2.2.3.2 Bayesian Measurement Models

According to West et al. (2012), modern measurement models rely on two essential properties: latent variables and the use of probabilistic models. Latent variables are unobservable variables, which represent the construction of the students' knowledge. The probabilistic models provide a formal mathematical process to measure the latent variables based on the available observed data.

Probabilistic models are broadly classified into two distinct approaches: Bayesian and frequentist approaches. In short, these two approaches differ in two basic ways. First, a Bayesian approach treats parameters in the models as random variables, while a frequentist approach treats parameters as fixed unknown quantities. Second, a Bayesian approach includes prior information in the estimation parameters, while a frequentist approach does not.

The differences between Bayesian and frequentist approaches have implications for modelling the observed and latent variables. In frequentist approach, the relation between observed variables and the latent variables is modelled through a conditional probability of the observed variables given the latent variables which is expressed as $P(X|\theta)$. This conditional probability shows a deductive reasoning which goes from the latent variables (cause) to the observed variables (effect) as discussed before in Section 2.2.2.3.1.

The estimation of the latent variable θ is performed through maximizing the likelihood function of $(X|\theta)$. In this case, the observed variables X are treated as random and the latent variables θ are treated as unknown fixed quantities. The latent variables θ are estimated through repeated samples of the experiment (Almond et al., 2015). This estimation shows an inductive inference process in a frequentist approach because the logic of inference goes from specific events, i.e., random quantities of data X to general i.e., fixed (constant) quantities of θ.

Bayesian approaches also use the likelihood function in conjunction with the prior θ, $P(\theta)$. However, Bayesian approaches treat latent variables θ as random quantities that reflect the uncertainties of the interest variables being estimated (e.g., students' competency).

The likelihood function $P(X|\theta)$, expresses the researcher's belief about students' competency (θ) computed from their responses on the given items. The plot in Figure 2.5 is an example of the likelihood function generated from 10 items where the number of correct answers is 8. This figure follows the illustration of the likelihood function in Almond et al. (2015). The results show that the likelihood function gives a range of θ that determines which value of θ is likely to be responsible for the number of correct responses. The likelihood function is maximised at 0.8 from the range of 0.4-1.0 of θ. This maximised value was the estimate of the student's θ.

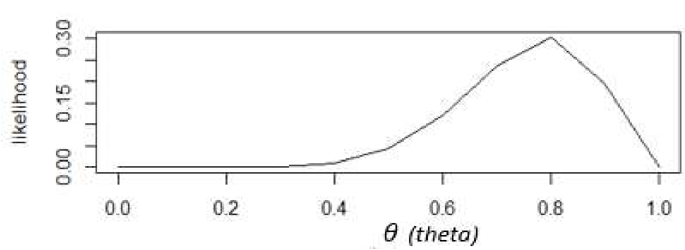


Figure 2.5 The likelihood function for $\boldsymbol{\theta}$ generated from 8 correct responses out of 10 items (Adapted from Almond et al., 2015, p. 65)

In contrast with frequentist approaches, Bayesian approaches draw inferences about the latent variables θ from a posterior probability. This posterior probability is generated from the conditional probability of the latent variables θ given the data *X*, $P(\theta|X)$. Using Bayesian theorem, $P(\theta|X)$ is calculated as follows:

$$P(\theta|X) = P(X|\theta)\, P(\theta)/P(X). \qquad (2.1)$$

In the context of this research, $P(\theta|X)$ is the probability of students' competency conditional on their responses; this is called the posterior probability. $P(X|\theta)$ is the probability of students' responses (correct/incorrect) conditional on their competency; this is known as the likelihood function. $P(\theta)$ is the prior probability which is "the distribution specified by the analyst to possibly reflect substantive, *a priori* knowledge, beliefs, or assumptions about the parameters" (Levy & Mislevy, 2016, p. 27). Moreover, $P(X)$ is the probability of the observed data unconditional with the model parameters, which is known as a normalizing constant. Because the normalizing constant is unconditional with the interest parameters, it is usually ignored in the calculation (Levy & Mislevy, 2016). As a result, the concern around Bayesian modelling is centred on the likelihood, the prior and the posterior probability. Hence, equation (2.1) can be presented as follows.

$$P(\theta|X) \infty P(X|\theta)\, P(\theta). \qquad (2.2)$$

From equation (2.2), it can be seen that the posterior density depends on the likelihood and the prior. There are two types of prior in Bayesian modelling: the informative and the non-informative prior. The informative prior can obtained from experts' information or their prior knowledge about the parameter being studied (Garthwaite, Kadane, & O'Hagan, 2005). On the other hand, the non-informative prior is, "a prior with minimal influence on the inference" (Syversveen, 1998, p. 1), for example, by assuming that a parameter of θ follows a uniform distribution.

The informative prior is the subjective part of the Bayesian inference, which often leads to a central criticism that the inference is not objective enough. Indeed, this criticism originally comes from the debate between objectivist and subjectivist perspectives in defining probability. Objectivism interprets probability "as real-world attributes of the events they refer to, unrelated to and unaffected by the extent of our knowledge" (Cowell, Dawid, Lauritzen, & Spiegelhalter, 1999, p. 12). In contrast, subjectivism considers that probability is "a numerical measure of a particular person's subjective degree of belief in A, with probability 1 representing certain belief in the truth of A, and probability 0 expressing certainty that A is false"(Cowell et al., 1999, p. 12). Moreover Cowell, Dawid, Lauritzen, and Spiegelhalter (2006) state that objectivists calculate probability from repeated experiments (events), while subjectivists estimate the probability from "singular propositions", which is opposite to the repeated experiments posed by objectivists (Cowell et al., 1999, p. 11).

Responding to the debate between subjectivists and objectivists in defining probability, Almond et al. (2015) argued that probability lies between the subjectivists' and the objectivists' perspectives. They defined probability as "representing a state of information about an unknown event" (p.46). Using this definition, the notion of objectivism and subjectivism can be accommodated when implementing the Bayesian inference. The prior probability can come from the subjective information, but then it is updated through the likelihood function that may come from the objective data. Hence, the posterior probability as the objective inference in a Bayesian approach can be objective enough in stating students' knowledge when the evidence from the students' performance is used to update the subjective probability of the prior.

The conditional probability and the prior in the Bayesian modelling discussed above are important in building the measurement model of learning progression in this study. Let X be the students' responses and θ the level of the students in the learning progression model. The likelihood $P(X|\theta)$ can be interpreted as the probability of the students' answers X (e.g. correct or incorrect responses), dependent on the students' level θ. In this case, the researcher might assume that the students at a particular level and above have a high probability of answering the items, while the students at the lower levels may have a low probability. This assumption is expressed through the prior $P(X|\theta)$ by setting the value of the prior low enough for the students at the lower levels and high enough for the students at the level and above the level.

The conditional probability of the likelihood $P(X|\theta)$ can be interpreted as the probability of the students' answers X (e.g. correct or incorrect responses), dependent on the students' level θ. In this case, the researcher might assume that the students at a particular level and above have a high probability of answering the items, while the students at the lower levels may have a low probability. This assumption is expressed through the prior $P(\theta)$ by setting the value of $P(\theta)$ low enough for the students at the lower levels and high enough for the students at the level and above the level.

The modelling discussed above allows the researcher to make two types of inferences: the item level and the students' level inferences (West et al., 2010; West et al., 2012). Item level inference is the inference that items at a certain level would be answered correctly by the students at that level or the upper level, whereas the students at the levels below would not answer these items successfully. The students' level inference is the inference that a student at a certain level would have sufficient competency at that level and below, but would not have enough competency at the upper level(s). These two types of inferences are the advantage of using Bayesian modelling, which was used to validate the learning progression model in the present study. This advantage is not found in frequentist approaches.

The next section introduces Bayesian Networks, which are specified in greater detail in Chapter 5, for measuring students' learning progression of fractions in the present research.

2.2.2.3.3 Bayesian Networks

Bayesian Networks combine Bayesian measurement models with graph theory. They provide a graphical probabilistic network of competencies which underpin students' performances (observed variables). Using Bayesian Networks, the interconnection (inter-dependency) among competencies in complex cognition models can be demonstrated and presented in (probabilistic) profiles (Mislevy, 1994b; Mislevy et al., 2000; West et al., 2012).

The statistical model of Bayesian Networks is constructed based on the joint distribution of random variables, by specifying the conditional distribution to be recursive (Levy & Mislevy, 2016; Mislevy, 1994b). The recursive property refers to the joint distribution that "can be expressed as a product of a distribution for the first variable, a distribution for the second variable conditional on the first, a distribution for the third variable conditional on the first and second, and so on" (Levy & Mislevy, 2016, p. 345). The recursive representation of the joint probability for Bayesian Networks is formulated as follows:

$$p(X_1, \dots, X_n) = p((X_n|X_{n-1}, \dots, X_1) \dots p(X_2|X_1)p(X_1) \qquad (2.3)$$
$$= \prod_{j=1}^{n} p\left(X_j \middle| X_{j-1}, \dots, X_1\right),$$

whereby, $X_1 \dots X_n$ is a set of *n* random variables (Mislevy, 1994b). The term *random variables* in this research refers to both latent and observables variables. Latent variables are variables indirectly observed, such as students' progression levels. Observed variables are variables that refer to directly observed behaviour, such as students' responses to a particular item.

Bayesian Networks are constructed based on a Directed Acyclic Graph (DAG), which corresponds to the latent and observed variables structured in equation (2.3). The graph is "directed", which means that each variable has a conditional relationship with the other. "Acyclic" means that the conditional relationship never goes back to the variable itself. For example, there are four random variables X_1, X_2, X_3 and X_4, where X_1 is conditional on X_2; X_2 is conditional on X_3; and X_3 is conditional on X_4. The acyclic property prohibits X_4 from being conditional on X_1.

The graph in Bayesian Networks consists of nodes and edges. The nodes represent the categorical variable in the model, while the edges are the arrows, which represent conditional relationships among variables. The nodes from which the arrows are originated are called "the parent", while the nodes to which the arrows are directed are "the child". The child node is a node which is conditional on the parent (Mislevy, 1994b). Therefore the equation 2.3 can be written as follows (Schwarz, Xie, & Yao, 2005):

$$p(X_1, \dots, X_n) = \prod_{j=1}^{n} p\left(X_j \middle| \text{"parents" of } X_j\right) \qquad (2.4)$$

The multivariate structures of Bayesian Networks in the DAG are estimated based on a conditional independence property. Mislevy (1994b) defined conditional independence as "one subset of variables which may be related in a population, but they are independent given the values of another subset of variables" (p.4). For example, variables X_1 and X_2 may be related, but once X_1 and X_2 are conditional on X_3, then X_1 and X_2 become independent. The following figure is a simple DAG structure to illustrate this relationship.

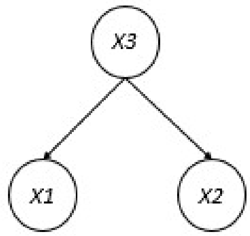


Figure 2.6 Conditional independence between *X1* and *X2*.

From Figure 2.6, it can be seen that X_1 and X_2 are conditional on X_3. The relationship of those variables can be denoted as follows:

$$p(X_1, X_2|X3) = p(X_1|X_3)p(X_2|X_3) \quad (2.5)$$

In practice, the relationship among complex variables can be structured and simplified using a conditional property. For example, the joint distribution of variables X_1, X_2, X_3 and X_4 is denoted as follows (Adapted from Levy & Mislevy, 2016)

$$p(X_1, X_2, X_3, X_4) = p(X_4|X_3, X_2, X_1)p(X_3|X_2, X_1)p(X_2|X_1)p(X_1) \quad (2.6)$$

Suppose that X_1 and X_2 refer to the basic skills that should be mastered in a certain domain of learning. X_3 is a skill which is developed based on skills X_1 and X_2, and skill X_4 is developed based on skill X_3. Hence, using the conditional independent property, the joint distribution of equation (2.6) can be simplified as follows.

$$p(X_1, X_2, X_3, X_4) = p(X_3|X_2, X_1)p(X_4|X_3)p(X_2)p(X_1) \quad (2.7)$$

The DAG for the joint distribution in equation 2.7 is presented in Figure 2.7.

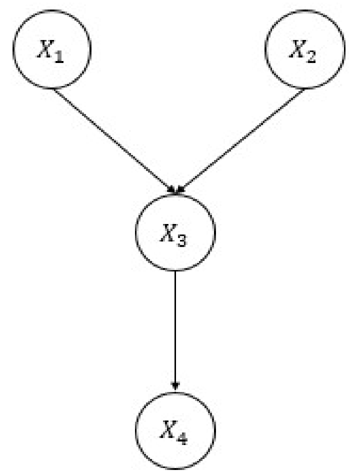


Figure 2.7 A DAG diagram representing skills X_1, X_2, X_3 and X_4, where X_1 and X_2 are independent; X_3 is dependent on X_1 and X_2; and X_4 is dependent on X_3 (Adapted from Levy & Mislevy, 2016, p. 346)

The conditional independent property is an important feature in structuring variables in complex networks. However, the most important thing in developing a Bayesian Networks model is that structuring the joint probability of the variables should be built based on a “theory-driven” or “deductive-reasoning” approach (Mislevy, 1994b, p. 474). As a result, the sound theoretical deductive-reasoning model of the joint probability in Bayesian Networks supports sound inductive-reasoning drawn from the data (Mislevy, 1994b). The development and model specifications of Bayesian Networks for modelling learning progressions in this research are discussed further in Chapter 5.

2.2.3. Summary of the Rationale

In summary, the present assessment is developed based on a cognitive model of learning. This cognitive model provides a framework to design assessment tasks and to interpret students’ responses. A learning progression is the cognitive model developed in the present research. Learning progressions can guide assessment to identify where the students are situated in their learning journey. Hence, the feedback about learning can be generated based on students’ progression levels. Such feedback can be used by educators to improve students’ learning.

The dependency between the levels in the learning progression model add to the complexity of the measurement model. The Bayesian Networks model is a probabilistic model that can handle the complexity and dependency of variables (levels). This

Bayesian model also provides deductive and inductive reasoning, which is consistent with the principle of assessments based on cognitive models.

2.3. Content Domain: Fractions

2.3.1. Fraction Learning Progression

Existing fraction learning progressions have been based on the theoretical work of Kieren (Kieren, 1976, 1980). Kieren, however, proposed five sub-constructs in the interpretation of fractions. These sub-constructs are “part-whole”, which expresses the fraction concept as the equal parts of a larger whole (unit); “measure” which expresses fractions as measurement points in a number line; “quotient” which views a fractions as division; “operators” which defines fractions as a function that transforms a quantity (number) into another quantity (number) with a smaller or bigger value; and “ratio” which interprets rational numbers as a ratio to compare the two entities.

Arieli-Attali and Cayton-Hodges (2014) adopted Kieren’s fraction sub-constructs as big ideas to develop a rational number learning progression. Big ideas are the “central concepts and principles of a discipline”, (Smith, Wiser, Anderson, & Krajcik, 2006, p. 2). They also used some other big ideas such as half and halving procedures, unit fraction, decimals, place value, and equivalent fractions. From these big ideas, they constructed the following progress variables: fractional unit, measure/fraction as number, additive structure, multiplicative structure, and strategic thinking/flexibility. Based on these progress variables, they structured the progression of students in rational number learning into six levels: prior knowledge (half and halving), early part-whole understanding, fraction as unit, fraction as single number and fraction as measure, representational fluency, and a general model of a rational number.

Confrey, Nguyen, and Maloney (2011) developed a fraction learning trajectory based on the common core state standards of the American Curriculum (CCSS). The learning trajectory begins at grade 3 by introducing the relationship between parts and their referenced whole. They used equipartitioning to build a unit fraction (1/b, where b is a whole number). After that, a fraction a/b is introduced based on the unit fraction 1/b. Next, students are introduced to equivalent fractions and fraction comparison with the same numerator or denominator. Next in grade 4, students learn how to compare two fractions with different numerators and begin to understand the rule of fraction addition and subtraction with the same denominator. At this level, students are also introduced to fraction multiplication. At grade 5, comparisons of fractions with different denominators are introduced. Moreover, the multiplication of a fraction by a fraction and the division of fractions are also introduced at this level.

In contrast to the work Arieli-Attali and Cayton-Hodges (2014) and Confrey et al. (2011), the proposed model of fraction learning progression combines Kieren's theoretical framework and the empirical research performed by Stafylidou and Vosniadou (2004) to develop a learning progression of students' understanding of both the numerical and symbolic notation of fractions. In their research, Stafylidou and Vosniadou (2004) proposed three explanatory frameworks for understanding the symbolic notation of fractions. The first explanatory framework is a "fraction as two independent natural numbers". Within this category, students' understanding of the fraction symbol is influenced by the notations of whole numbers, so that they believe that the fraction symbol consists of two independent numbers (the numerator and the denominator) and that the value of fractions increases when either the numerator or the denominator of a fraction increase. The second explanatory framework is associated with the idea that "a fraction is a part of a whole". Within this category, students conceive the relationship between the numerator and the denominator of fractions as that of a part of a whole, where the value of a fraction is always smaller than 1. The third explanatory framework is the "relationship between the numerator and denominator". Within this explanatory framework, students begin to understand the relationship between the numerator and the denominator so that they start to see a fraction as a number that can be bigger than one. They understand that if the numerator is bigger than the denominator, then the value of the fraction is also bigger and vice versa (Stafylidou & Vosniadou, 2004).

Another principal understanding about fractions that is not captured explicitly in Kieren's sub-constructs is the density concept of rational numbers, i.e., that "between any two different rational numbers there are infinitely many rational numbers" (Vamvakoussi and Vosniadou (2004, p. 456). Understanding the density of rational numbers presupposes that students understand that fractions are numbers that can be located on the number line but goes beyond understanding fraction as measure.

Combining the theoretical framework of Kieren's fraction sub-constructs and that of Vosniadou and her colleagues results in building a learning progression of students' understanding of the symbolic notation of fractions, which is more comprehensive than the previous work by Arieli-Attali and Cayton-Hodges (2014) and Confrey et al. (2011). This progression starts from students seeing the symbolic notation of fractions as representing two independent numbers until they understand the density property of fractions.

2.3.2. Two-dimensional Knowledge of Fraction Learning Progression

As detailed in Section 2.1, current learning progression models do not distinguish between the development of conceptual and procedural knowledge in students' mathematical learning. However, mathematics learning includes two essential knowledge dimensions: conceptual and procedural knowledge (Hibert & Lefevre, 1986; Rittle-Johnson & Schneider, 2014). Rittle-Johnson and Alibali (1999) defined conceptual knowledge as "explicit or implicit understanding of the principles that govern a domain and of the interrelations between pieces of knowledge in a domain" (p. 175). Accordingly, Hibert and Lefevre (1986) defined conceptual knowledge as:

> "a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all the pieces of information are linked to some network" (p. 3).

Both definitions are complimentary in conceiving conceptual knowledge as consisting of interconnected pieces of information about ideas and principles.

In terms of understanding the symbolic notation of fractions, conceptual knowledge includes understanding fraction properties (e.g. understanding the magnitudes of fractions), understanding fraction principles (e.g. understanding the density of fractions) and understanding the value of fractions (Bailey et al., 2015). Related to understanding the (symbolic) notation of fractions, conceptual knowledge includes understanding the meaning/interpretation of fraction sub-constructs, as discussed in Section 2.3.1.

Rittle-Johnson et al. (2001) defined procedural knowledge as “the ability to execute action sequences to solve problems” (p. 346). Similarly, Star and Stylianides (2013) expressed procedural knowledge as “knowledge of procedures, including action sequences and algorithms used in problem solving” (P. 6). In learning fractions, Bailey et al. (2015) described procedural knowledge as the knowledge of fraction operations. They defined procedural knowledge as “fluency with the four fraction arithmetic operations: addition, subtraction, multiplication, and division” (Bailey et al., 2015, p. 69).

Current practices for teaching and assessing mathematics emphasize procedural learning (Joersz, 2017; Sullivan, 2011). Teachers often introduce mathematics to students as procedures to solve mathematical tasks, without explaining the conceptual understanding underlying the procedures. In fact, some of the students’ procedural mistakes happen precisely because of students’ inadequate conceptual understanding. For example, students who perform addition across the numerator and denominator in fraction addition tasks see fractions as two independent whole numbers (Smith III, 2002; Stafylidou & Vosniadou, 2004).

The balance of teaching and assessing conceptual and procedural knowledge can be supported by the cognitive model of learning, which can cover the progression of these two types of knowledge. This can be done by placing the cognitive model as the foundation of the development of the curriculum, assessment, and instructions, as suggested by Pellegrino et al. (2001) and Wilson (2009a). Figure 2.8 shows the interconnections between the curriculum, assessment and instruction, based on a cognitive model of learning.

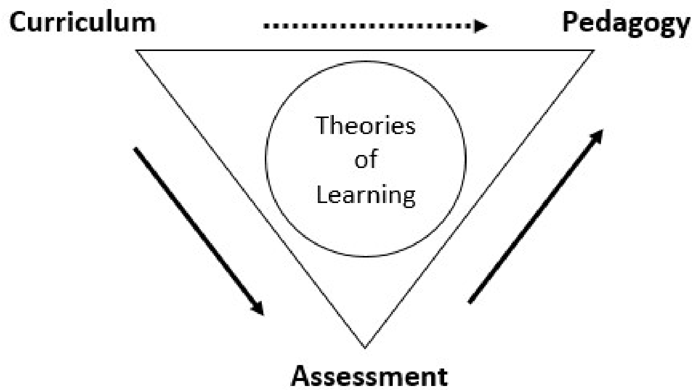


Figure 2.8 The relationship between curriculum, pedagogy, assessment and theories of learning (Adapted from Wilson, 2009a, p. 6)

The cognitive model of learning can shape the curriculum, assessment and pedagogy (see Figure 2.8). Hence, the construction of a cognitive model of learning, based on the conceptual and procedural dimension of fraction learning, can produce a more balanced curriculum, assessment and pedagogy.

2.3.3. Summary of the Rationale

In previous research, the theoretical framework of Kieren's sub-constructs were used as a foundation from which to develop fraction learning progressions of the development of the concept of fractions. The present research differs from previous work, first because it developed a learning progression of the development of students' understanding of the symbolic notation of fractions. Second, it used, in addition to Kieren's sub-constructs, Vosniadou and colleagues' explanatory frameworks for understanding fraction notation based on empirical research. Third, previous learning progressions did not differentiate between the conceptual and procedural knowledge of fractions, as was done in the present work.

2.4 Summary of the Chapter

This chapter discussed the literature relating to assessment, comparing traditional methods of assessment with assessments based on cognitive models. It was argued that assessments built based on cognitive models are superior to traditional assessments in terms of providing diagnostic information about student learning that can be used in instruction. Learning progression is a promising cognitive model that can be used as a foundation to develop diagnostic assessments because it provides information about the developmental levels of students' learning. A Bayesian Network is a statistical model that can be used for modelling the complexity and the hierarchical dependency between the levels of a learning progression. The hierarchical dependency can be modelled through conditional probability.

This chapter also discussed previous research on fraction learning progressions. The theoretical framework from Kieren and empirical research from Vosniadou on fraction learning have been combined to create a more comprehensive foundation for building a learning progression in fractions. Moreover, distinguishing conceptual and procedural knowledge in the learning progression is important in order to provide adequate information about these different types of mathematical knowledge in assessment to guide both the curriculum and the instruction.

The next chapter discusses the development of the cognitive model of a fraction learning progression based on conceptual and procedural knowledge, and their corresponding items.

CHAPTER 3 : THE HYPOTHESIZED LEARNING PROGRESSION MODEL AND ITEM TASK DEVELOPMENT

3.1 Introduction

The purpose of this chapter is to discuss the development of the hypothetical model of fraction learning progression and the corresponding item tasks. The hypothetical model of fraction learning progression was developed based on two dimensions of knowledge: conceptual and procedural. Furthermore, this chapter discusses the item tasks used to assess competencies for each level of the conceptual and procedural knowledge dimensions. It is organized into two main sections, which are:

1. The Proposed Model of Fraction Learning Progression
2. The Development of the Item Tasks

3.2 The Proposed Model of Fraction Learning Progression

Fractions are parts of the set of rational numbers, which are expressed in the form a/b, where a and b are integers and b is not zero (Bronshtein, Semendayev, Musiol, & Mulig, 2015; Rosen, 2007). In the present research, Lamon (2012) definition of fractions, which restricts fractions to “non-negative rational numbers" (p.29) is used. Hence negative fractions are not included in the proposed model. Furthermore, the model focuses on students’ understanding of the symbolic notation of fractions, rather than on the development of the fraction concept as a whole.

Fractions have a distinct symbolic notation and properties, which are different from those of natural numbers (Stafylidou & Vosniadou, 2004). First, natural numbers consist of one number only, while fractions consist of two numbers separated by a line. The top number is called the numerator and the number at the bottom is called the denominator. Second, fractions are unbounded-infinite numbers, meaning that there are no smallest or biggest fractions. This is different from natural numbers, where the unit is the smallest number. Next, no unique fraction precedes or follows another fraction, which means that there is always another fraction between two fractions. This is in contrast with natural numbers, where there is a unique number that precedes and follows all natural numbers.

Research shows that in the beginning of fraction instruction, students may not understand the symbolic notation of fractions. They may see fractions as two distinct (independent) whole numbers (Hartnett & Gelman, 1998; Smith III, 2002; Stafylidou & Vosniadou, 2004). They do not understand the meaning of the numerator and the denominator and their relationship. They treat fractions as they treat natural numbers. For example, in ordering fractions, they may consider that a fraction with a bigger numerator or denominator is a bigger fraction; in adding fractions, they may add across the numerator and denominator (2/3 + 1/4 = 3/7).

As discussed in Chapter 2, fractions have several sub-constructs, and this study focuses on the sub-constructs of the conceptual dimension underpinning the hypothesized model. These sub-constructs are: first, fractions can be seen as expressing the relationship of part-whole, which is "a comparison between the number of parts of the partitioned unit to the total number of parts in which the unit is partitioned" (Charalambous & Pitta-Pantazi, 2007, p. 296). Second, fractions can be seen as expressing "the measure assigned to some interval or region"(Lamon, 2012, p. 210). Hence, if there is a length l and then it is divided equally into *b* sub-divisions in which each sub-division has a length *1/b,* the fraction *a/b* can be interpreted as "*a* intervals of length *l/b"* (Lamon, 2012, p. 210). Finally, fractions can be seen to represent an operation of division so that the fraction *a/b* is "used as a way of writing *a* ÷ *b*"(Behr, Lesh, Post, & Silver, 1983, p. 95).

The hypothesized model of fraction learning progression that is proposed in this study expands the modelling approach of learning progressions from a unidimensional into a multidimensional model of learning progressions. The models of learning progressions in mathematics learning developed so far have been based on one dimensional knowledge, as discussed in Chapter 2. By differentiating the conceptual and procedural knowledge dimensions, the present hypothesized model provides a more detailed roadmap of students' learning. Mathematics instructors can use this model to assess students' knowledge and skills better and thus to improve their instruction of fractions at a classroom level. Educators can also use this model to develop or refine a mathematics curriculum at a state or national level.

Conceptual knowledge of fractions is defined as understanding the meaning of the symbolic notation of fractions. According to Hibert and Lefevre (1986), a symbol can express mathematical ideas and concepts. For fractions, the symbol a/b can be used to refer to several sub-constructs, as discussed before. Thus, understanding the meaning of the symbolic notation of fractions is related to the understanding of the various fraction sub-constructs. Procedural knowledge in this study is defined as knowledge of the series of steps or rules required to perform fraction addition, subtraction, multiplication and division. This definition is similar to the definition used by other researchers in the field, such as Rittle-Johnson and Schneider (2014), who defined a procedure as "a series of steps, or actions, done to accomplish a goal", and Bailey et al. (2015) who identified "the four fraction arithmetic operations: addition, subtraction, multiplication, and division" as the procedural knowledge of fractions. Within the hypothesized model, the procedural levels were specified based on knowledge of the rules that govern fraction operations.

The hypothesized model is constructed based on the development of several sub-constructs which are categorized into conceptual and procedural knowledge dimensions. Based on the development of these fraction sub-constructs (discussed in Section 4.2.1), the hypothesized model of fraction learning progression is developed in Section 4.2.2.

3.2.1 The Development of the Fraction Sub-constructs

The proposed hypothetical model of fraction learning progression attempts to capture the development of the students' understanding of the symbolic notation of fraction and of the rules and procedures which are used in fraction operations (fraction addition, subtraction, multiplication, and division). This hypothetical model distinguishes two dimensions of fraction knowledge: conceptual and procedural. The distinction between the two types of knowledge in this model is important because mathematics competencies rest on these two types of knowledge (Hibert & Lefevre, 1986; Rittle-Johnson & Schneider, 2014). It means that mathematical competencies rest on an understanding of mathematical symbols which are "connected to the conceptual knowledge they represent" (Hibert & Lefevre, 1986, p. 9), and on an understanding of the rules and procedures which are needed to execute mathematical tasks. Within each dimension, a number of fraction sub-constructs are proposed. These sub-constructs are similar to the "progress variables" stated in Pellegrino (2014) and Arieli-Attali and Cayton-Hodges (2014). The development of the sub-constructs for the conceptual and procedural dimensions underpinning the proposed learning progression is discussed in this section.

3.2.1.1 The Development of the Conceptual Sub-constructs

Within the conceptual knowledge dimension, there are five fraction sub-constructs which develop across the proposed levels. Those sub-constructs are: fraction as part-whole, fraction as measure, density of fractions, understanding fraction additive structure, and understanding fraction multiplicative structure.

3.2.1.1.1 Fraction as Part-whole

The part-whole sub-construct refers to the understanding of the symbolic notation of fractions as a representation of the part-whole relationship between the numerator and the denominator. The construction of this meaning is based on students' experiences of the partitioning of continuous objects into sets of discrete and equal parts (Behr et al., 1983; Kieren, 1980) and the linking of these experiences to fraction notation through instruction. In the hypothesized model, it is assumed that students may not fully understand the symbolic notation of fractions at the beginning of instruction, despite the fact they may have some understanding of part-whole relationships and may know that familiar fractions such as 1/2 and 1/3 refer to parts of a whole.

A more advanced understanding of the meaning of fraction notation is hypothesized to develop at level 2. At level 2, students may understand that the denominator refers to the total number of the parts that a whole consists of and that the numerator refers to a sub-set of these parts. Fraction understanding at level 2 is limited to fractions less than 1 (proper fractions) because students think that the numerator, which represents the sub-set of selected parts, should always be smaller than or equal to the denominator, which represents the total parts of the whole. However, at level 2, students may not understand that the denominator refers to equal size partitions of the whole. This understanding is hypothesised to be a competence achieved at level 3. At level 3, students also begin to understand improper fractions. They understand that the size of a part (the numerator) can exceed the size of the referenced whole (the denominator). In other words, they understand that the numerator can be greater than the denominator and if so, that the fraction is greater than the unit of the whole.

3.2.1.1.2 Fraction as Measure

The measure sub-construct refers to the meaning of fractions as "the measure assigned to some interval or region," which can "measure the distance of a certain point on the number line from zero" (Lamon, 2005, p. 170). Hence, fractions *a/b* can be interpreted as "a measure of *a* of *b* congruent parts" (Kieren, 1976, p. 131). This sub-construct includes understanding fractions as single-numbers on the number line, fraction order, and equivalent fractions. Understanding fractions as measures is hypothesized to be established at Level 3. At this level, students must be able to recognize the magnitude of fractions and the scales on the number line. At the next level (Level 4) they can order several fractions on the number line, including improper fractions and mixed numbers.

3.2.1.1.3 Infinity and density of fractions

The infinity part of this sub-construct refers to the fraction properties of unbounded infinity and density. Unbounded infinity means that fractions are seen as infinite numbers; there is no smallest or biggest fraction (Stafylidou & Vosniadou, 2004). The density part of this sub-construct refers to the fact that there are infinite numbers between two fractions (Bronshtein et al., 2015). The property of density of rational numbers is "radically different" from the discreteness property in whole numbers (Vamvakoussi & Vosniadou, 2004, p. 456), and it is difficult for students to understand.

Students at Level 3 and below may believe that there is a smallest and a biggest fraction, still influenced by their part-whole understanding of fractions. They also are not expected to understand the notion of density. They see fractions as discrete quantities, like whole numbers. Level 4 fraction understanding assumes that fractions are seen as infinite numbers, an idea that can lead students to understand the concept of density. They begin to see that there can be many numbers between two-pseudo successive fractions, but they still think that there are finite numbers between two fractions. Finally, students at Level 5 should have a complete understanding of the density property i.e. that there are infinite numbers between two-pseudo successive fractions.

3.2.1.1.4 Understanding Additive Fraction Operations

The sub-construct of additive fraction operations refers to students' understanding of the meaning of addition and subtraction of fractions. In fraction operations, the meaning of addition and subtraction is similar to the meaning of the additive operations with natural numbers. Addition of fractions always produces a bigger number and subtraction of fractions always produces a smaller number. This is also the case for natural numbers, where the addition of two natural numbers always produces a bigger number and thee subtraction of two natural numbers always produces a smaller number.

In the beginning, students do not understand fraction addition and subtraction. They may understand that adding fractions means joining parts of the object, or subtracting fractions means separating parts of the object, but they cannot translate this understanding into operations using the symbolic notation of fractions. Conversely, they do not understand the meaning of addition and subtraction when these operations are represented using the symbolic notation of fractions. Next, at Level 2, students understand fraction addition and subtraction based on their understanding of the symbolic notation fraction *a/b,* where b represents the "size of the parts" and a represents the "number" of those parts (Clarke, Roche, & Mitchell, 2008, p. 375).

3.2.1.1.5 Understanding Multiplicative Fraction Operations

The sub-construct of understanding multiplicative fraction operations refers to students' conceptual knowledge of fraction multiplication and division. The meaning of fraction multiplication is different from the meaning of natural number multiplication. In fractions, multiplication means "how much of" (Van de Walle, Karp, Bay-Williams, & Wray, 2015). So for example, a multiplication of 2/3 by 3/4 means how much of 2/3 in 3/4 (see Chinnappan & Forrester, 2014). As a result, fraction multiplication does not always produce bigger numbers (especially when the multiplication involves a proper fraction, it always produces a smaller number), while in natural numbers multiplication always produces a bigger number. The meaning of fraction division is similar to the meaning of natural number division, which is finding "how many" parts of a divisor are in a dividend. For example, 2/3 divided by 3/4 means finding how many 3/4 in 2/3. However, fraction division can produce a bigger value (i.e. when the divisor is a proper fraction), while in natural numbers division always produces a smaller value. Therefore, fraction multiplication and division are counter-intuitive for students who have prior knowledge of natural number operations. An understanding of fraction multiplication is hypothesized to emerge at the highest level of the proposed model.

3.2.1.2 The Development of the Procedural Sub-Constructs

The procedural knowledge dimension of the hypothesized fraction learning progression consists of two sub-constructs: additive operations and multiplicative operations.

3.2.1.2.1 Additive Operations

The sub-construct of additive operations refers to the procedural knowledge required in order to perform fraction addition and subtraction correctly. At level 1, students are not expected to have the procedural knowledge of adding or subtracting fractions, but they may be able to do a simple fraction addition and subtraction for fractions with the same denominator by transferring their knowledge of addition and subtraction from natural numbers – i.e., they may simply add the numerators or the denominators to get the answer. At Level 2, students must have developed their procedural knowledge of fraction addition and subtraction and must be able to perform additive fraction operations with unlike denominators. They are expected to know the rule that when adding or subtracting fractions the denominators should be the same. If the denominators are different, then they should manipulate the fractions (by transforming the fractions with a common denominator) to get the same denominator before they add or subtract them. At Level 3, they should expand their knowledge of the rules and procedures of adding and subtracting fractions at Level 2 and should be capable of adding and subtracting improper fractions and mixed numbers.

3.2.1.2.2 Multiplicative Operations

The sub-construct of multiplicative fraction operations refers to knowledge of the rules for fraction multiplication and division. Students at Level 3 are expected to be able to perform fraction multiplication and division, but not when the operations involve improper fractions and mixed numbers. At level 4, they must be able to handle more complex multiplicative fraction operations, which involve improper fractions and mixed numbers. Their fluency has emerged at this level and is at Level 5.
The summary of the sub-construct progression is presented in Table 3.1.

Table 3.1 The sub-construct progressions of the proposed model of fraction learning progression

Sub-Construct	Level 1	Level 2	Level 3	Level 4	Level 5
Conceptual					
Part-whole	Do not understand the relationship between the numerator and denominator.	See fractions as a comparison between the number of selected equal parts to the number of all parts of their referenced whole, but consider fractions to be smaller than then the unit	Understand improper fractions		
Measure			Understand a fraction as a point on the number line, but still limited to fractions smaller than the unit	Understand a fraction as a point on the number line, including improper fractions and mixed numbers	
Infinity (Unbounded-infinity and Density)		See fractions as discrete quantities	See fractions as discrete quantities	Understand that there is no smallest or biggest fraction	Recognize that there are infinite numbers between two-pseudo successive fractions.
Understanding of additive structure		Understand the meaning of fraction addition/subtraction operations			
Understanding of multiplicative structure					Understand the meaning of multiplicative fraction operations

Sub-Construct	Level 1	Level 2	Level 3	Level 4	Level 5
Procedural					
Additive Operations	Do not have procedural knowledge of fractions	Can do fraction additions and subtractions including fractions with unlike denominators but are limited to fractions less than 1	Can do additive fraction operations with improper fractions or mixed numbers		
Multiplicative Operations			Can do multiplicative fraction operations	Can do multiplicative fraction operations with improper fractions or mixed numbers	

3.2.2 The Hypothesized Model of Fraction Learning Progression

This section discusses the levels of the proposed model of fraction learning progression and the competencies for each level. The model is hypothesized to consist of five levels of conceptual knowledge dimension and four levels of procedural knowledge based on the development of the fraction sub-constructs, as discussed in the previous section.

3.2.2.1 Conceptual Knowledge Dimension

3.2.2.1.1 Level 1 – No Fraction Understanding

No conceptual understanding of the symbolic notation of fractions found at Level 1. Fraction notation may be interpreted as consisting of two independent numbers with the exception of some familiar fractions such as 1/2 and 1/4.

The symbols in mathematics (including the symbolic notation of fraction *a/b*) present mathematical ideas and concepts (Hibert & Lefevre, 1986). As discussed before, fractions *a/b* have been interpreted in various ways to accommodate fractional ideas and concepts (See Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007; Kieren, 1976, 1980; Lamon, 2012). Students may not understand the meaning of the symbolic notation of fraction *a/b* when first exposed to fractions. They may see fraction *a/b* as two independent natural numbers (see Hartnett & Gelman, 1998; Smith III, 2002; Stafylidou & Vosniadou, 2004), except for familiar fractions such as 1/2 and 1/4. However, they may understand part-whole relationships of objects in the world on the basis of their prior knowledge obtained from daily life experience. A study by Mack (1990) showed that students had prior knowledge of fractions such as "knowledge about parts of wholes in real world situations …[that] was based upon knowledge of whole numbers" (p.3). Those students who may have no understanding of fractions and may treat fractions as two independent numbers are hypothesized at the lowest level in the proposed model.

3.2.2.1.2 Level 2 – Part-Whole

Students understand a fraction as a part-whole. At this level, they understand that the numerator represents the number of selected parts and the denominator represents the number of all parts of their referenced whole. They also believe that a fraction is always smaller than 1 (a whole).

Students begin to understand the relationships between the numerator and denominator in fractions as a part-whole relationship. They understand the basic idea of fractions as part-whole which is, "some whole is broken into equal parts" (Kieren, 1980, p. 134). However, at this level they may not really understand the meaning of "equal size", and may sometimes generate pictorial representations of fractions consisting of unequal parts (see Arieli-Attali & Cayton-Hodges, 2014). In addition, they believe that fractions should be smaller than 1 (whole) because the number of parts (the numerator) cannot exceed the numbers of all the parts of the whole (the denominator) (Stafylidou & Vosniadou, 2004). Arieli-Attali and Cayton-Hodges (2014) gave the example of one student who read 6/4 and said that, "there cannot be 6 out of 4, so it must be 4 out of 6" (p.25). This is the next development of students' fraction understanding from level 1. In level 1 they do not understand the symbolic notation of fractions and may perceive the numerator and denominator as two independent numbers. In level 2, they understand fractions as part-whole and understand fractions smaller than 1. This roadmap of fraction learning may be highly related to the fact that fractions as part-whole are taught at the beginning of fraction instruction (Amato, 2005; Kieren, 1976).

The students at this level are hypothesized to have the following conceptual competencies:

1. *They generate the symbolic notation of proper fractions from pie diagrams.*

   Students are able to generate fraction notation from pie or other area model representations. However, their understanding is limited to fractions smaller than 1. They do not understand that fractions can be bigger than the unit. Previous research has revealed some of the problems students at this level may have with improper fractions.

2. *They map proper fractions onto pie diagrams.*

   At this level, students understand the symbolic notation of fraction as representing part-whole. This understanding allows them to map fraction notation onto pie diagrams, but it is limited to fractions smaller than 1. Students have difficulties with understanding fractions that are greater than 1 because they do not understand why a part can be bigger than the whole.

3. *They generate the symbolic notation of equivalent fractions from pie diagrams.*

   Students can understand equivalent fractions but are limited to fractions less than 1. They can demonstrate this understanding by generating equivalent fraction notations from pie diagrams. According to (Wong & Evans, 2011), equivalent fractions can be represented using pie diagrams. The area of the whole is constant, but the number of partitions within the whole can be varied. Thus, different fractions (which are equivalent) can be generated corresponding to the number of partitions of the whole (the denominator) and the number of the shaded partitions (the numerator) that are created.

4. *They can order proper fractions*

   Students understand the size of fractions based on part-whole understanding and can order proper fractions using pie diagrams. This is aligned with the study by Stafylidou and Vosniadou (2004), which showed that students at the explanatory framework of fractions as a part-whole were able to order fractions smaller than the unit 1.

5. *They can demonstrate correct fraction addition and subtraction using diagram representations.*

   Arieli-Attali and Cayton-Hodges (2014) highlighted that students who have a part-whole understanding are able to perform fraction addition and subtraction by adding and separating the selected (i.e. shaded) parts of diagram representations. Thus, students at this level are expected to be able to demonstrate fraction addition and subtraction with fractions with different denominators. Hence the conceptual understanding of adding and subtracting proper fractions with different denominators using diagram representations is hypothesized to emerge at this level.

3.2.2.1.3 Level 3 – Improper Fractions and Fractions as Measures

Students understand that if the numerator is greater than the denominator, then the fraction is greater than the referenced whole and vice versa. They also understand fractions as measures and conceive fractions as numbers on the number line.

Students begin to understand the relationship between the numerator and denominator beyond part-whole understanding. They understand that if the numerator is greater than denominator, then the fraction is greater than the referenced whole and vice versa. In other words, they understand that the numerator, which presents the size of the part, can exceed the denominator, which presents the size of the whole. On the other hand, they also understand fractions as measures. Behr et al. (1983) highlighted that understanding fractions as measures can be seen as understanding fractions as "a subset of the real numbers". Hence, this is a conceptual jump from learning fractions by understanding fractions as a relationship between parts and the whole of objects at Level 2, to understanding fractions as numbers on the number line. Stafylidou and Vosniadou (2004, p. 513) also argued that students who understand the relationship between the numerator and the denominator of fractions, including fractions greater than 1 (the referenced unit/whole), have "radically changed their beliefs about the concept of fraction". Therefore, the conceptual knowledge of fractions as measures and understanding fractions greater than 1 is at a different (higher) level compared with part-whole understanding at Level 2.

The following are the students' competencies hypothesized to emerge at this level.

1. *They can generate the notation of fractions from a pie diagram, which represents equal size partitions*

   Students advance their understanding of fractions as representing part-whole at Level 2. In this level, they understand that the denominator should represent equal size partitions of the whole. Understanding "equal size" is identified as the next step in learning after understanding the symbolic notation of fractions as a representation of part and whole (as established at Level 2) (see Arieli-Attali & Cayton-Hodges, 2014).

2. *They can generate an improper fraction notation from a pie diagram*

   Students advance their understanding of fractions as part-whole at level 2, which is limited to fractions less than 1, to understanding improper fractions at level 3. They can generate improper fractions from two circles of a pie diagram. They understand that the denominator of improper fractions represents the number of all parts of the referenced unit (whole), which is the number of all parts from a pie diagram. They also understand that the numerator of improper fractions refers to the number of all shaded (selected) parts, even though it is larger than the number of all parts of the referenced unit. Hence, they understand that if a fraction has a numerator greater than the denominator, then it is greater than the whole.

3. *They generate equivalent fraction notation, greater than 1, from pie diagrams*
   The students can generate a notation of equivalent fractions for fractions greater than 1 from two circles of a pie diagram. This is the development of students' competency on equivalent fractions that in the previous level (Level 2), their competency is limited to proper fractions.
4. *They generate a pie diagram to represent improper fraction notation*
   This competency is similar to the competency at point 2. The difference is that this competency requires students to translate a symbolic notation of improper fractions into a pie diagram, as compared to generating a fraction from a pie diagram.
5. *They can correctly order fractions, including improper fractions and mixed numbers*
   Students advance their understanding of improper fractions and mixed numbers at this level. They fully understand the relationship between the numerator and the denominator that a fraction symbol represents. Wenrick (2003) highlighted that understanding the relationship between a numerator and a denominator can produce a "quantitative notion" of fractions, which is extremely useful because it allows students to compare and order fractions.
6. *They can place a fraction on a number line*
   Students at this level begin to understand fractions as a measure. They understand that fractions can be used to represent the distance between zero and a certain point on the number line. They also understand that the numerator represents the distance - how many scales from zero to the certain point on the number line - while the denominator represents the total number of scales (equal intervals) within the unit. Without this understanding, students will have difficulty in locating fractions on a number line (Wong, 2013).

3.2.2.1.4 Level 4 – Unbounded Infinite Numbers of Fractions

Students view fractions as unbounded infinite numbers (there is no smallest or biggest fraction).

Stafylidou and Vosniadou (2004, p. 513) showed that students at the explanatory framework of "relation between numerator and denominator" and the sub-category "relation of two numbers with infinity" believed that fractions are unbounded infinite numbers. This belief emerged as a result of understanding the relationship between the numerator and denominator of fractions as division (Stafylidou & Vosniadou, 2004). This sub-category is higher than the sub-category "relation of two numbers without infinity" which is hypothesized to emerge at Level 3.

The following are the competencies which are hypothesized to emerge at this level.

1. *Students can order improper fractions on a number line*

   Students advance their measure understanding of fractions at Level 3 to the case of improper fractions. At this level they can locate and order fractions (including improper fractions) with different denominators on the number line.
2. *Students understand that there is no biggest and smallest fraction*
   At this level, students understand that fractions are unbounded infinite numbers

3.2.2.1.5 Level 5 – Density of Fractions and Understanding Multiplicative Fraction Operations

Students understand the density property of fractions, i.e. that there are unlimited numbers between any two fractions. They also have conceptual understanding of multiplicative fraction operations.

Students are able to understand that there are infinite numbers between any two fractions. Understanding the density concept at this level completes the students' understanding of fractions as numbers. Moreover, they also begin to understand fraction multiplication and division, which is different from multiplication and addition with whole numbers.

The following are the competencies which are hypothesized to emerge at this level

1. *Students are able to demonstrate that there are infinite numbers between any two fractions.*

At this level, students are expected to understand that there are unlimited numbers between two pseudo-successive fractions. Vamvakoussi and Vosniadou (2004) grouped the students who understand the density of fractions into two categories: “naïve density” and “sophisticated density”. The former included students who believed that there are infinite numbers between two fractions or between two decimals or believed that there are infinite numbers between both fractions and decimals but not between a decimal and a fraction. Sophisticated density is achieved when students understand that there are infinite numbers between any two rational numbers, regardless of their symbolic representations. Because the study focuses only on fractions, the present research cannot differentiate between naïve and sophisticated density.

2. *Students can represent multiplicative fraction operations using diagram representations.*

   Students at this level can demonstrate a conceptual understanding of fraction multiplication and division using pictorial representations like pie diagrams or number lines.

3.2.2.2 Procedural Knowledge Dimension

3.2.2.2.1 Level 1 – No Procedural Knowledge

No procedural knowledge is expected at this level. At Level 1, students may be able to add or subtract fractions with like denominators. However, their procedural knowledge at Level 1 may not depend on knowledge of fraction addition and subtraction, but knowledge of addition and subtraction with natural numbers. Since they consider fractions to be two independent natural numbers, they transfer their knowledge of natural numbers to addition with fractions. In such a situation, they can be correct on fraction addition or subtraction with fractions with like denominators but not when the denominators are different. Hansen, Jordan, and Rodrigues (2015) found that many students in the low growth procedures group treated numerators and denominators of two fractions in fraction addition and subtraction as “four separate whole numbers”(p.11). This is consistent with also the notion of whole number bias highlighted by Ni and Zhou (2005), according to which students apply whole number properties in cases where whole numbers do not apply, as in the case of fractions.

3.2.2.2.2 Level 2 – Additive Fraction Operation

Students know the procedures of fraction additions and subtractions but are limited to proper fractions. Students begin to know the rules and procedures of fraction addition and subtraction, including fractions with unlike denominators. Confrey et al. (2011) pointed out a trajectory of learning additive operations, which begins from additive operations with fractions with like denominators and finishes with those with unlike denominators.

The competencies which are hypothesized to emerge at this level are addition and subtraction with proper fractions, including fractions with unlike denominators. The students at this level know that the denominators should be the same when they add or subtract fractions. Moreover, if the fractions have unlike denominators, they know the procedure of how to transform the fractions to get equivalent fractions with a common denominator. However, their procedural knowledge of fraction addition and subtraction is limited to proper fractions. They have trouble with adding or subtracting improper fractions and mixed numbers.

3.2.2.2.3 Level 3 – Additive and Multiplicative Fraction Operations

Students expand their procedural knowledge of additive operations to include improper fractions and mixed numbers. They also begin to develop procedural knowledge of multiplicative fraction operations. The following are the procedural competencies which are hypothesized to emerge at this level.

1. *Students can add and subtract improper fractions and mixed numbers.*

   Students develop their procedural knowledge of additive operations, which was limited to fractions less than 1 at level 2, to additive operations which involve improper fractions and mixed numbers. These competencies require procedural knowledge of how to transform mixed numbers into common fractions and vice versa, and how to transform fractions with unlike denominators into equivalent fractions with a common denominator. Understanding fractions greater than 1, which emerges at this level, helps students to learn additive operations that involve improper fractions and mixed numbers.

2. *Students can multiply and divide fractions.*

   Students develop their procedural knowledge from additive fraction operations (fraction addition and subtraction) in Level 2 to multiplicative fraction operations (fraction multiplication and division) at level 3. Multiplicative operations usually are taught in schools after additive operations (see Balitbang, 2013a; Initiative, 2011). Therefore, the procedural knowledge of multiplicative fraction operations is hypothesized to emerge at this level after students learn the procedure of additive fraction operations at level 2. They can perform well in multiplying and dividing fractions for fractions less than 1, but they may have trouble when the operations involve improper fractions and mixed numbers. For example, when they multiply 2 ¼ and 1/2 they may multiply 1/4 by 1/2 and keep the whole number 2. They make an error by not transforming the mixed number 2 ¼ into a common fraction form (*a/b*) before proceeding to the multiplication of 2 ¼ and 1/4.

3.2.2.2.4 Level 4 – Advanced procedural knowledge of additive and multiplicative fraction operations

Students advance their procedural knowledge of multiplicative fraction operations from the previous level. The fluency of performing fraction operations (for both additive and multiplicative operations) emerges at this level. Students' procedural knowledge of fraction multiplication and division is developed at this level. In the previous level they begin to recognize the procedures of fraction multiplication and division, but they commit procedural errors when the operations involve mixed numbers, as discussed at level 4. At this level they can perform more complex multiplicative fraction operations (with not only greater complexity, but fewer procedural errors) that involve improper fractions and mixed numbers. They demonstrate fluency in performing fraction multiplication and division. The procedural competencies that are hypothesized to emerge at this level are that students are able to perform fraction multiplication and division that involves improper fractions and mixed numbers.

3.3 The Development of Item Tasks

This section discusses the item tasks that are developed to test students' level competencies within the proposed model of fraction learning progressions. The discussion is organized into two parts: conceptual item tasks and procedural item tasks. Some of the items were adapted from other research, while some were crafted by the researcher itself.

The items are organized into groups of tasks. There are eight tasks in the conceptual dimension: Task 1 Generating a Fraction from a Pie Diagram; Task 2 Shading a Pie Diagram to Represent a Fraction; Task 3 Ordering Fractions; Task 4 Locating Fractions on a Number Line; Task 5 Finding the Smallest and Biggest Fractions; Task 6 Finding how many Fractions lie between two Fractions; Task 7 Adding Fractions using Diagram Representation; and Task 8 Multiplying and Dividing Fractions using Diagram Representation. Next, the procedural knowledge dimension consists of two tasks: Task 1 Performing Additive Fraction Operations; and Task 2 Performing Multiplicative Fraction Operations.

Within each task there are items. The items are labelled using seven-digit codes. The first three digits of the codes refer to the conceptual and procedural dimension (Con for conceptual and Pro for procedural); the fourth and fifth digits refer to the task; and the last two digits refers to the items within the task. For example, the item code ConT1Q2 refer to the conceptual dimension, Task 1, Item 2. These codes are useful for the quantitative analysis performed in Chapter 6.

3.3.1 Conceptual Item Tasks

The conceptual item tasks are developed to test students' conceptual understanding of the symbolic notation of fractions (which is represented in the bi-partite notation *a/b* where *a* and *b* are whole numbers) and the meaning of fraction operations. In order to test students' understanding of the symbolic notation of fractions, item tasks based on the sub-constructs of part-whole, measure and infinity are developed. Likewise, in order to test students' understanding of the meaning of fraction operations, item tasks based on the sub-constructs of conceptual additive and multiplicative fraction operations are crafted.

The understanding of fractions as representing part-whole is tested using items that ask students to map fraction notation into pie diagrams and the opposite. Students are asked to generate fractions that represent the shaded parts of a pie diagram or are asked to shade the part of a pie diagram that corresponds to a given fraction. Pie diagrams are used to test students' understanding of proper, improper and equivalent fractions. Students' responses to these tasks can demonstrate whether they understand the symbolic notation of fractions when that symbolic notation is used to represent part-whole relationships. In addition, students are also asked to use part-whole representations to compare fractions and demonstrate which fraction is bigger. These tasks are used to reveal students' understanding of the numerical value of fractions based on part-whole representations.

The understanding of fractions as representing measure is tested using items that ask students to map fraction notation into number lines. Students are asked to put a fraction or several fractions on the number line. In order to be able to put a fraction on the number line, students should understand that the denominator of fractions represents the number of intervals within one unit and the numerator represents the number of intervals from zero to the point/mark of the fraction on the number line. Students' understanding of fractions as measure is inferred from their responses as to how they create the intervals/scales within one unit (when the fraction is an improper fraction) or within more than one unit (when the fraction is an improper fraction).

The infinity property of fractions is tested using items that ask students how many fractions are present between two fractions. There are two types of the infinity property in the hypothesized model of fraction learning progression: the unbounded infinity of fractions and the density of fractions. To reveal students' understanding of the unbounded infinity of fractions, students are asked to write the biggest and the smallest fractions they know. Students who answer this item correctly may say that there are no biggest and smallest fractions, meaning that fractions are unlimited or infinite. The density of fractions is tested using item tasks which ask students to find how many fractions are present between two fractions (both pseudo-successive and non-successive fractions). From their responses, it can be inferred whether they have a discreteness understanding of fractions or a density understanding of fractions, or whether they have misconceptions.

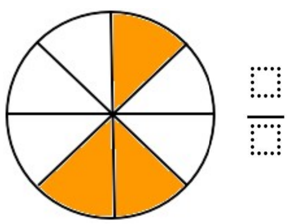
3.3.1.1 Understanding Fractions as Representing Part-Whole Relationships

Understanding fractions as part-whole is tested using Task 1 Generating a Fraction from a Pie Diagram, Task 2 Shading a Pie Diagram to Represent a Fraction and Task 3 Ordering Fractions.

3.3.1.1.1 Task 1 Generating a Fraction from a Pie Diagram

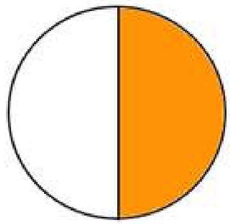
To test students' understanding of the symbolic notation of fractions as part-whole, students are asked to write a fraction that represents the shaded part of pie diagrams.

Item 1 - Write the fraction for the shaded part below (Adapted from Scanlon, 2013) (ConT1Q1)



Task 1 Item 1 asks students to write the symbolic notation of the fraction that represents the shaded parts of the pie diagram. To answer this item correctly, students should understand that the numerator represents the number of the shaded parts and the denominator represents the number of all parts of the pie diagram (the whole). The correct answer for this item 3/8. Students who answer this item correctly are put on level 2 (part-whole) of the hypothesized learning progression model, while students who answer this item incorrectly are put on level 1 (do not understand the symbolic notation of fractions). This item is developed to address Level 2 Competency 1 (generate the symbolic notation of proper fractions (fractions less than 1) from a pie diagram).

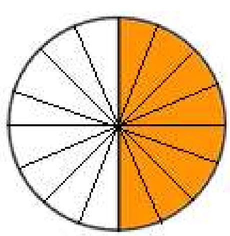
Item 2 - Write the numerator of the fraction for the shaded parts below (ConT1Q2)



$$\frac{\dots}{16}$$

Task 1 Item 2 tests the students' understanding of equivalent fractions within the part-whole sub-construct. The next step of learning fractions after recognizing fraction notation as a representation of parts and a whole is learning equivalent fractions (Confrey et al., 2011). Within this concept, students recognize that the selected part of the object (the whole) can be represented by different fractions, which are all equivalent: it means that these fractions have the same numerical value (size).

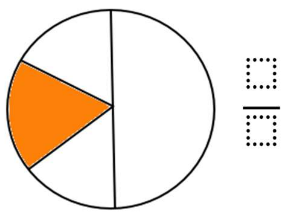
Task 1 Item 2 asks students to give a fraction that represents a half of the shaded area but has the denominator 16. Hence, students should understand that if a half shaded area, which is 1/2, is to be represented by another fraction which has the denominator 16, then this fraction should have the numerator 8. To answer this item correctly, students should know that the whole number of partitions is now 16 (because the denominator now is 16), and there are 8 partitions covering the shaded area, which is half of 16. These 8 partitions cover the same area as the area of the previous 1 partition, which means that the 8 eight partitions are equal to the previous 1 partition. Hence, students should be able to infer that the fraction representing the shaded 8 partitions is equal to the fraction which represents the previous 1 shaded partition. In other words, the correct answer for this item is 8 which is the numerator of the fraction representing a half-shaded area of the pie diagram with the denominator 16. This answer is illustrated as follows:



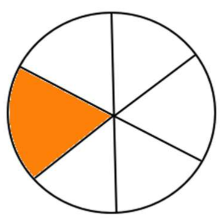
$$\frac{8}{16}$$

By answering this item correctly, students have demonstrated that 1/2 and 8/16 have the same value because they represent the same size of the shaded area of the pie diagram. Students who are able to answer this item correctly are put in Level 2 because understanding fractions as part-whole includes understanding equivalent fractions (Arieli-Attali & Cayton-Hodges, 2014; Kieren, 1980). This item is used to address level 2 competency 3 (generate equivalent fractions from a pie diagram).

Item 3 - Write the fraction for the shaded part below (Adapted from Pantziara & Philippou, 2012; Scanlon, 2013) (ConT1Q3)

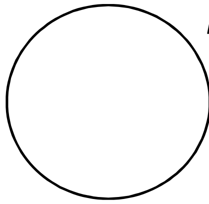


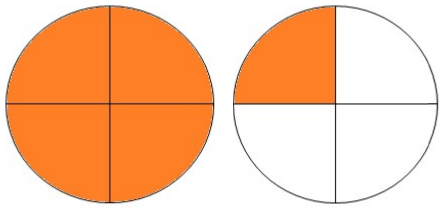
The correct answer for this item is 1/6 as represented in the following pie diagram.



$\frac{1}{6}$

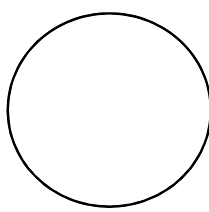
Task 1 Item 3 tests the students' understanding of equal partitions as a foundation of part-whole sub-construct (Kieren, 1976, 1980; Lamon, 2005). This is a development of students' understanding of the symbolic notation of fractions as part-whole at level 2. Students at level 2 understand the symbolic notation of fractions that represents part-whole relationships, but they may not understand that the denominator should represent parts in equal size. Arieli-Attali and Cayton-Hodges (2014) put students who understand equal partition one-level higher than those who had not acquired this concept. Therefore, these items are placed at level 3 of the proposed model. This item is used to address level 3 competency 3 (generate a fraction as equal-parts of a whole from a pie diagram).

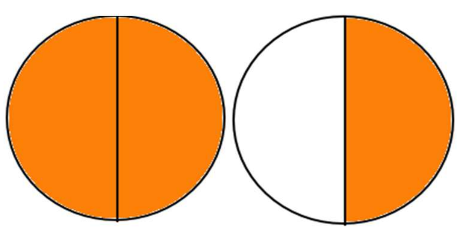
Item 4 If the figure  *is the whole, write the fraction for the shaded part below (ConT1Q4)*



$\frac{\square}{\square}$

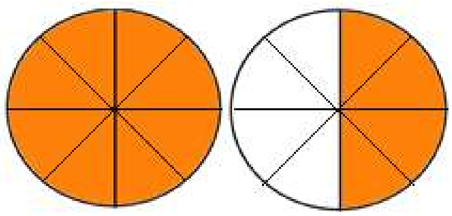
Task 1 Item 4 tests the students' ability to generate the symbolic notation of an improper fraction from a pie representation. The correct answer for this item is 5/4. In order to provide the correct answer, students should understand the symbolic notation of fractions, i.e., that the numerator represents the number of selected parts (the size of part) and the denominator represents the number of all parts of the whole (the size of whole). However, they should also understand that the size of a part can exceed the size of the referenced whole. For this item, students should understand that the circle is the referenced whole, as stated in the task. Hence the denominator is the number of all parts in one circle which is 4, while the numerator is the number of all the shaded parts from two circles, which is 5. By understanding the part and the referenced whole, students can understand why the numerator is bigger than the denominator. This task is used to address Level 3 competency 1 (generate improper fractions (fractions greater than 1) from pie representations).

Item 5- If the figure  *is the whole, write the numerator for the shaded part below (ConT1Q5)*



$\frac{\dots}{8}$

Task 1 Item 5 is similar to Task 1 Item 2, which asks students to find an equivalent fraction for 1 ½ shaded areas of the circles with the given denominator 8. However, Task 1 Item 5 asks students to generate an equivalent fraction for an improper fraction, while Task 1 Item 2 asks students to generate an equivalent fraction for a proper fraction. The correct answer for this item is 12 which represents the numerator of the fraction 12/8 as illustrated on the pie diagram below.



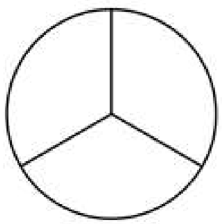
$\frac{12}{8}$

To answer this item correctly, students should understand both improper fractions and equivalent fractions. From the pie diagram above, students who understand improper fractions should know that the diagram represents 3/2 or 1 1/2 because there are three shaded parts and 2 partitions on each circle, where one circle is the referenced whole. However, they are asked to find another fraction with the denominator 8 which is equivalent to 3/2 or 1 1/2. Students should understand that the denominator 8 means that there are 8 partitions within the referenced whole, i.e., one circle. By partitioning each circle into 8 parts, they can see that there are 12 parts in the shaded areas of the two circles and can conclude that the fraction that represents the shaded area is 12/8, a fraction equivalent to 3/2 or 1 1/2. Because this task requires an understanding of improper fractions, students are expected to answer this item correctly at level 3. This item addresses Level 3 competency 2 (generate equivalent fractions greater than 1 from pie representations).

3.3.1.1.2 Task 2 Shading Pie Diagrams to Represent Fractions

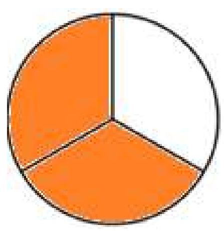
Previous items asked students to generate fractions from pie diagrams, while the following tasks ask students to shade the area of pie diagrams to represent fractions. The aim of these tasks is the same as the aim of the previous tasks, which is testing the students' understanding of the symbolic notation of fractions as part-whole. Students who understand fractions as part-whole should be able to generate fractions from the pie diagrams and they also should be able to shade the pie diagrams to represent the value expressed by the symbolic notation of fractions. So, these item tasks are created to gain more evidence of the students' understanding of fractions as part-whole.

Item 1 Shade the shape to show the fractions below (ConT2Q1).



$\frac{2}{3}$

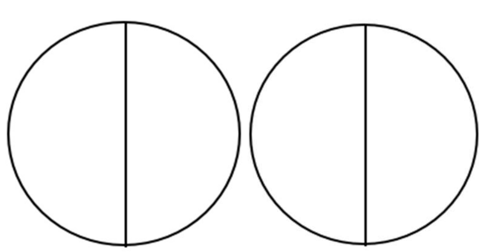
Task 2 Item 1 tests the students' understanding of fraction notations (within the part-whole sub-construct) by asking them to shade the area of the circle that the fraction denotes. The following pie diagram is the correct answer of this item.



$\frac{2}{3}$

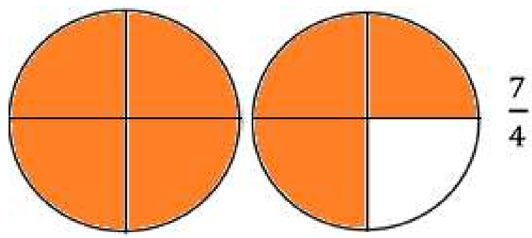
To answer this item correctly, students should know that the denominator represents the number of all parts in the circle, while the numerator represents the shaded parts of the circle. If they know this, the students should be able to understand that they are supposed to shade two of the 3 parts of the circle. Students who answer this item correctly provide evidence that they understand the symbolic notation of fractions as part-whole. Therefore, they are put in level 2 (part-whole), while students who fail to answer this item correctly are put in level 1 (do not understand the symbolic notation of fractions) of the hypothesized model of fraction learning progression. This item is used to address Level 2 competency 2 (generate a pie representation from a proper fraction).

Item 2 Shade the shape to show the fractions below (ConT2Q2).



$\frac{7}{4}$

ask 2 Item 2 tests the students' understanding of improper fractions. The item asks students to shade the fraction $\frac{7}{4}$ on two units (wholes) of circular representations, where each unit has two equal partitions. To answer this item correctly, students should understand that the referenced whole is represented by one circle, and they should understand that the denominator 4 refers to the number of partitions of one circle. Hence, they should create four partitions within each circle, and shade seven parts. The following pie diagram is the correct answer of this item.



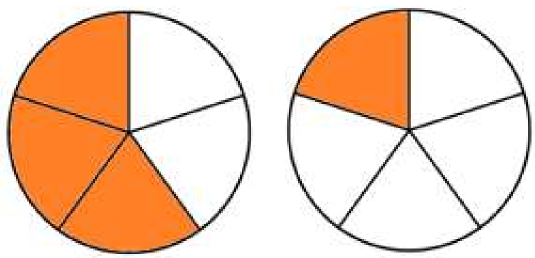
Students who successfully answer this item show that they understand that the denominator of improper fractions represents the number of parts of the referenced whole, i.e., of one circle in the present case. They also provide evidence that they understand that the numerator of the fraction represents the number of selected parts, which can be greater than the total number of parts of the referenced whole (the denominator). The students who answer this item correctly are put in level 3. This item is used to address level 3 competency 4 (generate a pie diagram to represent an improper fraction).

3.3.1.1.3 Task 3 Comparing Fractions

Task 3 Items 1 to 3 ask students to compare fractions based on a part-whole model. They test the students' understanding of the numerical value of fractions. Students who do not understand the symbolic notation of fractions may compare fractions based on their prior knowledge of whole numbers. Stafylidou and Vosniadou (2004) found that students in the explanatory framework of fractions as two independent natural numbers thought that "The numerical value of a fraction increases when either the numerator or the denominator increase" (p.511). Hence, students who see fractions as two independent numbers either compare fractions based on the size of the numerator only, or based on the size of the denominator only. On the other hand, students who understand fractions as part-whole should be able to demonstrate the numerical value of fractions using a part-whole model.

Item 1 Which is larger $\frac{3}{5}$ *or* $\frac{1}{5}$*? Illustrate how you got your answer by using a model such as a picture or a diagram representation (Adapted from Scanlon, 2013) (ConT3Q1).*

The correct answer is 3/5 because it has more shaded parts compared to that of 1/5 as illustrated on the pie diagrams below.



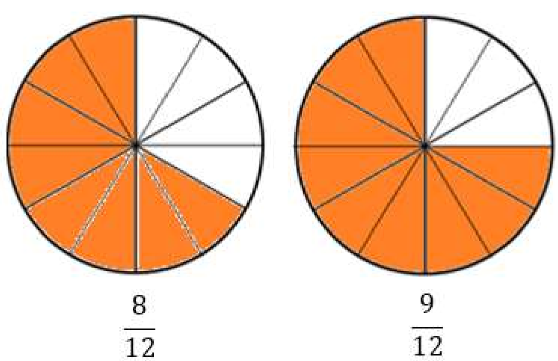
$\frac{3}{5}$ $\frac{1}{5}$

Task 3 Item 1 tests the students' understanding of the numerical value of fractions based on their understanding of fractions as part-whole by asking them to compare fractions. Students who understand that the numerator represents the number of selected parts and the denominator represent the number of all parts may answer that 3/5 is greater than 1/5 because 3/5 has more selected parts compared to 1/5. Students may also answer this item using their procedural knowledge. They may do a cross-product technique to determine which fraction is larger. This is because students may use their procedural knowledge and conceptual knowledge to solve a problem (Hallett, Nunes, & Bryant, 2010). This item asks students to compare fractions using pie or rectangle diagrams so that they can demonstrate their conceptual understanding of the numerical value of fractions. To compare 3/5 and 1/5 using pie or rectangle representations, students should compare the size of the shaded area of the pie or rectangle diagrams that corresponds to the fraction 3/5 and 1/5. This item is used to test Level 2 Competency 4 (order proper fractions using part-whole representation).

Item 2 Which is larger $\frac{2}{3}$ or $\frac{3}{4}$? Illustrate how you got your answer by using a model such as a picture or diagram representation (Adapted from Scanlon, 2013) (ConT3Q2).

Task 3 Item 2 asks the students to compare two fractions smaller than 1. This item requires more advanced understanding than Task 3 Item 1 because the fractions compared have different (unlike) denominators. To compare these fractions, students may use their procedural knowledge by performing a cross-product technique to determine which fraction is bigger, or they may use previous knowledge of whole numbers to find the answer (3/4 is greater than 2/3 because 3 is greater than 2 and 4 is greater than 3). However, this item asks the students to use diagram representations (pies or rectangles) to demonstrate their conceptual understanding of which fraction is bigger. Those answers that do not address this instruction (using procedural knowledge or whole number knowledge) are regarded as not demonstrating the required competency of ordering fractions using part-whole representation and so will be coded as incorrect answers.

The correct answer is 3/4 which is equivalent to 9/12. The pie diagram representing 9/12 has more shaded parts compared to that of 8/12 (which is equivalent to 2/3).

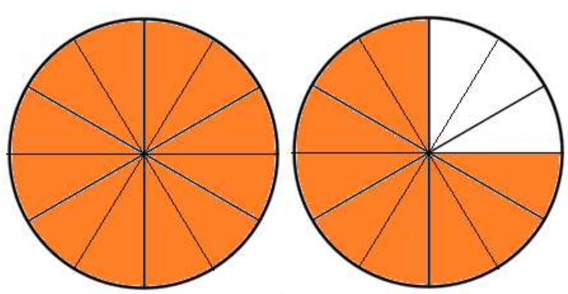


To answer correctly on this item, students should compare the sizes of the shaded parts of the representation of these two fractions. These shaded parts can be compared directly when these two fractions have the same denominators. Thus, these fractions should be transformed into equivalent fractions with a common denominator. The fraction 2/3 becomes 8/12, and 3/4 becomes 9/12. Students should draw diagram representations for 8/12 and 9/12. After that students can compare the number of shaded parts from these two fractions to determine which fraction is bigger. Such representations demonstrate that students have a conceptual understanding of the symbolic notation of fractions and their numerical values, based on a part-whole model. Therefore, this item is used to test students' understanding of fractions as part-whole at level 2, particularly Competency 4 (order proper fractions using part-whole representation).

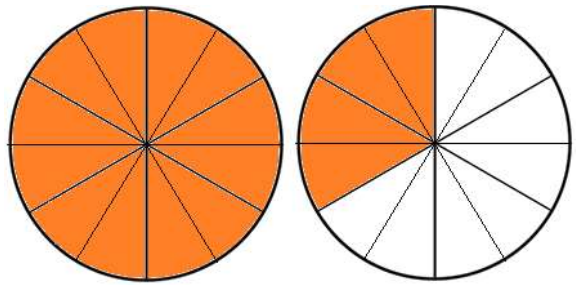
Item 3: Which is larger $\frac{7}{4}$ *or* $\frac{8}{6}$*? Illustrate how you got your answer by using a model such as a picture or diagram representation (Adapted from Scanlon, 2013) (ConT3Q3).*

Task 3 Item 3 tests the students' understanding of the value of fractions greater than 1 by asking them to compare two improper fractions. Similar to Task 3 Item 2, students can directly determine which fraction is greater using a cross-product technique. However, this item asks students to demonstrate their conceptual understanding of which fraction is larger using diagram representation. They are asked to use pie or rectangle diagrams in order to reveal their conceptual understanding of the numerical values of fractions based on a part-whole model. Task 3 Item 3 is more complex than the previous items on fraction comparison. It requires students to be able to draw equivalent fractions for fractions greater than 1 to justify which fraction is larger.

The correct answer is 7/4 because the equivalent fraction of 7/4 which is 21/2 has more shaded parts compared to the equivalent fraction of 8/6 which is 16/12 as shown on the pie diagrams below.



$$\frac{7}{4} = \frac{21}{12}$$



$$\frac{8}{6} = \frac{16}{12}$$

To answer this item correctly, first, students should understand the symbolic notation of improper fractions. To show 7/4 and 8/6, they should be able to draw the referenced whole for each and draw another pie diagram so that they can indicate the number of selected parts. In this case, students should know that the denominator represents the number of all parts within the referenced whole, while the numerator represents the number of all selected (shaded) parts. Second, they should be able to understand equivalent fractions for fractions greater than 1, because the comparison of the numerical values of 7/4 and 8/6 should be done when they have a common denominator to make sure that these fractions are compared for the same size of the whole. This item is used to test students' part-whole understanding at level 3, because it requires an understanding of improper fractions which is hypothesized to emerge at this level. Specifically, this item is used to address Level 3 Competency 3 (order fractions including improper fractions and mixed numbers).

3.3.1.2 Understanding Fractions as Measures

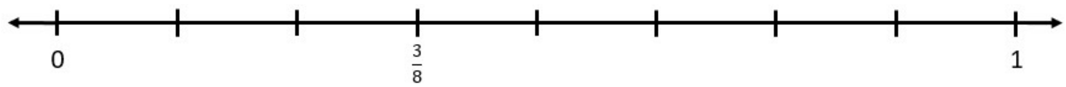
To test the students' understanding of fractions as measures, students are asked to map the symbolic notation of fractions into number lines. Students are asked to put a fraction or several fractions on the number line.

3.3.1.2.1 Task 4. Locating Fractions on the Number Line

Item 1 Show the fraction $\frac{3}{8}$ *on the number line below (ConT4Q1).*

Task 4 Item 1 tests the students' understanding of fractions as a point on a number line. Students at Level 2 may have some understanding of fractions as measures. To answer this item correctly, students should understand that the denominator of 3/8 indicates that the unit (1) is divided into 8 intervals, and the numerator indicates that

the location of the fraction on the number line is on the third interval from zero. Thus, the students must identify where 1 is located on the number line then divide the number line into 8 equal partitions and finally write the fraction 3/8 on the third interval. The correct answer for this item was the location of 3/8 shown on the number line below.

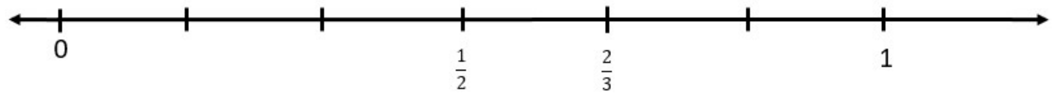


This item is used to address Level 3 Competency 6 (place a fraction on a number line).

Item 2 Show the fraction $\frac{1}{2}$ *on the number line below (Adapted from Scanlon, 2013) (ConT4Q2).*

Task 4 Item 2 is similar to Task 4 Item 1, but it requires students to put a proper fraction with the constraint of another fraction (with unlike denominators) on the number line. This item can be solved conceptually, or it can be solved using some procedures. To solve this item conceptually, students should find the unit of 2/3 and indicate where it is on the number line. Then they can divide the unit into two equal intervals in order to put 1/2 on the number line. The other way is to use procedural knowledge to find the equivalent fractions of 1/2 and 2 /3, which are 3/6 and 4/6. After that, they should indicate where the unit is on the number line and divided it into 6 intervals. After that the fractions 3/6 and 4/6 can be put on the third and fourth interval from zero respectively on the number line.

The number line representation is used to test students' understanding of the symbolic notation of fractions as measures by asking students to map the fraction notation into number lines. Although students may use some of their procedural knowledge to help them in mapping the fractions on the number line, their responses are still considered as evidence of their conceptual understanding of fractions as measures. The correct answer for this item is the location of 1/2 presented on the number line below.

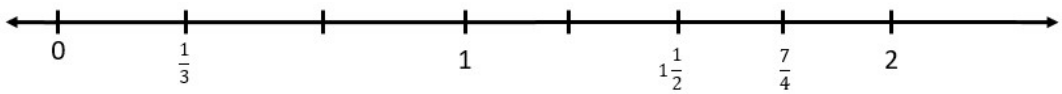


This item is used to address Level 3 Competency 6 (locate a proper fraction on a number line).

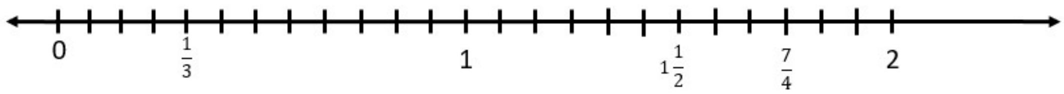
Item 3 Order the fractions $\frac{7}{4}$, $\frac{1}{3}$ *and* $1\frac{1}{2}$ *on the number line below (Adapted from Scanlon, 2013) (ConT4Q3)*

Task 4 Item 3 tests the students' understanding of the magnitude and order property of fractions on the number line. Understanding the magnitude and order property of fractions is essential to understanding fractions as numbers. This item requires students to be able to put fractions (including improper fractions and mixed numbers) on the same number line. Similar to Task 4 Item 2, students may use their conceptual knowledge, or they may use some procedures to help them put the given fractions on the number line.

To answer this item using conceptual knowledge, students should find the unit (1) on the number line and then divide the unit into three equal intervals to put 1/3 on the first interval from zero. Then, they should find the second unit (2) on the number line and they should divide the interval from the first unit to the second unit into four equal intervals and divide the interval from zero to the first unit into four equal intervals. Then, 7/4 is put on the seventh interval from zero. To put 1 ½ on the number line, the interval between the first unit and the second unit should be divided into 2, and the 1 ½ is put on the first interval between the first unit and the second unit because 1 ½ means that the distance of the fraction from zero is one and a half units. Alternatively, this item can be answered using procedural knowledge by finding equivalent fractions with the same common denominator. Using this procedure, the students will find the equivalent fractions for 7/4, 1/3, and 1 ½ are 21/12, 4/12, and 1 6/12. After that, they can determine a unit which has 12 intervals, and put the second unit 12 intervals from the first unit. Then 7/4 is put on the 21st interval from zero, 1/3 is put on the fourth interval from zero and 1 ½ is put in the sixth interval from the first unit. The correct answer for this item is the location of $\frac{7}{4}$, $\frac{1}{3}$ and $1\frac{1}{2}$ shown on the number line below.



or



This item is used to test Level 4 Competency 2 (order improper fractions on a number line).

3.3.1.3 Unbounded Infinity

To test the students' understanding of the infinity property of fractions, students are asked to write the biggest and the smallest fractions that they know and then explain their answers.

3.3.1.3.1 Task 5 Writing the Smallest and Biggest Fractions that They Can

Item 1 Write the biggest fraction that you know. Explain your answer (Adapted from Stafylidou & Vosniadou, 2004) (ConT5Q1).

Item 2 Write the smallest fraction that you know. Explain your answer (Adapted from Stafylidou & Vosniadou, 2004) (ConT5Q2).

Task 5 Items 1 and 2 test the students' understanding of the unbounded infinity concept of fractions. The correct answer for these items is fractions are infinite that there are no smallest or biggest fractions.

Understanding the unbounded infinity of fractions is beyond students' understanding of fractions as measures at Level 3. Students who understand the relationship between the numerator and the denominator of fractions as division are able to answer these items correctly (see Stafylidou & Vosniadou, 2004). By understanding fractions as division, students should understand that the numerator that becomes the dividend is infinite, meaning that there is always another bigger number that can be the numerator. From this, they should know that there is always another bigger fraction from a given fraction, meaning that there is no biggest fraction. Similarly, the denominator which becomes the divisor is also infinite, so there is always a smaller fraction from a given fraction, meaning that there is no smallest fraction. These item tasks are designed to examine the students' understanding of the unbounded infinity of fractions, which is hypothesized to emerge at level 4. Therefore, these items are used to address Level 4 Competency 1 (write the biggest and the smallest fraction they can).

3.3.1.4 Density

To demonstrate the students' understanding of the density property of fractions, students are asked to identify how many numbers between two pseudo-successive and non-successive fractions and then explain their answer.

3.3.1.4.1 Task 6 Finding How Many Fractions lie between Two Fractions

Item 1 How many numbers lie between $\frac{2}{5}$ *and* $\frac{4}{7}$ *? Explain your answer (Adapated from Vamvakoussi & Vosniadou, 2004) (ConT6Q1)*

Item 2 How many numbers lie between $\frac{4}{7}$ *and* $\frac{5}{7}$ *? Explain your answer (Adapated from Vamvakoussi & Vosniadou, 2004) (ConT6Q2)*

Task 6 Items 1 and 2 test the students' understanding of the density concept of fractions. The correct answers for these items are there are infinite numbers between $\frac{2}{5}$ and $\frac{4}{7}$ and between $\frac{4}{7}$ and $\frac{5}{7}$.

Task 6 Item 1 is developed to examine the students' understanding of density on non-successive fractions, while Task 6 Item 2 is on pseudo-successive fractions. Understanding density on two pseudo-successive fractions tends to be more difficult for students than for two non-successive fractions. Students who have a discreteness understanding of whole numbers may think that there are no fractions between 4/7 and 5/7 (Task 6 Item 2*).* In order to understand the density of fractions, students should understand several sub-constructs including fractions as division, fractions as measures, and equivalent fractions. Understanding fractions as division may help the students to understand density because it gives the students an understanding of fractions as single quantities (the result of the division of a numerator by the denominator). Understanding fractions as measures can give students the insight that fractions are quantities and can be ordered and treated as numbers (which can be multiplied, divided and so on). Understanding equivalent fractions can help students to find s(s) between two fractions by enlarging the common denominators. Hence, understanding fractions as division and as measure, and understanding equivalent fractions may help students to understand that between any two fractions there are an unlimited number of fractions. These items are used to address Level 5 Competency 1 (demonstrate that there are unlimited numbers between two fractions).

3.3.1.5 Conceptual Additive Fraction Operations

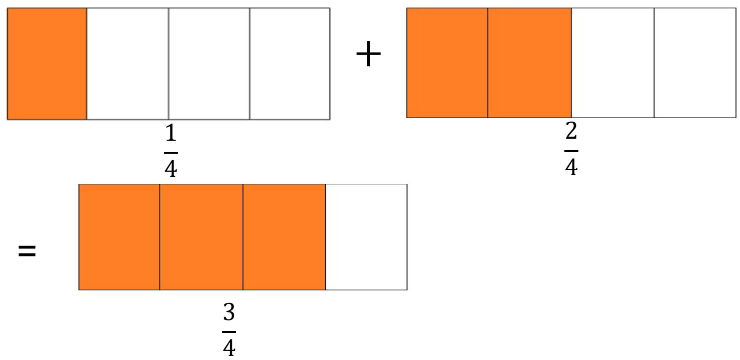
The students are asked to draw a representational model of additive fraction operations to demonstrate that they understand the meaning of additive fraction operations. As discussed before, addition and subtraction in fractions are similar to the addition and subtraction in whole numbers in which addition makes bigger while subtraction makes smaller. Using a representational model, students can show their conceptual understanding of addition, i.e., adding fractions means joining the (selected) parts towards the referenced whole, which produces bigger fractions, while subtracting fractions means separating parts, which produces smaller fractions .

3.3.1.5.1 Task 7 Adding Fractions Using Diagram Representation

Item 1 Draw a pictorial representation for the addition of fractions below. Explain your answer (ConT7Q1).

$\frac{1}{4} + \frac{2}{4}$

Task 7 Item 1 tests the students' understanding of a simple fraction addition. Pictorial representations (pies or rectangles) are used to demonstrate the students' conceptual understanding of the meaning of fraction addition. Students are asked to draw a fraction addition using diagram representation. The correct representation of the addition $\frac{1}{4} + \frac{2}{4}$ is illustrated as follow:

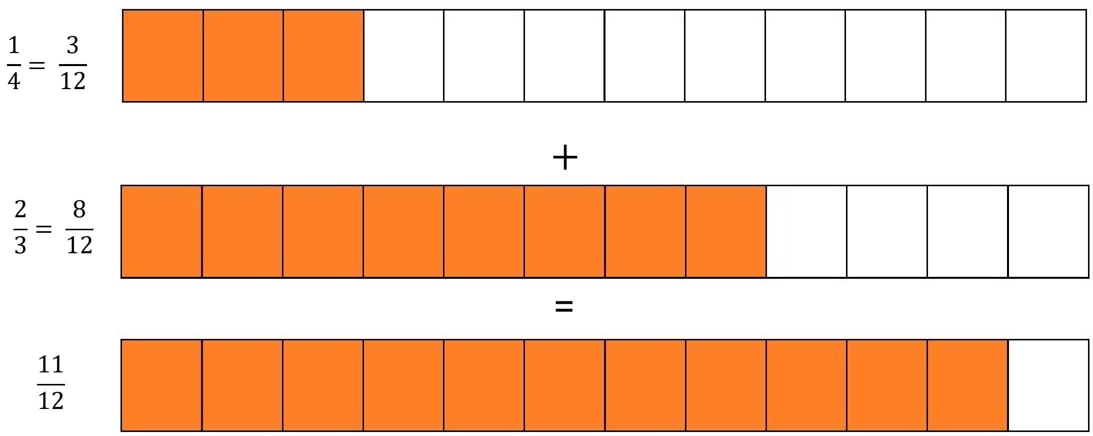


To answer this item correctly, students should understand that the numerators represent the selected parts while the denominator represents the number of all parts of the whole. Next, they should understand that adding fractions 1/4 and 2/4 means that they should join the selected parts of 1/4 with the selected parts of 2/4. Using a pictorial representation, they should be able to demonstrate how 1 selected part from 1/4 is added to 2 selected parts from 2/4 to produce 3 selected parts of 4 parts (which is 3/4 as the result). No procedural knowledge is needed to solve this item when students solve this item using a pictorial representation, but their procedural knowledge may inform students that the answer is 3/4. However, students should represent this fraction addition using pie or other area model representations, which can be used as evidence that they also understand this fraction addition conceptually. This item task is used to address Level 2 Competency 5 (demonstrate fraction addition and subtraction using diagram representation).

Item 2 Draw a pictorial representation for the addition of fractions below. Explain your answer (ConT7Q2).

$$\frac{1}{4} + \frac{2}{3}$$

Task 7 Item 2 tests the students' understanding of a fraction addition with unlike denominators. This item is more complex than the previous item (Task 7 Item 1) because the students cannot add the fractions from the diagram/pictorial representation of 1/4 and 2/3 directly, but should draw equivalent fractions of 1/4 and 1/3 with a common denominator. The correct representation of the addition $\frac{1}{4} + \frac{2}{3}$ is illustrated as follow:



To answer this item correctly, the students should know the meaning of fraction addition, which is joining the selected parts of the same whole. However, they cannot add fractions if the size of the whole is different. Hence, they should know about equivalent fractions with a common denominator, so that they can convert the fractions to equivalent fractions, which have the same size of whole (a common denominator). Student may answer this item in different ways, particularly when finding the equivalent fractions. Some students may use their conceptual knowledge to find the equivalent fractions by drawing a diagram representation, or they may use procedural knowledge to find the equivalent fractions. However, they are asked to demonstrate their understanding of fraction addition in this item by drawing pictorial representations to produce evidence that they have conceptual understanding of additions with unlike denominators. Because this item needs the students to understand equivalent fractions, a competency that emerges at level 2, this item is used to address Level 2 Competency 5 (demonstrate fraction addition and subtraction using diagram representation).

3.3.1.6 Conceptual Multiplicative Fraction Operations

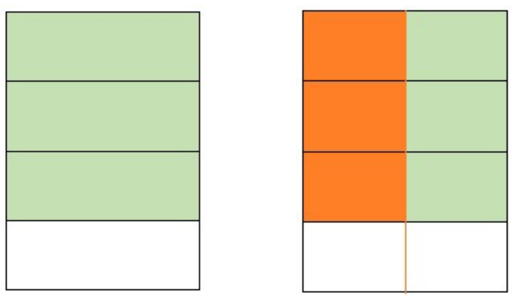
Similarly, to test students' understanding of additive fraction operations, students are asked to draw a representational model of fraction multiplication and division to show that they understand the meaning of multiplicative fraction operations conceptually.

3.3.1.6.1 Task 8 Multiplying and dividing fractions using diagram representation.

Item 1 Draw a pictorial representation for the multiplication of fractions below. Explain your answer (ConT8Q1)

$\frac{1}{2} \times \frac{3}{4}$

Task 8 Item 1 tests the students' understanding of the meaning of fraction multiplication. The correct answer for this item is presented as follow:



$\frac{3}{4}$ $\quad$ $\frac{1}{2}$ of $\frac{3}{4}$

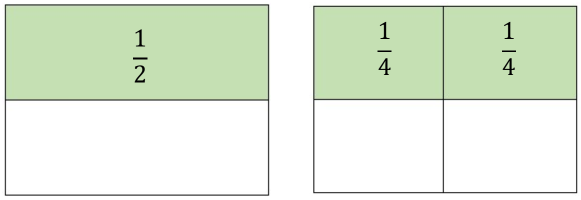
It can be observed that $\frac{1}{2}$ of $\frac{3}{4}$ is three shaded (orange) parts which is $\frac{3}{8}$ of the whole (all parts). Hence, the result of multiplication $\frac{1}{2}$ and $\frac{3}{4}$ is $\frac{3}{8}$.

To solve this item, the students should know the meaning of multiplication in fractions, which is different from the meaning of multiplication in whole numbers. In this case, students should understand that 1/2 multiplied by 3/4 means "how much is 1/2 of 3/4". To find out how much is 1/2 of 3/4, students can draw 3/4 using a pie or rectangle diagram and then draw 1/2 on the 3/4 diagram. The intersection between the area of 1/2 and 3/4 shows how much is 1/2 of 3/4. Students at Level 4 are hypothesized to have conceptual knowledge of fraction multiplication. This item is used to address Level 4 Competency 2 (represent multiplicative fraction operations using diagram representation).

Item 2 Draw a pictorial representation for the division of fractions below. Explain your answer (ContT8Q2)

$\frac{1}{2} \div \frac{1}{4}$

Task 8 Item 2 tests the students' understanding of fraction division. The correct answer for the division $\frac{1}{2} \div \frac{1}{4}$ is illustrated as follows:



$\frac{1}{2} \div \frac{1}{4}$ is finding how many $\frac{1}{4}$ in a half $(\frac{1}{2})$. It can be observed that there are 2 parts of $\frac{1}{4}$ in $\frac{1}{2}$. Hence $\frac{1}{2}$ divided by $\frac{1}{4}$ is 2.

This item is counterintuitive for students who have prior knowledge of division operations with whole numbers because the result produces a greater value. Dividing 1/2 by 1/4 is finding how many 1/4s are in 1/2. To represent this understanding of fraction division, the students can draw a picture where the 1/2 area is shaded. Then, this picture is partitioned into 4 parts. The answer is found by counting how many 1/4 parts are in 1/2 the shaded area of the picture. Students at Level 4 are expected to be able to demonstrate their conceptual understanding of fraction division using diagram representations. Hence, this item is used to address Level 4 Competency 2 (They represent multiplicative fraction operations using diagram representation).

3.3.2 Procedural Item Tasks

The procedural item tasks are developed based on two type of tasks which are: Task 1 Performing Additive Fraction Operations; and Task 2 Performing Multiplicative Fraction Operations. The items for testing both procedural additive operations and procedural multiplicative operations are adapted and extended from Newton (2008) and Newton, Willard, and Teufel (2014).

3.3.2.1 Additive Operations

3.3.2.1.1 Task 1 Performing Additive Fraction Operations

The items are developed to demonstrate students' procedural knowledge of fraction addition and subtraction. These items require the students to apply the formal mathematical procedures of adding and subtracting fractions to get the solutions.

Item 1 Find the sum of the fraction addition below (ProT1Q1)

$\frac{3}{8} + \frac{2}{8}$

Task 1 Item 1 is a fraction addition with the same denominator. The correct answer for this item is presented as follow:

$$\frac{3}{8} + \frac{2}{8} = \frac{5}{8}$$

To solve this item, the students should understand the rule that adding fractions can be executed directly by adding the numerators of fractions while keeping the denominator the same, in the case that the fractions have the same (like) denominator. If they do not know this rule, they may add across the numerators and denominators so that 3/8 + 2/8 is equal to 5/16. This item task is used to address Level 2 Competency 1 (add or subtract proper fractions).

Item 2 Find the sum of the fraction addition below (ProT1Q2)

$$\frac{14}{15} + \frac{2}{3}$$

Task 1 Item 2 tests the students' procedural knowledge of fraction addition with unlike denominators. The correct answer for this item is presented as follow:

$$\frac{14}{15} + \frac{2}{3} = \frac{14}{15} + \frac{10}{15} = \frac{24}{15}$$

To solve Task 1 Item 2 correctly, students need to know that adding fractions can be executed if they have a common denominator. This means that if the fractions have different denominators, they should transform those fractions into equivalent fractions with a common denominator. Thus, they should also understand the procedure of transforming fractions into equivalent fractions with a common denominator. Students may make mistakes in this instance, such as adding across the numerator and denominator because they do not know how to equate the denominators. Students recognize equivalent fractions at Level 2. Hence, the students at Level 2 in the procedural knowledge progressions are expected to answer this item correctly. This item is used to address Level 2 Competency 1 (add or subtract proper fractions).

Item 3 Find the difference of subtraction below (ProT1Q3)

$$5 - \frac{3}{8}$$

Task 1 Item 3 tests the students' procedural knowledge of fraction subtraction with a whole number. The correct answer for this item is presented as follow:

$$5 - \frac{3}{8} = \frac{5}{1} - \frac{3}{8} = \frac{40}{8} - \frac{3}{8} = \frac{47}{8}.$$

In order to answer this item correctly, students need to convert the whole number 5 into a fraction 5/1 before they do a fraction subtraction. After that, they need to transform 5/1 and 3/8 into equivalent fractions with a common denominator. This item involves an improper fraction which is 5/1, and therefore belongs in level 3, because this is the level in which students are hypothesized to understand improper fractions. Therefore, this item is used to address Level 3 Competency 1 (add or subtract improper fractions or mixed numbers).

Item 4 Find the sum of the fraction addition below (ProT1Q4)

$$2\frac{3}{5} + \frac{1}{2}$$

Task 1 Item 4 tests the students' procedural knowledge of fraction addition that involves a mixed number. The correct answer for this item is presented as follow:

$$2\frac{3}{5} + \frac{1}{2} = \frac{13}{5} + \frac{1}{2} = \frac{26}{10} + \frac{5}{10} = \frac{31}{10}$$

There are two ways to answer this item correctly. First, the mixed number 2 3/5 should be transformed into improper fractions and then added to 1/2. Second, the whole number in the mixed number is kept, and the fractions of 3/5 and 1/2 are added. The answer is produced by adding the whole number 2 with the result of the addition of 3/5 and 1/2. In answering this item, some students may make a mistake when transforming 2 3/5 into improper fractions, or they may only add the fractions without adding the whole number 2 in the mixed number 2 3/5.

Students begin to recognize fractions greater than 1 and mixed numbers at Level 3 in the proposed model. At this level, they are expected to expand their procedural skills at Level 2 (adding fractions less than 1 with unlike denominators) to be able to perform fraction addition that involves mixed numbers and fractions greater than 1. In particular, they should be able to transform mixed numbers to improper fractions. Therefore, this item is used to address Level 3 Competency 1 (add or subtract improper fractions or mixed numbers).

3.3.2.1 Multiplicative Operations

3.3.2.1.1 Task 2 Performing Multiplicative Fraction Operations

Task 2 Items 1-6 are developed to test the students' procedural knowledge of fraction multiplication and division. Students should apply the formal procedure of multiplying and dividing fractions in order to get the correct answers.

Item 1 Find the result of the fraction multiplication below

$\frac{2}{15} \times \frac{7}{15}$ *(ProT2Q1)*

Task 2 Item 1 tests the students' procedural knowledge of multiplying fractions by fractions. The correct answer for this item is presented as follows:

$$\frac{2}{15} \times \frac{7}{15} = \frac{14}{225}.$$

To answer this item correctly, students should understand the rules of fraction multiplication, which is that the numerator is multiplied by another numerator, and the denominator is multiplied by another denominator. Therefore, students should multiply 2 by 7 and 15 by 15 directly to get the answer. This item is set to have the same denominator which is 15. The students who have procedural knowledge of fraction multiplication will not be affected by this situation. They will multiply a numerator with a numerator and multiply a denominator with a denominator. However, the students who do not have sufficient knowledge of the fraction multiplication procedure may retain the same denominator for the result. This item is used to address Level 3 Competency 2 (multiply and divide fractions).

Item 2 Find the result of the fraction multiplication below

$\frac{1}{8} \times 24$ *(ProT2Q2)*

The item ProL3MT2 tests the students' procedural knowledge of fraction multiplication with a whole number. The correct answer for this item is presented as follows:

$$\frac{1}{8} \times 24 = \frac{1}{8} \times \frac{24}{1} = \frac{24}{8} = 3$$

This item asks the students to multiply a fraction (1/8) by a whole number (24). To answer this item correctly, students should understand the rules of fraction multiplication, which are that that the numerator is multiplied by the numerator, and the denominator is multiplied by the denominator. To answer this item, students may convert 24 into a fraction which is 24/1 and then multiply it by 1/8, or they may multiply 24 by 1 directly and then divide it by 8 to get the answer. This item involves the whole number 24, which can be represented as an improper fraction, 24/1. This item is used to address the procedural knowledge of fraction multiplication, which is hypothesized to emerge for students at Level 3. In particular, this item is used to address Level 3 Competency 2 (multiply and divide fractions).

Item 3 Find the result of the fraction division below

$\frac{9}{10} \div \frac{3}{10}$ *(ProT2Q3)*

Task 2 Item 3 tests the students' procedural knowledge of fraction division. The correct answer for this item is presented as follow:

$$\frac{9}{10} \div \frac{3}{10} = \frac{9}{10} \times \frac{10}{3} = \frac{90}{30} = 3.$$

To solve this item, the students should understand the rule of flipping the divisor in fraction division. For this item, students should flip 3/10 to be 10/3. After that, they should be able to multiply 9/10 by 10/3 to get the answer. Students may make some mistakes in this item. For example, they may flip the dividend instead of the divisor, or they may directly divide 9 by 3 because the denominators are the same (10). The latter mistake can occur because students misapply the rules of fraction addition or subtraction to the case of fraction division, in which the fractions have the same denominator. This item is used to address Level 3 Competency 2 (multiply and divide fractions).

Item 4 Find the result of the fraction multiplication below (ProT2Q4)

$3\frac{5}{7} \times 4\frac{3}{7}$

The item Task 2 Item 4 is similar to the item Task 2 Item 1, but it involves mixed numbers. The correct answer for this item is presented as follow:

$$3\frac{5}{7} \times 4\frac{3}{7} = \frac{26}{7} \times \frac{31}{7} = \frac{26 \times 31}{7 \times 7} = \frac{806}{49}.$$

To solve this item, students should understand the rule that two mixed numbers cannot be multiplied directly. They should transform the mixed numbers into improper fractions first, before multiplying them. Students may make mistakes in this item by multiplying a whole number by another whole number (3X4), and by multiplying a fraction by another fraction (5/7 X 3/7). This item is used to address Level 4 Competency 1 (multiply and divide improper fractions or mixed numbers).

Item 5 Find the result of the fraction division below (ProT2Q5)

$2\frac{1}{9} \div 3$

Task 2 Item 5 tests the students' procedural knowledge, which is similar to Task 2 item 3, but is applied to a mixed number and a whole number. The correct answer for this item is presented as follow:

$$2\frac{1}{9} \div 3 = \frac{19}{9} \div \frac{3}{1} = \frac{19}{9} \times \frac{1}{3} = \frac{19 \times 1}{9 \times 3} = \frac{19}{27}$$

To answer this item correctly, students should understand the rule that in fraction division, mixed numbers should be transformed into improper fractions. Moreover, they should know that the divisor (3 or 3/1) should be flipped to become 1/3. Students may make mistakes in this item. They may not transform 2 1/9 into improper fractions or they may not flip the divisor. They may also make a mistake by only dividing 1/9 by 3 and excluding the whole number 2. This item is used to address Level 4 Competency 1 (multiply and divide improper fractions or mixed numbers).

Table 3.2 The hypothesized order of acquisition of competencies corresponding to items and tasks for fraction learning progression

	Conceptual	Competencies	TASK	Procedural	Competencies	TASK
Level 1 No Fraction Understanding	No or incomplete conceptual and procedural understanding of fractions, less than what is expected at Level 2					
Level 2 Part-Whole	*Students understand a fraction as representing a part-whole but consider fractions always to be smaller than 1. They do not see a fraction as single number, but they see a fraction as a representation of parts and their referenced whole.*	They generate the symbolic notation of proper fractions from pie diagrams They generate a pie diagram to represent a proper fraction They order proper fractions. They can demonstrate proper fraction addition and subtraction using a diagram representation	Task 1: Items 1, 2 Task 2: Item 1 Task 3: Items 1, 2 Task 7: Items 1, 2	*Students have knowledge about the procedure for fraction addition and subtraction but are limited to proper fractions, for which the total is less than 1.*	They add or subtract proper fractions	Task 1: Items 1, 2
Level 3 Improper Fractions and Fractions as Measures	*Students understand if the numerator is greater than the denominator, then the fraction is greater than the referenced whole and vice versa. They also*	They generate the symbolic notation of proper fractions from pie representations with unequal partitions and the symbolic representation of improper fractions from pie diagrams.	Task 1: Items 1, 2, 3, 4, 5	*Students expand their procedural knowledge of additive and multiplicative fraction operations including improper fractions and mixed numbers.*	They add and subtract improper fractions or mixed numbers. They multiply and divide a fraction by another fraction.	Task 1: Items 1,2 3, 4 Task 2: Items 1, 2, 3

	Conceptual	Competencies	TASK	Procedural	Competencies	TASK
	understand fractions as measures and conceive a fraction as a single number (not two independent numbers) on the number line.	They shade a pie diagram to represent an improper fraction They order improper fractions and mixed numbers They place a proper fraction on a number line	Task 2: Items 1, 2 Task 3: Items 1, 2, Task 4: Item 1			
Level-4 Unbounded infinite numbers of fractions	*Students view fractions as unbounded infinite numbers (there is no smallest or biggest fraction)*	They write the biggest and the smallest fraction that they can They order improper fractions on a number line	Task 1: Items 1, 2, 3, 4, 5 Task 2: Items 1, 2 Task 3: Items 1, 2, 3 Task 5: Items 1, 2 Task 4: Items 1, 2, 3	*Students advance their procedural knowledge of multiplicative fraction operations from the previous level. The fluency of performing fraction operations emerges at this level*	They multiply and divide improper fractions or mixed numbers	Task 1: Items 1, 2, 3, 4 Task 2: Items 1, 2, 3, 4, 5 and 6
Level-5 Understanding the density of fractions and Fractional Fluency	*Students understand the density property of fractions, i.e. that there are unlimited numbers between any two fractions*	They demonstrate that there are unlimited numbers between two fractions They represent multiplicative fraction operations using a diagram representation	Task 1: Items 1, 2, 3, 4, 5 Task 2: Items 1, 2 Task 3: Items 1, 2, 3 Task 5: Items 1, 2 Task 4: Items 1, 2, 3 Task 6: Items 1, 2 Task 8: Items 1, 2	*Students have fluency when performing additive and multiplicative fraction operations as they are demonstrated at Level 4*	They perform complex fraction operations, which involve additive and multiplicative operations	Task 1: Items 1, 2, 3, 4 Task 2: Items 1, 2, 3, 4, 5 and 6 Task 3: Items 1, 2, 3

3.4 Comparison with Previous Work on Fraction Learning Progressions

This chapter developed the hypothesized model of fraction learning progression and the item tasks which are used to assess competencies within each level of the learning progression. The proposed model is hypothesized to consist of two knowledge dimensions underlying the progression of students' learning in fractions: conceptual and procedural knowledge. The conceptual knowledge dimension is developed to capture the emergence of students' understanding of the symbolic notation of fractions and the meaning of fraction operations, while the procedural knowledge dimension is developed to capture the development of students' procedural knowledge of fraction operations (addition, subtraction, multiplication and division).

A typical learning progression model is usually developed based on big ideas that summarize the essential knowledge and skills of a particular domain of learning. After that, progress variables are developed based on these big ideas to describe the progressions of specific knowledge and skills, which are organized into several hierarchical levels or blocks of learning development (e.g. Arieli-Attali & Cayton-Hodges, 2014; Gunckel, Mohan, Covitt, & Anderson, 2012; Jin & Anderson, 2012). Hence, in such models, a learning progression is constructed based on the content knowledge of a domain of learning, and usually has a unidimensional knowledge progression. The proposed model of the fraction learning progression is developed in a different way. It is crafted based on the dimensions of knowledge that are supposed to underlie the development of students' learning in mathematics. This knowledge is distinguished into two dimensions: conceptual and procedural knowledge. These two dimensions of mathematical knowledge are structured into the hierarchical levels of a learning progression. Thus, the proposed model expands the typical learning progression model from a unidimensional into a two-dimensional progression of knowledge.

There are several advantages to having two knowledge dimensions - conceptual and procedural - in the proposed model of fraction learning progression, compared with the previous general learning progression models. In particular, the assessment of students' knowledge and skills in this model can be more detailed and can provide information about the specific areas of strength and of weakness of students' learning, in terms of whether they are found in the area of conceptual or procedural knowledge. This type of detailed information can give more effective feedback to teachers in order to improve their instruction, and to students themselves to direct their self-study.

There is some previous work that developed a fraction (rational number) learning progression model. Arieli-Attali and Cayton-Hodges (2014) developed a rational number learning progression model (including fractions) which has five levels, namely early part-whole understanding, fractions as units, fractions as single numbers and fractions as measures, representational fluency, and a general model of rational numbers. Another work was performed by Wright (2014), who developed the Hypothetical Learning Trajectory (HLT) of rational numbers. He developed four hierarchical levels of rational number learning, which applied Kieren's sub-constructs (measure, operator, quotient, and ratio). These levels (from lowest to highest) are unit forming, unit coordination, equivalence, and comparison. Finally, Confrey et al. (2011) developed a learning trajectory to capture the development of the fraction concept, based on the common core state standards in the American Curriculum (CCSS). They begin to introduce fractions as part-whole at grade 3, equivalent fractions and fraction comparison at grades 3 and 4, and fraction additive fraction operations at grades 4 and 5.

All the fraction learning progression models from the previous work discussed above were developed to capture the development of the concept of fractions, where fractions can be interpreted in several sub-constructs, such as part-whole, measure, operator, quotient, and ratio. In contrast, the present research developed the hypothetical model of learning progression in order to capture the development of students' understanding of the symbolic notation of fractions. Hence, for example, the present learning progression did not include the development of students' concept of part-whole, which is a concept related to partitioning an object (continuous or discrete) into the same size of parts (Behr et al., 1983), and which is a prerequisite for understanding the concept of fractions. The present study investigated only students' understanding of the symbolic notation of fractions.

The present research also differs from prior fraction learning progressions in that it proposes two new levels of fraction understanding: understanding the unbounded infinity and density of fractions. These additional levels capture important properties of fractions that are radically different from the properties of whole numbers and need to be included in fraction learning progressions.

Another difference between the present research and the earlier work lies in the way the model was developed. For example, Arieli-Attali and Cayton-Hodges (2014) decided on the progress variables and structured them into the hierarchical levels of a learning progression, and then defined what students know and can do for each level. In contrast, in developing the model for this research, the sub-construct progressions were structured into two-dimensional knowledge progressions. Subsequently, conceptual competencies and procedural competencies for each level were developed. Hence, the present research developed a two-dimensional knowledge learning progression (conceptual and procedural), while the previous work developed a unidimensional knowledge progression.

In the proposed model, the development of the students' conceptual and procedural knowledge is differentiated, and the competencies which correspond to conceptual and procedural knowledge are also produced for each level. Hence, the progression of the students' learning in fractions can easily be tracked, based on the essential knowledge in mathematics learning: conceptual and procedural knowledge. Identifying the development of the students' learning in terms of conceptual and procedural knowledge is important for diagnostic assessment purposes and curriculum development.

In summary, the proposed model of fraction learning fraction progression followed a different approach from the prior work in modelling the development of learning fractions. In the proposed model, the development of fraction knowledge is structured into two essential dimensions of knowledge in mathematics: conceptual and procedural knowledge. The conceptual knowledge dimension focuses on describing the development of the students' understanding of the symbolic notation of fractions and the meaning of fraction operations, while the procedural knowledge dimension focuses on describing the development of students' knowledge of rules or procedures for fraction operations. This two-dimensional knowledge of learning fractions makes the proposed model different from the previous work. The information on the students' progression in terms of the conceptual and procedural knowledge dimensions in the proposed model gives more detailed information than the previous work about student competencies, enabling accurate diagnostic assessment, instruction and curriculum development.

3.5 Summary of the Chapter

This chapter presented the proposed model of fraction learning progression and the item task development. The proposed model was developed based on two knowledge dimensions of mathematics learning: conceptual and procedural knowledge. The conceptual knowledge dimension captured the emergence of the students' understanding of the symbolic notation of fractions and the meaning of fraction operations, while the procedural knowledge dimension captured the emergence of the students' understanding of the rules and procedures of fraction operations. The conceptual knowledge dimension consisted of a five level progression, which were, from lowest to highest: no understanding of fractions, part-whole, improper fractions and fractions as measures, unbounded infinity, and density. Meanwhile, the procedural knowledge dimension consisted of four level progression which were, from lowest to highest: no procedural knowledge, additive fraction operations, additive and multiplicative fraction operations, and advanced procedural knowledge of additive and multiplicative fraction operations.

The item tasks were developed to address the conceptual and procedural competencies for each level. For the conceptual competencies, the symbolic notation of fractions as representations of part-whole were tested using tasks which asked students to map the fraction notation into pie diagrams and vice versa. The pie diagrams were used to assess the students' understanding of proper fractions, improper fractions and equivalent fractions. The symbolic notation of fractions as a representation of measure were tested using number lines. The students were asked to map fraction notation into number lines. The infinity of fractions was tested by asking students to write the biggest and the smallest fractions they know, and how many fractions are present between two fractions. Finally, the conceptual understanding of additive and multiplicative fraction operations was tested by asking students to draw a representational model of additive and multiplicative fraction operations. The procedural competencies were tested using items that required students to apply formal mathematical procedures to solve additive and multiplicative fraction operations.

The proposed model of fraction learning progression developed in this research is different from previous work in at least in three aspects. First, the present research developed a learning progression of the students' development of the symbolic notation of fractions, while the previous work developed fraction learning progressions of the students' development of the concept of fractions. Second, the present research covered properties of fractions, such as unbounded infinity and density, which were not covered in the previous work. Finally, the present research developed a two-dimensional knowledge of fraction learning progression, namely conceptual and procedural, while the previous research developed unidimensional knowledge progressions.

CHAPTER 4 : ANALYSIS FROM THE COGNITIVE INTERVIEW

4.1 Introduction

The proposed model of fraction learning progression and the items have been developed and were described in Chapter 3. There are five hierarchical levels of learning fractions hypothesized, which are (from lowest to highest): no understanding of fractions, part-whole, improper fractions and fractions as measures, unbounded infinity, and density. These hierarchical levels are structured into two dimensions of knowledge: conceptual knowledge and procedural knowledge. The items have been developed to identify the students' level of fraction knowledge in the learning progression. These items were developed based on the competencies within each level of the conceptual and procedural knowledge dimensions.

The purpose of the present chapter is to present empirical evidence from a cognitive interview to validate the hypothesized model of fraction learning progression and to improve the item tasks. To validate the model, the hypothesized order of acquisition of fraction conceptual and procedural knowledge is examined through students' responses to the items, on the tasks in which the items are classified, on the order of the items in the tasks, and finally on the hypothesized level progression for the conceptual and procedural knowledge dimensions respectively.

Accordingly, the analysis of the results is structured into two sections: the conceptual knowledge dimension and the procedural knowledge dimension. Each section consists of four subsections. The first sub-section presents the within-item analysis, the purpose of which is to investigate whether the participants understood the instructions and the items as intended by the investigators, and whether the responses provided by the participants reflect the hypothesized competencies. The second sub-section presents a within-task analysis of the items. The purpose of this analysis is to investigate whether the order of acquisition of the items within each task is consistent with the hypothesized order. The next sub-section discusses how the learning progression was used to assign the participants into levels, and to examine whether the within-level results agree with the hypothesized order of acquisition of the items. More specifically, the researcher is interested in finding out whether there were some participants who were able to respond correctly to some items at the upper levels of the progression but failed to exhibit understanding of the hypothesized competencies at the lower levels. This type of analysis will be done first for the items in the conceptual knowledge dimension and then for the items at the procedural knowledge dimension. At the end, the relationship between conceptual and procedural knowledge is discussed.

4.2 Method

4.2.1 Participants

Fifteen students from a junior high school in Bogor, Indonesia, participated in the cognitive interviews. They comprised 4 students at grade 7, 6 students at grade 8, and 5 students at grade 9. The participants were approximately 13, 14, and 15 years of age for grades 7, 8, and 9 respectively. For each grade, the participants were selected to represent low, medium, and high achieving students, based on information from their teacher. The distribution of the participants is presented in Table 4.1

Table 4.1 The distribution of participants across the levels and their achievement in mathematics

No	Participant	Grade	Achievement in Mathematics
1	Participant 7-IS	7	Low
2	Participant 14-DE	7	Medium
3	Participant 4-JA	7	High
4	Participant 5-RI	7	Low
5	Participant 9-OK	8	Low
6	Participant 8-NA	8	Medium
7	Participant 12-AU	8	Medium
8	Participant 13-FI	8	medium
9	Participant 5-LA	8	High
10	Participant 10-BA	8	Medium
11	Participant 17-FA	9	Medium
12	Participant 6-JO	9	High
13	Participant 11-RE	9	High
14	Participant 3-JI	9	High
15	Participant 16-AKh	9	High

4.2.2 Materials

The materials that were used in the cognitive interview were the conceptual and procedural items developed in Chapter 3.

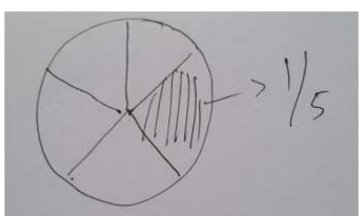
4.2.3 Procedure

During the interview, the participants received one item at a time on a card. The participants were asked to think aloud while responding to the item on the card. In the introduction of the interview (Exhibit 4.1), the researcher gave an example of how to think aloud while answering the item.

Exhibit 4.1 Instructions and an example of thinking aloud in the cognitive interview

RESEARCHER: Thank you for your participation in this interview. Today I will give you some cards with mathematics problems. I want you to solve these problems and explain how you get the answers. If you find any words that you don't understand, please let me know. Please keep talking aloud while answering the questions and describing what you think. You can make any notes and draw on the cards. I will give you an example.

If a pizza is divided for five people, what portion of the pizza will each person get? I would answer the question like this. For example, there is a pizza which is usually in a circle shape (the researcher made a circle). Then, it is shared by 5 people. In order to get a fair share, I divide the pizza into 5 equal sizes (the researcher drew lines to make 5 partitions of the circle). It means that each person will get 1/5 of the pizza (the researcher shaded one part of the five partitions of the pizza to show the final answer).



The length of the interview was limited to 30 minutes. To optimize the 30 minute interview, the items were given adaptively to each participant. For the first item, all participants received Task 1 Item 1 (generating a proper fraction from a given pie diagram), but for the following items, the participants received different items, depending on their answers to the first task. If their answer was correct, they received an item from a higher level derived from the hypothetical model. If their answer was not correct, then they received another item from the other competencies at the same level. The interview was terminated after 30 minutes, or earlier if there was enough evidence to identify the participant's level for both the conceptual and procedural knowledge dimensions.

4.3 Results

The results from the cognitive interviews are organized into two consecutive parts: the conceptual and the procedural knowledge dimension. Four types of analysis are conducted for each part: within-item analysis; within-task analysis; assigning students to levels; and within-level analysis.

4.3.1 Conceptual Knowledge Dimension

4.3.1.1 Within-Item Analysis

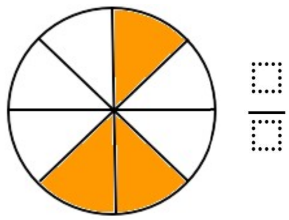
The aims of the within-item analysis are: 1) to examine whether the instructions for each item were understood as intended by the participants; and 2) to examine whether the participants' responses to each item could be used to infer the student's competency level. To achieve these goals, the correct and (or) incorrect answers for each item are discussed.

By looking at the participants' responses, overall, it can be inferred that the participants understood the instructions for the items as intended, and there were no responses that indicated that the students did not understand what they were intended to do. Furthermore, the participants' responses seemed to reflect the intended competencies underlying each item accurately, as hypothesized in the proposed model. Detailed examples will be given below to demonstrate the above claims.

As discussed in Chapter 4, section 4.3, the conceptual items are classified into eight tasks, namely: Task 1: generating a fraction from a pie diagram; Task 2: shading a pie diagram to represent a fraction; Task 3: comparing fractions; Task 4: locating fractions on a number line; Task 5: finding the smallest and biggest fractions; Task 6: finding how many fractions lie between two fractions; Task 7: adding fractions using diagram representation; and Task 8: multiplying and dividing fractions using diagram representation. The within-item analyses are presented by following the structure of the tasks.

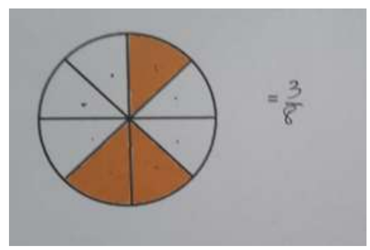
4.3.1.1.1 Task 1 Generating a Fraction from a Pie Diagram

Item 1 - Write the fraction for the shaded part below (Adapted from Scanlon, 2013) (ConT1Q1)



All the participants, with one exception, answered this item correctly. Exhibit 4.2 presents the response from participant 14-DE (medium achieving student), who successfully generated a fraction from a pie diagram, when responding to item 1 in task 1.

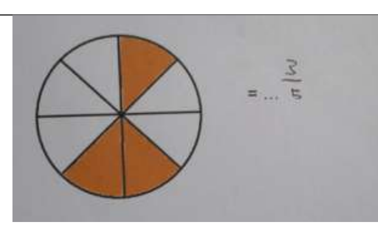
Exhibit 4.2 The answer of participant 14-DE on Task 1 Item 1 of the conceptual knowledge dimension

PARTICIPANT: Write the fraction for the shaded part below. (the participant counted all parts of the circle) PARTICIPANT: 3/8 RESEARCHER: Can you tell me how you got 3/8? PARTICIPANT: There are 8 parts, and three are shaded so it's 3/8	

The participant determined the numerator by counting the number of shaded parts in the diagram, and determined the denominator by counting the number of all the parts in the pie diagram. From this response, it can be inferred that the participant understood the symbolic notation of fractions as a representation of part and whole of a diagram/object.

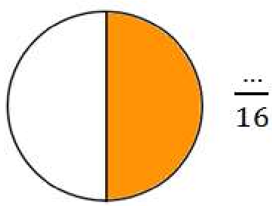
Only one participant did not give the correct answer on item 1 in task 1. Exhibit 4.3 shows the participant's answer.

Exhibit 4.3 The answer of the participant 9-OK on Task 1 Item 1 of the conceptual knowledge dimension

PARTICIPANT: "These are one, two, three, four, five parts (the participant counted the five unshaded parts of the circle) and these are three parts (the participant pointed to the three shaded parts of the circle) so there are 8 parts. These are 3 shaded parts which shows the top number, and these are 5 unshaded parts which shows the bottom number."	

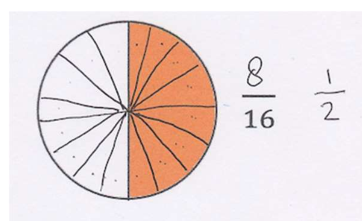
From Exhibit 4.3, it can be seen that the participant did not know how the fraction symbol should be generated from the pictorial representation. The participant considered the number of shaded parts as the numerator and the unshaded parts as the denominator. Hence, the participant's mistake was not caused by a misinterpretation of the task (including the instruction for the task and the picture), but by his misunderstanding of how to generate the fraction notation (numerator and denominator) from the part whole representation.

Item 2 - Write the numerator of the fraction for the shaded parts below (ConT1Q2)



From the 15 participants who received Item 2 in Task 1, 11 participants correctly answered the item. Exhibit 4.4 demonstrates one of the answers from Participant 17-FA who answered Item 2, Task 1 correctly.

Exhibit 4.4 The answer from participant 17_FA on Task 1 Item 2 of the conceptual knowledge dimension

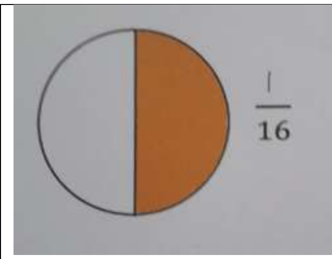
PARTICIPANT: Write the numerator of the fraction for the shaded parts below PARTICIPANT: One, two … (the participant counted all parts of the circle). Draw the picture? RESEARCHER: Yes, you can. PARTICIPANT: Make more partitions like this (the participant drew lines to make 8 partitions on the shaded area of the circle). This one is also the same (the participant drew lines again to make 8 partitions on the unshaded area of the circle). So we got 8/16.	

From Exhibit 4.4 it can be seen that, to find the numerator, the participant made 16 partitions on the pie diagram (8 partitions on the shaded are and 8 partitions on the unshaded area), and successfully determined 8 as the numerator. This evidence shows that the participant understood equivalent fractions that she could generate a fraction 8/16 for the half-shaded area of the pie diagram.

On the other hand, there were four participants who answered this item incorrectly. Exhibit 4.5 shows one of the answers.

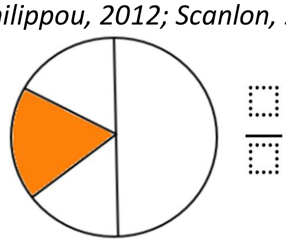
Exhibit 4.5 The answer from participant 12-AU on Task 1 Item 2 of the conceptual knowledge dimension

PARTICIPANT: Write the numerator of the fraction for the shaded part below
PARTICIPANT: One (1)
RESEARCHER: Can you tell me how you got the answer "one"?
PARTICIPANT: Because one is the number for this part (the participant pointed to a half-shaded area of the circle), and 1 circle is 16 here (she pointed to the denominator of 16). What is being asked is only the numerator, which is this shaded part, one.



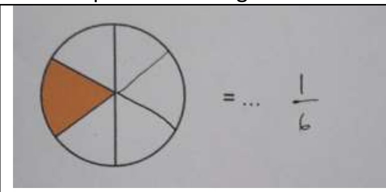
The participant could not find the right numerator for the (equivalent) fraction to represent the half-shaded area of the pie diagram. The participant answered 1 as the numerator without considering that the denominator was 16. The participant did not understand the relationship between the numerator and the denominator in representing a shaded part of the diagram. As a result, the participant simply put 1 as the numerator because it represents the number of the shaded part of the pie diagram without taking into account the given denominator 16. Hence, it can be inferred that the participant's incorrect answer in this item is because of insufficient knowledge on the part of the participant about the relationship between the numerator and denominator in representing equivalent fractions.

Item 3 - Write the fraction for the shaded part below (Adapted from Pantziara & Philippou, 2012; Scanlon, 2013) (ConT1Q3)



From 15 participants, 10 participants answered the item correctly. Exhibit 4.6 demonstrates one of the participants' answer.

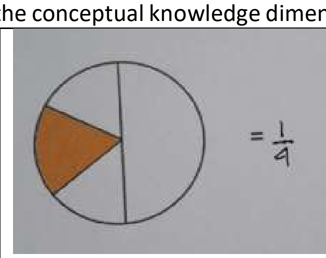
Exhibit 4.6 The answer of participant 13-JI for Task 1 Item 3 of the conceptual knowledge dimension

PARTICIPANT: Write the fraction for the shaded part below. PARTICIPANT: These parts have different sizes. We have to draw lines to make equal sizes. RESEARCHER: What do you mean equal sizes? PARTICIPANT: The parts (Then the participant drew lines to make equal partitions of the circle) RESEARCHER: Okay PARTICIPANT: There are 1, 2, 3, 4, 5, 6 parts (the participants counted all of the parts of the circle); the denominator is 6, and only one part is shaded, so the answer is 1/6	

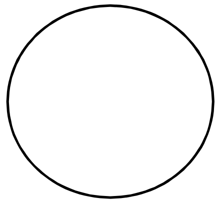
The participant recognized that the partitions on the pie diagram were not equal sizes. To solve this problem, the participant drew additional lines to make six equal parts of the diagram. The participant determined the denominator by counting the number of all the equal parts in the diagram, and determined the numerator by counting the number of the shaded part of the diagram. From this response, it can be inferred that the participant understood that a fraction (both numerator and denominator) should be generated from the number of equal partitions of the diagram (object).

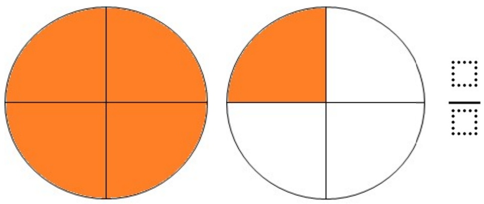
Five participants incorrectly answered Task 1 Item 3. Exhibit 4.7 shows one of the participants' answers.

Exhibit 4.7 The answer of participant 7-IS for Task 1 Item 3 of the conceptual knowledge dimension

PARTICIPANT: Write the fraction for the shaded part below. PARTICIPANT: The shaded only 1, 1/4 (the participant wrote down 1/4 as the answer on the card) RESEARCHER: Why is this 1/4? PARTICIPANT: Because the shaded part is only 1. The 4 is all of the parts	

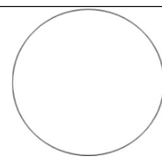
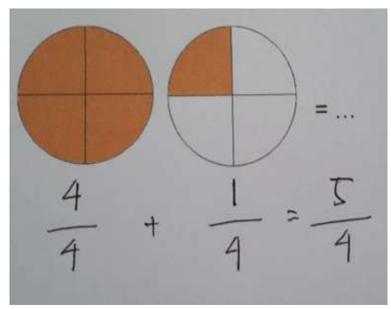
The evidence from Exhibit 4.7 shows that the participant did not understand that all the parts in the pie diagram should be equal in size. The participant counted all the parts, regardless of their sizes, to represent the denominator. From this response, it can be inferred that the participant did not understand that a fraction (in this case the denominator) should be generated from the number of equal partitions of the diagram (object). Hence, the participant's mistake in this item reveals that he did not understand the equal-size principle in representing fractions.

Item 4 If the figure  *is the whole, write the fraction for the shaded part below (ConT1Q4)*



From 12 participants who received this item, five participants answered the item successfully. Exhibit 4.8 shows one of the participants' answers.

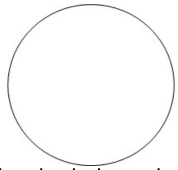
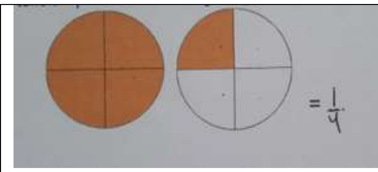
Exhibit 4.8 The answer of participant 3-JI for Task 1 Item 4 of the conceptual knowledge dimension

<table>
<tr>
<td>PARTICIPANT: If the figure  is the whole, write the fraction for the shaded part below
PARTICIPANT: The first circle consists of 4 parts, all parts are shaded. So the fraction is 4/4 or 4 shaded parts of the 4 parts. The second circle consists of 4 parts and only 1 part is shaded. This is 1/4. They are joined together and it becomes 5/4.</td>
<td>
</td>
</tr>
</table>

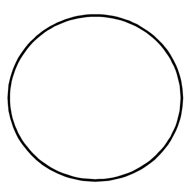
The participant considered that the denominator is represented by the number of all parts in one circle (a pie diagram), and the numerator is represented by the number of the shaded part(s) within each circle. The participant translated the shaded part(s) for each circle into the symbolic notation of fractions, and joined them to get an improper fraction. This evidence demonstrates that the participant understood the symbolic notation of improper fractions as a representation of part and whole.

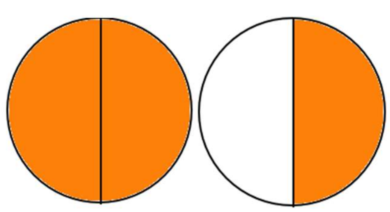
Seven participants failed to answer the item correctly. Exhibit 4.9 demonstrates one of the participants' responses.

Exhibit 4.9 The answer of participant 14-DE for Task 1 Item 4 of the conceptual knowledge dimension

PARTICIPANT: If the figure  is the whole, write the fraction for the shaded part below PARTICIPANT: Is it the whole? (the participant pointed to the circle in the task) RESEARCHER: Yes (The participant was silent for a moment and then wrote down 1/4 on the card) PARTICIPANT: 1/4 RESEARCHER: Can you tell me how you got the answer 1/4? PARTICIPANT: From the other picture (the participant pointed to the circle which represents 1/4) RESEARCHER: How about this picture? (the researcher pointed a circle which was fully shaded) PARTICIPANT: Ehmm … RESEARCHER: I mean, why is this fraction determined by this circle only (the researcher pointed to the circle where one part is shaded) and does not involve this fully shaded circle? PARTICIPANT: Because if they are fully shaded, the values are the same. RESEARCHER: What do you mean? PARTICIPANT: The circle is divided by 4, and 4 parts are shaded, so the numerator and the denominator are the same. So the fraction is only determined by the other circle which is 1/4.	

The evidence in Exhibit 4.9 indicates that the participant believed that fractions are always smaller than one. The participant ignored the pie representation which was fully shaded because it would have the same number for both the numerator and denominator. From this evidence, it can be inferred that the participant did not understand improper fractions as a representation of part and whole, and the participant's understanding is still limited to fractions smaller than 1.

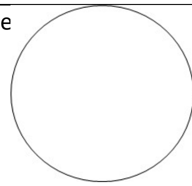
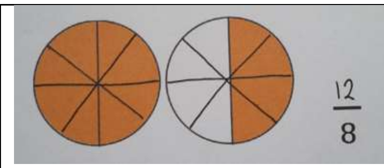
Item 5- If the figure  *is the whole, write the numerator for the shaded part below (ConT1Q5)*



$\frac{\dots}{8}$

Of the six participants who received this item, five of them answered the item correctly. Exhibit 4.10 represents the response from Participant 5-LA.

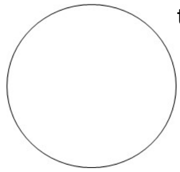
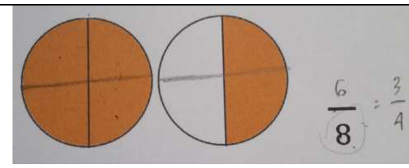
Exhibit 4.10 The answer of participant 5-LA for Task 1 Item 5 of the conceptual knowledge dimension

PARTICIPANT: If the figure  is the whole, write the numerator of the fraction for the shaded parts below PARTICIPANT: Eh.. This is a full circle and it is fully shaded. Therefore it is 1. Since there are 8 parts, it is written here 8 (the denominator of the given fraction), so it is 8/8. While this one, this is a half part. A half part of the 8 parts is 4. Hence, we just add 8 to 4 which equals 12. The result is 12/8 (the participant wrote down the answer). RESEARCHER: Could you demonstrate the answer using diagram representations? (the participant drew lines to make 8 partitions for each circle).	

The participant understood that the shaded parts of the pie diagram should be represented in a fraction with the denominator 8. Hence, she represented the value of the fully shaded circle as 1 and converted it to 8/8. After that she represented the value of a half shaded circle as 4/8, and joined the 8/8 with 4/8 to get the answer 12/8. From this response, it can be inferred that the participant understood equivalent fractions for improper fractions.

There was one participant who answered this item incorrectly. Exhibit 4.11 presents the answer for Participant 6-JO.

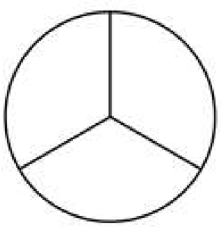
Exhibit 4.11 The answer of participant 6-JO on Task 1 Item 5 of the conceptual knowledge dimension

PARTICIPANT: If the figure  the whole PARTICIPANT: This is already 4 parts so they are made into 8 parts. So … these are 8 parts as the denominator. 1, 2, 3, 4, 5, 6 parts of which are shaded. So this is 6/8 , or it can be simplified to 3/4.	

From Exhibit 4.11, it can be seen that the participant translated the denominator 8 as the number of all parts from the two circles, and determined the numerator as the number of the shaded parts from both circles (after adding additional lines to separate each circle into four parts). This response shows that the participant did not understand improper fractions because the participant made an error in translating the denominator 8 into a pie representation.

4.3.1.1.2 Task 2 Shading Pie Diagrams to Represent Fractions

Item 1 Shade the shape to show the fractions below (ConT2Q1).



$\frac{2}{3}$

Of the two participants who received this item, one of them answered the item correctly. Exhibit 4.12 demonstrates the answer from Participant 7-IS, who correctly shaded the pie diagram to represent a proper fraction.

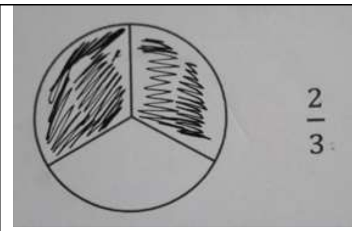
Exhibit 4.12 The answer of participant 7-IS for Task 2 Item 1 of the conceptual knowledge dimension

PARTICIPANT: Shade the shape to show the fractions below

(Then the participant shaded two parts of the circle)

RESEARCHER: Can you explain how you got the answer?

PARTICIPANT: The parts of the circle are 3, the shaded parts are only 2. The unshaded part is only one. If all the parts are shaded, then it is 3/3. If the shaded parts are only two, then it is 2/3.



The participant shaded two parts of the pie diagram which has three equal partitions to represent the fraction 2/3. This indicates that the participant understood the symbolic notation of fractions as representing part and whole.

The other student (Participant 8-OK) answered the item incorrectly. Exhibit 4.13 demonstrates his answer.

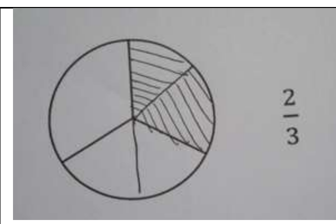
Exhibit 4.13 The answer of participant 8-OK for Task 2 Item 1 of the conceptual knowledge dimension

PARTICIPANT: Shade the shape to show the fractions below

PARTICIPANT: Two-thirds. There are 3 parts in this circle. (The participant looked confused. Then, the participant drew lines to make additional partitions so that there are 5 partitions, and then he shaded two parts)

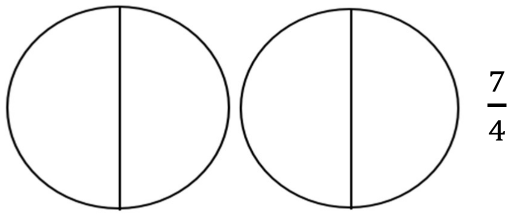
RESEARCHER: Can you tell me why you made additional lines here?

PARTICIPANT: Because there are only 2 and 3 here (the participant pointed to the numerator and denominator of the fraction 2/3). Previously this picture only had 3 parts. So if the two parts are shaded than the remaining part is only one. I drew these additional lines to make 2 shaded parts, and 3 unshaded parts.



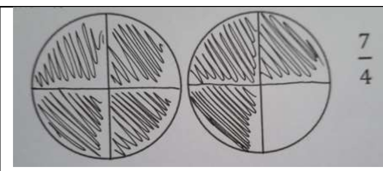
The participant's response in Exhibit 4.13 showed that the participant did not know how to translate symbolic notation into a pie diagram. The participant thought that the numerator represents the number of the shaded parts, while the denominator represents the unshaded parts. Hence the participant's incorrect answer is not because of misunderstanding of the instruction, but because the participant did not understand the top and the bottom number of a fraction notation.

Item 2 Shade the shape to show the fractions below (ConT2Q2).



From the six participants who received this item, four participants answered the item correctly. Exhibit 4.14 presents the response from Participant 5-LA, who shaded the pie diagrams correctly to represent the improper fraction in Task 2 Item 2.

Exhibit 4.14 The answer of participant 5-LA for Task 2 Item 2 of the conceptual knowledge dimension

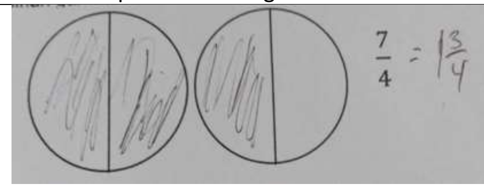
PARTICIPANT: Shade the shape to show the fractions below PARTICIPANT: 7/4 means that the denominator is 4, so it should be divided by 4 first (the participant drew lines to make 4 partitions for each circle). Since it is 7/4, 7 parts are shaded (the participant shaded the 7 parts of the circle).	

The participant made four partitions for each circle and shaded seven parts from both circles to represent the improper fraction 7/4. This response showed that the participant understood that the denominator 4 represented four partitions for each whole (circle) and the numerator 7 represents the number of shaded parts from these two circles. Hence, it can be inferred that the participant understood the symbolic notation of improper fractions.

Two participants did not give a correct answer for this item. Exhibit 4.15 demonstrates the response from Participant 10-BA, who failed to represent the improper fraction corresponding to Task 2 Item 2.

The participant found the numerator bigger than the denominator. The participant converted the improper fraction 7/4 to a mixed number 1 3/4. To represent this mixed number, the participant shaded three parts of the two circles to represent the numerator 3. The participant considered that the two circles are the whole, and only represented the proper fraction (3/4) from this whole and ignoring the whole number in the mixed number 1 3/4. From this response it can be inferred that the participant did not understand the symbolic notation of either improper fractions or mixed numbers.

Exhibit 4.15 The answer of participant 10-BA for Task 2 Item 2 of the conceptual knowledge dimension

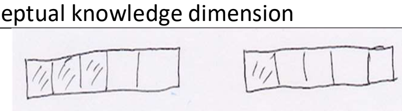
PARTICIPANT: Shade the shape to show the fractions below RESEARCHER: Can you tell me what the problem is in this task? PARTICIPANT: The problem is the numerator is greater than the denominator (The participant was silent) RESEARCHER: Keep talking please PARTICIPANT: So the fraction should be converted to a mixed number, 7/4 is equal to 1 3/4. (After that, the participant shaded full one of the circle and shaded a half for the other circle) RESEARCHER: Can you explain why you shaded 3 parts of the circles? PARTICIPANT: In order to get 1 3/4 RESEARCHER: Can you tell me how you got 1 3/4 from this drawing? PARTICIPANT: This is a mixed number, and the numerator is 3, so 3 parts of the circles are shaded out of all 4 parts.	

4.3.1.1.3 Task 3 Comparing Fractions

Item 1 Which is larger $\frac{3}{5}$ or $\frac{1}{5}$? Illustrate how you got your answer by using a model such as a picture or a diagram representation (Adapted from Scanlon, 2013) (ConT3Q1)..

From the nine participants who received this item, seven participants correctly answered the item. Exhibit 4.16 represents the answer from participant 17-FA, who successfully compared two fractions with the same denominator, using a part-whole representation.

Exhibit 4.16 The answer of participant 17-FA for Task 3, Item 1 of the conceptual knowledge dimension

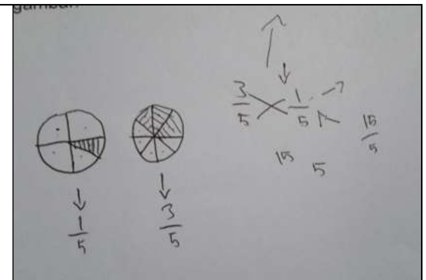
PARTICIPANT: Which is larger $\frac{3}{5}$ or $\frac{1}{5}$? Illustrate how you got your answer using a picture. PARTICIPANT: Can I use a picture other than a circle? RESEARCHER: Yes, you can PARTICIPANT: Draw a rectangle with 5 parts where 3 parts are shaded. Draw a second rectangle with the same parts as before, but where only 1 part is shaded. RESEARCHER: So which one is greater? PARTICIPANT: 3/5, because it has more shaded parts.	

The participant drew a part-whole representation using rectangle diagrams to compare the fractions. The participant determined the bigger fraction based on the number of the shaded parts. This response demonstrates that the participant understood the instruction and understood the size of fractions based on part-whole understanding.

There were two participants who answered this item incorrectly. Exhibit 4.17 demonstrates the answer of participant 9-OK.

Exhibit 4.17 The answer of participant 9-OK for Task 3 Item 1 of the conceptual knowledge dimension

PARTICIPANT: Which is larger $\frac{1}{5}$ or $\frac{3}{5}$? Illustrate how you got your answer by using a picture.
PARTICIPANT: 3/5 and 1/5. A cross product, 3 multiplied by 5 is 15, 5 multiplied by 1 is 5. So 1/5 is the biggest.
RESEARCHER: Can you explain again your answer?
PARTICIPANT: If I do not use a cross product, the biggest is 3/5 but if I use a cross product the bigger is 1/5
RESEARCHER: Can you tell me what it means?
PARTICIPANT: The one which does not use a cross product, 3, multiplied by 5, which is 15 (the participant multiplied the numerator and the denominator of 3/5), is greater than 1 multiplied by 5 (the participant multiplied the numerator and the denominator of 1/5).
RESEARCHER: How about the cross product?
PARTICIPANT: 3 multiplied by 5 is 15 (the participant multiplied the numerator of 3/5 with the denominator of 1/5) and 1 multiplied by 5, 5 (the denominator of 3/5 was multiplied with the numerator of 1/5). So the biggest of them is 1/5
RESEARCHER: So which one do you choose, a top-bottom product or a cross product?
PARTICIPANT: A cross product
RESEARCHER: Can you determine which one is bigger using pictures?
PARTICIPANT: Yes
(The participant drew two circles to represent 3/5 and 1/5, but he made 3/5 with unequal 5 partitions and drew 3/5 with 8 partitions of the circle).
RESEARCHER: So which one is bigger? 3/5 or 1/5?
PARTICIPANT: From the pictures it is 3/5
RESEARCHER: Why?
PARTICIPANT: Because for 1/5 there are 1, 2, 3, 4, 5 (he counted all of the parts of the circle representing 1/5). While in this one there are 1, 2, 3, 4, 5, 6, 7, 8 (he counted all of the parts of the circle representing 3/5).

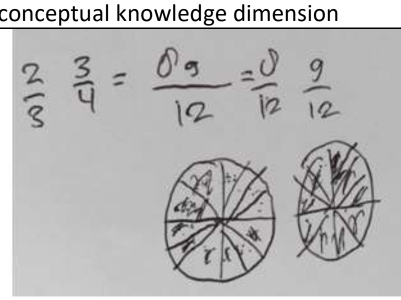


In the beginning, the participant used procedural rules to compare the fractions. The participant used a cross-product technique, but he made an incorrect conclusion to determine which fraction is bigger. Next, the participant also used an incorrect method i.e., multiplying the top and the bottom number. Finally, the participant used a diagram to compare the fractions, but he made mistakes in representing the fraction using a part-whole diagram. He generated a pie diagram with eight partitions and three of them were shaded to represent 3/5. The participant made an error when drawing a conclusion from the circle. This indicates that that the participant did not understand the size of fractions based on part-whole understanding.

Item 2 Which is larger $\frac{2}{3}$ *or* $\frac{3}{4}$ *? Illustrate how you got your answer by using a model such as a picture or diagram representation (Adapted from Scanlon, 2013) (ConT3Q2).*

Seven participants from low to medium achieving students received this item. Two of them answered the item correctly. Exhibit 4.18 shows the answer from participant 10-BA, who successfully generated pie diagrams to compare the fractions in Task 3 Item 2.

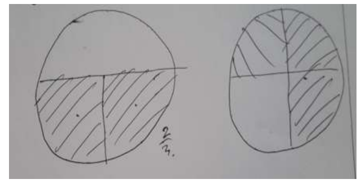
Exhibit 4.18 The answer of participant 10-BA for Task 3 Item 1 of the conceptual knowledge dimension

PARTICIPANT: Which is larger $\frac{2}{3}$ or $\frac{3}{4}$? Illustrate how you got your answer using a picture. PARTICIPANT: The fractions should be transformed with a common denominator 12. So 12 divided by 3 is 4, and 4 times 2 is 8. Then 12 divided by 3 is 3, and 3 times 3 is 9. So we have the fractions 8/12 and 9/12. So 3/4 is larger. RESEARCHER: Can you explain your answer using diagram? PARTICIPANT: Drew 12 parts … (The participant drew 2 circles to describe 8/12 and 9/12). For 8/12, 8 parts are shaded, while for 9/12, 9 parts are shaded RESEARCHER: So which one is larger? PARTICIPANT: 3/4 RESEARCHER: Why? PARTICIPANT: Because it has more shaded parts RESEARCHER: Okay, thank you for your answer.	

The participants used procedural knowledge to transform the fractions to equivalent fractions with a common denominator. Next, the participants used conceptual knowledge to map the equivalent fractions to part-whole representation. Finally, the participant successfully determined which fraction is bigger. This response demonstrates that the participant understood the instruction and understood the size of fractions based on part-whole understanding.

Five participants from low achieving students and some of the medium achieving students gave an incorrect answer for this item. Exhibit 4.19 demonstrates the response from participant 8-NA, who incorrectly answered the item.

Exhibit 4.19 The answer of participant 8-NA for Task 3 Item 2 of the conceptual knowledge dimension

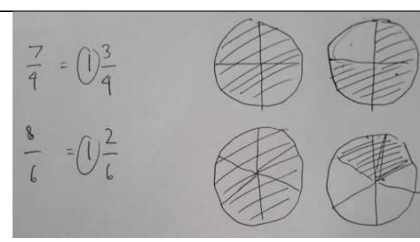
<table>
<tr>
<td>
PARTICIPANT: Which is larger $\frac{2}{3}$ or $\frac{3}{4}$? Illustrate how you got your answer using a picture.
PARTICIPANT: There is a circle (the participant drew a circle). This is for 2/3, and this is a circle again for 3/4 (the participant drew another circle). I divide this circle into 3 parts, and divide another one into 4. Here, there are two shaded parts, so this is 2/3. Then in here the numerator of 3/4 is 3, so I shade 3 parts. I think 2/3 is bigger than 3/4.
RESEARCHER: Why?
PARTICIPANT: Because if 2/3 is shared, the shared part is bigger
</td>
<td></td>
</tr>
</table>

The participant generated two pie diagrams to represent 2/3 and 3/4, but with unequal partitions representing 2/3. The participant determined that 2/3 was bigger than 3/4 because the participant thought that the shared part of 2/3 was bigger than 3/4. The participant made several errors. First, the participant did not accurately represent 2/3 using a pie diagram. Second, the participant compared 2/3 and 3/4 directly without making the number of parts of both pie diagrams the same. These errors caused participant to have a misleading conclusion to determine which fraction is bigger.

Item 3 Which is larger $\frac{7}{4}$ or $\frac{8}{6}$? Illustrate how you got your answer by using a model such as a picture or diagram representation (Adapted from Scanlon, 2013) (ConT3Q3).

Five participants, all high achieving students, answered the item successfully. Exhibit 4.20 shows participant 3-JI, who successfully answered the item.

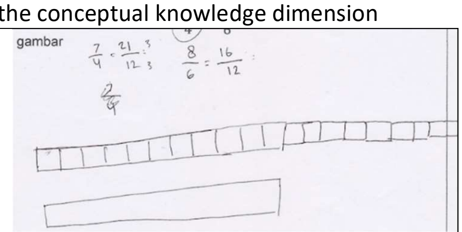
Exhibit 4.20 The answer of participant 3-JI for Task 3 Item 3 of the conceptual knowledge dimension

<table>
<tr>
<td>
PARTICIPANT: Which is larger $\frac{7}{4}$ or $\frac{8}{6}$? Illustrate how you got your answer using a picture.
(The participant drew two circles to represent 7/4 and the other two circles to represent 8/6)
PARTICIPANT: 7/4 means 1 3/4, and 8/6 is 1 2/6. This is a whole circle, and this is too, and they should be shaded (the participant fully shaded one of the circles for 7/4, and one of the circles for 8/6). This one is 3/4 (she shaded the other circle of 7/4), and this one is 2/6 (she shaded the other circle of 8/6). So the biggest is this one (she pointed to the two circles which represent 7/4).
RESEARCHER: Why are these ones bigger? (The researcher pointed to the two circles which represent 7/4)
PARTICIPANT: Because the remaining part is only this (the participant pointed to the unshaded area of the circles for 7/4), while the other one is larger (she pointed the unshaded area of the circles for 6/8)
</td>
<td>
</td>
</tr>
</table>

The participant converted the improper fractions into mixed numbers before mapping them to part-whole representations using pie diagrams. The participant successfully generated pie diagrams to represent the improper fractions, and correctly determined which fraction is bigger. This evidence shows that the participant understood the instruction, and understood the value (size) of fractions greater than 1.

Two participants answered the item incorrectly. Exhibit 4.21 shows the answer by participant 17-FA.

Exhibit 4.21 The answer of participant 17-FA for Task 3 Item 3 of the conceptual knowledge dimension

PARTICIPANT: Which is larger $\frac{7}{4}$ or $\frac{8}{6}$? Illustrate how you got your answer using a picture. PARTICIPANT: The denominators are different so we transformed them with a common denominator. 7/4 equals 21/12 and 8/6 equals 16/21. So we can draw now. (The participant drew a rectangle with 21 partitions) RESEARCHER : Why did you draw 21 partitions? PARTICIPANT: Because 21 is the numerator which is larger than the denominator RESEARCHER: Okay, so what is the next step? PARTICIPANT: If 21 parts are shaded, then there are no spaces for the 12 parts (The participant looked confused) RESEARCHER: Okay, we can discuss this again later. PARTICIPANT: But, the largest is 7/4, because after they are transformed with the common denominator, 21/12, which is 7/4, is larger than 16/12. RESEARCHER: Oh okay, thank you.	

The participant used procedural knowledge to transform the fractions into equivalent fractions with a common denominator. The participant tried to generate a rectangle diagram to represent the fraction, but was not successful because the participant was confused as to how to represent the numerator in a situation where the numerator is greater than the denominator. From this response, it can be inferred that the participant did not understood the meaning of the numerator and denominator of fractions greater than 1, so the participant had a difficulty in representing and comparing the improper fractions in Task 3 Item 3.

4.3.1.1.4 Task 4. Locating Fractions on the Number Line

Item 1 Show the fraction $\frac{3}{8}$ *on the number line below (ConT4Q1).*

Six participants from the high achieving students answered the item correctly. Exhibit 4.22 demonstrates the response from Participant 6-JO.

Exhibit 4.22 The answer of participant 6-JO for Task 4 Item 1 of the conceptual knowledge dimension

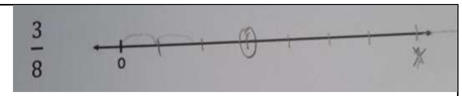
PARTICIPANT: Show the fractions on the number lines below

PARTICIPANT: 3/8 is less than 1. This is 1 and this null, so to get 3/8 distance from 0 to 1 is it should be divided into eight jumps, so 1, 2, 3, 4, 5, 6, 7, 8 (the participant makes 8 scales from 0 to 1). This is not enough, so we move 1 here. So this is 1, which means there are 1, 2, 3, 4, 5, 6, 7, 8. To get 1, there are 8 parts. What you requested is 3 of 8, so there are only 1, 2, 3 (the participant circled the point 3/8 on the number line).

RESEARCHER: Should the scales be the same size?

PARTICIPANT: Yes, they should be.

RESEARCHER: Okay, thank you.



The participant successfully placed the fraction on the number line. The participant used the denominator 8 to divide the length between 0 and 1 (the unit) into equal intervals (scales). After that, the participant used the numerator 3 to place the fraction on the third interval from 0. This response shows that the participant understood the symbolic notation of fractions as representation of measures.

Nine participants from the low and medium achieving students did not give a correct answer. Exhibit 4.23 demonstrates the response from Participant 14-DE.

Exhibit 4.23 The answer of participant 14-DE for Task 4 Item 1 of the conceptual knowledge dimension

PARTICIPANT: Show the fractions on the number lines below

(The participant was silent)

RESEARCHER: Can you tell me what the problems are in this task?

PARTICIPANT: Euh… the order, so for 3/8 … if 1/2, it's located in the middle. If I do counting, it starts from the right or the left of the number line…

(The participant was silent for a moment and then she made dot points on the number line)

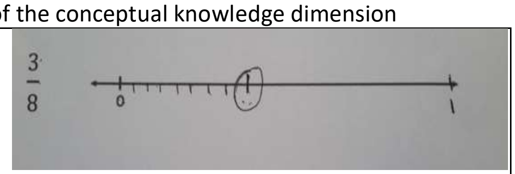
PARTICIPANT: Here. (The participant circled the location of 3/8)

RESEARCHER: Can you tell me why you put 3/8 there?

PARTICIPANT: This is 1, I counted the dots, this is null then 1, 2, 3, 4, 5, 6, 7, 8. I put it on the 8th dot.

RESEARCHER: Why did you put 3/8 on the 8th dot?

PARTICIPANT: I am not sure, I put it there because I think I should put it on the last dot

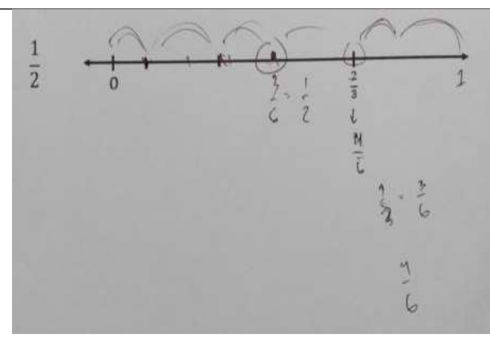


From the evidence in Exhibit 4.23, it can be seen that initially the participant was confused about whether to start numbering on the number line whether from the left or the right. The participant put a number 1 at the end of the right side of the number line, and made eight dots on the number line. The participant put the fraction on the 8th interval, which is the denominator of the fraction 3/8. Hence, it can be inferred that the incorrect response from the participants is because the participant did not understand the symbolic notation of fractions representing measures.

Item 2 Show the fraction $\frac{1}{2}$ *on the number line below (Adapted from Scanlon, 2013) (ConT4Q2)*

Only three participants from the high achieving students received this item. They answered the item correctly. Exhibit 4.24 shows the answer from participant 11-RE.

Exhibit 4.24 The answer of participant 11-RE for Task 4 Item 2 of the conceptual knowledge dimension

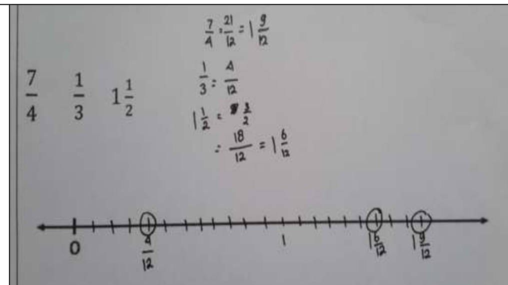
PARTICIPANT: Show the fractions on the number lines below PARTICIPANT: 1/2, oh we can take this 2/3 as 4/6. So this is 6. 1, 2, 3, 4, 5, 6. This is 1 (the participant put 1 on the sixth scale of the number line). So 3/6 is here. 3/6 is the same with 1/2.	

The participant converted the fractions (1/2 and 2/3) into equivalent fractions with a common denominator of 6. This common denominator was used to determine how many intervals (scales) should be created between 0 and 1 (the unit). After that, the fraction 1/2 (which is equal to 3/6) was put on the number line, one interval to the left side of the fraction 2/3 (which is equal to 4/6). This response demonstrates that the participant understood the instruction and understood the symbolic notation of fractions (smaller than 1) as representing measure.

Item 3 Order the fractions $\frac{7}{4}$ *,* $\frac{1}{3}$ *and* $\frac{1}{2}$ *on the number line below (Adapted from Scanlon, 2013) (ConT4Q3).*

Only five participants from the high achieving students received this item. All of them answered the item correctly. Exhibit 4.25 demonstrates the response from participant 5-LA for Task 4 Item 3.

Exhibit 4.25 The answer of participant 5-LA for Task 4 Item 3 of the conceptual knowledge dimension

PARTICIPANT: Order these fractions from the smallest to the largest on the number line below. PARTICIPANT: Eh..the denominators can be equated first, so for 7/4, the denominator becomes 12. 12 divided by 4, 3 and 3 is multiplied by 7, 21. So it is 21/12. For 1/3, 12 divided by 3 equals to 4, and 4 multiplied by 1 is 4, so it is 4/12. The 1 1/2 is the same as 3/2, so 12 divided by 2 is 6; and 6 multiplied by 3, 18, so it is 18/12. (the participant made scales (and put 4/12, 1, 1 6/12, and 1 9/12 on the number line)	

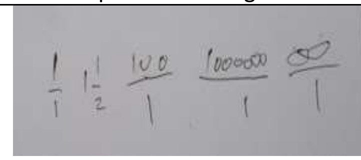
The participants converted the fractions (including improper fraction and mixed numbers) into equivalent fractions with a common denominator. The participant created scales based on this common denominator and put all of the fractions on these scales. From this evidence, it can be inferred that the participant understood fractions (including improper fractions and mixed numbers) as representing measures.

4.3.1.1.5 Task 5 Writing the Smallest and Biggest Fractions that they Can

Item 1 Write the biggest fraction that you know. Explain your answer (Adapted from Stafylidou & Vosniadou, 2004) (ConT5Q1).

Only six participants from the high achieving students received this item. Four participants answered the item correctly. Exhibit 4.26 shows the answer from participant 16-AK.

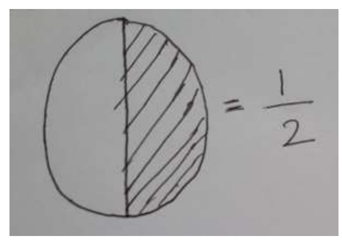
Exhibit 4.26 The answer of participant 16-AK for Task 5 Item 1 of the conceptual knowledge dimension

PARTICIPANT: Write the biggest fraction that you know. PARTICIPANT: Hm… it could be 1/1, 1 1/2, 100/1, 1000000/1, an infinite number per 1 RESEARCHER: So, what is your conclusion? PARTICIPANT: Infinite	

The participant demonstrated his understanding of the infinity property of fractions from this item. The participant successfully concluded that there was no biggest fraction after he demonstrated an increasing pattern (size/value) of fractions by increasing the value of the numerators, while the denominator is kept constant at 1. From this response, it can be inferred that the participant understood that the biggest fraction did not exist.

Two participants gave an incorrect answer for this item. Exhibit 4.27 demonstrates the response from participant 4-JA.

Exhibit 4.27 The answer of participant 4-JA for Task 5 Item 1 of the conceptual knowledge dimension

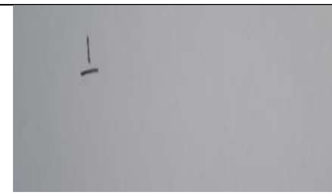
PARTICIPANT: Write the biggest fraction that you know. PARTICIPANT: First, a circle is drawn. The meaning of this circle is 1. Then, we divide it into 2 so that if we shade one of them, it will become 1/2. This means that the biggest fraction is 1 divided by 2, 1/2. RESEARCHER: Can you explain why 1/2 is the biggest fraction? PARTICIPANT: Euh … because… (the participant was silent) RESEARCHER: Keep talking please PARTICIPANT: Euh because 1/2 is the biggest fraction. RESEARCHER: Okay, let's discuss this task later.	

The participant generated a half-shaded pie diagram (representing 1/2) to show the biggest fraction. The participant could not explain why 1/2 was the biggest fraction. From this response, it can be inferred that the participant understood the instruction, but did not understand the unbounded infinity of fractions.

Item 2 Write the smallest fraction that you know. Explain your answer (Adapted from Stafylidou & Vosniadou, 2004) (ConT5Q2).

Only six participants from high achieving students received this item. Four participants answered the item correctly. Exhibit 4.28 demonstrates the answer of participant 11-RE.

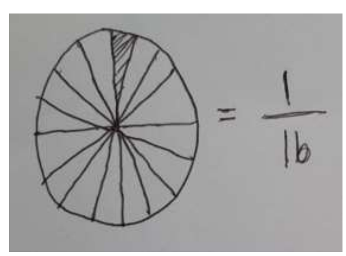
Exhibit 4.28 The answer of participant 11-RE for Task 5 Item 1 of the conceptual knowledge dimension

PARTICIPANT: Write the smallest fraction that you know. Explain your answer. PARTICIPANT: Unlimited… RESEARCHER: Can you explain why this is unlimited? PARTICIPANT: Because a fraction, e.g. 1 per… (the participant wrote down 1 / on the card) … the denominator can be anything, it could be 1 million or 1 billion or also 1 trillion, depending on the person.	

The participant considered that fractions were unlimited. The participant represented a fraction with the numerator 1 and allowed the denominator to be filled with any number. The participant took an example for the denominator with the increasing number such as 1 million, 1 billion, and 1 trillion to show that the fraction can be very small, and it can be smaller than these numbers if the denominator were increased again. It indicates that the participant understood the relationship between the numerator and denominator; that the greater the increase in the denominator, the smaller the fraction will be. However, because the denominator can be set arbitrarily, the participant considered that fractions are unlimited. From this response, it can be inferred that the participant understood the unbounded infinity of fractions.

Two participants answered the item incorrectly. Exhibit 4.29 demonstrates the response from participant 4-JA.

Exhibit 4.29 The answer of participant 4-JA for Task 5 Item 2 of the conceptual knowledge dimension

PARTICIPANT: Write the smallest fraction that you know. PARTICIPANT: First, draw a circle, euh 1/2, then it is divided into 1/4, and it is divided again into 1/8, and it is divided again into 1/16. So the smallest fraction is 1/16. RESEARCHER: Can you tell me why 1/16 is the smallest fraction? PARTICIPANT: Because euh… (The participant was silent)… PARTICIPANT: Because, euh... because the circle is divided into 16, the shaded part is 1, and there are 15 remaining parts, so 1/16 is the smallest fraction. RESEARCHER: Oh okay, let's discuss it later again	

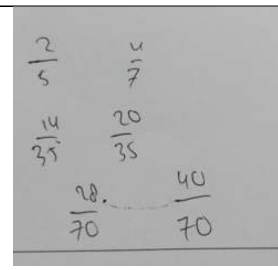
From Exhibit 4.29, it can be seen that the participant used a part-whole understanding by generating a pie diagram to determine the smallest fraction. From this response it can be inferred that the participant's mistake is not because she misinterpreted the instruction, but because she did not understand the unbounded infinity of fractions.

4.3.1.1.6 Task 6 Finding How Many Fractions lie between Two Fractions

Item 1 How many numbers lie between $\frac{2}{5}$ and $\frac{4}{7}$? Explain your answer (Adapated from Vamvakoussi & Vosniadou, 2004) (ConT6Q1)

Only six participants from the high achieving students received this item. Three participants answered the item correctly. Exhibit 4.30 demonstrates the answer from participant 11-RE.

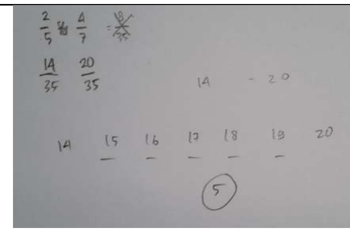
Exhibit 4.30 The answer of participant 11-RE for Task 6 Item 1 of the conceptual knowledge dimension

PARTICIPANT: How many numbers are there between $\frac{2}{5}$ and $\frac{4}{7}$? PARTICIPANT: It depends on the denominator that we use… RESEARCHER: Oh, can you explain more about this? PARTICIPANT: If for example … it's also infinite … if we use 2/5 and 4/7 and we use 35 as the denominator. We can say that there are fractions which are 14/35, 16/35, 17/35, 18/35, and 19/35. There are five fractions. But it can be extended, for example, 14/35 becomes 28/70 and 20/35 becomes 40/70. So there are gaps between them, and if the question is how many numbers are there, it can be more numbers again, so it's infinite too…	

The participant could determine the unlimited number of fractions between two fractions (2/5 and 4/7). The participant transformed the fractions to equivalent fractions with a common denominator. By increasing the common denominator for equivalent fractions, the participant could see there are unlimited numbers of fractions between 2/5 and 4/7. From this response, it can be inferred that the participant understood the density property of fractions.

Three participants gave an incorrect answer for this item. Exhibit 4.31 demonstrates the answer from participant 6-JO.

Exhibit 4.31 The answer of participant 6-JO for Task 6 Item 1 of the conceptual knowledge dimension

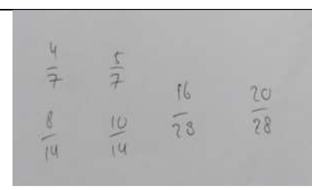
PARTICIPANT: How many numbers are there between $\frac{2}{5}$ and $\frac{4}{7}$? PARTICIPANT: 2/5 and 4/7, hm… these denominators need to be equal. So, the common denominator is 5 times 7 which is 35. 35 divided by 5 is 7, and 7 multiplied by 2 is 14. This is 14/35. While for this one, 35 divided by 7 is 5, and 5 multiplied by 4 is 20. The question is how many numbers lie between 14/35 and 20/35. So this is the same as how many numbers lie between 14 and 20. There are 15, 16, 17, 18, and 19. There are five numbers.	

The participant found limited (finite) numbers between 2/5 and 4/7. The participant transformed the fractions with a common denominator, 35, and found there are five fractions between 2/5 and 4/7. This evidence demonstrates that the participant did not understand the density property of fractions.

Item 2 How many numbers lie between $\frac{4}{7}$ and $\frac{5}{7}$? Explain your answer (adapated from Vamvakoussi & Vosniadou, 2004) (ConT6Q2)

Six participants from the high achieving students received this item, and four of them gave a correct answer for this item. Exhibit 4.32 shows the answer from participant 11-RE.

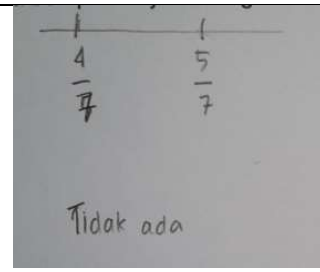
Exhibit 4.32 The answer of participant 11-RE for Task 6 Item 2 of the conceptual knowledge dimension

PARTICIPANT: How many numbers are there between $\frac{4}{7}$ and $\frac{5}{7}$? PARTICIPANT: This is the same. The problem is if we only look at this without paying deeper attention, there are no numbers between 4/7 and 5/7 if the denominator is only 7. But if we transform them to 8/14 and 10/14, there is 9/14. If, for example, they become 16/28 and 20/28, there are more numbers again, so this is infinite too.	

The participant found that there are unlimited numbers between 4/7 and 5/7. The participant transformed the fractions into equivalent fractions with a common denominator. The participant found that if the common denominator is increased (getting larger), then the number of fractions between 4/7 and 5/7 is also increased. This evidence shows that the participant understood the density property of fractions.

Two participants gave an incorrect answer. Exhibit 4.33 demonstrates the answer from participant 6-JO.

Exhibit 4.33 The answer of participant 6-JO for Task 6 Item 1 of the conceptual knowledge dimension

PARTICIPANT: How many numbers are between $\frac{4}{7}$ and $\frac{5}{7}$? PARTICIPANT: No numbers RESEARCHER: Can you tell me why there are no numbers? PARTICIPANT: This is 4/7 and this is 5/7 (the participant created a number line and then put 4/7 and 5/7 on the line). On this number line, from here, 4/7 directly jumps to 5/7, there are no numbers between them.	

The participant represented the fraction 4/7 and 5/7 on the number line, and considered that there was "a jump" between 4/7 and 5/7. This response indicates that the participant had a discrete understanding of the numerical value of fractions.

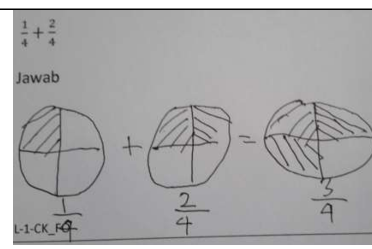
4.3.1.1.7 Task 7 Adding Fractions Using Diagram Representation

Item 1 Draw a pictorial representation for the addition of the fractions below. Explain your answer (ConT7Q1).

$$\frac{1}{4}+\frac{2}{4}$$

Eight participants from the low and medium students received the item. All of these participants answered the item correctly. Exhibit 4.34 shows the answer from participant 15-RI.

Exhibit 4.34 The answer of participant 15-RI for Task 6 Item 1 of the conceptual knowledge dimension

PARTICIPANT: Draw a pictorial representation for the fraction addition below (The participant drew circles to represent the fraction addition of 1/4 + 2/4) RESEARCHER: Can you tell me how you got the answer? PARTICIPANT: The first circle is 1/4, the second circle is 2/4, then the shaded parts from these two circles are added to get the result, which is 3/4.	

The participants drew pie diagrams to represent a fraction addition with the same denominator. The participants added the number of the shaded parts from both pie diagrams to generate the pie diagram, which shows the result of this fraction addition. From this response, it can be inferred that the participant understood fraction addition with the same denominator.

Item 2 Draw a pictorial representation for the addition of the fractions below. Explain your answer (ConT7Q2).

$$\frac{1}{4}+\frac{2}{3}$$

Fourteen participants received this item. Nine of them answered the item correctly. Exhibit 4.35 shows the answer from participant 17-FA.

Exhibit 4.35 The answer of participant 17-FA for Task 7 Item 1 of the conceptual knowledge dimension

PARTICIPANT: Draw a pictorial representation for the fraction addition below. Explain your answer

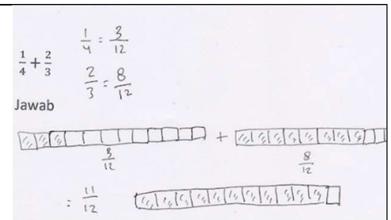
$\frac{1}{4}+\frac{2}{3}$

PARTICIPANT: These should be transformed again using a common denominator. So 1/4 to be 3/12, and 2/3 to be 8/12.

(The participant drew two rectangles with 12 partitions for each. Three parts are shaded in the first rectangle, and 8 parts are shaded for the second rectangle. After that, she drew one rectangle again to represent the result with 12 partitions where 11 parts are shaded)

RESEARCHER: How did you get the result?

PARTICIPANT: The shaded parts of 3/12 and 8/12 are added.

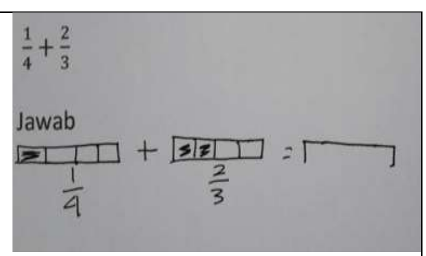


The participant transforms the fractions into equivalent fractions with a common denominator and draw rectangle diagrams to show the fraction addition. The participant added the shaded parts of the fractions (which had already the same number of partitions for each whole of the fractions) to get the result. From this response, it can be inferred that the participant understood the meaning of fraction addition, which involves fractions with different denominators.

Five participants answered the item incorrectly. Exhibit 4.36 shows the answer from participant 7-IS.

Exhibit 4.36 The answer of participant 7-IS for Task 7 Item 2 of the conceptual knowledge dimension

PARTICIPANT: Draw a pictorial representation for the fraction addition below.
(The participant drew rectangles to represent 1/4 and 2/3, and suddenly stopped her drawing when she tried to draw the result)
PARTICIPANT: I do not understand this
RESEARCHER: Could you tell me why you do not understand?
PARTICIPANT: I don't understand why these are different (the participant pointed to the denominators of 1/4 and 2/3 which are 4 and 3).
RESEARCHER: Oh okay, what can you do to solve this problem?
PARTICIPANT: I don't know



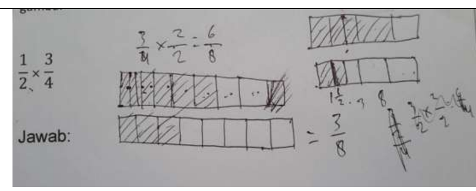
The participant drew rectangle diagrams to show fraction addition of 1/4 and 2/3, but the participant did not continue the process because the denominators were different. This evidence showed that the participant did not understand fraction addition, which involves fractions with different denominators.

4.3.1.1.8 Task 8 Multiplying and dividing fractions using diagram representation

Item 1 Draw a pictorial representation for the multiplication of the fractions below. Explain your answer (ConT8Q1)

Eight participants received Task 8 Item 1. Only one participant answered the item correctly. Exhibit 4.37 shows the answer from participant 16-AK, who successfully represented fraction multiplication using diagram representation.

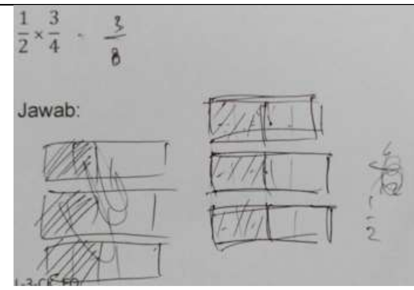
Exhibit 4.37 The answer of participant 16-AK for Task 8 Item 1 of the conceptual knowledge dimension

PARTICIPANT: Draw a pictorial representation for the fraction multiplication below PARTICIPANT: So this is 1/2 of 3/4. For example there are 1, 2, 3, 4, 5, 6, 7, 8 (the participant drew a rectangle with 8 partitions). 3/4 is equal to 6/8, 6 parts are shaded. If this is multiplied by 1/2, it means that a half of these 6 parts. So, 1, 2, 3, parts are shaded or this is the same as 3/8.	

The participant interpreted the multiplication of 1/2 and 3/4 as “1/2 of 3/4". Based on this interpretation, the participant developed a diagram representation to show the meaning of “1/2 of 3/4". The participant converted 3/4 to 6/8 and drew the diagram representing 6/8. Because “1/2 of 3/4” means a half of 3/4, the participant take a half of the shaded parts of 6/8, which is three shaded parts. The participant considered these three shaded parts as the result of the fraction multiplication of 1/2 and 3/4. These three shaded parts were from eight parts of the whole, so the fraction for these shaded parts were 3/8 which is the answer of the multiplication of 1/2 and 3/4. This response indicates that the participant understood fraction multiplication.

Seven participants, including several high achieving students, could not answer Task 8 Item 1. Exhibit 4.38 shows the answer from participant 11-RE.

Exhibit 4.38 The answer of participant 11-RE for Task 8 Item 1 of the conceptual knowledge dimension

PARTICIPANT: Draw a pictorial representation for the fraction multiplication below PARTICIPANT: 1/2 is multiplied by 3/4, the answer is 3/8 but how I can draw it? Oh, this is 3/4 so this one is multiplied by 3 (the participant drew three rectangles with a half-shaded for each rectangle). Then eh … I don't know how to draw this. Hmm…, it's divided by 4. How to draw this … RESEARCHER: Can you tell me what the problems are in this task? PARTICIPANT: Eh … the drawing. The answer is already known, 3/8, but how to draw this, hmm ... I don't know. Wait so this is … eh… I don't know.	

The participant tried to draw diagram representations to represent the multiplication of 1/2 and 3/4, but was not successful. The participant drew a three rectangles diagram of a half (1/2) to represent the multiplication of 1/2 and the numerator of 3/4. After that the participant tried to divide the result of multiplication 1/2 and 3 (the numerator of 3/4) by 4 (the denominator of 3/4) using diagram representation, but was not successful. From this response, it can be inferred that the participant did not understand fraction multiplication.

Item 2 Draw a pictorial representation for the division of the fractions below. Explain your answer (ContT8Q2)

$$\frac{1}{2} \div \frac{1}{4}$$

Only one participant (who successfully answered Task 8 Item 1) received the item. The participant answered the item correctly. Exhibit 4.39 demonstrates the answer from participant 16-AK, who successfully represented a fraction division using a diagram representation.

The participant drew rectangle diagrams to represent the fraction division of 1/2 and 1/4. The participant successfully demonstrated how many 1/4 in a half (1/2) using diagram representation. This is the interpretation of fraction division (See Chinnappan & Forrester, 2014; Van de Walle et al., 2015). This response demonstrates that the participant understood fraction division.

Exhibit 4.39 The answer of participant 16-AK for Task 8 Item 2 of the conceptual knowledge dimension

PARTICIPANT: Draw a pictorial representation for the fraction division below

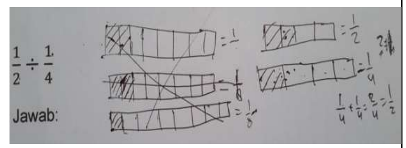
PARTICIPANT: 1/2 divided by 1/4. This is a half, firstly there are 4 parts and 1 part is shaded which is 1/4, than a half of 1/4 is taken, because the number is not nice, so it is multiplied by 2 which is 1, 2, 3, 4, 5, 6, 7, 8 (The participant created a rectangle with 8 partitions). If divided by 1/2, how many, ah …1, 2, 3, 4. How many of this fraction to become … (the participant looked confused)

RESEARCHER: Can you tell me what the meaning of 1/2 divided by 1/4 is?

PARTICIPANT: How many 1/4 to become 1/2.

(The participant drew a rectangle with 4 partitions and two of them are shaded to represent 1/2). It is the same with 2 (he pointed to the 2 shaded partitions). In order to become 1/2, so this one (he pointed one shaded area of the rectangle which represent 1/4) needs 2 times of itself, so the answer is 2, which is 2 times of this part (he pointed to the rectangle which represent 1/4). 1/4 plus 1/4 equals 2/4 or 1/2, meaning that it needs 2 times of 1/4 so the result is 2.

RESEARCHER: Okay, thank you, well done



4.3.1.2 Within-Task Analysis

The within-task analysis was conducted for all items within a task. The aim of this analysis was to examine whether the obtained order of acquisition of the items is consistent with the hypothesized order. The within-task analysis focuses on finding whether there is evidence that some of the participants answered the items at the higher levels in the task successfully but could not answer the items at the lower levels of the task. This evidence would show that that the order of acquisition of the items within a task are not consistent with the hypothesized order.

Table 4.2 shows the participants' responses structured within task and level. The responses are coded as 0, 1, and blank. The code 1 refers to the correct response; the code 0 refers to the incorrect response; and the blank means that the participant did not receive the item. The same code is used to score the items in all conceptual and procedural tasks.

Task 1 (generating a fraction from a pie diagram) consists of five items that address conceptual competencies at Levels 2 to 3. From Table 4.2, it can be observed that all the participants who answered items 3, 4, and 5 (Level 3) correctly gave correct answers for items 1 and 2 (Level 2). Likewise, there were no participants who could answer items 3, 4, and 5, which belong to level 3, correctly but could not answer item 1 or item 2 of Level 2 correctly. This means that there are no cases where the participants answered the items at the lower level (Level 2) incorrectly while they answered the items at the higher level (Level 3) successfully.

Task 2 (Shading a pie diagram to represent a fraction) has two items that address conceptual competencies at Levels 2 to 3. From the table, it can be observed that there are no cases where the participants answered the item at the upper level (Item 2 at Level 3) correctly, but answered the item at lower level (Item 1 at Level 2) incorrectly. Similar findings also can be observed from the participants' responses to the other tasks (Task 3 – 8) of the conceptual dimension presented in Table 4.2. This result shows that the order of acquisition of the items within each task is consistent with the hypothesized order.

Table 4.2 The distribution of the participants' responses within the task of the conceptual dimension

Task	Item	Level	Participant														
			9-OK	**7-IS**	**5-RI**	**8-NA**	**12-AU**	**13-FI**	**10-BA**	**14-DE**	**17-FA**	**4-JA**	**5-LA**	**6-JO**	**11-RE**	**3-JI**	**16-Ak**
Task 1	Item 1	Level 2	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	Item 2	Level 2	0	0	1	0	0	1	1	1	1	1	1	1	1	1	1
	Item 3	Level 3	0	0	1	0	0	1	0	1	1	1	1	1	1	1	1
	Item 4	Level 3		0	0			0	0	0	0	1	1	0	1	1	1
	Item 5	Level 3										1	1	0	1	1	1
Task 2	Item 1	Level 2	0	1													
	Item 2	Level 3		0		0	0					1	1	1		1	
Task 3	Item 1	Level 2	0	1	1	1	0	1	1	1	1						
	Item 2	Level 2		0	0	0		0	1	0							
	Item 3	Level 3									0		1	1	1	1	1
Task 4	Item 1	Level 3	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
	Item 2	Level 4										1			1	1	
	Item 3	Level 4										1	1	1	1	1	1
Task 5	Item 1	Level 4										0	0	1	1	1	1
	Item 2	Level 4										0	0	1	1	1	1
Task 6	Item 1	Level 5										0	0	0	1	1	1
	Item 2	Level 5										0	1	0	1	1	1
Task 7	Item 1	Level 2	1	1	1	1	1	1	1	1							
	Item 2	Level 2	0	0	0	0	0	0	1	0	1	1		1	1	1	1
Task 8	Item 1	Level 5							0		0	0	0	0	0	0	1
	Item 2	Level 5															1

4.3.1.3 Assigning Participants into the Levels of the Conceptual Dimension and Changes in the Model

The participants were assigned into the levels of the proposed model by examining their responses on all the conceptual items. The analysis was implemented for all 15 participants by comparing the order of their obtained responses with the hypothesized order, as shown in Table 4.3. The purpose of this analysis was to examine whether the participants' profiles were in agreement with the proposed model. Moreover, the participants' responses can be used to improve the models by adjusting the proposed levels, or the items within the levels, so that the proposed model fits well with the participants' responses.

Table 4.3 The hypothesized order of acquisition of items and tasks for the conceptual dimension of the learning progression

Level	Tasks
Level 1	--
Level 2	Task 1: Items 1, 2 Task 2: Item 1 Task 3: Items 1, 2 Task 7: Items 1, 2
Level 3	Task 1: Items 3, 4, 5 Task 2: Item 2 Task 3: Item 3 Task 4: Items 1, 2
Level 4	Task 4: Item 3 Task 5: Items 1, 2
Level 5	Task 6: Items 1, 2 Task 8: Items 1, 2

The profile for all 15 participants' responses to the conceptual level tasks is presented in Table 4.4, below. The participants are assigned to a certain level if they have all the competencies at that level and below, but they do not have enough competencies at the upper level.

Table 4.4 The distribution of participants' responses across the level of the conceptual knowledge dimension

Level	Task	Item	Description	9-OK	7-IS	5-RI	8-NA	12-AU	13-FI	10-BA	14-DE	17-FA	4-JA	5-LA	6-JO	11-RE	3-JI	16-AK
Level 1	-	-																
Level 2	Task 1	Item 1	Writing a proper fraction from a pie diagram	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	Task 1	Item 2	Write the numerator of an equivalent fraction for a fraction less than 1	0	0	1	0	0	1	1	1	1	1	1	1	1	1	1
	Task 2	Item 1	Shade pie diagram to represent a proper fraction	0	1													
	Task 3	Item 1	Compare a proper fraction with the same denominator	0	1	1	1	0	1	1	1	1						
	Task 3	Item 2	Compare proper fractions with different denominators		0	0	0		0	1	0							
	Task 7	Item 1	Add fractions with the same denominator	1	1	1	1	1	1	1	1							
	Task 7	Item 2	Add fractions with different denominators	0	0	0	0	0	0	1	0	1	1		1	1	1	1
Level 3	Task 1	Item 3	Write a proper fraction from a pie diagram with unequal partitions	0	0	1	0	0	1	0	1	1	1	1	1	1	1	1
	Task 1	Item 4	Write an improper fraction from a pie diagram		0	0			0	0	0	0	1	1	0	1	1	1
	Task 1	Item 5	Write the numerator of an equivalent fraction for a fraction greater than 1										1	1	0	1	1	1
	Task 2	Item 2	Shade a pie diagram to represent an improper fraction		0		0	0					1	1	1		1	
	Task 3	Item 3	Compare improper fractions with different denominators using part-whole diagrams									0		1	1	1	1	1
	Task 4	Item 1	Put a proper fraction on a number line	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
	Task 4	Item 2	Put a proper fraction on a number line with a constraint*										1			1	1	
Level 4	Task 4	Item 3	Put fractions, including an improper fraction and a mixed number, on a number line										1	1	1	1	1	1
	Task 5	Item 1	Write the biggest fraction they can										0	0	1	1	1	1
	Task 5	Item 2	Write the smallest fraction they can										0	0	1	1	1	1
Level 5	Task 6	Item 1	Find out how many fractions lie between two fractions										0	0	0	1	1	1
	Task 6	Item 2	Find out how many fractions lie between two-pseudo successive fractions										0	1	0	1	1	1
	Task 8	Item 1	Multiply fractions using a diagram representation							0		0	0	0	0	0	0	1
	Task 8	Item 2	Divide fractions using a diagram representation															1

Table 4.4 shows the distribution of participants' responses across the items, tasks and levels. There is one participant (9-OK) who did not answer most of the items at Level 2 correctly. This evidence strongly indicates that the participant had no fraction understanding, and he should be placed at Level 1. On the other hand, two participants (10-BA and 17-FA) answered all the given items at Level 2 correctly. This result shows that the participants understood the symbolic notation of fractions as a representation of part-whole, equivalent fractions, fractions' order (the size of fractions) and fraction addition. Thus, the participants had all the competencies required at Level 2. However, some participants answered several items correctly and answered the other items at level 2 incorrectly. For example, participants 7-IS and 8-NA answered Task 1 item 1 (generating a fraction from pie diagram), Task 3 Item 1 (comparing proper fraction with the same denominator), and Task 7 Item 1 (adding fractions with the same denominator) correctly, but answered the other items at Level 2 such as Task 1 Item 2 (generating an equivalent fraction), Task 2 Item 2 (comparing proper fractions with the same denominator), and Task 7 Item 2 (adding fractions with different denominators) incorrectly. This evidence suggests that understanding part-whole with the same denominator has a different level of learning from understanding part-whole with different denominators and equivalent fractions.

Next, at Level 3, several participants (participants 5-RI 13-FI 14-DE, and 17-FA) who could not answer Task 1 Item 4 (generating improper fractions from a pie diagram) and Task 4 Item 1 (putting a proper fraction on a number line) correctly, were able to answer Task I Item 3 (generating a fraction from a diagram with unequal partitions) correctly. This evidence suggests that understanding improper fractions and fractions as measures requires different levels of learning from understanding the equal size principle of fractions. The latter seems to have the same level of learning as level 2 because several participants who could answer most of the items at level 2 correctly could also generate a fraction from a pie diagram with unequal partitions correctly.

At Level 4, two participants (participants 4-JA and 5-LA) were unable to answer Task 5 Items 1 and 2 (writing the biggest fraction and the smallest fraction respectively) correctly, but correctly answered Task 4 Item 3 (putting fractions, including an improper fraction and a mixed number, on a number line). This evidence suggests that understanding the unbounded infinity of fractions requires different levels of learning compared with understanding fractions as measures, even though this measure requires advanced understanding (it incorporates improper fractions and mixed numbers). Indeed, the responses from participants 4-JA and 5-LA indicate that Task 4 Item 3 has the same level of learning as level 3, because they could answer all the given items at level 3 and this item (Task 4 Item 3) correctly. This result shows that understanding improper fractions and fractions as measures are on the same level of learning.

At level 5, three participants (participants 3-JI, 11-RE, and 16-AK) answered both Task 6 Items 1 and 2 (finding how many fractions lie between two fractions, and finding how many fractions lie between two-pseudo successive fractions respectively) correctly, but could not answer Task 8 Item 1 (multiplying fractions using a diagram representation) and Task 8 Item 2 (dividing fractions using a diagram representation) correctly. This evidence shows that understanding the density property of fractions is likely to have a different level of learning from understanding multiplicative fraction operations. Understanding of the density of fractions also tends to require different learning from understanding the unbounded infinity of fractions at level 4, because participant 6-JO, who answered Task 5 Item 1 and 2 (writing the biggest fraction and the smallest fraction respectively) correctly, could not answer Task 6 Items 1 and 2 (finding how many fractions lie between two fractions, and finding how many fractions lie between two-pseudo successive fractions respectively) correctly.

Based on the findings discussed above, the hypothesized conceptual knowledge dimension is revised as follows. Level 0 was created to capture those students who have no fraction understanding or do not have enough competencies at Level 1. At this level, the students do not understand fraction notation, nor any relationships between a numerator and a denominator. They conceive fractions as two unrelated (independent) numbers. Next, at Level 1, students begin to understand the symbolic notation of fractions as a representation of part-whole but are still limited to proper fractions with the same denominator. At this level, students can generate a proper fraction from diagram representations, order proper fractions with the same denominator, and demonstrate fraction addition with the same denominator. At Level 2, students advance their part-whole understanding into equivalent fractions and fractions with different denominators, but are still limited to proper fractions. They can generate equivalent fractions from diagram representations, order fractions with different denominators, and add fractions with different denominators using diagram representations. At Level 3, students advance their part-whole understanding at Level 2 into understanding improper fractions and fractions as measures. At this level, students are able to generate improper fractions and their equivalence fractions from diagram representations, and put fractions on a number line. At Level 4, students understand the unbounded infinity of fractions, such that they can show that there is no smallest or biggest fraction. Next, at level 5, they advance their infinity understanding of fractions into density, such that they can show that there are unlimited numbers between two fractions. Finally, at level 6, students understand multiplication and division of fractions, such that they can represent these operations using diagram representations.

The changes to the model are followed by a revision of the order of acquisition of items, tasks and levels. Table 4.5 shows the changes to the order of acquisition of items, tasks, and levels from the hypothesized model into the obtained (revised) model suggested by the pattern of participants' responses. Table 4.6 shows the distribution of participants' responses on the revised levels of the conceptual knowledge dimension.

Table 4.5 The hypothesized and the revised order of acquisition of items, tasks and levels for the conceptual dimension of the learning progression

<table>
<tr><th colspan="2">Hypothesized</th><th colspan="2">Revised</th></tr>
<tr><th>Level</th><th>Tasks</th><th>Level</th><th>Tasks</th></tr>
<tr><td rowspan="2">Level 1</td><td rowspan="2">-</td><td>Level 0</td><td>-</td></tr>
<tr><td>Level 1</td><td>Task 1: Item 1
Task 2: Item 1
Task 3: Item 1
Task 7: Item 1</td></tr>
<tr><td>Level 2</td><td>Task 1: Items 1, 2
Task 2: Item 1
Task 3: Items 1, 2
Task 7: Items 1, 2</td><td>Level 2</td><td>Task 1: Items 2, 3
Task 3: Item 2
Task 7: Item 7</td></tr>
<tr><td>Level 3</td><td>Task 1: Items 3, 4, 5
Task 2: Item 2
Task 3: Item 3
Task 4: Items 1, 2</td><td>Level 3</td><td>Task 1: Items 4, 5
Task 2: Item 2
Task 3: Item 3
Task 4: Items 1, 2, 3</td></tr>
<tr><td>Level 4</td><td>Task 4: Item 3
Task 5: Items 1, 2</td><td>Level 4</td><td>Task 5: Items 1,2</td></tr>
<tr><td rowspan="2">Level 5</td><td rowspan="2">Task 6: Items 1, 2
Task 8: Items 1, 2</td><td>Level 5</td><td>Task 6: Items 1,2</td></tr>
<tr><td>Level 6</td><td>Task 8: Items 1,2</td></tr>
</table>

Table 4.6 The distribution of participants' responses across the levels of the *revised* conceptual knowledge dimension

Level	Task	Item	Description of item	9-OK	7-IS	5-RI	8-NA	12-AU	13-FI	10-BA	14-DE	17-FA	4-JA	5-LA	6-JO	11-RE	3-JI	16-AK
Level - 0	-	-																
Level 1	Task 1	Item 1	Write a proper fraction from a pie diagram	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	Task 2	Item 1	Shade a pie diagram to represent a proper fraction	0	1													
	Task 3	Item 1	Compare proper fractions with the same denominator	0	1	1	1	0	1	1	1	1						
	Task 7	Item 1	Add fractions with the same denominator using diagram representations	1	1	1	1	1	1	1	1							
Level 2	Task 1	Item 2	Write an equivalent fraction for a fraction less than 1	0	0	1	0	0	1	1	1	1	1	1	1	1	1	1
	Task 1	Item 3	Write a proper fraction from a pie diagram with unequal partitions	0	0	1	0	0	1	0	1	1	1	1	1	1	1	1
	Task 3	Item 2	Compare proper fractions with different denominators		0	0	0		0	1	0							
	Task 7	Item 2	Add fractions with different denominators via diagram representations	0	0	0	0	0	0	1	0	1	1		1	1	1	1
Level 3	Task 1	Item 4	Write an improper fraction from a pie diagram		0	0			0	0	0	0	1	1	0	1	1	1
	Task 1	Item 5	Write an equivalent fraction for a fraction greater than 1										1	1	0	1	1	1
	Task 2	Item 2	Shade a pie diagram to represent an improper fraction		0		0	0					1	1	1		1	
	Task 3	Item 3	Compare improper fractions with different denominators using part-whole diagrams									0		1	1	1	1	1
	Task 4	Item 1	Put a proper fraction on a number line	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
	Task 4	Item 2	Put a proper fraction on a number line with a constraint										1			1	1	
	Task 4	Item 3	Put fractions, including an improper fraction and a mixed number, on a number line										1	1	1	1	1	1
Level 4	Task 5	Item 1	Write the biggest fraction they can										0	0	1	1	1	1
	Task 5	Item 2	Write the smallest fraction they can										0	0	1	1	1	1
Level 5	Task 6	Item 1	Find how many fractions lie between two fractions										0	0	0	1	1	1
	Task 6	Item 2	Find how many fractions lie between two-pseudo successive fractions										0	1	0	1	1	1
Level 6	Task 8	Item 1	Multiply fractions using a diagram representation							0		0	0	0	0	0	0	1
	Task 8	Item 2	Divide fractions using a diagram representation															1

From Table 4.6, it can be seen that most of the participants' responses well fit with the revised model. They could be assigned at a certain level where they have competencies at that level and below, but they do not have the competencies to be placed at the upper level. The distribution of participants' levels based on the revised order of acquisition of items, tasks, and levels is presented in Table 4.7.

Table 4.7 The distribution of participants' Levels based on the revised conceptual knowledge dimension

No	Participant	Level
1	Participant 9-OK	0
2	Participant 7-IS	1
3	Participant 5-RI	1
4	Participant 8-NA	1
5	Participant 12-AU	1
6	Participant 13-FI	1
7	Participant 10-BA	2
8	Participant 14-DE	1
9	Participant 17-FA	2
10	Participant 4-JA	3
11	Participant 5-LA	3
12	Participant 6-JO	4
13	Participant 11-RE	5
14	Participant 3-JI	5
15	Participant 16-AKh	6

However, there are some cases that show the patterns of participants' responses are not in agreement with the proposed model. For example, participant 6-JO could answer all the given items at the upper level (Level 4), but made some errors in the lower level (Level 3). We cannot find any justifiable reasons to revise the model. In all other cases, the participants who had competencies x also had competencies y. Such deviations are sometimes unavoidable and show the limitations of a deterministic approach to scoring when used with human subjects and suggests that using a probabilistic model might have some advantages.

A probabilistic approach will be explored in Chapter 6, with a larger and more complete dataset of students' responses. The probabilistic response model will be employed to examine the fit of students' responses to the proposed model and to estimate how likely a particular student is to be at a certain level. This probabilistic model will show whether some "noise" and slight deviations from the model are acceptable. In other words, the probabilistic model "would enable one to decide whether a diagnosis with less than perfect fit should be considered enough" (Nichols et al., 1995, p. 6). A probabilistic model can take into account the stochastic aspects of students' responses e.g. students have a competency on a particular item but they can slip in answering the item, or students do not have competency at a particular item but can guess the answer correctly (Almond et al., 2015; Nichols et al., 1995) . In short, the probabilistic response model can estimate how likely it is for the model to fit with the data, which is essential for empirical validation of the proposed model in this study. In the meantime, because the revised model tends to be well fitted with most of the participants' responses on the raw data presented in Table 4.5, the revised model of the conceptual knowledge dimension is used as a cognitive model (see the assessment triangle introduced by Pellegrino et al. (2001)) to perform a large scale test and its analysis is discussed in Chapter 6.

4.3.2 Procedural Dimension

4.3.2.1 Within-Item Analysis

The aims of within-item analysis in the procedural dimension are the same as those in the conceptual dimension, which are: 1) to examine whether the instruction for each item is understood as intended by the participants; and 2) to examine whether the participants' responses to each item can be used to infer about the students' competencies.

As discussed at Chapter 4, there are two tasks within the procedural dimension: Task 1 Performing Additive Fraction Operations, and Task 2 Performing Multiplicative Fraction Operations. The discussion of within-item analysis is organised within these tasks.

Overall, after examining the responses from all the participants, there were no cases showing that participants misinterpreted the items, meaning that they understood the procedural items as intended. Moreover, it can be concluded that the participants' responses to the procedural items reflected the intended competencies underlying each item, as hypothesized in the proposed model.

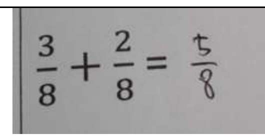
4.3.2.1.1 Task 1 Performing additive fraction operations

Item 1 – Find the sum of the fraction addition below (ProT1Q1)

$$\frac{3}{8}+\frac{2}{8}$$

From four participants (three from the low achieving students and one from the medium achieving students) who received this item, all of them answered the item correctly. Exhibit 4.40 demonstrates the answer from participant 9-OK, who successfully added proper fractions with the same denominator.

Exhibit 4.40 The answer of participant 9-OK on Task 1 Item 1 of the procedural knowledge dimension

PARTICIPANT: Find the results of the fraction addition below. PARTICIPANT: 5/8 RESEARCHER: Can you explain your answer? PARTICIPANT: If the bottom numbers are the same, they can be added directly, so 3 is added to 2, 5. So 5/8. RESEARCHER: Okay, thank you	

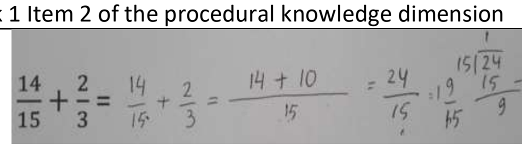
The participant added the numerators and kept the denominator the same to get the result of a fraction addition with the same denominator. This evidence shows that the participant understood the rule of adding fractions with the same denominator.

Item 2 – Find the sum of the fraction addition below (ProT1Q2)

$$\frac{14}{15}+\frac{2}{3}$$

From nine participants who received this item, six of them (medium achieving students) answered the item correctly. Exhibit 4.41 represents the response from Participant 12-AU, who successfully added proper fractions with different denominators.

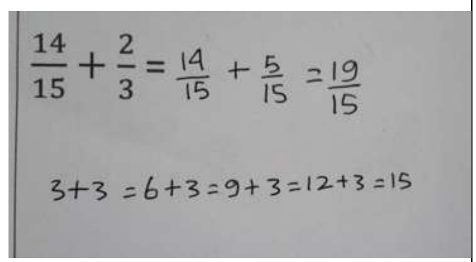
Exhibit 4.41 The answer of participant 12-AU for Task 1 Item 2 of the procedural knowledge dimension

PARTICIPANT: Find the sum of the fraction addition below PARTICIPANT: These denominators should be equated first. The LCD (Least Common Denominator) from 3 and 15 is 15, so 15 divided by 15 is 1, and 1 times 14 is 14. Then, 15 divided by 3 is 5, and 5 multiplied by 2 is 10. So the result is 14 plus 10, per 15, which is 24/15. The result is simplified, so it becomes 1 9/15.	

The participant used a least common denominator to convert the fractions into equivalent fractions before adding them. Then, the participant added the numerators and kept the common denominator the same in the result. This evidence demonstrates that the participant understood the rule of adding fractions with different denominators.

Three participants did not give the correct answer to Task 1 Item 2. Exhibit 4.42 shows the answer from participant 7-IS.

Exhibit 4.42 The answer from participant 7-IS for Task 1 Item 2 of the procedural knowledge dimension

PARTICIPANT: Find the results of the addition below (The participant was silent) PARTICIPANT : I don't know RESEARCHER : Could you tell me what the problems are in this task? PARTICIPANT : The bottom numbers are not the same (the participant was silent, and then did a calculation and wrote the answer on the card) RESEARCHER: Could you explain the process as to why this is 15? (the researcher pointed to the common denominator) PARTICIPANT : It is a repeated addition, 3 is added to 3, 6, 6 is added to 9, 9 is added to 3, 12, 12 is added to 3, 15. RESEARCHER: And then… PARTICIPANT : I transformed 2/3 to 5/15 RESEARCHER: Could you tell me why this is 5? (The researcher pointed the numerator 5 in 5/15) PARTICIPANT : The number 3s are counted, 1, 2, 3, 4, 5 (the participant counted how many 3s in the equation of 3+3=6+3=9+3=12+3=15) RESEARCHER: Okay, thanks for your answer, we can discuss it again later.	

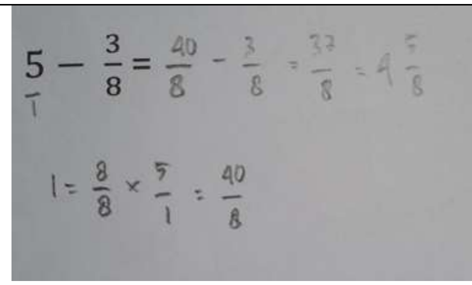
The participant tried to convert the fractions into equivalent fractions with a common denominator. However, the participant made a mistake in transforming 2/3 to an equivalent fraction with a common denominator of 15. From this response, it can be inferred that the participant understood the instruction, but the participant did not know the procedure for adding fractions with different denominators, especially the procedure for transforming the fractions to their equivalent fractions with a common denominator.

Item 3 – Find the difference of the fraction subtraction below (ProT1Q3)

$$5 - \frac{3}{8}$$

From the 3 participants who received Item 3 in Task 1 of procedural knowledge, two participants demonstrated that they understood the items and answered them correctly. Exhibit 4.43 demonstrates one of the answers from Participant 6-JO, who answered correctly the item.

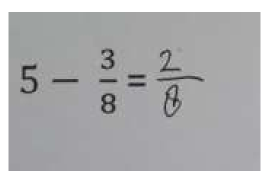
Exhibit 4.43 The answer from participant 6-JO for Task 1 Item 3 of the procedural knowledge dimension

PARTICIPANT: *Find the difference of the fraction subtraction below* PARTICIPANT: For 5 subtracted by 3/8, it should use per 8. 1 equals 8/8. So if it is 5, it is equal to 5/1. Then 8/8 multiplied by 5/1 is 40/8. This 40/8 is subtracted by 3/8, so the result is 37/8 or 4 5/8.	

The participant converted the whole number 5 into an improper fraction (5/1) before being subtracted by 3/8. After that, the participant converted the improper fraction into an equivalent fraction with a common denominator 8 successfully, and therefore performed this fraction subtraction successfully. From this response, it can be inferred that the participant understood the procedure for subtracting fractions that involve a whole number.

One participant gave an incorrect response for Task 1 Item 3 of procedural knowledge. Exhibit 4.44 shows the answer of participant 5-RI.

Exhibit 4.44 The answer of participant 5-RI for Task 1 Item 3 of the procedural knowledge dimension

PARTICIPANT: Find the difference of the fraction subtraction below (The participant directly wrote down the answer 2/8 on the card) RESEARCHER: Can you tell me why the answer is 2/8? PARTICIPANT: The numerator was got from 5 minus 3, and the denominator doesn't change.	

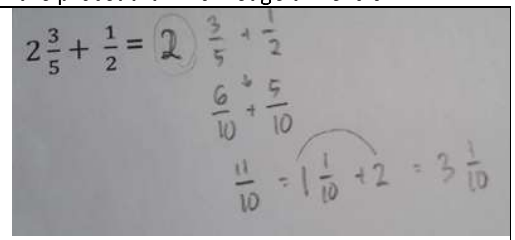
From Exhibit 4.44, it can be seen that the participant subtracted the whole number from the numerator, and kept the denominator of the fraction the same in the result. This response indicates that the participant's mistake is not due to misinterpretation of the item, but because the participant did not know how to subtract a whole number from a fraction.

Item 4 Find the sum of the fraction addition below (ProT1Q4)

$$2\frac{3}{5} + \frac{1}{2}$$

The two participants who received Item 4 in Task 1 of procedural knowledge answered the item correctly. Exhibit 4.45 shows the response from participant 6-JO, who successfully answered the item.

Exhibit 4.45 The answer of participant 6-JO for Task 1 Item 4 of the procedural knowledge dimension

PARTICIPANT: Find the sum of the fraction addition below PARTICIPANT: There are two ways to answer this question. The easy one is keeping the 2, then the denominators of 3/5 and 1/2 are equated. So 3/5 is added to 1/2. The denominators are equated with 10, so this one is 10, and also this one. If this is 10 and this is 5, 3 multiplied by 2 is 6. This is 10, this is 2, so it is multiplied by 5. Hence, this is 11/10 or 1 1/10.Next, this 1 1/10 is added to 2 to get 3 1/10. RESEARCHER: Where does this 2 come from? PARTICIPANT: From here (the participant showed the number 2 which is kept from before), and it is added to 1 because they are the same as whole numbers.	

The participant took the fraction part from the mixed number and then added it to the other fraction. At the end, the participant added the whole number (which was kept from the mixed number before) to the result of the fraction addition to get the solution. From this response, it can be inferred that the participant understood the procedure for adding a mixed number with a fraction.

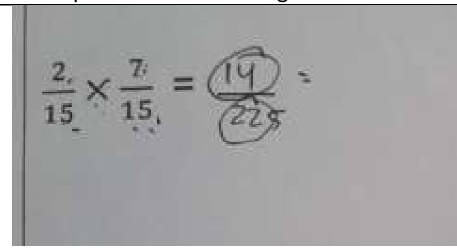
4.3.2.1.2 Task 2 Performing multiplicative fraction operations

Item 1 Find the result of the fraction multiplication below (ProT2Q1)

$$\frac{2}{15} \times \frac{7}{15}$$

From eight participants who received this item, four participants answered the item correctly. Exhibit 4.46 shows the response from participant 14-DE, who multiplied a fraction by a fraction successfully.

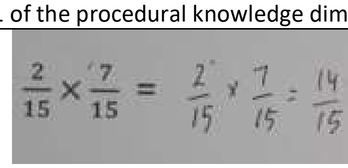
Exhibit 4.46 The answer of participant 14-DE for Task 2 Item 1 of the procedural knowledge dimension

PARTICIPANT: Find the result of the fraction multiplication below PARTICIPANT: 2/15 times 7/15. Euh… so 2 times 7, and 15 times 15. 2 times 7 equals 14, and 15 times 15 equals 225. So the answer is 14/225. RESEARCHER: Can you tell me how you did that? PARTICIPANT: The numerator is multiplied by the numerator and the denominator is multiplied by the denominator	

The participant multiplied the numerator by the other numerator, and multiplied the denominator by the other denominator to get the solution to the fraction multiplication problem. This response demonstrates that the participant understood the procedure for multiplying a fraction by a fraction.

Four participants answered Task 2 Item 2 of the procedural knowledge incorrectly. Exhibit 4.47 shows the response from participant 12-AU, who made a procedural error in answering the item.

Exhibit 4.47 The answer of participant 12-AU for Task 2 Item 1 of the procedural knowledge dimension

PARTICIPANT: Find the results of the fraction multiplication below PARTICIPANT: Ehm.. (the participant wrote down 2/15 multiplied by 7/15, and she put the answer 14/15) RESEARCHER: Can you tell me how you got the answer 14/15? PARTICIPANT: Ehm … They are just multiplied, because the denominators are already the same, so only the numerators are multiplied, the result is 14.	

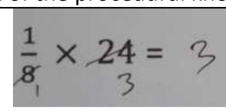
From Exhibit 4.47, it can be seen that the participant only multiplied the numerator with the other numerator and kept the denominator the same in the result. It seems that the participant misapplied the procedure for fraction addition with the same denominator to the case of fraction multiplication, because in fraction addition with the same denominator, only the numerators are added and the denominator remains the same in the result. From this response it can be inferred that the participant understood the instruction, but did not know the procedure for multiplying fractions with the same denominator.

Item 2 Find the result of the fraction multiplication below

$\frac{1}{8} \times 24$ *(ProT2Q2)*

Two participants received the item, and one of them answered the item correctly. Exhibit 4.48 shows the response from participant 10-BA, who successfully multiplied a fraction with a whole number.

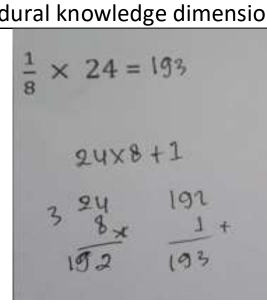
Exhibit 4.48 The answer of participant 10-BA for Task 2 Item 2 of the procedural knowledge dimension

PARTICIPANT: Find the result of the fraction multiplication below PARTICIPANT: 24 can be divided by 8, which is 3, and 1/8 becomes 1/1. So the result is 3.	

The participant divided the whole number by the denominator to get the solution for fraction multiplication with a whole number. This response shows that the participant understood the instruction and knew the procedure for multiplying a fraction by a whole number.

One participant answered the item incorrectly. Exhibit 4.49 shows the response from participant 12-AU, who made a procedural error in answering the item.

Exhibit 4.49 The answer of participant 13-FI for Task 2 Item 2 of the procedural knowledge dimension

PARTICIPANT: Find the result of the fraction multiplication below PARTICIPANT: Hm … 193 RESEARCHER: Can you tell me how you got the answer 193? PARTICIPANT: 24 times 8 plus 1. 24 times 8 equals 192, plus 1 is the same with 193.	

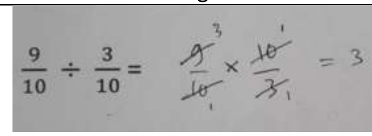
From Exhibit 4.49, it can be seen that the participant multiplied the whole number by the denominator and added it to the numerator to get the solution. This response shows that the participant understood the instruction, but made a procedural error in multiplying a fraction by a whole number.

Item 3 Find the result of the fraction division below (ProT2Q3)

$$\frac{9}{10} \div \frac{3}{10}$$

Eleven participants received the item, and seven participants answered the item correctly. Exhibit 4.50 shows the response from participant 3-JI who found the solution to Task 2 Item 3 of the procedural knowledge successfully.

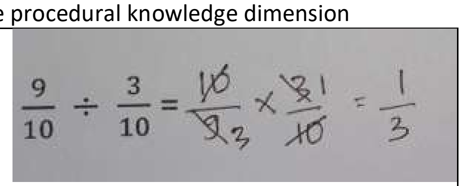
Exhibit 4.50 The answer of participant 3-JI for Task 2 Item 3 of the procedural knowledge dimension

PARTICIPANT: In fraction division, the numbers after the division sign should be flipped, so 3/10 becomes 10/3. After that we multiply these fractions. 10 divided by 10, 1, and 9 divided by 3, 3 so the answer is 3.	

The participant flipped the divisor and multiplied the dividend by the flipped-divisor to get the answer. From this response, it can be inferred that the participant understood the procedure for dividing a fraction by a fraction.

Two participants answered Task 2 Item 3 incorrectly. Exhibit 4.51 shows the response from participant 13-FI, who made a procedural error in answering the item.

Exhibit 4.51 The answer of participant 13-FI for Task 2 Item 2 of the procedural knowledge dimension

PARTICIPANT: Find the result of the fraction division below PARTICIPANT: 9 per 10 divided by 3 per 10. Euh this division is converted to multiplication than this one is flipped (the participant flipped the dividend) so 10 per 9 times 3 per 10. 10 divided by 10, 1 then 9 divided by 3, 3. Here is 1 so this is 1/3.	

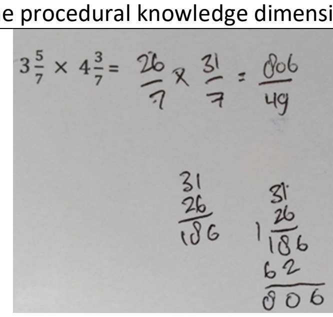
The participant flipped the dividend-fraction and multiplied by the divisor-fraction to get the solution for the fraction division. The participant made a procedural error by flipping the dividend. This response showed that the participant understood the item, but did not know the correct procedure for fraction division.

Item 4 – Find the result of the fraction multiplication below (ProT2Q4)

$$3\frac{5}{7} \times 4\frac{3}{7}$$

Four participants received this item, and three of them answered the item correctly. Exhibit 4.52 shows the response from participant 10-BA, who answered the item successfully.

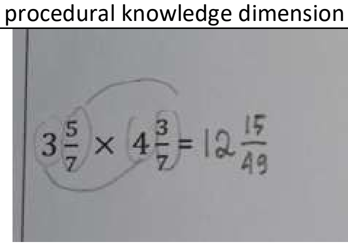
Exhibit 4.52 The answer of participant 16-AK for Task 2 Item 4 of the procedural knowledge dimension

PARTICIPANT: These are mixed numbers that should be converted first into common fractions. For 3 5/7, 7 times 3 is 21, and 21 plus 5 is 26. So this is 26/7. While for 4 3/7, 7 times 4 is 28, and 28 plus 3 is 31. So this is 31/7. Now they can be multiplied directly. 26 times 37, per 7 times 7. (The participant did the calculation for 26 times 31 and 7 times 7). The result is 806/49.	

The participant transformed the mixed numbers into improper fractions, and then performed fraction multiplication on these improper fractions. This response demonstrates that the participant understood the procedure for multiplying mixed numbers.

One participant answered the item incorrectly. Exhibit 4.53 show the participant’s response, revealing a procedural error in multiplying a mixed number by a mixed number.

Exhibit 4.53 The answer of participant 6-JO for Task 2 Item 4 of the procedural knowledge dimension

<table>
<tr>
<td>PARTICIPANT: Find the result of the fraction multiplication below
PARTICIPANT: 3 5/7 multiplied by 4 3/7, so 3 times 4 is 12, and 5 times 3 is 15. And then, 7 times 7 is 49.
RESEARCHER: Can you explain how you get 12?
PARTICIPANT: 3 times 4, because they are in the same as whole numbers, while 3/5 and 4/7 are the same as fractions, so a whole number is multiplied by a whole number, while a fraction is multiplied by a fraction.</td>
<td>
</td>
</tr>
</table>

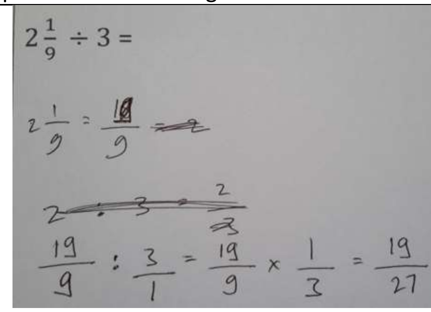
The participant multiplied a whole number with another whole number, and a fraction multiplied by another fraction to get the solution of the mixed number multiplication. It looks like the participant misapplied a mixed number addition algorithm to the mixed number multiplication. When adding mixed numbers, it is allowed to add a whole number to another whole number, and to add a fraction to another fraction, and then join both of the results to get the solution. From this response, it can be inferred that the participant’s mistake in adding mixed numbers was not due to misinterpretation of the result but because the participant did not know the correct procedure for multiplying a mixed number by a mixed number.

Item 5 – Find the result of the fraction division below (ProT2Q5)

$$2\frac{1}{9} \div 3$$

One participant received Task 2 Item 5, and answered the item correctly. Exhibit 4.54 shows the response from participant 4-JA, who divided a mixed number by a whole number successfully.

Exhibit 4.54 The answer of participant 4-JA for Task 2 Item 4 of the procedural knowledge dimension

PARTICIPANT: 2 1/9 is divided by 3. Because 2 1/9 is a mixed number so we transform this to common fraction form. 2 1/9 is equal to 2 multiplied by 9, 18, 18 added to 1, 19, so 19/9 divided by 3. We change to multiplication; 3 is the same as 3/1, but because it is changed to multiplication, we flip this. So 19/9 multiplied by 1/3 is 19; 19 multiplied by 1 is 19; and 9 multiplied by 3 is 27, so the result is 19/27.	

The participant transformed a mixed number (2 1/9) and a whole number (3) into an improper fraction before performing a fraction division. After that, the participant flipped the divisor (3/1) and multiplied by the dividend (19/9). This response demonstrates that the participant understood the procedure for multiplying a mixed number with a whole number.

4.3.2.2 Adding New Items during the Cognitive Interview

Task 2 Items 4 to 6 were designed to test the participants at a high level of the proposed model (Level 4). However, the participants from the medium and high achieving students answered the items easily. This evidence suggests a need to create more complex items to test the competency of high level students, and to differentiate between high achieving students and medium achieving students. For these reasons, a new task with several items was developed during the cognitive interviews, as follows.

4.3.2.2.1 Task 3 Complex Fraction Operations

The items within task 3 are different from the previous tasks. In task 1 and task 2, each item tests competency in additive and multiplicative fraction operations separately. In task 3, the participants need to combine both additive and multiplicative fraction operations to solve the items. The operations are "nested" in the numerator or the denominator of fractions. For example, consider a fraction *a/b*. In the previous task, the numerator *a* and the denominator *b* are whole numbers. In Task 3, the numerator *a* or denominator *b* could be a fraction operation, for example if *a* is *p/q – x/y*, the fraction operation becomes $\frac{p/q - x/y}{b}$. The competency of performing such a fraction operation is essential in further study, especially in Algebra (See Karr, Massey, & Gustafson, 2015).

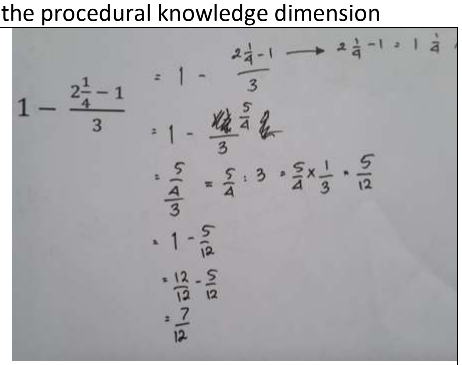
Item 1 – Find the result of the fraction operation below (ProT3Q1)

$$1-\frac{2\frac{1}{4}-1}{3}$$

Task 3 Item 1 tests the students' procedural knowledge of fraction operations that involve fraction subtraction with a whole number and a fraction division. Division of fraction is needed to simplify the fraction $\frac{2\frac{1}{4}-1}{3}$. To answer this item correctly, students should understand several rules. First, students should understand the rules of fraction subtraction, especially where the mixed number is subtracted by a whole number, which is needed to solve the operation in the numerator of the fraction $\frac{2\frac{1}{4}-1}{3}$. Secondly, students should understand the rule of simplifying the fraction $\frac{2\frac{1}{4}-1}{3}$ by dividing the result of $2\frac{1}{4} - 1$ by the whole number 3. Finally, the third rule is similar to the first rule, but in this case a whole number is subtracted by a fraction (1 minus $\frac{2\frac{1}{4}-1}{3}$, in which $\frac{2\frac{1}{4}-1}{3}$ could be simplified to a fraction). This item is used to address the competency of procedural knowledge at Level 5.

The results from the cognitive interviews show that two students gave evidence that they understood the instruction and answered the item correctly. Exhibit 4.56 shows the response from participant 5-LA, who successfully solved the item.

Exhibit 4.55 The answer of participant 5-LA for Task 3 Item 1 of the procedural knowledge dimension

PARTICIPANT: Find the result of the fraction operation below (The participant was silent and then did some calculations) RESEARCHER: Keep talking please PARTICIPANT: Euh this is from the item that 2 ¼ minus 1 is 5/4. So 1 is subtracted by 5/4 minus 1 per 3 ah.. 5/4 per 3 means that 5/4 is divided by 3, so that it becomes 5/4 multiplied by 1/3, which is 5/12. Hence, 1 is subtracted by 5/12. 1 subtracted by 5/12 is equivalent with 12/12, subtracted by 5/12, equals 7/12	

At first, the participant solve the operation of $2\frac{1}{4} - 1$ in the numerator of the fraction $\frac{2\frac{1}{4}-1}{3}$, which gave the result 5/4. Next, the participant simplified the fraction $\frac{5/4}{3}$, by dividing 5/4 with 1/3 which produced 5/12. Finally, the participant subtracted 5/12 from 1 to get the answer. From this response, it can be inferred that the participant understood several rules to solve the item, such as fraction subtraction and fraction division.

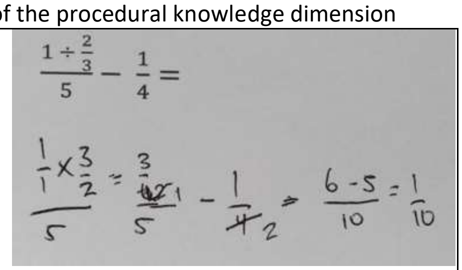
Item 2 – Find the result of the fraction operation below (ProT3Q2)

$$\frac{1 \div \frac{2}{3}}{5} - \frac{1}{4}$$

Task 3 Item 2 tests the students' procedural knowledge of fraction operations that involve fraction division and fraction subtraction. To solve this item correctly, students should understand several rules. First, students should understand the rule of dividing a whole number by a fraction, which is needed to solve the operation in the numerator of $\frac{1 \div \frac{2}{3}}{5}$. Second, students should understand the rule of simplifying a fraction $\frac{1 \div \frac{2}{3}}{5}$, in which the numerator is also a fraction (the result of $1 \div \frac{2}{3}$ can be transformed to a fraction). Finally, the participants should understand the rule of fraction subtraction with different denominators to subtract a fraction from the result of $\frac{1 \div \frac{2}{3}}{5}$ by 1/4 to get the final result. This item is used to address competency in procedural knowledge at Level 5.

The result from the cognitive interview shows that of the two participants who received this item, one participant gave evidence that he understood the instruction and answered the item correctly. Exhibit 4.57 shows the response from participant 10-BA, who solved the item successfully.

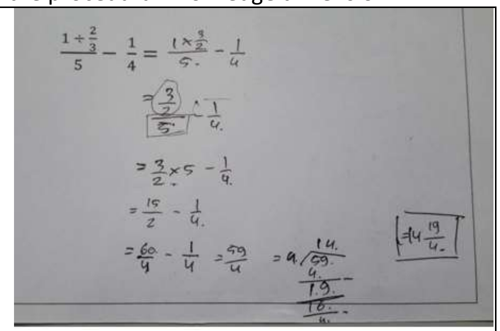
Exhibit 4.56 The answer of participant 10-BA for Task 3 Item 1 of the procedural knowledge dimension

PARTICIPANT: Find the result of the fraction operation below PARTICIPANT: 1 divided by 2/3 per 5 minus 1/4. This 1 divided by 2/3 should be calculated first. So 1 per 1 times 3/2 per 5 is equal to 3/2/5. Hm… what's the next step (the participant was silent) RESEARCHER: Could you tell me what the problems are in this task? PARTICIPANT: Ehm this is double fractions (After that the participant crossed out the denominator 2 of 3/2 and the denominator 4 of 1/4). This means 3/5 minus 1/2 are transformed with the denominator 10. So 10 divided by 5 is 2, and 2 times 3 is 6. 10 divided by 2 is 5, and 5 times 1 is 5. Now 6 minus 5 per 10 or 1/10.	

The participant solved the operation on the numerator of the fraction $\frac{1 \div \frac{2}{3}}{5}$ and successfully simplified this fraction and subtracted by 1/4 to get the result. This response demonstrates that the participant understood several procedures, such as fraction division and fraction subtraction with different denominators.

One participant could not answer Task 3 Item 2 correctly. Exhibit 4.58 shows the response from participant 8-NA who failed to solve the item.

Exhibit 4.57 The answer of participant 8-NA for Task 3 Item 2 of the procedural knowledge dimension

PARTICIPANT: Find the result of the fraction operation below PARTICIPANT: There are mixed numbers and common fractions and there is a fraction here but it is joined so 1 divided by 2/3 per 5, subtracted by 1/4 is equal to 1 multiplied by 3/2 per 5, and subtracted by 1/4. So this is equal to 3/2/5 ehm … . Hm… the problem is that 3/2 is already one unit, while the denominator, but per 5, I don't know what this number belongs to. While this 1/4, I have already know. RESEARCHER: Do you want to try first? PARTICIPANT: this 3/2 per 5 means that this is divided, so 3/2 multiplied by 5 is equal to 5 multiplied by 3, 15. So 15/2 is subtracted by 1/4. Its denominator is not the same, so it should be equated first with 4. 4 multiplied by 15 is 60 so 60/4 is subtracted by ¼ which is 59/4. If it is simplified so 59 divided by 4 (the participant divided 59 by 4 using the whole number division technique). The result is 14 19/4.	

The participant successfully executed the operation $1 \div \frac{2}{3}$ as the numerator of $\frac{1 \div \frac{2}{3}}{5}$ to produce a fraction $\frac{3/2}{5}$. However, the participant had difficulty in simplifying the fraction $\frac{3/2}{5}$. Next, the participant made a mistake by multiplying 3/2 by 5 to simplify $\frac{3/2}{5}$. From this response, it can be inferred that the participant understood the instruction but did not understand the rule of simplifying a fraction in which the numerator is in a fraction form.

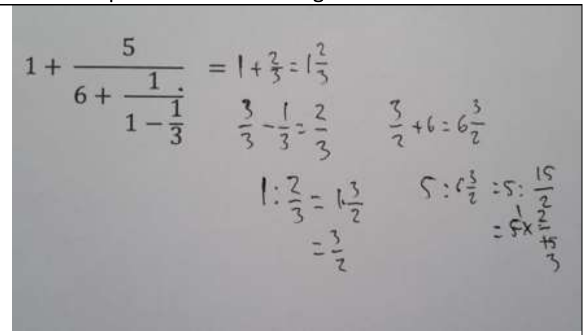
Item 3 – Find the result of the fraction operation below (ProT3Q3)

$$1 + \frac{5}{6 + \frac{1}{1 - \frac{1}{3}}} \quad =$$

Task 3 Item 3 tests the students' procedural knowledge of fraction operations that involve more nested fraction operations than Task 3 Items 1 and 2. To solve this item, the participants should understand the rules of fraction addition and subtraction with a whole number, and understand how to use fraction division to simplify a fraction where the numerator or denominator contains a fraction or a fraction operation.

The results from the cognitive interviews show that of the six participants who received this item, three participants gave evidence that they understood the instruction and solved the item successfully. Exhibit 4.59 shows the response from participant 16-AK, who solved the item successfully.

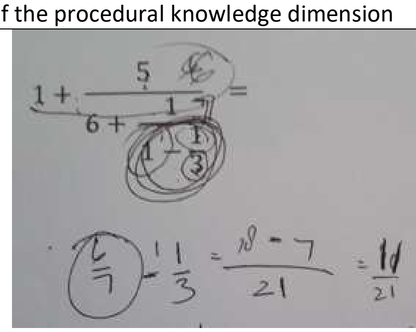
Exhibit 4.58 The answer of participant 16-AK for Task 3 Item 3 of the procedural knowledge dimension

PARTICIPANT: Find the result of the fraction operation below PARTICIPANT: Firstly, we do the operation in the bottom which is 1 or 3/3 minus 1/3, which is equal to 2/3. Then 1 divided by 2/3 or 1 times 3/2 which is 3/2. Next, 3/2 plus 6 equals 6 3/2, then 5 divided by 6 3/2 which is the same as 5 divided by 15/2. It is the same as 5 times 2/15 which is 2/3. Finally, 1 plus 2/3 which is equal to 1 2/3	

The participant executed the fraction operation from the denominator in the bottom of the operation and moved to the upper level operation by dividing the numerator with the denominator. For example, after calculating 1-1/3 to get 2/3 as the denominator of $\frac{1}{1-\frac{1}{3}}$, the participant moved to the higher level of operation by dividing 1 with 2/3 to get 3/2. This 3/2 is added to 6 to get 15/2, which has become the denominator in which the numerator is 5. The participant divided 5 by 15/2 to get 2/3. Finally 2/3 is added to 1 to get the final answer. This response demonstrates that the participant understood the instruction and understood several rules to solve complex fraction operations, such as fraction subtraction and fraction division that were used to simplify the nested fraction.

Three participants answered Task 3 Item 3 incorrectly. Exhibit 4.59 shows the response from participant 17-FA.

Exhibit 4.59 The answer of participant 17-FA for Task 3 Item 3 of the procedural knowledge dimension

PARTICIPANT: Find the result of the fraction operation below (The participant was silent for a moment) RESEARCHER: Can you tell me what the problems are in this task? PARTICIPANT: Euh … I do not understand 5 and 1 here, but I think 1 is added to 5, so 1 plus 5 equals 6. 6 plus 1 equals 7 so this is 6/7. Then 6/7 is subtracted by 1/3. The denominators are equated to 21. 21 divided by 7 is 3, and 3 times 6 is 18. 21 divided by 7 is 7, and 7 times 1 is 7 so the result is 11/21.	

From Exhibit 4.59, it can be seen that the participant looked confused with the whole number 1 and the numerator 5. The participant started the calculation by adding the whole number 1 to 5, and took the result as the numerator. Next, the participant added the whole number 6 to 1, and took the result as the denominator. Hence the participant got a fraction 6/7 which is subtracted later by 1/3 to get the final result. From this response, it can be inferred that the participant understood the instruction, but did not understand the rule and procedure to execute a complex (nested) fraction operation.

In summary, the results of within-task analysis of the procedural knowledge dimension show that there are no cases where the participants made a mistake because of misinterpreting the instructions of the items, meaning that the participants could understand all the items as intended. The results also show that the participants' responses reflect accurately the hypothesized competencies. Hence, the responses can be used to infer the participants' procedural knowledge. Next, the hierarchical order of the items examined in within-task analysis are presented in the following section.

4.3.2.3 Within-Procedural Task Analysis

The within-task analysis was conducted for all items within a task in the procedural knowledge dimension. The aim of the within-task analysis of the procedural knowledge dimension is the same as that of the conceptual knowledge dimension, which is to examine whether the hierarchical order of the items within the task is consistent with the hypothesized order. To achieve this goal, the analysis is focused on finding whether there is evidence that some of the participants answered the items at the higher levels in the task successfully but could not answer the items at the lower levels of the task. Such evidence would show that there is an inconsistency between the order of the items within a task and the hypothesized order.

Table 4.8 demonstrates the participants' responses, which are structured within the tasks and levels of the procedural knowledge dimension. There are three tasks within the procedural knowledge dimension, which are: 1) performing additive fraction operations; 2) performing multiplicative fraction operations; and 3) performing complex fraction operations. From Table 4.8, it can be seen that there are no cases within tasks where students could answer an upper item correctly, but not a lower item. Therefore, it can be inferred that the hierarchical order of the procedural items is consistent with the hypothesized order.

Table 4.8 The distribution of the participants' responses within task of the conceptual dimension

Task	Item	Level	9-OK	7-IS	5-RI	8-NA	12-AU	13-FI	10-BA	14-DE	17-FA	4-JA	5-LA	6-JO	11-RE	3-JI	16-AK
Task 1	Item 1	Level 2	1	1	1	1											
	Item 2	Level 2	0	0	0	1	1	1	1	1	1						
	Item 3	Level 3			0	1								1			
	Item 4	Level 3											1	1			
Task 2	Item 1	Level 3	0	0		1	0	0		1	1			1			
	Item 2	Level 3						0	1								
	Item 3	Level 3				1	0	0		1	1		1	1		1	1
	Item 4	Level 4			0				1					0		1	1
	Item 5	Level 4										1					
	Item 6	Level 4				1											
Task 3	Item 1	Level 5										1	1				
	Item 2	Level 5				0			1								
	Item 3	Level 5							0	0	0			1	1		1

4.3.2.4 Assigning the Participants to the Levels of the Procedural Dimension and Changes in the Model

The purpose of this analysis is to examine whether the participants' profiles were in agreement with the proposed model. To achieve this goal, the order of the participants' responses for the procedural tasks were compared with the hypothesized order of items and tasks nested in levels, as shown in Table 4.9 below.

Table 4.9 The hypothesized order of acquisition of items and tasks for the procedural dimension of the learning progression

Level	Task
Level 1	-
Level 2	Task 1: Items 1, 2
Level 3	Task 1: Items 3, 4 Task 2: Items 1, 2, 3
Level 4	Task 2: Items 4, 5, 6
Level 5	Task 3: Items 1, 2, 3

The profile for all 15 participants' responses on the procedural knowledge dimension are presented in Table 4.10 below. The criteria for assigning participants into the procedural level is the same as that for conceptual knowledge, which is: the participants are assigned to a certain level if they have all the competencies at that level and below, but they have not enough competencies at the upper level.

From Table 4.10, one can see the distribution of the participants' responses across the items, tasks and levels in the hypothesized procedural knowledge dimension. At level 2, three participants (participants 9-OK, 7-IS and 5-RI) could answer Task 1 Item 1 (adding proper fractions with the same denominator) correctly, but could not answer Task 1 Item 2 (adding proper fractions with different denominators) correctly. This evidence indicates that adding proper fractions with the same denominator is much easier than adding fractions with different denominators and it might be better to assign it to a different level in the progression

As a result, Level 2 is revised by differentiating the levels of additive operations of proper fractions with the same denominator (level 1) and those of additive operations with different denominators.

Table 4.10 The distribution of participants' responses across the levels of the procedural knowledge dimension

Level	Task	Item	Description of item	9-OK	7-IS	5-RI	8-NA	12-AU	13-FI	10-BA	14-DE	17-FA	4-JA	5-LA	6-JO	11-RE	3-JI	16-Akh
Level 1	-	-	-															
Level 2	Task 1	Item 1	Add fractions with the same denominator	1	1	1	1											
	Task 1	Item 2	Add fractions with different denominators	0	0	0	1	1	1	1	1	1						
Level 3	Task 1	Item 3	Subtract a fraction from a whole number			0	1								1			
	Task 1	Item 4	Add a fraction with a mixed number											1	1			
	Task 2	Item 1	Multiply a fraction with a fraction	0	0		1	0	0		1	1			1			
	Task 2	Item 2	Multiply a fraction with a whole number						0	1								
	Task 2	Item 3	Divide a fraction with a fraction				1	0	0		1	1		1	1		1	1
Level 4	Task 2	Item 4	Multiply a mixed number with a mixed number			0				1					0		1	1
	Task 2	Item 5	Divide a mixed number with a whole number										1					
	Task 2	Item 6	Divide a mixed number with a mixed number				1											
Level 5	Task 3	Item 1	Solve a nested fraction operation where the numerator is a fraction subtraction										1	1				
	Task 3	Item 2	Solve a nested fraction operation where the numerator is a fraction division				0			1								
	Task 3	Item 3	Solve a fraction operation with a two-level nested fraction							0	0	0			1	1		1

At level 2, two participants (participants 12-AU and 13-FI) could add proper fractions with different denominators (Task 1 Item 2), but could not multiply or divide a proper fraction by a proper fraction (Task 2 Item 1 and 3). This evidence indicates that additive fraction operations (for proper fractions) are easier than multiplicative fraction operations (for proper fractions), and therefore should belong to a different level in the progression. For this reason, the levels of additive proper fraction operations (Level 2) is differentiated from multiplicative proper fraction operations (Level 3).

At level 3, participant 6-JO could answer both items of proper fraction multiplication and division (Task 2 Item 1 and Item 3 at Level 3) correctly but could not answer multiplication of mixed numbers (Task 2 Item 4) correctly. This result suggests that it might be better to separate the competency of multiplying with a proper fraction from the competency of multiplying with mixed numbers. Therefore, the competency of multiplicative fraction operations for proper fractions is placed at level 3 and that of multiplicative fraction operations with mixed numbers or improper fractions is placed at level 4.

At levels 4 and 5, participant 8-NA could answer Task 2 Item 6 (dividing a mixed number with a mixed number) correctly, but could not answer Task 3 Item 2 (solving a nested fraction operation with the numerator is a fraction division) correctly. This evidence suggests that multiplicative fraction operations require a different level of learning from a nested fraction operation as hypothesized in the model. This result confirms the order of acquisition of levels 4 and 5.

From level 5, participant 10-BA could solve Task 3 Item 2 (solving a nested fraction operation with the numerator is a fraction division) correctly, but could not answer Task 3 Item 3 (Solving a fraction operation with two-level nested fraction) correctly. This indicates that performing one-level nested fraction operations requires a different level of learning from that of two-level or more nested fractions. This result suggests that level 5 should be split to separate the competency of performing two-level nested fraction operations from the competency of performing one-level nested fraction operations.

Based on the findings discussed above, the hypothesized procedural knowledge dimension was changed as follows. Level 0 was created to capture students who have no procedural knowledge of fraction operations. Although there are no cases of participants who represent this level from the cognitive interview, this level was still created to accommodate the possibility of such cases appearing in the large study discussed in the next chapter (Chapter 6). At level 1, students begin to know the procedure for additive proper fraction operations with the same denominator. Next, at level 2, they advanced their procedural knowledge of additive fraction operations into additive proper fraction operations with different denominators. At level 3, they advanced their additive operations into operations that involve improper fractions and mixed numbers. In addition, their competency with multiplicative fraction operations emerges at this level but is still limited to proper fractions. Next, at level 4, they advance their multiplicative fraction operations to multiplicative operations which involve improper fractions and mixed numbers. At level 5, they have the competency to solve more complex fraction operations that involve one-level nested fraction operations. Finally, at Level 6, they have sufficiently advanced procedural competency that they can solve two or more nested complex fraction operations.

Proceeding from the changes to the hypothesized model of the procedural knowledge dimension, the revised order of acquisition of items, tasks, and levels is presented in Table 4.11 below, and the distribution of participants' responses is presented in Table 4.12

Table 4.11 The hypothesized and revised order of acquisition of items, tasks and levels for the procedural knowledge dimension

Hypothesized		**Revised**	
Level	**Tasks**	**Level**	**Tasks**
Level 1	-	Level 0	-
		Level 1	Task 1: Item 1
Level 2	Task 1: Items 1, 2	Level 2	Task 1: Item 2
Level 3	Task 1: Items 3, 4 Task 2: Items 1, 2, 3	Level 3	Task 1: Items 3, 4 Task 2: Items 1, 2, 3
Level 4	Task 2: Items 4, 5, 6	Level 4	Task 2: Items 4, 5, 6
Level 5	Task 3: Items 1, 2, 3	Level 5	Task 3: Items 1, 2
		Level 6	Task 3: Item 3

Table 4.12 The distribution of participants' responses across level of the *revised* procedural knowledge dimension

Level	Task	Item	Description of item	9-OK	7-IS	5-RI	8-NA	12-AU	13-FI	10-BA	14-DE	17-FA	4-JA	5-LA	6-JO	11-RE	3-JI	16-Akh
Level 0	-	-	-															
Level 1	Task 1	Item 1	Add fractions with the same denominator	1	1	1	1											
Level 2	Task 1	Item 2	Add fractions with different denominators	0	0	0	1	1	1	1	1	1						
Level 3	Task 1	Item 3	Subtract a fraction from a whole number			0	1								1			
	Task 1	Item 4	Add a fraction with a mixed number											1	1			
	Task 2	Item 1	Multiply a fraction with a fraction	0	0		1	0	0		1	1			1			
	Task 2	Item 2	Multiply a fraction with a whole number						0	1								
	Task 2	Item 3	Divide a fraction with a fraction				1	0	0		1	1		1	1		1	1
Level 4	Task 2	Item 4	Multiply a mixed number with a mixed number			0				1					0		1	1
	Task 2	Item 5	Divide a mixed number with a whole number										1					
	Task 2	Item 6	Divide a mixed number with a mixed number				1											
Level 5	Task 3	Item 1	Solve a nested fraction operation where the numerator is a fraction subtraction										1	1				
	Task 3	Item 2	Solve a nested fraction operation where the numerator is a fraction division				0			1								
Level 6	Task 3	Item 3	Solve a fraction operation with a two-level nested fraction							0	0	0			1	1		1

From Table 4.12, it can be seen that most of the participants' responses are in agreement with the revised model. The participants were assigned to a certain level if they demonstrated competencies at that level and below, but they did not have enough competencies at the upper level. The participants' levels are presented in Table 4.13. However, there are some cases which show the pattern of participants' responses are not perfectly in agreement with the proposed model. For example, participant 6-JO, who could answer an item at level 6, but made an error in answering Task 2 Item 4 at Level 4.

This case is similar to the case in the conceptual knowledge dimension where there is slight deviation from the student's answer to the hypothesized model. As discussed earlier, a probabilistic response model should test the goodness of fit of the hypothesized model with the data from the participants' responses. The purpose of the analysis is to test whether slight deviations from the model are accepted. This discussion will be conducted in Chapter 6 using the complete data set.

Table 4.13 The distribution of participants' Levels of the procedural knowledge dimension

No	Participant	Level
1	Participant 9-OK	1
2	Participant 7-IS	1
3	Participant 5-RI	1
4	Participant 8-NA	4
5	Participant 12-AU	2
6	Participant 13-FI	2
7	Participant 10-BA	4
8	Participant 14-DE	3
9	Participant 17-FA	3
10	Participant 4-JA	5
11	Participant 5-LA	5
12	Participant 6-JO	6
13	Participant 11-RE	6
14	Participant 3-JI	4
15	Participant 16-AK	6

4.4 Discussion of the Results

The within-item analysis shows that the conceptual and procedural items were understood by the participants and they tested the intended competencies. The participants' responses indicated that they understood the instructions and their mistakes were caused by incomplete or no knowledge. The responses from the participants also show that the items test the intended competencies successfully.

The within-task analysis also confirmed that the hierarchy of items within the tasks was consistent with the hypothesized order. Most of the participants who could answer correctly at the upper level, could also answer the items at the lower level. There were almost no cases where the participant successfully answered the items at the upper level but could not answer the items at the lower level. This evidence shows that the order of the items within each task is consistent with the hypothesized order.

The analysis that tested the fit of the tasks/items with the hypothesized levels showed that several changes in the model were needed in order to get a better fit with the participants' responses. Within the conceptual knowledge dimension, the part-whole level of the hypothesized model was split into two levels (level 1 and level 2) because the participants' responses showed that there were two different constructs within this level, which could not be placed on the same level of learning. One is a level 1 part-whole understanding, which shows some understanding of part-whole relations but limited to fractions with the same denominator. Level 2 in the revised model, represents a more advanced part-whole understanding, which extends to fractions with different denominators and to equivalent fractions. In addition, there was a competency at Level 3 – i.e., generating a fraction from a pie diagram with unequal partitions – which was more likely to have the same construction as the construction at Level 2 of the revised model. This competency was moved to level 2.

The competency of generating improper fractions at Level 4 was moved to level 3, so level 3 became improper fractions and fractions as measures, and level 4 became the unbounded infinity of fractions. Next, understanding multiplicative fraction operations tends to require a different level of learning than the construction of understanding of the density of fractions. Hence, understanding density was placed at level 5, and understanding multiplicative fraction operations was placed at level 6. Level 0 is created to accommodate participants who have no fraction understanding.

For the procedural knowledge dimension, participants' responses show that knowing the procedure for fraction addition with the same denominator requires a different level of learning than the procedure for fraction addition with different denominators. Hence, level 1 is adding fractions with the same denominator, while level 2 is adding fractions with different denominators. Level 3 and level 4 remain the same as the hypothesized model. Level 3 covers additive improper fractions and mixed numbers operations, and multiplicative fraction operations limited to proper fractions, while level 4 is multiplicative fraction operations which involve improper fractions and mixed numbers. Levels 5 and 6 were created during the cognitive interview to recognise the procedural knowledge of high achieving students. Level 5 is one-level nested fraction operations, while level 6 is two-level nested fraction operations. Level 0 is created to accommodate the possibility of having participants who do not have procedural knowledge of fraction operations.

Most of the participants fitted well into the revised model for both the conceptual and procedural knowledge dimensions. Table 4.14 demonstrates the profile of the participants' levels for both the conceptual and procedural knowledge dimensions.

The results show that the participants look varied in their levels for learning fractions. At a low level, several participants (participants 7-IS and 5-RI) have the same level in both the conceptual and procedural knowledge dimensions. Similarly, at a high level, participant 17-AK has the same level in both the conceptual and procedural knowledge dimensions. In the medium level, most of the participants have a higher level of procedural than conceptual knowledge. This evidence shows that during the development of fraction learning, students tend to have more knowledge about algorithms/procedures for fraction operations than understanding of the symbolic notation of fractions and the meaning underlying fraction operations.

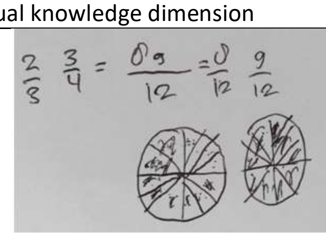
Table 4.14 The distribution of participants' Levels in the procedural knowledge dimension

No	Participant	Conceptual	Procedural
1	Participant 9-OK	0	1
2	Participant 7-IS	1	1
3	Participant 5-RI	1	1
4	Participant 8-NA	1	4
5	Participant 12-AU	1	2
6	Participant 13-FI	1	2
7	Participant 10-BA	2	4
8	Participant 14-DE	1	3
9	Participant 17-FA	2	3
10	Participant 4-JA	3	5
11	Participant 5-LA	3	5
12	Participant 6-JO	4	6
13	Participant 11-RE	5	6
14	Participant 3-JI	5	4
15	Participant 16-AK	6	6

In some cases, procedural knowledge helped participants to find the solutions for conceptual items. For example, participant 10-BA used his procedural knowledge of computing equivalent fractions with a common denominator to compare fractions with different denominators. The following exhibit (Exhibit 4.61) shows the participant's responses.

Exhibit 4.60 The answer of participant 10-BA for Task 3 Item 1 of the conceptual knowledge dimension

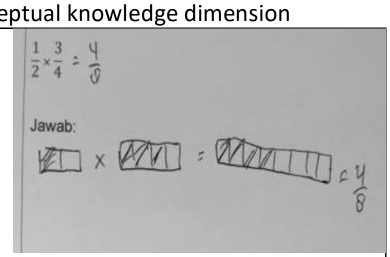
PARTICIPANT: Which is larger $\frac{2}{3}$ or $\frac{3}{4}$? Illustrate how you got your answer using a picture.
PARTICIPANT: The fractions should be transformed with a common denominator 12. So 12 divided by 3 is 4, and 4 times 2 is 8. Then 12 divided by 3 is 3, and 3 times 3 is 9. So we have the fractions 8/12 and 9/12. So 3/4 is larger.
RESEARCHER: Can you explain your answer using diagram?
PARTICIPANT: Drew 12 parts …
(The participant drew 2 circles to describe 8/12 and 9/12).
For 8/12, 8 parts are shaded, while for 9/12, 9 parts are shaded
RESEARCHER: So which one is larger?
PARTICIPANT: 3/4
RESEARCHER: Why?
PARTICIPANT: Because it has more shaded parts
RESEARCHER: Okay, thank you for your answer.



The participant activated procedural knowledge to transform the fractions 2/3 and 3/4 into equivalent fractions with a common denominator (8/12 and 9/12). Next, the participant drew pie diagrams for these equivalent fractions to compare 2/3 and 3/4. This result shows how procedural knowledge was used effectively in tandem with conceptual knowledge to solve a conceptual item. The participants' understanding of comparing two fractions, that the denominator should be the same, induced the participant to use procedural knowledge to calculate the equivalent fractions with a common denominator. After that, the participant used conceptual knowledge to compare the equivalent fractions using diagram representations to determine which fraction is bigger.

In another case, the participant used a correct procedure for fraction multiplication but could not understand the procedure.

Exhibit 4.61 The answer of participant 10-BA for Task 8 Item 1 of the conceptual knowledge dimension

PARTICIPANT: Draw a pictorial representation for the fraction multiplication below PARTICIPANT: 1/2 times 3/4 is equal to 4/8. The denominator should not be the same, so we can draw 1/2 and ¾ directly. For 1/2, there are two parts and one is shaded, and for 3/4 there are four parts and three are shaded. So the result is 4/8 in which there are 8 parts and 4 parts are shaded.	

The participant successfully calculated 1/2 multiplied by 3/4 procedurally. However, the participant could not represent the fraction multiplication using a diagram representation. This result shows that the participant had procedural knowledge of fraction multiplication but did not understand the procedure.

From the cases discussed above, the relationship between conceptual and procedural knowledge can be summarized as follows. In some cases, conceptual and procedural knowledge seem unrelated. In other cases, conceptual and procedural knowledge are intertwined. From all these cases, it may be argued that procedural knowledge is learned first or that conceptual knowledge is learned first. However, these results do not suggest that a type of knowledge (either conceptual or procedural) necessarily leads to or causes an increase in the other form of knowledge. These findings are consistent with the hypothesis of individual differences proposed by Hallett et al. (2010). Further investigation of the relationship between conceptual and procedural knowledge will be performed in the next chapter using a more complete data set from the 516 student-data test.

4.5 Summary of the Chapter

This chapter discussed the empirical evidence from the cognitive interviews to validate the proposed model and to improve the item tasks. To achieve this goal, four type of analyses were implemented and discussed for each conceptual and procedural knowledge dimension, namely within-item analysis, within-task analysis, assigning participants to levels, and within-level analysis. The results showed that: 1) the instructions for the items were understood by the participants; 2) the participants' responses reflected the intended (hypothesized) competencies; 3) the order of acquisition of the items was consistent with the hypothesized order; 4) the responses of the individual participants were consistent with the proposed levels for both the conceptual and the procedural knowledge dimensions; and 5) the order of acquisition of the levels was consistent with the hypothesized order.

The proposed model and items revised in this chapter were used to conduct a larger study with 516 students at Junior Secondary School, from grades 7 to 9. The results of the study are presented and discussed in the following chapters.

CHAPTER 5 : BAYESIAN NETWORKS MODELLING FOR MEASURING LEARNING PROGRESSIONS

5.1 Introduction

The proposed model of fraction learning progression has been revised for both conceptual and procedural knowledge dimensions, based on the results of the cognitive interview, as discussed in Chapter 4. The revised model of the conceptual items now consists of the following seven levels which are (from lowest to highest): no fraction understanding; part-whole of proper fractions with the same denominator; part-whole of proper fractions with different denominator and equivalent fractions; improper fractions and fractions as measures; unbounded infinity; density; and understanding multiplicative fraction operations. The revised model of the procedural dimension also has seven levels, which are (from lowest to highest): no procedural knowledge; additive operations of proper fractions with the same denominator; additive operations of proper fractions with different denominators; multiplicative operations of proper fractions; multiplicative operations of improper fractions/mixed numbers; one-nested complex fraction operations; and two or more nested complex fraction operations.

There are two main aims in this chapter. The first aim is to develop a measurement model to assist in the validation of the proposed model using Bayesian statistical approaches. Two types of Bayesian Networks models were developed to assess the students' learning progression. The first model is a simple Bayesian Networks model with a single parameter. This model is commonly used in Bayesian Latent Class Analysis, which is usually applied for measuring learning progression (Jefrey Thomas Steedle, 2008; West et al., 2012). In the second model, a more complex Bayesian networks model with several parameters was developed to capture the progression levels in the proposed model of fraction learning progression (adapted from Rutstein, 2012).

The second aim is to perform a model evaluation of the Bayesian Networks models that are used to measure and validate the learning progression model. Posterior

Predictive Model Checking (PPMC) and Entropy Statistics (Levy & Mislevy, 2016) were employed to perform the model validation.

To achieve the two aims, this chapter is organized into two main sections. Section 5.2 discusses the Bayesian Networks Model with a single latent variable (referred to as Model 1) followed by the Bayesian Networks Model with multiple latent variables (referred to as Model 2). Section 5.3 details the model evaluation of the Bayesian Networks using the PPMC and the Entropy Statistics. In summary, this chapter is designed to answer the following research question: "To what extent does the hypothetical model that we developed based on the distinction between conceptual and procedural knowledge capture the emergence of student competencies in fraction learning?"

5.2 Bayesian Networks Modelling

The general introduction of Bayesian inference in the context of educational assessment was presented in Chapter 2, which includes the review of the Bayesian Networks model. In this section, the specification of the development of Bayesian Networks model for measuring learning progressions is discussed. We propose two models, namely Model 1 and Model 2 with the following details.

5.2.1 Model 1: Bayesian Networks with a Single Latent Variable θ

The notation and the development of the Bayesian Networks Modelling in this study follow those of Bayesian networks as described in Levy and Mislevy (2016). Let $\boldsymbol{x} = (x_{ij})$ be the matrix data of the responses of *n* students on *J* items in which all the items have the same number of response categories, where x_{ij} is the response of the i^{th}-student on item *j* for i=1,.., *n* and j=1,…, *J*. Two sets of parameters $\boldsymbol{\theta}$ and $\boldsymbol{\pi}$ represent the latent variable of students' level and the conditional probability of students correctly answer the items given that the students have the competencies at a particular level respectively.

Model 1 is developed based on the assumption that the students' level in the learning progression is represented by a single latent variable $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$, where θ_i

is the level of the i^{th} student for $i = 1, \ldots, n$. In a simple [2]DAG, Model 1 is represented in Figure 5.1.

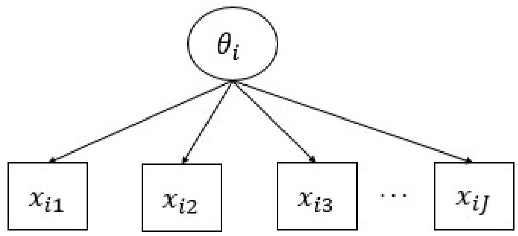


Figure 5.1 A simple DAG of a Latent Class Analysis (Adapted from Levy & Mislevy, 2016)

The simple DAG in Figure 5.1 states that the probability of every observable x_{ij} (a response of the i^{th}-student on item *j*) is conditional on θ_i (the level of the i^{th}-student. This conditional probability can be expressed as

$$p(x_{ij}|\theta_i).$$

In latent class analysis, suppose that every observable x_{ij} has k possible values (from k =1,…, K, where K is the number of the response categories), then the sum of probabilities of the i^{th}-student for all possible k values is 1 for a given level of $\theta_i = c$, where *c* is the level of the i^{th}-student. This sum of probabilities can be expressed as follows

$$\sum_{k=1}^{K} p(x_{ij} = k|\,\theta_i = c) = 1. \qquad (5.\,1)$$

The relationship between the response of x_{ij} and the students' category (level) in Equation (5.1) expresses the measurement model of the DAG. This measurement model can now be denoted as

$$\pi_{cjk} = p(x_{ij} = k|\theta_i = c),$$

$$\sum_{k=1}^{K} \pi_{cjk}=1, \text{ where } c=1,\ldots,C.$$

In our study, the proposed model of learning progression for both the conceptual and the procedural knowledge dimension has six levels (*C*=6) and two response

[2] A DAG is a directed acyclic graph, as discussed in Chapter 2, the Literature Review.

categories k of the students' answers: incorrect and correct answers (denoted as k=0,1 respectively). Hence the conditional probability of π_{cjk} for k=1 (a correct answer) is simplified as π_{cj}, and for *k*=0 (an incorrect answer) is denoted as $1 - \pi_{cj}$. Thus π_{cjk} in the above can now be simplified as

$$\pi_{cj} = p(x_{ij} = 1|\theta_i = c), \quad (5.2)$$

$\sum_{k=1}^{K} \pi_{cj}$=1, where *c*=1,…,*C*.

The conditional probability of π_{cj} is represented in Table 5.1. Table 5.1 is called a conditional probability table (CPT), which is one of the main important features in Bayesian networks computation (Almond et al., 2015; Levy & Mislevy, 2016).

Table 5.1 Conditional probability π_{cj} table with two response categories and six classes of θ_i

θ_i	Response Category	
	0	1
1	1-π_{1j}	π_{1j}
2	1-π_{2j}	π_{2j}
3	1-π_{3j}	π_{3j}
4	1-π_{4j}	π_{4j}
5	1-π_{5j}	π_{5j}
6	1-π_{6j}	π_{6j}

Let $\boldsymbol{\pi} = (\boldsymbol{\pi_1}, \dots, \boldsymbol{\pi_J})$, where $\boldsymbol{\pi_j}$ is the collection of π_{cj} for *c*=1,…,*C*, and *j*=1,…,*J*. Moreover, let $\boldsymbol{x_i}$ be the collection of the i^{th}-student's answers on *J* items for the i^{th}-student. The joint likelihood function of the observables x_{ij} conditional on both θ_i and π_{cj} can be denoted as follows:

$$p(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{\pi}) = \prod_{i=1}^{n} p(\boldsymbol{x_i}|\theta_i, \boldsymbol{\pi}) = \prod_{i=1}^{n} \prod_{j=1}^{J} p\left(x_{ij}|\theta_i = c, \boldsymbol{\pi_j}\right), \quad (5.3)$$

where $(x_{ij}|\theta_i = c, \boldsymbol{\pi_j})$ ~ Bernoulli(π_{cj}) for all possible dichotomous response values (incorrect and correct answers).

As discussed in Chapter 2, Bayesian inference requires a prior specification of the parameters. In Model 1, there are two parameters which need prior specification, θ_i and π_{cj} respectively. Because θ_i and π_{cj} are assumed to be independent, then the joint prior density function of θ_i and π_{cj} can be denoted as

$$p(\theta_i, \pi_{cj}) = p(\theta_i)p(\pi_{cj}).$$

In our case, θ_i has six categories which are defined through a distribution with hyper-parameter $\boldsymbol{\gamma}$, where $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_6)$. Hence, the distribution of θ_i is assumed as categorical conditional on $\boldsymbol{\gamma}$, which is

$$\theta_i | \boldsymbol{\gamma} \sim \text{categorical } (\boldsymbol{\gamma}). \quad (5.\ 4)$$

As $\boldsymbol{\gamma}$ is unknown, the hyperparameter of $\boldsymbol{\gamma}$ should be specified using a conjugate prior density for categorical responses which have values of [0,1], such as a Dirichlet distribution (Levy & Mislevy, 2016). Hence, the distribution of $\boldsymbol{\gamma}$ can be denoted as

$$\boldsymbol{\gamma} \sim \text{Dirichlet } (\boldsymbol{\alpha_\gamma}) \text{ where } \boldsymbol{\alpha_\gamma} = (\alpha_1, \dots, \alpha_6). \quad (5.\ 5)$$

The values of $\boldsymbol{\alpha_\gamma}$ are set to 1s which give uninformative priors of the students' level. Hence $\boldsymbol{\alpha_\gamma}$ is denoted as $\boldsymbol{\alpha_\gamma}$ = (1,1,1,1,1,1) which gives the same prior probability (about 16.67%) for all levels. These uninformative priors are chosen to allow that the estimates of students' levels were produced more from the likelihood of the students' responses (data) than the effects of the prior.

Next, as the parameter of the measurement model π_{cj} is unknown, the prior distribution of π_{cj} should be specified. Because π_{cj} represents the probability of dichotomous possible outcomes of correct and incorrect response given the students' class in θ_i, then π_{cj} is specified to have the conjugate prior of beta distribution, as follows:

$$\pi_{cj} \sim \text{Beta}\left(\alpha_{\pi_{cj}}, \beta_{\pi_{cj}}\right), \alpha_{\pi_{cj}} > 0, \beta_{\pi_{cj}} > 0\ . \quad (5.\ 6)$$

Given that we have defined priors in (5.4), (5.5), and (5.6) and the joint likelihood functions in (5.3), the joint posterior distribution for Model 1 can now be defined follows:

$$p(\boldsymbol{\theta}, \boldsymbol{\gamma}, \boldsymbol{\pi}|\boldsymbol{x}) \propto \; p(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{\gamma}, \boldsymbol{\pi})p(\boldsymbol{\theta}, \boldsymbol{\gamma}, \boldsymbol{\pi})$$

$$= p(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{\pi})p(\boldsymbol{\theta}|\boldsymbol{\gamma})\boldsymbol{p}(\boldsymbol{\gamma})\boldsymbol{p}(\boldsymbol{\pi}),$$

$$= \prod_{i=1}^{n}\prod_{j=1}^{J} p\left(x_{ij}\middle|\theta_i, \boldsymbol{\pi_j}\right)p\,(\theta_i|\boldsymbol{\gamma})p(\boldsymbol{\gamma})\prod_{c=1}^{C} p(\pi_{cj}), \qquad (5.\,7)$$

where $(x_{ij}|\theta_i = c, \boldsymbol{\pi_j})$~Bernoulli (π_{cj})

$\theta_i|\, \boldsymbol{\gamma}$ ~ categorical $(\boldsymbol{\gamma})$,

$\boldsymbol{\gamma}$~Dirichlet $(\boldsymbol{\alpha_\gamma})$ where $\boldsymbol{\alpha_\gamma}$ = $(\alpha_1, \dots, \alpha_6)$,

π_{cj}~Beta$(\alpha_{cj}, \beta_{cj})$ for i = 1, … , n; j=1, … , J; c=1,…,C.

The complete DAG representing the joint posterior density in Equation (5.7) is shown in Figure 5.2.

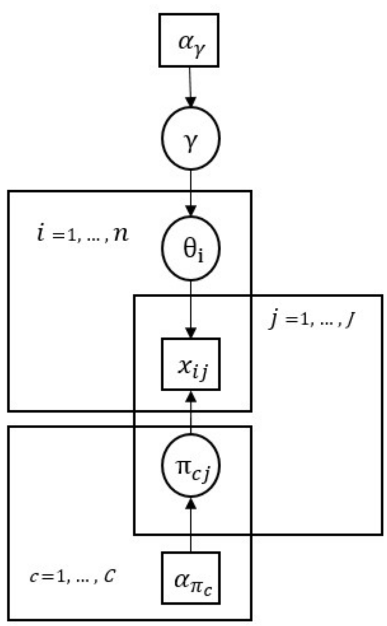


Figure 5.2 The complete DAG of a Bayesian Network for the Latent Class Analysis in Model 1; adapted from Levy and Mislevy (2016).

Having obtained the joint posterior distribution of the parameters $(\boldsymbol{\theta}, \boldsymbol{\gamma}, \boldsymbol{\pi})$ for Model 1, as in Equation (5.6), the parameters will then be estimated using the MCMC method discussed later in Section 5.2.3. However, as commonly found in statistical latent models, Model 1 is an unconstrained model which has an indeterminacy problem in labelling the class (Levy & Mislevy, 2016). Indeterminacy in LCA (Latent Class Analysis) refers to the problem of identifying which labels are for the class because there is no information obtained from the model to identify which class is called class 1, class 2 and so forth (Levy & Mislevy, 2016).

One of the strategies to solve the indeterminacy in latent class model is by specifying the prior distribution for measurement model parameters π_{cj} (See Levy & Mislevy, 2016, p. 319). We set the prior distribution π_{cj} using an informative prior with the belief that students at a certain level *c* have a high probability (with a mode around 80%) to get correct answers for the items at that level and below, and with the belief that the students have low probability (with a mode around 20%) to correctly answer the items at levels higher than level *c*. To translate these beliefs into our Bayesian model, we use the properties of Beta distribution by setting the items at a certain level (class) and below as follows:

$$\pi_{cj} \sim \text{Beta}(80,20).$$

Similarly for the items at a higher level than level *c*, we set the Beta distribution as follows

$$\pi_{cj} \sim \text{Beta}(20,80).$$

The density plots of Beta distribution for Beta (80,20) and Beta (20,80) are presented in Figures 5.3 and 5.4 respectively.

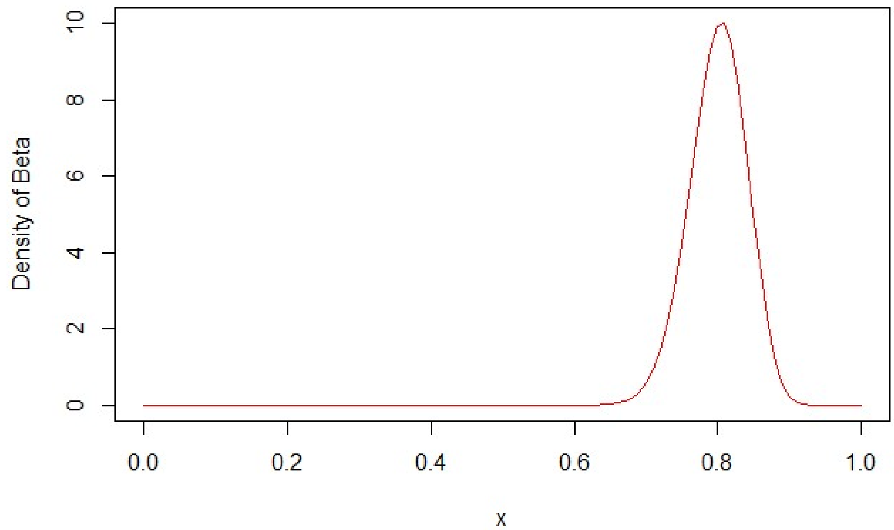


Figure 5.3 The density plot of Beta distribution Beta (80,20)

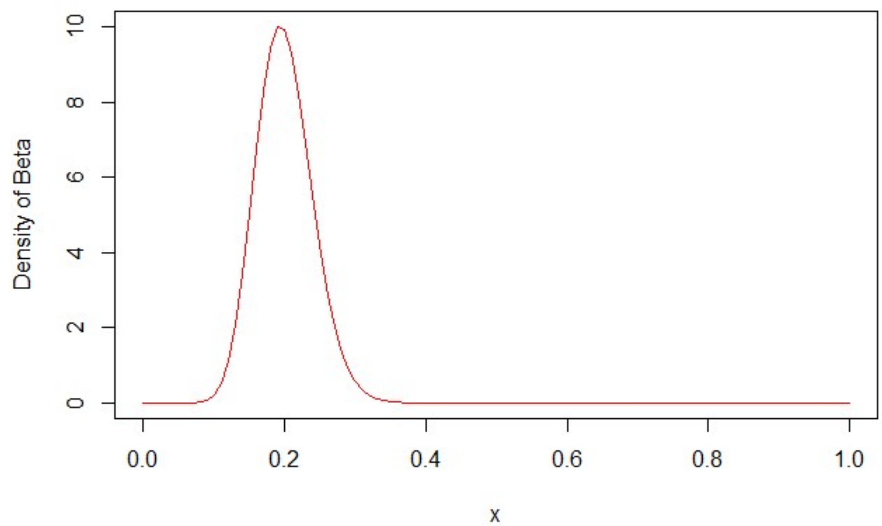


Figure 5.4 The density plot of Beta distribution Beta (20,80)

The distribution of Beta (α, β) has a mean $\alpha/(\alpha+\beta)$ and variance $\alpha\beta/[(\alpha+\beta)^2(\alpha+\beta+1)]$. From Figures 5.3 and 5.4, it can be seen that the density function of Beta (80,20) is centered around its mean of 0.8, while the density function of Beta (20,80) is centered around its mean of 0.2. Both shapes of the densities are narrow, which indicate small variances around the means. These small variances reflect the strong belief of the researcher that students at the upper level should be able to answer the items at that level and below, but the students at the lower level are unlikely to answer the items at the upper level(s) correctly.

By incorporating the Beta priors into the model, the Bayesian Latent Class Analysis performed in this study combines the researcher's knowledge and the data through the joint posterior distribution to group students to classes or levels by imposing the assumption of the hierarchical levels of the proposed model of fraction learning progression into the model. This is different with Classical Latent Class Analysis (a Frequentist approach) which groups students based on the data only (Levy & Mislevy, 2016).

5.2.2 Model 2: Bayesian Networks with Multiple Latent Variables θ

Given that the research setting on learning progression in this thesis assumes a hierarchy, Model 2 is extended from Model 1. Instead of modeling the levels of the proposed model of a learning progression in a single latent variable θ_i with discrete-independent C classes/categories (as in Model 1), Model 2 constructs the learning progression with multiple latent variables $\boldsymbol{\theta} = (\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_C})$, where $\boldsymbol{\theta_c}$ is the collection of θ_{ci} for c=1,…,C and i=1,…,n.

Model 2 assigns a latent variable θ_i for each level c (θ_{ci}), in which θ_i in the upper level is conditional on the θ_i from the lower level. This conditional setting of θ_i is created to reflect the hierarchical levels of the knowledge/skills in the proposed model of learning progression for both conceptual and procedural knowledge dimensions.

In a simple DAG representation, Model 2 is now presented in Figure 5.5 as follows.

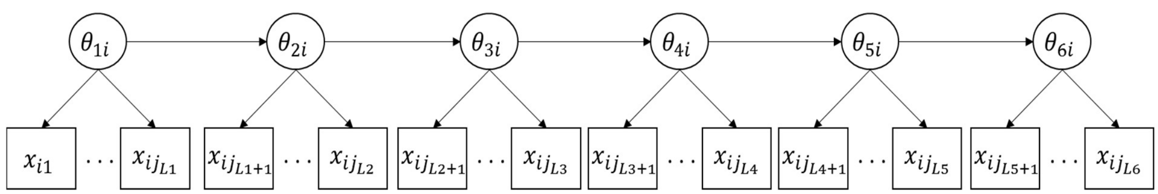


Figure 5.5 A simple DAG representation of a Bayesian Network for model 1 (Adapted from Rutstein, 2012)

Figure 5.5 represents a simple DAG of Model 2 for both the conceptual and the procedural knowledge dimensions. It can be seen from this Figure that the arrows come from the lower level (θ_i) to the upper level (θ_i), reflecting the dependency between the proposed levels of the learning progression model. Consequently, we now detail the variables of Model 2, the observed and the latent variables respectively.

The Latent Variables of Model 2

The dependency between the levels is expressed through the conditional probabilities of the latent variables θ_{ci}. Let θ_{1i} be the parameter indicates the knowledge at Level 1 as the parent of the other θ's. This is due to the setting in the model in Figure 5.5, which implies that students cannot proceed to the higher levels without having the knowledge at Level 1 (Levy & Mislevy, 2016).

Suppose that θ_{1i} has two *z* categories: 1 is the code for a student i at level 1 (has the knowledge of that level), and 0 is the code for the students who are not at that level (do not have knowledge of that level). Because θ_{1i} has binary categories, it is then appropriate to assign θ_{1i} with a Bernoulli distribution such that

$$\theta_{1i}|\gamma_1 \sim \text{Bernoulli } (\gamma_1).$$

Next, we assume that the students can be at Level 2 if they have the knowledge at Level 1. This dependency can be expressed as θ_{2i} conditional on θ_{1i}. We denote this conditional distribution as

$$(\theta_{2i}|\theta_{1i} = \mathrm{z}, \gamma_2) \sim \text{Bernoulli } (\gamma_{2z}) \text{ for } z\text{=0,1, where } \gamma_{2z} = (\gamma_{20}, \gamma_{21}).$$

The notation γ_{2z} expresses the probability of θ_{2i} to have value 1 given θ_{1i} has a value *z* (either 0 or 1). In other words, γ_{2z} implies the probability of the students at Level 2 conditional on the situation whether they have the knowledge at Level 1 or not.

Likewise, we define the conditional distributions for the higher levels (Level 3 to Level 6) as follows:

$$(\theta_{3i}|\theta_{2i} = z, \gamma_{3z}) \sim \text{Bernoulli } (\gamma_{3z}) \text{ for z = 0,1,} \quad (5.8)$$

$$(\theta_{4i}|\theta_{3i} = z, \gamma_{4z}) \sim \text{Bernoulli } (\gamma_{4z}), \text{ for } z \text{ = 0,1,}$$

$$(\theta_{5i}|\theta_{4i} = z, \gamma_{5z}) \sim \text{Bernoulli } (\gamma_{5z}), \text{ for } z \text{ = 0,1,}$$

$$(\theta_{6i}|\theta_{5i} = z, \gamma_{6z}) \sim \text{Bernoulli } (\gamma_{6z}), \text{ for } z \text{ = 0,1.}$$

Similar to Rutstein (2012), based on the conditional distribution detailed above, we can identify the conditional probabilities table (CPT) of θ. The CPT for level 1 only has two conditions that depend on the values of θ_1 itself, which are 0 and 1. This situation is summarized in Table 5.2, below.

Table 5.2 The probability table of θ_{1i} (level 1) in the learning progression model, where γ_1 is the probability of θ_{1i}=1

θ_{1i}	$P(\theta_{1i})$
0	1-γ_1
1	γ_1

For the next levels (Levels 2 to 6), the probability of θ_{ci} is conditional on the values of $\theta_{c-1,i}$. To capture this conditional probability in our model, denote the probability of θ_{ci}=1 conditional on $\theta_{c-1,i}$=1 as γ_{c1}, and the probability of θ_{ci}=1 conditional on $\theta_{c-1,i}$=0 as γ_{c0}. The CPT for Levels 2 to 6 are presented in Table 5.3, below.

Table 5.3 The conditional probability table (CPT) of θ_{ci} conditional on $\theta_{c-1,i}$ in the learning progression model, where γ_{c1} is the probability of θ_{ci}=1 given the value $\theta_{c-1,i}$=1, and γ_{c0} is the probability of θ_{ci}=1 given the value $\theta_{c-1,i}$=0 for *c*=2,…,6

$\theta_{c-1,i}$	$P(\theta_{ci}\|\theta_{c-1,i})$	
	0	1
0	1-γ_{20}	γ_{20}
1	1- γ_{21}	γ_{21}

Let us denote $\boldsymbol{\gamma} = (\gamma_1, \gamma_{20}, \gamma_{21}, \gamma_{30}, \gamma_{31}, \gamma_{40}, \gamma_{41}, \gamma_{50}, \gamma_{51}, \gamma_{60}, \gamma_{61})$. From Table 5.3, it can be observed that the values of $\boldsymbol{\gamma}$ are not known. Hence, the conjugate prior of $\boldsymbol{\gamma}$ should be specified. Because θ's are the dichotomous variables, then each element of $\boldsymbol{\gamma}$ is assumed to have the conjugate prior of beta distribution, as follows (Levy & Mislevy, 2016):

$$\gamma_1 \sim \text{Beta}(\alpha_{\gamma_1},\ \beta_{\gamma_1})$$

$$\gamma_{20} \sim \text{Beta}(\alpha_{\gamma_{20}}, \beta_{\gamma_{20}}),\ \gamma_{21} \sim \text{Beta}(\alpha_{\gamma_{21}}, \beta_{\gamma_{21}})$$

$$\gamma_{30} \sim \text{Beta}(\alpha_{\gamma_{30}}, \beta_{\gamma_{30}}),\ \gamma_{31} \sim \text{Beta}(\alpha_{\gamma_{31}}, \beta_{\gamma_{31}})$$

$$\gamma_{40} \sim \text{Beta}(\alpha_{\gamma_{40}}, \beta_{\gamma_{40}}),\ \gamma_{41} \sim \text{Beta}(\alpha_{\gamma_{41}}, \beta_{\gamma_{41}})$$

$$\gamma_{50} \sim \text{Beta}(\alpha_{\gamma_{50}}, \beta_{\gamma_{50}}),\ \gamma_{51} \sim \text{Beta}(\alpha_{\gamma_{51}}, \beta_{\gamma_{51}})$$

$$\gamma_{60} \sim \text{Beta}(\alpha_{\gamma_{60}}, \beta_{\gamma_{60}}),\ \gamma_{61} \sim \text{Beta}(\alpha_{\gamma_{61}}, \beta_{\gamma_{61}})$$

In a similar way to Model 1, we use an informative prior in order to reflect the belief that the students who have knowledge at the lower level have a high chance to have the knowledge at the upper level. However, if they do not have the knowledge at the lower level, it is believed that they have little chance to master the knowledge at the upper level. To reflect this belief, we set a Beta (21,6) prior distribution (i.e. with the mean of 21/27≈0.8) for the students who have the knowledge at the lower level and a Beta (6,21) prior distribution for the students who do not have the knowledge at the lower level (adopted from Levy & Mislevy, 2016). This Beta distribution is different from the Beta distribution in Model 1 because of the different parameters in Model 2. Beta distribution is the prior density of $\boldsymbol{\gamma}$ as in Equation (5.8), while the Beta distribution in Model 1 is the prior density for the conditional probability of the observable variables π_{cj}, as presented in Equation (5.6).

Applying the conjugate prior of the Beta distribution to our learning progression model, we set the prior distribution of γ_1 for level 1 using the density Beta (21,6). However, for Level 2, because θ_{2i} is conditional on θ_{1i} which has two outcomes (0 represent no knowledge at Level 1 and 1 represent has knowledge at level 1), the prior distribution of γ_{2z} depends on the value of θ_{1i}. As discussed before, γ_{20} is the value of γ_{2z} when θ_{1i}=0, and γ_{21} is the value of γ_{2z} when θ_{1i}=1. To reflect the belief of the dependency of the knowledge discussed before, we set $\gamma_{20} \sim$ Beta(6,21) and $\gamma_{21} \sim$ Beta(21,6) (adapted from Levy & Mislevy, 2016).

Similarly, the prior distributions for $\boldsymbol{\gamma}$ at levels 3 to 6 are assigned as

$$\gamma_{30} \sim \text{Beta}(6,21),\ \gamma_{31} \sim \text{Beta}(21,6),$$

$$\gamma_{40} \sim \text{Beta}(6,21),\ \gamma_{41} \sim \text{Beta}(21,6),$$

$$\gamma_{50} \sim \text{Beta}(6,21),\ \gamma_{51} \sim \text{Beta}(21,6),$$

$$\gamma_{60} \sim \text{Beta}(6,21),\ \gamma_{61} \sim \text{Beta}(21,6).$$

Figures 5.6-5.7 are the density plots of the Beta distribution, Beta (6, 21) and Beta (21,6) respectively. From the plots we can see that both distributions are centered around 0.2 for Beta (6,21) and 0.8 for Beta (21,6). The range of the distribution is quite wide, spreading from 0 to 0.4 for Beta (6,21) and from 0.6 to 1 for Beta (21,6). This indicates that both distributions have moderate variances. These moderate variances are suitable within the context of our study, as they can accommodate the response uncertainties of students at a certain level, who have the knowledge at the lower level (Beta (21,6)). Similarly, they can also accommodate the response variances of students who do not have knowledge at the lower level (Beta (6,21)). A sensitivity analysis (which is beyond the scope of this study) needs to be performed to investigate further the effects of choosing different prior densities for the γ parameters, and for the other prior densities used in this thesis.

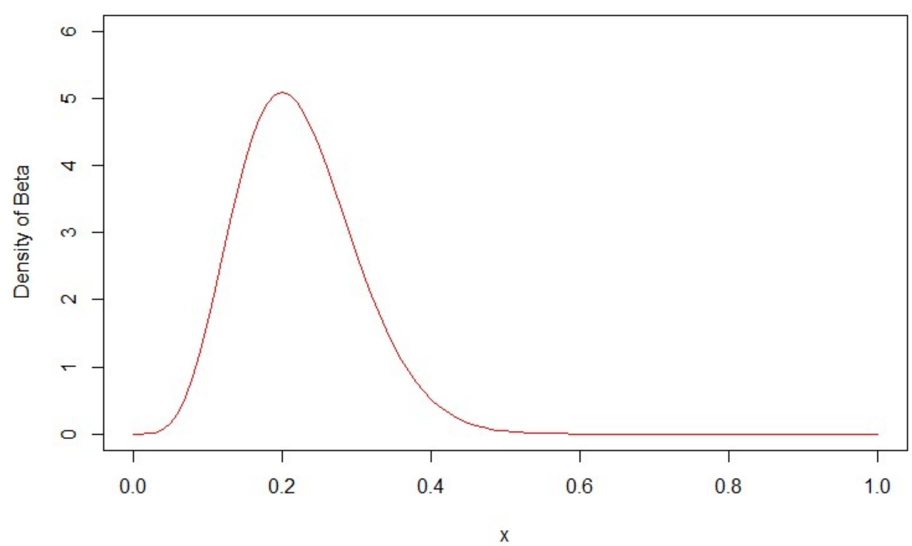


Figure 5.6 plot of Beta distribution Beta (6,21) for prior γs

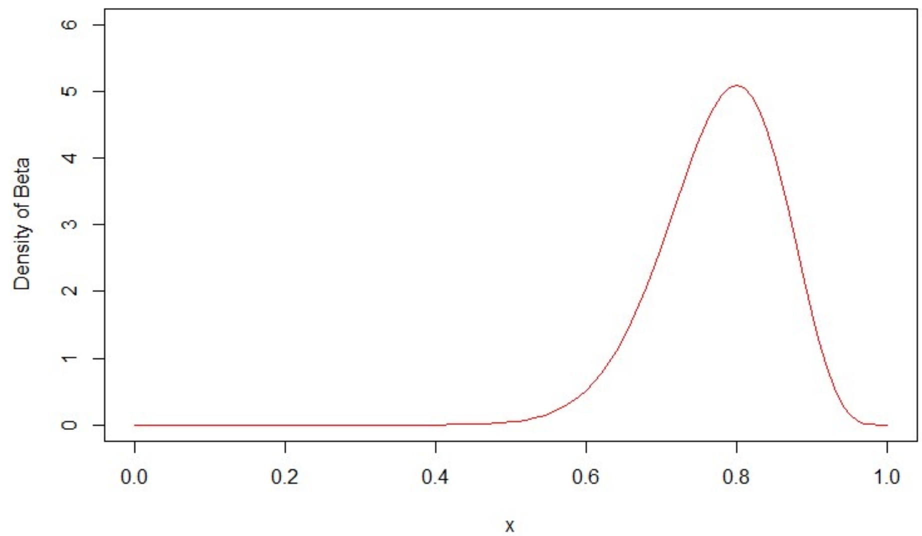


Figure 5.7 The density plot of Beta distribution Beta (21,6) for prior γs

Having completed the latent variable for Model 2, we now explain the observed variables.

The Conditional Probability of the Observed Variables of Model 2

Suppose that we have *J* items which are distributed based on the levels as follows:

Items at Level 1: Items 1,…,j_{L1}

Items at Level 2: Items j_{L1+1},…, j_{L2}

Items at Level 3: Items j_{L2+1},…, j_{L3}

Items at Level 4: Items j_{L3+1},…, j_{L4}

Items at Level 5: Items j_{L4+1},…, j_{L5}

Items at Level 3: Items j_{L5+1},…,*J*

As discussed in Model 1, for each item we have two response categories which are a correct response (*k*=1) and an incorrect response (*k*=0).

The simple DAG in Figure 5.5 shows that the students' responses on items at every level (observed variables x_{ij}) are conditional on the knowledge at that level (latent variable θ_{ci}). In Model 2, this conditional probability is denoted as follows:

$$\pi_{cjzk} = p(x_{ij} = k|\theta_{ci} = z),\ k= 0,1\ z = 0,1.$$

As we only have two response categories (K=2), then π_{cjzk} can be simplified as follows:

$$\pi_{cjzk} = \pi_{cjz1} = \pi_{cjz} = p(x_{ij} = 1|\theta_{ci} = z),\ z = 0,1. \tag{5. 9}$$

The z values in Equation (5.9) are used to encode the situation that students have the knowledge at a certain level (z= 1), or do not have the knowledge at that level (z=0). The conditional probability of π_{cjz} is represented in Table 5.8, below.

Suppose that $\boldsymbol{\pi_{cj}}$ be the collection of π_{cjz} for c=1,…,C, j=1,…,J and z = 0.1. Because the observed variables x_{ij} have dichotomous outcomes, we set the distribution of x_{ij} as a Bernoulli distribution, expressed as

$$(x_{ij}|\theta_{ci} = z, \boldsymbol{\pi_{cj}}) \sim \text{Bernoulli}(\pi_{cjz}).$$

As the values of π_{cjz} are not known, π_{cjz} is generated through a Beta distribution. Therefore, the prior distribution for the measurement model for each level is denoted as follows

Level 1: $\pi_{1j0} \sim \text{Beta}(\alpha_{\pi_{1j0}}, \beta_{\pi_{1j0}}),\ \pi_{1j1} \sim \text{Beta}(\alpha_{\pi_{1j1}},\ \beta_{\pi_{1j1}})$

Level 2: $\pi_{2j0} \sim \text{Beta}(\alpha_{\pi_{2j0}}, \beta_{\pi_{2j0}}),\ \pi_{2j1} \sim \text{Beta}(\alpha_{\pi_{2j1}},\ \beta_{\pi_{2j1}})$

Level 3: $\pi_{3j0} \sim \text{Beta}(\alpha_{\pi_{3j0}}, \beta_{\pi_{3j0}}),\ \pi_{3j1} \sim \text{Beta}(\alpha_{\pi_{3j1}},\ \beta_{\pi_{3j1}})$

Level 4: $\pi_{4j0} \sim \text{Beta}(\alpha_{\pi_{4j0}}, \beta_{\pi_{4j0}}),\ \pi_{4j1} \sim \text{Beta}(\alpha_{\pi_{4j1}},\ \beta_{\pi_{4j1}})$

Level 5: $\pi_{5j0} \sim \text{Beta}(\alpha_{\pi_{5j0}}, \beta_{\pi_{5j0}}),\ \pi_{5j1} \sim \text{Beta}(\alpha_{\pi_{5j1}},\ \beta_{\pi_{5j1}})$

Level 6: $\pi_{6j0} \sim \text{Beta}(\alpha_{\pi_{6j0}}, \beta_{\pi_{6j0}}),\ \pi_{6j1} \sim \text{Beta}(\alpha_{\pi_{6j1}},\ \beta_{\pi_{6j1}})$

Table 5.4 Conditional Probability Table for π_{cjz} of (5.8) for θ_{ci}=0,1 and z=0,1

Level	$\theta_{ci} = z$	Response Category (k)	$(x_{ij} = k\|\theta_{ci} = z)$
Level 1	0	0	1-π_{1j0}
		1	π_{1j0}
	1	0	1-π_{1j1}
		1	π_{1j1}
Level 2	0	0	1-π_{2j0}
		1	π_{2j0}
	1	0	1-π_{2j1}
		1	π_{2j1}
Level 3	0	0	1-π_{3j0}

		1	π_{3j0}
	1	0	1-π_{3j1}
		1	π_{3j1}
Level 4	0	0	1-π_{4j0}
		1	π_{4j0}
	1	0	1-π_{4j1}
		1	π_{4j1}
Level 5	0	0	1-π_{5j0}
		1	π_{5j0}
	1	0	1-π_{5j1}
		1	π_{5j1}
Level 6	0	0	1-π_{6j0}
		1	π_{6j0}
	1	0	1-π_{6j1}
		1	π_{6j1}

The prior density of π_{cjz} enables us to express our belief about the students' responses, given that they have or have not the required knowledge. The conditional probability π_{cj0} presented in Table 5.4 represents the probability of getting a correct answer for item *j* where θ_{ci} is 0 (no knowledge at that level), while π_{cj1} states the probability of getting a correct answer for item *j* where θ_{ci} is 1 (that is, the students have knowledge at that level). Hence, to reflect our belief that students who have the knowledge at that level will be highly likely to answer the items at that level correctly, we set the probability at 80% for $\pi_{cj1,}$ as follows

$$\pi_{cj1} \sim \text{Beta}(80{,}20).$$

In contrast, to reflect our belief that students who do not have the knowledge at a certain level are unlikely to answer the items at that level correctly, we set the probability at 20% for $\pi_{cj0,}$ as follows

$$\pi_{cj0} \sim \text{Beta}(20{,}80).$$

Having completed both the latent and observed variables, then we detail the joint posterior distribution of Model 2.

The Joint Posterior Distribution of Model 2

The following is the joint posterior distribution for Model 2:

$$p(\boldsymbol{\theta}, \boldsymbol{\gamma}, \boldsymbol{\pi}|\boldsymbol{x}) \infty\, p(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{\gamma}, \boldsymbol{\pi})p(\boldsymbol{\theta}, \boldsymbol{\gamma}, \boldsymbol{\pi})$$

$$= p(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{\pi})p(\boldsymbol{\theta}|\boldsymbol{\gamma})p(\boldsymbol{\gamma})p(\boldsymbol{\pi})$$

$$= \prod_{c=1}^{6} \prod_{i=1}^{n} \prod_{j=1}^{J} p\left(x_{ij}\middle|\theta_{ci}, \boldsymbol{\pi_{cj}}\right)p\,(\theta_{ci}|\boldsymbol{\gamma})p(\boldsymbol{\gamma}) \prod_{z=0}^{1} p\left(\pi_{cjz}\right)$$

where, $(x_{ij}|\theta_{ci} = z, \boldsymbol{\pi_{cj}})$ ~ Bernoulli(π_{cjz}) for z=0,1; i=1,…,n ; j=1, … , j ; and c=1,…,6.

$\theta_1|\gamma_1 \sim$ Bernoulli (γ_1),

$(\theta_{2i}|\theta_{1i} = \mathrm{z}, \gamma_{2z}) \sim$ Bernoulli (γ_{2z}),

$(\theta_{3i}|\theta_{2i} = \mathrm{z}, \gamma_{3z}) \sim$ Bernoulli (γ_{3z}),

$(\theta_{4i}|\theta_{3i} = z, \gamma_{4z}) \sim$ Bernoulli (γ_{4z}),

$(\theta_{5i}|(\theta_{4i} = z, \gamma_{5z}) \sim$ Bernoulli (γ_{5z}),

$(\theta_{6i}|\theta_{5i} = z, \gamma_{6z}) \sim$ Bernoulli (γ_{6z}).

$\gamma_1 \sim$Beta$(\alpha_{\gamma_1},\ \beta_{\gamma_1})$

$\gamma_{20} \sim$ Beta$(\alpha_{\gamma_{20}}, \beta_{\gamma_{20}})$, $\gamma_{21} \sim$ Beta$(\alpha_{\gamma_{21}}, \beta_{\gamma_{21}})$

$\gamma_{30} \sim$ Beta$(\alpha_{\gamma_{30}}, \beta_{\gamma_{30}})$, $\gamma_{31} \sim$ Beta$(\alpha_{\gamma_{31}}, \beta_{\gamma_{31}})$

$\gamma_{40} \sim$ Beta$(\alpha_{\gamma_{40}}, \beta_{\gamma_{40}})$, $\gamma_{41} \sim$ Beta$(\alpha_{\gamma_{41}}, \beta_{\gamma_{41}})$

$\gamma_{50} \sim$ Beta$(\alpha_{\gamma_{50}}, \beta_{\gamma_{50}})$, $\gamma_{51} \sim$ Beta$(\alpha_{\gamma_{51}}, \beta_{\gamma_{51}})$

$\gamma_{60} \sim$ Beta$(\alpha_{\gamma_{60}}, \beta_{\gamma_{60}})$, $\gamma_{61} \sim$ Beta$(\alpha_{\gamma_{61}}, \beta_{\gamma_{61}})$

$\pi_{1j0} \sim$Beta$(\alpha_{\pi_{1j0}}, \beta_{\pi_{1j0}})$, $\pi_{1j1} \sim$Beta$(\alpha_{\pi_{1j1}},\ \beta_{\pi_{1j1}})$

$\pi_{2j0} \sim$Beta$(\alpha_{\pi_{2j0}}, \beta_{\pi_{2j0}})$, $\pi_{2j1} \sim$Beta$(\alpha_{\pi_{2j1}},\ \beta_{\pi_{2j1}})$

$\pi_{3j0} \sim$Beta$(\alpha_{\pi_{3j0}}, \beta_{\pi_{3j0}})$, $\pi_{3j1} \sim$Beta$(\alpha_{\pi_{3j1}},\ \beta_{\pi_{3j1}})$

$\pi_{4j0} \sim$Beta$(\alpha_{\pi_{4j0}}, \beta_{\pi_{4j0}})$, $\pi_{4j1} \sim$Beta$(\alpha_{\pi_{4j1}},\ \beta_{\pi_{4j1}})$

$\pi_{5j0} \sim$Beta$(\alpha_{\pi_{5j0}}, \beta_{\pi_{5j0}})$, $\pi_{5j1} \sim$Beta$(\alpha_{\pi_{5j1}},\ \beta_{\pi_{5j1}})$

$\pi_{6j0} \sim$Beta$(\alpha_{\pi_{6j0}}, \beta_{\pi_{6j0}})$, $\pi_{6j1} \sim$Beta$(\alpha_{\pi_{6j1}},\ \beta_{\pi_{6j1}})$

The complete DAG for Model 2 is now presented in Figure 5.8 below.

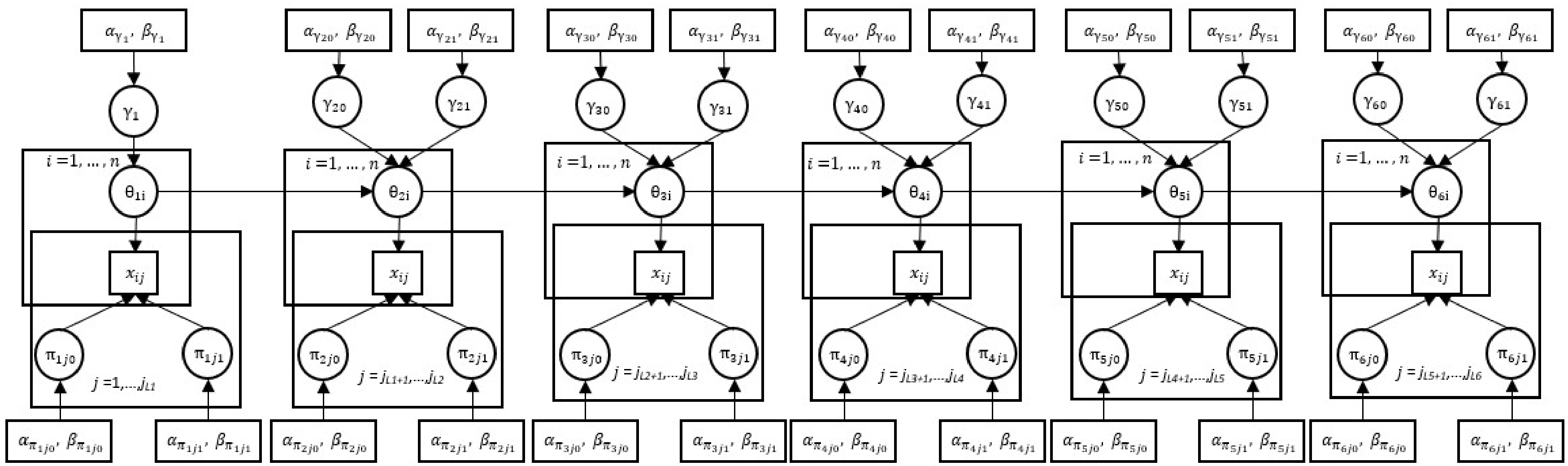


Figure 5.8 The complete DAG for Model 2 of the Bayesian Networks Modelling for measuring learning progression.

5.2.3 Parameter estimation using MCMC

5.2.3.1 MCMC Estimation Using the Gibbs Sampler Method

Within the Bayesian Paradigm, the information of interests regarding the parameters in Models 1 and 2 is contained within its joint posterior distributions. Markov Chain Monte Carlo (MCMC) algorithms approximate the joint posterior distribution of Models 1 and 2.

The term Markov Chain refers to, "random variables that are generated sequentially over time" (Cowles, 2013, p. 123), while Monte Carlo refers to, "the process [that] will involve simulating (sampling, generating, drawing) values from distributions" (Levy & Mislevy, 2016, p. 94). To perform MCMC on our data, we use the Gibbs Sampler algorithm, which is the underlying parameter estimation method of BUGS (Bayesian Estimation using Gibbs Sampler) Software.

Briefly, the following are the general steps in MCMC algorithms using Gibbs Sampler with *t* iterations (*t*=1, …, T, where T be the total number of iterations) (See Cowles, 2013; Levy & Mislevy, 2016). Suppose we have *R* parameters of θ denoted as $\theta_1, \dots, \theta_R$ and matrix data **X.**

Step 1

Generate the initial values for the parameters, which are $\theta_1^{(0)}, \dots, \theta_R^{(0)}$, either deterministically or randomly.

Step 2

Draw values for each iteration *t (t=1,…,T* where *T* is the maximum number of iterations) from the full conditional probability, given the most current values of the other parameters θ. Hence for each iteration *t,* we draw the values as follows:

$$\theta_1^{(t)} \text{ from } p\left(\theta_1 \middle| \theta_2^{(t-1)}, \theta_3^{(t-1)} \dots, \theta_R^{(t-1)}, \mathbf{X}\right)$$

$$\theta_2^{(t)} \text{ from } p\left(\theta_2 \middle| \theta_1^{(t)}, \theta_3^{(t-1)}, \dots, \theta_R^{(t-1)}, \mathbf{X}\right)$$

$$\theta_3^{(t)} \text{ from } p\left(\theta_3 \middle| \theta_1^{(t)}, \theta_2^{(t)}, \theta_3^{(t-1)}, \dots, \theta_R^{(t-1)}, \mathbf{X}\right)$$

.

.

$\theta_R^{(t)}$ from $p\left(\theta_R \middle| \theta_1^{(t)}, \theta_2^{(t)}, \dots, \theta_{R-1}^{(t)}, \mathbf{X}\right)$

Step 3

Repeat *Step 2* until the MCMC chains converge jointly to the joint posterior distribution.

5.2.3.2 MCMC Convergence Check

An MCMC convergence check is an important part of MCMC estimation (Levy & Mislevy, 2016). Although theoretically MCMC estimation is converged to a target distribution under some conditions, no one knows "when" the estimation is converged (Levy & Mislevy, 2016). According to Gelman et al. (2014), if the MCMC iterations are not run long enough, then the simulated values may not represent the target distribution of the estimation. One of prominent methods to check MCMC convergence is the Geweke test, proposed by Geweke (1992). The range of values in the Geweke test run from +2 to -2, showing a 95% confidence interval for the estimation to be converged, while the values out of the range indicate that the estimation has not yet converged.

Another tool to check that MCMC iterations have already achieved good estimates of the probabilities of the model is using the autocorrelation of the draws generated from the iterative simulation in MCMC estimation. Gelman et al. (2014) highlighted that estimations based on correlated draws are less accurate when compared with estimations using independent draws. To check this, the autocorrelation function on various lags of iterations can be used to detect the dependency between the draws (Levy & Mislevy, 2016). The autocorrelation at lag 0 is always 1 because the draw is correlated with itself. The autocorrelation is expected always to drop close to zero by increasing the lags. The autocorrelations which are close to zero indicate that the draws from the MCMC simulations are independent.

5.2.3.3 Software

There are three software packages required to perform the parameter estimation of Models 1 and 2, which are specified in the previous sections. These three packages are briefly explained below.

5.2.3.3.1 WinBUGS

The WinBUGS software is a commercial Bayesian estimation software run under Windows system. As it reveals through its name, this software mainly uses the Gibbs Sampler method to perform MCMC estimation (Lunn, Thomas, Best, & Spiegelhalter, 2000).

The WinBUGS software is used to perform an MCMC estimation to estimate the parameters (π, γ) of Models 1 and 2. The estimates of π are then used to perform *item analysis,* as discussed in the following chapter. Moreover, the estimates of the parameters (π, γ) are used as prior inputs for the Netica Software to estimate the students' levels $(\theta_{\boldsymbol{i}})$ in the learning progression model.

Table 5.5 shows the list of the parameters estimated by the WinBUGS software. These parameters are estimated for 516 students: 21 items for the conceptual dimension and 12 items for the procedural dimension.

Due to the total number of parameters in the models and a large number of iterations for MCMC, WinBUGS was not able to calculate the posterior distributions for θs. Therefore, we also use Netica.

5.2.3.3.2 Netica

Netica is a commercial Bayesian Networks Software developed by Norsys Sotware Corp. (Application, 2014). Netica is used to estimate the posterior probabilities for students to be at a certain level. The results from Netica estimation are used to perform the *students' level analysis* in Chapter 6.

The process of assigning students at the levels of the proposed learning progression model is implemented by entering each individual student's responses into the Networks of Bayesian Modelling created in Netica. The Netica software updated the prior probability of students' levels to get the posterior probability of the individual student's levels. Netica performs belief propagation using a Junction Tree Algorithm (see Neopolitan, 2004) to update the prior probabilities of the students' levels. The results generated from Netica are the posterior probabilities of $(\boldsymbol{\theta}|\boldsymbol{\gamma}, \boldsymbol{\pi}, \boldsymbol{x})$ for all students (516 students) for both Model 1 and Model 2.

5.2.3.3.3 R Software

The R software (Team, 2014) is an open source software for statistical computing and graphics analyses. This software is used to estimate the model fit/evaluation of Bayesian Networks for Model 1 and Model 2, as discussed in the following section.

5.3 Model Evaluation of the Bayesian Networks Model

We performed two methods to evaluate the models in this study, by comparing the observed data and the simulated/predicted data generated by the model. These methods are Posterior Predictive Model Checking (PPMC) and Entropy Statistics.

Table 5.5 The list of parameters estimated by the WinBUGS software

Model	Dimension	Parameter	Number of Parameters
Model 1	Conceptual (6 levels, 21 items)	$\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6)$	6
		$\pi_{cj} = (\pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}, \pi_{16}, \pi_{17}, \pi_{18}, \pi_{19}, \pi_{110}, \pi_{111}, \pi_{112}, \pi_{113}, \pi_{114}, \pi_{115}, \pi_{116}, \pi_{117}, \pi_{118}, \pi_{119}, \pi_{120}, \pi_{121},$ $\pi_{21}, \pi_{22}, \pi_{23}, \pi_{24}, \pi_{25}, \pi_{26}, \pi_{27}, \pi_{28}, \pi_{29}, \pi_{210}, \pi_{211}, \pi_{212}, \pi_{213}, \pi_{214}, \pi_{215}, \pi_{216}, \pi_{217}, \pi_{218}, \pi_{219}, \pi_{220}, \pi_{221},$ $\pi_{31}, \pi_{32}, \pi_{33}, \pi_{34}, \pi_{35}, \pi_{36}, \pi_{37}, \pi_{38}, \pi_{39}, \pi_{310}, \pi_{311}, \pi_{312}, \pi_{313}, \pi_{314}, \pi_{315}, \pi_{316}, \pi_{317}, \pi_{318}, \pi_{319}, \pi_{320}, \pi_{321},$ $\pi_{41}, \pi_{42}, \pi_{43}, \pi_{44}, \pi_{45}, \pi_{46}, \pi_{47}, \pi_{48}, \pi_{49}, \pi_{410}, \pi_{411}, \pi_{412}, \pi_{413}, \pi_{414}, \pi_{415}, \pi_{416}, \pi_{417}, \pi_{418}, \pi_{419}, \pi_{420}, \pi_{421},$ $\pi_{51}, \pi_{52}, \pi_{53}, \pi_{54}, \pi_{55}, \pi_{56}, \pi_{57}, \pi_{58}, \pi_{59}, \pi_{510}, \pi_{511}, \pi_{512}, \pi_{513}, \pi_{514}, \pi_{515}, \pi_{516}, \pi_{517}, \pi_{518}, \pi_{519}, \pi_{520}, \pi_{521},$ $\pi_{61}, \pi_{62}, \pi_{63}, \pi_{64}, \pi_{65}, \pi_{66}, \pi_{67}, \pi_{68}, \pi_{69}, \pi_{610}, \pi_{611}, \pi_{612}, \pi_{613}, \pi_{614}, \pi_{615}, \pi_{616}, \pi_{617}, \pi_{618}, \pi_{619}, \pi_{620}, \pi_{621})$	126
	Procedural (6 levels, 12 items)	$\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6)$	6
		$\pi_{cj} = (\pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}, \pi_{16}, \pi_{17}, \pi_{18}, \pi_{19}, \pi_{110}, \pi_{111}, \pi_{112},$ $\pi_{21}, \pi_{22}, \pi_{23}, \pi_{24}, \pi_{25}, \pi_{26}, \pi_{27}, \pi_{28}, \pi_{29}, \pi_{210}, \pi_{211}, \pi_{212},$ $\pi_{31}, \pi_{32}, \pi_{33}, \pi_{34}, \pi_{35}, \pi_{36}, \pi_{37}, \pi_{38}, \pi_{39}, \pi_{310}, \pi_{311}, \pi_{312},$ $\pi_{41}, \pi_{42}, \pi_{43}, \pi_{44}, \pi_{45}, \pi_{46}, \pi_{47}, \pi_{48}, \pi_{49}, \pi_{410}, \pi_{411}, \pi_{412},$ $\pi_{51}, \pi_{52}, \pi_{53}, \pi_{54}, \pi_{55}, \pi_{56}, \pi_{57}, \pi_{58}, \pi_{59}, \pi_{510}, \pi_{511}, \pi_{512},$ $\pi_{61}, \pi_{62}, \pi_{63}, \pi_{64}, \pi_{65}, \pi_{66}, \pi_{67}, \pi_{68}, \pi_{69}, \pi_{610}, \pi_{611}, \pi_{612})$	72
Model 2	Conceptual (6 levels, 21 items)	$\gamma = (\gamma_1, \gamma_{20}, \gamma_{21}, \gamma_{30}, \gamma_{31}, \gamma_{40}, \gamma_{41}, \gamma_{50}, \gamma_{51}, \gamma_{60}, \gamma_{61})$	11
		$\pi_{cjz} = (\pi_{110}, \pi_{120}, \pi_{130}, \pi_{140}, \pi_{250}, \pi_{260}, \pi_{270}, \pi_{280}, \pi_{390}, \pi_{3100}, \pi_{3110}, \pi_{3120}, \pi_{3130}, \pi_{3140}, \pi_{3150}, \pi_{4160}, \pi_{4170}, \pi_{5180}, \pi_{5190}, \pi_{6200}, \pi_{6210},$ $\pi_{111}, \pi_{121}, \pi_{131}, \pi_{141}, \pi_{251}, \pi_{261}, \pi_{271}, \pi_{281}, \pi_{391}, \pi_{3101}, \pi_{3111}, \pi_{3121}, \pi_{3131}, \pi_{3141}, \pi_{3151}, \pi_{4161}, \pi_{4171}, \pi_{5181}, \pi_{5191}, \pi_{6201}, \pi_{6211},$	42

	Procedural (6 levels, 12 items)	$\gamma = (\gamma_1, \gamma_{20}, \gamma_{21}, \gamma_{30}, \gamma_{31}, \gamma_{40}, \gamma_{41}, \gamma_{50}, \gamma_{51}, \gamma_{60}, \gamma_{61})$	11
		$\pi_{cjz} = (\pi_{110}, \pi_{120}, \pi_{130}, \pi_{140}, \pi_{250}, \pi_{260}, \pi_{270}, \pi_{280}, \pi_{390}, \pi_{3100}, \pi_{3110}, \pi_{3120},$ $\pi_{111}, \pi_{121}, \pi_{131}, \pi_{141}, \pi_{251}, \pi_{261}, \pi_{271}, \pi_{281}, \pi_{391}, \pi_{3101}, \pi_{3111}, \pi_{3121})$	24

5.3.1 Posterior Predictive Model Checking (PPMC)

PPMC is a prominent model evaluation for Bayesian modelling, which uses discrepancy measures to evaluate how the observed data differ from the data generated from the model (predicted data) (Sinharay, 2004). Levy and Mislevy (2016) described the discrepancy measures in PPMC in two categories. First, the discrepancy measures based on the observed data denoted as $D\ (\mathbf{x}; \boldsymbol{\theta})$ where $\mathbf{x} = x_{ij}$ and $\boldsymbol{\theta} = \theta_{\boldsymbol{i}}$. Second, the discrepancy measures based on the posterior predicted data denoted as $D\ (\mathbf{x}^{postpred}; \boldsymbol{\theta})$, where $\mathbf{x}^{postpred}$ are the predicted values of x_{ij}.

If we have *R* number of simulations, let us define the collection of discrepancy measures based on the data, as follows

$$D\ (\mathbf{x}; \boldsymbol{\theta}^{(\mathbf{1})}, \mathbf{x}; \boldsymbol{\theta}^{(\mathbf{2})}, \ldots, \mathbf{x}; \boldsymbol{\theta}^{(\boldsymbol{R})})\ ,$$

and the discrepancy measures based on the predictive values, as follows

$$D\ \left(\mathbf{x}^{postpred(1)}; \boldsymbol{\theta}^{(\mathbf{1})}, \mathbf{x}^{postpred(2)}; \boldsymbol{\theta}^{(2)}, \ldots,\ \mathbf{x}^{postpred(R)}; \boldsymbol{\theta}^{(R)}\ \right).$$

The discrepancy measure *D* is calculated using the following formula

$$D = V_{ij}\left(x_{ij}, \theta_i, \pi_j\right) = \frac{(x_{ij} - P_{ij})^2}{P_{ij}(1 - P_{ij})}\,, \tag{5. 10}$$

where *i* is the index for the *i*th-student, *j* is the index for the *j*th-item, x_{ij} is the students' responses, and P_{ij} is the probability of getting a correct answer for the student at a certain level, i.e.

$$P_{ij} = E(x_{ij} | \theta_i, \boldsymbol{\pi}_j). \tag{5. 11}$$

According to Yan, Mislevy, and Almond (2003), the discrepancy measure in Equation (5.10) expresses root mean square error (RMSE). The RMSE for the student fit can be calculated over all the items, whereas the RMSE for the item fit can be calculated over all the students. A lower discrepancy measure indicates a better fit. However, to discover to what degree this discrepancy measures indicates the model fit, we need to compare the discrepancy between the observed data and the posterior predictive values (Yan et al., 2003).

The posterior predictive p-values (PPP-values) are the statistics used to express the degree of the model fit based on the discrepancy measures (Sinharay, 2004). The PPP-values are estimated by comparing the discrepancy measures from the observed data and the discrepancy measures from the posterior predictive data, which is denoted as

$$\text{PPP-value} = p\Big(D\ (\mathbf{x}^{postpred}; \boldsymbol{\theta}\ ) \geq\ D\ (\mathbf{x}; \boldsymbol{\theta})\Big). \quad (5.\ 12)$$

A good fit is indicated by the PPP-values that are close to 0.5 (Gelman et al., 2014). A value of 0.5 means that the discrepancy measures based on the data are in the middle of the discrepancy measures calculated from the posterior predictive distribution. This implies that the data are consistent with the posterior predictive distribution. In contrast, the PPP-values that are close to 0 or 1 indicate that the discrepancy measures based on the data are on the lower tail or upper tail of the discrepancy measures based on the posterior predictive distribution, which suggests an inadequate fit (Levy, 2006). Gelman et al. (2014) highlighted that PPP-values between 0.05 and 0.95 are in a “reasonable range” for an adequate fit (p.151).

5.3.2 Entropy Statistic

The entropy statistic is used to measure the model improvement between the two models (Model 1 and Model 2). This method was originally proposed by Gilula and Haberman (2001). In applying the entropy to the Bayesian Networks, Levy and Mislevy (2016) described the formula to calculate the entropy based on the data (matrix) $\mathbf{x}$ for a particular model M (either Model 1 or Model 2) as follows:

$$Ent(M) = -\sum_{i=1}^{n} p(\mathbf{x_i}) \log\big(p(\mathbf{x_i})\big). \quad (5.\ 13)$$

where $\mathbf{x_i} = (x_{i1}, \dots, x_{iJ})$ and $p(\mathbf{x_i})$ is the probability of $\mathbf{x_i}$ specified in the model M.

Suppose we aim to compare two models (Model 1 and Model 2). Using the formula in Equation (5.13), we calculate the entropies of Model 1 ($Ent(M1)$) and Model 2 ($Ent(M2)$). The positive values of the difference between entropy Model 1 and entropy Model 2 ($Ent(M1) - Ent(M2)$) suggest that Model 2 has a better prediction of a new observation compared with Model 1 (Levy & Mislevy, 2016).

The proportional improvement of Model 2 from Model 1 can be estimated using the following formula

$$dEntropy = \frac{Ent(M1) - Ent(M2)}{Ent(M1)}. \quad (5.14)$$

The positive values of $dEntropy$ (between 0 and 1) indicate the proportion of improvement created by Model 2 relative to Model 1, suggesting that Model 2 is better than Model 1 in terms of model prediction.

5.4 Summary of the Chapter

This chapter has discussed the specifications of the Bayesian Networks Models, which are used to measure students' fraction learning progression on both conceptual and procedural knowledge dimensions (in the next chapter). Two types of Bayesian Networks models are developed in this chapter: Model 1 and Model 2. Model 1 is developed based on an assumption that the levels of the learning progressions are presented using independent categories of a single latent variable. In contrast, Model 2 is developed based on an assumption that the levels of the learning progressions are presented by multiple latent variables. Model 2 is better in terms of modelling the hierarchical dependency between the levels by setting up the conditional probabilities between the latent variables.

In the next chapter, these two Bayesian Network Models are used to measure students' levels in the learning progression model and to locate the items to the appropriate levels. Furthermore, an empirical comparison of Model 1 and Model 2 is performed in the context of validating the learning progression models, based on 516 students' responses on a fraction test.

CHAPTER 6 : BAYESIAN NETWORKS ANALYSES

6.1 Introduction

The main objective of the present chapter is to describe the empirical validation of the hypothesized model of fraction learning progression which was developed in Chapter 3 and revised in Chapter 4, based on the results of the cognitive interview. The validation was conducted on students' responses, obtained from the administration of the fraction learning progression instrument to a large number of middle-school students in Indonesia.

The purpose of this study was to validate the proposed model of fraction learning using Bayesian Network analysis. The Bayesian Networks models developed in Chapter 5 (Model 1: Bayesian Networks with a single latent variable, and Model 2: Bayesian Networks with multiple latent variables) were used to perform the analysis.

Two levels of statistical inferences of the Bayesian Network analysis were undertaken (adapted from West et al., 2010). The first inference concerned the validation of the instrument at *items* level. The objective of this was to examine the hypothesis that the items at the given levels would be answered correctly by those students who were found to belong to this level or an upper level, but not by the students at lower levels. The second inference concerned the validation of the instrument at *student* level. The purpose of this was to examine the hypothesis that those students who were at certain level in the progression would have sufficient competencies at that level and below but would not have competencies at the upper level(s).

This chapter is now structured into five sections: The method is presented in Section 6.2. This section provides details about the participants, the materials and the procedure of the test. In Section 6.3, the analysis of the results, which used two different Bayesian Network models, is presented. The discussion of the results is presented in Section 6.4, which includes a comparison of Models 1 and 2 and the contribution of these models in the field of educational assessment and measurement. Section 6.5 provides a summary of the chapter.

6.2 Method

6.2.1 Participants

The participants in this study were 516 students (232 male and 284 female) from a total of 26 classes from a public junior high school in Bogor, Indonesia. The distribution of the participants by school grade is presented in Table 6.1 below. The participants were sampled randomly, using a stratified sampling method. First, the population of students at the school was stratified based on grade levels. After that, several intake classes were drawn from each grade to participate in the study.

Table 6.1 The number of students per grade who participated in the study

Grade	Approximate Age	Number of Students
Grade Seven	13 years old	174
Grade Eight	14 years old	147
Grade Nine	15 years old	195
Total		516

The project received approval from the Social and Behavioural Research Ethics Committee (SBREC), Flinders University (reference approval number: 7200).

6.2.2 Materials

The fraction instrument developed in Chapter 3 and validated in Chapter 4 was used to test all the participants. The fraction instrument is shown in the Appendix. All the items in the instrument were collected in one booklet, which consisted of two main sections; namely conceptual and procedural. All the conceptual items were presented first, followed by the procedural items. The items were ordered based on their levels in the progression. For example, in the conceptual section, the items which represented fractions as part-whole (Levels 1 and 2) were shown at the beginning, followed by the items which represented improper fractions and fractions as measures (Level 3). Similarly, in the procedural section, fraction additions (Levels 1 and 2) were presented first, followed by fraction multiplications (Level 3).

6.2.3 Procedure

Testing took place in the students' classrooms and lasted approximately 90 minutes. The teacher told the students that the test aimed to investigate students' knowledge of fractions and to diagnose students' learning difficulties in fraction learning. The teacher also clarified that performance on this test would not be taken into consideration in students' grades. Subsequently, the students were instructed as follows:

1. The students were not allowed to cooperate or discuss the test with each other. The students were told to work on the questions independently
2. The students were told to use a ball point pen to answer questions with clear handwriting
3. The students could fill in the answers on the question sheet and do calculations behind the questions page
4. The students were not allowed to use calculation aids such as calculators and mobile phones
5. The student had to leave the classroom if they finished working on the questions before the end of the testing period test time (90 minutes).

6.3 Results from Bayesian Network Analysis

The analysis of the results was performed in four steps, as follows:

Step 1. Coding the Participants' Responses

The participants' responses were coded as 0, 1, and missing for all conceptual and procedural items. Code 1 refers to a correct response; code 0 refers to an incorrect response; and code missing means that the participant did not answer the item. There were a total of 5.57% missing responses (977 missing and 16567 valid responses).

Step 2. Modelling the participants' responses using Bayesian Networks.

As described in Chapter 5, the notations $\boldsymbol{x} = (x_{ij})$ represent the matrix data of the responses of *n* students on *J* items, i=1,…,*n*, and *j*=1,…,*J*; $\boldsymbol{\theta}$ represents the collection of θ_i for Model 1 and θ_{ci} for Model 2 - they represent students' levels; $\boldsymbol{\gamma}$ represents the hyper-parameter of $\boldsymbol{\theta}$; and $\boldsymbol{\pi}$ represents the measurement model of the students' responses and their levels in the learning progression models, i.e., the conditional probability of the students at a particular level correctly answer an item *j*.

The Bayesian Networks models developed in Chapter 5 (Model 1 and Model 2) were applied to analyse the students' responses based on the proposed model of fraction learning progression. The flow charts in Figures 6.1 and 6.2 illustrate the processes of the Bayesian Network Analysis for Models 1 and 2 respectively. The joint posterior distributions of the parameters for Models 1 and 2 were estimated based on the student's responses ($\boldsymbol{x}$), and the prior distributions for the parameters $(\boldsymbol{\theta}, \boldsymbol{\gamma}, \boldsymbol{\pi})$. Subsequently, the parameters $(\boldsymbol{\gamma}, \boldsymbol{\pi})$ of Model 1 and Model 2 were estimated with the MCMC method (as described in Section 5.2.3) using WinBUGS software (Spiegelhalter, Thomas, & Best, 2000). The results from the WinBUGS analysis were then used for the item analysis. They were also used by the Netica Software to estimate the posterior probabilities of the students' levels $(\boldsymbol{\theta}|\boldsymbol{\gamma}, \boldsymbol{\pi}, \boldsymbol{x})$.

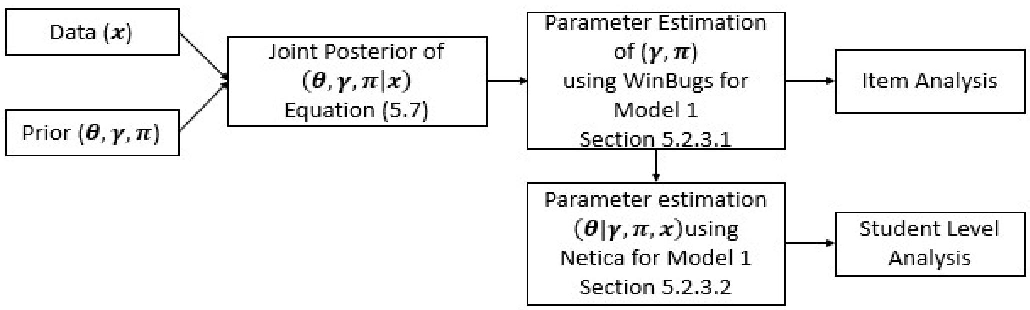


Figure 6.1 The flow chart of Bayesian Networks using Model 1 for the conceptual and procedural knowledge dimensions.

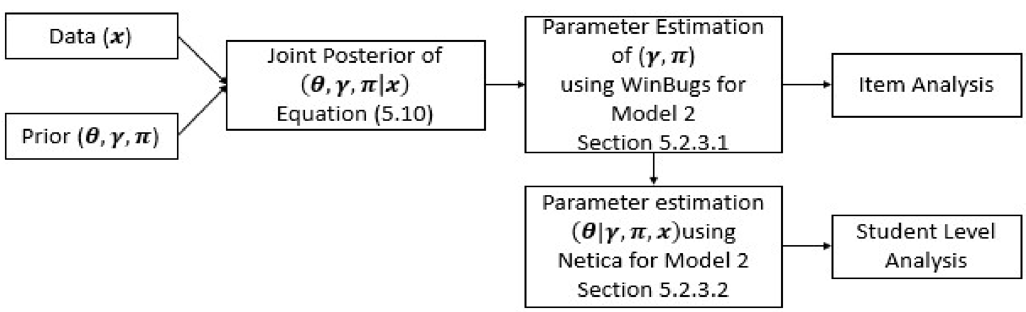


Figure 6.2 The flow chart of Bayesian Networks using Model 2 for the conceptual and procedural knowledge dimensions.

Step 3. Convergence Check of the Bayesian Networks Estimation

A convergence check (as described in Section 5.3.2.3) was performed for the results of the MCMC estimations generated from WinBugs. The trace of the last 10000 iterations with thinning of 10, the autocorrelation functions and Geweke tests were used to evaluate the convergences of the parameters $(\boldsymbol{\gamma}, \boldsymbol{\pi})$ produced from WinBugs. Once the convergence was achieved, estimates of the parameters $(\boldsymbol{\gamma}, \boldsymbol{\pi})$ were then presented for item analysis.

Step 4. Updating Posterior Probabilities for the Individual Students

Next, the convergence parameters generated from the WinBugs were used to build prior probabilities of the Bayesian Networks Model in Netica software (Corporation, 2017). Netica updated the posterior probabilities $(\boldsymbol{\theta}|\boldsymbol{\gamma}, \boldsymbol{\pi}, \boldsymbol{x})$ for individual students when their responses were entered into the networks of Bayesian models. The students' level analysis was performed based on the results from Netica.

6.3.1 Bayesian Network Analysis: The MCMC Estimation

The estimation was run using WinBugs Software to obtain the estimates of the parameters $(\boldsymbol{\gamma}, \boldsymbol{\pi})$ in Model 1 and Model 2. Based on Table 5.5 in Chapter 5, 132 parameters (6 γ's and 126 π's) and 54 parameters (6 γ's and 48 π's) were estimated for the conceptual and procedural dimensions of Model 1 respectively. For Model 2, 53 parameters (11 γ's and 42 π's) and 35 parameters (11 γ's and 24 π's) were estimated for the conceptual and procedural knowledge dimensions respectively. The scores derived from the medians from the posterior distribution $\boldsymbol{\pi}$, generated from Models 1 and 2, were used further for the item analysis.

The length of the MCMC iterations were varied, depending on whether the MCMC chains of Model 1 and Model 2 had achieved convergence. Table 6.2 shows the length of iterations for each model for the conceptual and the procedural knowledge dimensions.

Table 6.2 The length of iterations of the MCMC estimation

Model	Conceptual	Procedural
Model 1	100,000	150,000
Model 2	200,000	250,000

From the total number of MCMC iterations presented in Table 6.2, only the last 10000 iterations were used, discarding the previous iterations as burn-in iterations to ensure convergence.

Figures 6.3 and 6.4 represent a sample of the trace of iterations for the parameters π and γ from Model 1 of the conceptual knowledge dimension. The trace-plots for all parameters $(\boldsymbol{\gamma}, \boldsymbol{\pi})$ of the conceptual and procedural knowledge dimensions generated from Model 1 and Model 2 are presented in the Appendix. The trace-plots in Figure 6.3 exhibit the last 10000 iterations of the specific parameters π_{24} until π_{29}, which are the probabilities of correctly answering items 4-9, given that the students are at Level 2. The results showed that the MCMC iterations converged to certain values. These results were consistent with the Geweke test which shows the Geweke values for π_{24},…, π_{29} are 0.784541, 0.818438, 0.726941, -1.616439, 0.913738, -0.521404 (Appendix H). These values are between +2 and -2, showing a 95% confidence interval. Therefore, we conclude that the estimation of π_{24},…, π_{29} has converged."

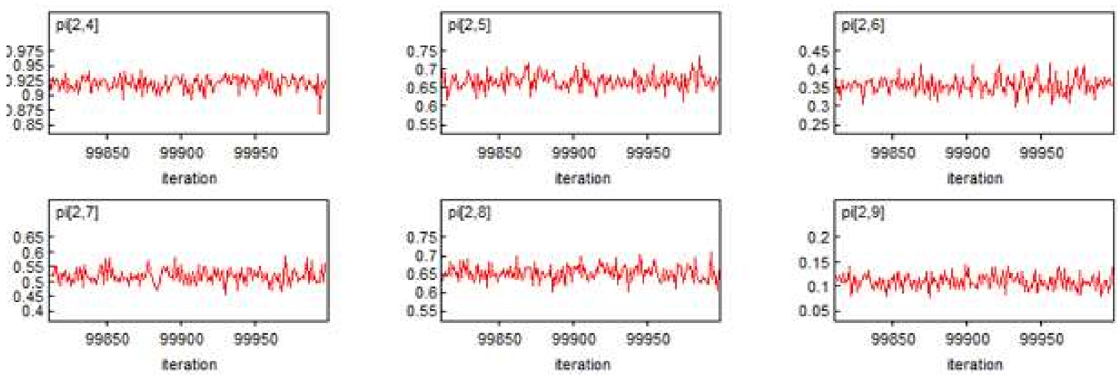


Figure 6.3 A sample of the last 10000 iterations of MCMC for π_{24},…, π_{29}

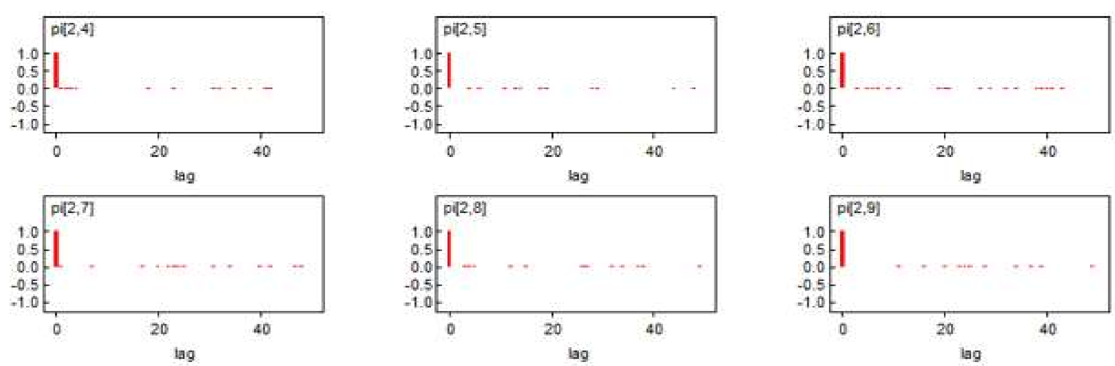


Figure 6.4 A sample of the autocorrelation plots of the last 10000 iterations of MCMC for π_{24},…,π_{29}

As indicated in the correlation plots in Figure 6.4, the results showed that the parameters π_{24},…, π_{29} had around zero autocorrelations as the lag increased. These results demonstrated that the draws from MCMC estimation was now independent. It was important to check this independence that correlated draws are less accurate when compared with estimations using independent draws (Gelman et al., 2014). Similar results could be found on the trace and the autocorrelation plots of $\gamma_1, \ldots, \gamma_6$, which are presented in Figures 6.5 and 6.6 respectively.

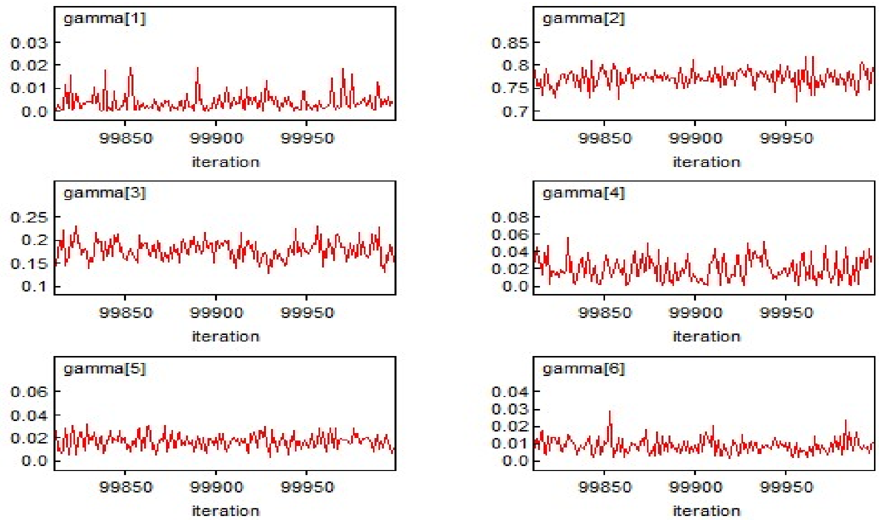


Figure 6.5 The trace of the last 10000 iterations of MCMC for the parameters $\gamma_1, \ldots, \gamma_6$

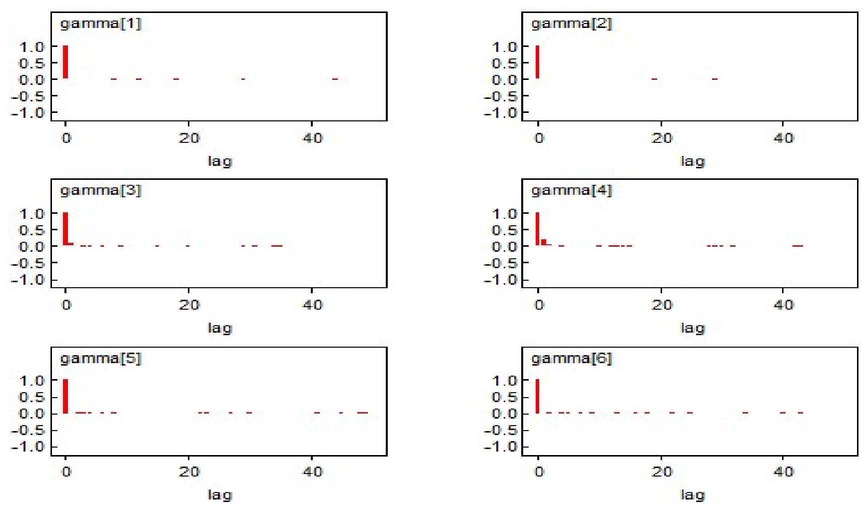


Figure 6.6 The autocorrelation of the last 10000 iterations of MCMC for the parameters $\gamma_1, \ldots, \gamma_6$

6.3.1 Model 1: Analysis of the Conceptual Knowledge Dimension

As detailed in Chapter 5, Model 1 was developed based on the assumption that the students' levels in the learning progression were represented by one latent variable $\boldsymbol{\theta}$ (Equation (5.4)). The parameter $\boldsymbol{\theta}$ had six categories, which reflected the students' levels in the learning progression model.

In Model 1, students were assumed to be at a certain level if they had a high probability of obtaining correct answers for the items at that level and below, and had a low probability of correctly answering the items at the upper level(s). This assumption was encapsulated in the parameter π_{cj} of Model 1 (Equation 5.2). The parameters π_{cj} represented the conditional probability of the students being able to answer the j^{th}-item correctly, given the level *c* of the students in the learning progression model.

This section presents the *item* and the *students' level* analyses for the conceptual knowledge dimension. The following is the item analysis based on the results of the Bayesian Networks estimation generated from the WinBUGS software.

6.3.1.1 Item Analysis

The purpose of the items analyses was to examine whether the items that were hypothesized to be at a certain level would be correctly answered by the students at that level and above, but incorrectly answered by the students at the level(s) below. From Equation 5.2, π_{cj} was $p(x_{ij} = k|\theta_i = c)$. The estimates of π_{cj} were computed using the MCMC method detailed in Section 5.2.3. This method represented the conditional probability of a student being able to answer item *j* "correctly", given that the student was at level *c* in the learning progression model. Hence, the conditional probability of the student being able to answer item *j* "incorrectly", given the student was at level *c* in the learning progression model was 1-π_{cj}. The findings of the analyses were used to locate the items along the progression levels of the model and to validate the competencies that were hypothesized for each level. Thus, the estimates of π_{cj} were used to achieve the purpose of the item analyses.

For interpretation purposes, the cut-off points of the conditional probability π_{cj} were defined as shown in Table 6.3. These cut-off points were developed based on how likely the students who had knowledge at Level *c* correctly answered item *j.*

Table 6.3 The cut-off points of π_{cj} to consider the items to be placed into the levels of the proposed model of fraction learning progression

Interval	Description
$0.65 \leq \pi_{cj} \leq 1$	Item *j* is placed at Level *c*. Students at level *c* are highly likely to answer the item correctly
$0.45 \leq \pi_{cj} < 0.65$	Item *j* is ambiguous to be placed at Level *c* because of the uncertainty of the students at level *c* being able to answer the item correctly
$0 \leq \pi_{cj} < 0.45$	Item *j* is too difficult to be placed at Level *c*. The students at this level are less likely to answer the item correctly

6.3.1.1.1 Item Analysis at Level 1

Four conceptual items were hypothesized at Level 1 (*c*=1, *j*=1,2,3,4) for the conceptual knowledge dimension. These were Items ConT1Q1 (generating a fraction from a part-whole (pie) diagram), ConT2Q1 (shading a pie diagram to represent a fraction less than 1), ConT3Q1 (comparing two fractions less than 1 using part-whole representation), and ConT7Q1 (adding fractions less than 1 using a part-whole representation diagram). These items were designed to test students' conceptual understanding of fractions as a representation of part-whole.

The conditional probabilities π_{1j} were estimated and are shown in Table 6.4. The results showed that the students from the lowest to the highest levels had a high probability (above 0.65) to answer these items correctly. It was concluded that Items Cont1Q1, ConT2Q1, ConT3Q1, and ConT7Q1 were suitable to be placed at Level 1.

Table 6.4 The estimates of the conditional probabilities π_{1j} of the conceptual knowledge items of Level 1 for Model 1

Item: ConT1Q1 (*j*=1)	Correct $\pi_{1,1}$	Incorrect 1-$\pi_{1,1}$	Item: ConT2Q1 (*j=2*)	Correct $\pi_{1,2}$	Incorrect 1-$\pi_{1,2}$
Level 1	0.8013	0.1987	Level 1	0.8011	0.1989
Level 2	0.9435	0.0565	Level 2	0.9533	0.0467
Level 3	0.8974	0.1026	Level 3	0.8975	0.1025
Level 4	0.8164	0.1836	Level 4	0.8162	0.1838
Level 5	0.8170	0.1830	Level 5	0.8168	0.1832
Level 6	0.8103	0.1897	Level 6	0.8096	0.1904

Item: ConT3Q1 (*j=3*) Item: ConT7Q1 (*j=4*)

	Correct $\pi_{1,3}$	Incorrect 1-$\pi_{1,3}$
Level 1	0.7992	0.2008
Level 2	0.7365	0.2635
Level 3	0.8851	0.1149
Level 4	0.8106	0.1894
Level 5	0.8167	0.1833
Level 6	0.8088	0.1912

	Correct $\pi_{1,4}$	Incorrect 1-$\pi_{1,4}$
Level 1	0.7995	0.2005
Level 2	0.9200	0.0800
Level 3	0.8921	0.1079
Level 4	0.8172	0.1828
Level 5	0.8167	0.1833
Level 6	0.8087	0.1913

The above results support the hypothesis that the competencies underlying the items which require the generation of a fraction from a part-whole (pie) diagram, shading a pie diagram to represent a fraction less than 1, comparing two fractions less than 1 using a part-whole representation, and adding fractions less than 1 using a part-whole representation diagram, are established at Level 1. This provides evidence that students' part-whole understanding emerges at Level 1.

6.3.1.1.2 Item Analysis at Level 2

Items ConT1Q2 (generating an equivalent fraction from a pie diagram), ConT1Q3 (generating a fraction from an unequal partition of a pie diagram), ConT3Q2 (comparing fractions of less than 1 with a different denominator using a part-whole diagram), ConT7Q2 (adding fractions with different denominators using a part-whole representation diagram) were hypothesized at Level 2 (c=2, j=5,6,7,8). These items were created to assess students' conceptual understanding of equivalent fractions, unequal partitions, and fractions as part-whole with different denominators.

The conditional probabilities π_{2j} for the conceptual items at Level 2 are presented in Table 6.5. The results showed that students at Level 2 had a probability of 0.6668 of answering item Cont1Q2 correctly. This was in contrast with the probability of students at Level 1 doing the same, which was only 0.1971. All the students at the higher levels (Level 3-6) had a considerably high probability of answering this item correctly. This result shows that Item Cont1Q2 could differentiate students at level 1 from those at Levels 2 and above. The students at Level 1 were unlikely to answer this item correctly, while the students at Level 2 and above were highly likely to answer the item correctly.

In contrast, for Item ConT1Q3, students at Level 2 had a low probability of answering this item correctly. The probability of getting a correct answer for this item for students at Level 1 was 0.3555, while the probability of getting an incorrect answer was 0.6445. This means that this item was too difficult for students placed at Level 2. In contrast, students at Level 3 and above, had a high probability of answering this item correctly. The results showed that this item was more suitable to be placed in Level 3, because it could discriminate between those students at Level 2 and those at Level 3.

Next, for Item ConT3Q2, students at Level 2 had a probability of 0.5222 of answering this item correctly. This item was considered ambiguous because its probability lies in the range between 0.45 and 0.65, as defined in Table 6.3.

Finally, for item ConT7Q2, students at Level 2 were likely to answer this item correctly with a probability of 0.6528. This was different from students at Level 1, who had a low probability, of 0.1973, of answering this item correctly. Hence, Item ConT7Q2 could discriminate those students at Level 1 and those at Level 2.

Table 6.5 The estimates of the conditional probabilities π_{2j} of the conceptual knowledge items at Level 2 for Model 1

Item: ConT1Q2 (*j*=5)	Correct $\pi_{2,5}$	Incorrect $1-\pi_{2,5}$
Level 1	0.1971	0.8029
Level 2	0.6668	0.3332
Level 3	0.8899	0.1101
Level 4	0.8177	0.1823
Level 5	0.8172	0.1828
Level 6	0.8091	0.1909

(a)

Item: ConT1Q3 (*j*=6)	Correct $\pi_{2,6}$	Incorrect $1-\pi_{2,6}$
Level 1	0.1960	0.8040
Level 2	0.3555	0.6445
Level 3	0.7780	0.2220
Level 4	0.7993	0.2007
Level 5	0.8020	0.1980
Level 6	0.8096	0.1904

(b)

Item: ConT3Q2 (*j*=7)	Correct $\pi_{2,7}$	Incorrect $1-\pi_{2,7}$
Level 1	0.1963	0.8037
Level 2	0.5222	0.4778
Level 3	0.8421	0.1579
Level 4	0.7951	0.2049
Level 5	0.8158	0.1842
Level 6	0.8088	0.1912

(c)

Item: ConT7Q2 (*j*=8)	Correct $\pi_{2,8}$	Incorrect $1-\pi_{2,8}$
Level 1	0.1973	0.8027
Level 2	0.6528	0.3472
Level 3	0.8623	0.1377
Level 4	0.8102	0.1898
Level 5	0.8103	0.1897
Level 6	0.8084	0.1916

(c)

Based on the results above, items Cont1Q2 and ConT7Q2, were suitable for placement at Level 2. Item ConT1Q3 was too difficult for students at Level 2, therefore it was more suitable for placement at Level 3, while Item ConT3Q2 fell into the 'ambiguous item' category.

The results support the hypothesis that the competencies underpin items Cont1Q2 and ConT7Q2: generating equivalent fractions and adding fractions with different denominators using part-whole representation/diagrams, which emerge at Level 2. Meanwhile, the competency underpins item ConT1Q3, generating a fraction from an unequal partition, as more likely to be established at level 3.

6.3.1.1.3 Item Analysis at Level 3

Seven items were hypothesized at Level 3 (*c*=3, *j*=9,10,11,12,13,14,15). These items were items ConT1Q4 (generating an improper fraction from a pie representation), ConT1Q5 (Generating an equivalent of an improper fraction from a pie diagram), ConT2Q2 (shading a pie diagram to represent an improper fraction), ConT3Q3 (comparing two improper fractions using part-whole representation), ConT4Q1 (generating a fraction less than 1 on a number line), ConT4Q2 (generating a fraction less than 1 on a number line with a constraint), and ConT4Q3 (generating fractions greater than 1 on a number line). These items were designed to test students' understanding of improper fractions and fractions as measurements.

The conditional probabilities π_{3j} are presented in Table 6.6. The results showed that the items which tested improper fractions (ConT1Q4, ConT1Q5, ConT2Q2, and ConT3Q3) discriminated well between Levels 2 and Level 3. For example, it can be seen that that students at the lower levels (Levels 1 and 2) had a low probability of answering item ConT1Q4 correctly, while students at Level 3 and above were highly likely to answer this item correctly, with the probability being equal to or greater than 0.75. Similarly, the probability of students at Levels 1 and 2 answering Item ConT1Q5 correctly were also low (0.1964 and 0.0799 respectively), however students at Level 3 and above had a high probability of answering this item to correctly (0.6933, 0.7922, 0.8061, 0.8076).

Table 6.6 The estimates of the conditional probabilities π_{3j} of the conceptual knowledge items of Level 3 for Model 1

Item: ConT1Q4 (*j*=9)

	Correct $\pi_{3,9}$	Incorrect 1-$\pi_{3,9}$
Level 1	0.1984	0.8016
Level 2	0.1107	0.8893
Level 3	0.7504	0.2496
Level 4	0.7979	0.2021
Level 5	0.8068	0.1932
Level 6	0.8087	0.1913

Item: ConT1Q5 (*j*=10)

	Correct $\pi_{3,10}$	Incorrect 1-$\pi_{3,10}$
Level 1	0.1964	0.8036
Level 2	0.0799	0.9201
Level 3	0.6933	0.3067
Level 4	0.7922	0.2078
Level 5	0.8061	0.1939
Level 6	0.8076	0.1924

Item: ConT2Q2 (*j*=11)

	Correct $\pi_{3,10}$	Incorrect 1-$\pi_{3,10}$
Level 1	0.1982	0.8018
Level 2	0.3237	0.6763
Level 3	0.8372	0.1628
Level 4	0.8094	0.1906
Level 5	0.8160	0.1840
Level 6	0.8090	0.1910

Item: ConT3Q3 (*j*=12)

	Correct $\pi_{3,10}$	Incorrect 1-$\pi_{3,10}$
Level 1	0.1973	0.8027
Level 2	0.1345	0.8655
Level 3	0.8368	0.1632
Level 4	0.8000	0.2000
Level 5	0.8161	0.1839
Level 6	0.8087	0.1913

Item: ConT4Q1 (*j*=*13*)

	Correct $\pi_{3,13}$	Incorrect 1-$\pi_{3,13}$
Level 1	0.1969	0.8031
Level 2	0.0711	0.9289
Level 3	0.8538	0.1462
Level 4	0.8144	0.1856
Level 5	0.8105	0.1895
Level 6	0.8092	0.1908

Item: ConT4Q2 (*j*=14)

	Correct $\pi_{3,14}$	Incorrect 1-$\pi_{3,14}$
Level 1	0.1960	0.8040
Level 2	0.0429	0.9570
Level 3	0.8682	0.1318
Level 4	0.8076	0.1924
Level 5	0.8101	0.1899
Level 6	0.8088	0.1912

Item: ConT4Q3 (*j*=15)

	Correct $\pi_{3,15}$	Incorrect 1-$\pi_{3,15}$
Level 1	0.1965	0.8035
Level 2	0.0480	0.9520
Level 3	0.7796	0.2204
Level 4	0.8026	0.1974
Level 5	0.8088	0.1912
Level 6	0.8091	0.1909

Likewise, the items related to fractions as measures (ConT4Q1, ConT4Q2, and ConT4Q3) also demonstrated a good discriminatory power between students at Level 2 and Level 3. These items had a low probability of students at Level 1 and Level 2 answering them correctly (below 0.20), whereas students at Level 3 and above had a high probability of obtaining a correct answer for this item, with the probability being about 0.80.

The results discussed above indicate that items ConT1Q4, ConT1Q5, ConT2Q2, ConT3Q3, ConT4Q1, ConT4Q2, and ConT4Q3 are suitable for placement at Level 3. This supports the hypothesis that the competencies underlying these items are established at this level, namely generating an improper fraction from a pie diagram representation, generating an equivalent of an improper fraction from a pie diagram, shading a pie diagram to represent an improper fraction, comparing two improper fractions using part-whole representation, generating a fraction less than 1 on a number line, generating a fraction less than 1 on a number line with a constraint, and generating fractions greater than 1 on a number line.

6.3.1.1.4 Item Analysis at Level 4

Two items were hypothesized at Level 4 (c=4, j=16,17). These items were item ConT5Q1 (writing the biggest fraction they can) and item ConT5Q2 (writing the smallest fraction they can). These items were used to test students' understanding of the unbounded infinity of fractions (there are no smallest or biggest fractions).

The results showed that students at Level 3 and below were likely to have an incorrect answer for both items ConT5Q1 and ConT5Q2 with the probabilities for being correct lying at lower than 0.20. In contrast, students at Level 4 and above were highly likely to answer these questions correctly, with the probabilities being above 0.80. These results showed that Item ConT5Q1 and Item ConT5Q2 could discriminate students at Level 3 and Level 4 effectively. The conditional probabilities π_{4j} are presented in Table 6.7.

The results indicate that items ConT5Q1 and ConT5Q2 are suitable for placement at Level 4. These results support the hypothesis that students' understanding of the unbounded infinity of fractions emerges at Level 4.

Table 6.7 The estimates of the conditional probabilities of the conceptual knowledge items of Level 4 for Model 1

Item: (j=16)	ConT5Q1		Item: (j=17)	ConT5Q2	
	Correct $\pi_{4,16}$	Incorrect 1-$\pi_{4,16}$		Correct $\pi_{4,17}$	Incorrect 1-$\pi_{4,17}$
Level 1	0.1957	0.8043	Level 1	0.1971	0.8029
Level 2	0.0642	0.9358	Level 2	0.0638	0.9362
Level 3	0.1893	0.8107	Level 3	0.1971	0.8029
Level 4	0.8119	0.1881	Level 4	0.8132	0.1868
Level 5	0.8101	0.1899	Level 5	0.8161	0.1839
Level 6	0.8077	0.1923	Level 6	0.8090	0.1910

6.3.1.1.5 Item Analysis at Level 5

Item ConT6Q1 (finding how many fractions lie between two fractions) and item ConT6Q2 (finding how many fractions lie between two pseudo successive fractions) were hypothesized as lying at Level 5 (c=5, j=18,19). These items were created to test students' understanding of the density of fractions.

The conditional probabilities π_{5j} were estimated and are shown in Table 6.8. The results showed that the students at Level 5 and 6 were more likely to answer item ConT6Q1 and ConT6Q2 correctly, with a probability greater than 0.8, while students at Level 4 and below were unlikely to answer correctly, with a probability less than 0.2. These results showed that items ConT6Q1 and ConT6Q2 could discriminate effectively between those students at Level 4 and those at Level 5.

The results discussed above indicate that items ConT6Q1 and ConT6Q2 are suitable for placement at Level 5. These results support the hypothesis that the competency of understanding the density of fractions emerges at this level.

Table 6.8 The estimates of the conditional probabilities π_{5j} of the conceptual knowledge items of Level 5 for Model 1

Item: ConT6Q1 (j=18)			Item: ConT6Q2 (j=19)		
	Correct $\pi_{5,18}$	Incorrect 1-$\pi_{5,18}$		Correct $\pi_{5,19}$	Incorrect 1-$\pi_{5,19}$
Level 1	0.1978	0.8022	Level 1	0.1971	0.8029
Level 2	0.0412	0.9588	Level 2	0.0393	0.9607
Level 3	0.1047	0.8953	Level 3	0.1277	0.8723
Level 4	0.1872	0.8128	Level 4	0.1904	0.8096
Level 5	0.8102	0.1898	Level 5	0.8120	0.1880
Level 6	0.8076	0.1924	Level 6	0.8077	0.1923

6.3.1.1.6 Item Analysis at Level 6

Two items were hypothesized as being at Level 6 (c=6, j=20,21), namely ConT8Q1 (multiplying fractions using a diagram representation) and ConT8Q2 (dividing fractions using a diagram representation). These items were designed to test students' understanding of multiplicative fraction operations.

The results showed that students at the top level of the conceptual dimension were likely to answer Items ConT8Q1 and ConT8Q2 correctly, with the probability being greater than 0.80. In contrast, students at Level 5 and below were unlikely to answer this item correctly, with the probability being less than 0.2. This indicates that both items can discriminate effectively between those students at Level 6 and those students who fall below this level. The conditional probabilities π_{6j} are presented in Table 6.9.

Table 6.9 The estimates of the conditional probabilities π_{6j} of the conceptual knowledge items of Level 6 for Model 1

Item: ConT8Q1 (j=20)			Item: ConT8Q2 (j=21)		
	Correct $\pi_{6,20}$	Incorrect 1-$\pi_{6,20}$		Correct $\pi_{6,21}$	Incorrect 1-$\pi_{6,21}$
Level 1	0.1960	0.8040	Level 1	0.1948	0.8052
Level 2	0.0395	0.9606	Level 2	0.0391	0.9609
Level 3	0.1072	0.8928	Level 3	0.1120	0.8880
Level 4	0.1843	0.8157	Level 4	0.1850	0.8150
Level 5	0.1877	0.8123	Level 5	0.1884	0.8116
Level 6	0.8074	0.1926	Level 6	0.8065	0.1935

The results demonstrate that items ConT8Q1 and ConT8Q2 are suitable for placement at Level 6. These results indicate that that the level of competency for understanding multiplicative fractions operations is established at this level.

Like the results generated from other statistical models, the posterior probabilities of students' correctly answering the items (π_{cj}) also contain some degree of uncertainty (random errors). Tables 6.4, 6.7 – 6.9 show that a few of the students at the higher levels have smaller probability to correctly answer the items at the lower levels compared to students at the lower levels.

For example, in Table 6.7, the probability of students at Levels 2 and 3 to correctly answer item ConT5Q1 is 0.0642 and 0.1893 is lower than the students at Level 1 which is 0.1957.

However, these variations do not really matter in the context of discrete hierarchical analysis, because the results show that the students at the lower levels (Levels 1,2, and 3) were very unlikely to correctly answer item ConT5Q1, while the students at Level 4 were highly likely answer correctly this item with the probability 0.8119.

In order to incorporate such variations, cut-off criteria were developed in Table 6.3. Using the cut-off points, the variations of the probabilities at Levels 1-3 presented in Table 6.7 did not affect locating item ConT5Q1 on the progression levels of the conceptual knowledge dimension.

6.3.1.2 *Analysis to Estimate Students' Levels in The Progression*

The purpose of the analyses is to estimate students' levels in the conceptual knowledge dimension using Model 1. Netica Software was used to estimate the posterior probabilities of the students' levels $P(\boldsymbol{\theta}|\boldsymbol{\gamma}, \boldsymbol{\pi}, \boldsymbol{x})$, as described in Section 5.2.3.3.2.

As described in Chapter 5, each individual student's responses, $x_{ij,}$ were entered into the network using Netica. Using Netica, the responses were compiled to update the prior probability $\boldsymbol{\gamma}$ of the student being in the network. The estimates of $\boldsymbol{\gamma}$ as in equation (5.5) and π_{cj} in equation (5.2) (generated from the WinBugs software) were used as priors in the networks.

Figure 6.7 presents the prior probabilities $\boldsymbol{\gamma}$ of the Netica graph for Model 1 on the conceptual knowledge dimension. This was the prior where there was no student data entered into the network. From this figure, the prior probabilities of students' levels, i.e. the estimates of $\boldsymbol{\gamma}$ (displayed in the node of "Conceptual_LP") were 77.70% (γ_1), 17.90% (γ_2), 1.68% (γ_3),, 1.57% (γ_4), 0.84%(γ_5), and 0.29% (γ_6). These prior estimates were different from the priors set for equation (5.5) which were 16.67% for all $\boldsymbol{\gamma}$ values. The priors of $\boldsymbol{\gamma}$ presented in Figure 6.7 showed that most of the students were at Levels 1 and 2, and a small number of students were at the remaining levels.

Moreover, the prior probabilities for each item were compiled in the Netica software based on the conditional probabilities π_{cj} obtained from the WinBugs estimation (as shown in Figure 6.7). The results showed that the probabilities of getting correct answers decreased as the levels increased. For example, the prior probability to get a correct answer for the items at Level 1 (Items ConT1Q1, ConT2Q1, ConT3Q1, and ConT7Q1) were about 80%, while the probability to get a correct answer for the items at Level 6 (ConT8Q1 and ConT8Q2) were about 16%. The results showed there was an increasing level of difficulty of the items, which was consistent with the hierarchical levels of the learning progression.

In the next step, the posterior probabilities of the students' levels $P(\boldsymbol{\theta}|\boldsymbol{\gamma},\boldsymbol{\pi},\boldsymbol{x})$ of 516 students were estimated. As examples of the cohort, only those results obtained by two particular students are presented. Figures 6.8 and 6.9 show the Netica estimations for students ID 187 and 424, respectively. The raw scores of students ID 187 and 424 are presented in Tables 6.10 and 6.11, respectively.

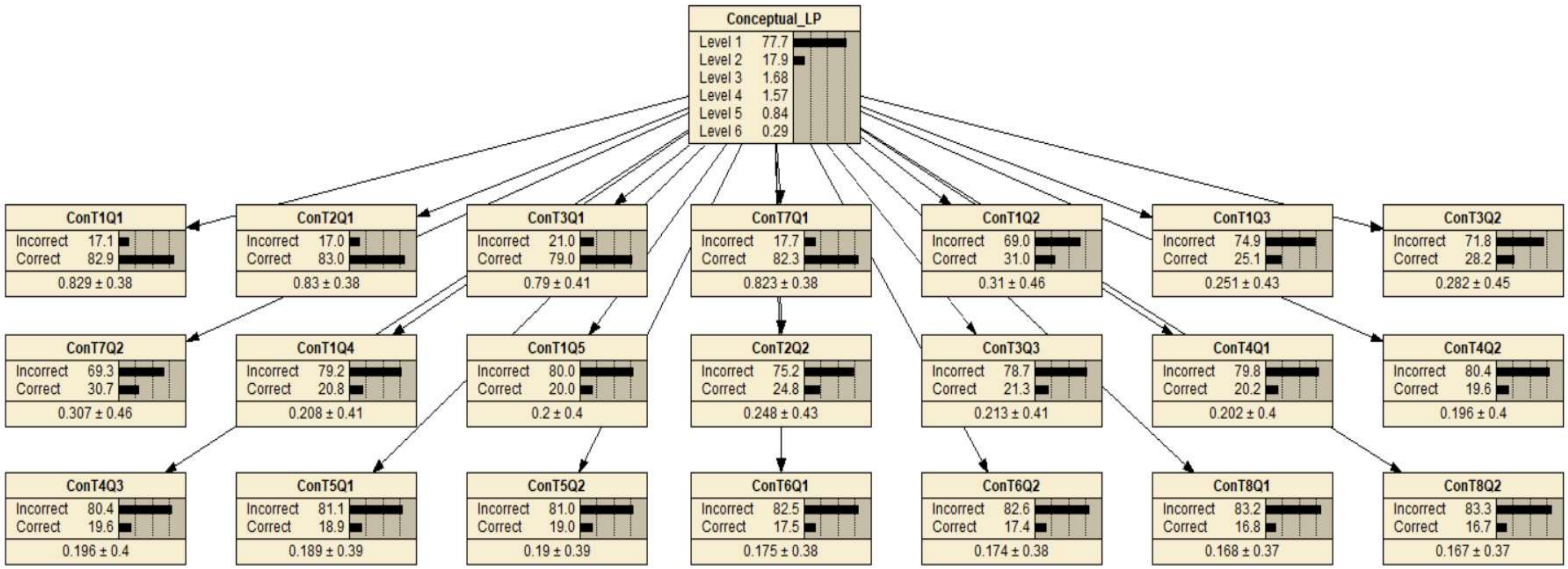


Figure 6.7 A Netica Graph of the prior probability of the conceptual knowledge dimension.

Table 6.10 The raw sores of student with ID 187

Level	Level 1				Level 2				Level 3							Level 4		Level 5		Level 6	
Item	ConT1Q1	ConT2Q1	ConT3Q1	ConT7Q1	ConT1Q2	ConT1Q3	ConT3Q2	ConT7Q2	ConT1Q4	ConT1Q5	ConT2Q2	ConT3Q3	ConT4Q1	ConT4Q2	ConT4Q3	ConT5Q1	ConT5Q2	ConT6Q1	ConT6Q2	ConT8Q1	ConT8Q2
Response	1	1	0	1	1	1	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0

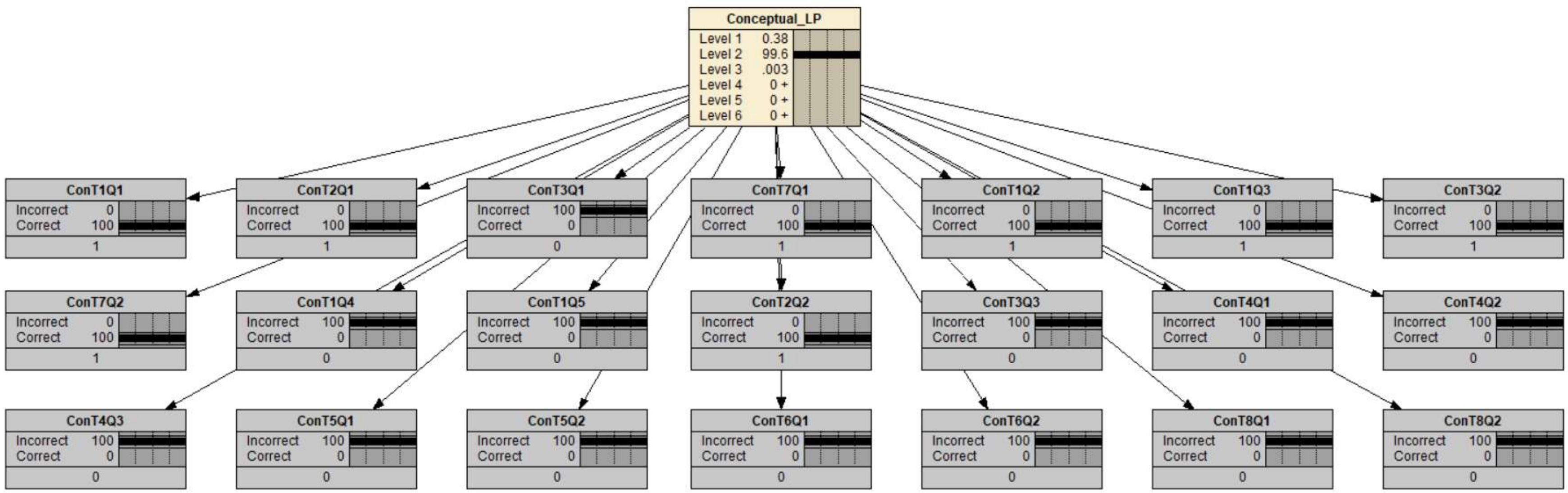


Figure 6.8 A Netica Graph of the posterior probability $P(\theta_i|\boldsymbol{\gamma}, \boldsymbol{\pi}, x_{ij})$ for the student with ID 187 (i=187, j=1,…,21).

Table 6.11 The raw sores of the student with ID 424

Level	Level 1				Level 2				Level 3							Level 4		Level 5		Level 6	
Item	ConT1Q1	ConT2Q1	ConT3Q1	ConT7Q1	ConT1Q2	ConT1Q3	ConT3Q2	ConT7Q2	ConT1Q4	ConT1Q5	ConT2Q2	ConT3Q3	ConT4Q1	ConT4Q2	ConT4Q3	ConT5Q1	ConT5Q2	ConT6Q1	ConT6Q2	ConT8Q1	ConT8Q2
Response	1	1	1	1	1	1	0	0	1	1	1	0	1	1	1	1	1	0	0	0	0

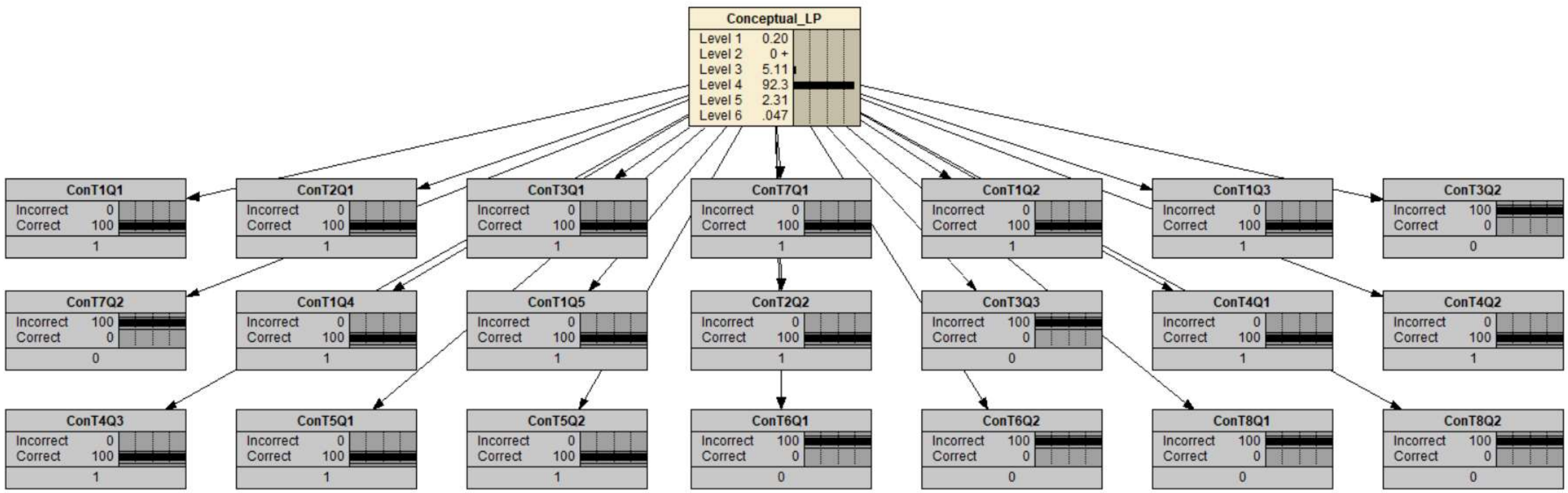


Figure 6.9 A Netica Graph of the posterior probability $P(\theta_i|\boldsymbol{\gamma}, \boldsymbol{\pi}, x_{ij})$ for the student with ID 424 (i=424, j=1,…,21).

From Table 6.10, it can be seen that student ID 187 could answer most of the items at Level 1, all the items at Level 2, and one item out of 7 items at Level 3. Based on these responses, the Netica software generated probabilities of each level for this student as follows (from level 1 to level 6): 0.38%, 99.6%, 0.003%, 0%, 0%, and 0%. As the highest probability of the level is at Level 2, the student with ID 187 is assigned to Level 2 in our model. Similarly, student ID 424, as can be observed in Table 6.11, could answer all the items at Level 1, most of the items at Levels 2 and 3, and all the items at Level 4. The Netica software produced the probability of each level for this student (from the lowest to the highest) as follows: 0.2%, 0%, 5.11%, 92.3%, 2.31%, and 0.47%. Hence, the student was assigned to Level 4 because the student had the highest probability to be placed at this level.

The same procedure as was discussed above was then applied to all 516 students' responses to estimate their levels. The results were summarized in Figure 6.10, which shows the distribution of the students' levels in the conceptual knowledge dimension. The results showed that the largest percentage of the students were at Level 2 (59.11%), followed by the percentages at Level 1 (20.16%), Level 3 (14.92%), Level 4 (3.68%), Level 5 (1.36%) and Level 6 (0.78%).

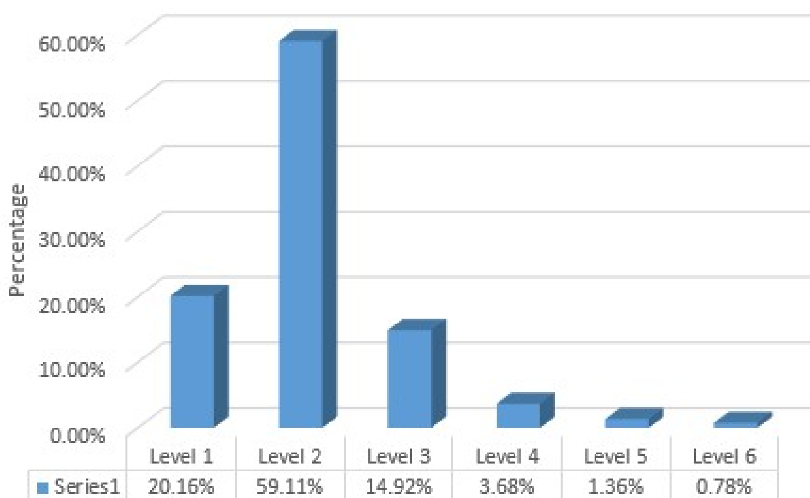


Figure 6.10 The distribution of the students' levels in the conceptual knowledge dimension

6.3.2 Model 1: Analysis of the Procedural Knowledge Dimension

6.3.2.1 Item Analysis

Similar to the item analysis of the conceptual knowledge dimension using Model 1, the purpose of the analysis of the items included in the procedural knowledge dimension of the fraction progression instrument was to validate the items in the various levels of this knowledge dimension.

6.3.2.1.1 Item Analysis at Level 1

Item ProT1Q1 (adding fractions with the same denominator) was hypothesized to be located at Level 1 (c=1, j=1). This item was used to test students' procedural knowledge of adding fractions with the same denominator. The results showed that the students from the lowest to the highest level were likely to answer item ProT1Q1 correctly with a probability above 75% at Level 1 and above 80% for Levels 2-6. Only 23% students at Level 1 will answer the item incorrectly. The results are shown in Table 6.12.

The results indicate that item ProT1Q1 is suitable to be placed at Level 1 of the procedural knowledge dimension, supporting the hypothesis that the competency of adding fractions with the same denominator that underpins item ProT1Q1, is established at Level 1.

Table 6.12 The estimates of the conditional probabilities π_{1j} of the procedural knowledge item falling at Level 1 for Model 1

Item: ProT1Q1 (j=1)

	Correct π_{11}	Incorrect 1-π_{11}
Level 1	0.7696	0.2304
Level 2	0.8028	0.1972
Level 3	0.8873	0.1127
Level 4	0.9095	0.0905
Level 5	0.8040	0.1960
Level 6	0.8897	0.1103

6.3.2.1.2 Item Analysis at Level 2

Item ProT1Q2 (adding fractions with different denominators) was hypothesized to test students' procedural knowledge at this level (c=2, j=2). The results showed that students at level 2 and above had a probability of over 80% of answering this item correctly. In contrast, students at Level 1 were unlikely to answer these items correctly, with a probability of about 23%. The results showed that this item discriminated effectively between the students at level 1 and the students at level 2.

The results indicate that Item ProT1Q2 is suitable to be placed at Level 2 in the proposed model of the procedural knowledge dimension. The results support the hypothesis that the competency of adding fractions with the same denominator, which underpins item ProT1Q2, is established at level 2.

Table 6.13 The estimates of the conditional probabilities π_{2j} of the procedural knowledge item falling at Level 2 for Model 1

Item: ProT1Q2 (*j*=2)		
	Correct π_{22}	Incorrect 1-π_{22}
Level 1	0.2287	0.7713
Level 2	0.8022	0.1978
Level 3	0.8439	0.1561
Level 4	0.8838	0.1162
Level 5	0.804	0.196
Level 6	0.875	0.125

6.3.2.1.3 Item Analysis at Level 3

Five items were hypothesized at Level 5 of the procedural knowledge dimension (*c*=3, *j*=3,4,5,6,7). These items were ProT1Q4 (adding fractions with a mixed number), ProT1Q3 (subtracting fraction with a whole number), ProT2Q1 (multiplying a fraction with a fraction), ProT2Q2 (multiplying a fraction with a whole number), and ProT2Q3 (dividing a fraction with a fraction).

The results showed that items ProT1Q4, ProT1Q3, ProT2Q2 and ProT2Q3 discriminated well between students at Level 2 and Level 3. Students at Level 1 and 2 had low probabilities (less than 31%) to answer these items correctly, while students at Level 3 and above had a probability of 67% of answering item ProT1Q3 correctly, and above an 80% probability to answer Items ProT1Q4, ProT2Q2 and ProT2Q3 correctly. The results are shown in Table 6.14

Table 6.14 The estimates of the conditional probabilities π_{3j} of the procedural knowledge item falling at Level 3 for Model 1

Item: ProT1Q3 (*j*=3)

	Correct π_{33}	Incorrect 1-π_{33}
Level 1	0.1635	0.8365
Level 2	0.2000	0.8000
Level 3	0.6693	0.3307
Level 4	0.8267	0.1733
Level 5	0.8043	0.1957
Level 6	0.8628	0.1372

Item: ProT1Q4 (*j*=4)

	Correct π_{34}	Incorrect 1-π_{34}
Level 1	0.3108	0.6892
Level 2	0.2019	0.7981
Level 3	0.8642	0.1358
Level 4	0.8998	0.1002
Level 5	0.8053	0.1947
Level 6	0.8931	0.1069

Item: ProT2Q1 (*j*=5)

	Correct π_{35}	Incorrect 1-π_{35}
Level 1	0.1210	0.8790
Level 2	0.1953	0.8047
Level 3	0.5634	0.4366
Level 4	0.8201	0.1799
Level 5	0.8020	0.1980
Level 6	0.8527	0.1473

Item: ProT2Q2 (*j*=6)

	Correct π_{36}	Incorrect 1-π_{36}
Level 1	0.3036	0.6964
Level 2	0.1978	0.8022
Level 3	0.8389	0.1611
Level 4	0.9100	0.09
Level 5	0.8055	0.1945
Level 6	0.8827	0.1173

Item: ProT2Q3 (*j*=7)

	Correct π_{37}	Incorrect 1-π_{37}
Level 1	0.1827	0.8173
Level 2	0.1977	0.8023
Level 3	0.8186	0.1814
Level 4	0.9142	0.0858
Level 5	0.8045	0.1955
Level 6	0.8922	0.1078

However, for item ProT2Q1, the probability for the students at Level 3 to answer this item correctly was 56% (referring to the ambiguous items between 0.45 and 0.65). The students at the lower levels (levels 1 and 2) had probabilities of less than 20% of answering this item correctly. In contrast, the students at level 4 and above were highly likely to answer this item correctly, with the probabilities lying above 80%. The results did not indicate that this item was better to be placed at Level 4, because the students at Level 3 were likely to answer this item correctly with a probability only slightly above 50%. Hence, there was ambiguity with placing this item at Level 3.

The results discussed above indicate that items ProT1Q4, ProT1Q3, ProT2Q2 and ProT2Q3 are suitable to be placed at Level 3. These results support the hypothesis that the competencies that underpin these items, adding fractions with a mixed number; subtracting fractions with a whole number; multiplying a fraction with a whole number; and dividing a fraction with a fraction, are established at this level.

6.3.2.1.4 Item Analysis at Level 4

Two items were hypothesized at Level 4 (*c*=4, *j*=8,9). These items were ProT2Q4 (multiplying a mixed number with a mixed number) and ProT2Q5 (dividing a mixed number with a whole number). These items were designed to test students' procedural knowledge of multiplicative fraction operations which involve mixed numbers.

The results showed that students at level 4 and above had probabilities of answering these items correctly at above 72% for item ProT2Q4 and above 78% for item ProT2Q5. In contrast, students at Level 3 and below were unlikely to answer these items correctly, with the probabilities lying at less than 20%. The results showed that these items discriminated well between those students at level 3 and those students at level 4. The results are shown in Table 6.15

Table 6.15 The estimates of the conditional probabilities π_{4j} of the procedural knowledge item falling at Level 4 for Model 1

Item: ProT2Q4 (*j*=8)			Item: ProT2Q5 (*j*=9)		
	Correct π_{48}	Incorrect 1-π_{48}		Correct π_{49}	Incorrect 1-π_{49}
Level 1	0.1227	0.8773	Level 1	0.1112	0.8888
Level 2	0.1959	0.8041	Level 2	0.1967	0.8033
Level 3	0.1898	0.8102	Level 3	0.197	0.803
Level 4	0.7357	0.2643	Level 4	0.7809	0.2191
Level 5	0.8002	0.1998	Level 5	0.8024	0.1976
Level 6	0.7289	0.2711	Level 6	0.8589	0.1411

The results indicate that items ProT2Q4 and ProT2Q5 are suitable to be placed at Level 4. These results support the hypothesis that the competencies for performing multiplicative fraction operations with a mixed number, which underpin these items, emerge at this level.

6.3.2.1.5 Item Analysis at Level 5

The items hypothesized at Level 5 (*c*=5, *j*=10, 11) were Item ProT3Q1 (solving a nested fraction operation where the numerator is a fraction subtraction) and ProT3Q2 (solving a nested fraction operation where the numerator is a fraction division). These items were created to test students' procedural knowledge of complex fractions with one-level nested fraction operations.

The results showed that the students at level 4 and below were unlikely to answer item ProT3Q1 correctly, with the probability being less than 20%. Students at Level 5 and above were highly likely to answer this item correctly, with the probability being above 80%. Similarly, the probabilities for students at Level 4 and below were less than 24% to answer item ProT3Q2 correctly, while students at Levels 5 and 6 have a probability of about 80% of answering this item correctly. Hence, it is shown that these items could discriminate the students at Levels 4 and 5. The results are shown in Table 6.16

Table 6.16 The estimates of the conditional probabilities π_{5j} of the procedural knowledge item falling at Level 5 for Model 1

Item: ProT3Q1(*j*=10)			Item: ProT3Q2(*j*=11)		
	Correct π_{510}	Incorrect $1-\pi_{510}$		Correct π_{511}	Incorrect $1-\pi_{511}$
Level 1	0.0952	0.9048	Level 1	0.1010	0.8990
Level 2	0.1950	0.8050	Level 2	0.1957	0.8043
Level 3	0.1400	0.8600	Level 3	0.1343	0.8657
Level 4	0.1875	0.8125	Level 4	0.2386	0.7614
Level 5	0.8025	0.1975	Level 5	0.8034	0.1966
Level 6	0.8732	0.1268	Level 6	0.7961	0.2039

The results indicate that items ProT3Q1 and ProT3Q2 are suitable to be placed at Level 5. This supports the hypothesis that the competencies underlying items ProT3Q1 and ProT3Q2, solving a nested fraction operation where the numerator is a fraction subtraction, and solving one-level nested fraction operations where the numerator is a fraction division, are established at this level.

6.3.2.1.6 Item Analysis at Level 6

Item ProT3Q3 (solving a fraction operation with a two-level nested fraction) was hypothesized as falling at Level 6 (*c*=2, *j*=12). This item was designed to test students' procedural knowledge of complex fraction operations with two or more nested operations.

The results showed that the students at level 6 had a high probability of answering item ProT3Q3 correctly (about 78%), whereas students at level 5 and below had low probabilities of doing so, at less than 20%. Hence, Item ProT3Q3 could discriminate effectively between students at Level 6 and the students at the levels below.

The results indicate that Item ProT3Q3 is suitable to be placed at Level 6. This confirms the hypothesis that the competency of performing a complex fraction operation with two level nested fraction operations is established at Level 6.

Table 6.17 The estimates of the conditional probabilities π_{6j} of the procedural knowledge item falling at Level 6 for Model 1

Item: ProT3Q3(*j*=12)	Correct π_{612}	Incorrect 1-π_{612}
Level 1	0.0945	0.9055
Level 2	0.1949	0.8051
Level 3	0.0907	0.9093
Level 4	0.0922	0.9078
Level 5	0.1971	0.8029
Level 6	0.7819	0.2181

6.3.2.2 Analysis to Estimate Students' Levels in the Progression

Similar to the analysis in the conceptual knowledge dimension, the aim of the procedural level analysis was to estimate students' levels $(\boldsymbol{\theta}|\boldsymbol{\gamma}, \boldsymbol{\pi}, \boldsymbol{x})$ in the procedural dimension. The prior of the networks, which was compiled by the Netica software from the estimates of $\boldsymbol{\gamma}$ as in Equation (5.5) and π_{cj} in Equation (5.2), is presented in Figure 6.11. The results showed that the prior probabilities for the students' levels (estimates of $\boldsymbol{\gamma}$ from Level 1 to Level 6) were 33.90%(γ_1), 26.30%(γ_2), 21.10%(γ_3), 17.90%(γ_4), 0.46%(γ_5), and 0.37%(γ_6).

The results also showed that the prior probabilities of items decreased with the increasing levels of the learning progression. For example, the prior probability for item ProT1Q1 at Level 1 was 82.9 %, and the prior probability for item ProT1Q1 at Level 2 was 63.1%. The smallest prior probability was on item ProT3QT at Level 6, which was 12.3%. The smallest prior probability in the network showed that this item was the most difficult for students, which was consistent with the hierarchical level of the proposed model of fraction learning progression.

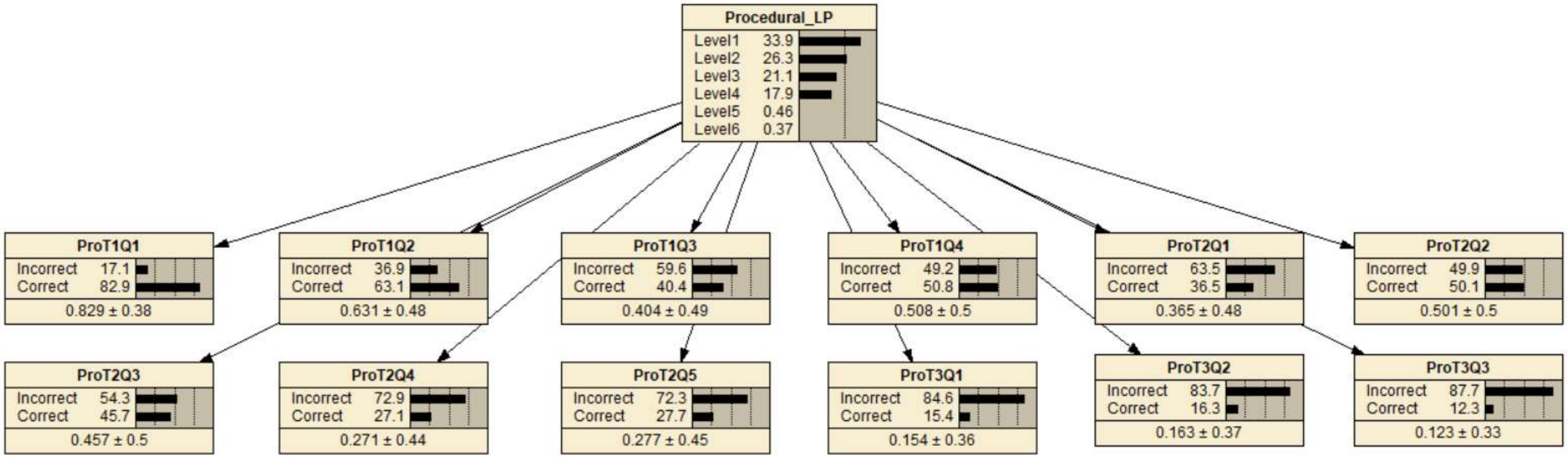


Figure 6.11 Netica Graph of the prior probability of the procedural knowledge dimension

Using Netica, the posterior probabilities of the levels $P(\boldsymbol{\theta}|\boldsymbol{\gamma}, \boldsymbol{\pi}, \boldsymbol{x})$ of 516 students were generated. Figures 6.12 and 6.13 show an example of Bayesian Network estimation, using Netica from the students with ID 452 and 261, with their responses in Tables 6.18 and 6.19 respectively. The results showed that the student with ID 452 answered the items at Level 1 and Level 2 correctly, alongside most of the items at Level 3. The Netica software generated posterior probabilities of the levels for the student with ID 452 as follows: Level 1 is 0.67%; Level 2 is 0.57%; Level 3 is 94.7%; Level 4 is 4.09%, Level 5 is 0.002% and Level 6 is close to 0%. From these results, it can be inferred that the student with ID 452 was estimated at Level 3, as this student has the highest probability to be placed at this level.

In the same way, the results showed that the student with ID 261 answered correctly the items at Level 1, Level 2, and most of the items at Level 3. The student made a mistake in one of the items at Level 4, and correctly answered all the items at Level 5 and 6. The following were the posterior probabilities generated from Netica, from Levels 1 to 6, which are: Level 1 was 0.022%, Level 2 was 0.43%, Level 3 was 12.6%; Level 4 was 23%, Level 5 was 8.42%, and Level 6 was 55.5%. Thus, student 261 was estimated as being at Level 6. The probability for student 261 being at Level 6 is not particularly high (about 55%), which reflects the student's errors in some items at the lower levels.

Table 6.18 Raw Scores of Student 452

Level 1	Level 2	Level 3					Level 4		Level 5		Level 6
ProT1Q1	ProT1Q2	ProT1Q3	ProT1Q4	ProT2Q1	ProT2Q2	ProT2Q3	ProT2Q4	ProT2Q5	ProT3Q1	ProT3Q2	ProT3Q3
1	1	1	1	0	1	1	0	0	0	0	0

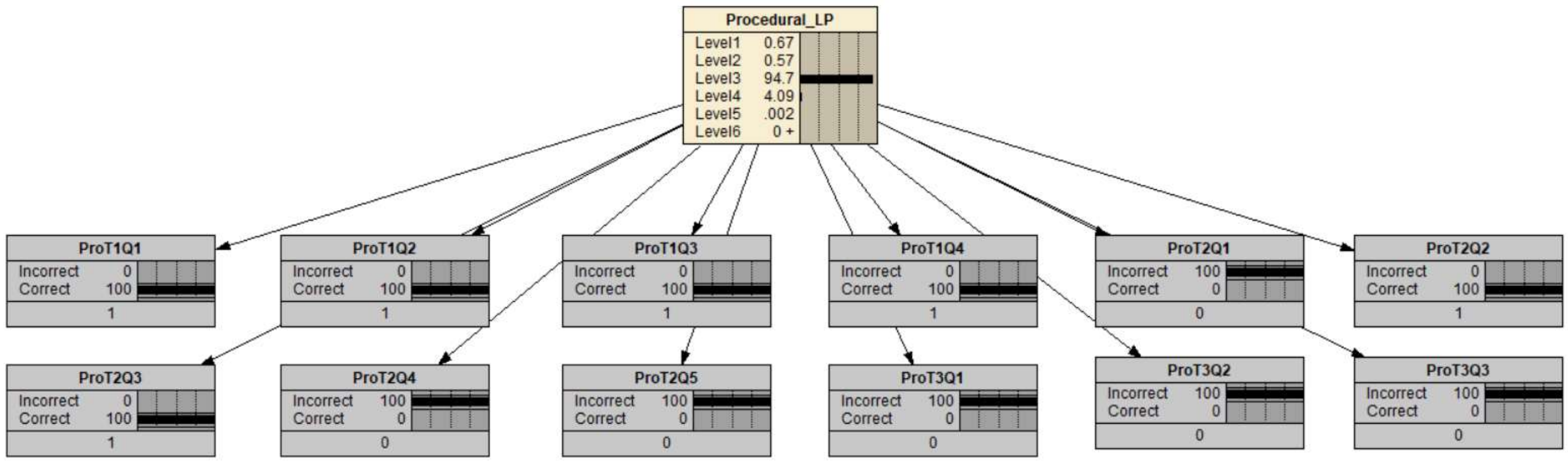


Figure 6.12 A Netica Graph of the posterior probability $(\theta_i|\boldsymbol{\gamma},\boldsymbol{\pi},x_{ij})$ for student 452 (i=452, j=1,…,12).

Table 6.19 Raw Scores of Student 261

Level 1	Level 2	Level 3					Level 4		Level 5		Level 6
ProT1Q1	ProT1Q2	ProT1Q3	ProT1Q4	ProT2Q1	ProT2Q2	ProT2Q3	ProT2Q4	ProT2Q5	ProT3Q1	ProT3Q2	ProT3Q3
1	1	1	1	0	1	1	0	1	1	1	1

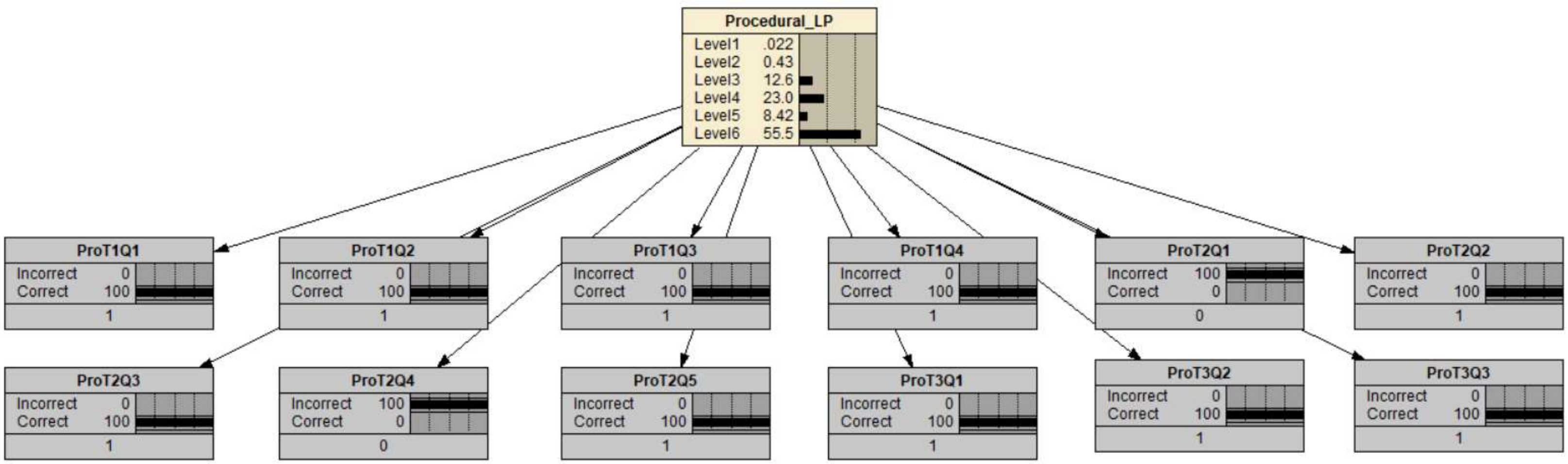


Figure 6.13 A Netica Graph of the posterior probability $(\theta_i|\boldsymbol{\gamma}, \boldsymbol{\pi}, x_{ij})$ for student 261 (i=261, j=1,…,12)..

The same procedure of estimation was applied to all 516 students. Figure 6.14 depicts the distribution of the students' procedural level generated from Model 1 of the Bayesian Network Modeling.

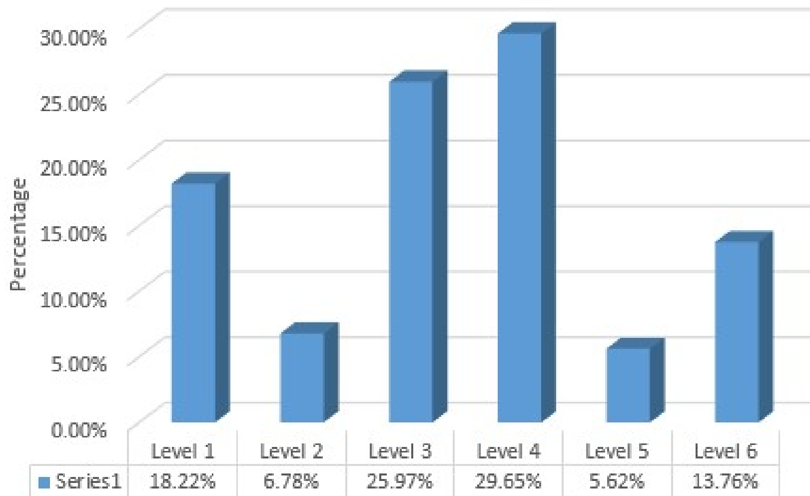


Figure 6.14 The distribution of students' level in the procedural knowledge dimension

The results showed that the highest percentage of students was at Level 4 (29.65%), followed by students at Level 3 (25.97%). At the high levels, the percentage of students at Level 6 was 13.76%, and at Level 5 was 5.62%. At the low levels, 18.22% of students were at Level 1 and 6.78% at Level 2. Most of the students were concentrated in the medium levels of the procedural knowledge dimension, and only a small number of students were at Level 2 and Level 5, which connected the students from the lowest level to the medium level, and from the medium level to the highest level, respectively.

6.3.3 Model 2: Analysis of the Conceptual Knowledge Dimension

As described in Chapter 5, the levels of the learning progression model were represented by several parameters $\boldsymbol{\theta} = (\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_C})$, where $\boldsymbol{\theta_c}$ was the collection of θ_{ci} for c=1,…,C and $i=1,\dots,n$. The dependency between the levels was reflected in the way that the θ_{ci} at the upper level was conditional (dependent) on the θ_{ci} in the lower level. This dependency reflected the assumption that to achieve the competencies at the upper level, the students should already have the competencies at the lower level, which was consistent with the hierarchical levels in the learning progression.

For Model 2, it was assumed that the students who had competencies at a certain level would have a high probability of answering the items at that level correctly. In contrast, students who had no competencies at that level were unlikely to answer the items correctly. In other words, Model 2 estimated two types of conditional probabilities to answer the questions correctly, which were: the probabilities of students who had competencies (θ_{ci}= 1) and the probabilities of the students who had no competencies (θ_{ci}= 0) for each level. These conditional probabilities were incorporated into Model 2 by the parameters π_{cjz} i.e. π_{cj1} and π_{cj} respectively, as shown from Equation (5.9).

6.3.3.1 Item Analysis

The objective of the item analysis in Model 2 was similar to that of Model 1, which was to place the items along the progression levels of the proposed model. However, in Model 2, the item analysis was performed based on two values of π_{cjz}, which are π_{cj1} and π_{cj} (Equation 5.9). The estimates of π_{cj1} represented the conditional probability of the students being able to answer item *j* "correctly" given that the student had competencies at a level c ($\theta_{ci} = 1$). The conditional probability π_{cj1} was similar to the conditional probability π_{cj} in Model 1 in terms of measuring the difficulty of the items. Table 6.20 represents the intervals of π_{cj1} that are used as criteria to assign the items into the levels of the learning progression. These criteria were developed based on how likely the students who had competencies at Level *c* were to answer item *j* correctly.

Table 6.20 The cut-off points of π_{cj} to consider the items to be placed into the levels of the proposed model of the fraction learning progression

Interval	Description
$0.65 \le \pi_{cj1} \le 1$	Item *j* is placed at Level *c*. Students at level *c* are highly likely to answer the item correctly
$0.45 \le \pi_{cj1} < 0.65$	Item *j* is ambiguous to be placed at Level *c* because of the uncertainty of the students at level *c* to answer the item correctly
$0 \le \pi_{cj1} < 0.45$	Item *j* is too difficult to be placed at Level *c*. The students at this level is less likely to answer the item correctly

The estimate of π_{cj0} represents the conditional probability of the student answering item *j* "correctly", given that the student has no competencies at level c

($\theta_{ci} = 0$). In other words, π_{cj} measures the probability of the students answering the item correctly by chance. A large value of π_{cj} indicates that the students are highly likely to answer the item by chance or guesswork. Table 6.21 shows the intervals of π_{cj0}, which are used to consider the location of the items in the proposed levels of fraction learning progression.

Table 6.21 The cut-off points of π_{cj0} to consider the location of the items to be placed into the levels of the proposed model of fraction learning progression

Interval	Description
$0.65 \leq \pi_{cj0} \leq 1$	Highly likely the students at Level *c* answer the item correctly by chance
$0.45 \leq \pi_{cj0} < 0.65$	Uncertain condition whether the students at Level *c* will answer the item by chance correctly or not
$0 \leq \pi_{cj0} < 0.45$	Less likely that the students at Level *c* will answer the item correctly by chance

Based on the criteria developed in Tables 6.20 and 6.21, the item analysis for each level was performed.

6.3.3.1.1 Item Analysis at Level 1

Four items were hypothesized at Level 1 for the conceptual knowledge dimension (*c*=1, *j*=1, 2, 3, 4). These items were items ConT1Q1 (generating a fraction from a part-whole (pie) diagram), ConT2Q1 (shading a pie diagram to represent a fraction less than 1), ConT3Q1 (comparing two fractions less than 1 using part-whole representation), and ConT7Q1 (adding fractions less than 1 using part-whole representation (diagram)).

The results showed that the students who had competencies at Level 1 (θ_{1i} = 1) were likely to answer items ConT1Q1, ConT2Q1, ConT7Q1 correctly, with the probabilities being above 90%, and item ConT3Q1 correctly, with a probability of 78% (above the cut off point of 65%). Moreover, the students who did not have competencies at Level 1 (θ_{1i} = 0) had low probabilities (less than 21%) of answering the items correctly by chance. The results are shown in Table 6.22.

From the results above, it can be inferred that all the items are suitable for placement at Level 1. These results indicate that the competencies underlying these items, generating a fraction from a part-whole (pie) diagram, shading a pie diagram to represent a fraction less than 1, comparing two fractions less than 1 using part-whole representation, and adding fractions less than 1 using part-whole representation (diagram), are established at Level 1.

Table 6.22 The estimates of the conditional probabilities π_{1j} of the conceptual knowledge items falling at Level 1 for Model 2

Items	θ_{1i} = 1		θ_{1i} = 0	
	Correct π_{1j1}	Incorrect $1-\pi_{1j1}$	Correct π_{1j0}	Incorrect $1-\pi_{1j0}$
ConT1Q1	0.9567	0.0433	0.2045	0.7955
ConT2Q1	0.9657	0.0343	0.1982	0.8018
ConT3Q1	0.7837	0.2163	0.1917	0.8083
ConT7Q1	0.9356	0.0644	0.2041	0.7959

6.3.3.1.2 Item Analysis at Level 2

Four items were hypothesized at Level 2 (*c*=2, *j*=4, 5, 6, 8). These items were items Cont1Q2 (generating an equivalent fraction from a pie diagram), ConT1Q3 (generating a fraction from an unequal partition of a pie diagram), ConT3Q2 (comparing fractions less than 1 with different denominators using a part-whole diagram), and ConT7Q2 (adding fractions with different denominators using a part-whole representation/diagram).

The results showed that the students who had competencies at Level 2 (θ_{2i} = 1) were highly likely to answer Items ConT1Q2, ConT3Q2, and ConT7Q2 correctly, with the probabilities being greater than 80%, while item ConT1Q3 was about 65% (the cut-off point of π_{cj1}). These probabilities were used to place each item at a particular level, as presented in Table 6.23. On the other hand, students who did not have competencies at Level 2 (θ_{2i} = 0) were less likely to answer the items correctly by chance, with the probability being less than 38%. The results are shown in Table 6.23.

Table 6.23 The estimates of the conditional probabilities π_{cjz} of the conceptual knowledge items falling at Level 2 for Model 2

Items	θ_{2i} = 1		θ_{2i} = 0	
	Correct π_{2j1}	Incorrect $1-\pi_{2j1}$	Correct π_{2j0}	Incorrect $1-\pi_{2j0}$
ConT1Q2	0.9075	0.0925	0.3792	0.6208
ConT1Q3	0.6558	0.3442	0.1254	0.8746
ConT3Q2	0.8076	0.1924	0.2439	0.7561
ConT7Q2	0.8817	0.1183	0.3666	0.6334

The results discussed above indicate that item ConT1Q3 is rather ambiguous at Level 2 and this item could be moved to the upper level. In contrast, the results show that items ConT1Q2, ConT3Q2, and ConT7Q2 are suitable to be placed at Level 2. These results support the hypothesis that the competencies underpinning items ConT1Q2, ConT3Q2, and ConT7Q2 emerge at this level, namely writing an equivalent fraction for a fraction less than 1, comparing fractions less than 1 with different denominators using a part-whole diagram, and adding fractions with the different denominators using diagram representations.

6.3.3.1.3 Item Analysis at Level 3

Seven items were hypothesized at Level 3 (c=2, j=9,10,11,12,13,14,15,16,17,18). These items were items ConT1Q4 (generating an improper fraction from a pie representation), ConT1Q5 (generating an equivalent of an improper fraction from a pie diagram), ConT2Q2 (shading a pie diagram to represent an improper fraction), ConT3Q3 (comparing two improper fractions using part-whole representation), ConT4Q1 (generating a fraction less than 1 on a number line), ConT4Q2 (generating a fraction less than 1 on a number line with a constraint), and ConT4Q3 (generating fractions greater than 1 on a number line).

The results showed that the probability for students who had competencies at Level 3 (θ_{3i} = 1) would answer all the items correctly were greater than 70%. Moreover, the probability that the students who did not have the competencies at Level 3 (θ_{3i} = 0) would answer the items by chance correctly were small (less than 33%). The results are shown in Table 6.24.

Table 6.24 The estimates of the conditional probabilities π_{3jz} of the conceptual knowledge items falling at Level 3 for Model 2

Level 3

Items	θ_{3i} = 1		θ_{3i} = 0	
	Correct π_{3j1}	Incorrect 1-π_{3j1}	Correct π_{3j0}	Incorrect 1-π_{3j0}
ConT1Q4	0.7573	0.2427	0.1087	0.8913
ConT1Q5	0.7007	0.2993	0.0783	0.9217
ConT2Q2	0.8482	0.1518	0.3213	0.6787
ConT3Q3	0.8416	0.1584	0.1321	0.8679
ConT4Q1	0.8585	0.1415	0.0691	0.9309
ConT4Q2	0.8656	0.1344	0.0419	0.9582
ConT4Q3	0.7828	0.2172	0.0474	0.9525

Therefore, items ConT1Q4, ConT2Q2, ConT1Q5, ConT3Q3, ConT4Q1, ConT4Q2, and ConT4Q3 are suitable for placement at level 3. These results indicate that the competencies corresponding with these items, shading a pie diagram to represent an improper fraction, comparing improper fractions with different denominators using a part-whole diagram, putting a proper fraction on a number line, putting a proper fraction on a number line with a constraint, and putting fractions, including an improper fraction and a mixed number, on a number line, emerge at this level.

6.3.3.1.4 Item Analysis at Level 4

Two items were hypothesized at Level 4 (*c*=4, *j*=16,17). These items are ConT5Q1 (writing the biggest fraction they can) and Item ConT5Q1 (writing the smallest fraction they can).

The results showed that the students who had competencies at Level 4 (θ_{4i} = 1) are likely to answer items ConT5Q1 and ConT5Q2 correctly with a probability greater than 80%. The probability students who did not have Level 4 competencies (θ_{4i} = 0) would answer these items correctly were very small; about 5%. The results are presented in Table 6.25.

From the results discussed above, it can be inferred that Items ConT5Q1 and ConT5Q2 are suitable for placement at Level 4. The results support the hypothesis that the competencies of writing the biggest fraction they can and writing the smallest fraction they can, are established this level.

Table 6.25 The estimates of the conditional probabilities π_{cjz} of the conceptual knowledge items falling at Level 4 for Model 2

Items	θ_{4i}= 1		θ_{4i} = 0	
	Correct π_{4j1}	Incorrect 1-π_{4j1}	Correct π_{4j0}	Incorrect 1-π_{4j0}
ConT5Q1	0.8205	0.1795	0.0516	0.9484
ConT5Q2	0.8356	0.1644	0.0513	0.9487

6.3.3.1.5 Item Analysis at Level 5

Two items (*c*=5, *j*=18,19) were hypothesized at this level: item ConT6Q1 (finding how many fractions lie between two-fractions) and item ConT6Q2 (finding how many fractions lie between two-pseudo successive fractions).

The results show that the students who had competencies at Level 5 ($\theta_5 = 1$) had a high probability (greater than 80%) of answering items ConT6Q1 and ConT6Q2 correctly. The students were unlikely to get correct answers by guessing because the probability of the students who had no competencies at this level of answering these items correctly were very small, less than 5%. The results are shown in Table 6.26.

Table 6.26 The estimates of the conditional probabilities π_{5jz} of the conceptual knowledge items falling at Level 5 for Model 2

Items	θ_{5i} = 1		θ_{5i} = 0	
	Correct π_{5j1}	Incorrect 1-π_{cj1}	Correct π_{5j0}	Incorrect 1-π_{5j0}
ConT6Q1	0.8111	0.1889	0.0345	0.9655
ConT6Q2	0.8164	0.1836	0.0400	0.9600

The results from Table 6.26 indicate that items ConT6Q1 and ConT6Q2 are suitable for placement at Level 5. The results support the hypothesis that the competencies of finding how many fractions lie between two fractions, and finding how many fractions lie between two-pseudo in successive fractions are established at this level.

6.3.3.1.6 Item Analysis at Level 6

Item ConT8Q1 (multiplying fractions using a diagram representation) and ConT8Q2 (dividing fractions using a diagram representation) were hypothesized at Level 6.

The results showed that the students who had competencies at Level 6 (θ_6=1) had probabilities greater than 80% of answering items ConT8Q1 and ConT8Q2 correctly. In contrast, the probability for students who had no competencies at Level 4 (θ_6 = 0) meant that they were unlikely to answer items ConT8Q1 and ConT8Q2 correctly, with a probability of less than 4%.

Table 6.27 The estimates of the conditional probabilities π_{6jz} of the conceptual knowledge items falling at Level 6 for Model 2

Items	θ_{6i} = 1		θ_{6i} = 0	
	Correct π_{6j1}	Incorrect 1-π_{6j1}	Correct π_{6j0}	Incorrect 1-π_{6j0}
ConT8Q1	0.8037	0.1963	0.0323	0.9677
ConT8Q2	0.8052	0.1948	0.0340	0.9661

From the results discussed above, items ConT8Q1 and ConT8Q2 are suitable for placement at Level 6. The results support the hypothesis that the competencies of multiplying fractions and dividing fractions using a diagram representation are established at this level.

6.3.3.2 Analysis to Estimate Students' Levels in the Learning Progression

The purpose of the analysis was to estimate students' levels in the conceptual knowledge dimension based on Model 2 using Bayesian Networks Modelling. As described in Chapter 5, the Netica software was used to estimate the posterior probabilities of the students' levels ($\boldsymbol{\theta}|\boldsymbol{\gamma},\boldsymbol{\pi},\boldsymbol{x}$) (Section 5.2.3.3.2).

Figure 6.15 shows the prior of Bayesian Networks Model 2 in the Netica graph. The prior probabilities in the nodes of levels were compiled from the estimates of $\boldsymbol{\gamma}$ in Equation (5.8). The prior probabilities in the nodes of items were compiled from the estimates of π_{cjz} in Equation (5.9). The estimates of $\boldsymbol{\gamma}$ and π_{cjz} were generated from the WinBugs software using MCMC estimation as detailed in Section 5.2.3.

The results showed that the prior probabilities $\boldsymbol{\gamma}$ for level 1 to level 6 are about 98%, 53%, 25%, 13%, 7%, and 6%. These prior probabilities reflected the belief about the proportion of students' levels in the population. These prior probabilities were updated when individual student's responses, x_{ij}, were entered into the network to produce the posterior probabilities γ of the student. The posterior probabilities γ for each individual student showed the probabilities of the student having the competencies required for those levels.

The results showed that the prior probabilities of getting correct answers for each item (compiled in Netica, based on the estimates of π_{cjz}) were decreased from the items at the lowest level to the highest level. For example, the prior probabilities to answer items at Level 1 correctly were between 77% - 95%. However, the prior probabilities to answer items at Level 2 correctly were between 34%-59%; Level 3 are between 23%-45;, Level 4 are about 15%; Level 5 are about 10% and Level 6 are about 8%. These prior probabilities were consistent with the hierarchical levels of the proposed model of learning progression, which the items in the upper levels being more difficult than the lower items.

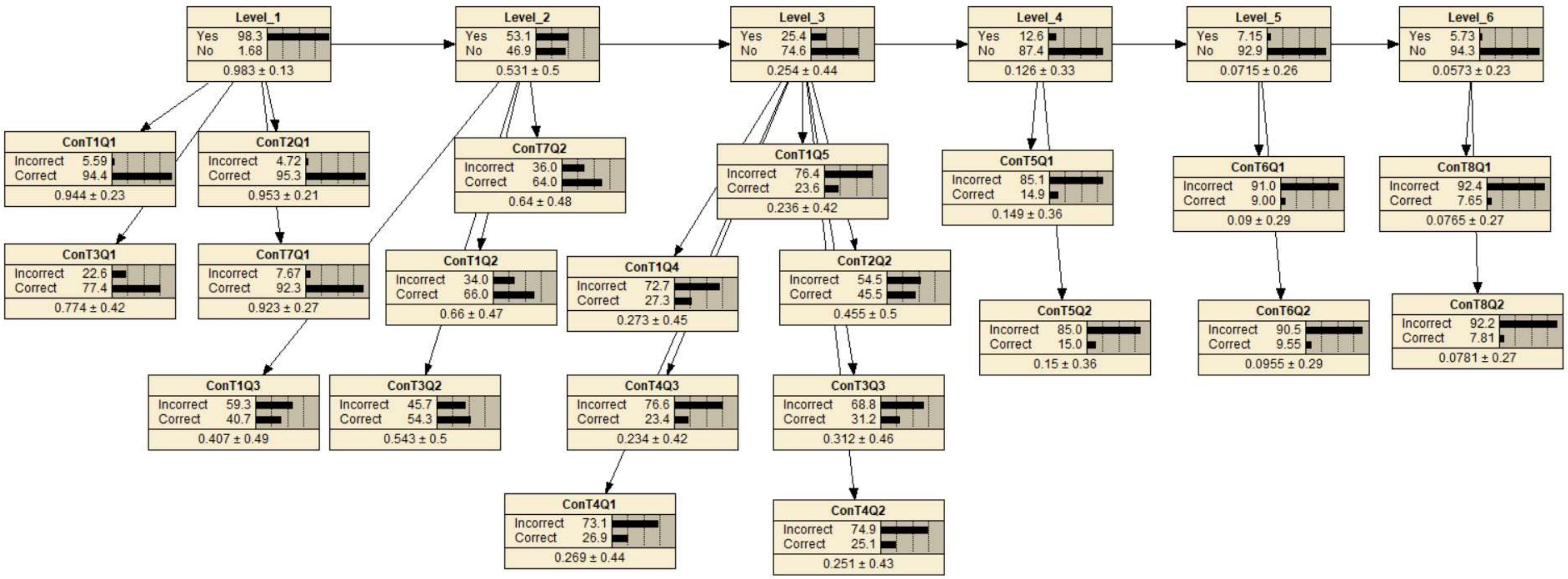


Figure 6.15 A Netica Graph of the prior probability for the Conceptual Knowledge Dimension generated from Model 2 of Bayesian Networks Modelling.

The posterior probabilities of the levels $P(\boldsymbol{\theta}|\boldsymbol{\gamma}, \boldsymbol{\pi}, \boldsymbol{x})$ of 516 students were estimated using the Netica software. As examples, the results of the Netica estimation for two particular students with IDs 44 and 301 are presented in Figures 6.16 and 6.17 respectively. The student with ID 44 correctly answered all the items at Level 1, most of the items at Level 2, and only 1 item at Level 3. The scores are presented in Table 6.28. From these scores, the Netica generated posterior probabilities of the levels as follows: Level 1 was 100%, Level 2 was 83%, and Level 3 to Level 6 were less than 1%. These results showed that the student had a high probability of having the competencies at Levels 1 and 2, but small probability to have the competencies at the upper levels. Based on these results, the student was assigned to Level 2.

The results showed that the student with ID 301 correctly answered all the items at Level 1, made a mistake at Level 2, and correctly answered the items from Levels 3 to 5 (Table 6.29). Netica produced posterior probabilities of the levels for student 301 as follows: for Levels 1 to 4 they were close to 100%, while for Levels 5 and 6 they were less than 2%. These results showed that the student had a high probability of having the competencies at Levels 1 to 4, but had low probability of having the competencies at Levels 5 and 6. Hence, the student with ID 301 was assigned to Level 4.

In a few extreme cases, a few students did not demonstrate their competencies for all the proposed levels. For example, a student with ID 61 had a correct answer for one item at Levels 1, 2 and 3. The posterior probabilities of the levels for this student are presented in Figure 6.18. The results showed that the posterior probabilities for this student were 13% for Level 1 and below 10% for Levels 2 to 6. These results showed that the student had low probabilities of having the competencies of all the levels. Hence, the student was assigned to Level 0. Level 0 was the level for the group of students who did not have sufficient competencies at Level 1 and above.

Table 6.28 The raw sores of the student with ID 44

Level 1				Level 2				Level 3							Level 4		Level 5		Level 6	
ConT1Q1	ConT2Q1	ConT3Q1	ConT7Q1	ConT1Q2	ConT1Q3	ConT3Q2	ConT7Q2	ConT1Q4	ConT1Q5	ConT2Q2	ConT3Q3	ConT4Q1	ConT4Q2	ConT4Q3	ConT5Q1	ConT5Q2	ConT6Q1	ConT6Q2	ConT8Q1	ConT8Q2
1	1	1	1	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0

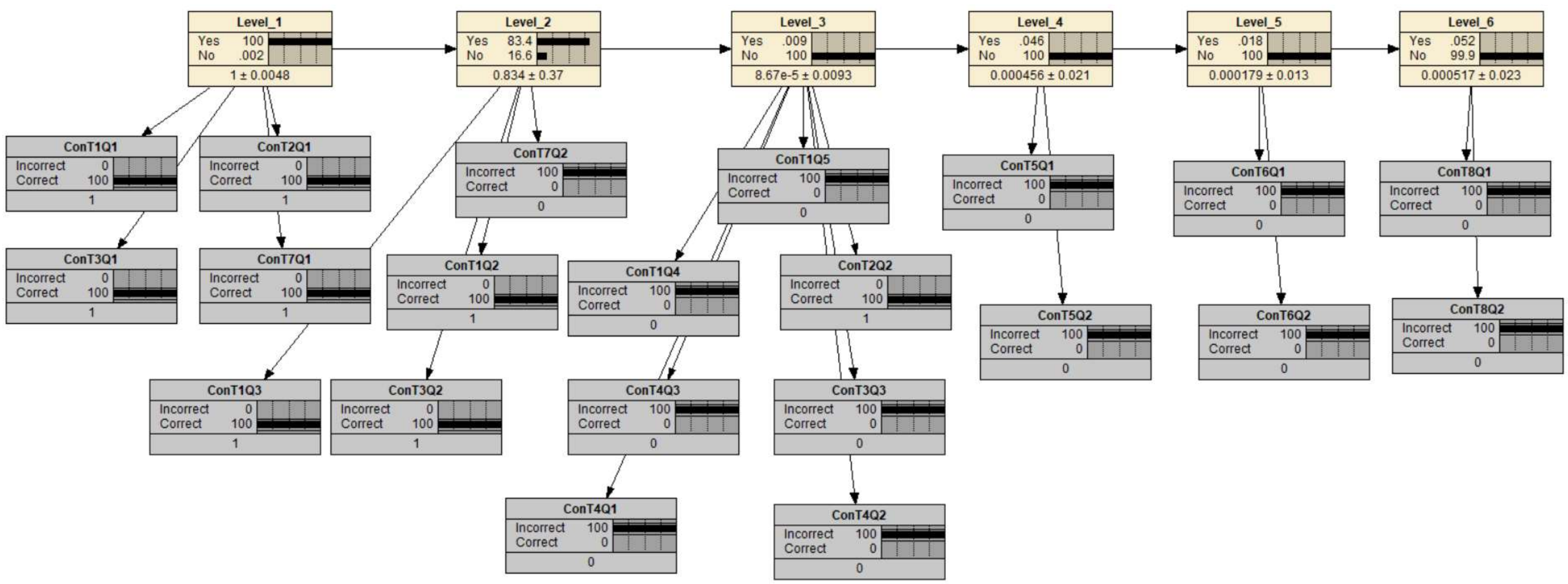


Figure 6.16 A Netica Graph of the posterior probability $(\theta_i|\boldsymbol{\gamma}, \boldsymbol{\pi}, x_{ij})$ for the student with ID 44 (i=44, j=1,…,21).

Table 6.29 The raw sores of student 301

Level 1				Level 2				Level 3							Level 4		Level 5		Level 6	
ConT1Q1	ConT2Q1	ConT3Q1	ConT7Q1	ConT1Q2	ConT1Q3	ConT3Q2	ConT7Q2	ConT1Q4	ConT1Q5	ConT2Q2	ConT3Q3	ConT4Q1	ConT4Q2	ConT4Q3	ConT5Q1	ConT5Q2	ConT6Q1	ConT6Q2	ConT8Q1	ConT8Q2
1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0

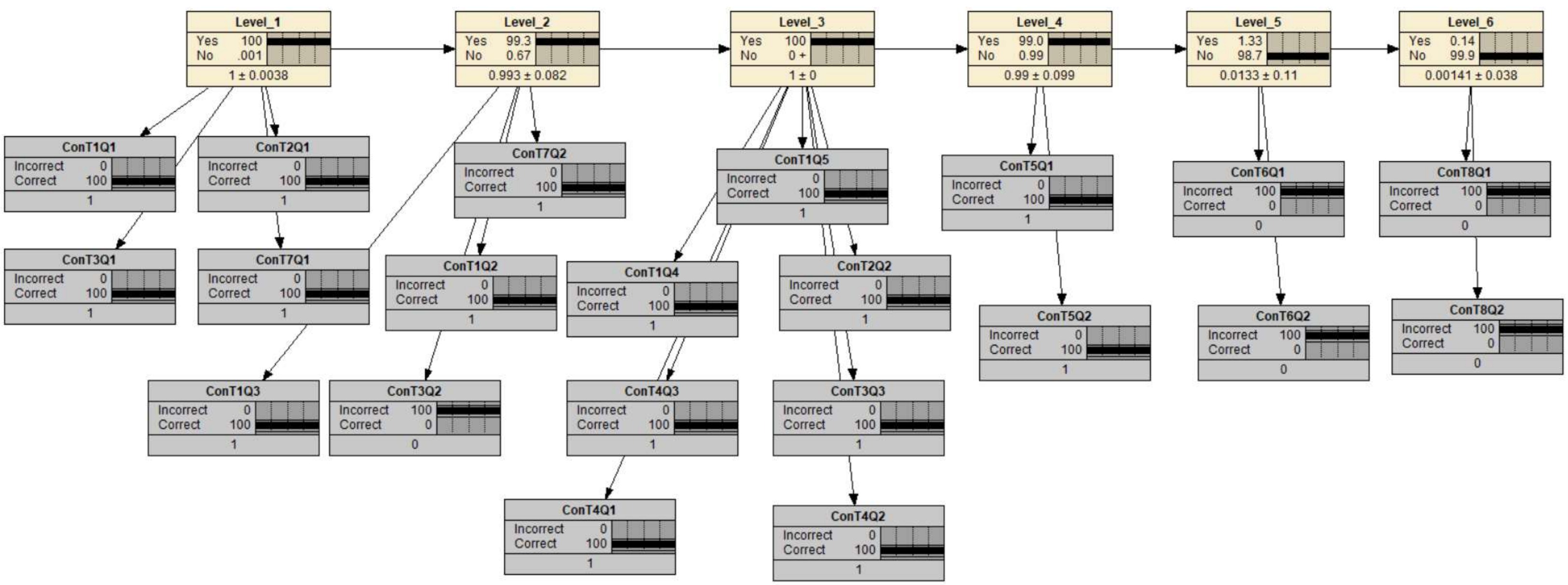


Figure 6.17 A Netica Graph of the posterior probability $(\theta_i|\boldsymbol{\gamma}, \boldsymbol{\pi}, x_{ij})$ for the student with ID 301 (i=301, j=1,…,21).

Table 6.30 The raw scores of student 61

Level 1				Level 2				Level 3							Level 4		Level 5		Level 6	
ConT1Q1	ConT2Q1	ConT3Q1	ConT7Q1	ConT1Q2	ConT1Q3	ConT3Q2	ConT7Q2	ConT1Q4	ConT1Q5	ConT2Q2	ConT3Q3	ConT4Q1	ConT4Q2	ConT4Q3	ConT5Q1	ConT5Q2	ConT6Q1	ConT6Q2	ConT8Q1	ConT8Q2
1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0

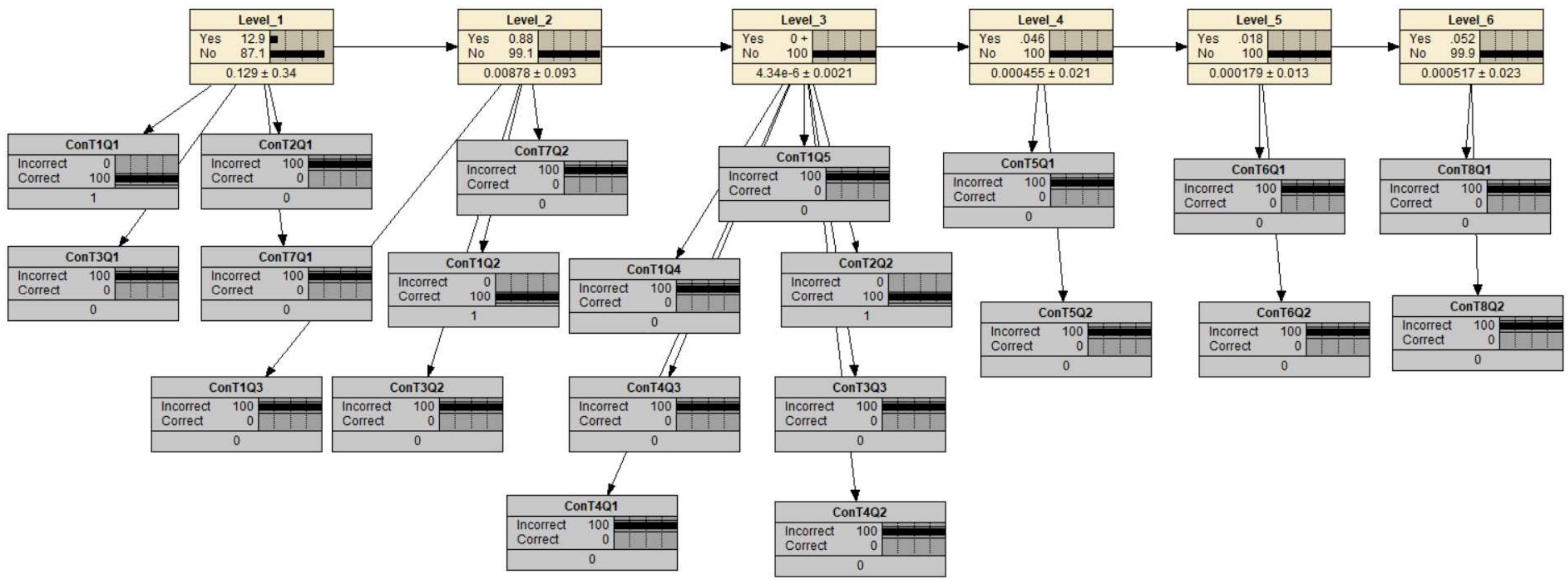


Figure 6.18 A Netica Graph of the posterior probability $(\theta_i|\boldsymbol{\gamma}, \boldsymbol{\pi}, x_{ij})$ for the student with ID 61 (i=61, j=1,…,21).

A similar procedure for assigning students to the conceptual knowledge dimension discussed above was applied to assign all 516 students. The distribution of the students' levels estimated using Model 2 is summarized in Figure 6.19.

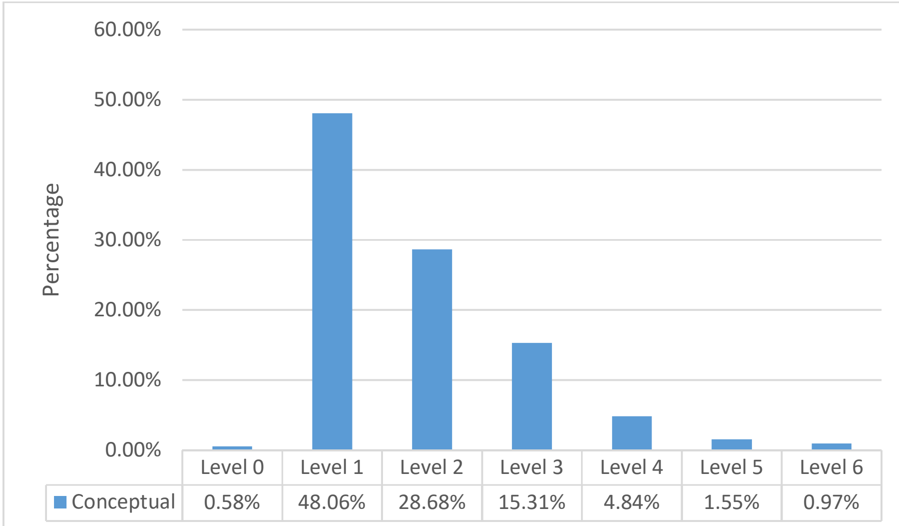


Figure 6.19 The distribution of students' levels on the conceptual knowledge dimension generated from Model 2

The results showed that the highest percentage of students was at Level 1, which was 48.06%. After that, the number of students from Level 2 to Level 6 decreased as follows: 28.68%, 15.31%, 6.40%, 43.84%, 1.55% and 0.97%. The lowest percentage was at Level 0 which is only 0.58%.

6.3.4 Model 2: Analysis of The Procedural Knowledge Dimension

The previous processes being applied for Model 2 were applied for the procedural knowledge dimension.

6.3.4.1 Item Analysis

This section focuses on allocating the items across the levels of the procedural knowledge dimension, based on the conditional probability item analysis obtained from Model 2. The cut-off points presented in Tables 6.20 and 6.21 were used to assign the items into the levels of the learning progression.

6.3.4.1.1 Item Analysis at Level 1

Item ProT1Q1 (adding fractions with the same denominator) was hypothesized at Level 1 (*c*=1, *j*=1). This item was used to test students' procedural competency in adding fractions with the same denominator. The results showed that the students who had competency at this level (θ_{1i} = 1) were highly likely to answer item ProT1Q1 correctly, with the probability being about 94%, while students who did not have the competency at Level 1 (θ_{1i} = 0) were unlikely to answer this item correctly, with a probability of about 21%. The results are shown in Table 6.31.

The results indicate that item ProT1Q1 is suitable to be placed at Level 1 of the procedural knowledge dimension. These results support the hypothesis that the competency of adding fractions with the same denominator that underpins item ProT1Q1 is established at Level 1.

Table 6.31 The estimates of the conditional probabilities π_{1jz} of the procedural knowledge items at Level 1 for Model 2

Items	θ_{1i} = 1		θ_{1i} = 0	
	Correct π_{1j1}	Incorrect $1-\pi_{1j1}$	Correct π_{1j0}	Incorrect $1-\pi_{1j0}$
ProT1Q1	0.9410	0.0590	0.2070	0.7930

6.3.4.1.2 Item Analysis at Level 2

Item ProT1Q2 (adding fractions with different denominators) was hypothesized as being at Level 2 (*c*=2, *j*=2). This item was used to test students' competency in adding proper fractions with different denominators.

The results showed that the students who had competency at Level 2 (θ_{2i} = 1), had a probability of about 91% to answer the item correctly. In contrast, students who did not have competency at Level 2 (θ_{2i} = 0) only had about 17% probability of answering this item correctly. These results indicate that Item ProT1Q2 is suitable to be placed at Level 2, to support the hypothesis that the competency of adding proper fractions with different denominators is established at this level. The results are shown in Table 6.32.

Table 6.32 The estimates of the conditional probabilities π_{2jz} of the procedural knowledge items at Level 2 for Model 2

Items	θ_{2i} = 1		θ_{2i} = 0	
	Correct π_{2j1}	Incorrect $1-\pi_{2j1}$	Correct π_{2j0}	Incorrect $1-\pi_{2j0}$
ProT1Q2	0.9112	0.0888	0.1720	0.8280

6.3.4.1.3 Item Analysis at Level 3

Four items were hypothesized at Level 3 for the procedural knowledge dimension (*c*=3, *j*=3, 4, 5, 6, 7). These items were ProT1Q4 (adding fractions with a mixed number), ProT1Q3 (subtracting a fraction from a whole number), ProT2Q1 (multiplying a fraction with a fraction), ProT2Q2 (multiplying a fraction with a whole number), and ProT2Q3 (dividing a fraction with a fraction).

The results showed that the students who had competencies at Level 3 ($\theta_{3i} = 1$) were likely to answer items ProT1Q3, ProT1Q4, ProT2Q2, and ProT2Q3 correctly, with the probabilities being greater than 80%, while the probability for item ProT2Q1 was about 77%. Moreover, the students who had no competencies at Level 3 (θ_{3i} = 0) were unlikely to answer the items correctly, with the probabilities being less than 39%. The results are presented in Table 6.33.

The results indicate that items ProT1Q3, ProT1Q4, ProT2Q1, ProT2Q2 and ProT2Q3 are suitable to be placed at Level 3. These results support the hypothesis that the competencies of subtracting a fraction from a whole number, adding a fraction with a mixed number, multiplying a fraction with a fraction, multiplying a fraction with a whole number, and dividing a fraction with a fraction, are established at Level 3.

Table 6.33 The estimates of the conditional probabilities π_{3jz} of the procedural knowledge items at Level 3 for Model 2

Items	θ_{3i} = 1		θ_{3i} = 0	
	Correct π_{3j1}	Incorrect $1-\pi_{3j1}$	Correct π_{3j0}	Incorrect 1-π_{3j0}
ProT1Q3	0.8037	0.1963	0.2017	0.7983
ProT1Q4	0.9255	0.0745	0.3887	0.6113
ProT2Q1	0.7685	0.2315	0.1143	0.8857
ProT2Q2	0.9277	0.0723	0.356	0.644
ProT2Q3	0.9351	0.0649	0.2322	0.7678

6.3.4.1.4 Item Analysis at Level 4

Two items were hypothesized at Level 4 (*c*=4, *j*=8, 9). These items were ProT2Q4 (multiplying a mixed number with a mixed number) and ProT2Q5 (dividing a mixed number with a whole number).

The results showed that the students who had competencies at Level 4 (θ_{4i} = 1) were likely to answer Items ProT2Q4 (about 70%) and ProT2Q5 (about 80%) correctly, while students who did not have competencies at Level 4 (θ_{4i}= 0) were unlikely to answer the items correctly, with the probability being less than 13%. The results are shown at Table 6.34.

The results indicate that Items ProT2Q4 and ProT2Q5 are suitable to be placed at Level 4. These results support the hypothesis that the competencies of multiplying a mixed number with a mixed number, and dividing a mixed number with a whole number are established at this level.

Table 6.34 The estimates of the conditional probabilities π_{4jz} of the procedural knowledge items at Level 5 for Model 2

Items	θ_{4i} = 1		θ_{4i} = 0	
	Correct π_{4j1}	Incorrect $1-\pi_{4j1}$	Correct π_{4j0}	Incorrect $1-\pi_{4j0}$
ProT2Q4	0.6917	0.3083	0.1283	0.8717
ProT2Q5	0.8032	0.1968	0.1071	0.8929

6.3.4.1.5 Item Analysis at Level 5

Items ProT3Q1 (solving a nested fraction operation with the numerator is a fraction subtraction) and ProT3Q2 (solving a nested fraction operation with the numerator is a fraction division) were hypothesized at Level 5 (*c*=5, *j*=10,11).

The results showed that the students who had competencies at Level 5 (θ_{5i} = 1) were highly likely to answer items ProT3Q1 and ProT3Q2 correctly, with the probabilities being about 85% and 77% , while students who had no procedural competencies (θ_{5i} = 0) were unlikely to answer the items correctly with the probability being less than 12%. The results are presented in Table 6.35.

The results indicate that items ProT3Q1 and ProT3Q2 were fit to be placed at Level 5. The results support the hypothesis that the competencies underlying these items, solving a nested fraction operation with the numerator is a fraction subtraction, and solving a nested fraction operation with the numerator is a fraction division, are established at this level.

Table 6.35 The estimates of the conditional probabilities π_{5jz} of the procedural knowledge items at Level 5 for Model 2

Items	θ_{5i} = 1		θ_{5i} = 0	
	Correct π_{5j1}	Incorrect $1-\pi_{5j1}$	Correct π_{5j0}	Incorrect 1-π_{5j0}
ProT3Q1	0.8451	0.1549	0.0905	0.9095
ProT3Q2	0.7664	0.2336	0.1243	0.8757

6.3.4.1.6 Item Analysis at Level 6

Item ProT3Q3 (solving a fraction operation with a two-level nested fraction) was hypothesized at Level 6 (*c*=6, *j*=12). This item was designed to test students' fluency in a complex fraction operation.

The results showed that the students who had competencies at Level 6 (θ_{6i} = 1) had a high probability of answering this item correctly, at about 78%, while students who did not have the competencies at this level (θ_{6i} = 0) had a low probability (less than 5%) of answering Item ProT3Q3 successfully. The results are presented in Table 6.36.

The results indicate that item ProT3Q3 is suitable to be placed at Level 6. These results support the hypothesis that the competency underlying item ProT3Q3, solving complex fraction operations with two or more nested fraction operations, is established at this level.

Table 6.36 The estimates of the conditional probabilities π_{6jz} of the procedural knowledge items at Level 5 for Model 2

Items	θ_{6i} = 1		θ_{6i} = 0	
	Correct π_{cj1}	Incorrect $1-\pi_{cj1}$	Correct π_{cj0}	Incorrect 1-π_{cj0}
ProT3Q3	0.7821	0.2179	0.0454	0.9546

6.3.4.2 Analysis to Estimate Students' Levels in the Progression

Similar to the process of estimating students' levels in the conceptual knowledge dimension generated from Model 2 (Section 6.3.2.1.2), the prior probabilities of

Bayesian Networks for procedural knowledge were also built based on the estimates of $\boldsymbol{\gamma}$ in Equation (5.8) and π_{cjz} in Equation (5.9). Using the Netica software, the prior probabilities in the nodes of levels were compiled from the estimates of $\boldsymbol{\gamma}$ in Equation (5.8), and the prior probabilities in the nodes of items were compiled from the estimates of π_{cjz} in Equation (5.9). Both $\boldsymbol{\gamma}$ and π_{cjz} were generated from the WinBugs software using MCMC estimation, as detailed in Section 5.2.3.1. The priors of Bayesian Network generated from the Netica are presented in Figure 6.20.

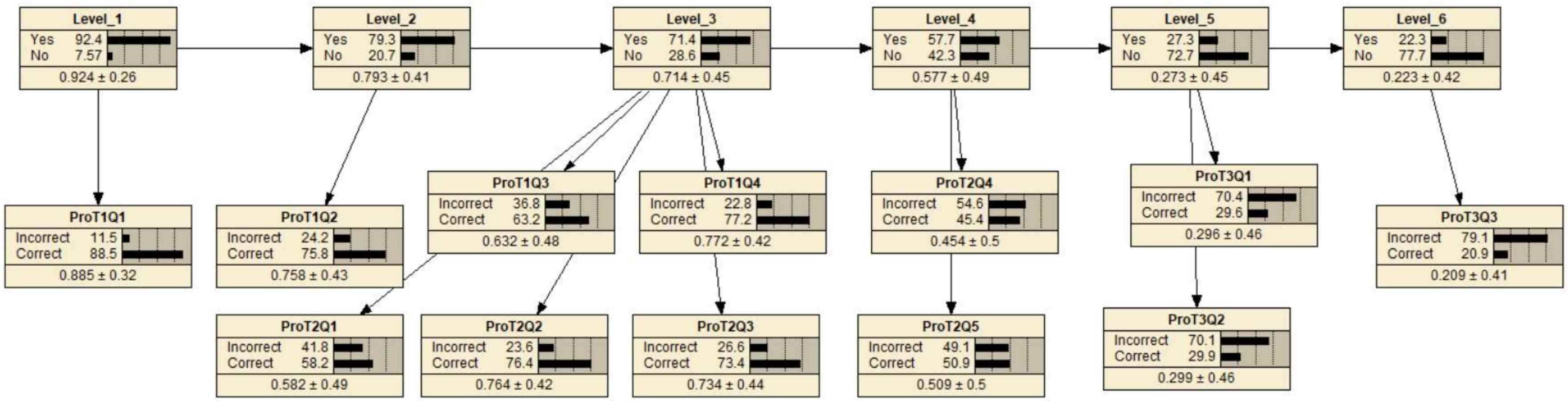


Figure 6.20 A Netica Graph of the prior probability for the Procedural Knowledge Dimension generated from Model 2 of Bayesian Networks Modelling.

Based on the students' responses, the Netica software updated the priors in Figure 6.20 to estimate the students' levels as described in Section 5.2.3.3.2. Netica estimated the posterior probabilities of the levels $P(\boldsymbol{\theta}|\boldsymbol{\gamma},\boldsymbol{\pi},\boldsymbol{x})$ of 516 students. Figures 6.21 and 6.22 show the examples of Bayesian Networks' estimation using the Netica software for students with IDs 110 and 376. Their raw scores are presented in Table 6.37 and 6.38 respectively.

The results showed that the student with ID 110 correctly answered the items at Level 1 and Level 2, and most of the items at Level 3. The student did not correctly answer all the items at Levels 4, 5 and 6. The results showed that the posterior probabilities for the student were: 99.6%, 99.3%, and 98.4% for Level 1 to Level 3 respectively; 15% at Level 4; and less than 6% at Levels 5 and 6. Hence, the student was assigned to Level 3 because the student had high probabilities to have the competencies at Level 3 and below, but had low probabilities to have the competencies at the upper levels. The student posterior probabilities are presented in the node level of the Netica graph in Figure 6.21.

Similarly, the student with ID 376 correctly answered all the items at Level 1, Level 2, Level 3 and Level 5 but a made a mistake in Levels 4 and 6. Using the Netica software, the posterior probabilities for the student with ID 376 were generated and presented in Figure 6.22. The results showed that the posterior probabilities for the student with ID 376 were greater than 90% for Level 1 to Level 5, and 39.5% for Level 6. Hence, because the student had a high probability to have the competencies at Level 5 and below, but had a low probability to have the competencies at the upper level (Level 6), the student with ID 376 was assigned to Level 5.

In practice, students might not have competencies from all the proposed levels. In this situation, the students were assigned to level 0. For example, the student with ID 9 had zero scores for all the given items (presented in Table 6.39). The posterior probabilities for student 9 were generated using the Netica software and are presented in Figure 6.23. The results showed that the posterior probabilities for the students were about 15% at Level 1 and less that 2 % for the upper levels. Hence, the student was assigned to Level 0.

Table 6.37 The raw sores of student with ID 110

Level 1	Level 2	Level 3					Level 4		Level 5		Level 6
ProT1Q1	ProT1Q2	ProT1Q3	ProT1Q4	ProT2Q1	ProT2Q2	ProT2Q3	ProT2Q4	ProT2Q5	ProT3Q1	ProT3Q2	ProT3Q3
1	1	0	1	1	1	1	0	0	0	0	0

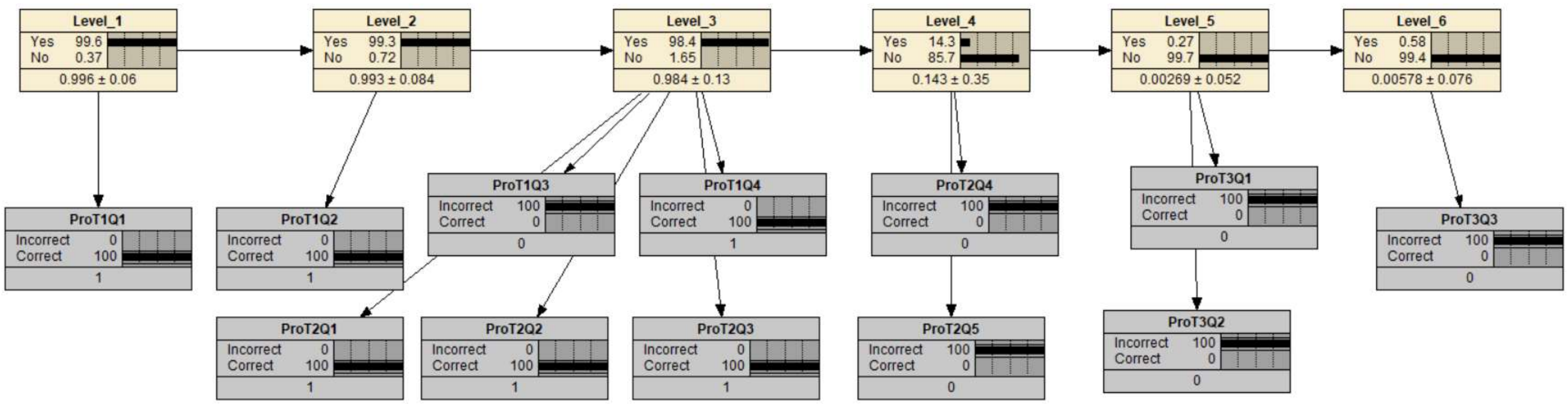


Figure 6.21 A Netica Graph of the posterior probability $P(\theta_i|\boldsymbol{\gamma}, \boldsymbol{\pi}, x_{ij})$ for the student with ID 110 (i=110, j=1,…,12).

Table 6.38 The raw sores of the student with ID 376

Level 1	Level 2	Level 3					Level 4		Level 5		Level 6
ProT1Q1	ProT1Q2	ProT1Q3	ProT1Q4	ProT2Q1	ProT2Q2	ProT2Q3	ProT2Q4	ProT2Q5	ProT3Q1	ProT3Q2	ProT3Q3
1	1	1	1	1	1	1	1	0	1	1	0

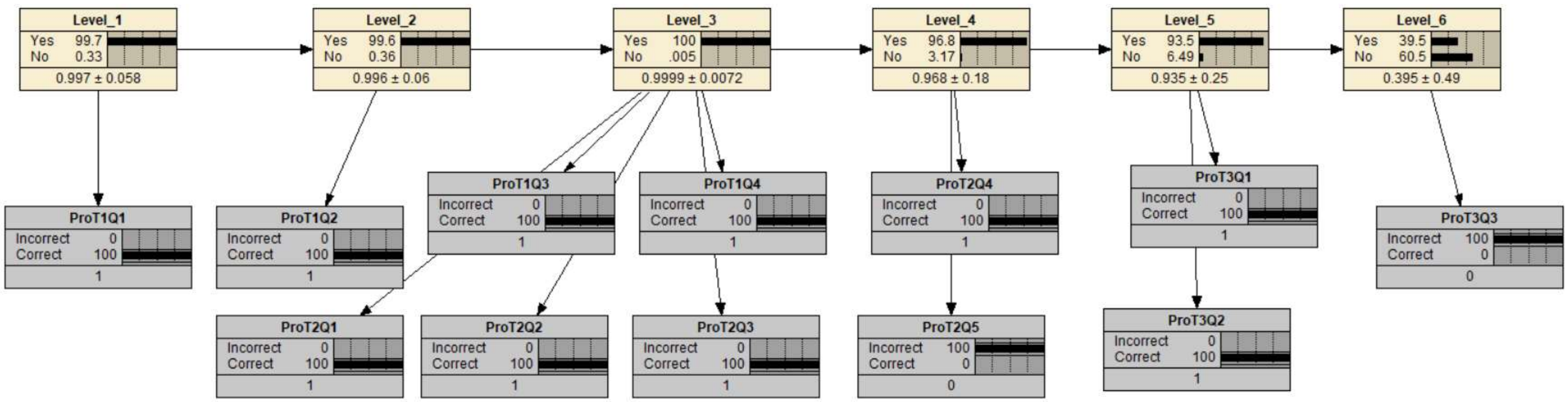


Figure 6.22 A Netica Graph of the posterior probability $P(\theta_i|\boldsymbol{\gamma},\boldsymbol{\pi},x_{ij})$ for the student with ID 376 (i=110, j=1,…,12)..

Table 6.39 The raw sores of the student with ID 9

Level 1	Level 2	Level 3					Level 4		Level 5		Level 6
ProT1Q1	ProT1Q2	ProT1Q3	ProT1Q4	ProT2Q1	ProT2Q2	ProT2Q3	ProT2Q4	ProT2Q5	ProT3Q1	ProT3Q2	ProT3Q3
0	0	0	0	0	0	0	0	0	0	0	0

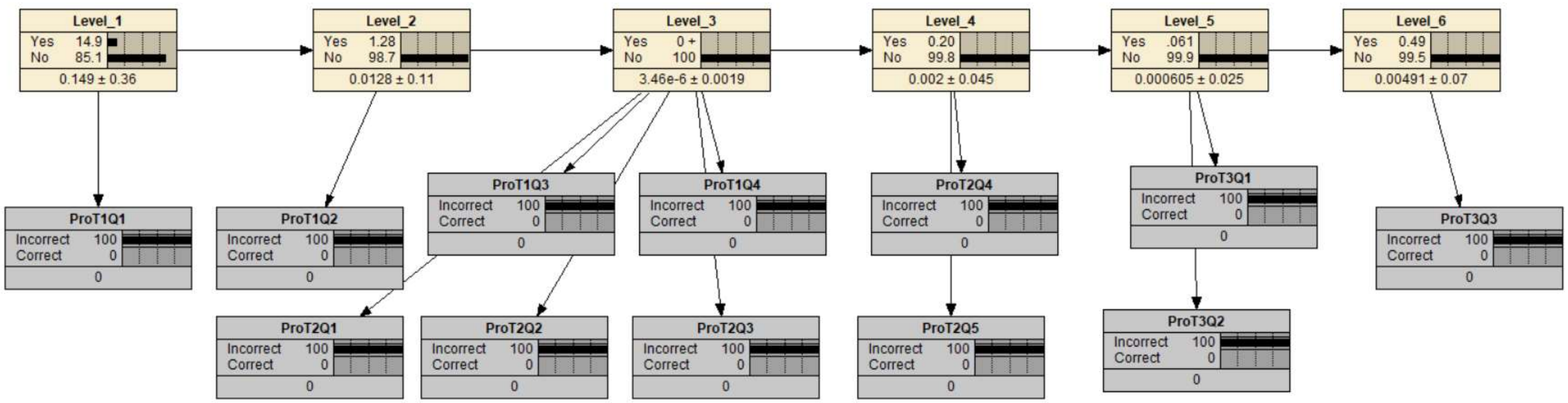


Figure 6.23 A Netica Graph of the posterior probability $P(\theta_i|\boldsymbol{\gamma}, \boldsymbol{\pi}, x_{ij})$ for the student with ID 9 (i=9, j=1,…,12).

The same procedure for estimating students' levels discussed above was applied to all the 516 students who participated in this study. The distribution of the students' levels are presented in percentage values in Figure 6.24.

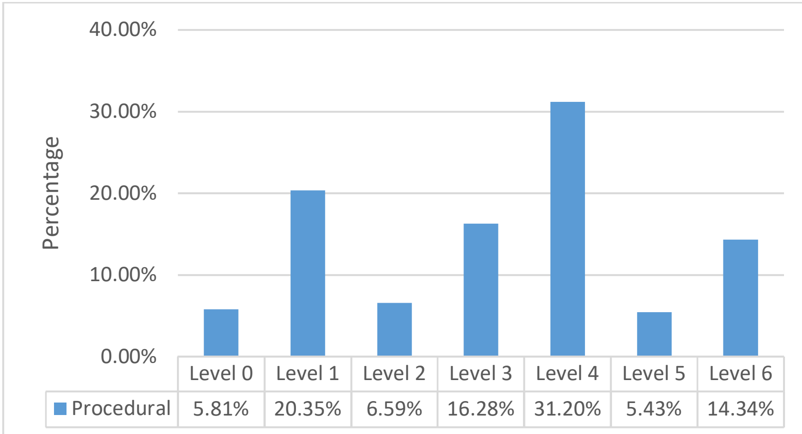


Figure 6.24 The distribution of the students' levels on the procedural knowledge dimension generated from Model 2

From Figure 6.24, the majority of students were at Level 4 (about 31.20%), followed by students at Level 1 (about 20.35%). The percentages for students at Levels 3 and 6 were 16.28% and 14.34% respectively, while for students at Levels 2 and 5 were 6.59% and 5.43%. Only about 5.81% of students were at Level 0.

6.3.5 Validation of Fraction Learning Progression

The aim of the empirical validation for the proposed learning progression was to seek evidence to support the hypothesis that the progression of students in learning fractions followed the hypothesized levels of the conceptual and procedural knowledge dimensions. Two types of analyses were performed to validate the proposed level of fraction learning progression: the item analysis and the analysis at student level. In the section that follows, the results of the item analysis produced from Models 1 and 2 were compared and used to validate the competencies in the progression levels. Subsequently, the analysis at student level was performed to examine how many students deviated from the order hypothesized by the learning progression.

6.3.5.1 Validation using Item Analysis

The item analysis collected evidence of the location of the items on the hypothesized levels of the conceptual and procedural knowledge dimensions. Allocating the items into the hypothesized levels supports the inferences about the competencies underlying the items for each level. The analysis is presented in two consecutive sections, the item analysis for the conceptual knowledge dimension and the item analysis for the procedural knowledge dimension.

6.3.5.2 Item Analysis for the Conceptual Knowledge Dimension

The results of the analysis using Model 1 and Model 2 showed that items Cont1Q1, onT2Q1, ConT3Q1, and ConT7Q1 were placed at Level 1. These items reflected the competencies of generating a fraction from a part-whole (pie) diagram, shading a pie diagram to represent a fraction less than 1, comparing two fractions less than 1 using part-whole representation, and adding fractions less than 1 using part-whole representation respectively. Hence, these results support the hypothesis that understanding fractions as a representation of part-whole emerge at the lowest level of the conceptual knowledge dimension. The results are presented in Table 6.40

Table 6.40 Item analysis of Levels 1 to 6 of the conceptual knowledge dimension based on Model 1 and Model 2

Item	Model 1- Assigned Level	Model 2- Assigned Level
Cont1Q1	Level 1	Level 1
ConT2Q1	Level 1	Level 1
ConT3Q1	Level 1	Level 1
ConT7Q1	Level 1	Level 1
Cont1Q2	Level 2	Level 2
ConT1Q3	**Level 3**	**Level 2**
ConT3Q2	**ambiguous**	**Level 2**
ConT7Q2	Level 2	Level 2
ConT1Q4	Level 3	Level 3
ConT1Q5	Level 3	Level 3
ConT2Q2	Level 3	Level 3
ConT3Q3	Level 3	Level 3
ConT4Q1	Level 3	Level 3
ConT4Q2	Level 3	Level 3
ConT4Q3	Level 3	Level 3
ConT5Q1	Level 4	Level 4

ConT5Q2	Level 4	Level 4
ConT6Q1	Level 5	Level 5
ConT6Q2	Level 5	Level 5
ConT8Q1	Level 6	Level 6
ConT8Q2	Level 6	Level 6

The results from the item analysis from both models indicated that items ConT1Q2 (generating an equivalent fraction from a pie diagram) and ConT7Q2 (adding fractions with different denominators using a part-whole representation/diagram) should be placed at Level 2. However, Models 1 and 2 produced different results on placing items ConT1Q3 and ConT3Q2. For item ConT1Q3 (generating a fraction from an unequal partition of a pie diagram), the results from Model 1 indicated that this item should be placed at Level 3, while Model 2 showed that this item should be placed at Level 2. From the previous tables (Tables 6.5 and 6.23), the results showed that the conditional probability for students at Level 2 to answer this item correctly was 35.55% for Model 1 and 65.58% for Model 2. The result from Model 1 showed that this item was too difficult to be placed at Level 2, while the result from Model 2 showed an uncertainty of about 34.42% (obtained from 100%-65.58%) to place this item at Level 2. Combining these results, it was decided to move item ConT1Q3 to the upper level (Level 3).

For item ConT3Q2 (compare proper fractions with different denominators), the result from Model 1 showed that this item was ambiguous, while the result from Model 2 indicated that this item should be placed at Level 2. The results from Model 1 and Model 2, which are presented in Tables 6.5 and 6.23, showed that the conditional probability for students at Level 2 to solve this item correctly was 52.22 % and 80.76% respectively. As the result from Model 1 did not show that Item ConT3Q2 should be placed at 2 or 3 (ambiguous), while the result from Model 2 clearly showed that the item should be placed at Level 2, it was decided that this item should be retained at Level 2.

Based on the discussion above, items ConT1Q2, ConT3Q2, and ConT7Q2 are placed at Level 2. These results support the hypothesis that the competencies underpin the items writing an equivalent fraction for a fraction less than 1, comparing proper fractions with different denominators, and adding fractions with different denominators in diagram representations emerge at Level 2 of the conceptual knowledge dimension.

Table 6.41 The validated competencies for each level of the conceptual knowledge dimension

Level	Competency
Level 1	Generating a fraction from a part-whole (pie) diagram
	Shading a pie diagram to represent a fraction less than 1
	Comparing two fractions less than 1 using part-whole representation
	Adding fractions less than 1 using part-whole representation
Level 2	Writing an equivalent fraction for a fraction less than 1
	Comparing proper fractions with different denominators
	Adding fractions with different denominators using a diagram
Level 3	Generating a fraction from an unequal partition of a pie diagram
	Generating an improper fraction from a pie diagram representation
	Generating an equivalent of an improper fraction from a pie diagram
	Shading a pie diagram to represent an improper fraction
	Comparing improper fractions with different denominators using a part-whole diagram
	Putting a proper fraction on a number line,
	Putting a proper fraction on a number line with a constraint
	Putting fractions including an improper fraction and a mixed number on a number line
Level 4	Writing the biggest fraction they can
	Writing the smallest fraction they can
Level 5	Finding how many fractions lie between two fractions
	Finding how many fractions lie between two pseudo successive fractions
Level 6	Multiplying fractions using a diagram representation
	Dividing fractions using a diagram representation

For the items at Levels 3 to 6, Model 1 and Model 2 produced the same results. For Level 3, the results from Models 1 and 2 showed that items ConT1Q4, ConT1Q5, ConT2Q2, ConT3Q3, ConT4Q1, ConT4Q2, and ConT4Q3 should be placed at Level 3. However, there was an additional item, ConT1Q3, at this level which came from Level 2. These results indicate that the competencies for generating a fraction from an unequal partition of a pie diagram (ConT1Q3), generating an improper fraction from a pie representation (ConT1Q4), generating an equivalent of improper fraction from a pie diagram (ConT1Q5); shading a pie diagram to represent an improper fraction (ConT2Q2), comparing improper fractions with different denominators using a part-whole diagram (ConT3Q3), putting a proper fraction on a number line (ConT4Q1), putting a proper fraction on a number line with a constraint (ConT4Q2), and putting fractions including an improper fraction and a mixed number on a number line (ConT4Q3) emerge at this level.

Similarly, Models 1 and 2 had the same results, placing items ConT5Q1 and ConT5Q2 at Level 4, items ConT6Q1 and ConT6Q2 at Level 5, and items ConT8Q1 and ConT8Q2 at Level 6. These results support the hypothesis that the competencies for writing the biggest fraction they can (ConT5Q1), and writing the smallest fraction they can (ConT5Q2) emerge at Level 4; the competencies of finding how many fractions lie between two fractions (ConT6Q1) and finding how many fractions lie between two pseudo successive fractions (ConT6Q2) emerge at Level 5; and the competencies of multiplying fractions using a diagram representation (ConT8Q1)and dividing fractions using a diagram representation (ConT8Q1) emerge at Level 6.

From the results discussed above, the revised competencies in the conceptual knowledge dimension, based on the location of the items in the model of fraction learning progression, are summarized in Table 6.41 above.

6.3.5.3 Item Analysis for the Procedural Knowledge Dimension

The results of procedural item analysis using Model 1 and Model 2 are presented in Table 6.42. The results show that both Models 1 and 2 placed item ProT1Q1 at the lowest level of the procedural knowledge dimension. This supports the inference that the competency for adding fractions with the same denominator emerges at level 1. Similarly, the results from both models also placed item ProT1Q2 at Level 2. These results support the inference that the competency of adding fractions with different denominator is established at Level 2.

Table 6.42 Item analysis of Levels 1 to 6 based on Model 1 and Model 2 in the procedural knowledge dimension

Item	Model 1- Assigned Level	Model 2- Assigned Level
ProT1Q1	Level 1	Level 1
ProT1Q2	Level 2	Level 2
ProT1Q3	Level 3	Level 3
ProT1Q4	Level 3	Level 3
ProT2Q1	**Ambiguous**	**Level 3**
ProT2Q2	Level 3	Level 3
ProT2Q3	Level 3	Level 3
ProT2Q4	Level 4	Level 4
ProT2Q5	Level 4	Level 4
ProT3Q1	Level 5	Level 5
ProT3Q2	Level 5	Level 5
ProT3Q3	Level 6	Level 6

Next, for Level 3, the results from Model 2 placed items ProT1Q3, ProT1Q4, ProT2Q1, ProT2Q2, and ProT2Q3 at Level 3. However, the results from Model 1 showed that item ProT2Q1 was ambiguous because the students at this level had only a 56% probability of answering the item correctly. In contrast, Model 2 estimated students who had competencies at Level 3 had a 76.85% chance of answering the item correctly with a small probability of guessing, 11.43%. Based on these results, item ProT2Q1 was retained at Level 3. These results support the inference that the competencies underlying the items, which are adding fractions with a mixed number, subtracting a fraction from a whole number, multiplying a fraction with a fraction, multiplying a fraction with a whole number, and dividing a fraction with a fraction, emerge at this level.

The analyses from Models 1 and 2 had the same results in locating the items for Levels 4 to 6. The results from both Models indicate that items ProT2Q4 and ProT2Q5 should be placed at Level 4; items ProT3Q1 and ProT3Q2 at Level 5; and item ProT3Q3 at Level 6. These results support the hypothesis that the competencies underlying items ProT2Q4 and ProT2Q5, multiplying a mixed number with a mixed number, and dividing a mixed number with a whole number, are established at Level 4; the competencies underlying items ProT3Q1 and ProT3Q2, solving a nested fraction operation with the numerator is a fraction subtraction, and solving a nested fraction operation with the numerator is a fraction division, are established at Level 5; and the competency underlying item ProT3Q3, solving a fraction operation with two-level nested fractions, is established at Level 6.

The competencies in the procedural knowledge dimension which have been revised based on the results of the analyses from Models 1 and 2 are presented in Table 6.43.

Table 6.43 The validated competencies for each level of the procedural knowledge dimension

Level	Competency
Level 1	Adding fractions with the same denominator
Level 2	Adding fractions with a different denominator
Level 3	Adding fractions with a mixed number
	Subtracting a fraction from a whole number
	Multiplying a fraction with a fraction
	Multiplying a fraction with a whole number
	Dividing a fraction with a fraction.
Level 4	Multiplying a mixed number with a mixed number
	Dividing a mixed number with a whole number
Level 5	Solving a nested fraction operation with the numerator is a fraction subtraction
	Solving a nested fraction operation with the numerator is a fraction division
Level 6	Solving a fraction operation with two level nested fractions

6.3.5.4 Validation using Students' Level Analysis

The proposed model of fraction learning progressions provided a hypothetical pathway for students learning fractions through two-dimensional knowledge dimensions, conceptual and procedural knowledge. In each dimension, it was hypothesized that students learn fractions sequentially from the lower level to the upper level. Hence, in the proposed model, students at the upper level should also have competencies from the lower levels.

From the hypothesis underlying the proposed model discussed above, one of the effective ways to evaluate the hierarchical levels of the proposed model was to collect empirical evidence from students' responses that students had the competencies at the upper level but did not have the competencies at the lower level(s). Model 2 of the Bayesian Networks has an important feature to facilitate this validation.

As discussed previously, Model 2 represented a parameter θ_{ci} for each level of the proposed model of fraction learning progression. Hence, there were six parameters θ_{ci} for each conceptual and procedural knowledge dimension. These parameters θ_{ci} were interrelated so that θ_{ci} at the upper level was conditional on θ_{ci} at the lower level. A simple DAG in Figure 6.25 reflects the dependency between the levels of θ_{ci} in Model 2.

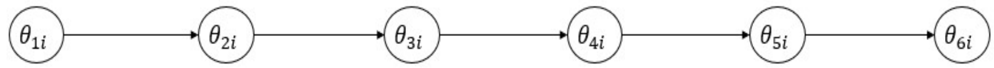


Figure 6.25 A simple DAG which represents the dependency between the levels θ_{ci} in Model 2 of the Bayesian Networks Modelling

The posterior probabilities of θ_{ci} represented the probability of students having competencies at the corresponding level of θ_{ci}. Hence, to perform a model validation for a student who was assigned to a certain level, the posterior probabilities of the student at the lower level could be checked. The low posterior probabilities indicated that the student did not have sufficient competencies at the lower level, which showed the deviation of the proposed model. If there were too many cases that showing a deviation from the model, it would diminish the validity of the interpretation drawn from the proposed model of fraction learning progression.

As an example, Figure 6.26 depicts the student with ID 358 who deviated from the proposed model of fraction learning progression for the conceptual knowledge dimension. From Figure 6.29, it can be observed that the student had a posterior probability of about 98.2% at Level 3, but only had a posterior probability of about 45.7% at Level 2. The results showed that the student had competencies at Level 3 but did not have sufficient competencies at Level 2. These results challenge the validity of the model of fraction learning progression.

Based on the posterior probabilities of 516 students, an R program was developed to assess how many students showed competencies at the upper level yet had insufficient competencies at the lower level(s). A cut-off point of 70% of the posterior probabilities was created to discriminate between students who had sufficient competencies at a certain level ($\geq 70\%$) and students who had no or insufficient competencies at that level ($< 70\%$). The cut-off point of 70% was chosen to get ample evidence that the students with a probability of 70% or above were highly likely to have competencies at that level, while students with a probability below 70% were less likely to have the competencies at that level. The results are presented in Table 6.44.

Table 6.44 The distribution of students who fit and deviate from the proposed levels of fraction learning progression using Model 2.

	Conceptual	Procedural
Fit with the Hypothesis	506	512
Deviate from the Hypothesis	10 (1.94%)	4 (0.78%)

The results demonstrated that ten students (students with IDs 103, 149, 204, 273, 279, 309, 358, 423, 468, and 516) were not consistent with the assumption of the fraction learning progression on the conceptual knowledge dimension. It means that they showed competencies at the upper level but not at the lower level(s). Similarly, four students (student with IDs 22, 177, 451, and 459) had procedural competencies at the upper level but did not have sufficient competencies at the lower levels. The number of deviations from both the conceptual and the procedural knowledge dimension was relatively small, at less than 5%. Therefore, it can be concluded that the students' responses are consistent with the hierarchical assumption of the proposed levels of the learning progression model.

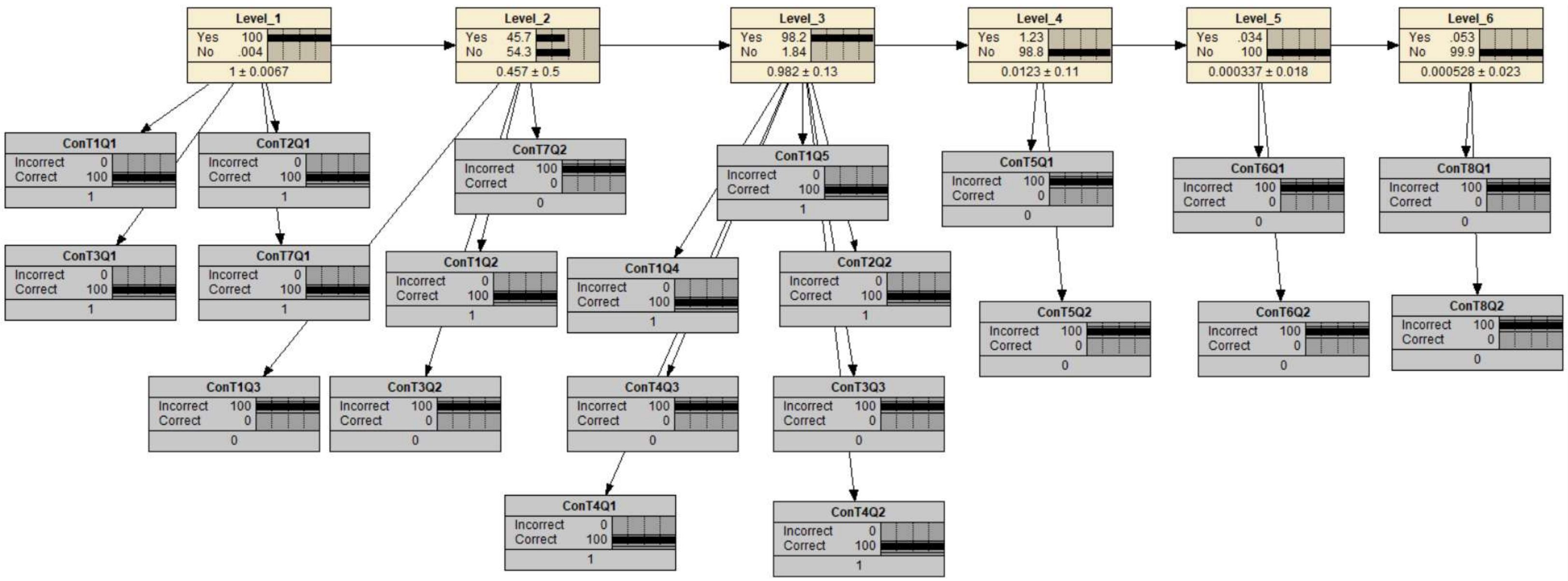


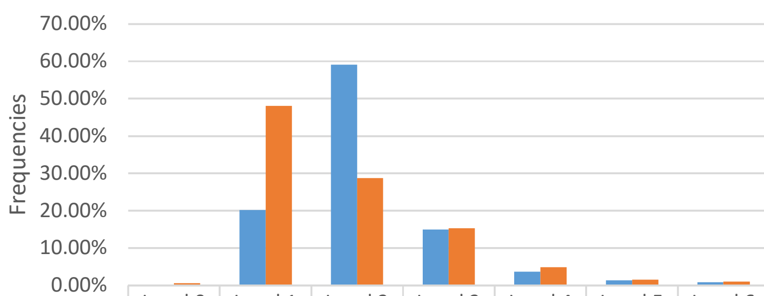
Figure 6.26 Netica Graph for the student with ID 358 who has deviated from the proposed model of fraction learning progression

6.4 Discussion

The discussion of the research findings is organized into two main sections: the comparison of the Bayesian Networks Models 1 and 2 (Section 6.4.1) and the research contribution for educational measurement and assessment (Section 6.4.2).

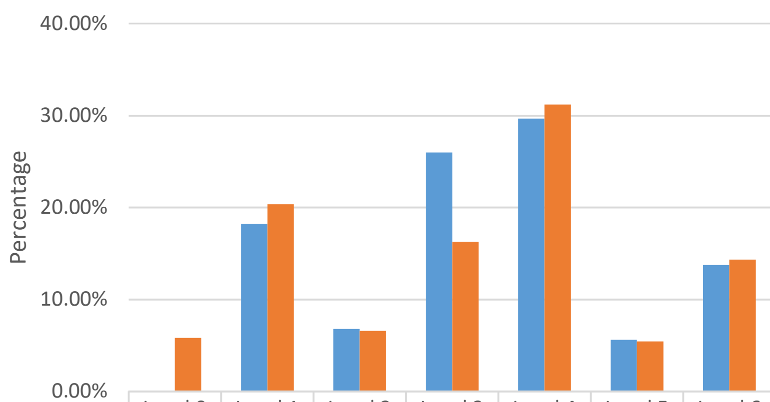
6.4.1 Comparison of the Bayesian Network Models 1 and 2

The distribution of the students' levels for both the conceptual and the procedural knowledge dimensions are presented in Figures 6.27 and 6.28. The results showed that Models 1 and 2 produced different estimates of the students' levels.



	Level 0	Level 1	Level 2	Level 3	Level 4	Level 5	Level 6
Model 1		20.16%	59.11%	14.92%	3.68%	1.36%	0.78%
Model 2	0.58%	48.06%	28.68%	15.31%	4.84%	1.55%	0.97%

Figure 6.27 The distribution of the students' conceptual levels based on Model 1 and Model 2



	Level 0	Level 1	Level 2	Level 3	Level 4	Level 5	Level 6
Model 1		18.22%	6.78%	25.97%	29.65%	5.62%	13.76%
Model 2	5.81%	20.35%	6.59%	16.28%	31.20%	5.43%	14.34%

Figure 6.28 The distribution of the students' procedural levels based on Model 1 and Model 2

From Figure 6.27, it can be observed that Level 0 was detected by Model 2 but not Model 1. For the conceptual knowledge dimension (Figure 6.27, the peak was at level 2 for Model 1 but at Level 1 for Model 2. For Level 3 to Level 6, both models produced similar estimates. Next, for the procedural knowledge dimension (Figure 6.28), Model 1 and Model 2 produced similar results for estimating students at most of the levels, except at Levels 0 and 3. Model 1 categorized more students in Level 3 than Model 2.

In the following section we will compare the fit of Models 1 and 2 in order to decide which one was better. Two types of statistical methods were used to compare the model fit: The Posterior Predictive Model Check (PPMC) and the Entropy statistic. Subsequently, Models 1 and 2 will be compared based on their capacity to diagnose students' competencies at each level of the learning progression (diagnostic-analytic), and in performing item analyses such as item difficulty, item discrimination, and pseudo-guessing item analysis.

6.4.1.1 Model Fit Analyses for Models 1 and 2

The Posterior Predictive Model Checking (PPMC) analysis using discrepancy measures (Section 5.3.1) was used first to estimate the model fit for Models 1 and 2. The PPMC evaluated how the observed data differ when compared with the data generated from the model (the predicted data) (Sinharay, 2004). The PPMC was computed based on the last 1000 iterations of the MCMC for Model 1 and Model 2 to compute the PPP-values. The PPP-values were estimated by comparing the discrepancy measures from the observed data and the posterior predictive data, generated from the PPMC, as presented in Equation (5.13). As described in Section 5.3.1, PPP-values greater than 0.05 and less than 0.95 indicate a good fit, otherwise the PPP-values indicate an inadequate fit (Gelman et al., 2014).

Table 6.45 summarizes the PPP-values computed for 516 students obtained from Model 1 and Model 2 for both the conceptual and procedural knowledge dimensions.

Table 6.45 Person Fit of Model 1 and Model 2 for both conceptual and procedural knowledge dimensions

PPP Values	Person FIT			
	Conceptual		Procedural	
	Model 1	Model 2	Model 1	Model 2
> 0.05 and < 0.95 (adequate fit)	76.16%	92.83%	88.37%	92.44%

The results showed that the student fit of the conceptual knowledge dimension for Model 2 is 92.83%. This was considerably higher than the person fit of Model 1 which was only 76.16%. It means that Model 2 had a better fit with the students' responses when compared with Model 1 in the conceptual knowledge dimension. Model 2 in the procedural knowledge dimension also had a higher person fit which was 92.44% compared with that of Model 1 which is 88.37%. This indicates that Model 2 showed a better fit with the students' responses on the procedural knowledge dimension compared with that of Model 1.

The model fit of Model 1 and Model 2 discussed above was consistent with the fit estimated by using the entropy statistic on the conceptual and procedural knowledge dimensions. The entropy statistic (detailed in Section 5.3.2) was calculated for both Model 1 and Model 2 using Equation (5.14). Based on the proportional improvement of Model 2 from Model 1 described in Equation (5.13), the positive difference of the entropy statistics indicates that Model 2 makes a better prediction on a new observation compared with Model 1 (Levy & Mislevy, 2016).

The results show that the entropy of Model 1 in the conceptual knowledge dimension was 2296.23, and the entropy for Model 2 on the same dimension was 1243.38. Hence, the difference between entropy Model 1 and entropy Model 2 (Entropy Model 1-Entropy Model 2) in the conceptual knowledge dimension was 1052.85 (positive). Similarly, the difference between Entropy Model 1 and Model 2 in the procedural knowledge dimension was 204.75 (positive). Hence, it can be inferred that Model 2 makes a better prediction on a new observation compared with Model 1 for both the conceptual and procedural knowledge dimensions. The results are presented in Table 6.46.

Table 6.46 The entropy of Model 1 and Model 2 for the conceptual and procedural knowledge dimensions

Dimension	Entropy	
	Model 1	Model 2
Conceptual	2296.23	1243.38
Procedural	1293.11	1088.36

The relative improvement of Model 2 towards Model 1 (denoted as $dEntropy$) is calculated based on the Equation (5.15). The results are presented in Table 6.47 below.

Table 6.47 $dEntropy$ Model 1 and Model 2 for the conceptual and procedural knowledge dimensions

Dimension	$dEntropy$
Conceptual	0.4585
Procedural	0.1583

The results showed that the $dEntropy$ of the conceptual knowledge dimension was 0.4585. This indicates that Model 2 improved on model 1 by about 45.85% for the conceptual knowledge dimension. Likewise, the $dEntropy$ of the procedural knowledge dimension was 0.1583. This indicates that Model 2 made an improvement on Model 1 of about 15.83% for the procedural knowledge dimension.

6.4.1.2 Diagnostic-Analytic

Model 1 is deficient in information about the students' strengths and weaknesses in their learning at the levels of the learning progression model. This is because the low probabilities of the students at the lowest levels do not automatically point out that the students are weak at those levels. For example, the probabilities generated from Model 1 for student with ID 416 on the conceptual knowledge dimension are presented in Table 6.48 (the raw scores and Netica graph for this student are presented in Table 6.50 and Figure 6.29, respectively). This student was chosen to represent a typical result of the analysis generated from Model 1.

Table 6.48 The probability of student with ID 481 for each level in the conceptual knowledge dimension generated from Model 1

Level	Probability for Each Level
Level 1	0.036%
Level 2	0.003%
Level 3	83.100%
Level 4	16.400%
Level 5	0.470%
Level 6	0.008%

The results showed that Model 1 generated the highest probability at Level 3. Hence, student with ID 416 was obviously placed at Level 3. However, the low probabilities at the other levels below Level 3 did not necessarily show that the student had low competencies at those levels. For example, the student had low probabilities at Levels 1 and 2. In fact, from Table 6.50, the student had strong competencies at these two levels in that the student could correctly answer all the items at Level 1 and most of the items at Level 2. Hence, the results from Model 1 are counter-intuitive if they are used as a diagnostic-analytic tool.

In contrast, the analyses generated from Model 2 produced high probabilities for student with ID 416 at Level 1 and Level 2. These results showed that the student with ID 416 had strong competencies at these levels. The results are summarized in Table 6.49. The Netica graph for this student, generated from Model 2, is presented in Figure 6.30.

Table 6.49 The probability of student with ID 416 for each level in the conceptual knowledge dimension generated from Model 2

Level	Model 2
Level 1	100.00%
Level 2	99.30%
Level 3	100.00%
Level 4	54.40%
Level 5	0.74%
Level 6	0.10%

In contrast, the results showed that Model 2 generated high probabilities at Level 1 to Level 3 which were 100%, 99.3%, and 100%, and generated low probabilities for levels 4, 5 and 6 which are 54.40%, 0.74% and 0.10%. From these results, it can be inferred that the student was highly likely to have strong competencies at Levels 1, 2, and 3 but low competencies at Levels 4, 5 and 6. Hence, Model 2 could show the students' strength and weaknesses in the levels and that a high probability indicated that the student had strong competencies at that level, and a low probability showed that the student had weak competencies at that level. This information is essential to diagnose at which level the student had difficulties in learning.

In an extreme case, for example, student with ID 61 correctly answered one item at Level 1 and incorrectly answered most of the rest of the conceptual items. However, Model 1 generated a high probability for the student achieving at Level 1, 81%, as presented in Figure 6.31. This is because, in Model 1, the sum of the probabilities for all categories should reach 100% (see Equation (5.1) in Chapter 5). Hence, Model 1 imposed a high probability at Level 1 to obtain the total probabilities for all categories as 100%, despite student 61 only having one correct answer at Level 1. This evidence shows that Model 1 failed to represent any situation where a student did not have enough competency at any of the proposed levels. In other words, Model 1 failed to assign students who did not have enough competencies to achieve Level 1(+) into Level 0.

In contrast, Model 2 used different parameters to capture the different levels of the proposed model of learning progression, by setting the dependency between the lower level and the upper level where the upper level was conditional (depends) on the lower level. This dependency reflected the hypothesized hierarchical progression of fraction learning. For each level, the model had two categories which were "yes" or "no" and represented the probability of these categories for students at that level. The sum of the probability of "yes" or "no" at a level should reach 1, and the probability of students for all levels was not necessarily 1. Using this strategy of modelling, the function of the Bayesian Network of Model 2 was not necessarily to impose or distribute the probabilities through all the levels, which was different from Model 1. As a result, Model 2 could provide low probabilities for all levels for students who did not show competencies at the proposed levels. From the previous example of student 61, Model 2 generated the posterior probabilities for all the levels were less than 12%, which indicated that the student had not enough competencies at the proposed levels (presented in Figure 6.18 in section 6.3.2.1.2). Hence, using this result, student 61 was assigned at Level 0, unlike Model 1, which assigned this student at Level 1.

Table 6.50 The raw sores of student 416

Level 1				Level 2				Level 3							Level 4		Level 5		Level 6	
ConT1Q1	ConT2Q1	ConT3Q1	ConT7Q1	ConT1Q2	ConT1Q3	ConT3Q2	ConT7Q2	ConT1Q4	ConT1Q5	ConT2Q2	ConT3Q3	ConT4Q1	ConT4Q2	ConT4Q3	ConT5Q1	ConT5Q2	ConT6Q1	ConT6Q2	ConT8Q1	ConT8Q2
1	1	1	1	1	0	1	1	1	0	1	1	1	1	1	0	1	0	0	0	0

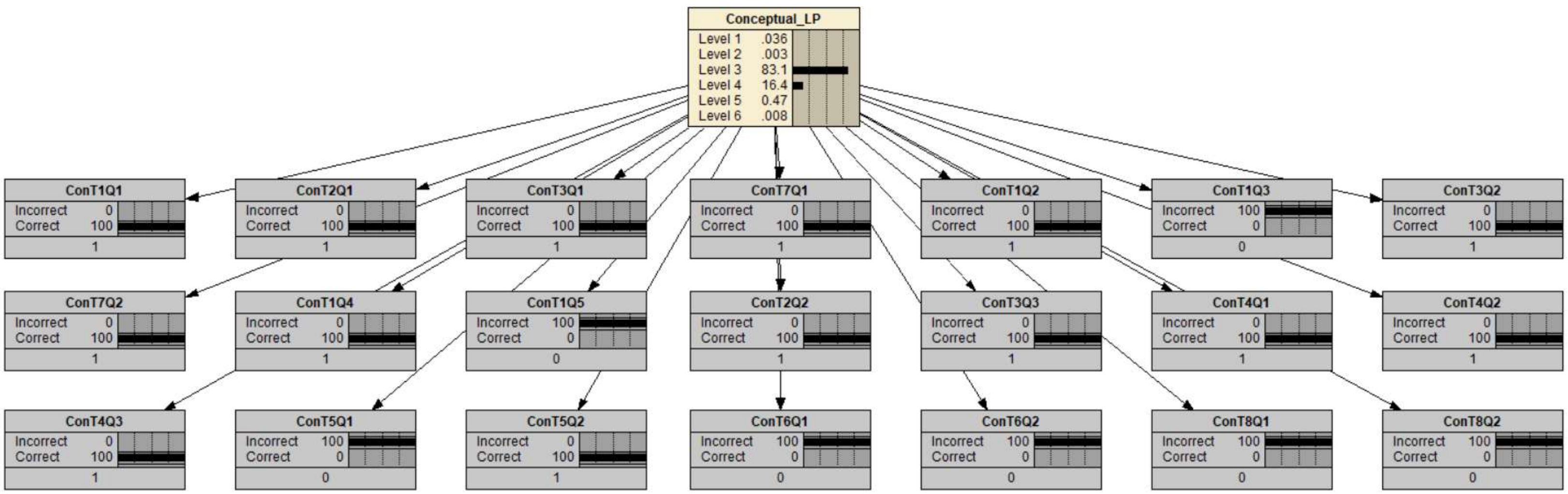


Figure 6.29 A Netica Graph of the posterior probability $P(\theta_i|\boldsymbol{\gamma}, \boldsymbol{\pi}, x_{ij})$ for student with ID 416 (i=416, j=1,…,21) generated from Model 1.

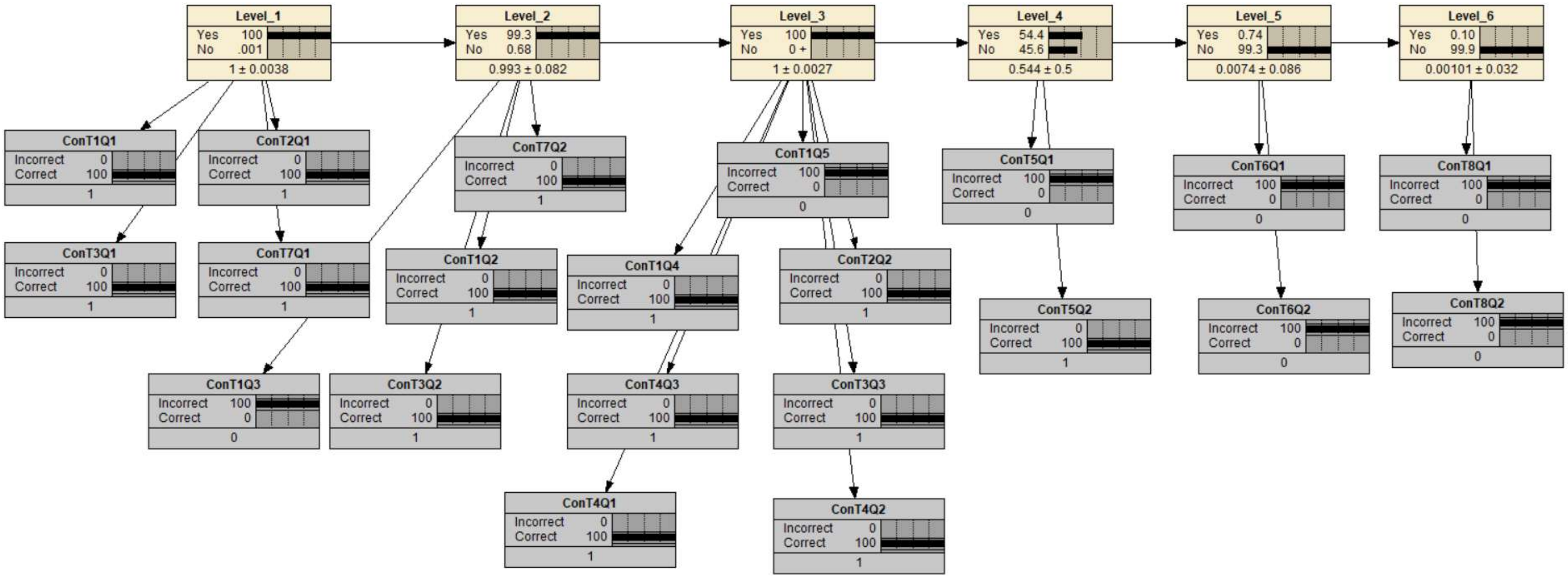


Figure 6.30 A Netica Graph of the posterior probability $P(\theta_i|\boldsymbol{\gamma}, \boldsymbol{\pi}, x_{ij})$ for student with ID 416 (i=416, j=1,…,21) generated from Model 2.

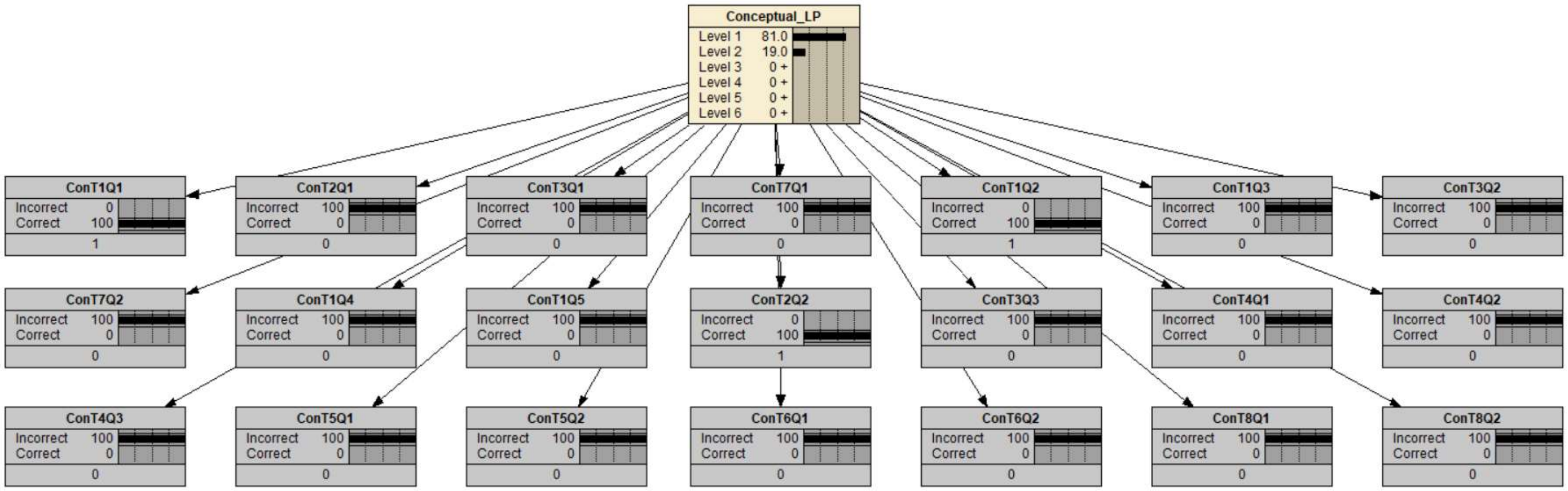


Figure 6.31 A Netica Graph of the posterior probability $(\theta_i|\boldsymbol{\gamma}, \boldsymbol{\pi}, x_{ij})$ for student with ID 61 (i=61, j=1,…,21) generated from Model 1

The raw scores and the posterior probabilities $(\theta_i|\boldsymbol{\gamma}, \boldsymbol{\pi}, x_{ij})$ for the student with ID 61 generated from Model 2 are presented in Table 6.30 and Figure 6.18, respectively.

6.4.1.3 Item Difficulty Analysis

Both Models 1 and 2 presented the conditional probabilities of the students, given the students had competencies at a particular level c to answer correctly an item j (represented by π_{cj} in Equation (5.2) for Model 1, and π_{cj1} in Equation (5.9) for Model 2). These conditional probabilities demonstrate the difficulty of the item for the students at that level. The larger the conditional probabilities of the students given the level, the easier the items are for the students at that level, and vice versa. For example, Table 6.51 presents the estimates of the conditional probabilities π_{3j} and π_{3j1} for the conceptual items at Level 3.

Table 6.51 The estimates of the conditional probabilities of correctly answering the items at Level 3 of the conceptual knowledge dimension generated from Model 1 (π_{3j}) and Model 2 (π_{3j1}).

Item	Model 1 π_{3j}	Model 2 π_{3j1}
ConT1Q4	0.7504	0.7573
ConT1Q5	0.6933	0.7007
ConT2Q2	0.8372	0.8482
ConT3Q3	0.8368	0.8416
ConT4Q1	0.8538	0.8585
ConT4Q2	0.8682	0.8656
ConT4Q3	0.7796	0.7828

The results showed that item ConT1Q5 was the most difficult item for students at Level 3. This was because item ConT1Q5 had the smallest conditional probabilities for the students at level 3 to answer this item (π_{3j} = 0.6933, and π_{3j1} = 0.7007) correctly. On the other hand, item ConT4Q2 was the easiest item because it has the largest conditional probabilities of π_{3j} (0.8682) and π_{3j1}(0.8656). Both Models 1 and 2 support the analysis of item difficulty within the discrete-levels of the learning progression.

6.4.1.4 Item Discrimination Analysis

Model 1 was capable of showing item discrimination of any one item across the levels. This was because in Model 1, all the items were set to correspond to a single parameter θ, which captured all the proposed levels. Hence the probabilities of the students answering an item correctly can be evaluated from the all different levels. For example, Table 6.52 show the conditional probabilities of item ConT4Q1 (c=3, j=13.

Table 6.52 The estimates of the conditional probabilities for item ConT4Q1 ($\pi_{3j}, j = 13$) of the conceptual knowledge items at Level 3 for Model 1

	Correct π_{3j}	Incorrect 1-π_{3j}
Level 1	0.1969	0.8031
Level 2	0.0711	0.9289
Level 3	0.8538	0.1462
Level 4	0.8144	0.1856
Level 5	0.8105	0.1895
Level 6	0.8092	0.1908

The results showed that the students at low levels (Level 1 and Level 2) were unlikely to answer Item ConT4Q1 correctly, but students at Level 3 and above were highly likely to answer the item correctly. Hence, item-discriminant analysis can be performed using Model 1.

Model 2 did not generate information for item-discriminant analysis. This was because specific items were only modeled to measure a specific level. For example, the results of the conditional probabilities that were estimated using Model 2 are presented in Table 6.53. The results showed that the probability of answering Item ConT4Q1 correctly was only estimated at Level 3, by hypothesizing θ_{3i} = 1 (have competencies at level 3) and θ_{3i} = 0 (do not have competencies at level 3). Hence, Model 2 only produced the probability of students correctly answering the items from the corresponding level (Level 3). As a result, there was no information about how likely it would be for students to answer the items from the other levels. Model 2 could only discriminate between those students who had competencies at that level and those who did not have competencies at that level.

Table 6.53 The estimates of the conditional probabilities for Item ConT4Q1 ($\pi_{3jz}, j = 13$) for the conceptual knowledge items at Level 3 for Model 2

Level 3

Items	θ_{3i} = 1		θ_{3i} = 0	
	Correct π_{2j1}	Incorrect 1-π_{2j1}	Correct π_{2j0}	Incorrect 1-π_{2j0}
ConT1Q4	0.6592	0.3408	0.0649	0.9351
ConT1Q5	0.5741	0.4259	0.0351	0.9649
ConT2Q2	0.8239	0.1761	0.2941	0.7059
ConT3Q3	0.8065	0.1935	0.0882	0.9119
ConT4Q1	0.8486	0.1514	0.0226	0.9774
ConT4Q2	0.8550	0.1450	0.0022	0.9978
ConT4Q3	0.7136	0.2864	0.0061	0.9940

6.4.1.5 Pseudo-guessing Item Analysis

In terms of the Item analysis, unlike Model 1, Model 2 provided information about the likelihood that the students who had competencies at a particular level could correctly answer the items at that level by chance. This analysis was not available in Model 1. This analysis is in some respects similar to the pseudo-guessing item analysis in Item Response Theory. Model 2 incorporated this analysis well by setting the parameter level θ_{ci} equal to 0, identifying those students who did not have competencies at that level. The results showed that there was a low probability of guessing. It should be pointed out that this innovative analysis is important in order to ensure that the students are not guessing the correct answers to the items. A finding that there is a high probability of the students with no competencies at a given level ($\theta_{ci} = 0$) could correctly answer the items at that level would indicate that the students can guess the answer or answer the items correctly by chance.

In summary, the Bayesian Network with multiple latent variables, Model 2, has a better fit compared with the Bayesian Networks with a single latent variable (Model 1). Moreover, Model 2 is better than Model 1 in terms of diagnosing students' competencies on the progression levels and detecting extreme cases where the students do not have sufficient competencies at any of the hypothesized levels of the learning progression. Both Models 1 and 2 can perform item difficulty analysis, which measures how difficult an item is to be answered by students at a particular level. Model 1 provides a more comprehensive item discrimination analysis compared with Model 2. Model 1 can provide information about how likely an item is to be answered correctly by the students from different levels. In contrast, Model 2 only provides information about how an item is answered correctly by the students who have a competency at a particular level and not the students who have no competencies at that level. Model 2 does not provide information about the likelihood of the students from different levels answering a given item correctly but does provide information about the likelihood of the students who have no competencies at a particular level answer a given item correctly by chance (pseudo-guessing analysis).

6.4.2 Research Contribution for Educational Measurement and Assessment

Two contributions of the present research will be discussed in the sections that follow: contributions for the development of Bayesian Network Models for learning progressions and contributions for item analyses using Bayesian Networks.

6.4.2.1 Contributions for the development of Bayesian Networks for Modelling Learning Progressions

In previous research, learning progressions have typically been modelled using Bayesian Networks with a single latent variable. This is known as Bayesian Latent Class Analysis (Jefrey Thomas Steedle, 2008; West et al., 2010; West et al., 2012). The present research performed a Bayesian Networks analysis with a single latent variable and extended this model into a Bayesian Networks analysis with multiple latent variables. The development of Bayesian Networks with multiple latent variables in the present research was inspired by the work of Rutstein (2012), who discussed several possible Bayesian Network models for measuring learning progressions. Three unique aspects of the model developed in the present research, compared with other existing models, will be discussed below.

6.4.2.1.1 Development of an Informative Prior for dichotomous responses in Bayesian Network modelling

West et al. (2010) developed Bayesian Networks with a single latent variable to measure learning progressions. However, they did not specify the information needed prior to encoding the levels of the learning progression. In contrast, the development of a Bayesian Network with a single latent variable (Model 1) and with multiple latent variables (Model 2) in this study used informative prior for dichotomous responses. This informative prior was used to incorporate the assumption, underlying the hierarchical model of learning progression, that students at a certain level would have a high probability of obtaining correct answers for the items at that level and below, but a low probability of answering items at the upper level(s) correctly. This informative prior was also used by Jefrey Thomas Steedle (2008) to build Bayesian Networks from polytomous responses. In the present research, however, the informative prior was used to develop Bayesian Networks for dichotomous variables. No previous studies have previously been undertaken to model learning progressions with informative prior on the dichotomous responses using Bayesian Network modelling.

6.4.2.1.2 Use of a Confirmatory Approach to validate the hypothesized learning progression

West et al. (2010) performed Bayesian Networks with a single latent variable on the different possible levels of a learning progression. Using this approach, they built several Bayesian Networks Models which corresponded to the different numbers of levels of the learning progression. The Bayesian Networks were applied to the data and then their model fits were compared. The model which had the best fit with the data was chosen to give information about how many levels there should be in the learning progression model. In this case, West et al. (2010) used an exploratory approach, namely they used the data to define the levels of the learning progression.

In contrast, the present research took a confirmatory approach. The Bayesian Network models were run strictly, based on the number of levels defined in the hypothetical cognitive model of the learning progression, after they were revised previously through a cognitive interview. Hence, in this research, the results of Bayesian networks analysis were used to confirm the hypothesized model of fraction learning progression and not to generate the levels of the learning progressions. This confirmatory approach is essential if we want to make sure that the Bayesian Network modelling applied to the data is theoretically driven, as suggested by Mislevy (Mislevy, 1994a, 1994b), as opposed to data driven. This is an important point, given that learning progressions developed based on the data may not be well supported by cognitive theory and research in learning.

6.4.2.1.3 Modelling the Hierarchical-Dependency of the Levels and Diagnostic Analytics

As discussed before, Bayesian Network models have been developed in previous research-modelled learning progressions, using a single latent variable with several classes/categories, reflecting the levels of the learning progression (Jefrey Thomas Steedle, 2008; Jeffrey T Steedle & Shavelson, 2009; West et al., 2012). These classes were assumed to be independent of each other. Consequently, the dependency between the levels in the learning progression were not formulated in a formal statistical model.

In contrast, the present research developed a Bayesian Network with multiple latent variables, i.e., Model 2. Using Model 2, the hierarchical dependency between the levels was expressed through the conditional probabilities of the latent variables. These latent variables corresponded to the levels hypothesized in the learning progression. Using this approach, the conditional probability of the students to have competencies at a particular level could be estimated. The likelihood of a student having competencies at a particular level can be used as a diagnostic-analytic tool to evaluate students' competencies in the levels of the learning progression. This is a significant contribution to educational measurement, particularly in developing a diagnostic-measurement model of learning progression using a Bayesian Networks approach.

In summary, the present research contributes to the development of Bayesian Networks in learning progressions in the following ways. First, this research developed a way to introduce informative prior for dichotomous responses. This informative prior is important to incorporate the assumption, which guided the construction of the learning progression, that students at a certain level would have a high probability of obtaining correct answers for the items at that level and below, but a low probability of answering items correctly at the upper level(s). Moreover, the present research developed a confirmatory approach for validating the learning progression. This confirmatory approach is essential in order to validate the hypothesized levels of learning progression with the empirical evidence deriving from the students' responses. Finally, this present research developed Bayesian Networks with multiple latent variables (Model 2) to model the hierarchical dependency of the levels into a formal statistical model. This model can be used as a diagnostic analytic tool for evaluating students' competencies in a learning progression.

6.5.2.2 Contributions to Item Analysis using Bayesian Networks

Current practice for item analyses is dominated by Item Response Theory (IRT), which assumes that the students' abilities are represented as a continuum on a latent scale (De Ayala, 2009). In contrast, learning progressions represent the progression of students in learning on the discrete-latent scale. Consequently, the typical item analyses using IRT cannot be performed directly on the hypothesized discrete levels of learning progression to support the inference that students at this level and above have a high probability of answering the item correctly, while the students at lower levels have a low probability of answering the item correctly.

Wilson (2012) performed item analysis using the Rasch Model (one of the IRT Models) to validate a learning progression. He plotted both the item difficulty and the students' abilities on the same latent continuum scale using the Wright Map (Wilson, 2005). Subsequently, items for each level were identified using cut off points on the scale. These cut off points discretized the latent continuum scale into several categories. These categories were then interpreted as the levels of the learning progression. Items which fell into these categories were placed on the corresponding levels of the learning progression. In this case, Wilson's approach did not perform item analyses based directly on the hypothesized levels of the learning progression. In fact, the item analyses were performed on the empirical levels generated from the students' responses. Consequently, the validation of the learning progression based on these item analyses seems to be more exploratory than confirmatory because the analyses of the items were not performed based on the hypothesized levels of the learning progression.

To date, however, there has been little discussion about item analysis developed for the discrete latent scale, as assumed in the hierarchical levels of the learning progressions. In previous research, West et al. (2010) used Bayesian Networks with a single latent variable (Bayesian Latent Class Analysis) to perform item analysis in the context of a learning progression to support the inference that items placed at a certain level would be answered correctly by the students at that level, but would not be answered correctly by the students at lower levels. In this case, West et al. (2010) performed an item discrimination analysis of the items at a given level and below. The present research expands the item analysis from West et al. (2010), using Bayesian Networks models with a single latent variable (Model 1) and multiple latent variables (Model 2). As discussed earlier, in Sections 6.4.1.3 - 6.4.1.5, three types of item analyses for discrete latent scales were developed based on these models: namely, item difficulty analysis, item discrimination analysis and pseudo-guessing analysis. These item analyses are similar to the typical item analyses in IRT models. However, the interpretation of the results of the analyses are different due to the differences in the assumptions of the latent scales (discrete versus continuum latent scales).

In this research, item difficulty refers to the probability of the students who have competencies at a particular level answering an item correctly. The higher the probability of a correct answer for this item, the easier the item is for the students at that level, and vice versa. The item is placed at a particular level if the students at that level have high probabilities for answering the item correctly. Hence, the item difficulty generated from Bayesian Networks is developed, based on the discrete levels of the learning progression and representing the uncertainties of item location at these levels. In contrast, item difficulty in IRT, known as the item parameter, is located on the continuum scale, so that the students' ability is also plotted in this scale (De Ayala, 2009; Wilson, 2005). If the ability of the students is equal to or higher than the item difficulty, then the students are likely to answer the items correctly. If the ability of the students is less than the item difficulty, they are likely to answer the item incorrectly. Items are located along a continuum from negative to positive infinity. The easier items are located at the lower end of the continuum and the more difficult items are located towards the higher end (De Ayala, 2009).

The item discrimination analysis developed in this research discriminates between students at different levels of the learning progression (a discrete scale). Students at a particular level have a high probability of answering the items at that level correctly, while students at the lower levels have less probability (are unlikely) to answer these items correctly (West et al., 2010). In contrast, the item discrimination analysis in IRT refers to the item discrimination parameters in the IRT model (IRT with a 2-parameter logistic model), which is used to differentiate students from different points on the continuum scale. For example, students who are at the lower end of the latent continuum are unlikely to answer an item that is located in the middle of the scale, but students at the middle and upper ends of the scale are highly likely to answer the item correctly. Hence, item discrimination analysis using Bayesian Networks is developed based on the discrete latent scale (levels in the learning progression) while item discrimination analysis in IRT is developed from the latent continuum scale.

The pseudo-guessing analysis using Bayesian Networks in this research estimates how likely the students who have no competencies at a particular level are to answer the items at that level correctly. This pseudo-guessing analysis resembles the pseudo-guessing in the IRT model (IRT with a three parameter model) (De Ayala, 2009). However, the pseudo-guessing in IRT is developed based on the assumption of the continuum scale. The pseudo-guessing in IRT analyses the probability of the students at the lower end of continuum scale answering the items located in the middle or upper ends of the scale.

In summary, the present research developed item analyses for the discrete latent scale of a learning progression. These item analyses are different from the item analyses in IRT, which are developed based on the latent continuum scale. The item analyses using Bayesian Networks are useful in order to analyze items on the hypothesized levels of the learning progression. These item analyses are essential in the context of an analytic approach that uses confirmatory analyses to validate the hypothesized learning progression models. This is a significant contribution to the present research in educational measurement, as little discussion about developing item analyses for the discrete latent scale of a learning progression has yet taken place.

6.7 Summary and Conclusions

This chapter has discussed the development of Bayesian Networks Analysis using Models 1 and 2 to validate a hypothesized model of fraction learning progression for a conceptual and a procedural knowledge dimension. The validation was performed using item level analysis and student level analysis to support the inferences that: (a) an item at a certain level would be answered correctly by the students at that level or the upper level, but would not be answered correctly by the students at the lower levels; and (b) students assigned to a certain level would have sufficient competencies at that level and the levels below, but would not have enough competencies at the levels above their competency level. The results of the analyses produced: 1) validated levels of the conceptual and procedural knowledge dimensions; 2) the location of conceptual and procedural items along the levels of the learning progression; and 3) the location of students along the levels of the conceptual and procedural knowledge dimensions of the learning progression.

The comparison of Model 1 and Model 2 showed that Model 2 had a better fit compared with Model 1. Furthermore, Model 2 had several desirable properties which are superior to Model 1, such as the measurement of competency, diagnostic analytic, and pseudo-guessing item analysis. However, Model 1 was better than Model 2 for item discrimination analysis. The item analysis generated from Models 1 and 2 is an important innovation in this study because it is applied to the discrete levels of learning progression. Previous models of item analysis based on IRT were developed based on the assumption of a continuum latent scale of students' ability, while the present item analyses, on Bayesian Network Models 1 and 2, were developed based on the assumption that the students' learning progression was a discrete latent scale. Using Bayesian network item analyses, the hypothesized learning progression can be validated empirically using confirmatory analysis. To conclude, this present research contributes to the development of Bayesian networks for measuring learning progression by employing informative prior on the dichotomous responses, performing a confirmatory approach of analysis, and modelling the hierarchical dependency of the levels in the learning progression model using a Bayesian Network with multiple latent variables (Model 2).

The next chapter discusses further the results of the Bayesian Networks analysis on the conceptual and procedural knowledge dimensions and the relationship between the two.

CHAPTER 7 : RESULTS OF STUDENT PERFORMANCE ON THE CONCEPTUAL AND PROCEDURAL DIMENSIONS OF THE LEARNING PROGRESSION AND INTERRELATIONS

7.1 Introduction

The purpose of this chapter is to examine and discuss the results of the Bayesian Networks analysis produced by Model 2 in more detail, showing the levels of student performance in the conceptual and procedural knowledge dimensions and the interrelationships between them. In Sections 1 and 2, the results from the Conceptual and Procedural Knowledge Dimensions of the fraction learning progression are discussed in the context of the 2013 Indonesian curriculum. In Section 3, the interrelationships between the performance of the students in the Conceptual and Procedural Knowledge Dimensions are discussed. A summary presentation of the 2013 Indonesian Curriculum is included in Appendix G.

7.2. Section 1: Student Performance on the Conceptual Knowledge Dimension

The results produced by Model 2 of the Bayesian Network Analysis (shown in Table 7.1) indicate that about half of the students (48.06%) were located at Level 1 of the conceptual fraction understanding validated progression. These students could only deal with fractions that had the same denominator and did not know how to compare fractions with different denominators. In other words, half of the students could not understand the meaning of the fraction symbol even when dealing with fractions smaller than the unit.

This result is very disappointing because it shows that these students were only reaching the basic fraction competence in the Indonesian curriculum for grade 3, even though they were in grades 7-9. The conceptual challenge these students face is understanding that the value of a fraction is represented by the relationship between the numerator and denominator and that different fractions can be equivalent, or have the same value (Lamon, 2012).

Table 7.1 Frequency and percentage of students in the different levels on the conceptual knowledge dimension based on the Bayesian Network Analysis (n=516)

Level	Frequency and Percent N=516
Level 0 – No fraction understanding	(3) 0.58%
Level 1 – Part-whole with the same denominator	(248) 48.06%
Level 2 – Part-whole with the different denominators	(148) 28.68%
Level 3 – Improper fractions and fractions as measures	(79) 15.31%
Level 4 – Unbounded infinity of fractions	(25) 4.84%
Level 5 – Density	(8) 1.55%
Level 6 – Understanding multiplicative fraction operations	(5) 0.97%

About a quarter of the students (28.68%) in the sample were found to belong to Level 2 in the conceptual dimension of fraction understanding. At Level 2, students have a good understanding of the part-whole representation of fractions and can compare fractions with different denominators. They can also illustrate fraction addition with different denominators using a part-whole diagram. Still, these students do not understand improper fractions. The students at Level 2 in the present learning progression were below the basic competence of fractions taught in the Indonesian curriculum at grade 4. At this grade, equivalent fractions are introduced and used to compare and order fractions, and to perform arithmetic fractions operations with different denominators. Obviously, understanding improper fractions is the main conceptual challenge for the students at this level. It is possible that students' difficulties in dealing with this conceptual challenge is related to the strong emphasis in the Indonesian curriculum on the part-whole teaching of fractions and also the fact that the curriculum does not introduce a conceptual understanding of improper fractions.

Findings from Arieli-Attali and Cayton-Hodges (2014) show that students who see fractions as part of a whole often find it challenging to understand how the number representing the part (the numerator) could be greater than the number representing the whole (the denominator). Consequently, the students have difficulty in accepting improper fractions. For example, students often do not understand an improper fraction such as 4/3, stating that four parts cannot be produced from dividing an object into three parts (Fazio & Siegler, 2011). Similar results have been obtained by Resnick et al.

(2016), who found that the students from grades 4, 5 and 6 in their sample consistently estimated both proper and improper fractions to have values between 0 and 1. Fazio and Siegler (2011) discussed the difficulties that might arise in the understanding of improper fractions when the curriculum emphasizes part-whole representations. In order to facilitate the understanding of improper fractions, it helps to introduce number line representations (fractions as measures), something which is not done in the Indonesian Curriculum.

Only 15.31% of the students were found to be at Level 3 of conceptual fraction understanding, meaning that these students were capable of understanding improper fractions and representing fractions on a number line. This group of the students resembled the cohort of students in the explanatory framework of fractions as Relation between Numerator/Denominator - sub-category C1: Relation of Two Numbers Without Infinity (Stafylidou & Vosniadou, 2004). Although they understood the relationship between the numerator and denominator, these students still perceived fractions as finite and thought that there exists a smallest and a biggest fraction. The conceptual challenge the students face at this level is to understand the unbounded infinity of fractions. Instruction on the unbounded infinity of fractions is not included in the Indonesian curriculum.

One way to assist students to understand the unbounded infinity of fractions is to introduce fractions as division. Stafylidou and Vosniadou (2004) found a group of students (Subcategory C2: Relation of Two Numbers with Infinity) who believed that fractions are infinite numbers because they saw fractions as the results of the division of the numerator with the denominator. Understanding fractions as division can facilitate the students' understanding of the numerical values of fractions. These values represent the results of the division of the numerator by the denominator. The values are getting bigger when the numerator is increased and getting smaller when the denominator is decreased (in the condition that one of the numerators or denominators is constant). The results of the cognitive interview showed that the students who demonstrated fractions as division could explain effectively that there were no biggest and smallest fractions. However, fractions as division is also not covered in the Indonesian Curriculum.

There were very few students at levels 4 and 5 in the conceptual dimension of the validated fraction learning progression. Approximately 5% of the students were placed at Level 4. This group was similar to the group of students in the explanatory framework of fractions as Relation between Numerator/Denominator - sub-category C2: Relation of Two Numbers With Infinity (Stafylidou & Vosniadou, 2004). These students understood that there were no smallest or biggest fractions, but they still believed that there were limited or no numbers between two pseudo-fractions. In other words, these students had not understood the 'no successor' principle of rational numbers. This is a difficult concept to understand. Vamvakoussi and Vosniadou (2012, p. 266) have proposed using a 'rubber line' analogy – namely, to think of numbers as placed on a number line which can be stressed and shrunk like rubber - to introduce the idea of fraction density. The results of their experiments show that a rubber line analogy can assist students to understand the no successor principle. Again, the no successor principle in fractions is not taught in the Indonesian curriculum.

Only a very small percentage of the students were placed at Level 5 (1.55%). These students had a complete understanding of the infinity of fractions, including unbounded infinity and density. In other words, they completely understood that fractions are infinite and dense, which is different from the finite and discreteness properties of whole numbers. Still these students had difficulties in understanding fraction multiplication and division. They had difficulties in translating the multiplication and division of fraction operations from symbolic notation into a diagram representation. The results from the cognitive interview showed that the students at this level tried to use their procedural knowledge to assist them in drawing the diagram representations but without success. Similar results were found by Chinnappan and Forrester (2014), who investigated the conceptual and procedural knowledge of fraction operations using a sample of pre-service teachers. In the case of fraction multiplication, the result indicated that 76.69% of the participants could solve the problems procedurally but could not demonstrate a conceptual understanding of fraction multiplication using diagram representations. These empirical results indicate that the students who can perform the operations of fraction multiplication and division do not understand the meaning of these operations. A conceptual understanding of multiplicative fraction operations is not included in the Indonesian curriculum. However, the materials which introduce fractions using diagrams/strips (area models) can be found in the Indonesian mathematics text book at grade 7 (see As'ari et al., 2014).

Fewer than 1 % of the students were placed at Level 6 in the learning progression. These students could demonstrate their understanding of multiplicative fraction operations using diagram representations. The students viewed fraction multiplication, for example 1/2 X 3/4, as finding how much is 1/2 of 3/4. They performed partitioning on 3/4 into two equal parts to get the solution of 1/2 multiplied by 3/4. The results from the cognitive interview show that this understanding helped the students to demonstrate fraction multiplication using a diagram representation. There was no issue of learning difficulties for the students at this level. They had a complete conceptual understanding of fractions, ranging from fractions as part-whole, improper fractions, fractions as measures, unbounded infinity of fractions, density, and understanding multiplicative fraction operations.

In summary, the distribution of students in the conceptual dimension of the fraction learning progression indicated that most of the students had a very low level of conceptual knowledge of fractions. About 3/4 of the students only effectively reached the basic competences of fractions taught in the Indonesian curriculum in grades 3 and 4, which is part-whole understanding. Their conceptual understanding of fractions was minimal, even though they were at grades 7-9. The learning challenges these students faced were understanding fractions greater than the unit (improper fractions) and fractions as measures. The students lacked a high-level conceptual understanding of fractions, such as unbounded infinity and density and could not understand multiplicative fraction operations.

The results indicate that the Indonesian curriculum (1) was ineffective in developing a conceptual understanding of fractions, even when the conceptual understanding was part of the curriculum and (2) did not cover important areas of fraction conceptual knowledge that should have been included in the curriculum.

Regarding the first point, a closer look at the Indonesian curriculum indicated that it did not provide clear guidance as to how teachers could develop conceptual understanding of fractions in their students, even when such understanding was part of the curriculum goals. For example, in introducing equivalent fractions, the basic competence in grade 4, it was stated that students should "recognize the concept of equivalent fractions …" (Appendix H). However, it was not possible to find a description of what equivalent fractions are anywhere in the Indonesian curriculum, nor the different kinds of equivalent fractions that exist (e.g., simple and not simple equivalent fractions), or how to introduce equivalent fractions to students (for example using visual fraction models). Moreover, equivalent fractions were only mentioned in grade 4 and were not related to other relevant fraction sub-concepts or operations, such as ordering fractions, or adding fractions with different denominators. Because of these limitations, the Indonesian curriculum did not provide enough guidance to teachers about how to promote student conceptual understanding.

Regarding the second point, the Indonesian curriculum did not cover important areas of fraction conceptual knowledge; i.e., it did not cover fractions as measures, improper fractions, fractions as division, and other higher-level conceptual understandings of fractions, such as the unbounded infinity of fractions, density and conceptual understanding of multiplicative fraction operations. On the contrary, the curriculum is dominated by a part-whole understanding of fractions. A number of researchers have argued that the emphasis on part-whole understanding of fractions can stand in the way of students' understanding of improper fractions. Indeed, understanding fractions as measures, by introducing, for example, number line representations, can facilitate this understanding (Fazio & Siegler, 2011), Similarly, introducing fractions as division can assist students to understand the density of fractions, as can a rubber line analogy, which can facilitate understanding of the no-successor principle (Vamvakoussi & Vosniadou, 2012). Understanding fraction multiplication and division should also be included in the curriculum and introduced to students using area models (diagram representations), so that the students can create a conceptual understanding of these procedures (Fazio & Siegler, 2011; Van de Walle et al., 2015).

7.3 Section 2. Student Performance on the Procedural Knowledge Dimension

The results of the Bayesian analysis Model 2 (Table 7.2) showed that about half of the students were at high levels of performance in the procedural knowledge dimension (50.90% of the students were at Levels 4, 5 and 6). These students had procedural skills such as addition, subtraction, multiplication, and division of fractions and they also could do fraction operations with mixed numbers. These students had effectively reached the goals of the Indonesian curriculum for grade 7. Moreover, about a fifth of the students could solve complex fraction operations, which are not specifically taught in the Indonesian curriculum.

Table 7.2 Frequency and percentage of students in the different levels on the procedural knowledge dimension based on the Bayesian Network Analysis (n=516)

Level	Percentage N=516
Level 0 – No procedural knowledge	(30) 5.81%
Level 1 – Additive operations with the same denominator	(105) 20.35%
Level 2 – Additive operations with different denominators	(34) 6.59%
Level 3 – Multiplicative fraction operations	(84) 16.28%
Level 4 – Multiplicative operations with mixed numbers	(161) 31.20%
Level 5 – One nested- Complex fraction operations	(28) 5.43%
Level 6 – Two or more nested- Complex fraction operations	(74) 14.34%

Let us describe the results shown in Table 7.2 in greater detail. Although about 50% of the students reached a procedural competence with the fractions expected at year 7 of the Indonesian curriculum, as mentioned above, the remaining 50% had considerable difficulty with fraction operations. A small number of students (6% percent) were found to be incapable of performing even the simplest operations with fractions correctly. They had difficulties adding fractions with the same denominator (Level 0). Their difficulties in performing simple fraction operations could have been affected by limited conceptual understanding of fractions, as discussed in the previous section.

About 20% of the students were at Level 1 of the procedural knowledge dimension. These students could add fractions with the same denominators, but they had difficulties adding fractions with different denominators. These students had only effectively reached the basic competences of fraction operations at grade 3 of the Indonesian curriculum. The cognitive interview data with students at this level revealed that they did not know the procedure for converting fractions with different denominators into fractions with similar denominators. This was the main obstacle for moving into higher levels of fraction procedural knowledge. Introducing fraction addition with a visual representation such as pie diagram could help these students to understand why a common denominator is required for addition of fractions with different denominators (Fazio & Siegler, 2011; Van de Walle et al., 2015)

Some students (about 7%) were at Level 2 in the procedural dimension, meaning that they could add fractions with different denominators but had difficulties adding fractions involving mixed numbers. This small group of students had achieved the Indonesian curriculum goals at grade 4 – i.e., additive fraction operations with different denominators - but did not know how to transform mixed numbers into improper fractions. This result is consistent with the finding that about 28.68% of the students were at Level 2 of the conceptual knowledge progression, meaning that they had difficulties in understanding improper fractions. This aligns with the study by Arieli-Attali and Cayton-Hodges (2014), which revealed that some of students' errors in fraction operations could be attributed to a lack of understanding of improper fractions.

The students at this level also had difficulties with multiplicative fraction operations. The results from the cognitive interview revealed various mistakes the students made when multiplying fractions. Some only multiplied the numerators and not the denominators if the denominators of the fractions were the same. This mistake indicates that they erroneously applied the principle of fraction addition to the case of fraction multiplication (Brown & Quinn, 2006), i.e., they did not understand why the denominators are kept the same in the results of fraction addition with the same denominator but are multiplied in fraction multiplication (Van de Walle et al., 2015). Visual representations, such as pie or rectangular diagrams, can be used to illustrate that in fraction addition, but not in multiplication, the denominator remains the same in the results.

Placing the competence of adding fractions with the same denominator at a different level (Level 1) from that of the competence of adding fractions with different denominators (Level 2) is an innovation of the present learning progression in the procedural knowledge dimension. Previous fraction learning progressions (Arieli-Attali & Cayton-Hodges, 2014) placed these competencies at the same level. The results from the present research indicate that this differentiation is useful and provides diagnostic information about students' procedural difficulties.

A considerable number of students were at level 3 (16.28%) in the procedural knowledge dimension. They could perform additive operations which involved mixed numbers and multiplicative operations of fractions. These students had reached basic competence for fraction operations at grades 5-6 of the Indonesian curriculum. However, these students had difficulties executing multiplicative operations involving mixed numbers. One of their prevalent mistakes was multiplying mixed numbers directly, without converting them into improper fractions. A direct operation of mixed numbers (without converting them to improper fractions) is correct in the case of addition of mixed numbers, but not in the case of mixed number multiplication (Newton, 2008; Van de Walle et al., 2015). Again this error is evidence that the students misapplied the algorithm of addition to the multiplication of fractions with mixed numbers (Brown & Quinn, 2006). Similar to that in Level 2, this error indicates a deficiency in the students' conceptual understanding of the procedures of addition and multiplication of fractions, which caused these two procedures to become swapped over easily.

The remaining group, approximately 50% of the students, had reached the basic competences of grade 7 in the Indonesian curriculum (Levels 4, 5 and 6 of the present learning progression). Many students were at Level 4 (31.20%). They could execute additive and multiplicative fraction operations, including operations with mixed numbers. These students had difficulties with complex fraction operations. Specifically, they could not simplify nested fraction operations of fractions where either the numerator or the denominator was not a whole number but another fraction operation, as in $1-\frac{2\frac{1}{4}-1}{3}$. The students struggled to figure out the sequence of operations to solve such tasks. The evidence from the cognitive interview showed both that the students were confused by the representation of nested fractions and that they could not identify which part of the operation is the numerator and the denominator. These students were thus unable to simplify the nested fractions into a common form. To assist the students to solve this problem, understanding the structure of complex fraction operations is important.

About 5% of the students were at Level 5. They could complete one-level nested fraction operations in which either the numerator or denominator contained a fraction operation. The students at this level could develop a procedure to solve such complex fraction operations, which demonstrated procedural skills in an unfamiliar context (Brown & Quinn, 2006). However, the Bayesian Network analysis showed that the students at this level had difficulties solving more complex fraction operations with two or more levels of nested fraction operations. According to Brown and Quinn (2006), memorizing algorithms to solve such complex fraction operations is not an effective method. Understanding the representation of the nested fraction operations is critical when simplifying the operations. However, the students also should have a high-level computational fluency to solve such a complex fraction operation. According to Russell (2000) this computational fluency refers to: 1) efficiency in calculation which is when the students are not trapped in irrelevant steps or computations in getting the solution; 2) accuracy, so the students can avoid making careless mistakes in computation; 3) flexibility, which is so the students can choose a suitable computation strategy, which is relevant to the problem given.

Finally, about 14% of students were at the highest level of the procedural knowledge dimension. The results from the cognitive interviews showed that the ability to perform sequential processes of fraction operations was the key in solving such complex operations. These results were consistent with those of Brown and Quinn (2006) who found that computational fluency, as discussed by Russell (2000) above, is required to solve complex fraction operations. Brown and Quinn (2006) found that only 15% of the students from an elementary algebra class of high school students could answer complex fraction operations similar to the ones used in the present study correctly.

In summary, the results showed that 50% of the students had achieved a relatively high-level of procedural knowledge of fraction operations, including additive and multiplicative fractions and operations with mixed numbers. A considerable number of students could also perform complex fraction operations. These students had effectively reached most of the procedural requirements of fraction learning included in the Indonesian curriculum. Nevertheless, the remaining 50% had not reached the goals of the Indonesian curriculum for grade 7. About one-fifth of students were still identified as being at Level 1, i.e., they had difficulties in adding fractions with different denominators.

Overall, the performance of the students in the procedural dimension of the learning progression was better than their performance in the conceptual knowledge dimension. This result can be explained in view of the fact that mathematics instruction in Indonesia is almost entirely procedural (Zulkardi, 2002). However, only about 50% of the students had reached the goals of the Indonesian curriculum at grade 7. This result might be attributable to several reasons. First, the Indonesian curriculum did not provide clear guidance that could help the development of students' procedural knowledge of fractions. For example, in developing multiplicative fraction operations, the Indonesian curriculum at grade 5 states "perform fraction multiplication and division" without specifically mentioning how this should be done. For example, it could be mentioned that there are a number of steps in introducing fraction multiplication or division to students, such as multiplying a unit fraction with whole numbers, which should be introduced first, followed by the procedure for multiplying a fraction by another fraction and so forth.

Another limitation of the Indonesian curriculum is that procedural information is only introduced procedurally, without regard for the students' conceptual understanding. The results in the present research indicate that it was easy for students to misapply the procedures for fraction operations because of a lack of conceptual understanding of the procedures. The students who understand that a common denominator is required to perform additive fraction operations are more likely to recall the correct procedures, compared with students who do not have this conceptual understanding (Fazio & Siegler, 2011). Hence, teaching conceptual understanding of fraction operations should go hand in hand with teaching procedural knowledge of fraction operations.

7.4 Section 3. Interrelations in Student Performance between the Conceptual and the Procedural Knowledge Dimensions

The results discussed in the previous section showed that 76% of the students were distributed at part-whole levels (Levels 1 and 2) on the conceptual knowledge dimension of the learning progression. Meanwhile, on the procedural knowledge dimension, the same students were distributed at several levels of performance (Levels 1, 2, 3, and 4), raising questions about the relationships between the students' conceptual and procedural knowledge.

The results of existing research on the relationship between conceptual and procedural knowledge can be summarized into four different positions or theoretical points of view (Hallett et al., 2010; Rittle-Johnson et al., 2001). The first position argues that students develop their conceptual understanding first and then use this understanding to learn procedures to solve problems in a particular domain of learning. The findings from several studies support this position. Among them is a study on fraction arithmetic conducted by Byrnes and Wasik (1991), which involved 72 students from grades four and six, and 51 students from grade five. The results showed that conceptual knowledge was required to support the learning of the procedure for finding the least common denominator (LCD) in fraction addition with different denominators. In another study, Byrnes (1992) investigated the relationship between conceptual and procedural knowledge in integer operations using a pretest-posttest design, which involved 27 students from grade seven. He found that the conceptual understanding of integers at the pretest was a good predictor of students' computational scores at the posttest. The results of these two studies indicate that conceptual knowledge influences the development of students' procedural knowledge.

The second position argues that students first learn procedures and then, from the repeated experience of applying these procedures, they acquire conceptual understanding in the specific area of mathematics involved (Robert S Siegler, 1991). In a study of counting with a sample of 10 three year olds, Briars and Siegler (1984) found that the counting procedure was developed first, before the children understood the counting principles. Another study by Robert S Siegler and Crowley (1994), with a sample of 23 kindergarten children (about 5 years old), showed that some of the children who could perform whole number addition understood the commutative principle – namely that the order of adding two whole numbers does not change the results – even though they were not taught this principle. This result indicates that they acquired conceptual understanding from their experiences of adding whole numbers. Both of these studies support the position that procedural knowledge is developed first, before the development of conceptual knowledge.

The two above-mentioned positions of the relationship between conceptual and procedural knowledge assume that the relationship between these two types of knowledge is only in one direction, either from conceptual knowledge to procedural, or vice versa. The third position states that there can be a bi-directional relationship between conceptual and procedural knowledge (Rittle-Johnson et al., 2001). This position argues that the development of conceptual and procedural knowledge is interrelated and reciprocal, and that conceptual knowledge can influence the development of procedural knowledge and vice versa (Rittle-Johnson & Alibali, 1999; Rittle-Johnson & Schneider, 2014; Rittle-Johnson et al., 2001). Some of the evidence that supports the third position comes from a study by Rittle-Johnson and Alibali (1999) in which a sample of 60 students from grade four and 29 students from grade five were used to investigate students' knowledge on equivalence problems in mathematics. They found that students who received procedural instructions had a better conceptual understanding of equivalence problems. Meanwhile, the students who received conceptual instruction also developed a correct and flexible procedure for solving these problems. The authors interpreted their results to indicate that the relationship between conceptual and knowledge is iterative, as increasing of one type of knowledge influences the other type and vice versa.

Hallett et al. (2010) proposed individual differences as the fourth position to explain the different and contradictory findings of the previous research. The hypothesis of individual differences highlights that students are different in the way they combine conceptual and procedural knowledge, and that some students rely more on their conceptual knowledge, while other students rely more on procedural knowledge. The individual differences position holds a similar assumption to the third position regarding the bi-directional relationship between procedural and conceptual knowledge but holds that, in addition, there are individual differences in students' preferences for a conceptual and/or procedural approach. The individual differences position is supported by the results of a study by Hallett et al. (2010) on fraction learning using a sample of 318 students from grades four and five. The results showed five clusters of students, reflecting different combinations of conceptual and procedural knowledge. Some of the students were strong in conceptual knowledge, while others were strong in procedural knowledge. Using a much smaller sample (seven students from grade nine), a case study by Bempeni and Vamvakoussi (2015) also investigated individual differences in conceptual and procedural knowledge in fraction learning. The results revealed three student profiles: students with strong conceptual and procedural knowledge, students with strong procedural knowledge but weak conceptual knowledge, and students with strong conceptual knowledge but weak procedural knowledge. The authors concluded that their results provided evidence for the individual differences position.

The present research used a more comprehensive test of fraction knowledge, a larger sample (516 students) and older students (grades 7 to 9; between 12 – 15 years old) compared with the Hallett et al. (2010) study. Therefore, the present research is in a position to provide evidence to better discriminate amongst the various positions. In order to examine the data, first, a cross-tab analysis was performed to explore the pattern of the distribution of the students across the different levels of conceptual and procedural knowledge dimensions. Next, a cluster analysis was employed to investigate whether the levels of the conceptual and procedural knowledge dimensions could be grouped into clusters.

The results of the cross-tab analysis (Table 7.3) showed the different profiles of the students in terms of their conceptual and procedural levels. Almost half of the students (248/560 – 48% of the sample) were at Level 1 in the conceptual dimension. Out of these 248, 106 students were either in Level 0 in the procedural dimension (23/248) or Level 1 (83/248). We could say that these students (about 20% of the whole sample) had both a very limited conceptual understanding of fractions and a very limited procedural knowledge of fraction operations.

A second group at Level 1 conceptual knowledge comprised 127/248 students, who were distributed at Levels 2 (19/248), Level 3 (44/248), and Level 4 (64/248) of the procedural knowledge dimension. Although the students in this group (about 25% of the whole sample) also had a very limited conceptual understanding of fractions, they demonstrated better procedural knowledge. They could perform additive (Level 2) and multiplicative fraction operations (Levels 3 and 4).

Table 7.3 Cross-tabulation of the conceptual and procedural levels of student performance on the fraction learning progression

			Procedural							Total
			0	1	2	3	4	5	6	
Conceptual	0	Count	2	0	1	0	0	0	0	3
		% of Total	0.4%	0.0%	0.2%	0.0%	0.0%	0.0%	0.0%	0.6%
	1	Count	23	83	19	44	64	8	7	248
		% of Total	4.5%	16.1%	3.7%	8.5%	12.4%	1.6%	1.4%	48.1%
	2	Count	5	14	11	30	61	6	21	148
		% of Total	1.0%	2.7%	2.1%	5.8%	11.8%	1.2%	4.1%	28.7%
	3	Count	0	3	3	9	23	11	30	79
		% of Total	0.0%	0.6%	0.6%	1.7%	4.5%	2.1%	5.8%	15.3%
	4	Count	0	5	0	1	7	2	10	25
		% of Total	0.0%	1.0%	0.0%	0.2%	1.4%	0.4%	1.9%	4.8%
	5	Count	0	0	0	0	5	1	2	8
		% of Total	0.0%	0.0%	0.0%	0.0%	1.0%	0.2%	0.4%	1.6%
	6	Count	0	0	0	0	1	0	4	5
		% of Total	0.0%	0.0%	0.0%	0.0%	0.2%	0.0%	0.8%	1.0%
Total		Count	30	105	34	84	161	28	74	516
		% of Total	5.8%	20.3%	6.6%	16.3%	31.2%	5.4%	14.3%	100.0%

The last group at Level 1 of the conceptual knowledge dimension were a small number of students (15/248) who were distributed at Level 5 (8/248) and Level 6 (7/248) in the procedural knowledge dimension. The students in this group (about 3% of the whole sample) exhibited the same limited conceptual fraction understanding as in the two previous groups but could still perform complex fraction operations.

The above results indicate that there can be substantial individual differences associated with the learning of fractions, given that the students who were at Level 1 of conceptual understanding ranged from no procedural knowledge of fractions to the performance of complex fraction operations.

Let us now examine the results for the 148/516 students at Level 2 of conceptual understanding. The results showed that these students (approximately 29% of the total sample) were also distributed across all the levels in the procedural knowledge dimension. However, a smaller percentage of the students, at Level 2 conceptual knowledge, were at the two lowest levels of the procedural knowledge dimension – only 19/148 – 12.84% – compared with 106/248 – 42.74% – in the case of Level 1 conceptual knowledge. The bulk of the students at level 2 conceptual knowledge were grouped at levels 2, 3 and 4 for procedural knowledge (102/148 - 68.92%).

The results from the analysis of the relationships between Level 2 conceptual and procedural knowledge again confirm a conclusion of individual differences. However, they also indicate some dependencies between conceptual and procedural understanding, given that the students at Level 2 of conceptual understanding seem to be a little more advanced in their procedural knowledge overall compared with the students at Level 1 conceptual knowledge.

The examination of the students at Level 3 of the conceptual knowledge dimension shows that these students were predominantly grouped at Levels 4 (23/79 – 29.11%), 5 (11/79 – 13.92%), and 6 (30/79 – 37.98%) of the procedural knowledge dimension. These results, again, confirm the presence of individual differences. However, some dependencies between conceptual understanding and procedural knowledge also can be observed. High level conceptual knowledge, understanding improper fractions and fractions as measures, was associated with high level procedural knowledge - Levels 4, 5, and 6.

A similar pattern can be observed when examining students at Levels 4, 5, and 6 of the conceptual knowledge dimension. Most of the students at these levels were grouped at Levels 4, 5, and 6 of the procedural knowledge dimension. Only 6 out of 25 students at Level 4 of the conceptual knowledge dimension were identified at Levels 1 (5/25) and 3 (1/25) of the procedural knowledge dimension respectively.

The above results support the hypothesis that there are individual differences in learning fractions. However, the results also indicate considerable dependencies between conceptual and procedural knowledge. In order to investigate these dependencies further, a cluster analysis was performed using the R Package ClustOfVar (Chavent, Kuentz, Liquet, & Saracco, 2011).

The results of the cluster analysis (Figure 7.1) identified three clusters. Cluster 1 consisted of conceptual level 1 (ConL1) and procedural levels 1, 2, 3 and 4 (ProL1, ProL1, ProL2, ProL3, and ProL4). Cluster 2 consisted of conceptual level 2 (ConL2) and procedural levels 5 and 6 (ProL5 and ProL6). Cluster 3 consisted of conceptual levels 4, 5, and 6 (ConL4, ConL5, and ConL6).

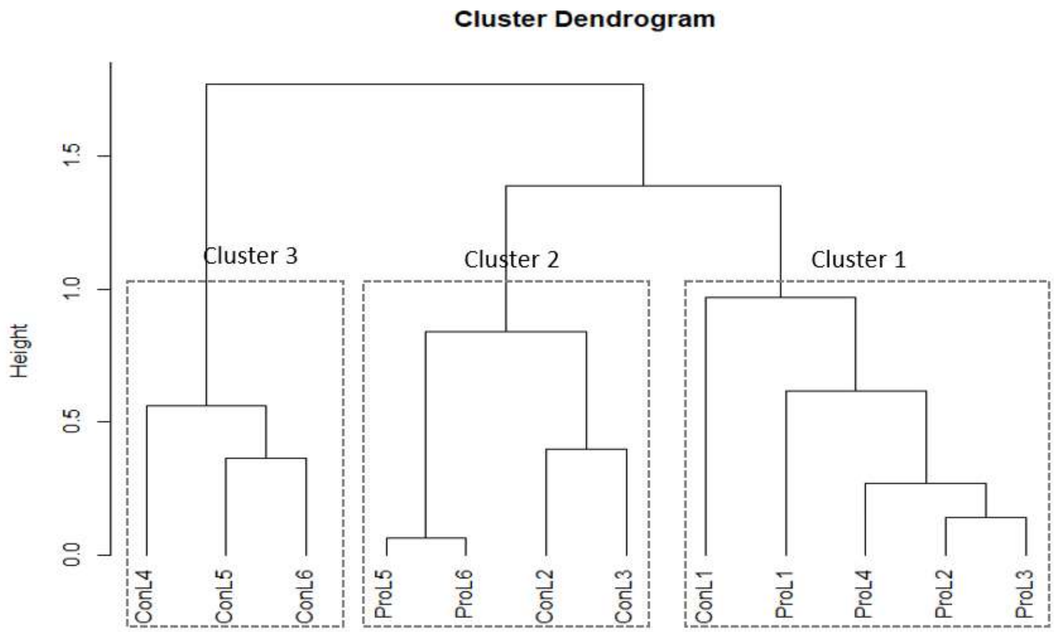


Figure 7.1 A dendrogram of the cluster analysis for the conceptual and procedural levels of the fraction learning progression

Cluster 1 showed that Level 1 of the conceptual knowledge dimension was highly correlated with the procedural knowledge of additive and multiplicative fraction operations (Levels 1 to 4). This result is consistent with the crosstab analysis, which revealed considerable numbers of students at Levels 1 to 4 in the procedural knowledge dimension were at Level 1 of the conceptual knowledge dimension. This finding indicates that an understanding of the symbolic notation of fractions as part-whole is capable of supporting additive and multiplicative fraction operations.

Clusters 2, 5 & 6 level of procedural knowledge (complex fraction operations) was highly correlated with Levels 2 and 3 of the conceptual knowledge dimension. These results align with the crosstab analysis, which showed that the number of students in procedural knowledge Levels 5 and 6 were concentrated in Levels 2 and 3 of the conceptual knowledge dimension. This finding indicates that understanding improper fractions and fractions as measures is associated with procedural knowledge of complex fraction operations.

Cluster 3 consisted of the three highest levels of conceptual knowledge (Levels 4, 5 and 6) only. It did not include any levels from the procedural knowledge dimension. This result indicates that conceptual knowledge above Level 3 did not help the students to answer the procedural questions in the test. It is of course possible that the students with the higher conceptual understanding could potentially solve other more difficult procedural problems which required understanding of the density of fractions and multiplicative fraction operations that were not included in the present fraction learning progression.

To conclude, the results confirmed that there are considerable individual differences in students' procedural knowledge within every individual level of the conceptual knowledge dimension. These individual differences could be related to differences in classroom instruction (Hecht & Vagi, 2012) and/or differences in individual students' learning approaches (Bempeni & Vamvakoussi, 2015). Despite the presence of individual differences, both the crosstab and cluster analyses indicated that there are also dependencies between the levels of the conceptual and procedural knowledge dimensions. A basic understanding of part-whole fractions was associated with the procedural knowledge of additive and multiplicative fraction operations, while conceptual understanding of improper fractions and fractions as measures was associated with complex fraction operations. Conceptual understanding at higher levels did not contribute to increased procedural knowledge in the context of the present learning progression. However, this conceptual understanding might be required for the development of more advanced procedural knowledge, for which an understanding of the density concept might be necessary.

7.5 Summary and Conclusions

The results of the Bayesian Network analysis showed that 48% of the students were grouped at part-whole levels of conceptual understanding of fractions and that only a small number of students could understand improper fractions, place a fraction on the number line or exhibit higher level conceptual knowledge, such as understanding the density of factions and multiplicative fraction operations. These results were explained on the grounds that the Indonesian curriculum did not cover important areas of fraction conceptual knowledge and/or because it was ineffective in developing conceptual understanding, even in the areas that it covered. It was recommended that the Indonesian curriculum needs to be revised to introduce alternatives to part-whole representations of fractions, such as number lines and also introduce fractions as division in order to facilitate students' understanding of improper fractions and the unbounded infinity of fractions.

The results showed that the performance of the students was better in the procedural knowledge dimension, compared with their conceptual knowledge. About 50% of the students had reached the goals of the Indonesian curriculum at grade 7 and could perform the basic fraction operations as well as more complex operations with mixed numbers. This finding was explained on the grounds that the mathematics instruction in Indonesia is mostly procedural. Despite that, the remaining 50% of the students had not reached the goals of the curriculum at grade 7 and had difficulties, particularly with fraction multiplication and division. This finding was explained on the grounds that the Indonesian curriculum did not provide clear guidance as to how to develop students' procedural knowledge effectively, with procedural skills being taught independently of conceptual understanding. Consequently, students often made mistakes in the application of the procedural processes and misapplied the algorithms for addition and multiplication.

The third important finding relates to the interrelationship between conceptual and procedural knowledge. The present research used a more comprehensive test of fraction knowledge, a larger sample and older students than prior research and was thus in a position to provide more conclusive evidence regarding the relationships between conceptual and procedural knowledge of fractions. The results supported the hypothesis that there are considerable individual differences in how students combine conceptual and procedural knowledge. However, there were also important dependencies between conceptual and procedural knowledge, indicating that a basic understanding of fractions as parts of a whole can support additive and multiplicative fraction operations but a higher conceptual understanding of fractions as measures and as division is associated with complex fraction operations. Knowledge of the density concept of fractions was not associated with additional procedural competencies in the present learning progression but it is possible that it might be associated with the performance of more complex fraction operations not tested in the present research.

CHAPTER 8 : DISCUSSION

8.1 Introduction

The research conducted in this dissertation designed and validated a two-dimensional learning progression of fractions. The learning progression was based on two types of mathematical knowledge i.e., conceptual and procedural. The empirical validation of the progression was conducted through two sequential stages: a qualitative study using a cognitive interview (Chapter 4) and a quantitative study using Bayesian Networks analysis (Chapter 6). The validation was guided by the assessment triangle (Pellegrino et al., 2001), based on a cognitive model of how students learn fractions.

In the conceptual knowledge dimension, there were seven validated levels ranging from no understanding of fractions to understanding the unbounded infinity and density of fractions and understanding multiplicative fraction operations. In the procedural knowledge dimension, there were seven validated levels ranging from a lack of any procedural knowledge of fractions to understanding nested complex fraction operations. The results revealed that most of the students were at the lower levels of the conceptual knowledge dimension but on the higher levels of the procedural knowledge dimension (Chapter 7).

An important result of the research is the development of the two measurement models using Bayesian Networks. These measurement models were used to assess and validate the learning progression at both the item and student levels. Two Bayesian Network models were developed, namely Model 1 - Bayesian Networks with a single latent variable, and Model 2 - Bayesian Networks with multiple latent variables (Chapter 5). The results showed that Model 2 had a better fit with the students’ responses than Model 1 and had more desirable properties for measuring and diagnosing students’ learning progression (Chapter 6).

8.2 Discussion of the Research Findings

8.2.1. The Development of a Cognitive Model of Two-Dimensional Fraction Learning Progression

The fraction learning progression developed in this research makes a significant contribution to existing research on the assessment of mathematics learning. It is the first learning progression that has developed a two-dimensional fraction learning progression that distinguishes conceptual and procedural knowledge in mathematics. Differentiating these two types of knowledge is important because it provides additional information on the basis from which the students' mathematical knowledge can be assessed. The resulting learning progression can provide more accurate profiles of students' progression levels than previous research, which did not distinguish conceptual knowledge and procedural knowledge (Arieli-Attali & Cayton-Hodges, 2014; Confrey et al., 2011).

The learning progression developed in this research also included essential aspects of fraction conceptual knowledge, such the unbounded infinity and density of rational numbers, and the conceptual understanding of multiplicative fraction operations. Prior research investigated students' understanding of the unbounded infinity of fractions (Stafylidou & Vosniadou, 2004) and of density (Vamvakoussi & Vosniadou, 2004). The results of this research have not been utilized so far in prior assessments of students' fraction knowledge. The present research utilized these research findings in order to develop an assessment of the students' conceptual knowledge of fractions. Items were created that tested students' understanding of the unbounded infinity of fractions and of density in the context of the fraction learning progression. As a result, it was possible to evaluate the emergence of this knowledge and compare it with the emergence of other aspects of conceptual understanding of the symbolic notation of fractions, such as part-whole, improper fractions, fractions as measures and understanding multiplicative fraction operations. The results of the Bayesian Networks analysis showed that understanding the unbounded infinity of fractions was conditional on students' understanding of improper fractions and of fractions as measures, while understanding fraction density was shown to be conditional on students' understanding of the unbounded infinity of fractions.

Another innovation of the present research is that the conceptual dimension included items to assess students' understanding of the operation of fraction multiplication. The results showed that this understanding was the last to be developed and that it was conditional on students' understanding of fraction density. The empirical sequence of the emergence of students' conceptual knowledge of fractions, covering aspects of conceptual knowledge not included in previous assessments (e.g., Arieli-Attali & Cayton-Hodges, 2014; Confrey et al., 2011) is a significant contribution of the present research.

Let us now discuss the development of the procedural knowledge dimension of the learning progression in the present research. The first innovation of the present research was the construction of a separate scale to assess this aspect of students' fraction knowledge. The second innovation was the extension of the procedural knowledge of fractions investigated, compared with previous research. Although students' procedural knowledge of fraction operations was not described in a separate scale in previous learning progressions, several aspects of this competence were included in previous research. These typically assessed students' additive and multiplicative fraction operations (Arieli-Attali & Cayton-Hodges, 2014; Confrey et al., 2011). Extending the progression of procedural learning from previous studies, the present research designed items and validated students' additional competencies to perform complex fraction operations empirically, for example for nested operations of fractions where the numerator or denominator were not a whole number but another fraction operation. The results showed that about 20% of the students were at levels indicating competencies in performing complex fraction operations (Levels 5 and 6). The significant number of students at these levels demonstrates the importance of including these competencies in mathematics assessments.

In terms of the investigation of the relationship between conceptual and procedural knowledge, the present research utilized a larger sample size and older students than the previous research. Moreover, it developed assessment instruments which covered comprehensive aspects of the conceptual and procedural knowledge of fraction learning, ranging from no understanding of the fraction symbols to understanding multiplicative fraction operations, and from no procedural knowledge to procedural knowledge of complex fraction operations, respectively. Specifically, this research examined the relationship between conceptual and procedural knowledge based on a validated cognitive model of learning progression. Hence, the information about the relationship between conceptual and procedural knowledge of learning fractions obtained from this research is unique because it was contextualized in the progression of students' learning. No previous studies investigated the relationsship between conceptual and procedural knowledge intertwined with the development of students' learning.

The results of the present research supported the hypothesis that there are significant individual differences in students' conceptual and procedural knowledge (Hallett et al., 2010; Hallett, Nunes, Bryant, & Thorpe, 2012). However, the findings also showed some dependencies between the various levels of conceptual and procedural knowledge. More specifically, a basic understanding of fractions as a representation of part-whole was highly correlated with the procedural knowledge of additive and multiplicative fraction operations, while a conceptual understanding of improper fractions and fractions as measures was highly correlated with the procedural knowledge of complex fraction operations. These findings enrich the results of previous research on the relationship between conceptual and procedural knowledge in learning fractions. For example, Byrnes and Wasik (1991) found that conceptual knowledge and procedural knowledge of fractions was moderately correlated (r=0.5, p<0.01), but did not give more information about the nature of the specific conceptual understanding that correlated with the procedural knowledge of additive or multiplicative fraction operations. In contrast, this research provided information about how the various levels of conceptual understanding of fractions (part-whole, improper fractions, or fractions as measures) correlated with students' procedural knowledge of fraction operations.

The results of the present research challenge the previous research on learning progressions in fractions and in mathematics in general. As discussed in Chapters 2 and 3, previous learning progressions did not differentiate conceptual and procedural competencies within the levels of the learning progressions (Arieli-Attali & Cayton-Hodges, 2014; Confrey et al., 2011). For example, in the rational number learning progression developed by Arieli‑Attali and Cayton‑Hodges (2014), conceptual knowledge of fractions as representations of part-whole emerged at Level 2, together with the emergence of procedural knowledge of fraction addition being limited to fractions with the same denominator. However, the findings from the present research showed that students who had limited part-whole level understanding (Level 1) in the conceptual knowledge dimension had large differences in their procedural knowledge. The findings of the present research indicate that the learning progressions developed from the previous studies, in which conceptual and procedural knowledge were placed at the same level in the progression, did not accurately represent differences in students' mathematical knowledge. Therefore, the cognitive model of the two-dimensional learning progression offers a better identification of the position of the students along their learning journey, which is the main goal of educational assessment (Masters, 2013).

The profiles of individual differences of conceptual and procedural levels can be useful in order to diagnose students' difficulties in their learning and provide information to teachers that enables them to identify gaps in students' conceptual and procedural knowledge. This information can then be used to tailor teaching to better meet students' needs (Black & Wiliam, 1998; Huff & Goodman, 2007; Pellegrino et al., 2001; Richard J Stiggins, 2002). Hence, the analysis of individual differences in the relationship between conceptual and procedural knowledge is one of the significant contributions of this research to the development of formative assessment.

The two-dimensional learning progression also can contribute to the development of assessment in modern environments such as online learning. Timms (2017) highlighted that the most important thing in developing assessment for online learning is the clarity of the learning goals. The two-dimensional learning progression not only provides clear learning goals in every level of students' progression but also guides both instruction and assessment to achieve the learning goals. Particularly, this learning progression provides a guidance to achieve learning goals on the specific knowledge dimensions of conceptual and procedural, which are the core knowledge of mathematics learning (Hiebert & Wearne, 1996; Rittle-Johnson & Schneider, 2014).

In summary, the development of a two-dimensional fraction learning progression in the present research has produced important results for (1) assessment (2) instruction and (3) curriculum development. In the area of assessment, the two-dimensional learning progression developed in this research is innovative because for the first time it differentiates students' conceptual and procedural knowledge. The results showed that separating these two-knowledge dimensions results in more accurate assessment than those learning progressions that do not distinguish these two key types of knowledge. Specifically, the results that showed the profiles of individual differences in the conceptual and procedural knowledge dimensions are useful to produce diagnostic information about students' progression and their learning challenges. This is a significant contribution to the development of formative assessments. Moreover, the present research developed assessment instruments which assess important aspects of conceptual knowledge (including the unbounded infinity and density and understanding multiplicative fraction operations) and procedural knowledge (including complex fraction operations) comprehensively. As a consequence, the emergence of essential aspects of conceptual and procedural knowledge in learning fractions and their relationships could be examined.

In the area of instruction, the two-dimensional learning progression provides a road map. In the conceptual knowledge dimension, teachers are informed about how students develop their conceptual understanding from no understanding of fractions until they reach high level conceptual understanding, such as infinity and understanding multiplicative fraction operations. Likewise, in the procedural knowledge dimension, teachers are informed about how students develop their procedural knowledge from no valid procedural knowledge until they can perform complex fraction operations. Moreover, this learning progression provides information about students' learning challenges at different levels of students' progression in learning, which can assist teachers to develop more effective instruction.

In the area of curriculum development, the two-dimensional learning progression covered many essential aspects of conceptual and procedural knowledge of learning fractions. In the conceptual knowledge dimension, the learning progression includes understanding the symbolic notation of fractions as a representation of part-whole, fractions as measures, the unbounded infinity of fractions and understanding multiplicative fraction operations. In the procedural knowledge dimension, the curriculum covered additive and multiplicative operations and complex fraction operations. All these materials are structured in the progression levels and can be used in the curriculum to organize these materials across grades in schools. This two-dimensional learning progression is useful to balance the conceptual and procedural aspects of fraction learning, given that many curricula favour the procedural aspect of learning and teaching fractions.

8.2.2 The Development of Bayesian Networks Models

Current practice for measuring students' learning progression typically uses Bayesian Networks with a single parameter/Bayesian Latent Class Analysis (Model 1) (Jeffrey T Steedle & Shavelson, 2009; West et al., 2012). Model 1 in the present research followed this tradition. The levels of learning progression were assumed to be independent of each other in Model 1. Consequently, the dependency between the levels in the learning progression were not formulated in a formal statistical model. A Bayesian Networks Model 2, with multiple latent variables, was developed in order to address the limitations of Model 1. Model 2 was developed to reflect the hierarchical dependency between the levels assumed in the learning progression model (Popham, 2007).

Model 2 combined a cognitive model of a learning progression with a Bayesian statistical approach. This modelling approach is a significant contribution to the development of measurement models in the context of cognitive assessments (Pellegrino et al., 2001). The results of the present research showed that Model 2 had a better fit than Model 1 and superior properties in terms of the diagnostic analytics of students' competencies, pseudo-guessing analysis and detecting extreme cases in students' responses.

With respect to the validation of the learning progression, the results showed that both Bayesian Networks, Models 1 and 2, could be used effectively to validate both the item level analysis and the students' level analysis. These analyses are important in order to support statistical inferences at the item level and at the student level. Item level inference supports the interpretation that items that were assigned to a certain level would be answered correctly by the students at that level or the upper level(s). Student level inference supports the interpretation that students who were assigned to certain levels would have sufficient competencies at that level and below, but would not have enough competencies at the upper level(s) respectively (adapted from West et al., 2010). These types of inferences provide a clear explanation as to why certain items or students were placed at certain levels. These inferences can only be performed if the measurement model is developed based on a discrete latent scale, so that the item and student analyses can be performed directly on the discrete levels of the learning progression. Developing these two types of inferences to validate a learning progression model is a significant contribution of this research. There are no previous studies that used a similar approach to perform the validation of a learning progression.

Corcoran, Mosher, and Rogat (2009) considered learning progressions as “testable hypotheses” (p.15) of students’ learning. In order to evaluate these hypotheses a confirmatory approach is preferable to an exploratory approach because such an approach makes it possible to examine whether the hypotheses are supported by the data. The Bayesian Networks approach adopted in the present research can be considered confirmatory because the analysis was performed based on the predetermined levels developed from both theory and previous empirical research on fraction learning. The two types of analyses that make Bayesian Networks confirmatory in this research are the item level analysis and the student level analysis. The item analysis was performed to examine whether the items that were hypothesized to be at a certain level would be correctly answered by the students at that level and above, but incorrectly answered by the students at the level(s) below; while the student level analysis was used to estimate students’ location on the hypothesized levels of the conceptual and procedural knowledge dimension.

This confirmatory analysis is different from the exploratory approach using Bayesian Latent Class Analysis by West et al. (2010), which produced several Bayesian Networks with different levels of learning progressions and then selected the model with the best fit. Such an exploratory approach is also used in other methods, such as the Rasch Model. For example, Wilmot et al. (2011) plotted item difficulty and students’ abilities on the same latent continuum scale. They then developed cut-off points based on the group items, which appeared on the scale to identify the levels of learning progression. They did not perform item analyses directly based on the hypothesized levels of the learning progression. In fact, the item analyses were performed on the empirical levels generated from students' responses. The advantage of performing confirmatory analysis in the present research is the capacity to test the hypothesized levels of the learning progression directly. The confirmatory approach developed in this research is a significant contribution to the assessment literature on learning progressions.

Another significant contribution of the present research is the development of item analyses, such as item difficulty, item discrimination, and pseudo-guessing analysis using Bayesian Network models. These item analyses have been well-established in CTT and IRT models (Crocker & Algina, 2008; De Ayala, 2009; Nitko & Brookhart, 2007). However, as discussed before in Chapter 6, the item analyses developed from CTT and IRT were different from the Bayesian Networks Item analyses developed in this study. The fundamental differences are on the assumptions of these models. The Item analyses in IRT and CTT were developed based on the continuum scale of latent ability, while the Item analyses in Bayesian Networks were developed from the assumption of a discrete scale of the students' latent progression in learning. In practice, Bayesian Network Item analysis is preferable for validating a learning progression because the analysis can be applied to the discrete levels of the learning progression directly. Developing Bayesian Network Item analysis for validating the hierarchical levels of a learning progression is the most important contribution of this study to the field of educational assessment and measurement.

In summary, the Bayesian Networks models (Models 1 and 2) developed in this research are a significant contribution to educational measurement in terms of measuring and validating learning progressions. First, the Bayesian Networks models have taken into account prior information about student knowledge into the models and have measured the uncertainties of each of the parameters in the models. Second, the Bayesian Networks Model 2, with multiple hierarchical latent variables, successfully addressed the limitations of Bayesian Networks Model 1 in representing the hierarchical dependency between the levels by accommodating this dependency into a formal statistical model. Next, both models support a confirmatory approach for evaluating the hypothesized levels of a learning progression through two types of analyses: item level and student level analyses. These analyses can test the hypothesized levels of the learning progression directly. Finally, the present research, for the first time, has developed Bayesian Networks item analyses such as item difficulty, item discrimination and pseudo-guessing analysis, which were originally developed in CTT and IRT, in order to validate the hierarchical, discrete levels of a learning progression.

8.3 Limitations and Recommendation for Further Studies

This study has several limitations. The sample in this study was taken from just one of the junior public schools in Indonesia. The research findings may not represent the whole population of students at grades 7-9 in Indonesia or in other countries. Hence, a larger study sampling student from other junior public schools across the country is required so that the findings can be generalized into the national context of Indonesia. The Bayesian Networks models developed in this research are specific for a hierarchical setting that fits into the current context of learning progressions. Further research is needed to investigate different types of hierarchical settings, depending on other contexts.

Moreover, the Bayesian Networks models developed here are based on specific, informative, prior information. The prior could affect the posterior estimates when the number of samples is small as found in the higher levels of the fraction learning progression model. Furthermore, the research context needs informative prior, not uninformative. It is therefore recommended to conduct a sensitivity analysis using different informative prior information in future research.

Moreover, the Bayesian Networks models developed here are based on specific and informative prior information, which were obtained from previous studies or expert opinion. For complex models with many parameters such as the models developed in this thesis, the choice of priors and conclusions of the subsequent Bayesian analysis are usually validated through a prior sensitivity analysis. Given the context of this research that prior information is usually informative, the prior sensitivity analysis can be conducted using different types of informative priors, weakly informative or strongly informative. This thesis used moderate informative priors. As a future work in this research, a prior sensitivity analysis can be performed using either weakly or strongly informative prior distributions.

In Bayesian perspectives, large sample sizes become more important especially in the context of strong prior being that applied in the analysis. The limitation of this research has shown that at the higher levels, the posterior estimates are getting close to the priors due to a small number of students at these levels. However, it is worth to note that the small number of students in the high levels of fraction learning progression is not surprising because it is consistent with the previous research that conceptual understanding of fractions at high levels were difficult for students (Vamvakoussi & Vosniadou, 2004, 2010). Therefore, by considering the context and the theory of development of fraction learning, the results of Bayesian estimation are retained in this study by acknowledging that in some cases the prior could affect the posterior estimates as a limitation which is inherent in the Bayesian Network approach when dealing with small number of samples. This issue can be addressed by simultaneously increasing the sample sizes of the study and using weakly informative prior for the Bayesian Network models proposed in this thesis.

The Bayesian Networks models were implemented for each dimension, procedural and conceptual. Future research is required to develop a bivariate analysis of Bayesian Networks for these two knowledge dimensions. Finally, the cut-off probabilities for placing items and students along the progression levels were not well-established in the literature. Future work is needed to set a threshold of probabilities to be used for validating and measuring students’ learning progressions.

APPENDICES

Appendix A. Research Ethics Approval

FINAL APPROVAL NOTICE

Project No.:	**7200**
Project Title:	Assessing Learning Progressions in the Domain of Rational Numbers
Principal Researcher:	Mr Bakir Haryanto
Email:	hary0006@flinders.edu.au

Approval Date:	19 May 2016	Ethics Approval Expiry Date:	**15 December 2018**

The above proposed project has been **approved** on the basis of the information contained in the application, its attachments and the information subsequently provided with the addition of the following comment(s):

Additional information required following commencement of research:

1. Permissions
   Please ensure that copies of the correspondence granting permission to conduct the research from the principals of the schools in Bogor, Indonesia are submitted to the Committee *on receipt*. Please ensure that the SBREC project number is included in the subject line of any permission emails forwarded to the Committee. Please note that data collection should not commence until the researcher has received the relevant permissions (item D8 and Conditional approval response – number 12).

Appendix B. Letter of Introduction for School Principals

David D Curtis
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LETTER OF INTRODUCTION
(For School Principals)

To whom it may concern

This letter is to introduce Bakir Haryanto who is a Doctor of Philosophy (PhD) student in the School of Education at Flinders University. He will produce his student card, which carries a photograph, as proof of identity.

He is undertaking research leading to the production of a thesis or other publications on the subject of "Mathematics Learning" which investigates the progression of students' fraction learning in terms of conceptual and procedural knowledge.

He would like to invite you to assist with this project by agreeing to allow students in your school to be involved in an interview, and test. The interview should last about 45 minutes, and the test should last 120 minutes. Be assured that any information provided will be treated in the strictest confidence and none of the participants will be individually identifiable in the resulting thesis, report or other publications.

Since he intends to make a digital recording of the interview, he will seek parents' consent, on the attached form, to record the interview, to use the recording or a transcription in preparing the thesis, report or other publications, on condition that children's names or identities are not revealed, or that the recording will not be made available to any other person. You may be assured that the confidentiality of the material will be respected and maintained at all time during the research process.

Any enquiries you may have concerning this project should be directed to me at by e-mail to: david.curtis@flinders.edu.au

Thank you for your attention and assistance.

Yours sincerely

Associate Professor Dr David Curtis
School of Education
Flinders University

This research project has been approved by the Flinders University Social and Behavioural Research Ethics Committee (Project number 7200.). For more information regarding ethical approval of the project the Executive Officer of the Committee can be contacted by telephone on 8201 3116, by fax on 8201 2035 or by email human.researchethics@flinders.edu.au

Appendix C. Letter of Introduction for Parents

Flinders UNIVERSITY

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He is undertaking research leading to the production of a thesis or other publications on the subject of "Mathematics Learning" which investigates the progression of students' fraction learning in terms of conceptual and procedural knowledge.

He would like to invite you to assist with this project by agreeing to allow your child to be involved in an interview and test. The interview should last about 45 minutes and the test should last 120 minutes. Be assured that any information provided will be treated in the strictest confidence and none of the participants will be individually identifiable in the resulting thesis, report or other publications. Your child is, of course, entirely free to discontinue your participation at any time or to decline to answer particular questions.

Since he intends to make a digital recording of the think aloud protocol and interview, he will seek parents' consent, on the attached form, to record the interview, to use the recording or a transcription in preparing the thesis, report or other publications, on condition that children's names or identities are not revealed, or that the recording will not be made available to any other person. You may be assured that the confidentiality of the material will be respected and maintained at all time during the research process.

Any enquiries you may have concerning this project should be directed to me at by e-mail to: david.curtis@flinders.edu.au

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Appendix D. Information Sheet for School Principals

Bakir Haryanto
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INFORMATION SHEET
School Principals

Title: Learning in Mathematics

Researcher:
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Professor Dr Stella Vosniadou
School of Education, Flinders University
GPO Box 2100 Adelaide SA 5001
Email : stella.vosniadou@flinders.edu.au

Description of the study:
This study is part of the project entitled 'Learning in Mathematics'. This project will investigate the development of students' fraction learning and their difficulties.
This project is supported by the Faculty of Education, Humanities, and Law, Flinders University, Australia.

Purpose of the study:
This project aims at

- To assess students' learning progression in the domain of fractions
- To diagnose students' difficulties in fractions

What will I be asked to do?
You will be asked to approve grade 7, 8, and 9 participations for the following activities:

- ***Interview***
  Children will be given several mathematics item tasks, and they will be asked to speak loudly to describe their thinking process when they solve the tasks. It will take no more than 45 minutes.

inspiring achievement

- ***Test***

  Children will be asked to complete the test of fractions. It will take 120 minutes.

What benefit will children gain from being involved in this study?
The sharing of children's experiences will improve the planning and delivery of fraction teaching-learning and inform the curriculum designers about the progression of fraction learning that can be used to refine mathematics curriculum in the future.

Will children be identifiable by being involved in this study?
We do not need children's' name and they will be anonymous. Once the think aloud protocol and interview has been typed-up and saved as a file, the voice file will then be destroyed. Any identifying information will be removed and the typed-up file stored on a password protected computer that only the coordinator (Mr Bakir Haryanto) will have access to. Your children's comments will not be linked directly to your children.

Are there any risks or discomforts if children are involved?
There will be no risks or discomfort in children's involvement. However, if you have any concerns regarding anticipated or actual risks or discomforts, please raise them with the researcher.

How do children agree to participate?
Participation is voluntary. Children may answer 'no comment' or refuse to answer any questions and you are free to withdraw from the focus group at any time without effect or consequences. A parent consent form accompanies this information sheet. If children agree to participate please read and sign the form with their parents and send it back to the researcher prior to the start of the study.

How will I receive feedback?
Outcomes from the project will be summarised and given to you by the investigator if you would like to see them.

Thank you for taking the time to read this information sheet and we hope that you will accept our invitation to be involved.

This research project has been approved by the Flinders University Social and Behavioural Research Ethics Committee (Project number 7200). For more information regarding ethical approval of the project the Executive Officer of the Committee can be contacted by telephone on 8201 3116, by fax on 8201 2035 or by email human.researchethics@flinders.edu.au

Appendix E. Information Sheet for Parents

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INFORMATION SHEET
Parents

Title: Learning in Mathematics

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Purpose of the study:
This project aims at

- To assess students' learning progression in the domain of fractions
- To diagnose students' difficulties in fractions

What will your child be asked to do?

- ***Interview***
  Your child will be given several mathematics item tasks, and they will be asked to speak loudly to describe their thinking process when he/she solves the tasks. It will take no more than 45 minutes.

inspiring achievement

- ***Test***
  Your child will be asked to complete the test of fractions. It will take 120 minutes.

What benefit will your child gain from being involved in this study?
The sharing of your child experiences will improve the planning and delivery of fractions teaching-learning and inform the curriculum designers about the progression of fractions that can be used to refine mathematics curriculum in the future.

Will your child be identifiable by being involved in this study?
We do not need your child's name and your childe will be anonymous. Once the think aloud protocol and interview has been typed-up and saved as a file, the voice file will then be destroyed. Any identifying information will be removed and the typed-up file stored on a password protected computer that only the coordinator (Mr Bakir Haryanto) will have access to. Your child's comments will not be linked directly to your child.

Are there any risks or discomforts if I am involved?
There will be no risks or discomfort in your child involvement. However, if you have any concerns regarding anticipated or actual risks or discomforts, please raise them with the researcher.

How do I agree to participate?
Participation is voluntary. Your child may answer 'no comment' or refuse to answer any questions and you are free to withdraw from the focus group at any time without effect or consequences. A consent form accompanies this information sheet. If you agree to participate please read and sign the form and send it back to the researcher prior to the start of the study.

How will I receive feedback?
Outcomes from the project will be summarised and given to you by the investigator if you would like to see them.

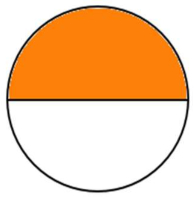
Thank you for taking the time to read this information sheet and we hope that you will accept our invitation to be involved.

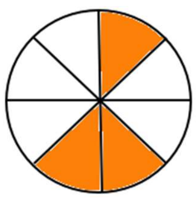
This research project has been approved by the Flinders University Social and Behavioural Research Ethics Committee (Project number 7200)). For more information regarding ethical approval of the project the Executive Officer of the Committee can be contacted by telephone on 8201 3116, by fax on 8201 2035 or by email human.researchethics@flinders.edu.au

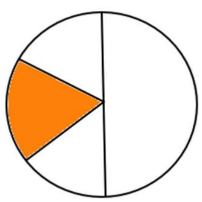
Appendix F. The English Version of the Fraction Learning Progression Assessment Instrument

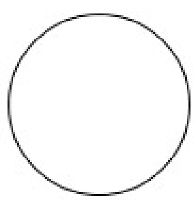
Please answer correctly all the following questions

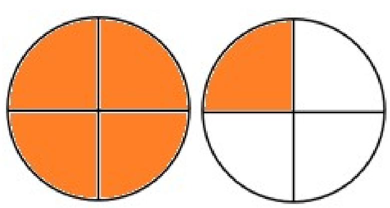
1. Write the fraction for the shaded part below

For example :  $= \frac{1}{2}$

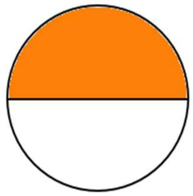
i)  = …

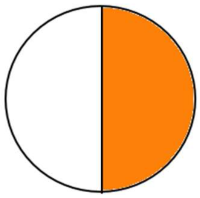
ii)  = …

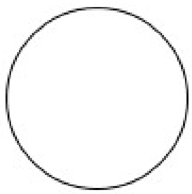
iii). If the figure  is the whole, write the fraction for the shaded part below

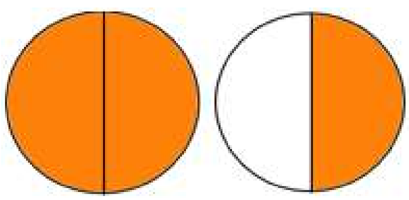
 $= \frac{\dots}{\dots}$

2. Write the numerator of the fraction for the shaded parts

For example :  $= \frac{2}{4}$

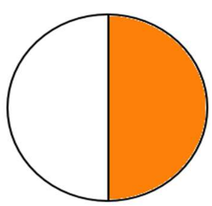
i)  $= \frac{\cdots}{16}$

ii) If the figure  is the whole, write the numerator of the fraction for the shaded parts below

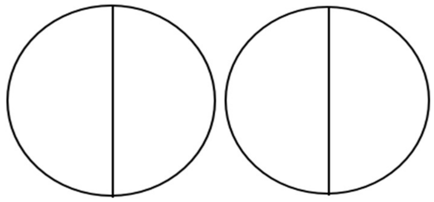
 $= \frac{\cdots}{8}$

3. Shade the shape to show the fractions below.

.

For example :  $\frac{1}{2}$

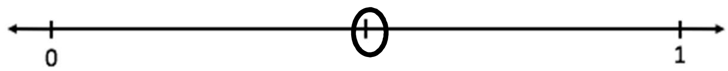
i)  $\frac{2}{3}$

ii)  $\frac{7}{4}$

4. Show the fraction on the number lines below.

For example:

$\frac{1}{2}$



$\frac{3}{8}$

5. Show the fraction $\frac{1}{2}$ on the number line below

6. Order the fractions $\frac{7}{4}$, $\frac{1}{3}$ and $1\frac{1}{2}$ on the number line below

7. Which is larger $\frac{3}{5}$ or $\frac{1}{5}$? Illustrate how you got your answer by using a model such as a picture or a diagram representation.

8. Which is larger $\frac{2}{3}$ or $\frac{3}{4}$? Illustrate how you got your answer by using a model such as a picture or a diagram representation.

9. Which is larger $\frac{7}{4}$ or $\frac{8}{6}$? Illustrate how you got your answer by using a model such as a picture or a diagram representation.

10. Write the biggest fraction that you know. Explain your answer

11. Write the smallest fraction that you know. Explain your answer

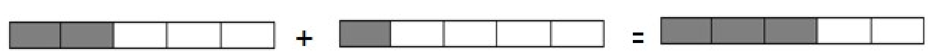
12. How many numbers between $\frac{2}{5}$ and $\frac{4}{7}$? Explain your answer

13. How many numbers between $\frac{4}{7}$ and $\frac{5}{7}$? Explain your answer

14. Draw a pictorial representation for the addition and multiplication of fractions below.

For example: *Draw a pictorial representation for the addition of fractions below.*

$\frac{2}{5} + \frac{1}{5}$



$\frac{2}{5}$ $\frac{1}{5}$ $\frac{3}{5}$

i) Draw a pictorial representation for the addition of fractions below. Explain your answer

$\frac{1}{4} + \frac{2}{4}$

ii) Draw a pictorial representation for the addition of fractions below. Explain your answer

$$\frac{1}{4} + \frac{2}{3}$$

iii) Draw a pictorial representation for the multiplication of fractions below. Explain your answer

$$\frac{1}{2} \times \frac{3}{4}$$

iii) Draw a pictorial representation for the division of fractions below. Explain your answer

$$\frac{1}{2} \div \frac{1}{4}$$

14. Find the sum, difference, product, or quotient of the fraction operations in the table below. Show your work and write your answer in simplest form (The questions in Level-1 to Level 3 are adapted and extended from Newton, 2008; Newton et al., 2014)

i) $\frac{3}{8}+\frac{2}{8}$ Answer:	ii) $\frac{14}{15}+\frac{2}{3}$ Answer:
iii) $5-\frac{3}{8}$ Answer:	iv) $2\frac{3}{5}+\frac{1}{2}$ Answer:

v) $\frac{2}{15} \times \frac{7}{15}$ Answer:	vi) $\frac{1}{8} \times 24$ Answer:
vii) $\frac{9}{10} \div \frac{3}{10}$ Answer:	viii) $3\frac{5}{7} \times 4\frac{3}{7}$ Answer:

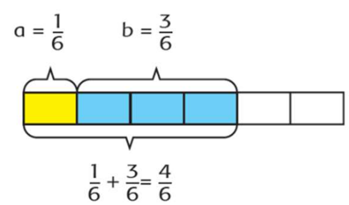
iX) $2\frac{1}{9} \div 3$ Answer:	x) $1 - \frac{2\frac{1}{4} - 1}{3}$ Answer:
xi) $\frac{1 \div \frac{2}{3}}{5} - \frac{1}{4}$ Answer:	xii) $1 + \cfrac{5}{6 + \cfrac{1}{1 - \frac{1}{3}}}$ Answer:

Good Luck !

Appendix G. Fraction Learning in the Indonesian Curriculum

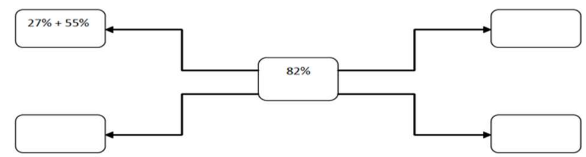
In the Indonesian New Curriculum which is called “The 2013 Curriculum”, (Balitbang, 2013a, 2013b) fractions are first introduced in the third grade and their instruction continues throughout elementary school until grade seven. In the seventh grade, fractions are not presented as an independent topic but are embedded in the topic of “numbers” which include whole numbers. After this, fractions are taught as ratios until grade nine. This section discusses the development of students’ learning of fractions (exclude ratios which are beyond the present research), based on the Indonesian curriculum. The discussion will be shaped in terms of the development of conceptual and procedural knowledge in learning fractions. Two main resources are used in this discussion which are the curriculum stating the basic competence and the books (teacher and student’s books) which accompany the curriculum. The analysis was presented in Table 1.

Table 1. The analysis of conceptual and procedural fraction competencies outlined in the 2013 Indonesian Curriculum and the mathematics textbooks published by the Ministry of Education and Culture, Indonesia.

Grade	Basic Competence (The 2013 Curriculum)	Conceptual	Procedural
Grade-3	3.3 Understand the concept of simple fractions using concrete objects / images, as well as determining the value of the smallest and largest fractions in these objects 3.14 Recognize fractions and decimals, and perform the addition and subtraction of fractions with the same	Grade 3 students begin to understand fractions as part-whole. They learn simple fractions such as 1/2, 1/3 and 1/4). Moreover, students begin to recognize which a fraction is bigger and smaller by comparing the sizes of the parts from the same whole. (Kurnianingsih, Assagaf, Muhibba, & Nurhasanah, 2015a) Students are introduced to decimal notation. They recognize that common fractions with the denominator 10 can be expressed in decimal notation e.g. 4/10 = 0.4 (Kurnianingsih, Assagaf, Muhibba, & Nurhasanah, 2015b). Moreover, they are introduced to simple fraction addition using representation as illustrated in Figure 1. 	Students can add and subtract fractions with the same denominator for fractions less than 1 e.g. 1/3 + 1/3 = ..., 3/5+1/5=..., 4/4 - 3/4=..., 5/6-1/6=..., etc (Kurnianingsih et al., 2015b)

Grade	Basic Competence (The 2013 Curriculum)	Conceptual	Procedural
	denominator	Figure 1. Fraction addition using diagram representation (Kurnianingsih et al., 2015b, p. 121)	
Grade-4	3.1 Recognize the concept of equivalent fractions and perform arithmetic operation of fractions using concrete objects / images	At grade 4, students begin to learn equivalent fractions using diagram representations, which are developed from the part-whole concept introduced in grade 3. Students use equivalent fraction understanding to compare and order fractions with different denominator, but this is still limited to fractions less than 1 (Afriki, Farani, Anggari, Wulan, Purnihastuti, Puspitawati, Destianti, Miga, & Maryanto, 2014). Students further learn how to perform fraction operations using diagram representations (which are introduced in Grade-3) including fractions with unlike- denominators (Afriki, Farani, Anggari, Wulan, Purnihastuti, Puspitawati, Destianti, Miga, Susilowati, et al., 2014).	Students are able to add and subtract fractions less than 1 including fractions with unlike-denominators. They can simplify the results of fraction addition and subtraction (Afriki, Farani, Anggari, Wulan, Purnihastuti, Puspitawati, Destianti, Miga, Susilowati, et al., 2014). Students are introduced to converting fractions into decimals in several ways: by changing the denominator into 10 or by dividing the numerator with the denominator. They also learn how to convert a fraction into percentage by changing the denominator into 100 (Afriki, Farani, Anggari,

Grade	Basic Competence (The 2013 Curriculum)	Conceptual	Procedural
	4.3 Express fractions in decimal and percentage form		Wulan, Purnihastuti, Puspitawati, Destianti, Miga, Susilowati, et al., 2014)
Grade-5	3.2 Understand the different forms of fractions (common fractions, mixed numbers, decimals and percentage), change fractions into decimals, and perform fraction multiplication and division	At grade-5, students are introduced to fractions greater than 1 and mixed numbers using diagram/pictorial representations. They are also taught that fractions can be expressed in different forms such as common fractions, mixed numbers, decimals, and percentage (Maryanto, Susilawati, Kusumawati, Subekti, & Karitas, 2014). Related to the conceptual knowledge underlying fraction operations, students are introduced to fraction multiplication and division using diagram representations. For example, a	At this level, students expand the additive operations of fractions that they learned in Grade 3 into multiplicative operations. They can multiply and divide fractions including fractions greater than 1. Students also learn further how to convert between fractions (a/b), decimals and percent which involve improper fractions (Maryanto et al., 2014).

Grade	Basic Competence (The 2013 Curriculum)	Conceptual	Procedural
	4.12 Express a decomposed fraction as two fractions which are expressed as a decimal and percent and which have a range of possible answers, using addition, subtraction, multiplication and division	fraction multiplication (1/2 X 2/3) is illustrated in figure 2.  Figure 2. A representation of a fraction multiplication 1/2 X 1/3 using rectangle diagrams (adopted from Maryanto et al., 2014, p. 30)	Students learn that a fraction can be expressed as the results of fraction operations as illustrated in Figure 3.  Figure 3. Decomposing a fraction as an addition of two fractions (adopted from Maryanto et al., 2014, p. 125)
Grade 6	3.1 Understand the arithmetic operation involving various forms of	At grade 6, students learn more about different representations of fractions including decimals and percent (Afriki et al., 2015a, 2015b).	Students learn fractions operations (that they learned in previous grades) in greater depth, which

Grade	Basic Competence (The 2013 Curriculum)	Conceptual	Procedural
	fractions (fractions, mixed numbers, decimals and percent)		involve decimals and percentage (Afriki et al., 2015a, 2015b).
Grade 7	3.1 Compare and order different types of numbers and apply arithmetic operations of integers and fractions by using a variety of operating properties	Students learn more intensively the concepts of fractions including part-whole, equivalent fractions, decimals and percent. They learn how to compare and order fractions, decimals and percent as illustrated below $\frac{3}{5}$, 70%, 0,55, 500‰ They also further learn the concept of fraction operations using fraction strips/diagrams (As'ari et al., 2014).	Students learn in greater depth additive and multiplicative fractions that they have already learned in primary school (grade 3 to 6) so that they can solve more complex fraction operations (As'ari et al., 2014). .

The Indonesian 2013 curriculum covers many topics regarding the learning of fractions. The topics concerning the conceptual knowledge of fractions are the following: At grade 3, students are introduced to part whole as an entry point into recognizing fractions. At this level, they are also taught how to add simple fractions using diagram representations. After that, they are introduced to equivalent fractions at grade 4 so that they can order and compare fractions with unlike-denominators. At grade 5, they are taught to recognize improper fractions, mixed numbers, decimals and percentage. Next, in grade 6, they are introduced more deeply on how to represent fractions in decimals and percentages. In grade 7, they review all the material taught on fractions in grades 3 to 6, and are taught how to compare and order fractions including decimals and percentages. They are also introduced to the concept of fraction operations using fraction strips/diagrams.

Concerning the procedural knowledge of fractions, students are taught additive operations from grade 3 to 4, and are introduced to multiplicative operations at grade 5. At grade, 6 they learn fraction operations which involve decimals and percentages. Finally, at grade 7, they are introduced in greater depth to more complex additive and multiplicative fraction operations.

However, there are several essential fraction concepts which are not covered in the 2013 curriculum. First, the curriculum does not introduce fractions as measures. All the fraction concepts introduced in the curriculum have an over-emphasis on part-whole understanding. Introducing fractions as measures is important in order for students to understand fractions as numbers (Arieli-Attali & Cayton-Hodges, 2014). Next, the curriculum does not include instruction on the “unbounded infinity” of fractions, meaning there is no smallest or biggest fraction (Stafylidou & Vosniadou, 2004). Finally, the curriculum does not cover the density concept of fractions, meaning that there are infinite numbers between two fractions (Vamvakoussi & Vosniadou, 2004). The 2013 curriculum could be improved by incorporating materials to further develop student learning in these areas.

Appendix H. Geweke Test Estimation for Convergence Test and gamma (γ) and pi (π) parameters generated using CODA Package in R

Geweke Test for Model 1 of the Conceptual Knowledge Dimension

```
 gamma[1]  gamma[2]  gamma[3]  gamma[4]  gamma[5]  gamma[6] 
 0.490712 -0.626240  0.470150  0.570294  0.005518 -1.662310 
  pi[1,1]   pi[1,2]   pi[1,3]   pi[1,4]   pi[1,5]   pi[1,6] 
 0.617808  1.286797 -0.804341  1.694752  0.527221  1.648695 
  pi[1,7]   pi[1,8]   pi[1,9]  pi[1,10]  pi[1,11]  pi[1,12] 
-0.004892 -0.860232 -0.060908  0.071053  0.625309  1.138121 
 pi[1,13]  pi[1,14]  pi[1,15]  pi[1,16]  pi[1,17]  pi[1,18] 
-0.700553 -0.005713  0.543834 -0.687961  1.172969 -0.812506 
 pi[1,19]  pi[1,20]  pi[1,21]   pi[2,1]   pi[2,2]   pi[2,3] 
-0.551563 -0.041621  1.351098  0.240884 -0.637400 -0.325650 
  pi[2,4]   pi[2,5]   pi[2,6]   pi[2,7]   pi[2,8]   pi[2,9] 
 0.784541  0.818438  0.726941 -1.616439  0.913738 -0.521404 
 pi[2,10]  pi[2,11]  pi[2,12]  pi[2,13]  pi[2,14]  pi[2,15] 
 1.130307  0.733221  0.483723 -0.093295 -0.044078 -1.642028 
 pi[2,16]  pi[2,17]  pi[2,18]  pi[2,19]  pi[2,20]  pi[2,21] 
 0.862408 -0.734300  0.145336 -0.427807 -1.337637  0.366285 
  pi[3,1]   pi[3,2]   pi[3,3]   pi[3,4]   pi[3,5]   pi[3,6] 
 1.712758  0.431907  0.444354  1.233695 -1.422345  0.877708 
  pi[3,7]   pi[3,8]   pi[3,9]  pi[3,10]  pi[3,11]  pi[3,12] 
-1.049826 -0.343016 -1.311626 -1.847013 -1.072481 -0.753578 
 pi[3,13]  pi[3,14]  pi[3,15]  pi[3,16]  pi[3,17]  pi[3,18] 
 0.120117  0.499039 -0.406753 -0.705981 -0.253896  0.101185 
 pi[3,19]  pi[3,20]  pi[3,21]   pi[4,1]   pi[4,2]   pi[4,3] 
 0.952138 -0.362436  0.841088 -0.194156 -0.418119 -0.632626 
  pi[4,4]   pi[4,5]   pi[4,6]   pi[4,7]   pi[4,8]   pi[4,9] 
 0.491605 -0.619785  0.031210 -0.661878 -0.997749 -1.790882 
 pi[4,10]  pi[4,11]  pi[4,12]  pi[4,13]  pi[4,14]  pi[4,15] 
 0.031437 -0.256421 -0.731369  0.718214  1.043495  0.110951 
 pi[4,16]  pi[4,17]  pi[4,18]  pi[4,19]  pi[4,20]  pi[4,21] 
-1.028516  0.759290 -0.295129  0.115957  0.140306 -1.236072 
  pi[5,1]   pi[5,2]   pi[5,3]   pi[5,4]   pi[5,5]   pi[5,6] 
-0.359621 -0.101860  0.188587 -1.631224 -0.147195 -1.442883 
  pi[5,7]   pi[5,8]   pi[5,9]  pi[5,10]  pi[5,11]  pi[5,12] 
 0.043249  0.800412  0.586067  0.001195  0.791922  0.348552 
```

```
 pi[5,13]  pi[5,14]  pi[5,15]  pi[5,16]  pi[5,17]  pi[5,18] 
 0.091012  1.450053 -0.507667  0.045625 -0.428344 -1.090335 
 pi[5,19]  pi[5,20]  pi[5,21]   pi[6,1]   pi[6,2]   pi[6,3] 
 1.159484  1.641590  0.024415 -0.583123  0.147721 -0.315051 
  pi[6,4]   pi[6,5]   pi[6,6]   pi[6,7]   pi[6,8]   pi[6,9] 
-0.373097  1.770135  1.544629 -1.353087  1.994993 -0.217660 
 pi[6,10]  pi[6,11]  pi[6,12]  pi[6,13]  pi[6,14]  pi[6,15] 
-1.099212  0.252504 -0.897223 -1.408613 -0.635260 -0.857159 
 pi[6,16]  pi[6,17]  pi[6,18]  pi[6,19]  pi[6,20]  pi[6,21] 
-0.793391 -0.702598  1.022452 -0.213616 -0.645353  1.435137 
```

Geweke Test for Model 1 of the Procedural Knowledge Dimension

```
 gamma[1] gamma[2] gamma[3] gamma[4] gamma[5] gamma[6] 
 -0.45834  0.05635  1.49298 -0.77578  0.27093 -0.94027 
 pi[1,1]  pi[1,2]  pi[1,3]  pi[1,4]  pi[1,5]  pi[1,6] 
 0.29045  0.83607 -0.83498 -1.17869 -0.73040 -0.50142 
 pi[1,7]  pi[1,8]  pi[1,9] pi[1,10] pi[1,11] pi[1,12] 
-0.04321 -0.24464 -0.28280  1.38373  0.12608  0.63585 
 pi[2,1]  pi[2,2]  pi[2,3]  pi[2,4]  pi[2,5]  pi[2,6] 
 0.24597  1.07106 -0.82652 -0.05438  0.51161  1.14149 
 pi[2,7]  pi[2,8]  pi[2,9] pi[2,10] pi[2,11] pi[2,12] 
-0.70418 -1.28812 -1.60968 -0.79841 -1.57595 -1.10159 
 pi[3,1]  pi[3,2]  pi[3,3]  pi[3,4]  pi[3,5]  pi[3,6] 
-0.11682 -0.45518  0.27498  0.01856  0.23486 -0.43008 
 pi[3,7]  pi[3,8]  pi[3,9] pi[3,10] pi[3,11] pi[3,12] 
-1.09016  1.07352  1.46621  1.46780  0.98966 -0.14422 
 pi[4,1]  pi[4,2]  pi[4,3]  pi[4,4]  pi[4,5]  pi[4,6] 
-0.50054  0.87073  1.93257 -0.24542 -0.05377  0.90543 
 pi[4,7]  pi[4,8]  pi[4,9] pi[4,10] pi[4,11] pi[4,12] 
 1.01740 -0.07739  0.07472  1.13069 -1.82578  0.87120 
 pi[5,1]  pi[5,2]  pi[5,3]  pi[5,4]  pi[5,5]  pi[5,6] 
 0.49469 -0.29565 -0.17534  0.92988  0.12676  0.40917 
 pi[5,7]  pi[5,8]  pi[5,9] pi[5,10] pi[5,11] pi[5,12] 
 0.24544  0.39709  0.22877 -1.64664 -0.37963 -1.43941 
 pi[6,1]  pi[6,2]  pi[6,3]  pi[6,4]  pi[6,5]  pi[6,6] 
-0.79094 -0.10340  0.22295  1.38070  1.11238 -1.44034 
 pi[6,7]  pi[6,8]  pi[6,9] pi[6,10] pi[6,11] pi[6,12] 
 0.55105  1.84607 -0.77276 -1.65914  0.32504  1.02250 
```

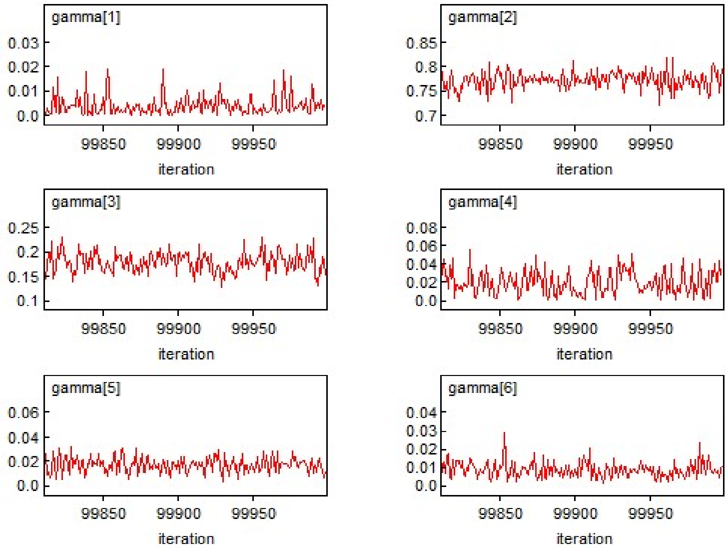
Geweke Test for Model 2 of the Conceptual Knowledge Dimension

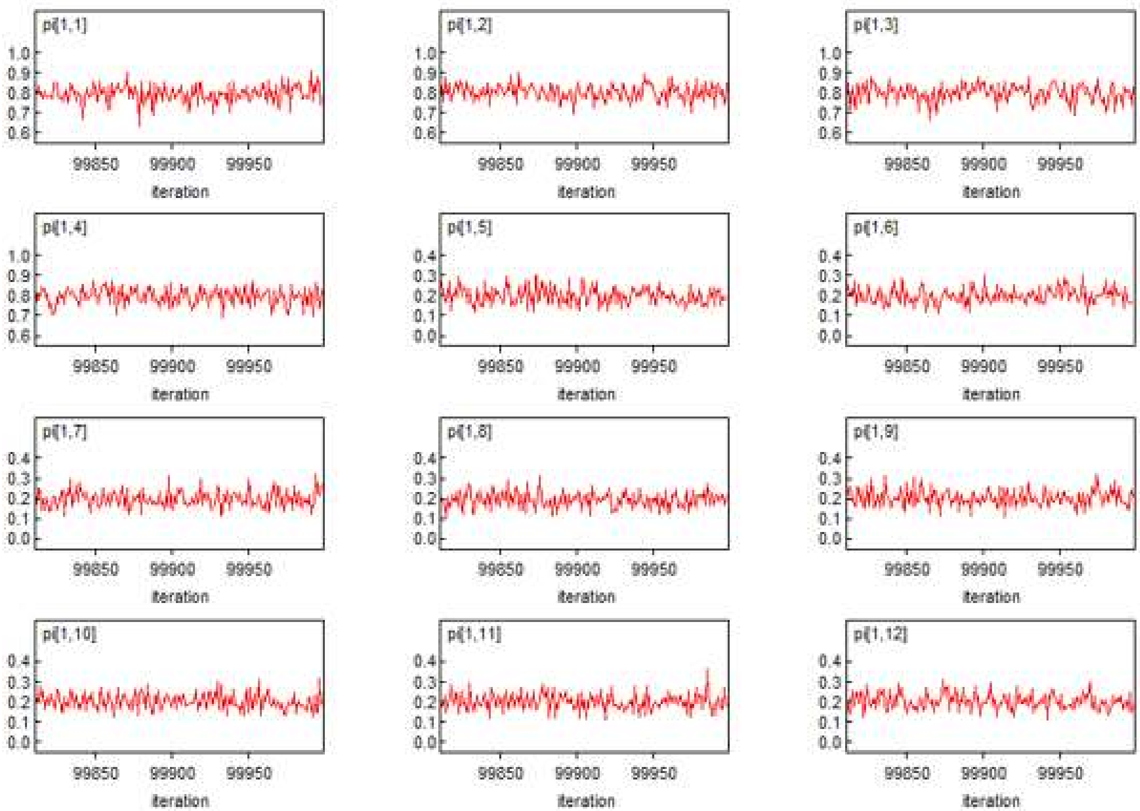
gamma_2[1]	gamma_2[2]	gamma_3[1]	gamma_3[2]	gamma_4[1]
0.1115	1.5192	-0.9774	0.5647	0.2747
gamma_4[2]	gamma_5[1]	gamma_5[2]	gamma_6[1]	gamma_6[2]
1.7981	0.7566	-0.7984	0.1601	0.8932
gamma_dummy	pi[1,1]	pi[1,2]	pi[1,3]	pi[1,4]
-0.4589	0.9674	0.7531	0.5015	-0.4957
pi[1,5]	pi[1,6]	pi[1,7]	pi[1,8]	pi[1,9]
-0.6912	-1.5393	-0.2141	-0.6964	-1.6167
pi[1,10]	pi[1,11]	pi[1,12]	pi[1,13]	pi[1,14]
-1.2346	-0.3172	-0.1220	1.6227	-1.8265
pi[1,15]	pi[1,16]	pi[1,17]	pi[1,18]	pi[1,19]
-0.8837	0.8121	1.2048	-0.4253	0.8857
pi[1,20]	pi[1,21]	pi[2,1]	pi[2,2]	pi[2,3]
-2.0952	-0.2844	-1.6287	-1.7158	-0.4705
pi[2,4]	pi[2,5]	pi[2,6]	pi[2,7]	pi[2,8]
-0.9119	0.6819	-1.0370	0.9131	-1.2923
pi[2,9]	pi[2,10]	pi[2,11]	pi[2,12]	pi[2,13]
-0.7593	0.3629	-0.4688	-0.7845	-0.7119
pi[2,14]	pi[2,15]	pi[2,16]	pi[2,17]	pi[2,18]
0.7136	-0.1845	1.1145	1.1720	-1.1514
pi[2,19]	pi[2,20]	pi[2,21]		
-1.2004	0.6283	0.1735		

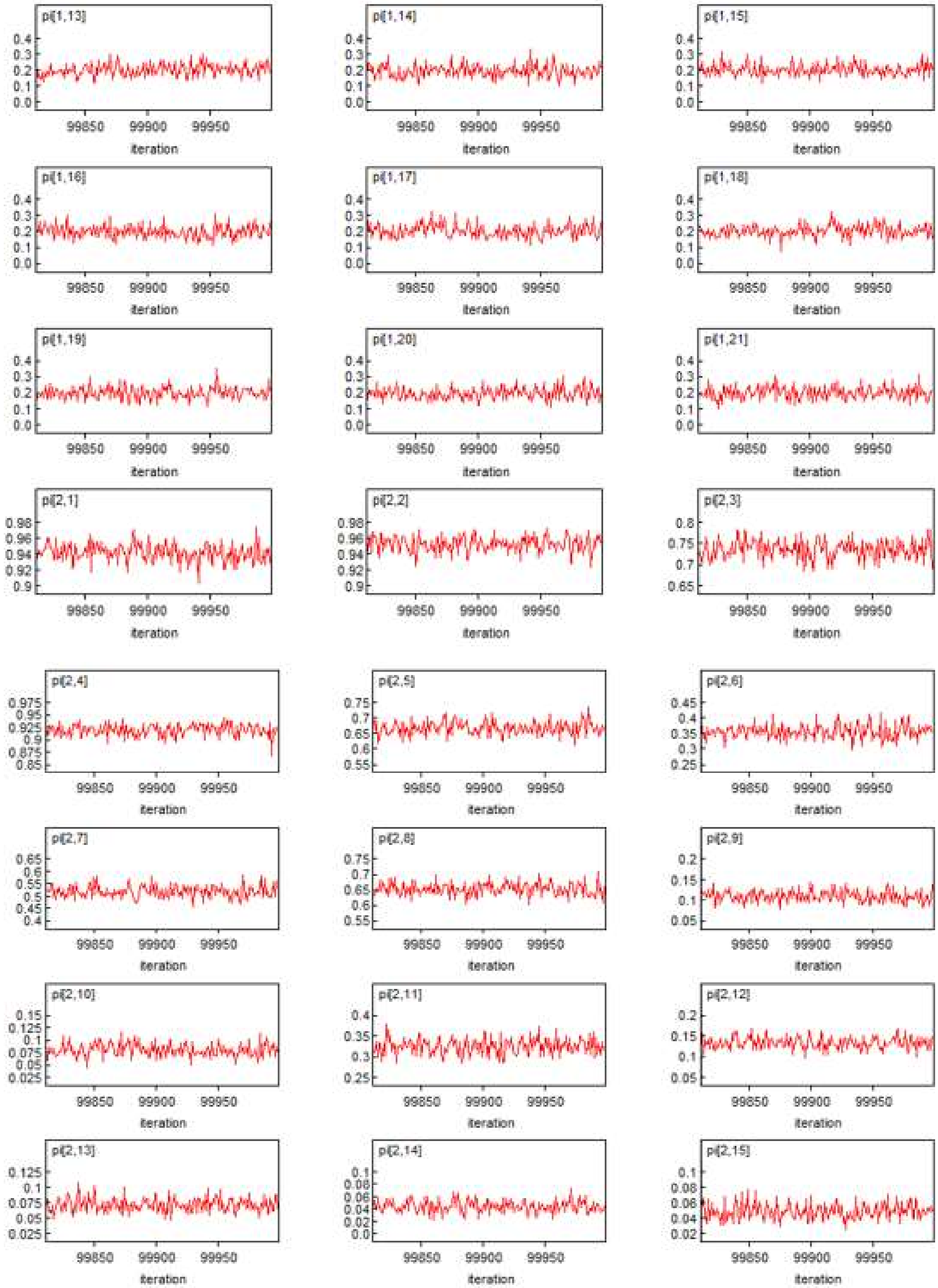
Geweke Test for Model 2 of the Procedural Knowledge Dimension

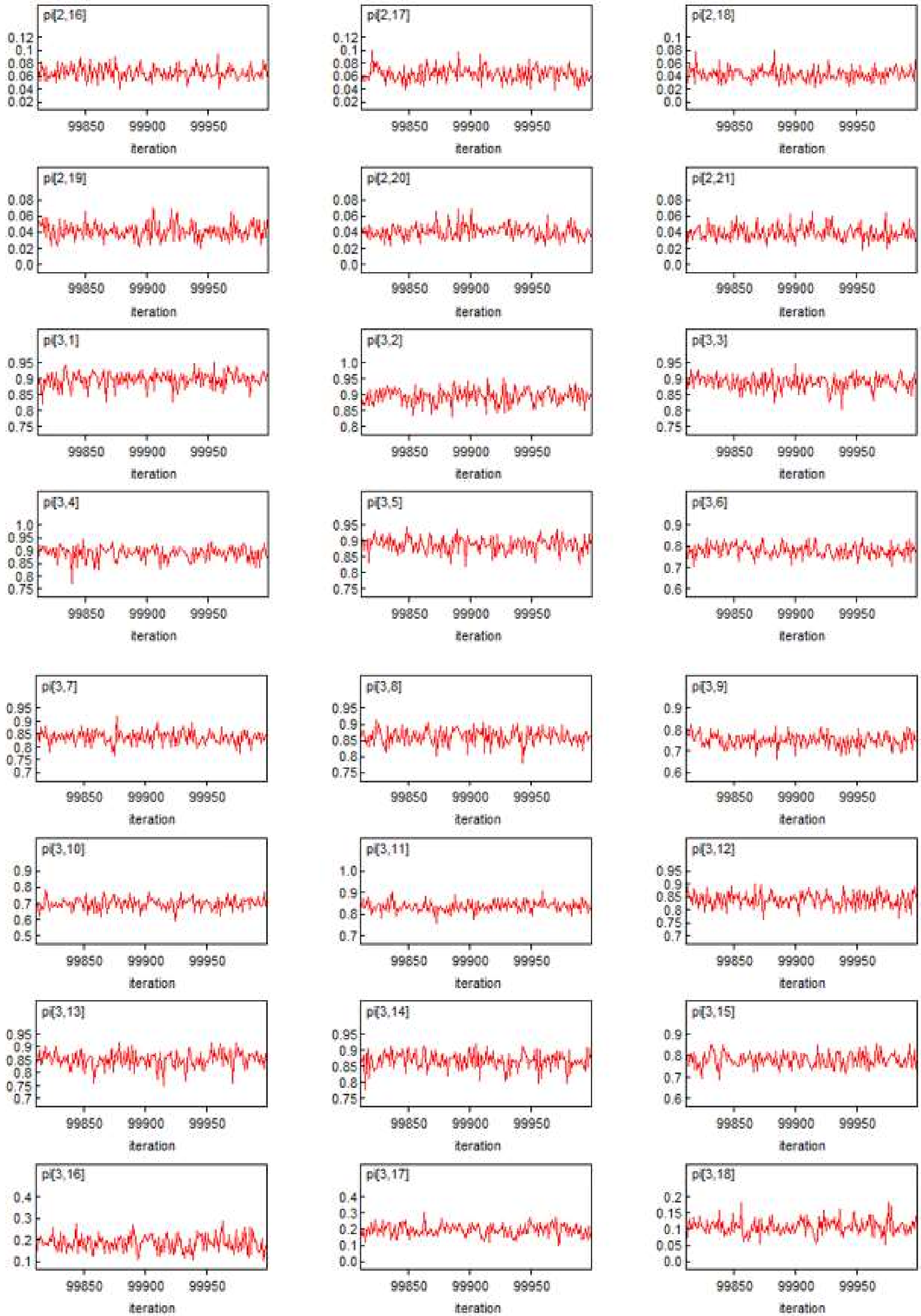
gamma_2[1]	gamma_2[2]	gamma_3[1]	gamma_3[2]	gamma_4[1]
-1.6572	0.2922	1.2874	0.4322	-0.4534
gamma_4[2]	gamma_5[1]	gamma_5[2]	gamma_6[1]	gamma_6[2]
-1.1384	1.0672	1.6197	0.1683	-1.7054
gamma_dummy	pi[1,1]	pi[1,2]	pi[1,3]	pi[1,4]
-0.2404	-0.0167	0.1739	0.4642	-0.2596
pi[1,5]	pi[1,6]	pi[1,7]	pi[1,8]	pi[1,9]
-1.5466	1.8681	-0.8296	-1.2827	1.4654
pi[1,10]	pi[1,11]	pi[1,12]	pi[2,1]	pi[2,2]
-1.1435	0.2722	0.7502	-1.4016	-0.9734
pi[2,3]	pi[2,4]	pi[2,5]	pi[2,6]	pi[2,7]
-1.0843	1.1791	0.3072	-1.1987	0.3351
pi[2,8]	pi[2,9]	pi[2,10]	pi[2,11]	pi[2,12]
0.5008	0.3837	-0.4126	0.5838	1.2276

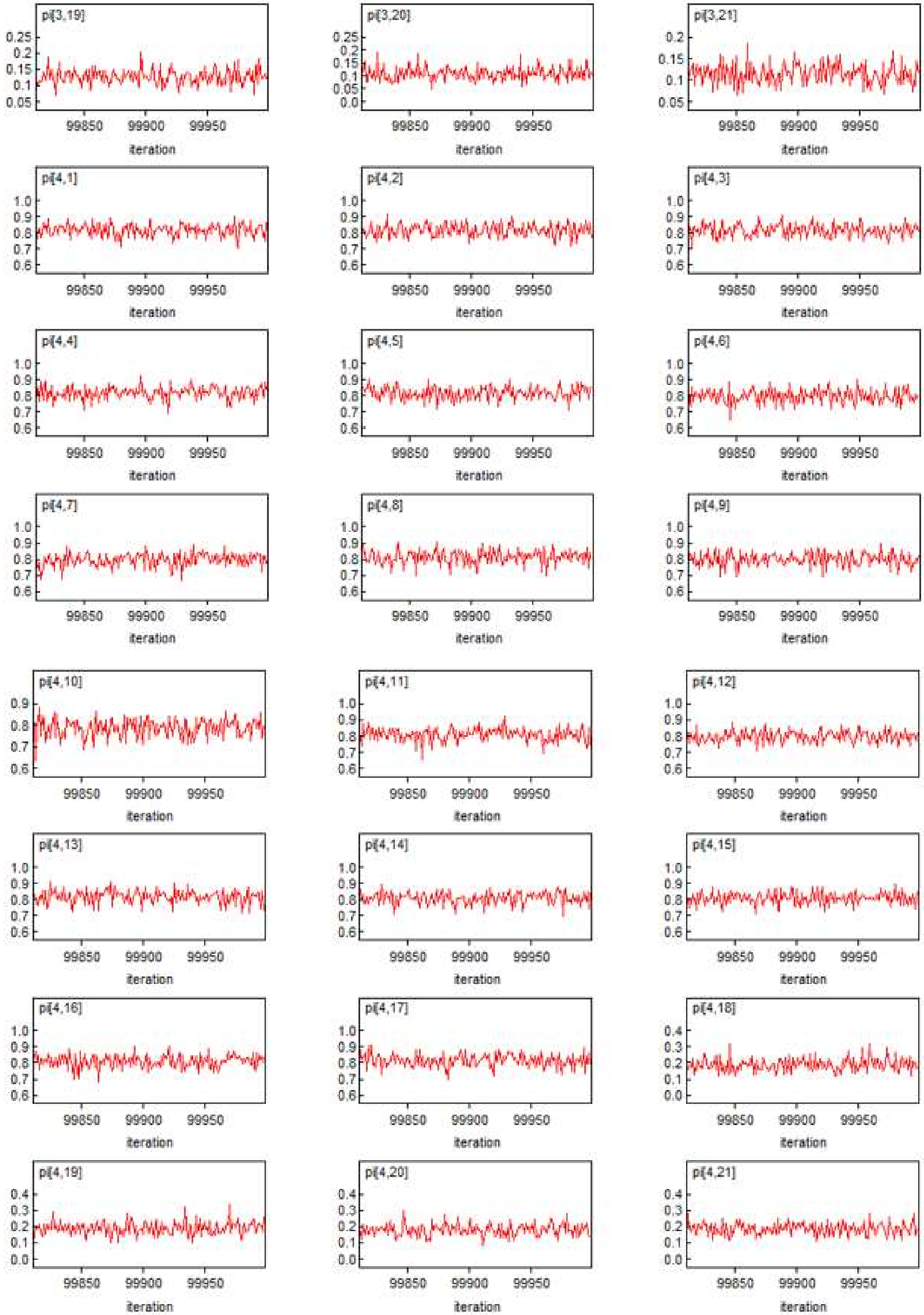
Appendix I. The last 10000 iterations of MCMC for all parameters (γ, π) generated from Model 1 for the conceptual knowledge dimension

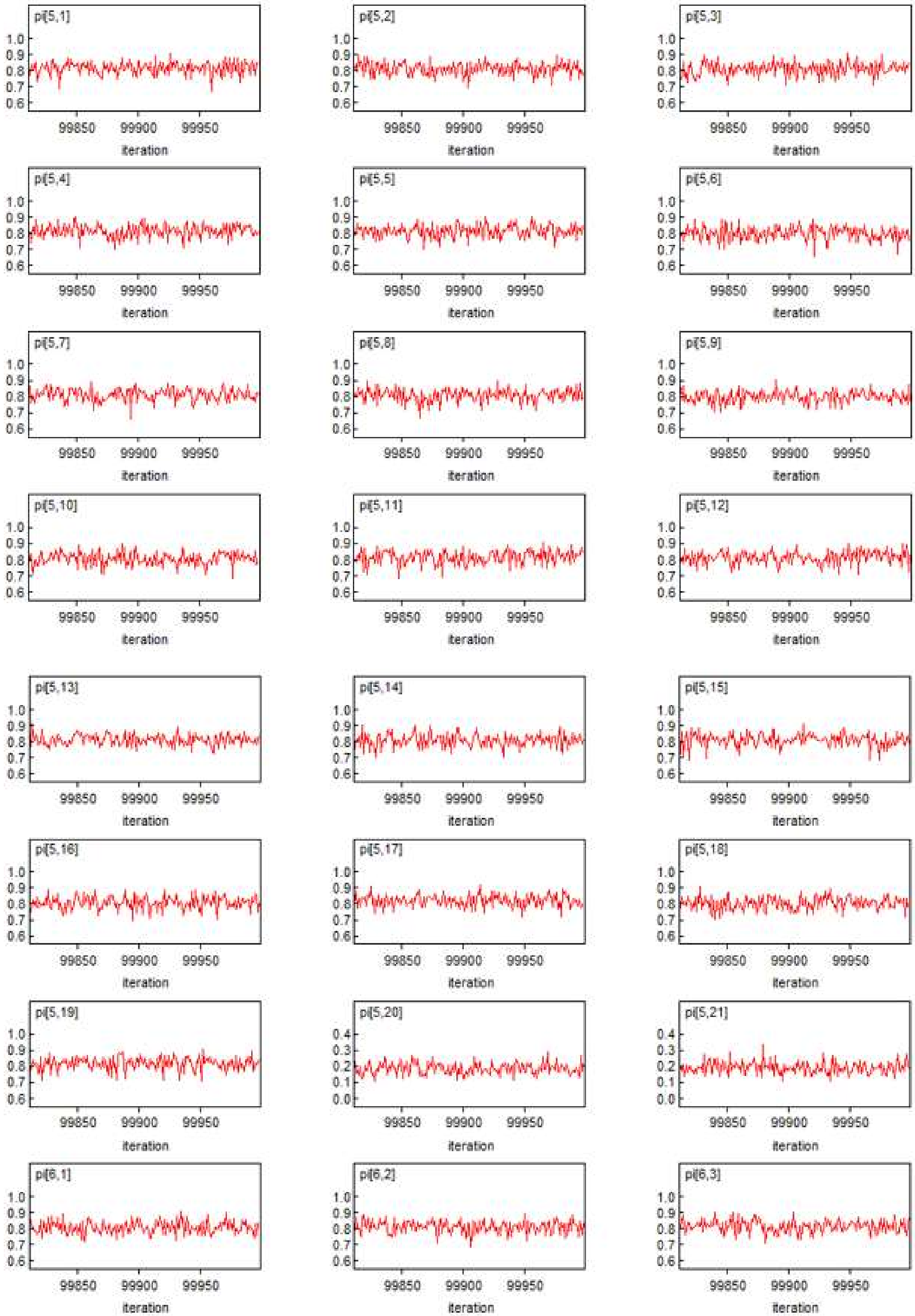





pi[1,13]
pi[1,14]
pi[1,15]
pi[1,16]
pi[1,17]
pi[1,18]
pi[1,19]
pi[1,20]
pi[1,21]
pi[2,1]
pi[2,2]
pi[2,3]
pi[2,4]
pi[2,5]
pi[2,6]
pi[2,7]
pi[2,8]
pi[2,9]
pi[2,10]
pi[2,11]
pi[2,12]
pi[2,13]
pi[2,14]
pi[2,15]
99850
99900
99950
iteration


pi[2,16]
pi[2,17]
pi[2,18]
pi[2,19]
pi[2,20]
pi[2,21]
pi[3,1]
pi[3,2]
pi[3,3]
pi[3,4]
pi[3,5]
pi[3,6]
pi[3,7]
pi[3,8]
pi[3,9]
pi[3,10]
pi[3,11]
pi[3,12]
pi[3,13]
pi[3,14]
pi[3,15]
pi[3,16]
pi[3,17]
pi[3,18]
99850
99900
99950
iteration


pi[3,19]
pi[3,20]
pi[3,21]
pi[4,1]
pi[4,2]
pi[4,3]
pi[4,4]
pi[4,5]
pi[4,6]
pi[4,7]
pi[4,8]
pi[4,9]
pi[4,10]
pi[4,11]
pi[4,12]
pi[4,13]
pi[4,14]
pi[4,15]
pi[4,16]
pi[4,17]
pi[4,18]
pi[4,19]
pi[4,20]
pi[4,21]
99850
99900
99950
iteration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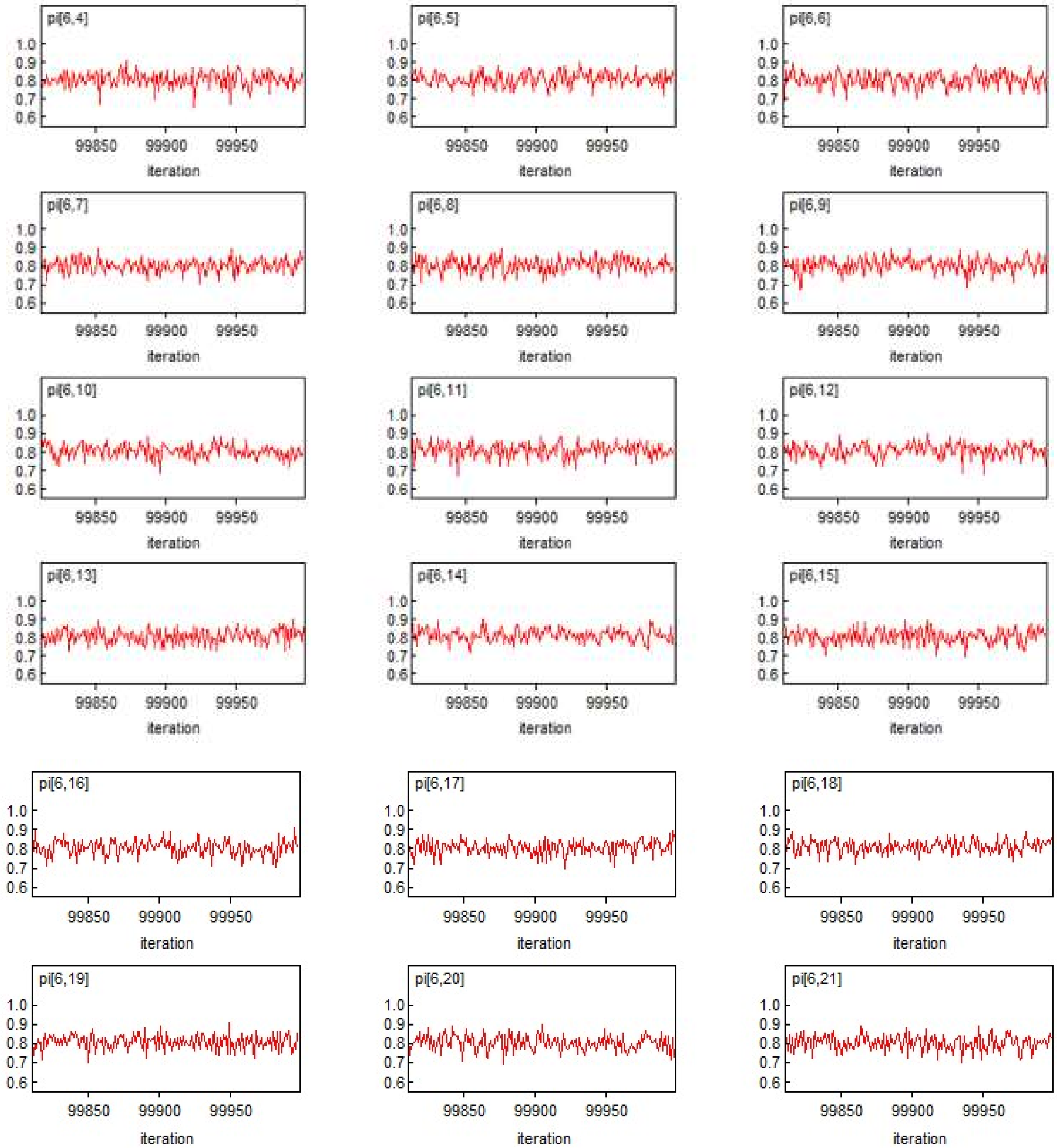
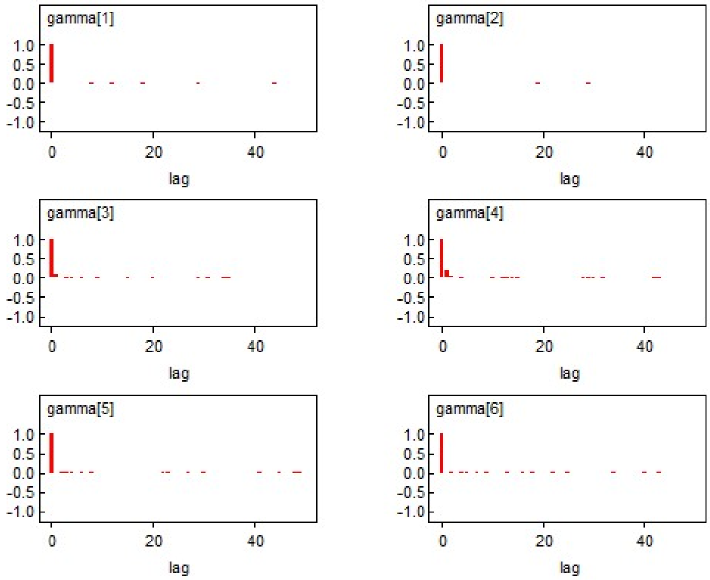
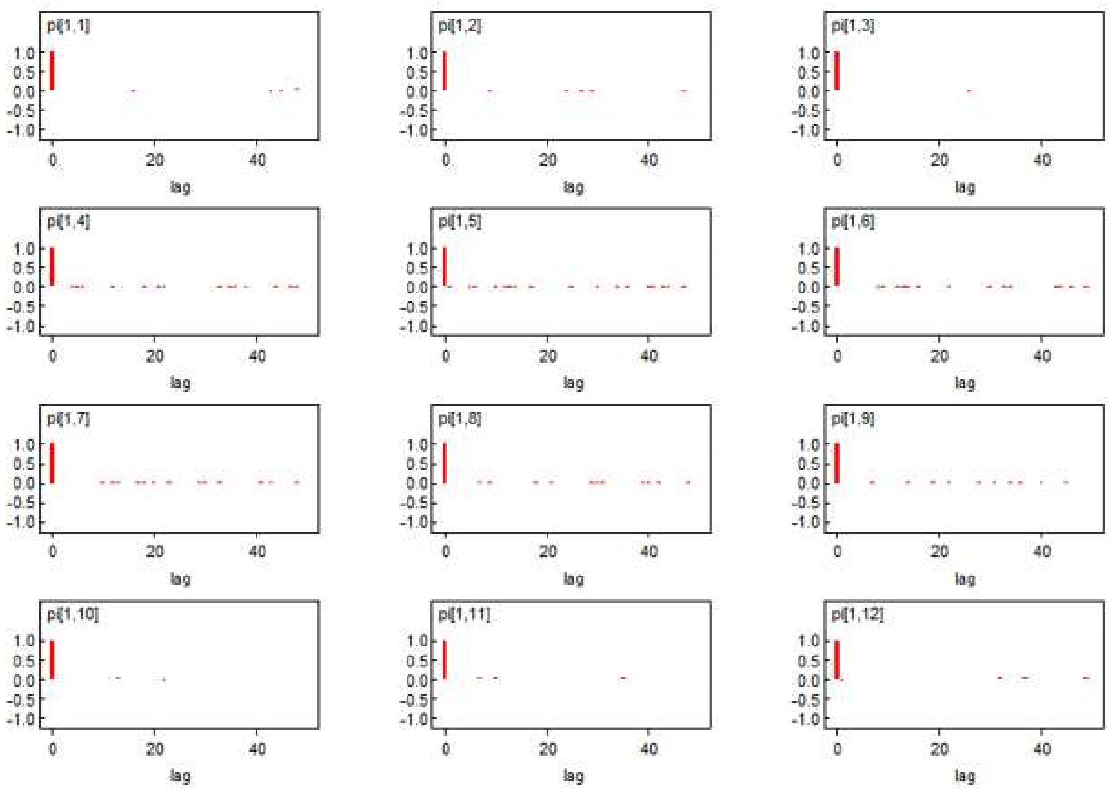
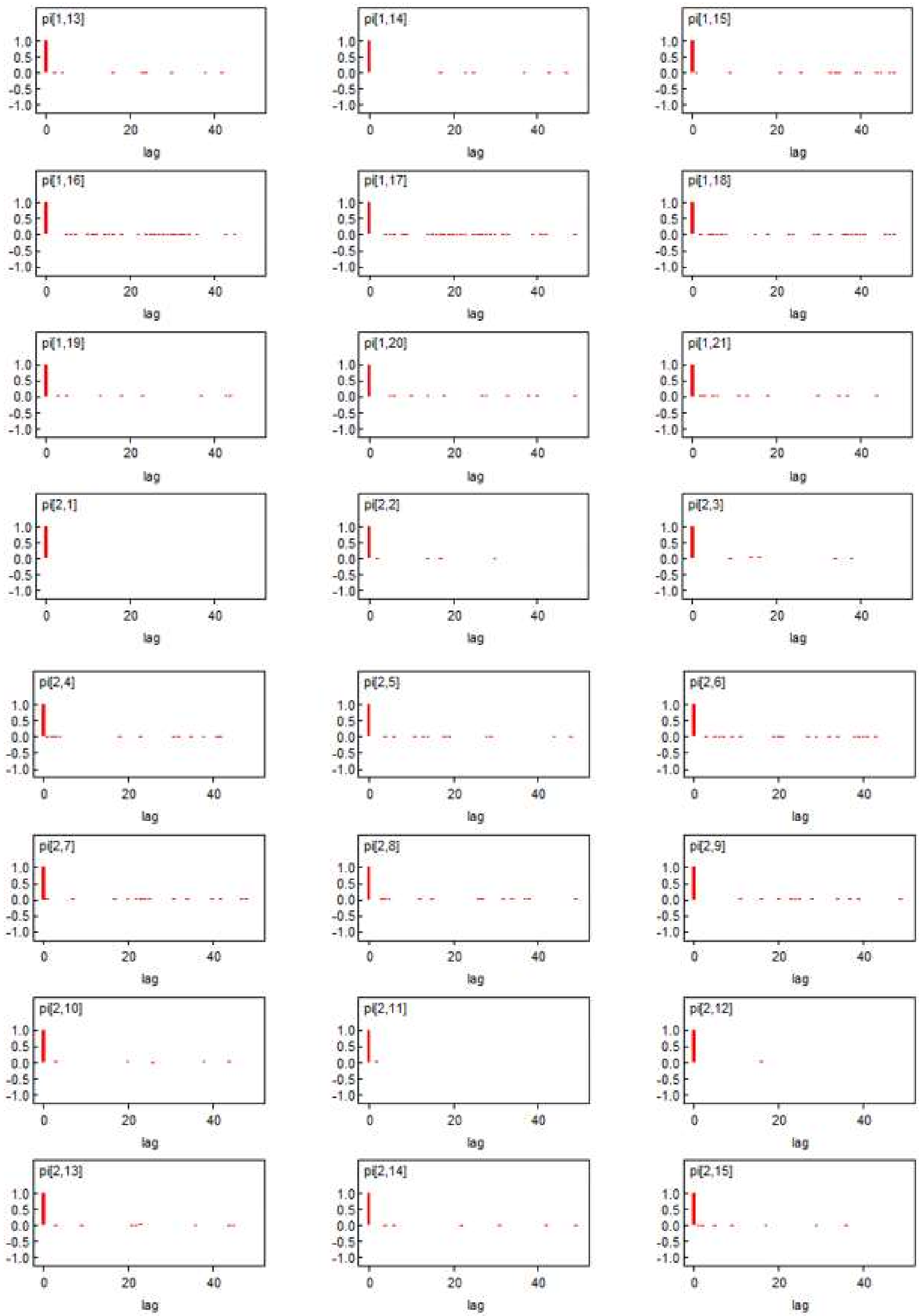
pi[5,2]
pi[5,3]
pi[5,4]
pi[5,5]
pi[5,6]
pi[5,7]
pi[5,8]
pi[5,9]
pi[5,10]
pi[5,11]
pi[5,12]
pi[5,13]
pi[5,14]
pi[5,15]
pi[5,16]
pi[5,17]
pi[5,18]
pi[5,19]
pi[5,20]
pi[5,21]
pi[6,1]
pi[6,2]
pi[6,3]
1.0
0.9
0.8
0.7
0.6
0.4
0.3
0.2
0.1
0.0
99850
99900
99950
iteration

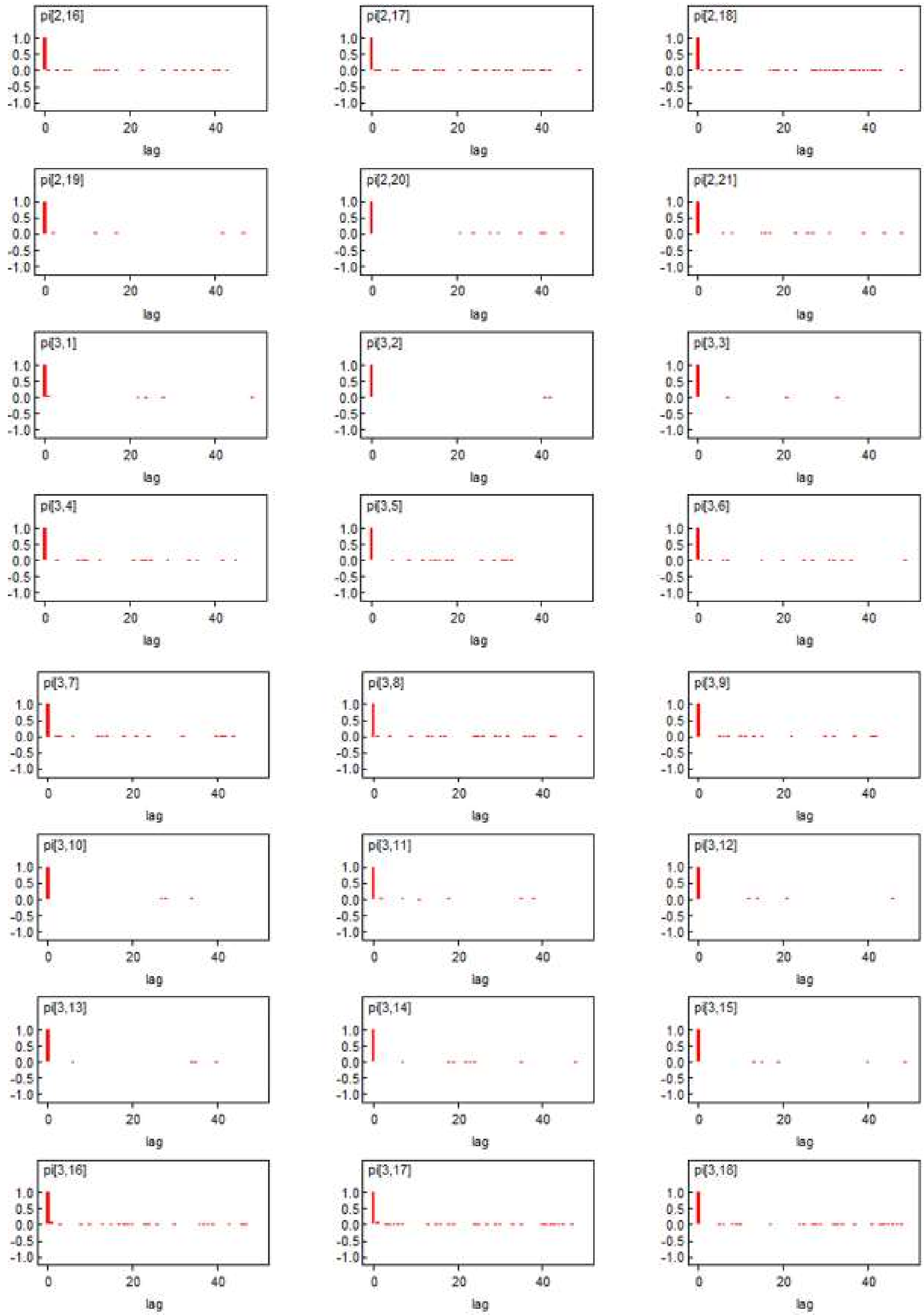

pi[6,4]
pi[6,5]
pi[6,6]
pi[6,7]
pi[6,8]
pi[6,9]
pi[6,10]
pi[6,11]
pi[6,12]
pi[6,13]
pi[6,14]
pi[6,15]
pi[6,16]
pi[6,17]
pi[6,18]
pi[6,19]
pi[6,20]
pi[6,21]
1.0
0.9
0.8
0.7
0.6
99850
99900
99950
iteration

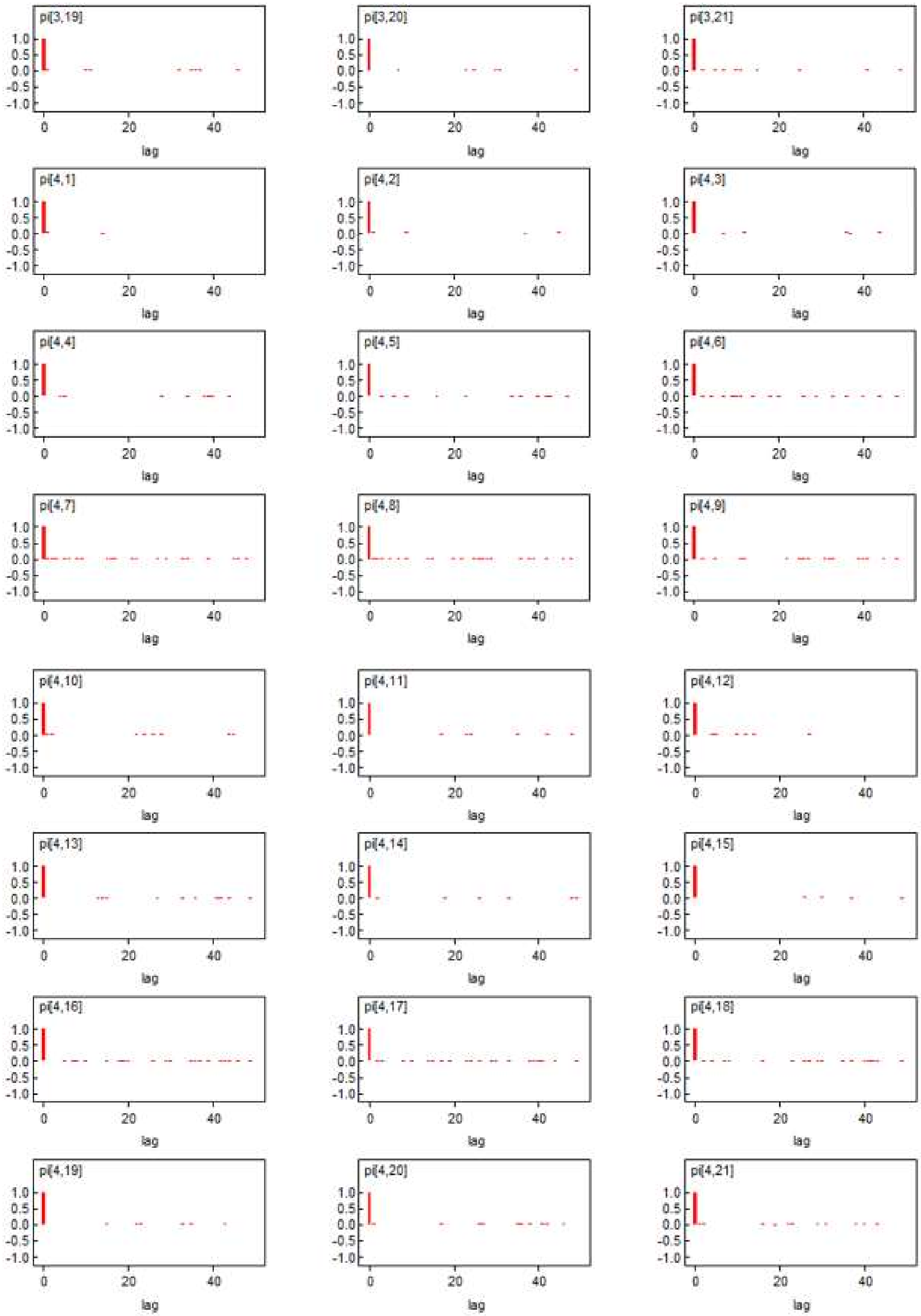
Appendix J. The autocorrelation plots of the last 10000 iterations of MCMC for for all parameters (γ, π) generated from Model 1 for the conceptual knowledge dimension

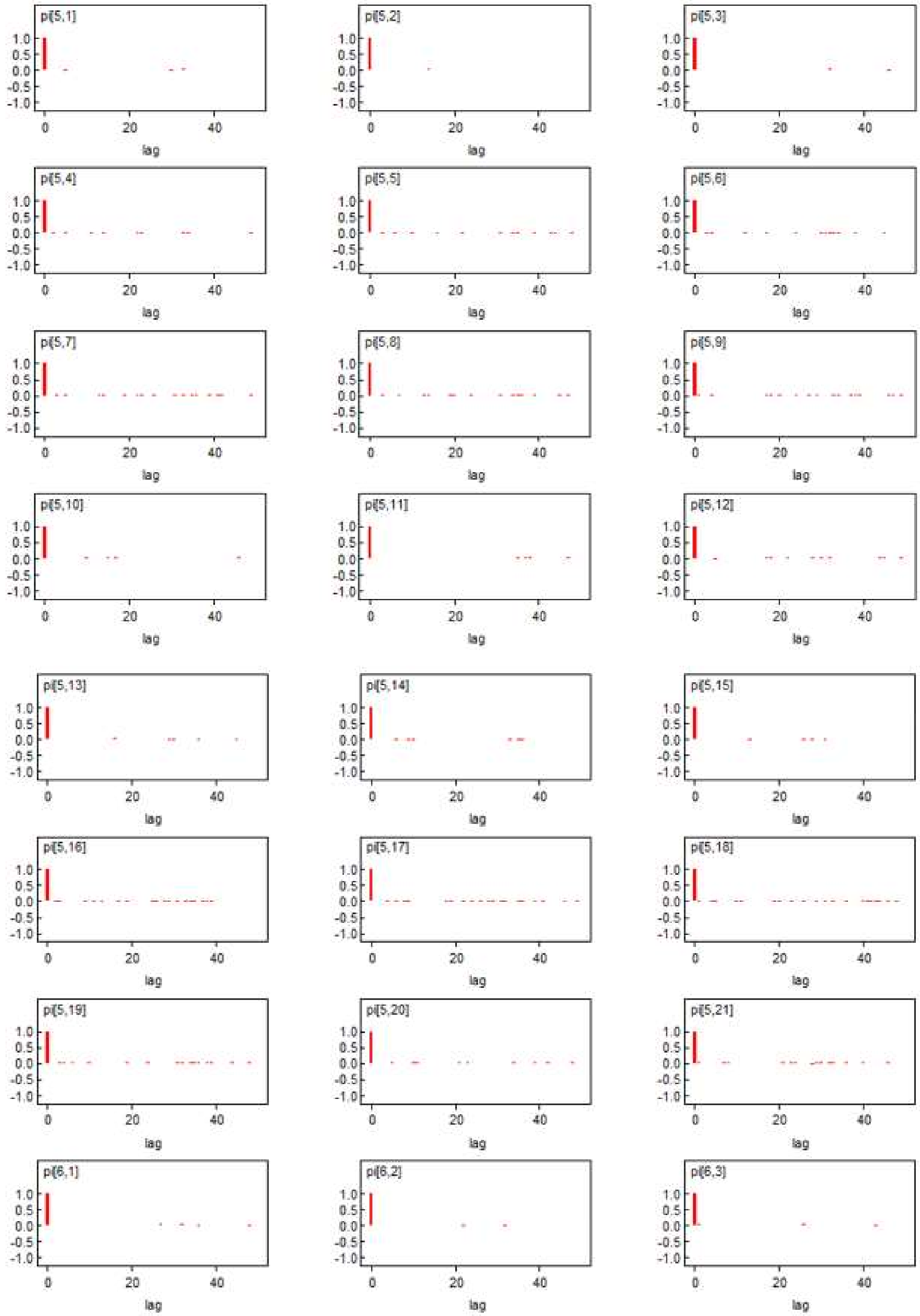


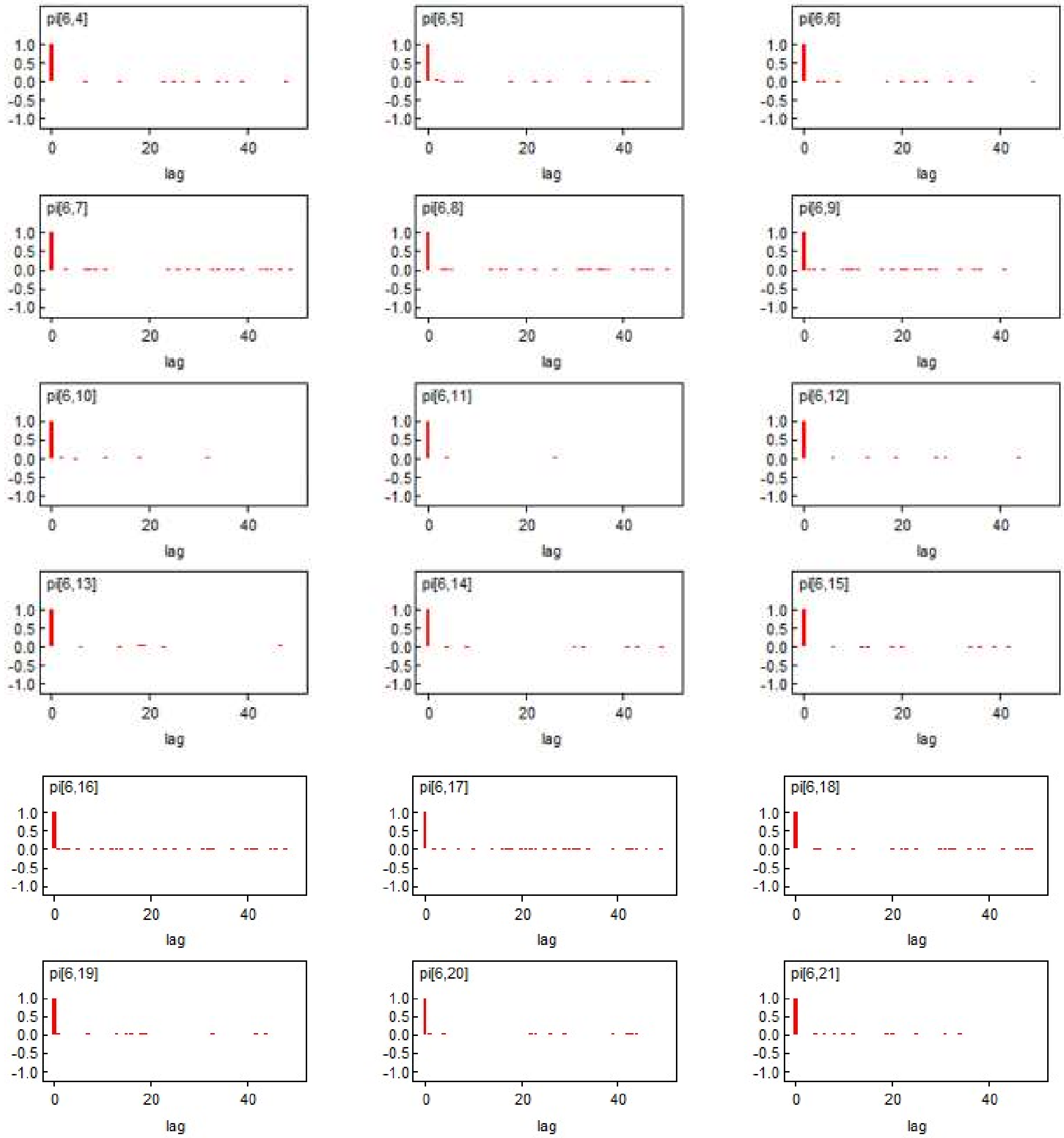



pi[1,13]
pi[1,14]
pi[1,15]
pi[1,16]
pi[1,17]
pi[1,18]
pi[1,19]
pi[1,20]
pi[1,21]
pi[2,1]
pi[2,2]
pi[2,3]
pi[2,4]
pi[2,5]
pi[2,6]
pi[2,7]
pi[2,8]
pi[2,9]
pi[2,10]
pi[2,11]
pi[2,12]
pi[2,13]
pi[2,14]
pi[2,15]
lag

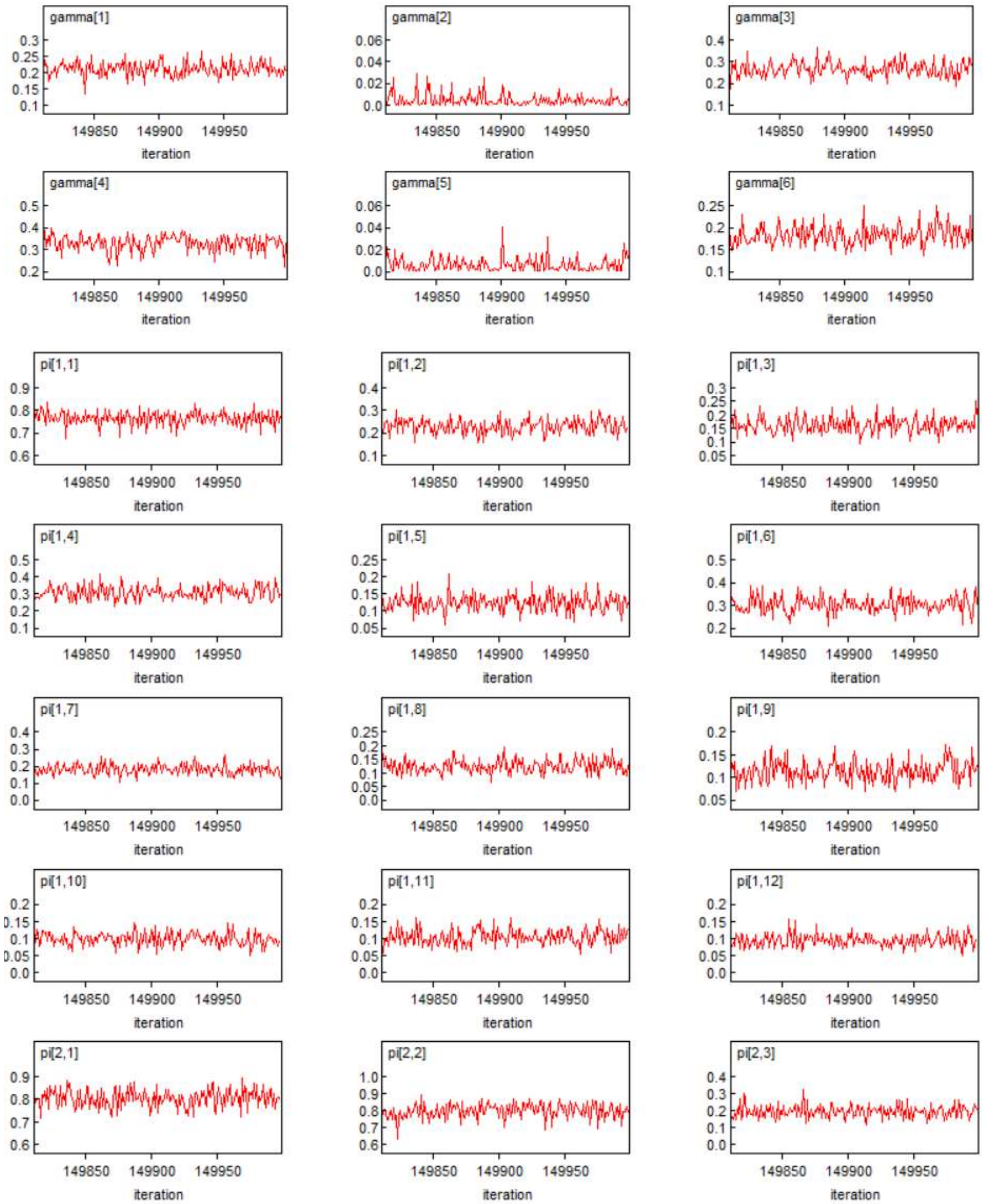

pi[2,16]
pi[2,17]
pi[2,18]
pi[2,19]
pi[2,20]
pi[2,21]
pi[3,1]
pi[3,2]
pi[3,3]
pi[3,4]
pi[3,5]
pi[3,6]
pi[3,7]
pi[3,8]
pi[3,9]
pi[3,10]
pi[3,11]
pi[3,12]
pi[3,13]
pi[3,14]
pi[3,15]
pi[3,16]
pi[3,17]
pi[3,18]
lag

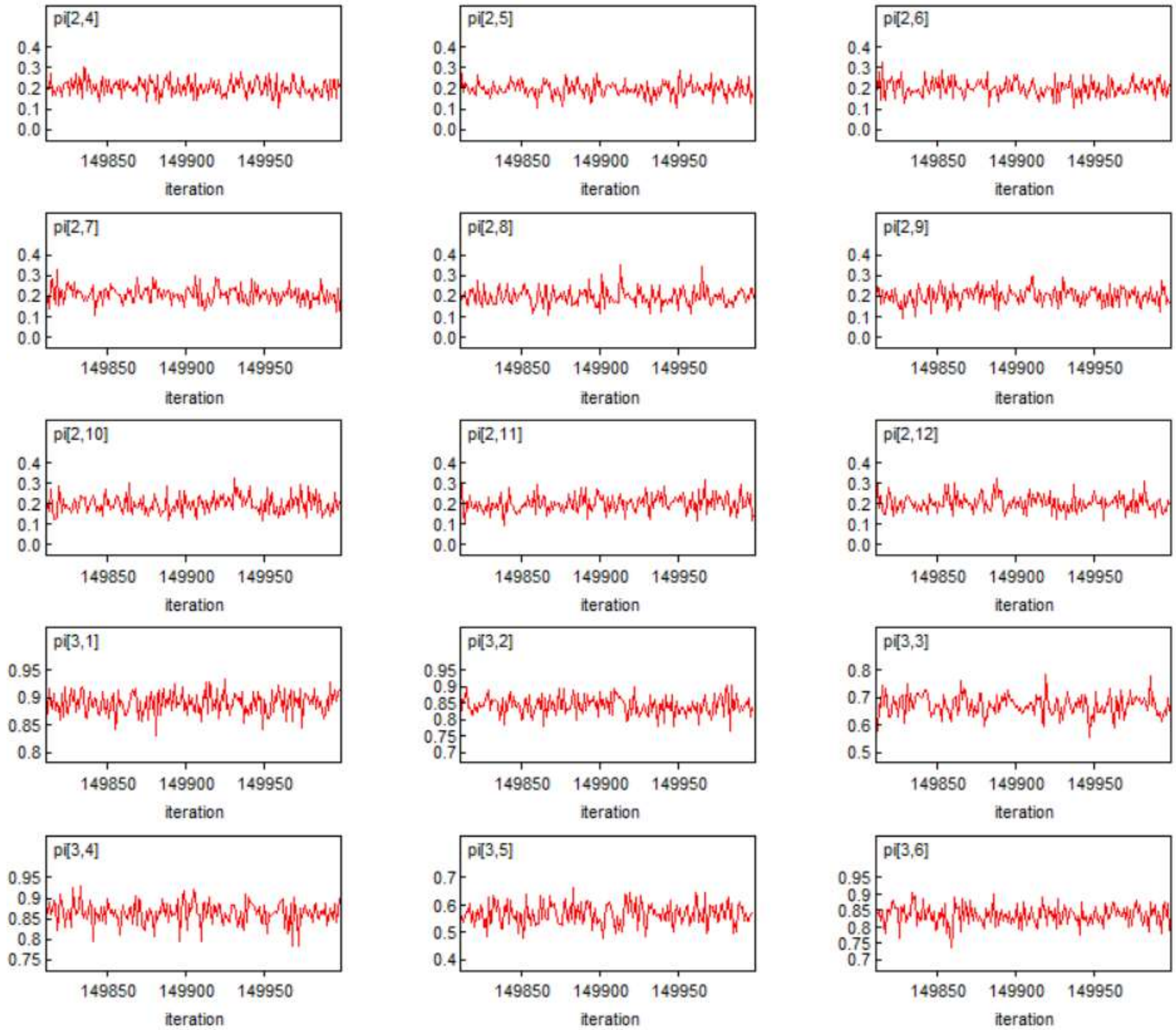

pi[3,19]
pi[3,20]
pi[3,21]
pi[4,1]
pi[4,2]
pi[4,3]
pi[4,4]
pi[4,5]
pi[4,6]
pi[4,7]
pi[4,8]
pi[4,9]
pi[4,10]
pi[4,11]
pi[4,12]
pi[4,13]
pi[4,14]
pi[4,15]
pi[4,16]
pi[4,17]
pi[4,18]
pi[4,19]
pi[4,20]
pi[4,21]
1.0
0.5
0.0
-0.5
-1.0
0
20
40
lag

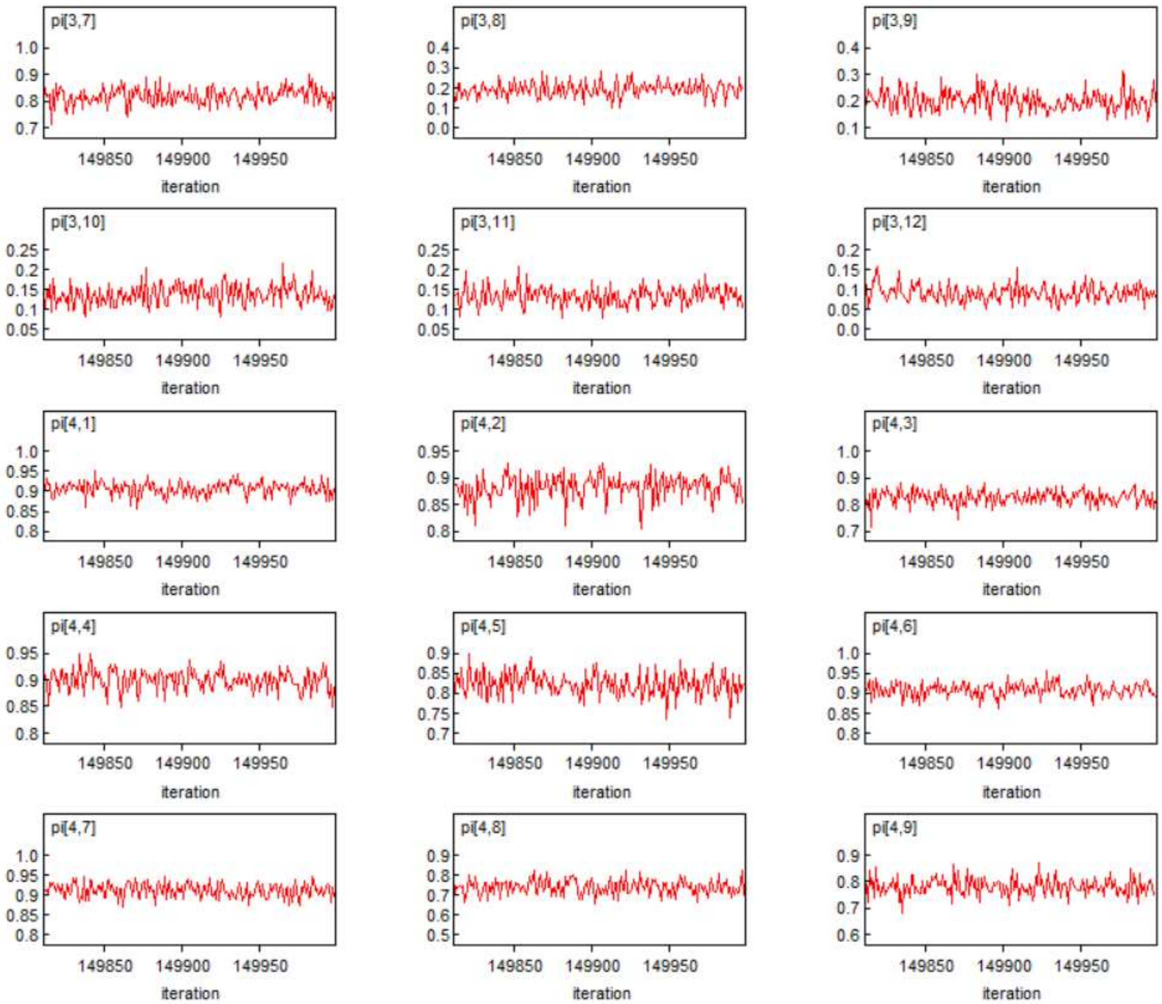

pi[5,1]
pi[5,2]
pi[5,3]
pi[5,4]
pi[5,5]
pi[5,6]
pi[5,7]
pi[5,8]
pi[5,9]
pi[5,10]
pi[5,11]
pi[5,12]
pi[5,13]
pi[5,14]
pi[5,15]
pi[5,16]
pi[5,17]
pi[5,18]
pi[5,19]
pi[5,20]
pi[5,21]
pi[6,1]
pi[6,2]
pi[6,3]
1.0
0.5
0.0
-0.5
-1.0
0
20
40
lag

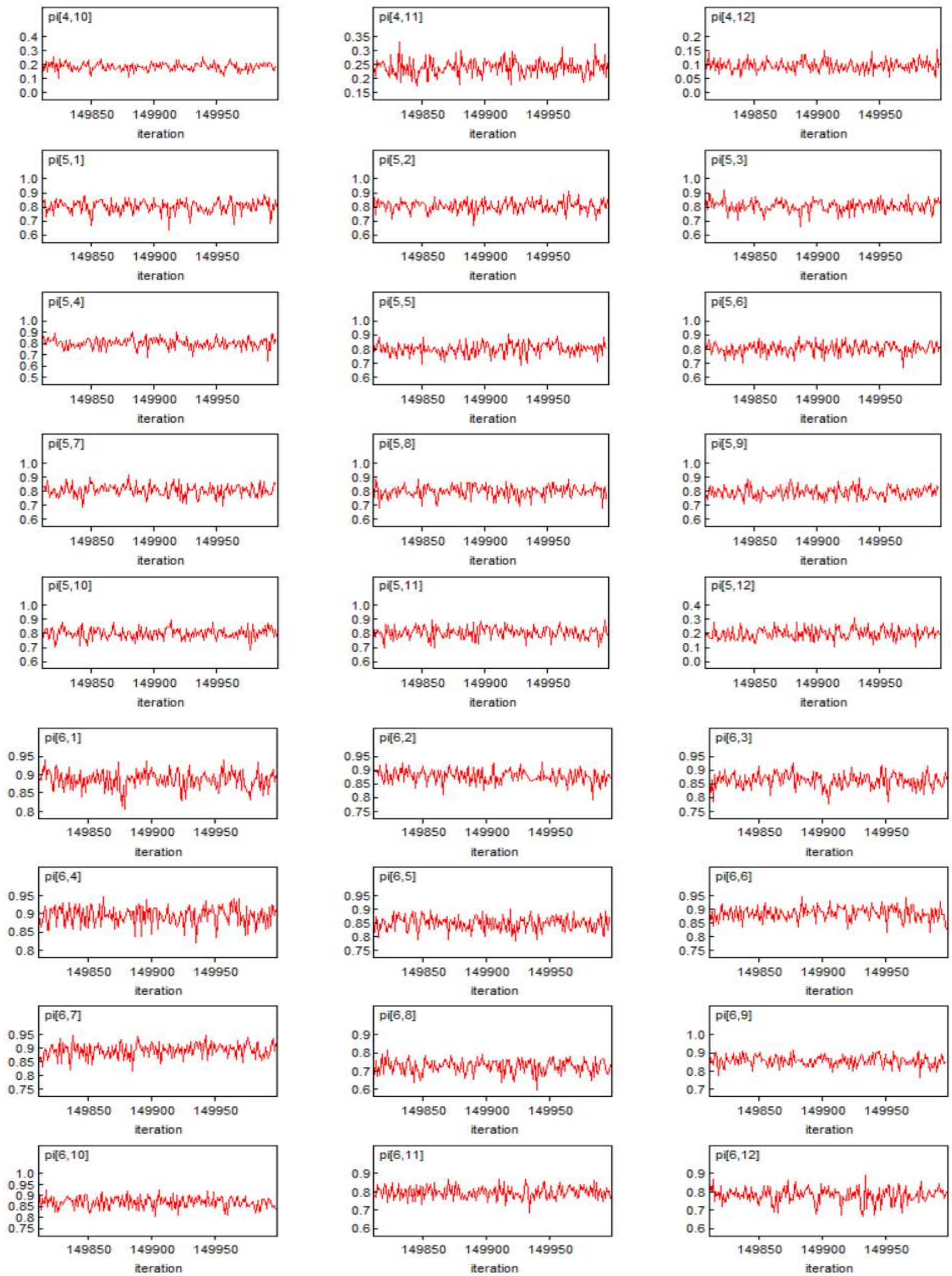

pi[6,4]
pi[6,5]
pi[6,6]
pi[6,7]
pi[6,8]
pi[6,9]
pi[6,10]
pi[6,11]
pi[6,12]
pi[6,13]
pi[6,14]
pi[6,15]
pi[6,16]
pi[6,17]
pi[6,18]
pi[6,19]
pi[6,20]
pi[6,21]
1.0
0.5
0.0
-0.5
-1.0
0
20
40
lag

Appendix K. The last 10000 iterations of MCMC for all parameters (γ, π) generated from Model 1 for the procedural knowledge dimension

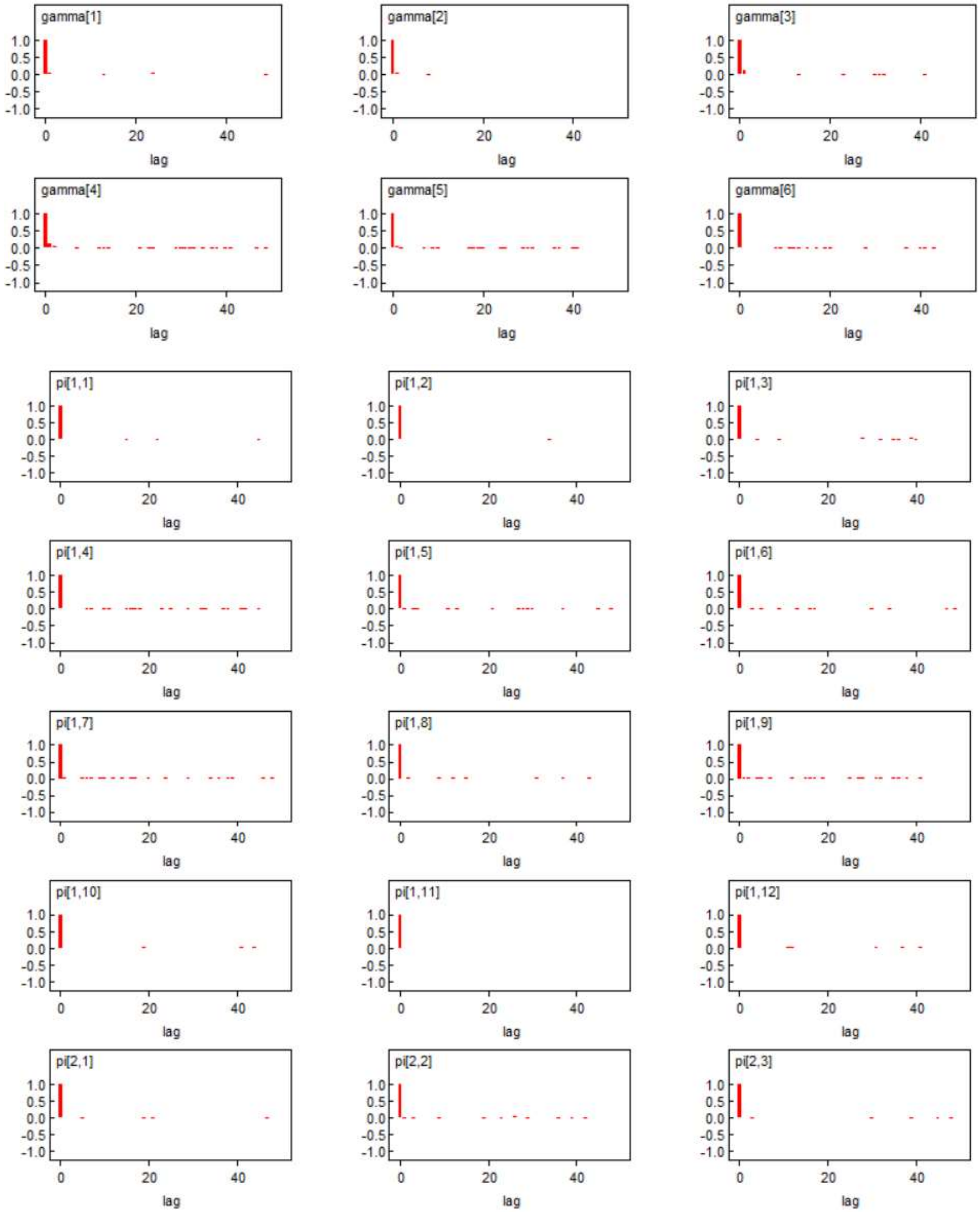


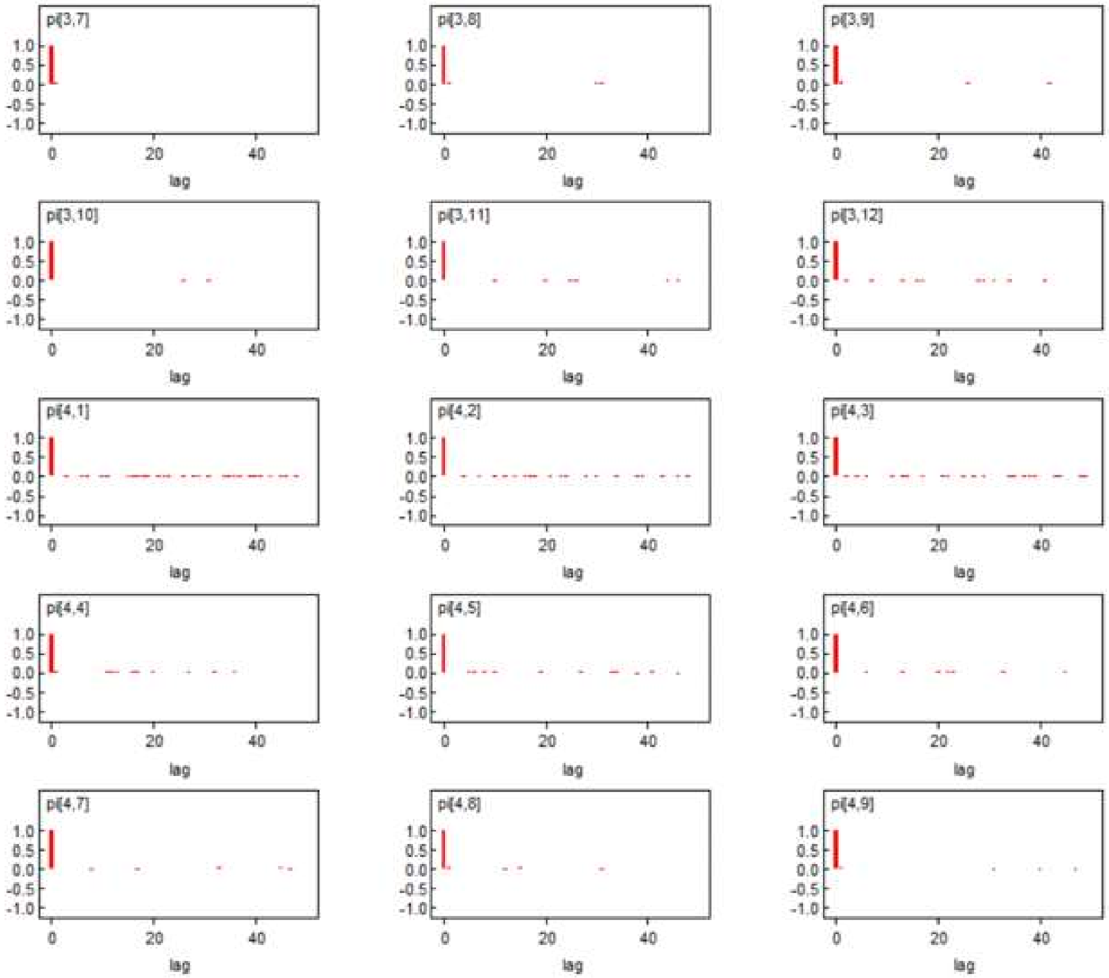

pi[2,4]
pi[2,5]
pi[2,6]
pi[2,7]
pi[2,8]
pi[2,9]
pi[2,10]
pi[2,11]
pi[2,12]
pi[3,1]
pi[3,2]
pi[3,3]
pi[3,4]
pi[3,5]
pi[3,6]
149850 149900 149950
iteration

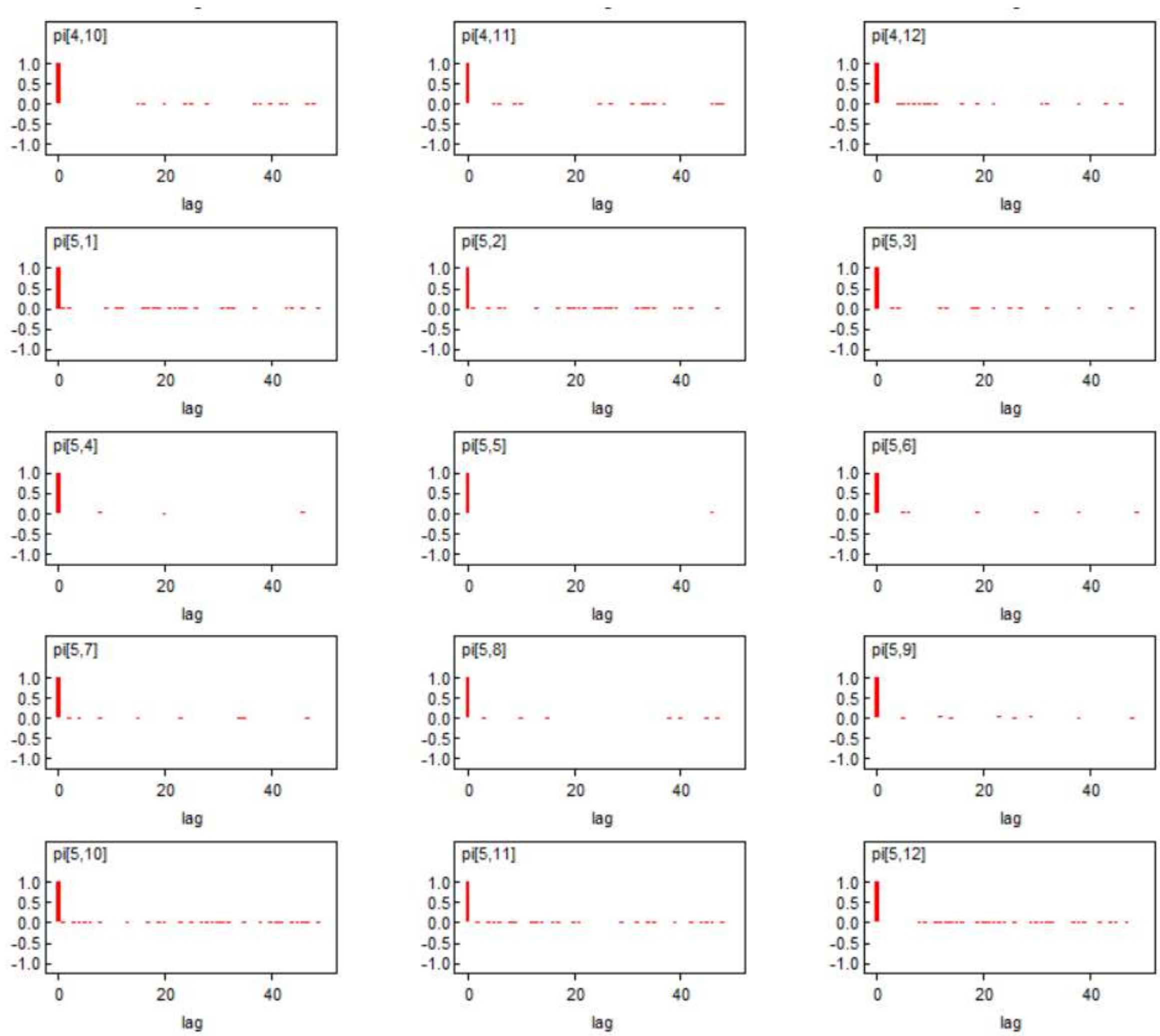

pi[3,7]
pi[3,8]
pi[3,9]
pi[3,10]
pi[3,11]
pi[3,12]
pi[4,1]
pi[4,2]
pi[4,3]
pi[4,4]
pi[4,5]
pi[4,6]
pi[4,7]
pi[4,8]
pi[4,9]
149850 149900 149950
iteration

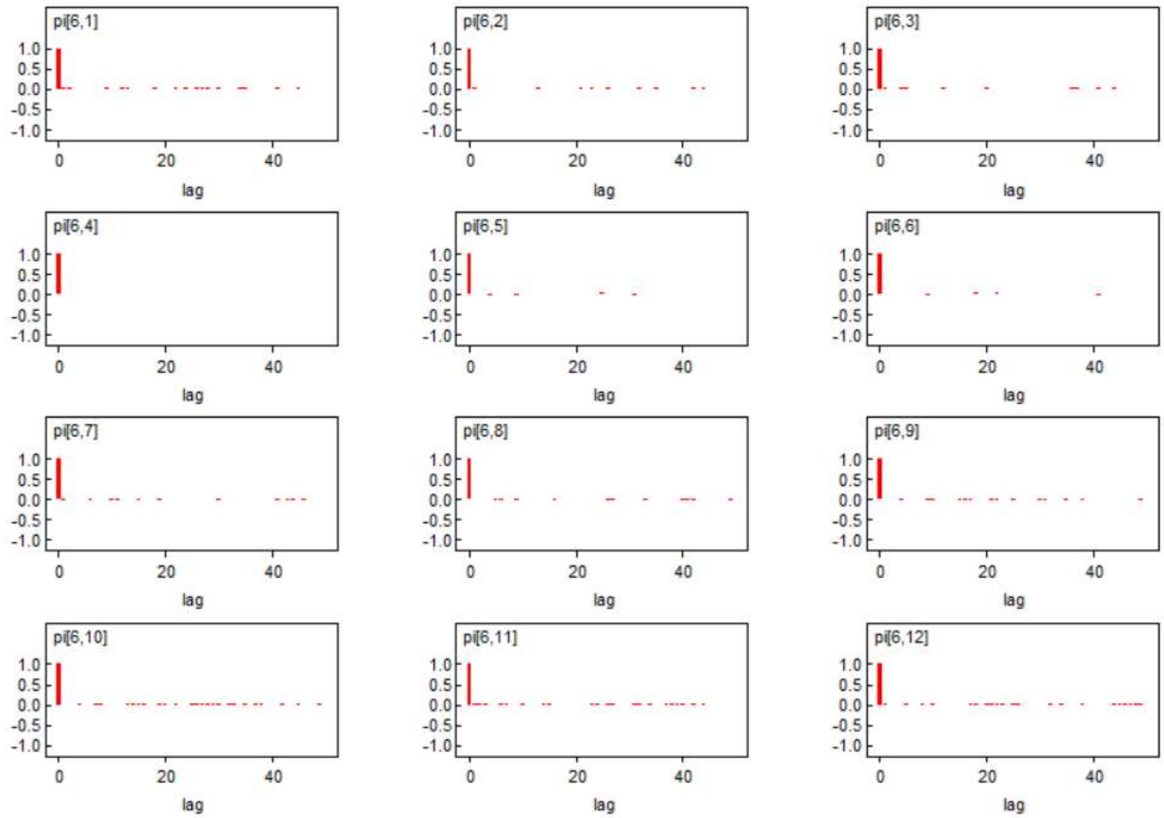

pi[4,10]
pi[4,11]
pi[4,12]
pi[5,1]
pi[5,2]
pi[5,3]
pi[5,4]
pi[5,5]
pi[5,6]
pi[5,7]
pi[5,8]
pi[5,9]
pi[5,10]
pi[5,11]
pi[5,12]
pi[6,1]
pi[6,2]
pi[6,3]
pi[6,4]
pi[6,5]
pi[6,6]
pi[6,7]
pi[6,8]
pi[6,9]
pi[6,10]
pi[6,11]
pi[6,12]
iteration

Appendix L. The autocorrelation plots of the last 10000 iterations of MCMC for for all parameters (γ, π) generated from Model 1 for the procedural knowledge dimension

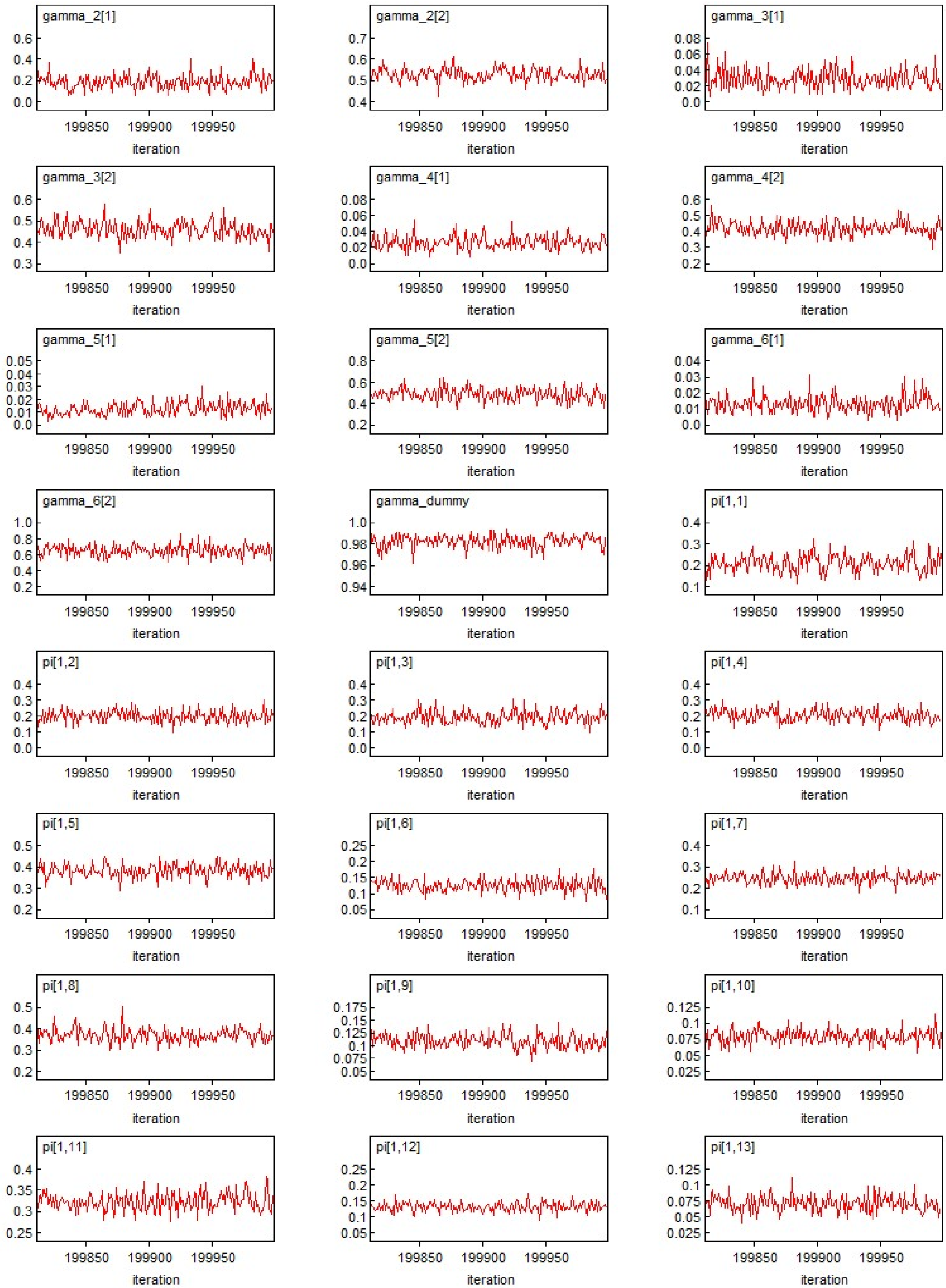


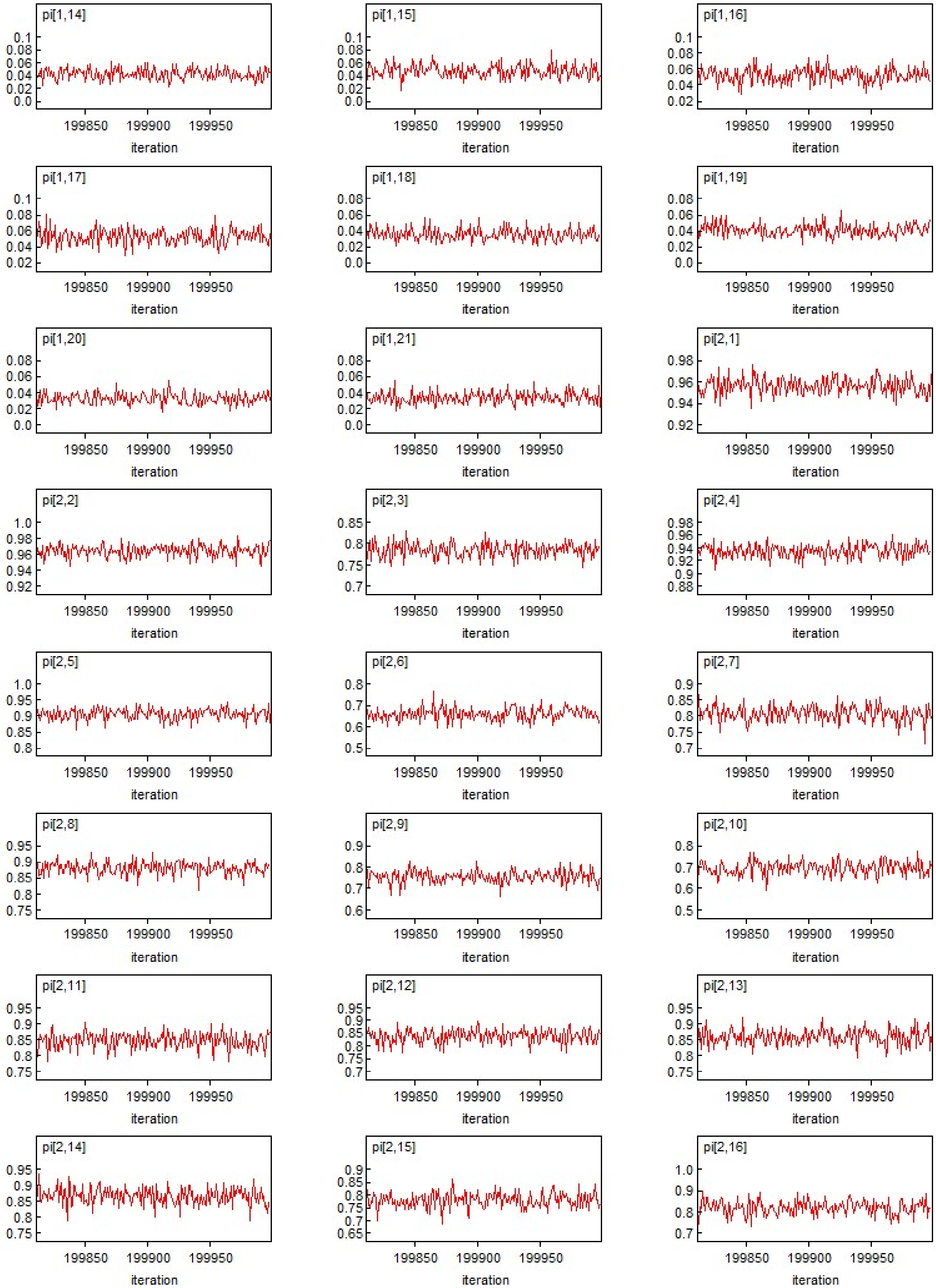

pi[3,7]
pi[3,8]
pi[3,9]
pi[3,10]
pi[3,11]
pi[3,12]
pi[4,1]
pi[4,2]
pi[4,3]
pi[4,4]
pi[4,5]
pi[4,6]
pi[4,7]
pi[4,8]
pi[4,9]
lag

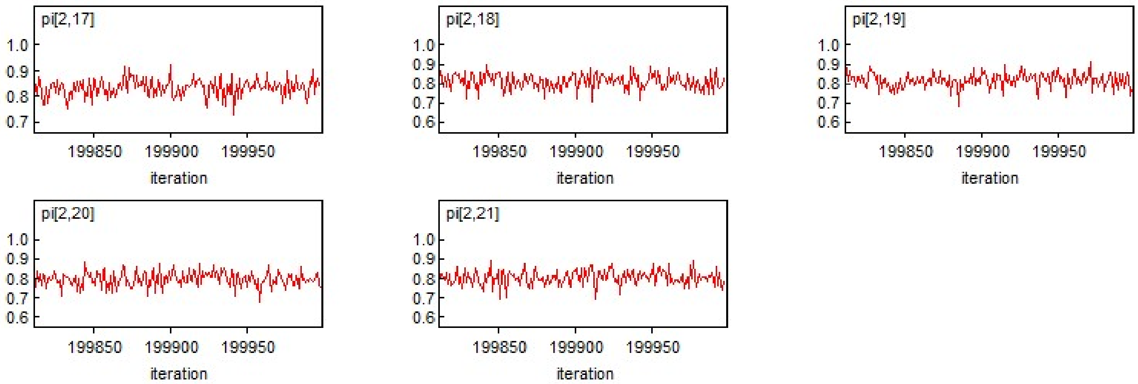

pi[4,10]
pi[4,11]
pi[4,12]
pi[5,1]
pi[5,2]
pi[5,3]
pi[5,4]
pi[5,5]
pi[5,6]
pi[5,7]
pi[5,8]
pi[5,9]
pi[5,10]
pi[5,11]
pi[5,12]
lag


pi[6,1]
pi[6,2]
pi[6,3]
pi[6,4]
pi[6,5]
pi[6,6]
pi[6,7]
pi[6,8]
pi[6,9]
pi[6,10]
pi[6,11]
pi[6,12]
1.0
0.5
0.0
-0.5
-1.0
0
20
40
lag

Appendix M. The last 10000 iterations of MCMC for all parameters (γ, π) generated from Model 2 for the conceptual knowledge dimension

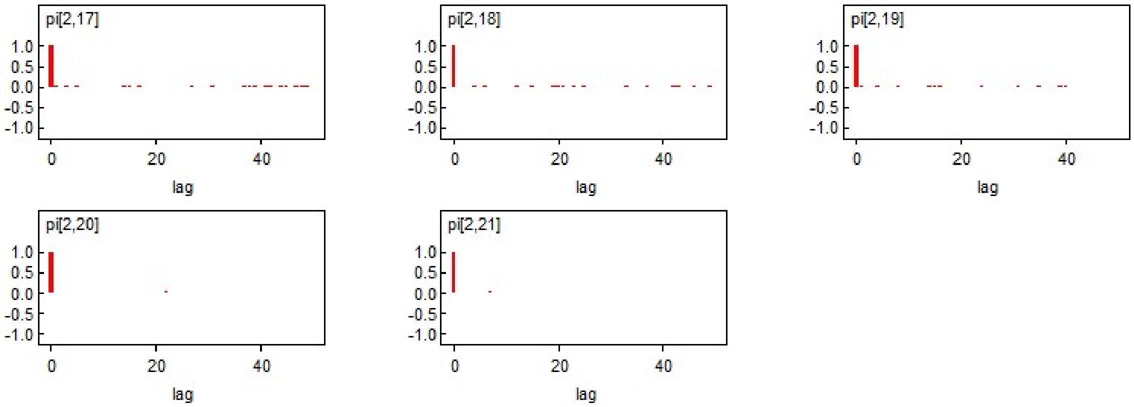



pi[1,14]
pi[1,15]
pi[1,16]
pi[1,17]
pi[1,18]
pi[1,19]
pi[1,20]
pi[1,21]
pi[2,1]
pi[2,2]
pi[2,3]
pi[2,4]
pi[2,5]
pi[2,6]
pi[2,7]
pi[2,8]
pi[2,9]
pi[2,10]
pi[2,11]
pi[2,12]
pi[2,13]
pi[2,14]
pi[2,15]
pi[2,16]
199850 199900 199950
iteration

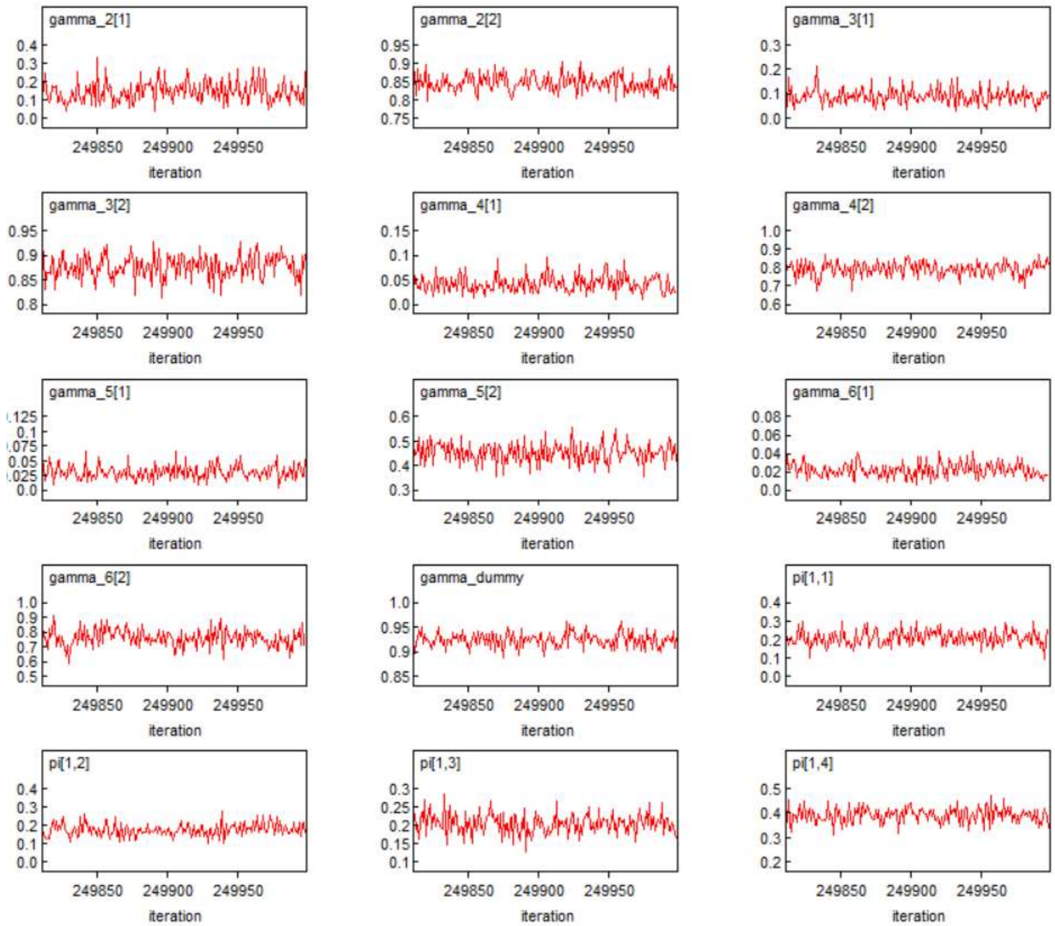

pi[2,17]
pi[2,18]
pi[2,19]
pi[2,20]
pi[2,21]
1.0
0.9
0.8
0.7
0.6
199850
199900
199950
iteration

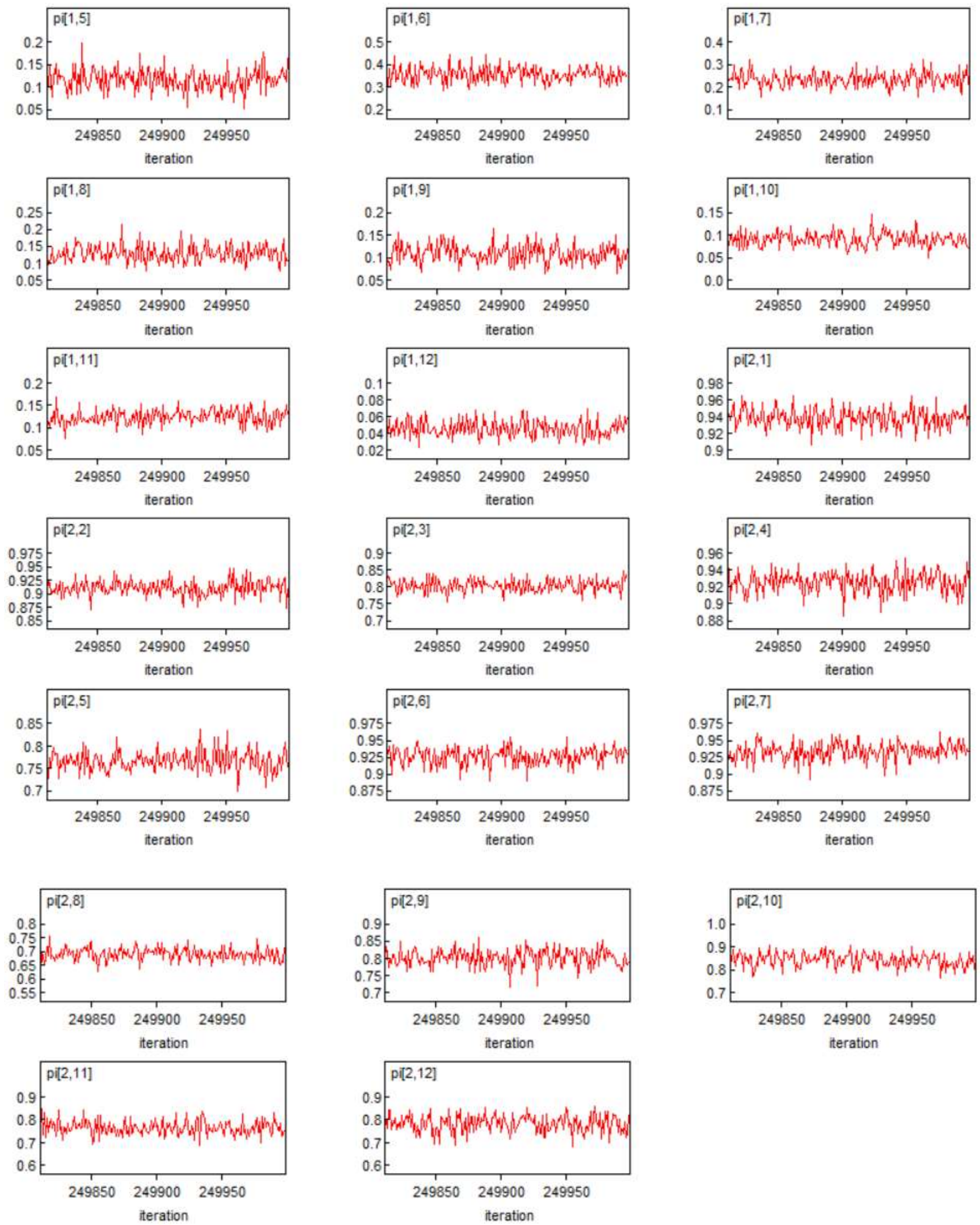
Appendix N. The autocorrelation plots of the last 10000 iterations of MCMC for for all parameters (γ, π) generated from Model 2 for the conceptual knowledge dimension


pi[1,14]
pi[1,15]
pi[1,16]
pi[1,17]
pi[1,18]
pi[1,19]
pi[1,20]
pi[1,21]
pi[2,1]
pi[2,2]
pi[2,3]
pi[2,4]
pi[2,5]
pi[2,6]
pi[2,7]
pi[2,8]
pi[2,9]
pi[2,10]
pi[2,11]
pi[2,12]
pi[2,13]
pi[2,14]
pi[2,15]
pi[2,16]
lag

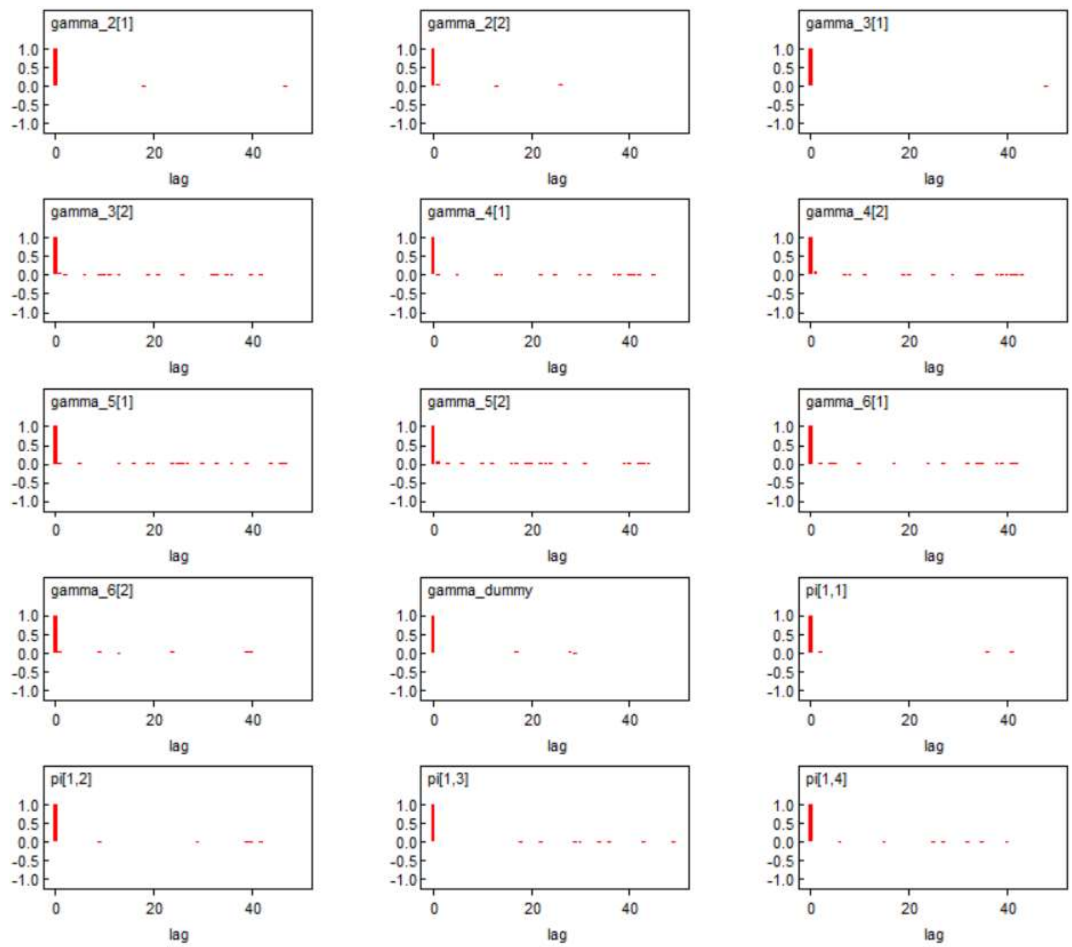

pi[2,17]
pi[2,18]
pi[2,19]
pi[2,20]
pi[2,21]
1.0
0.5
0.0
-0.5
-1.0
0
20
40
lag

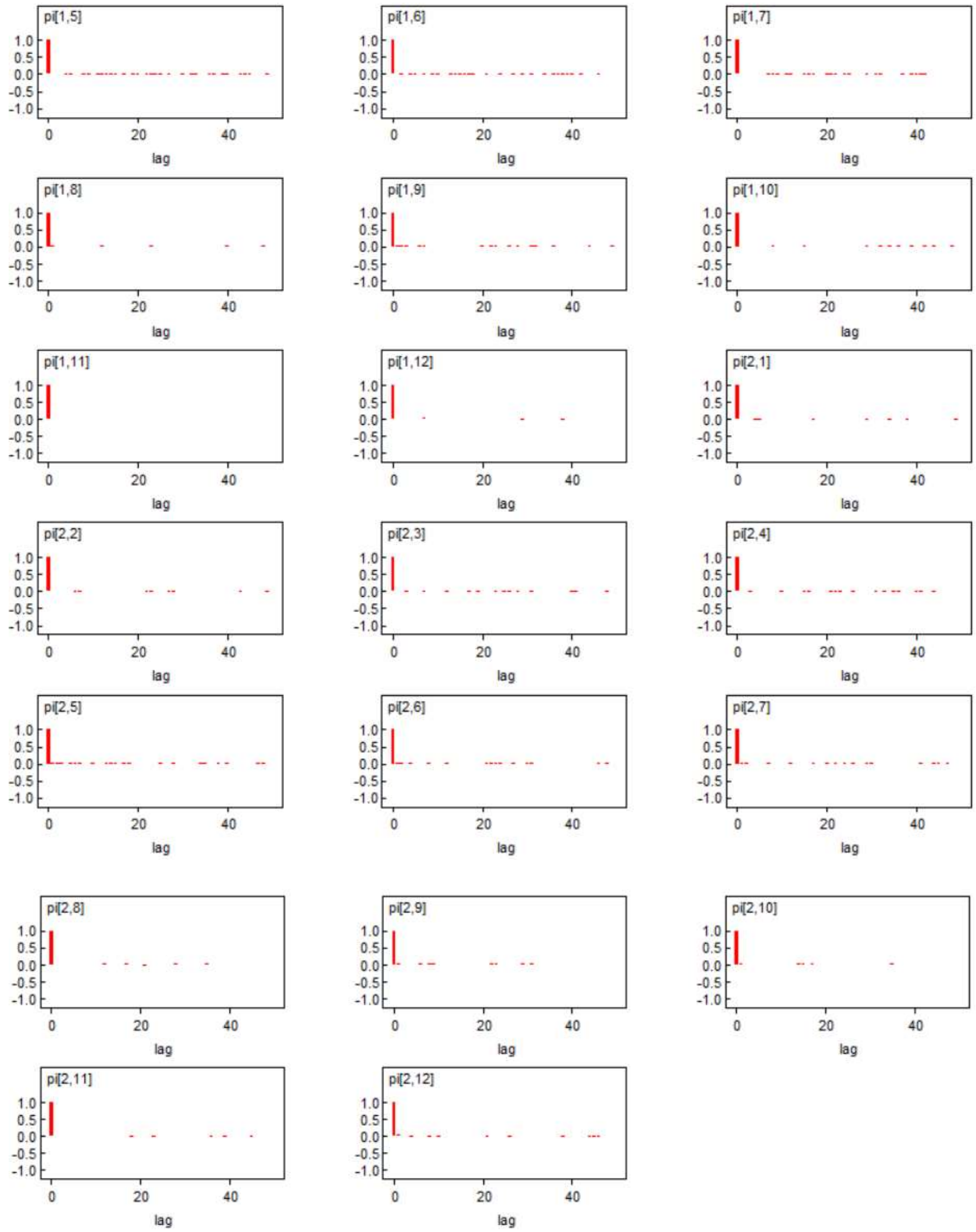
Appendix O. The last 10000 iterations of MCMC for all parameters (γ, π) generated from Model 2 for the procedural knowledge dimension




pi[1,5]
pi[1,6]
pi[1,7]
pi[1,8]
pi[1,9]
pi[1,10]
pi[1,11]
pi[1,12]
pi[2,1]
pi[2,2]
pi[2,3]
pi[2,4]
pi[2,5]
pi[2,6]
pi[2,7]
pi[2,8]
pi[2,9]
pi[2,10]
pi[2,11]
pi[2,12]
249850 249900 249950
iteration

Appendix P. The autocorrelation plots of the last 10000 iterations of MCMC for for all parameters (γ, π) generated from Model 2 for the procedural knowledge dimension




pi[1,5]
pi[1,6]
pi[1,7]
pi[1,8]
pi[1,9]
pi[1,10]
pi[1,11]
pi[1,12]
pi[2,1]
pi[2,2]
pi[2,3]
pi[2,4]
pi[2,5]
pi[2,6]
pi[2,7]
pi[2,8]
pi[2,9]
pi[2,10]
pi[2,11]
pi[2,12]
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Appendix Q. Translation of Interview with One of Participants

INTRODUCTION

RESEARCHER: Thank you for your participation in this interview. Today I will give you some cards of mathematics problems. I want you to solve those problems and explain how you get the answers. If you find any words that you don't understand, please let me know. Please keep talking loudly while answering the questions and describing what you think. You can make any notes and (draw) on the cards. I will give you an example.

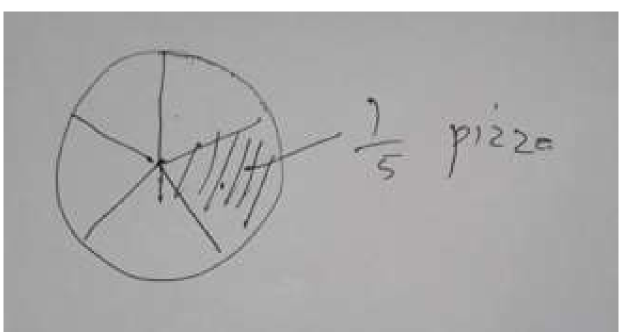
(the researcher gave an example how to speak loudly and make any notes/drawings when solving the problem

RESEARCHER: Read the question on the card.

(The researcher read loudly the question on the card)

RESEARCHER: If a pizza is divided for five people, what portion of pizza will each person get?

RESEARCHER: I answer the question like this, for example, there is a pizza which is usually in a circle shape (the researcher made a circle). Then, it is shared to 5 people. In order to get a fair share, I divide the pizza into 5 equal sizes (the researcher drew lines to make 5 partitions of the circle). It means that each person will get 1/5 of the pizza.



RESEARCHER: Let's begin with the first question.

(The researcher gave the participant the first Card (Card ConT1Q1))

PARTICIPANT: Write the fraction for the shaded part below.

PARTICIPANT: The total of all parts is 8, and 3 parts are shaded, so this is 3/8.

RESEARCHER: What do you think if the bottom number is getting bigger, is the fraction getting smaller or bigger?

PARTICIPANT: Is the top number fixed?

RESEARCHER: Yes

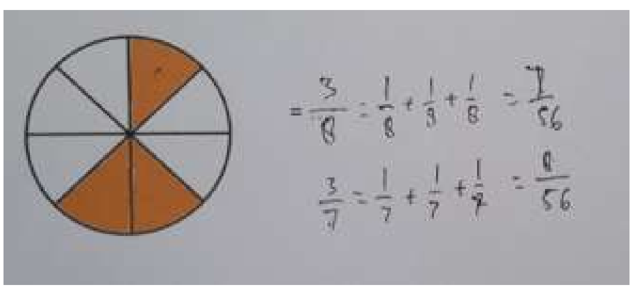
PARTICIPANT: It’s getting smaller

RESEARCHER: If the bottom number getting smaller than what happened with the fraction?

PARTICIPANT: It’s getting bigger

RESEARCHER: Can you tell me why if the bottom number is getting bigger than the fraction is getting smaller?

PARTICIPANT: Look at this example, 3/8 and 3/7. Suppose that 3/8 consists of three of 1/8, and 3/7 consists of three of 1/7. If the denominators are equated to 56, so 1/8 becomes 7/56 and 1/7 becomes 8/56. So if the bottom number is getting smaller, than the fraction will become greater than before.

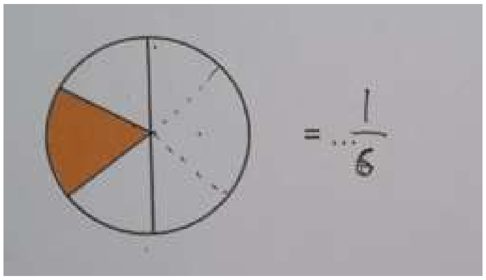


RESEARCHER: Thanks for answering the question well.

(The researcher gave the participant ConT1Q3 card)

PARTICIPANT: Write the fraction for the shaded part below.

PARTICIPANT: If this is a half, so the number of these parts should be the same. It means we can give 3 parts on the left and 3 on the right. Because 1 part is shaded, and the total is 1,2,3,4,5,6 , so this is 1/6.



RESEARCHER: Can you tell me why you drew additional lines?

PARTICIPANT: Just to make sure that the parts on the left are the same with those on the right

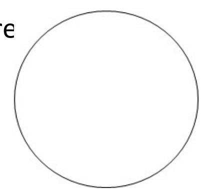
RESEARCHER: How about the sizes of the parts, should they be the same or not?

PARTICIPANT: Should be the same

RESEARCHER: Okay, thanks for answering the question well.

ConPWL3Q1(a)

(the researcher gave the participant ConT1Q4 card)

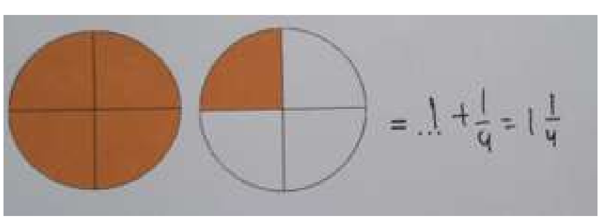
PARTICIPANT: If the figure  is the whole, write the fraction for the shaded part below

PARTICIPANT: If this is a whole, and this is a whole (the participant pointed the two circles below). It means there are two wholes, but one if fully shaded or 1, and the other is only 1/4 shaded. Then 1 is added to 1/4, which is 1 1/4

RESEARCHER: So, what do these two circles show?

PARTICIPANT: Two wholes.

RESEARCHER: Okay, thanks for answering the question well.



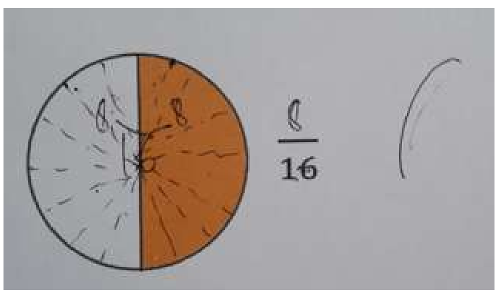
(The researcher gave the participant ConT1Q2 card)

PARTICIPANT: Write the numerator of the fraction for the shaded parts below

PARTICIPANT: Oh, if the total of parts is 16 and what have been drawn here is half, so this is 8 (The participant wrote down 8 as the answer).

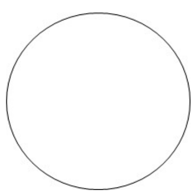
RESEARCHER: Can you show me how you got the answer using a diagram?

PARTICIPANT: Because this is a half part, oh for example, there are 16 parts (the participant drew lines to divide the circle into 16 parts), so there are 8 parts here (the participant pointed the shaded area of the circle) and also there are 8 here the participant pointed the unshaded area of the circle). So the numerator is 8, the number of the shaded parts.



RESEARCHER: Okay, thanks for answering the question well.

(the researcher gave the participant ConT1Q5 card)

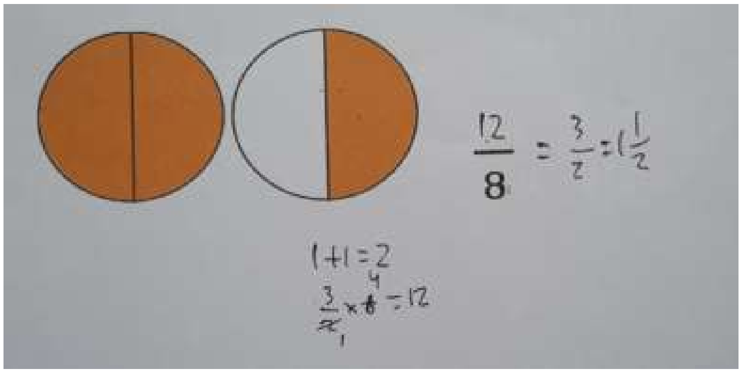
PARTICIPANT: If the figure  is the whole

Write the numerator of the fraction for the shaded parts below

PARTICIPANT: If this 1 circle representing 8, so 2 circles means 2 of 8 which is 16, but this circle is not fully shaded, a half of 8, meaning that this is 12.

RESEARCHER: Can you tell me how you got 12?

PARTICIPANT: These are two wholes which is 1 plus 1, but this one is not fully shaded, 1/2, which means there are 3/2 parts multiplied by 8, equals 12 which become the numerator.



RESEARCHER: Can you simplify the result?

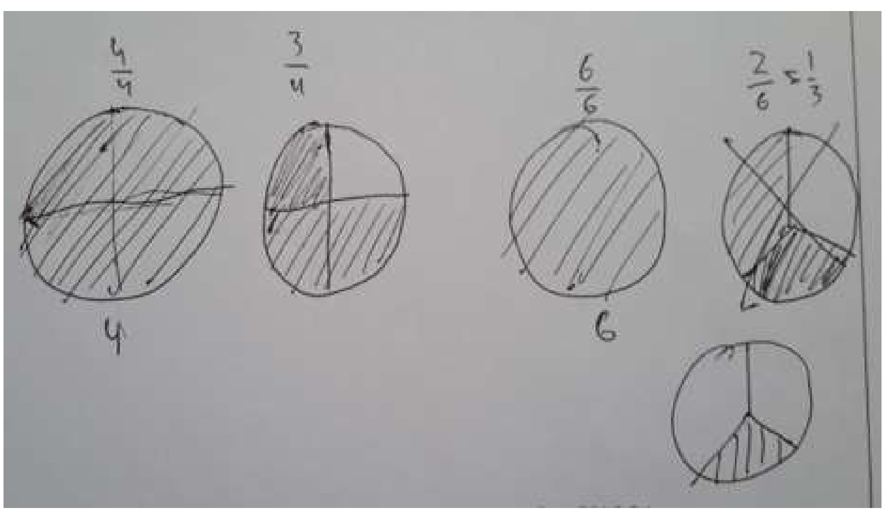
PARTICIPANT: 3/2 or 1 1/2

RESEARCHER: Okay, thanks for answering the question well.

(The researcher gave the participant ConT3Q3 card)

PARTICIPANT: Which is larger $\frac{7}{4}$ or $\frac{8}{6}$? Illustrate how you got your answer using a picture.

PARTICIPANT: Suppose there is 7/4, 4 parts which is fully shaded, 4/4 and 3/4. Next, 8/6, 6/6 is one circle which is fully shaded and the remaining is 2/6 or 1/3.



RESEARCHER: So based on these diagrams, which one is greater?

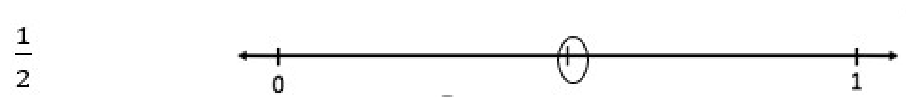
PARTICIPANT: This one (The participant pointed the diagrams representing 7/4)

RESEARCHER: Why?

PARTICIPANT: Because it has more shade parts.

RESEARCHER: Okay, thanks for your answer, we can discuss this task again later.

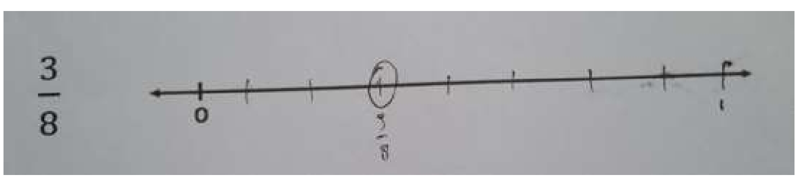
RESEARCHER: You know number lines, like this (the researcher demonstrated the example of a number line as follows)



(the researcher gave the participant ConT4Q1 card)

PARTICIPANT: Show the fractions on the number lines below

PARTICIPANT: 1,2,3,4,5,6,7,8 (The participant made 8 scales). From the left 1,2,3 (the participant circled the location of 3/8)



RESEARCHER: Can you tell me how you got the answer?

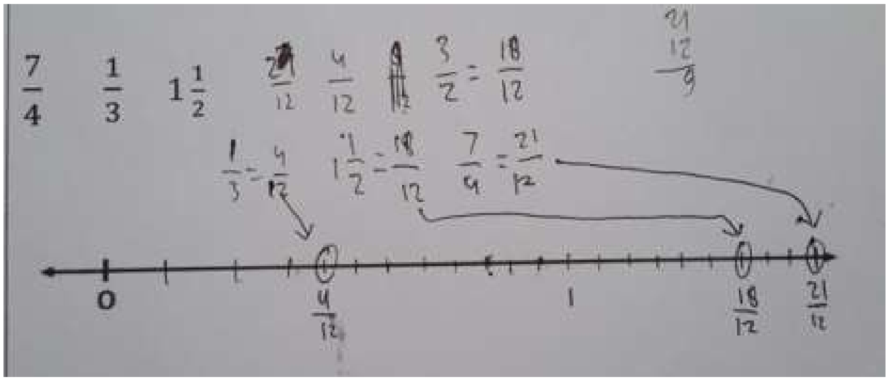
PARTICIPANT: 3/8 is smaller than 1, so we should create 8 points. Then, the third dot from null is the answer, 3/8

RESEARCHER: Oh okay, thank you, well done.

ConMSL3Q1

(the researcher gave the participant ConT4Q3 card)

PARTICIPANT: Order these fractions from the smallest to the largest on the number line below.



PARTICIPANT: Euh, first we should equate the denominators to 12. 7/4 equals 21/12, 1/3 equals 4/12, then 1 1/2 or 3/2 equals 18/12.

(After that the participant created 21 scales on the number line)

Then we put these fractions on the number line. First, the smallest fraction 1/3 or 4/12 is put here, then 18/12, finally 7/4 or 21/12

RESEARCHER: So which one is the greatest fraction?

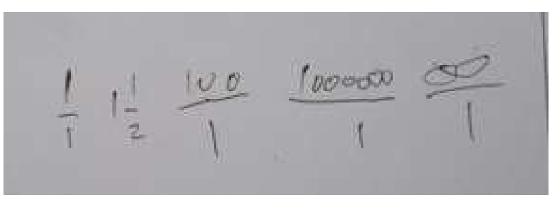
PARTICIPANT: 7/4

RESEARCHER: Oh okay, thank you, well done.

(The researcher gave the participant ConT5Q1 card)

PARTICIPANT: Write the biggest fraction that you know.

PARTICIPANT: Hm… it could be 1/1, 1 1/2, 100/1, 1000000/1, an infinite number per 1



RESEARCHER: So, what is your conclusion?

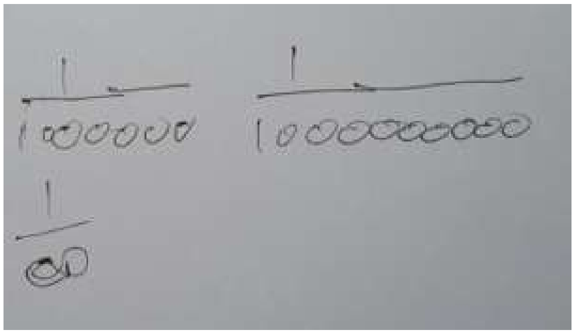
PARTICIPANT: Infinite

RESEARCHER: Okay, thanks for answering the question well.

(The researcher gave the participant ConT5Q2 card)

PARTICIPANT: Write the smallest fraction that you know.

PARTICIPANT: The smallest fraction that close to null. Maybe 1 per million, or 1 per billion, 1 per an infinite number



RESEARCHER: So, what is your conclusion?

PARTICIPANT: The smallest fraction that close to null with the denominator is an infinite number.

RESEARCHER: Thanks for answering the question well.

(The researcher gave the participant ConT6Q1 card)

PARTICIPANT: How many numbers are there between $\frac{2}{5}$ and $\frac{4}{7}$?

PARTICIPANT: This is for all numbers or only limited to the fraction with the denominator 5?

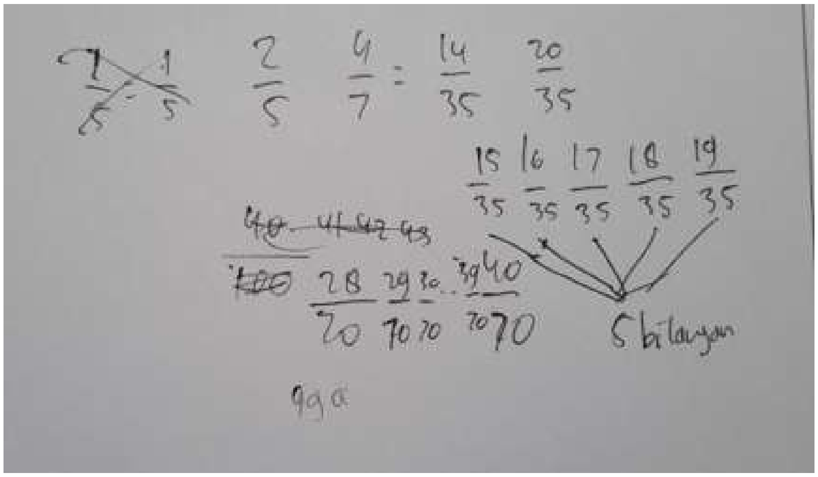
RESEARCHER: All numbers

PARTICIPANT: Oh if how many numbers, they can be infinite until the unknown unit, but if limited within the denominator 5, oh 2/5 and 4/7 so the denominators are equated first to 35. So, between 14/35 and 20/35 there are 15/35, 16/35, 17/35, 18/35, and 19/35. So there are only 5 numbers.

RESEARCHER: You mentioned infinite, what do you mean infinite?

PARTICIPANT: If the denominators are made very big for example, 40,41,42,43, until oh … (the participant crossed out 40,41,42,43), so the denominator is for example 70. It means there are 28/70, 29/70, 30/70 until 39/70. If this denominator is increased, there will be many numbers, cannot be counted.

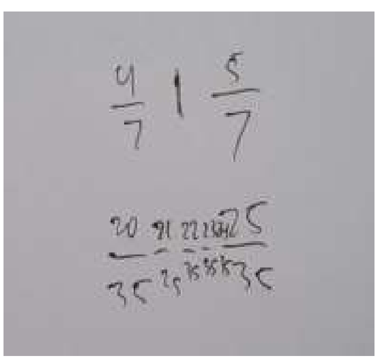
RESEARCHER: Thanks for answering the question well.



(The researcher gave the participant ConT6Q2 card)

PARTICIPANT: How many numbers are there between $\frac{4}{7}$ and $\frac{5}{7}$?

PARTICIPANT: If the denominator is not changed, than it looks there is no number between them, but if the denominator is made bigger than there are numbers between them. For example, the denominator is 35, so there will be 21/35, 22/35, 23/35, and 24/35, or if this denominator is made bigger again than there many numbers between them. So they also cannot be counted.

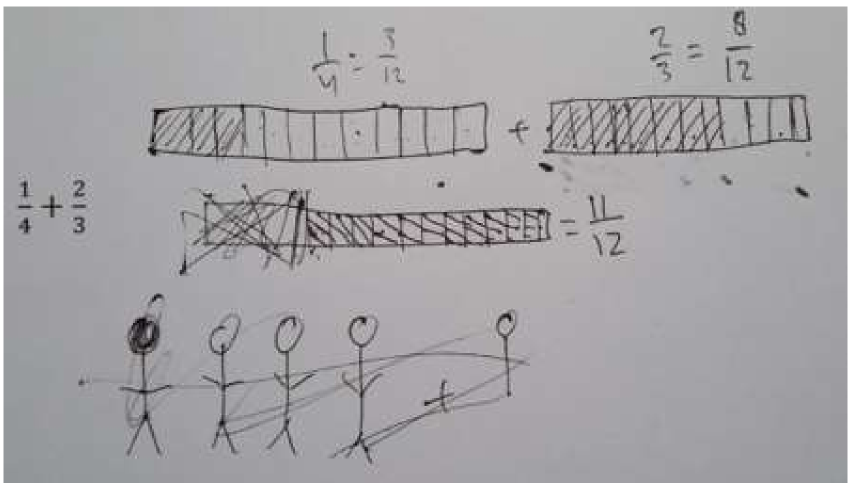


RESEARCHER: This is the example of fraction addition (the researcher demonstrated the example of fraction addition using diagram representations. After that, the researcher gave the participant ConFOL1Q1 card)

RESEARCHER: Thanks for answering the question well.

(The researcher gave the participant ConT7Q2card)
PARTICIPANT: Draw a pictorial representation for the fraction addition below

PARTICIPANT: 1/4 plus 1/3. For example there is 1 black man of 4 people and there are also.. Oh don't use people, it's not nice. This 1/4 is added to 2/3, so there are 12 parts (The participant created rectangles for each 12 partitions). 1/4 equals 3/12, while 2/3 equals 8/12. 1, 2, 3 (The participant shaded three parts of the first recatngle to represent 1/4 or 3/12). 1,2,3,4,5,6,7,8 (The participant shaded 8 parts of the second rectangle to represent 2/3 or 8/12). So the result is 1,2,3,4,5,6,7,8,9,10,11,12 (the participant created another rectangle with 12 partitions). There are 3 parts here (the participant shaded three parts of the rectangle), and there are 8 parts here (he continued shaded 8 parts of the same rectangle so that there were 11 part which were shaded). So the result is 11/12



RESEARCHER: What were you added in this diagram representation?

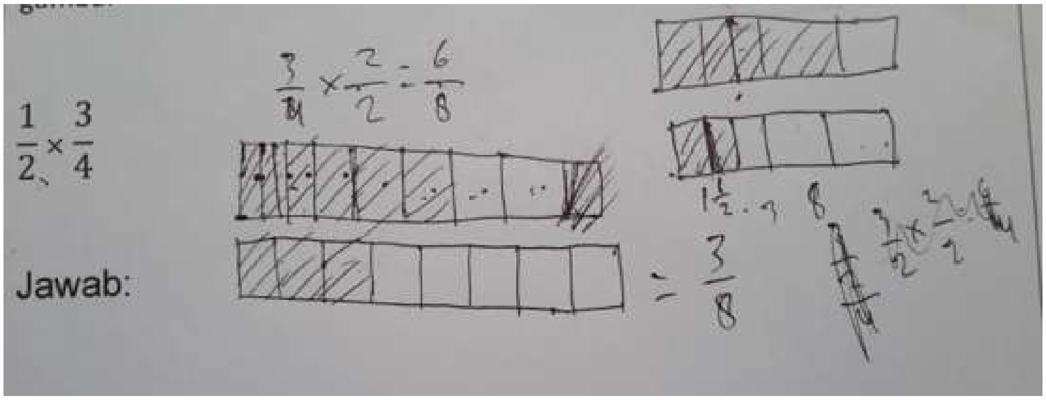
PARTICIPANT: The shaded parts or the numerator

RESEARCHER: Okay, thanks for your excellent answer.

(The researcher gave the participant ConT8Q1 card)

PARTICIPANT: Draw a pictorial representation for the fraction multiplication below

PARTICIPANT: So this is 1/2 of 3/4. For example there are 1,2,3,4,5,6,7,8 (the participant drew a rectangle with 8 partitions). 3/4 is equal to 6/8, 6 parts are shaded. If this is multiplied by 1/2, meaning that a half of these 6 parts. So, 1,2,3, there are parts are shaded or this is the same with 3/8.



RESEARCHER: Could you tell me how you got the answer in more detail?

PARTICIPANT: Firstly, there are 6 shaded parts of 8 parts. The number 8 is the result of 3/4 times 2/2

RESEARCHER: Why did you multiply this by 2/2?

PARTICIPANT: In order to get 8 so it become easy to be divided.

RESEARCHER: Please continue …

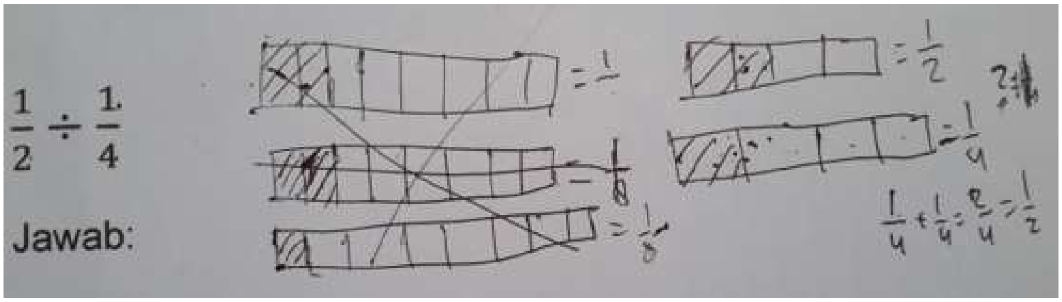
PARTICIPANT: Then 1/2 of 6 parts are 3 parts which are 3 shaded parts of the total of 8 parts.

RESEARCHER: Thanks for answering the question well.

(The researcher gave the participant ConT8Q2 card)

PARTICIPANT: Draw a pictorial representation for the fraction division below

PARTICIPANT: 1/2 divided by 1/4. This is a half, firstly there are 4 parts and 1 part is shaded which is 1/4, than a half of 1/4 is taken, because the number is not nice, so it is multiplied by 2 which is 1,2,3,4,5,6,7,8 (The participant created a rectangle with 8 partitions). If divided by 1/2, how many, ah …1,2,3,4. How many of this fraction to become … (the participant looked confused)



RESEARCHER: Can you tell me what the meaning 1/2 divided by 1/4 is?

PARTICIPANT: How many 1/4 to become 1/2.

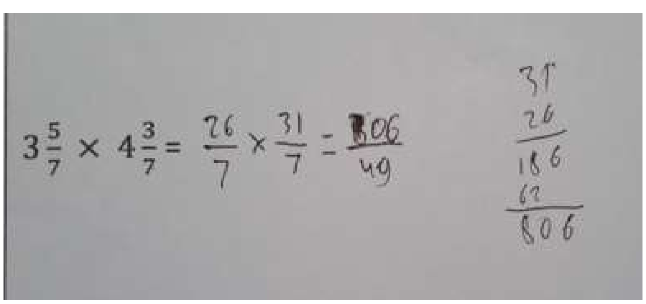
(The participant drew a rectangle with 4 partitions and two of them are shaded to represent 1/2). It is the same with 2 (he pointed to the 2 shaded partitions). In order to become 1/2, so this one (he pointed one shaded area of the rectangle which represent 1/4) needs 2 times of itself, so the answer is 2, which is 2 times of this part (he pointed to the rectangle which represent 1/4). 1/4 plus 1/4 equals 2/4 or 1/2, meaning that it needs 2 times of 1/4 so the result is 2.

RESEARCHER: Okay, thank you, well done

(The researcher gave the participant ProT2Q4 card)

PARTICIPANT: Find the result of the fraction multiplication below

PARTICIPANT: First, they should be converted into a common fraction form. 3 5/7 equals 26/7 and 4 3/7 equals 31/7, then 21 times 31 (The participant calculated 26 times 31) which is 806, then 7 times 7, 49 so the result is 806/49.



RESEARCHER: How do you solve this task?

PARTICIPANT: It's transformed into common fractions, then denominator times the denominator and the numerator times the numerator.

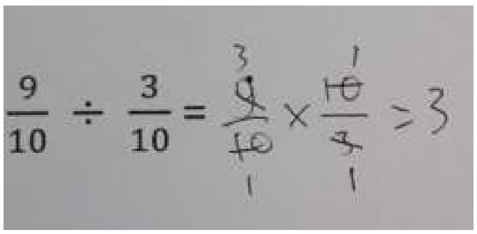
RESEARCHER: Thanks for answering the question well.

L3PKQ3

(The researcher gave the participant ProT2Q3 card)

PARTICIPANT: Find the result of the fraction division below

PARTICIPANT: 9/10 divided by 3/10. To become easier, it changed to multiplication. 9/10 times … because this is division so 3/10 is flipped to 10/3. Then, 9 divided by 3, 3 and 3 divided by 3, 1. 10 divided by 10, 1 so the result is 3.

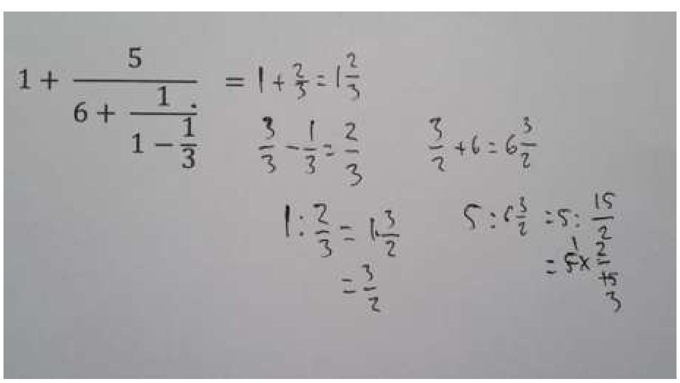


RESEARCHER: Okay, thank you …

(The researcher gave the participant ProT3Q3 card)

PARTICIPANT: Find the result of the fraction operation below

PARTICIPANT: Firstly, we do the operation in the bottom which is 1 or 3/3 minus 1/3, which is 2/3. Then 1 divided by 2/3 or 1 times 3/2 which is 3/2. Next, 3/2 plus 6 equals 6 3/2, then 5 divided by 6 3/2 which is the same with 5 divided by 15/2. It is the same with 5 times 2/15 which is 2/3. Finally, 1 plus 2/3 which is equal to 1 2/3



RESEARCHER: Okay, thank you, well done

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