A TOPOLOGICAL APPROACH TO SPECTRAL FLOW

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Contents

Declaration

I certify that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge and belief it does not contain any material previously published or written by another person except where due reference is made in the text.

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Abstract

It is a well-known result of T. Kato that given a continuous path of square matrices of a fixed dimension, the eigenvalues of the path can be chosen continuously. In this paper, we give an infinite-dimensional analogue of this result, which naturally arises in the context of unitary spectral flow. This provides a new approach to spectral flow, which seems to be missing from the literature. It is the purpose of this paper to fill in this gap.

1. Introduction

By "operators" we always mean bounded linear operators on a separable Hilbert space $\mathcal{H}$.

1.1. **Motivation.**

1.1.1. *T. Kato's finite-dimensional continuous enumeration.* The task of *continuous enumeration* is akin to tracking the individual movements of, for example, a swarm of bees. Our "bees" are utterly identical, they pass through one another, and they can make instant changes of direction infinitely many times per second (since we consider merely continuous paths), so that we cannot know which is which after a collision. However, it still seems intuitive that we should be able to assign (although not uniquely) a finite number of continuous functions which completely describe the movement of the "swarm".

Now, we give a rigorous formulation of finite-dimensional continuous enumeration due to T. Kato. The following exposition is directly taken from [Bha, §VI.1]. Let $\mathbb{C}^n_{\mathrm{sym}}$ be the quotient topological space obtained from $\mathbb{C}^n$ via the equivalence relation which identifies two n-tuples of complex numbers, if they are permutations of each other. That is, $\mathbb{C}^n_{\mathrm{sym}}$ can be viewed as the space of "unordered n-tuples" of complex numbers. Given an n-tuple $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, we denote its equivalence class in $\mathbb{C}^n_{\mathrm{sym}}$ by $(\lambda_1, \ldots, \lambda_n)^*$. The topological space $\mathbb{C}^n_{\mathrm{sym}}$ thus defined inherits a metric

$$\operatorname{dist}\left((\lambda_1, \ldots, \lambda_n)^*, (\mu_1, \ldots, \mu_n)^*\right) := \min_{\pi} \max_{1 \le i \le n} |\lambda_i - \mu_{\pi_i}|,$$

where the minimum is taken over all permutations π. The following result is Kato's selection theorem ([Kat2, Theorem II.5.2]):

Theorem 1.1. *Let $\lambda(\cdot)$ be a continuous mapping from an interval I of $\mathbb{R}$ into the space $\mathbb{C}^n_{\mathrm{sym}}$. Then there exist n continuous complex-valued functions $\lambda_1(\cdot), \ldots, \lambda_n(\cdot)$ on I, such that $\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t))^*$ for all $t \in I$.*

As is typical, although seemingly obvious, an existence theorem of this kind is not altogether straightforward to prove. Furthermore, the following example shows that the domain I cannot be replaced by a general metric space:

Example 1.2. Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices of complex entries equipped with the ordinary uniform norm. In [Bha, §VI.1], it is proved that the mapping

$$M_n(\mathbb{C}) \ni A \longmapsto (\lambda_1(A), \ldots, \lambda_n(A))^* \in \mathbb{C}^n_{\mathrm{sym}}, \tag{1}$$

where $\lambda_1(A), \ldots, \lambda_n(A)$ are the eigenvalues of A repeated according to their algebraic multiplicities, is continuous. Let us consider the case $n = 2$, and set $A(z) := \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$ for all $z \in \mathbb{C}$. The mapping $A(\cdot)$ is continuous on any open subset I of $\mathbb{C}$ and the eigenvalues of $A(z)$ are $\pm z^{1/2}$. Continuity of the mapping (1) implies that $I \ni z \longmapsto (\lambda_1(A(z)), \lambda_2(A(z)))^* \in \mathbb{C}^n_{\mathrm{sym}}$ is continuous. However, if the domain I contains the origin, then this continuous mapping cannot be represented by constituent continuous functions.

Given a square matrix $A \in M_n(\mathbb{C})$, we may identify the spectrum $\sigma(A)$ of A with the unordered tuple as in (1). The following result is an immediate consequence of Theorem 1.1 and the continuity of the mapping $M_n(\mathbb{C}) \ni A \longmapsto \sigma(A) \in \mathbb{C}^n_{\mathrm{sym}}$:

Theorem 1.3 (Kato's finite-dimensional continuous enumeration)**.** *If $A(\cdot)$ is a continuous path of square complex matrices of a fixed dimension n, then there exist continuous paths $\lambda_1(\cdot), \ldots, \lambda_n(\cdot)$ in $\mathbb{C}$, s.t. $\sigma(A(\cdot)) = (\lambda_1(\cdot), \ldots, \lambda_n(\cdot))^*$.*

In this paper, we give a certain infinite-dimensional analogue of Kato's continuous enumeration of eigenvalues, which naturally arises in the context of the so-called *unitary spectral flow*. This provides a new approach to spectral flow, which seems to be missing from the literature. It is the purpose of this paper to fill in this gap.

1.1.2. *Self-adjoint Fredholm spectral flow.* The origin of spectral flow goes back to Atiyah-Patodi-Singer [APS]. Spectral flow has since found many connections, famously for example to the Fredholm index (see [RS]). Given a continuous one-parameter family $\{F(t)\}_{t\in[0,1]}$ of self-adjoint Fredholm operators, we naively understand the spectral flow of the continuous path F to be the number of eigenvalues of $F(t)$ that cross 0 rightward minus the number that cross 0 leftward as t monotonically increases from 0 to 1. The usual way of making this idea rigorous involves the notion of intersection number: we precisely define the spectral flow of the path F to be the intersection number of the graph $\bigcup_{t\in[0,1]}\sigma(F(t))$ with the line $\lambda=-\epsilon$, where ϵ is any sufficiently small positive number. Spectral flow turns out to be a homotopy invariant.

1.1.3. *Unitary spectral flow.* The notion of unitary spectral flow is discussed in [Pus]. Let $\mathfrak{U}_p(\mathcal{H},I)$ be the set of all unitary operators U such that $U-I$ is in the p-Schatten class $\mathfrak{S}_p(\mathcal{H})$ (see below for definition), where I denotes the identity operator. Throughout this paper, we let p be a fixed number in $[1,\infty]$. The collection $\mathfrak{U}_p(\mathcal{H},I)$ thus defined admits a natural complete metric

$$\operatorname{dist}(U,U') := \|U-U'\|_{\mathfrak{S}_p}, \quad \forall U,U'\in\mathfrak{U}_p(\mathcal{H},I),$$

where $\|\cdot\|_{\mathfrak{S}_p}$ is the norm on $\mathfrak{S}_p(\mathcal{H})$. It follows from Weyl's theorem on the stability of essential spectrum that the essential spectrum[1] $\sigma_{\mathrm{ess}}(U)$ of any unitary operator $U\in\mathfrak{U}_p(\mathcal{H},I)$ is $\{1\}$. We can then understand the spectral flow of a continuous path $U(\cdot)$ of unitary operators in $\mathfrak{U}_p(\mathcal{H},I)$ to be the integer-valued function $\operatorname{sf}(-;U):(0,2\pi)\to\mathbb{Z}$ given by

$$\begin{aligned}(2)\qquad \operatorname{sf}(\theta;U) := {}&\langle\text{the number of eigenvalues of } U(t) \text{ that cross } e^{i\theta} \text{ anticlockwise}\rangle\\ &-\langle\text{the number of eigenvalues of } U(t) \text{ that cross } e^{i\theta} \text{ clockwise}\rangle\end{aligned}$$

as t monotonically increases from 0 to 1.

1.1.4. *Unitary spectral flow and spectral shift function.* In [Pus] the naive definition (2) is made precise and is used to express the spectral shift function (SSF) as the averaged spectral flow of a path of unitary operators. This path of unitary operators is obtained from the scattering matrix by analytic continuation of the spectral parameter (energy) into the complex plane: see [Pus, (4.9)] for details. Let us briefly recall the definition of SSF. If H,H_0 are two self-adjoint operators with a trace-class difference $H-H_0\in\mathfrak{S}_1(\mathcal{H})$, then the SSF $\xi(-;H,H_0)$ of this pair, introduced by [Lif] and [Kre] (see also [GM], [Yaf], [Sim1]), is a unique real-valued integrable function satisfying

$$\operatorname{Tr}(\phi(H)-\phi(H_0)) = \int_{\mathbb{R}}\phi'(\lambda)\xi(\lambda;H,H_0)d\lambda$$

for all compactly supported smooth functions ϕ on $\mathbb{R}$.

[1] Recall that given a normal operator N, the *discrete spectrum* $\sigma_{\mathrm{dis}}(N)$ is the set of all those eigenvalues of N which are isolated points of the spectrum $\sigma(N)$ and have finite multiplicities. The complement of the discrete spectrum in the spectrum is the *essential spectrum* $\sigma_{\mathrm{ess}}(N)$.

1.1.5. *Calculating unitary spectral flow via continuous enumeration.* Suppose for simplicity that $U(\cdot)$ is a loop in $\mathfrak{U}_p(\mathcal{H}, I)$ based at I. According to the naive definition (2), the spectral flow $\mathrm{sf}\,(-;U)$ in this case assumes some constant value $N \in \mathbb{Z}$ independent of the angle θ : the number N represents the net number of windings that the eigenvalues of $U(\cdot)$ make in the anti-clockwise direction. Perhaps, it should be possible to continuously enumerate the eigenvalues of $U(\cdot)$ as in the finite-dimensional setting. At this point, we recall the notion of extended enumeration due to Kato:

Definition 1.4. Given a normal operator N, a sequence $(\lambda_i)_{i\in\mathbb{N}}$ of complex numbers is called an *extended enumeration* of the discrete spectrum $\sigma_{\mathrm{dis}}(N)$, if $(\lambda_i)_{i\in\mathbb{N}}$ contains all eigenvalues of N in $\sigma_{\mathrm{dis}}(N)$ taking into account their multiplicities, and in addition, may contain some boundary points of the essential spectrum $\sigma_{\mathrm{ess}}(N)$ repeated arbitrarily often.

We propose the possibility of selecting a sequence $(\lambda_j(\cdot))_{j\in\mathbb{N}}$ of loops in $\mathbb{T}$ based at the boundary point 1 of the common essential spectrum, such that for each $t \in [0,1]$ the sequence $(\lambda_j(t))_{j\in\mathbb{N}}$ is an extended enumeration of $\sigma_{\mathrm{dis}}(U(t))$. It is necessary to consider extended enumerations by allowing λ_j's to take the boundary value 1. If such an enumeration is possible, an intuitive understanding of the number $\mathrm{sf}\,U := N$ would be the formal sum

$$\mathrm{sf}\,U = [\lambda_1]_{\pi_1} + [\lambda_2]_{\pi_1} + \dots, \tag{3}$$

where each $[\lambda_i]_{\pi_1}$ is the homotopy class in the fundamental group $\pi_1(\mathbb{T}, 1) \cong \mathbb{Z}$, representing the net number of windings that λ_i makes in the anti-clockwise direction.

1.2. **Infinite-dimensional continuous enumeration.** The infinite analogue of a finite unordered tuple is often called a *multiset*. Given a nonempty set X, a multiset in X is understood naively as a subset of X, whose elements can be repeated more than once. For instance, the multiset $\{x,x\}^*$ in X, where we are using * to distinguish multisets from ordinary subsets of X, is considered to be different from $\{x\}^*$ or $\{x,x,x\}^*$. Given any unitary operator $U \in \mathfrak{U}_p(\mathcal{H}, I)$, we may identify its spectrum $\sigma(U)$ with the following multiset in $\mathbb{T}$:

$$\sigma(U) = \sigma_{\mathrm{dis}}(U) \cup \{1\} \equiv \{z_1, z_2, z_3, \dots, 1, 1, 1, \dots\}^*, \tag{4}$$

where z_i's are the eigenvalues in $\sigma_{\mathrm{dis}}(U)$ taking multiplicities into account and 1's are repeated infinitely many times. The question which needs to be addressed next is the following: is there a natural topology in the set of multisets which makes the mapping $\mathfrak{U}_p(\mathcal{H}, I) \ni U \longmapsto \sigma(U)$ continuous? The answer is affirmative, and it is based upon the following estimates.

1.2.1. *The Hoffman-Wielandt inequality.* Hoffman-Wielandt proved the following well-known matrix inequality (see [Bha, Theorem VI.4.1] for details):

Theorem 1.5 (Hoffman-Wielandt)**.** *If N, N' are two $n \times n$ normal matrices, then we can enumerate the eigenvalues of N, N' as $(\lambda_1, \dots, \lambda_n)$, $(\lambda'_1, \dots, \lambda'_n)$ respectively, so that*

$$\left[\sum_{i=1}^{n} |\lambda_i - \lambda'_i|^2\right]^{\frac{1}{2}} \le \|N - N'\|_{\mathfrak{S}_2}.$$

We are interested in infinite-dimensional analogues of the Hoffman-Wielandt inequality: given a pair N, N' of normal operators with $N - N' \in \mathfrak{S}_\Phi$, can we choose a pair $(\lambda_i), (\lambda'_i)$ of extended enumerations of the discrete spectra of N, N' respectively, such that

$$\left[\sum_{i=1}^{\infty} |\lambda_i - \lambda'_i|^p\right]^{\frac{1}{p}} \le C\,\|N - N'\|_{\mathfrak{S}_p}, \tag{5}$$

where C is a positive constant which does not depend on N, N'? Kato ([Kat1, Theorem II]) proved (5) under the assumption that N, N' are self-adjoint operators and $C = 1$. Kato's result was extended to unitary N, N' with $C = \pi/2$ by Bhatia-Sinha ([BS]). Bhatia-Davis ([BD, Corollary 2.3]) proved (5) under the assumption that $N, N', N - N'$ are normal operators and $C = 1$.

1.2.2. *Summable multisets.* Formally, a *multiset* in $\mathbb{T}$ is a mapping $S\colon \mathbb{T} \to \{0, 1, 2, \ldots, \infty\}$, which assigns to each point $z \in \mathbb{T}$ a unique nonnegative integer or an infinity $S(z)$ which is understood as the *multiplicity* of the point z. A *countable multiset* in $(\mathbb{T}, 1)$ is a multiset S in $\mathbb{T}$ with the following properties:

1. The fixed point 1 is the only point having infinite multiplicity in S.
2. The *support* of S given by $\operatorname{supp} S := \{z \in \mathbb{T} \mid S(z) > 0\}$ is countable.

We shall make use of the trivial multiset $O_1 := \{1, 1, 1, \ldots\}^*$. A sequence $(z_i)_{i\in\mathbb{N}}$ in $\mathbb{T}$ is called an *enumeration* of a countable multiset S, if it contains each point of $\mathbb{T}$ according to its multiplicity in S. Evidently, S admits a representation $S = \{z_1, z_2, \ldots\}^*$. Given countable multisets $S = \{z_1, z_2, \ldots\}^*$ and $T = \{w_1, w_2, \ldots\}^*$ in $(\mathbb{T}, 1)$, we define their *p-distance* by

$$d_p(S, T) := \inf_{\pi} \left[\sum_{i=1}^{\infty} |z_i - w_{\pi_i}|^p \right]^{\frac{1}{p}}, \tag{6}$$

where the infimum is taken over all permutations π. A countable multiset S in $(\mathbb{T}, 1)$ is said to be *p-summable*, if $d_p(S, O_1) < \infty$. In this paper it is shown that the set of all p-summable multisets in $(\mathbb{T}, 1)$, denoted by $\mathcal{S}_p(\mathbb{T}, 1)$, forms a complete metric space with the metric d_p. In fact, we have chosen the metric d_p so that Bhatia-Sinha's result ([BS]) immediately implies:

1. The spectrum of each unitary operator $U \in \mathcal{U}_p(\mathcal{H}, I)$ can be viewed as a member of $\mathcal{S}_p(\mathbb{T}, 1)$ through (4). That is, $\sigma_{\mathrm{dis}}(U)$ can be shown to be p-summable.
2. The mapping $\mathcal{U}_p(\mathcal{H}, I) \ni U \longmapsto \sigma(U) \in \mathcal{S}_p(\mathbb{T}, 1)$ is continuous.

Indeed, we have

$$d_p(\sigma(U), \sigma(U')) \le \frac{\pi}{2} \|U - U'\|_{\mathfrak{S}_p} \qquad \forall U, U' \in \mathcal{U}_p(\mathcal{H}, I), \tag{7}$$

and setting $U' := I$ ensures the p-summability of each $\sigma(U)$ since $\sigma(I) = \{1, 1, 1, \ldots\}^*$.

1.2.3. *Continuous enumeration in the setting of unitary spectral flow.* In this paper, it is shown that any continuous path of the form $S : [0, 1] \to \mathcal{S}_p(\mathbb{T}, 1)$ admits a *continuous enumeration* $(\lambda_i(\cdot))_{i\in\mathbb{N}}$ in the sense that each λ_i is a continuous path in $\mathbb{T}$ with the property that for each $t \in [0, 1]$ the sequence $(\lambda_i(t))_{i\in\mathbb{N}}$ is an enumeration of the multiset $S(t)$. An immediate consequence of this result and (7) is the following unitary analogue of Kato's continuous enumeration:

Theorem 1.6. *Let $\mathcal{H}$ be a separable Hilbert space. If $U(\cdot)$ is a continuous path in $\mathcal{U}_p(\mathcal{H}, I)$, then there exists a sequence $(\lambda_j(\cdot))_{j\in\mathbb{N}}$ of continuous paths in $\mathbb{T}$, such that*

1. $\sigma(U(\cdot)) = \{\lambda_1(\cdot), \lambda_2(\cdot), \ldots\}^*$.
2. $(\lambda_j(\cdot))_{j\in\mathbb{N}}$ *is an extended enumeration of* $\sigma_{\mathrm{dis}}(U(\cdot))$ *pointwise.*

In fact, we obtain this result as a special case. More precisely, we generalise this setting by replacing the identity operator I by any fixed unitary operator U_0. Details are summarised below.

1.3. **Main results.**

1.3.1. *Generalisation to symmetric norms.* We have only considered the p-Schatten classes $\mathfrak{S}_p(\mathcal{H})$ so far, but they are only special types of the general Schatten-class $\mathfrak{S}_\Phi(\mathcal{H})$, where Φ is a so-called *symmetric norm* (see below for definition). In fact, the previously mentioned theorems by Bhatia-Shinha and Bhatia-Davis are concerned with symmetric norms:

Theorem 1.7 ([BS])**.** *Let $\mathcal{H}$ be a separable Hilbert space, and let Φ be a symmetric norm. For any pair U, U' of unitary operators on $\mathcal{H}$ with $U - U' \in \mathfrak{S}_\Phi(\mathcal{H})$, there exists a pair $(\lambda_i), (\lambda'_i)$ of extended enumerations of the discrete spectra of U, U' respectively, s.t.*

$$\Phi(|\lambda_1 - \lambda'_1|, |\lambda_2 - \lambda'_2|, \dots) \leq \frac{\pi}{2}\|U - U'\|_{\mathfrak{S}_\Phi}.$$

Theorem 1.8 ([BD, Corollary 2.3])**.** *Let $\mathcal{H}$ be a separable Hilbert space, and let Φ be a symmetric norm. For any pair N, N' of normal operators on $\mathcal{H}$ with $N - N'$ being normal Φ-Schatten class, there exists a pair $(\lambda_i), (\lambda'_i)$ of extended enumerations of the discrete spectra of U, U' respectively, s.t.*

$$\Phi(|\lambda_1 - \lambda'_1|, |\lambda_2 - \lambda'_2|, \dots) \leq \|N - N'\|_{\mathfrak{S}_\Phi}.$$

In this paper, we work with the general Schatten class $\mathfrak{S}_\Phi(\mathcal{H})$ for completeness.

1.3.2. *General multiset theory.* Sections §3-6 are devoted to general multiset theory about a metric space X and a fixed point $x_0 \in X$. Given a symmetric norm Φ, the definition of $\mathcal{S}_\Phi(X, x_0)$ requires the obvious modification (see §3.2 and §3.3 for details). As before, we make use of the multiset $O_{x_0} := \{x_0, x_0, x_0, \dots\}^*$. The following are our main results:

1. Theorem 3.7 asserts that $\mathcal{S}_\Phi(X, x_0)$ is metric space. In addition, it is shown in Theorem 3.20 that if X is complete and if Φ is a *regular symmetric norm* (see below for definition), then $\mathcal{S}_\Phi(X, x_0)$ is complete.
2. Theorem 5.1 asserts that any continuous path in $\mathcal{S}_\Phi(X, x_0)$ has a continuous enumeration.
3. In §6, under the assumption that Φ is a regular symmetric norm, and that X is a path-connected, locally simply connected metric space, we construct a group isomorphism

$$\Psi_\Phi : \pi_1(\mathcal{S}_\Phi(X, x_0), O_{x_0}) \simeq H_1(X), \tag{8}$$

where $\pi_1(\mathcal{S}_\Phi(X, x_0), O_{x_0})$ is the fundamental group and $H_1(X)$ is the first singular homology group. The formal sum (3) is used to define Ψ_Φ.

1.3.3. *Infinite-dimensional analogues of Kato's continuous enumeration.* One of our main results is Theorem 1.6 with the identity operator I replaced by a fixed operator U_0. To state this result, we consider the metric space $\mathcal{U}_\Phi(\mathcal{H}, U_0)$ whose definition is obvious (see §7.2 for details). Since the essential spectrum $K := \sigma_{\mathrm{ess}}(U_0)$ is no longer a point-set, we need to form the quotient space $\mathbb{T}/K = \{[z]_K\}_{z \in \mathbb{T}}$ via the equivalence relation which identifies points of K and leaves other points as they are. Let $\mathcal{K}$ denote the equivalence class represented by points of K. The topological space $X := \mathbb{T}/K$ is a metrizable space (see Theorem 7.1) with a fixed point $x_0 := \mathcal{K}$, and so we may consider

$$\mathcal{S}_\Phi(\mathbb{T}, K) := \mathcal{S}_\Phi(\mathbb{T}/K, \mathcal{K}).$$

As before, we can view the spectrum of each unitary operator $U \in \mathcal{U}_\Phi(\mathcal{H}, U_0)$ as a multiset in $\mathbb{T}/K$ through (39). With the notations introduced above in mind, Theorem 7.5 is our main theorem, which is an infinite-dimensional version of Kato's continuous enumeration. We also give an analogous result for self-adjoint perturbations (see Theorem 7.8).

1.3.4. *Unitary Spectral Flow.* In §8, we give an alternative approach to the unitary spectral flow. Note first that if we set $(X, x_0) := (\mathbb{T}, 1)$, then the isomorphism (8) is of the form

$$\Psi_\Phi : \pi_1(\mathcal{S}_\Phi(\mathbb{T}, 1), O_1) \simeq \mathbb{Z}.$$

If $U(\cdot)$ is a loop of unitary operators in $\mathcal{U}_\Phi(\mathbb{T}, 1)$ based at the identity operator I, then $\sigma(U(\cdot))$ is a loop in $\mathcal{S}_\Phi(\mathbb{T}, 1)$ based at O_1. We define the *spectral flow* of the path $U(\cdot)$ to be

$$\operatorname{sf} U := \Psi_\Phi([\sigma(U)]_{\pi_1}) \in \mathbb{Z},$$

where $[\,\cdot\,]_{\pi_1}$ denotes the homotopy class in $\pi_1(\mathcal{S}_\Phi(\mathbb{T}, 1), O_1)$. This definition is indeed consistent with the naive one (3), where the existence of each loop $\lambda_i(\cdot)$ in $(\mathbb{T}, 1)$ is asserted in Theorem 1.6.

2. Preliminaries

Here, we briefly recall standard facts about symmetric norms and Schatten class operators. Details can be found in [GK] and [Sim2].

2.1. **Symmetric norms.** Let c_0 be the set of all real-valued sequences converging to 0, and let c_{00} be the set of all real-valued sequences with a finite number of non-zero terms. Evidently, c_0 and c_{00} can be both viewed as vector spaces over $\mathbb{R}$.

Definition 2.1. A norm Φ on c_{00}, which assigns to each sequence $\xi = (\xi_i)_{i\in\mathbb{N}}$ in c_{00} a unique non-negative number $\Phi(\xi) = \Phi(\xi_1, \xi_2, \dots)$, is called a *symmetric norm*, if the following conditions are satisfied:

1. $\Phi(1, 0, 0, \dots) = 1$.
2. $\Phi(\xi_1, \xi_2, \dots) = \Phi(|\xi_{\pi_1}|, |\xi_{\pi_2}|, \dots)$ for any $\xi \in c_{00}$ and any permutation π.

Let Φ be a symmetric norm. A sequence $\xi \in c_0$ is said to be Φ-*summable*, if the limit

$$\Phi(\xi) := \lim_{n\to\infty} \Phi(\xi_1, \dots, \xi_n, 0, 0, \dots)$$

is finite. The vector space of all Φ-*summable* sequences, denoted by ℓ_Φ, is called the *natural domain* of the symmetric norm Φ. The pair (ℓ_Φ, Φ) turns out to be a Banach space (see [Sim2, Theorem 1.16 (d)]). The symmetric norm Φ is said to be *regular*, if

$$\lim_{n\to\infty} \Phi(\xi_{n+1}, \xi_{n+2}, \dots) = 0 \iff \lim_{n\to\infty} (\xi_1, \dots, \xi_n, 0, 0, \dots) = \xi \qquad \forall \xi \in \ell_\Phi.$$

Let ℓ_Φ^+ be the set of all those sequences in ℓ_Φ with non-negative terms.

Example 2.2. Given a fixed number $p \in [1, \infty]$, we define the regular symmetric norm Φ_p by

$$\Phi_p(\xi) = \begin{cases} \left(\sum_{i=1}^\infty |\xi_i|^p\right)^{1/p}, & \text{if } p < \infty, \\ \sup_{i\in\mathbb{N}} |\xi_i|, & \text{if } p = \infty, \end{cases} \tag{9}$$

where $\xi \in c_{00}$. The natural domain $\ell_p := \ell_{\Phi_p}$ is known as the set of *p-summable sequences* in $\mathbb{R}$. Evidently, $\ell_\infty = c_0$. See [GK, §III. 7] for more details.

Given a sequence $\xi = (\xi_i)_{i\in\mathbb{N}}$ of non-negative terms in c_0, we define the sequence $\xi^\downarrow = (\xi_i^\downarrow)_{i\in\mathbb{N}}$ to be the non-increasing rearrangement of $\xi_1, \xi_2, \dots$. That is, we define $\xi^\downarrow$ through

$$\xi_1^\downarrow = \max_{i\in\mathbb{N}} \xi_i, \quad \xi_1^\downarrow + \xi_2^\downarrow = \max_{i\neq j} (\xi_i + \xi_j), \quad \dots$$

The non-increasing rearrangement of a finite sequence of non-negative terms can be defined analogously.

2.2. **Schatten class operators.** Let Φ be a symmetric norm, and let $\mathcal{H}$ be a separable Hilbert space. The *singular numbers* of a compact operator A on $\mathcal{H}$, denoted by $s_1(A), s_2(A), \ldots$, are the eigenvalues of the positive operator $|A| := \sqrt{A^*A}$, that are repeated according to their multiplicities and arranged in the non-increasing order. The operator A is said to be Φ-*summable*, if $(s_i(A))_{i\in\mathbb{N}} \in \ell_\Phi$: that is,

$$\|A\|_{\mathfrak{S}_\Phi} := \lim_{n\to\infty} \Phi(s_1(A), \ldots, s_n(A), 0, 0, \ldots) < \infty. \tag{10}$$

The set $\mathfrak{S}_\Phi(\mathcal{H})$ of all Φ-summable operators, known as the Φ-*Schatten class*, forms a Banach space with the norm (10). Details can be found in [GK, §III.4]. The *p-Schatten class* is the Banach space $\mathfrak{S}_p(\mathcal{H}) := \mathfrak{S}_{\Phi_p}(\mathcal{H})$.

2.3. **Majorisation and inequalities.** Here, we introduce the notion of majorisation which allows us to develop useful inequalities involving symmetric norms. Let $\mathbb{R}^n_+$ be the set of all finite sequences of length n whose terms are non-negative real numbers. Given two finite sequences $\xi, \eta \in \mathbb{R}^n_+$, we say that ξ is *weakly majorized* by η, written $\xi \prec_w \eta$, if

$$\sum_{j=1}^{k} \xi^{\downarrow}_j \leq \sum_{j=1}^{k} \eta^{\downarrow}_j \quad \forall k = 1, \ldots, n,$$

A norm Φ on $\mathbb{R}^n$ is referred to as a *finite symmetric norm*, if the two conditions specified in Definition 2.1 are satisfied. It is a well-known fact (see [Bha, Example II.3.13]) that a finite symmetric norm Φ on $\mathbb{R}^n$ respects weak majorization. That is,

$$\xi \prec_w \eta \Rightarrow \Phi(\xi) \leq \Phi(\eta) \qquad \forall \xi, \eta \in \mathbb{R}^n_+.$$

We will make use of the following obvious lemma throughout this subsection:

Lemma 2.3. *If Φ is a symmetric norm, then the following is a finite symmetric norm:*

$$\mathbb{R}^n \in (\xi_1, \ldots, \xi_n) \longmapsto \Phi(\xi_1, \ldots, \xi_n, 0, 0, \ldots) \in \mathbb{R}$$

To begin we consider the following standard facts (see [GK, §III.3] for details), which will be freely used throughout the paper without any further comment:

Lemma 2.4. *Let Φ be a symmetric norm, and let $\xi, \eta \in \ell^+_\Phi$:*

1. *$\Phi(\xi_1, \xi_2, \ldots) = \Phi(\xi_{\pi_1}, \xi_{\pi_2}, \ldots)$ for any permutation π. In particular, $\Phi(\xi) = \Phi(\xi^{\downarrow})$.*
2. *If $\xi_i \leq \eta_i$ for each $i \in \mathbb{N}$, then $\Phi(\xi) \leq \Phi(\eta)$.*
3. *$\xi^{\downarrow}_1 \leq \Phi(\xi) \leq \sum_{i=1}^{\infty} \xi^{\downarrow}_i$.*

Note that the last assertion implies $\ell_1 \subseteq \ell_\Phi \subseteq \ell_\infty$.

Proof. For the first assertion, observe that for each $n \in \mathbb{N}$ there exists a large enough index N_n, s.t. $\xi_{\pi_1}, \ldots, \xi_{\pi_n}$ is among $\xi_1, \ldots, \xi_{N_n}$. Since a finite symmetric norm respects weak majorisation, we have $\Phi(\xi_{\pi_1}, \ldots, \xi_{\pi_n}, 0, 0, \ldots) \leq \Phi(\xi_1, \ldots, \xi_{N_n}, 0, 0, \ldots)$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ establishes $\Phi(\xi_\pi) \leq \Phi(\xi)$. A similar argument shows $\Phi(\xi) \leq \Phi(\xi_\pi)$, and the firs assertion follows. The second assertion follows from $(\xi_1, \ldots, \xi_n) \prec_w (\eta_1, \ldots, \eta_n)$ for all $n \in \mathbb{N}$. The last assertion follows from $\Phi(\xi) = \Phi(\xi^{\downarrow})$ and

$$(\xi^{\downarrow}_1, 0, \ldots, 0) \prec_w (\xi^{\downarrow}_1, \xi^{\downarrow}_2, \ldots, \xi^{\downarrow}_n) \prec_w \left(\sum_{i=1}^{n} \xi^{\downarrow}_i, 0, \ldots, 0\right) \qquad \forall n \in \mathbb{N}.$$

$\square$

We will conclude this section by obtaining an infinite analogue of the following inequality:

Lemma 2.5. *For any finite symmetric norm Φ on $\mathbb{R}^n$, we have*

$$\Phi(|\xi_1^\downarrow - \eta_1^\downarrow|, \ldots, |\xi_n^\downarrow - \eta_n^\downarrow|) \leq \Phi(|\xi_1 - \eta_1|, \ldots, |\xi_n - \eta_n|) \qquad \forall \xi, \eta \in \mathbb{R}_+^n. \tag{11}$$

Proof. The claim follows from the following non-trivial majorization:

$$(|\xi_1^\downarrow - \eta_1^\downarrow|, \ldots, |\xi_n^\downarrow - \eta_n^\downarrow|) \prec_w (|\xi_1 - \eta_1|, \ldots, |\xi_n - \eta_n|) \qquad \forall \xi, \eta \in \mathbb{R}_+^n.$$

See [MO, Theorem 6.A.2.a] for details. □

We believe that an infinite analogue of this inequality must be a standard result, but were unable to find an appropriate reference. Here, we will present our own proof for which we do not claim the originality. We prove the following lemma first.

Lemma 2.6. *Let Φ be a regular symmetric norm, and let $\xi \in \ell_\Phi^+$. If we let $\xi^{(n)} := (\xi_1, \ldots, \xi_n, 0, 0, \ldots)$ for each $n \in \mathbb{N}$, then $(\xi^{(n)})^\downarrow \to \xi^\downarrow$ as $n \to \infty$.*

Note that $(\xi^{(n)})^\downarrow \neq (\xi^\downarrow)^{(n)}$ in general (otherwise this claim would be trivial).

Proof. Here, we consider the non-trivial case where ξ is a sequence with infinitely many non-zero terms. For each $n \in \mathbb{N}$, we set $\xi_{(n)} := (\xi_{n+1}, \xi_{n+2}, \ldots)$. It follows from the regularity of Φ that for any $\epsilon > 0$ there exists an index n_0 s.t. $\Phi(\xi_{(n_0)}) < \epsilon/2$ and $\Phi[(\xi^\downarrow)_{(n_0)}] < \epsilon/2$. Furthermore, there exists an index $N > n_0$ s.t. for all $n > N$ we have $\Phi(\xi_{(n)}) < \xi_{n_0+1}^\downarrow$. It follows that for all $n > N$ the numbers $\xi_{n+1}, \xi_{n+2}, \ldots$ are all strictly less than $\xi_{n_0}^\downarrow$: that is, the first n_0 terms of $\xi^\downarrow, (\xi^{(n)})^\downarrow$ are identical. For all $n > N$ we have

$$\begin{aligned}
\Phi(\xi^\downarrow - (\xi^{(n)})^\downarrow) &= \Phi(0, \ldots, 0, \xi_{n_0+1}^\downarrow - (\xi^{(n)})_{n_0+1}^\downarrow, \ldots) \\
&\leq \Phi[(\xi^\downarrow)_{(n_0)}] + \Phi[((\xi^{(n)})^\downarrow)_{(n_0)}] \\
&< \frac{\epsilon}{2} + \Phi[((\xi^{(n)})^\downarrow)_{(n_0)}].
\end{aligned}$$

It remains to prove $\Phi[((\xi^{(n)})^\downarrow)_{(n_0)}] < \epsilon/2$ for all $n > N$. Let $n > N$ be fixed. Then there exists a permutation π of $\{1, \ldots, n\}$ s.t. $\xi_{\pi_1} \geq \ldots \geq \xi_{\pi_n}$. It is easy to observe that

$$((\xi^{(n)})^\downarrow)_{(n_0)} = (\xi_{\pi_{n_0+1}}, \ldots, \xi_{\pi_n}, 0, 0, \ldots).$$

Since $\xi_{\pi_{n_0+1}}, \ldots, \xi_{\pi_n}$ are the smallest $n - n_0$ terms of ξ^n, we have

$$\begin{aligned}
\Phi[((\xi^{(n)})^\downarrow)_{(n_0)}] &= \Phi(\xi_{\pi_{n_0+1}}, \ldots, \xi_{\pi_n}, 0, 0, \ldots) \\
&\leq \Phi(\xi_{n_0+1}, \ldots, \xi_n, 0, 0, \ldots) \\
&\leq \Phi(\xi_{n_0+1}, \ldots, \xi_n, \xi_{n+1}, \xi_{n+2}, \ldots) \\
&= \Phi(\xi_{(n_0)}) < \frac{\epsilon}{2}.
\end{aligned}$$

The proof is complete. □

We are now in a position to prove the following result:

Theorem 2.7. *Given a regular symmetric norm Φ, we have*

$$\Phi(|\xi_1^\downarrow - \eta_1^\downarrow|, |\xi_2^\downarrow - \eta_2^\downarrow|, \ldots) \leq \Phi(|\xi_1 - \eta_1|, |\xi_2 - \eta_2|, \ldots) \qquad \forall \xi, \eta \in \ell_\Phi^+. \tag{12}$$

That is, $\ell_\Phi^+ \ni \xi \longmapsto \xi^\downarrow \in \ell_\Phi$ is 1-Lipschitz continuous.

Proof. Let Φ be a regular symmetric norm. It follows from Inequality (11) that for any $\xi, \eta \in \ell_\Phi^+$

$$\Phi(|(\xi^{(n)})^{\downarrow} - (\eta^{(n)})^{\downarrow}|) \leq \Phi(|\xi^{(n)} - \eta^{(n)}|) \qquad \forall n \in \mathbb{N}.$$

By Lemma 2.6, taking the limit as $n \to \infty$ completes the proof. $\square$

3. Summable Multisets

3.1. **Countable multisets.** Let X be a nonempty set with a fixed point $x_0 \in X$. A *multiset* in X is understood naively as a subset of X, whose elements can be repeated more than once. For instance, the multiset $\{x, x\}^*$, where we use notation $\{\dots\}^*$ to distinguish it from ordinary subsets of X, is considered to be different from $\{x\}^*$. We shall make use of the following multiset throughout the paper:

$$O_{x_0} := \{x_0, x_0, x_0, \dots\}^*,$$

where x_0 is repeated infinitely many times. Formally, we define a multiset in X to be any mapping $S : X \to \{0, 1, 2, \dots, \infty\}$ assigning to each point $x \in X$ a unique non-negative integer of infinity, $S(x)$, which is understood as the multiplicity of x in S.

Definition 3.1. A *countable multiset in* (X, x_0) is a multiset S in X s.t.

1. The fixed point x_0 is the only point in S having the infinite multiplicity.
2. The *support* of S, defined by $\operatorname{supp} S := \{x \in X \mid S(x) > 0\}$, is a countable subset of X.

Evidently, O_{x_0} is a trivial example of a countable multiset in (X, x_0). Throughout this paper, we will only consider multisets of this kind, and freely make use of the following convention without any further comment. Given a finite or infinite sequence $(s_1, s_2, \dots)$ in X, we assume that the multiset $\{s_1, s_2, \dots\}^*$ contains the fixed point x_0 infinitely many times, so that it can always be viewed as a countable multiset in (X, x_0).

Example 3.2. With the above convention in mind, the correct interpretation of the multiset $S := \{x_1, x_1\}^*$, where $x_1 \neq x_0$, is the mapping $S : X \to \{0, 1, 2, \dots, \infty\}$ given by $S(x_1) = 2, S(x_0) = \infty$, and $S(x) = 0$ whenever $x \neq x_0$ and $x \neq x_1$.

Let us introduce the following terminology:

Definition 3.3. Let S be a countable multiset in (X, x_0):

1. A sequence $(s_i)_{i \in \mathbb{N}}$ is called an *enumeration* of S, if the representation $S = \{s_1, s_2, s_3, \dots\}^*$ holds. If the enumeration $(s_i)_{i \in \mathbb{N}}$ contains the fixed point x_0 infinitely many times, it is called a *proper enumeration* of S.
2. The *rank* of S, denoted by $\operatorname{rank} S$, is the sum of the multiplicities of all points in $\operatorname{supp} S$ except the fixed point x_0.

Remark 3.4. Let S be a countable multiset in (X, x_0). Any two proper enumerations of S are identical up to a permutation. Furthermore, given an enumeration $(s_i)_{i \in \mathbb{N}}$ of S, the sequence $(s_1, x_0, s_2, x_0, \dots)$ is a proper enumeration of S.

Given two countable multisets S, T in (X, x_0), we agree to write $T \leq S$ if $T(x) \leq S(x)$ for all $x \in X$. We define the *sum* $S + T$, and *difference* $S - T$ in case $T \leq S$, by

$$(S \pm T)(x) = \begin{cases} \infty, & \text{if } x = x_0, \\ S(x) \pm T(x), & \text{otherwise.} \end{cases}$$

Given a countable multiset S in (X, x_0) and an arbitrary subset U of X, we define their *intersection*, denoted by $S \cap U$, to be the multiset

$$(S \cap U)(x) := \begin{cases} \infty, & \text{if } x = x_0, \\ S(x), & \text{if } x \neq x_0 \text{ and } x \in U, \\ 0, & \text{if } x \neq x_0 \text{ and } x \notin U. \end{cases}$$

Note that the multiplicity of the fixed point x_0 in $S \cap U$ is always infinite, even if the fixed point x_0 does not belong to the set U. Thus, we can always view $S \cap U$ as a countable multiset in (X, x_0). We also define $S \setminus U := S \cap (X \setminus U)$.

3.2. **Summable multisets.**

Notation. We assume the following throughout the remaining parts of the current section:

1. Let Φ be a symmetric norm.
2. Let (X, d) be a metric space with a fixed point $x_0 \in X$.

Let S be a countable multiset in (X, x_0) with an enumeration $(s_i)_{i\in\mathbb{N}}$. The multiset S is said to be Φ-*summable*, if $(d(x_0, s_i))_{i\in\mathbb{N}} \in \ell_\Phi$. That is, $d(x_0, s_i) \to 0$ as $i \to \infty$ and

$$d_\Phi(O_{x_0}, S) = \Phi(d(x_0, s_1), d(x_0, s_2), \ldots) < \infty.$$

The set of all such multisets is denoted by $\mathcal{S}_\Phi(X, x_0)$. The Φ-*distance* between any two countable multisets $S, T \in \mathcal{S}_\Phi(X, x_0)$ is defined to be

$$d_\Phi(S, T) := \inf\ \Phi(d(s_1, t_1), d(s_2, t_2), \ldots), \tag{13}$$

where the infimum is taken over all pairs of enumerations (or equivalently over all pairs of proper enumerations[2]) $(s_i)_{i\in\mathbb{N}}, (t_i)_{i\in\mathbb{N}}$ of S, T respectively. Note that $(d(s_i, t_i))_{i\in\mathbb{N}} \in \ell_\Phi$ by triangle inequality.

Remark 3.5. Let S, T be multisets in $\mathcal{S}_\Phi(X, x_0)$ with enumerations $(s_i)_{i\in\mathbb{N}}, (t_i)_{i\in\mathbb{N}}$ respectively. Since the sequence $(d(s_i, t_i))_{i\in\mathbb{N}} \in \ell_\Phi$ is Φ-summable, we have

$$\sup_{i\in\mathbb{N}} d(s_i, t_i) \leq \Phi(d(s_1, t_1), d(s_2, t_2), \ldots) \leq \sum_{i=1}^{\infty} d(s_i, t_i). \tag{14}$$

Furthermore, if Φ is a regular symmetric norm, then we have

$$\lim_{i\to\infty} \Phi(d(x_0, s_{i+1}), d(x_0, s_{i+2}), \ldots) = 0. \tag{15}$$

We will prove that $\mathcal{S}_\Phi(X, x_0)$ forms a metric space with (13) using the following lemma:

Lemma 3.6. *If $S \in \mathcal{S}_\Phi(X, x_0)$, then the following assertions hold true:*

1. $\operatorname{supp} S$ *can have one and only one accumulation point x_0.*
2. $\operatorname{supp} S$ *is a compact subset of X.*

Proof. For the first part, assume that $\operatorname{supp} S$ is infinite. If $\operatorname{supp} S$ had an accumulation point other than x_0, then the sequence $(d(x_0, s_i))_{i\in\mathbb{N}}$ could converge to zero. This is a contradiction. The second part is now an immediate consequence. □

Theorem 3.7. $\mathcal{S}_\Phi(X, x_0)$ *forms a metric space with the distance function (13).*

[2] This immediately follows from the second part of Remark 3.4.

Proof. The symmetry $d_\Phi(S,T) = d_\Phi(T,S)$ is obvious. For the non-degeneracy, let $d_\Phi(S,T) = 0$. We assume $S \neq T$ and derive a contradiction. Without loss of generality, we may assume that there exists a point $x' \neq x_0$, s.t. $S(x') < T(x')$. Since x' cannot be an accumulation point of $\operatorname{supp} S$ by Lemma 3.6, we can choose a small enough open ϵ-ball $B_\epsilon(x')$ around x', whose intersection with $\operatorname{supp} S$ is either the empty set $\emptyset$ or the singleton $\{x'\}$. In either case, this leads to a contradiction $d_\Phi(S,T) \geq \epsilon > 0$ by (14).

To prove the triangle inequality $d_\Phi(S,T) \leq d_\Phi(S,U) + d_\Phi(U,T)$, we let $(s_i), (t_i)$ be proper enumerations of S, T respectively, and let $(u_i)_{i\in\mathbb{N}}, (u'_i)_{i\in\mathbb{N}}$ be two proper enumerations of U. Then there exists a permutation π satisfying $u'_{\pi_i} = u_i$ for each $i \in \mathbb{N}$. Now,

$$\begin{aligned}
\Phi[(d(s_i,u_i))_{i\in\mathbb{N}}] + \Phi[(d(u'_i,t_i))_{i\in\mathbb{N}}] &= \Phi[(d(s_i,u_i))_{i\in\mathbb{N}}] + \Phi[(d(u_i,t_{\pi_i}))_{i\in\mathbb{N}}] \\
&\geq \Phi(d(s_1,u_1) + d(u_1,t_{\pi_1}), d(s_2,u_2) + d(u_2,t_{\pi_2}), \ldots) \\
&\geq \Phi(d(s_1,t_{\pi_1}), d(s_2,t_{\pi_2}), \ldots) \\
&\geq d_\Phi(S,T),
\end{aligned}$$

where the the second inequality follows from the triangle inequality w.r.t. d. Since all the proper enumerations $(s_i)_{i\in\mathbb{N}}, (t_i)_{i\in\mathbb{N}}, (u_i)_{i\in\mathbb{N}}, (u'_i)_{i\in\mathbb{N}}$ were chosen arbitrarily, taking the infimum over these sequences establishes the triangle inequality. In particular, selecting $U := O_{x_0}$ ensures $d_p(S,T) < \infty$ for all $S, T \in \mathcal{S}_\Phi(X,x_0)$. The proof is now complete. $\square$

Example 3.8. Let Φ_p be the symmetric norm in Example 2.2. In this case, we use the short hand $(\mathcal{S}_p(X,x_0), d_p) := (\mathcal{S}_{\Phi_p}(X,x_0), d_{\Phi_p})$. The metric d_p is then given by

$$d_p(S,T) = \inf \begin{cases} \left(\sum_{i=1}^\infty d(s_i,t_i)^p\right)^{1/p}, & \text{if } p < \infty, \\ \sup_{i\in\mathbb{N}} d(s_i,t_i), & \text{if } p = \infty, \end{cases}$$

where the infimum is taken over all pairs of enumerations $(s_j), (t_j)$ of S, T respectively. It follows from (14) that $\mathcal{S}_1(X,x_0) \subseteq \mathcal{S}_\Phi(X,x_0) \subseteq \mathcal{S}_\infty(X,x_0)$ for any symmetric norm Φ.

Lemma 3.9. *If Φ is a regular symmetric norm, then the mapping*

$$\mathcal{S}_\Phi(X,x_0) \ni S = \{s_1, s_2, \ldots\}^* \longmapsto (d^\downarrow(x_0,s_i))_{i\in\mathbb{N}} \in l_\Phi,$$

where $(d^\downarrow(x_0,s_i))_{i\in\mathbb{N}}$ is the non-increasing rearrangement of the sequence $(d(x_0,s_i))_{i\in\mathbb{N}}$, is a 1-Lipschitz continuous mapping.

Proof. Let $S, T \in \mathcal{S}_\Phi(X,x_0)$, and let $(s_i), (t_i)$ be arbitrary enumerations of S, T respectively. For notational simplicity, we let $\xi := (d(x_0,s_i))_{i\in\mathbb{N}}$ and $\eta := (d(x_0,t_i))_{i\in\mathbb{N}}$. It follows that

$$\begin{aligned}
\Phi(d(s_1,t_1), d(s_2,t_2), \ldots) &\geq \Phi(|d(x_0,s_1) - d(x_0,t_1)|, |d(x_0,s_2) - d(x_0,t_2)|, \ldots) \\
&= \Phi(|\xi_1 - \eta_1|, |\xi_2 - \eta_2|, \ldots) \\
&\geq \Phi(|\xi_1^\downarrow - \eta_1^\downarrow|, |\xi_2^\downarrow - \eta_2^\downarrow|, \ldots),
\end{aligned}$$

where the last inequality follows from (12). Taking the infimum over $(s_i)_{i\in\mathbb{N}}, (t_i)_{i\in\mathbb{N}}$ establishes the 1-Lipschitz estimate $\Phi(\xi^\downarrow - \eta^\downarrow) \leq d_\Phi(S,T)$. The proof is complete. $\square$

3.3. **Some estimates.**

3.3.1. *Estimates involving sum.*

Lemma 3.10. *$d_\Phi(S + S', T + T') \leq d_\Phi(S,T) + d_\Phi(S',T')$ for all $S, S', T, T' \in \mathcal{S}_\Phi(X,x_0)$.*

Proof. Let $(s_i)_{i\in\mathbb{N}}, (s'_i)_{i\in\mathbb{N}}, (t_i)_{i\in\mathbb{N}}, (t'_i)_{i\in\mathbb{N}}$ be enumerations of S, S', T, T' respectively. Since

$$(s_1, s'_1, s_2, s'_2, s_3, s'_3, \ldots) \text{ and } (t_1, t'_1, t_2, t'_2, t_3, t'_3, \ldots)$$

are two enumerations of $S+S', T+T'$ respectively, we have

$$\begin{aligned}\Phi[(d(s_i,t_i))_{i\in\mathbb{N}}] + \Phi[(d(s'_i,t'_i))_{i\in\mathbb{N}}] &= \Phi(d(s_1,t_1),0,d(s_2,t_2),0,\dots) + \Phi(0,d(s'_1,t'_1),0,d(s'_2,t'_2),\dots)\\ &\geq \Phi(d(s_1,t_1),d(s'_1,t'_1),d(s_2,t_2),d(s'_2,t'_2),\dots)\\ &\geq d_\Phi(S+S',T+T').\end{aligned}$$

Since the enumerations $(s_i)_{i\in\mathbb{N}}, (s'_i)_{i\in\mathbb{N}}, (t_i)_{i\in\mathbb{N}}, (t'_i)_{i\in\mathbb{N}}$ were chosen arbitrarily, taking infimum over these enumerations establishes the claim. $\square$

3.3.2. *Estimates involving difference.* In general, an estimate analogous to Lemma 3.10,

$$d_\Phi(S-S',T-T') \leq d_\Phi(S,T) + d_\Phi(S',T'), \tag{16}$$

where $S,S',T,T' \in \mathcal{S}_\Phi(X,x_0)$ with $S' \subseteq S$ and $T' \subseteq T$, fails to hold as below:

Example 3.11. Let N be a natural number > 1. Here, we consider the space $\mathcal{S}_2(\mathbb{R}_+,0)$, where $\mathbb{R}_+$ is equipped with the standard metric $\rho(x,y) := |x-y|$. We define multisets S,S',T,T' through

$$S = S' = T = \left\{\tfrac{1}{N},\tfrac{2}{N},\dots,1\right\}^* \text{ and } T' = T - \{1\}^* = \left\{\tfrac{1}{N},\tfrac{2}{N},\dots,\tfrac{N-1}{N}\right\}^*.$$

Then $S-S' = O_0$ and $T-T' = \{1\}^*$, and so $\rho_2(S-S',T-T') = \rho_2(O_0,\{1\}^*) = 1$. On the other hand, since $\left(\frac{1}{N},\frac{2}{N},\dots,1,0,0,0,\dots\right), \left(0,\frac{1}{N},\frac{2}{N},\dots,\frac{N-1}{N},0,0,0,\dots\right)$ are enumerations of S',T' respectively,

$$\begin{aligned}\rho_2(S,T)+\rho_2(S',T') &\leq 0 + \left(\left|\frac{1}{N}-0\right|^2 + \left|\frac{2}{N}-\frac{1}{N}\right|^2 + \dots + \left|1-\frac{N-1}{N}\right|^2\right)^{\frac{1}{2}}\\ &\leq \left(\frac{1}{N^2}+\dots+\frac{1}{N^2}\right)^{\frac{1}{2}}\\ &\leq \frac{1}{\sqrt{N}} < 1 \leq \rho_2(S-S',T-T').\end{aligned}$$

That is, Inequality (16) fails to hold in general.

Nevertheless, the following weaker version turns out to be sufficient:

Lemma 3.12. *Let $S,S',T,T' \in \mathcal{S}_\Phi(X,x_0)$ be multisets satisfying $S' \subseteq S$ and $T' \subseteq T$. If S',T' are finite-rank multisets and if $n := \operatorname{rank} S' + \operatorname{rank} T'$, then*

$$d_\Phi(S-S',T-T') \leq 3n\left(d_\Phi(S,T)+d_\Phi(S',T')\right).$$

This result will be proved with the aid of the following lemma:

Lemma 3.13. *If $S,T,U \in \mathcal{S}_\Phi(X,x_0)$ and if $n := \operatorname{rank} U < \infty$, then*

$$d_\Phi(S,T) \leq 3n \cdot d_\Phi(S+U,T+U).$$

Proof. (A) Let us first prove the claim for $U = \{u\}^*$. Let $(s'_i)_{i\in\mathbb{N}}, (t'_i)_{i\in\mathbb{N}}$ be enumerations of $S+U, T+U$ respectively, s.t. $s'_{i_0} = u$ and $t'_{j_0} = u$ for some $i_0, j_0 \in \mathbb{N}$. If $i_0 \neq j_0$, we can then simultaneously renumber $(s'_i)_{i\in\mathbb{N}}, (t'_i)_{i\in\mathbb{N}}$, so that

$$(s'_i)_{i\in\mathbb{N}} = (s_1,u,s_2,s_3,\dots) \text{ and } (t'_i)_{i\in\mathbb{N}} = (u,t_1,t_2,t_3,\dots)$$

for some enumerations $(s_i)_{i\in\mathbb{N}}, (t_i)_{i\in\mathbb{N}}$ of S, T respectively. It follows that

$$\begin{aligned} d_\Phi(S,T) &\leq \Phi(d(s_1,t_1), d(s_2,t_2), \ldots) \\ &\leq \Phi(d(s_1,t_1),0,0,\ldots) + \Phi(0, d(s_2,t_2), d(s_3,t_3)\ldots) \\ &\leq d(s_1,u) + d(u,t_1) + \Phi(0, d(s_2,t_2), d(s_3,t_3)\ldots) \\ &= \Phi(d(s_1,u),0,0,\ldots) + \Phi(0,d(u,t_1),0,\ldots) + \Phi(0, d(s_2,t_2), d(s_3,t_3)\ldots) \\ &\leq 3\Phi(d(s_1,u), d(u,t_1), d(s_2,t_2),\ldots) \\ &= 3\Phi(d(s_1',t_1'), d(s_2',t_2'),\ldots). \end{aligned}$$

Note that the same estimate $d_\Phi(S,T) \leq 3\Phi[(d(s_i',t_i'))_{i\in\mathbb{N}}]$ also holds trivially in the case $i_0 = j_0$, and so taking the infimum over $(s_i')_{i\in\mathbb{N}}, (t_i')_{i\in\mathbb{N}}$ establishes the claim for $U = \{u\}^*$.

(B) For the general case, suppose $U = \{u_1,\ldots,u_n\}^*$. It follows from (A) that

$$3n \cdot d_\Phi(S+U, T+U) \geq 3(n-1)\cdot d_\Phi(S + \{u_1,\ldots,u_{n-1}\}^*, T + \{u_1,\ldots,u_{n-1}\}^*).$$

Continuing this way establishes the claim. □

Proof of Lemma 3.12. The multiset $S' + T'$ has finite rank n. By Lemma 3.13 we have

$$\begin{aligned} d_\Phi(S-S', T-T') &\leq 3n\cdot d_\Phi(S - S' + (S'+T'), T - T' + (S'+T')) \\ &\leq 3n\cdot d_\Phi(S+T', T+S') \\ &\leq 3n\left(d_\Phi(S,T) + d_\Phi(S',T')\right), \end{aligned}$$

where the last inequality follows from Lemma 3.10. □

3.3.3. *Estimates involving finite-rank multisets.* We shall make use of the following estimates:

Lemma 3.14. *Given $s_0, s_1, \ldots, s_n \in X$ and $t_1,\ldots,t_n \in X$, we have*

$$d_\Phi(\{s_1,\ldots,s_n\}^*, \{t_1,\ldots,t_n\}^*) \leq \sum_{i=1}^n d(s_i,t_i), \tag{17}$$

$$\sup_{1\leq i\leq n} d(s_0,s_i) \leq 2\, d_\Phi(\{\underbrace{s_0,\ldots,s_0}_{n\ times}\}^*, \{s_1,\ldots,s_n\}^*). \tag{18}$$

Proof. Inequality (17) immediately follows from Lemma 3.10: indeed,

$$d_\Phi(\{s_1,\ldots,s_n\}^*, \{t_1,\ldots,t_n\}^*) \leq d_\Phi(\{s_1\}^*,\{t_1\}^*) + \ldots + d_\Phi(\{s_n\}^*,\{t_n\}^*) \leq \sum_{i=1}^n d(s_i,t_i).$$

For (18), it follows from the triangle inequality w.r.t. d that

$$d(s_0,s_i) \leq d_\Phi(\{s_0,\ldots,s_0\}^*, \{s_1,\ldots,s_n\}^*) \qquad \forall i = 1,\ldots,n.$$

□

3.3.4. *Estimates involving intersection.* Given $S, T \in \mathcal{S}_\Phi(X, x_0)$ and a subset U of X, the following inequality does not hold in general:

$$d_\Phi(S\cap U, T\cap U) \leq d_\Phi(S,T). \tag{19}$$

Here, we establish a criterion under which estimate (19) holds true.

1. Given a subset U of X, we set $\mathcal{S}^U_\Phi(X,x_0) := \{S \in \mathcal{S}_\Phi(X,x_0) \mid \operatorname{supp} S \subseteq U\}$.

2. A finite tuple $(U_0, \ldots, U_n)$ of non-empty subsets of X is called *positively separated*, if

$$\operatorname{dist}(U_i, U_j) := \inf_{(u_i,u_j)\in U_i\times U_j} d(u_i, u_j) > 0 \quad \forall i \neq j.$$

By convention, whenever we speak of a positively separated tuple $(U_0, \ldots, U_n)$, we will always assume that the fixed point x_0 belongs to the first component U_0.

Lemma 3.15. *Let $(U_0, \ldots, U_k)$ be a positive-separated tuple of subsets of X. Let $\delta := \min_{i\neq j} \operatorname{dist}(U_i, U_j)$, and let $U := \bigcup_{i=0}^k U_i$. If $S, T \in \mathcal{S}_\Phi^U(X, x_0)$ satisfy $d_\Phi(S, T) < \delta$, then:*

1. $d_\Phi(S \cap U_i, T \cap U_i) \leq d_\Phi(S, T)$ *for all $i = 0, \ldots, k$.*
2. $\operatorname{rank}(S \cap U_i) = \operatorname{rank}(T \cap U_i)$ *for all $i = 1, \ldots, k$.*

Proof. Suppose that $S, T \in \mathcal{S}_\Phi^U(X, x_0)$ satisfy $d_\Phi(S, T) < \delta$, and that $(s_i)_{i\in\mathbb{N}}, (t_i)_{i\in\mathbb{N}}$ are arbitrary enumerations of S, T respectively satisfying $\Phi[(d(s_i, t_i))_{i\in\mathbb{N}}] < \delta$. That is,

$$\sup_{i\in\mathbb{N}} d(s_i, t_i) < \delta. \tag{20}$$

It follows that each neighborhood U_j has the property that $s_i \in U_j \iff t_i \in U_j$ for all $i \in \mathbb{N}$. The second assertion follows. The first assertion follows by taking the infimum over $(s_i)_{i\in\mathbb{N}}, (t_i)_{i\in\mathbb{N}}$. □

The following result is an immediate corollary:

Corollary 3.16. *If $(U_0, \ldots, U_k)$ is a positively separated tuple of subsets of X with $U := \bigcup_{i=0}^k U_i$, then each mapping*

$$\mathcal{S}_\Phi^U(X, x_0) \ni S \longmapsto S \cap U_i \in \mathcal{S}_\Phi(X, x_0), \quad i = 0, \ldots, k,$$

is continuous. Furthermore, the following function is locally constant:

$$\mathcal{S}_\Phi^U(X, x_0) \ni S \longmapsto (\operatorname{rank}(S \cap U_1), \ldots, \operatorname{rank}(S \cap U_n)) \in \mathbb{Z}^n$$

Lemma 3.17. *If U is an open subset of X, then $\mathcal{S}_\Phi^U(X, x_0)$ is an open subset of $\mathcal{S}_\Phi(X, x_0)$.*

This lemma will be used frequently with Corollary 3.16 under the assumption that $(U_0, \ldots, U_n)$ is a positively-separated tuple of *open subsets* of X, and that $U := U_0 \cup \ldots \cup U_n$.

Proof. Given $S \in \mathcal{S}_\Phi^U(X, x_0)$, we set $\delta := \operatorname{dist}(\operatorname{supp} S, X \setminus U) > 0$. Let $T \in \mathcal{S}_\Phi(X, x_0)$ be a multiset satisfying $d_\Phi(S, T) < \delta$. Then then there exist enumerations $(s_i), (t_i)$ of S, T respectively, s.t. (20) holds. It follows that $t_i \in U$ for all $i \in \mathbb{N}$, and so $T \in \mathcal{S}_\Phi^U(X, x_0)$. That is, the open δ-neighborhood of S is included in $\mathcal{S}_\Phi^U(X, x_0)$. Thus, $\mathcal{S}_\Phi^U(X, x_0)$ is open. □

3.4. **Canonical Lipschitz mappings.** Let (Y, ρ) be a metric space with a fixed point $y_0 \in Y$, and let $f : X \to Y$ be an L-Lipshiz continuous mapping s.t. $f(x_0) = y_0$. It is easy to see that f naturally induces an L-Lipschitz mapping

$$\mathcal{S}_\Phi(X, x_0) \ni \{s_1, s_2, \ldots\}^* \longmapsto \{f(s_1), f(s_2), \ldots\}^* \in \mathcal{S}_\Phi(Y, y_0). \tag{21}$$

3.5. **Separability.** The aim of the current subsection is to prove the following result:

Theorem 3.18. *If Φ is a regular symmetric norm and if X is a separable metric space, then $\mathcal{S}_\Phi(X, x_0)$ is a separable metric space.*

This is an immediate consequence of the following lemma:

Lemma 3.19. *If Φ is a regular symmetric norm, then the set of all finite-rank multisets in (X, x_0) is a dense subset of $\mathcal{S}_\Phi(X, x_0)$.*

Proof. If $S = \{s_1, s_2, \ldots\}^*$ belongs to $\mathcal{S}_\Phi(X, x_0)$, then Inequality (15) implies

$$\lim_{i\to\infty} d_\Phi(S, \{s_1, \ldots, s_i\}^*) \le \lim_{i\to\infty} \Phi(0, \ldots, 0, d(x_0, s_{i+1}), d(x_0, s_{i+2}), \ldots) = 0.$$

Since each $\{s_1, \ldots, s_i\}^*$ is a finite-rank multiset, the claim follows. □

Proof of Theorem 3.18. Let $\mathcal{S}_0(X, x_0)$ be the set of all finite-rank multisets in $\mathcal{S}_\Phi(X, x_0)$. By Lemma 3.19, it suffices to construct a countable dense subset of $\mathcal{S}_0(X, x_0)$. Let A be a countable dense subset of X. Without loss of generality, we may assume that $x_0 \in A$. Let

$$\mathcal{S}_0^A(X, x_0) := \{S \in \mathcal{S}_0(X, x_0) \mid \operatorname{supp} S \subseteq A\},$$

which is a countable set[3]. We show that $\mathcal{S}_0^A(X, x_0)$ is a dense dense subset of $\mathcal{S}_0(X, x_0)$. Let $S = \{s_1, \ldots, s_n\}^*$ be in $\mathcal{S}_0(X, x_0)$. Since A is a dense subset of X, there exist n sequences $(s_i^{(1)})_{i\in\mathbb{N}}, \ldots, (s_i^{(n)})_{i\in\mathbb{N}}$ in A converging to $s_1, \ldots, s_n$ respectively. It follows from (17) that $\{s_i^{(1)}, \ldots, s_i^{(n)}\}^* \to S$ as $i \to \infty$. The claim follows. □

3.6. **Completeness.** The aim of the current subsection is to prove the following result:

Theorem 3.20. *If (X, d) is a complete metric space and if Φ is a regular symmetric norm, then $\mathcal{S}_\Phi(X, x_0)$ is a complete metric space.*

Throughout the current subsection, we will assume that (X, d) is a complete metric space and that Φ is a regular symmetric norm. We will first prove the following special case:

Lemma 3.21. *$\mathcal{S}_\Phi(\mathbb{R}_+, 0)$ is a complete metric space.*

Proof. Let ρ be the standard metric on $\mathbb{R}_+$, and let $(S_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{S}_\Phi(\mathbb{R}_+, 0)$. Each S_n has an enumeration $\xi^n := (s_i^{(n)})_{i\in\mathbb{N}}$, s.t. $s_1^{(n)} \ge s_2^{(n)} \ge \ldots$. It follows from Lemma 3.9 that $(\xi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in ℓ_Φ. Since ℓ_Φ is a Banch space, $(\xi^n)_{n\in\mathbb{N}}$ converges to a limit $\xi^0 := (\xi_1^0, \xi_2^0, \ldots)$. Now,

$$\begin{aligned}
\rho_\Phi(S_n, \{\xi_1^0, \xi_2^0, \ldots\}) &= \rho_\Phi(\{s_1^{(n)}, s_1^{(n)}, \ldots\}^*, \{\xi_1^0, \xi_2^0, \ldots\}^*) \\
&\le \Phi(|s_1^{(n)} - \xi_1^0|, |s_2^{(n)} - \xi_2^0|, \ldots) \\
&= \Phi(\xi^n - \xi^0) \to 0.
\end{aligned}$$

That is, the Cauchy sequence $(S_n)_{n\in\mathbb{N}}$ has a limit $S_0 := \{\xi_1^0, \xi_2^0, \ldots\}^*$. Note that the Φ-summability of the multiset S_0 is equivalent to $\xi^0 \in \ell_\Phi$. □

[3] To see why this is true, we can take the following approach. Let $\mathcal{A}$ be the set of all finite subsets of A, which is clearly a countable set. We can then write

$$\mathcal{S}_0^A(X, x_0) = \bigcup_{A' \in \mathcal{A}} \{S \in \mathcal{S}_0(X, x_0) \mid \operatorname{supp} S = A'\},$$

where each set $\{S \in \mathcal{S}_0(X, x_0) \mid \operatorname{supp} S = A'\}$ is countable. It follows that $\mathcal{S}_0^A(X, x_0)$ is countable.

An immediate consequence of this lemma is as follows. Throughout this subsection, we let $(Y, y_0) := (\mathbb{R}_+, 0)$ and $\rho(x, y) := |x - y|$, and define $f : X \to \mathbb{R}_+$ by $f(x) := d(x_0, x)$. Since f is a 1-Lipschitz continuous mapping, it induces the 1-Lipschitz continuous mapping (21). We shall also make use of the shorthand

$$f(\{s_1, s_2, \ldots\}) := \{f(s_1), f(s_2), \ldots\}^* \qquad \forall \{s_1, s_2, \ldots\} \in \mathcal{S}_\Phi(X, x_0).$$

Evidently, if $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}_\Phi(X, x_0)$, then $(f(S_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}_\Phi(\mathbb{R}_+, 0)$. Since $\mathcal{S}_\Phi(\mathbb{R}_+, 0)$ is complete, the sequence $(f(S_n))_{n\in\mathbb{N}}$ has a limit in $\mathcal{S}_\Phi(\mathbb{R}_+, 0)$. With this fact in mind, we will prove Theorem 3.20 with the aid of the following two lemmas:

Lemma 3.22. *If $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}_\Phi(X, x_0)$ with the property that each term of it has a constant finite rank k, then it converges.*

Proof. Before we proceed to the induction on k, let us first observe that if the union $A := \bigcup_{n\in\mathbb{N}} \operatorname{supp} S_n$ is a finite subset of X, then the Cauchy sequence $(S_n)_{n\in\mathbb{N}}$ will eventually be constant, and so the claim follows. Suppose that A is an infinite subset of X. For the base step $k = 1$, there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of points in X, s.t. $S_n = \{s_n\}^*$ for all $n \in \mathbb{N}$. It follows from (18) that $(s_n)_{n\in\mathbb{N}}$ is Cauchy sequence in X, and so $(s_n)_{n\in\mathbb{N}}$ converges to some point $s_0 \in X$. It follows from Inequality (17) that $(S_n)_{n\in\mathbb{N}}$ converges to $S_0 := \{s_0\}^*$. This completes the base step.

For the induction step, we will assume that the claim has been proved for k replaced by any smaller number. Since $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, the infinite set A is totally bounded[4]. It follows that the closure $\overline{A}$ is a compact subset of X, and so A contains a limit point $s_0 \in X$. For each $n \in \mathbb{N}$, we choose a point $s_n \neq x_0$ in $\operatorname{supp} S_n$ that is closest to s_0. If $(s_n)_{n\in\mathbb{N}}$ does not converge to s_0, there exists a $\delta > 0$ such that $d(s_n, s_0) \geq 2\delta$ for infinitely many n's. At the same time, since s_0 is a limit point of A, the open ball $B_\delta(s_0)$ contains infinitely many points of A. This contradicts the fact that $(S_n)_{n\in\mathbb{N}}$ is Cauchy, and so $s_n \to s_0$ as $n \to \infty$. Now, $(S_n - \{s_n\}^*)_{n\in\mathbb{N}}$ is a Cauchy sequence of the constant finite rank $k - 1$ by Lemma 3.12, and so the claim follows by induction. □

Lemma 3.23. *Let (S_n) be a Cauchy sequence in $\mathcal{S}_\Phi(X, x_0)$ s.t. $R := \lim_{n\to\infty} f(S_n)$, and let $(I_0, \ldots, I_k)$ be a positively-separated tuple of open subsets of $\mathbb{R}_+$ s.t.*

$$\operatorname{supp} R \subseteq I_0 \cup \ldots \cup I_k.$$

Let $U_0, \ldots, U_k$ be the inverse images of $I_0, \ldots, I_k$ under f. Then $k + 1$ sequences $(S_n^{(0)})_{n\in\mathbb{N}}, \ldots, (S_n^{(k)})_{n\in\mathbb{N}}$ given by $S_n^{(i)} := S_n \cap U_i$ are all Cauchy sequences with the property that there exists an index N satisfying the following properties:

1. *$S_n = S_n^{(0)} + \ldots + S_n^{(k)}$ for all $n \geq N$.*
2. *$d_\Phi(S_m^{(i)}, S_n^{(i)}) \leq d_\Phi(S_m, S_n)$ for each $i = 0, \ldots, k$ and for each $m, n \geq N$,*
3. *$\operatorname{rank} S_m^{(i)} = \operatorname{rank} S_n^{(i)}$ for each $i = 1, \ldots, n$ and for each $m, n \geq N$.*

[4] To see why this is true, we let $\epsilon > 0$ be arbitrary and assume $S_n = \{s_1^{(n)}, \ldots, s_k^{(n)}\}^*$ for all $n \in \mathbb{N}$. Since $(S_n)_{n\in\mathbb{N}}$ is Cauchy, there exists an index N, s.t. for all $n \geq N$ we have $d_\Phi(S_n, S_N) < \epsilon$. That is,

$$\bigcup_{n\geq N} \operatorname{supp} S_n \subseteq B_\epsilon(x_0) \cup B_\epsilon(s_1^{(N)}) \cup \ldots \cup B_\epsilon(s_k^{(N)}).$$

Since $\bigcup_{n<N} \operatorname{supp} S_n$ is a finite subset of X, A is totally bounded.

Proof. Let $I := I_0 \cup \ldots \cup I_k$ and $U := U_0 \cup \ldots \cup U_k$. Let us first observe that $(U_0, \ldots, U_k)$ is positively-separated tuple of open subsets of X. Indeed, if $i \neq j$, then

$$\operatorname{dist}(U_i, U_j) = \inf_{(u_i,u_j)\in U_i\times U_j} d(u_i, u_j) \geq \inf_{(u_i,u_j)\in U_i\times U_j} |f(u_i) - f(u_j)| \geq \operatorname{dist}(I_i, I_j) > 0,$$

Note that $x_0 \in U_0$. Since $f(S_n) \to R$ as $n \to \infty$, there exists an index N s.t. for all $n \geq N$ we have $f(S_n) \in \mathcal{S}^I_\Phi(\mathbb{R}_+, 0)$ by Corollary 3.16 and Lemma 3.17. It follows that $S_n \in \mathcal{S}^U_\Phi(X, x_0)$ for all $n \geq N$. Since $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, the claim follows from Lemma 3.15. $\square$

Proof of Theorem 3.20. Let $(S_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{S}_\Phi(X, x_0)$, and let $R := \lim_{n\to\infty} f(S_n)$. Suppose that $\operatorname{supp} R = \{r_1, r_2, \ldots, 0\}$, where $r_1 > r_2 > \ldots > 0$, and that each r_i has the multiplicity m_i in R. Let $\{I_i\}_{i\in\mathbb{N}} = \{(\alpha_i, \beta_i)\}_{i\in\mathbb{N}}$ be a countable family of open intervals in $\mathbb{R}$, s.t. $r_i \in I_i$ for each $i \in \mathbb{N}$. We may assume that $\bigcap_{i\in\mathbb{N}}[\alpha_i, \beta_i] = \emptyset$.

(A) Given an arbitrary index $k \in \mathbb{N}$, we set $I_0 := [0, \beta_{k+1})$ and $A_0 := f^{-1}(I_0)$. It is easy to see that $(I_0, \ldots, I_k)$ is positively-separated open subsets of $\mathbb{R}_+$, and so Lemma 3.23 holds true. It follows from Lemma 3.22 that the k sequences $(S^{(1)}_n)_{n\in\mathbb{N}}, \ldots, (S^{(k)}_n)_{n\in\mathbb{N}}$ all converge to some multisets $S^{(1)}_0, \ldots, S^{(k)}_0 \in \mathcal{S}_\Phi(X, x_0)$. Now,

$$f(S^{(i)}_0) = \lim_{n\to\infty} f(S^{(i)}_n) = \lim_{n\to\infty} f(S_n \cap U_i) = \lim_{n\to\infty} (f(S_n) \cap I_i) = R \cap I_i \qquad \forall i = 1, \ldots, k, \tag{22}$$

where the last equality follows from Corollary 3.16 and Lemma 3.17. It follows that $\operatorname{rank} S^{(i)}_0 = m_i$ for each $i = 1, \ldots, k$. That is, each $S^{(i)}_0$ admits a representation

$$\operatorname{rank} S^{(i)}_0 = \{s^{(i)}_1, \ldots, s^{(i)}_{m_i}\}^*$$

for some $s^{(i)}_1, \ldots, s^{(i)}_{m_i} \in X$. Evidently, $f(S^{(i)}_0) = \{r_i, \ldots, r_i\}^*$.

(B) Part (A) allows us to define the multiset $S_0 := \{s^{(1)}_1, \ldots, s^{(1)}_{m_1}, s^{(2)}_2, \ldots, s^{(2)}_{m_2}, \ldots\}^*$, whose Φ-summability follows from $f(S_0) = R$. We show that $S_n \to S_0$ as $n \to \infty$. Let $\epsilon > 0$ be arbitrary. Then there exists a large enough index $k \in \mathbb{N}$, s.t.

$$\rho_\Phi(R \cap I_0, O_0) = \Phi(\underbrace{r_{k+1}, \ldots, r_{k+1}}_{m_{k+1} \text{ times}}, \underbrace{r_{k+2}, \ldots, r_{k+2}}_{m_{k+2} \text{ times}}, \ldots) < \frac{\epsilon}{4}, \tag{23}$$

where $I_0 := [0, \beta_{k+1})$ as in (A). We set $S^{(0)}_0 := S_0 \cap U_0$. Since the last equality in (22) is also true for $i = 0$, it follows that there exists an index N s.t. for all $n \geq N$ we have

$$\rho_\Phi(f(S_n) \cap I_0, O_0) < \frac{\epsilon}{4}. \tag{24}$$

As in (A), we can always increase the index N, if necessary, so that for all $n \geq N$

$$S_n = S^{(0)}_n + \ldots + S^{(k)}_n \text{ and } \sum_{i=1}^{k} d_\Phi(S^{(i)}_n, S^{(i)}_0) < \frac{\epsilon}{2}. \tag{25}$$

It follows from (23), (24) (25) that for all $n \geq N$

$$\begin{aligned} d_\Phi(S_n, S_0) &= d_\Phi(S_n^{(0)} + \ldots + S_n^{(k)}, S_0^{(0)} + \ldots + S_0^{(k)}) \\ &\leq d_\Phi(S_n^{(0)}, S_0^{(0)}) + d_\Phi(S_n^{(1)}, S_0^{(1)}) + \ldots + d_\Phi(S_n^{(k)}, S_0^{(k)}) \\ &< d_\Phi(S_n^{(0)}, O_{x_0}) + d_\Phi(O_{x_0}, S_0^{(0)}) + \frac{\epsilon}{2} \\ &= \rho_\Phi(f(S_n) \cap I_0, O_0) + \rho_\Phi(R \cap I_0, O_0) + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

where the first inequality follows from Lemma 3.10. Thus, $S_n \to S_0$ as $n \to \infty$, and so the proof is now complete. ☐

4. Continuity of Multiset-valued Mappings

Notation. We will assume the following throughout:

1. Let Φ be a symmetric norm unless otherwise stated.
2. Let I be a metric space unless otherwise stated.
3. Let (X, d) be a metric space with a fixed point $x_0 \in X$.

The purpose of the current section is to establish several results about continuity of multiset-valued mappings. We will make use of the following terminology:

Definition 4.1. The *rank* of a mapping $S : I \to \mathcal{S}_\Phi(X, x_0)$ is defined to be the smallest non-negative number N such that $\operatorname{rank} S(t) \leq N$ holds for all $t \in I$. The mapping S is called a *finite-rank mapping*, if it has a finite rank.

4.1. Continuity of sums.

Given $S, T : I \to \mathcal{S}_\Phi(X, x_0)$, their *sum* $S + T : I \to \mathcal{S}_\Phi(X, x_0)$ is defined by $(S + T)(\cdot) := S(\cdot) + T(\cdot)$.

Theorem 4.2. *If $S, T : I \to \mathcal{S}_\Phi(X, x_0)$ are continuous, then so is $S + T : I \to \mathcal{S}_\Phi(X, x_0)$.*

Proof. Lemma 3.10 establishes the estimate

$$d_\Phi((S + T)(t), (S + T)(t')) \leq d_\Phi(S(t), S(t')) + d_\Phi(T(t), T(t')) \qquad \forall t, t' \in I,$$

from which the continuity of $S + T$ follows. ☐

4.2. Continuity of differences.

Given $S, T : I \to \mathcal{S}_\Phi(X, x_0)$ with $T(t) \subseteq S(t)$ for all $t \in I$, their *difference* $S - T : I \to \mathcal{S}_\Phi(X, x_0)$ is defined by $(S - T)(\cdot) := S(\cdot) - T(\cdot)$.

Corollary 4.3. *Let $S, T : I \to \mathcal{S}_\Phi(X, x_0)$ be two continuous mappings with $T(t) \subseteq S(t)$ for all $t \in I$. If each point $t_0 \in I$ has a neighborhood I_0 s.t. the restriction $T|_{I_0}$ is a finite-rank mapping, then $S - T : I \to \mathcal{S}_\Phi(X, x_0)$ is continuous.*

Proof. Given an arbitrary point $t_0 \in I$, there exist a neighborhood I_0 of t_0 and a nonnegative integer n, s.t. $\operatorname{rank} T(t) \leq n$ for all $t \in I_0$. It follows from Lemma 3.12 that

$$d_\Phi((S - T)(t_0), (S - T)(t)) \leq 6n \left(d_\Phi(S(t_0), S(t)) + d_\Phi(T(t_0), T(t)) \right) \qquad \forall t \in I_0.$$

The continuity of $S - T$ at t_0 follows from that of S, T. ☐

4.3. **Continuity of intersections.** Given a mapping $S : I \to \mathcal{S}_\Phi(X, x_0)$ and a subset U of X, we define the mapping $S \cap U : I \to \mathcal{S}_\Phi(X, x_0)$ by $(S \cap U)(\cdot) := S(\cdot) \cap U$. We also define the mapping $S \setminus U : I \to \mathcal{S}_\Phi(X, x_0)$ by $S \setminus U := S \cap (X \setminus U)$.

Theorem 4.4. *Let $S : I \to \mathcal{S}_\Phi(X, x_0)$ be a continuous mapping, and let $t_0 \in I$ be fixed. Suppose that* $\operatorname{supp} S(t_0) \subseteq U_0 \cup \ldots \cup U_k$ *for some positively-separated tuple $(U_0, \ldots, U_k)$ of open subsets of X. Then there exists a neighborhood I_0 of t_0, s.t. the mappings $S \cap U_0, \ldots, S \cap U_k$ are all continuous on I_0. Furthermore, the neighborhood I_0 can be chosen in such a way that the following function is constant:*

$$I_0 \ni t \longmapsto (\operatorname{rank} S(t) \cap U_1, \ldots, \operatorname{rank} S(t) \cap U_k) \in \mathbb{Z}^n \tag{26}$$

Proof. This immediately follows from Corollary 3.16 and Lemma 3.17. □

4.4. **Continuity of induced mappings.** A finite collection of X-valued mappings $\lambda_1, \ldots, \lambda_n$ on the metric space I naturally induce the mapping

$$I \ni t \longmapsto \{\lambda_1(t), \ldots, \lambda_n(t)\}^* \in \mathcal{S}_\Phi(X, x_0),$$

which will be denoted by $\{\lambda_1, \ldots, \lambda_n\}^*$ from here on. It is true in general that if $\lambda_1, \ldots, \lambda_n$ are continuous, then so is the induced mapping $S := \{\lambda_1, \ldots, \lambda_n\}^*$ by (17). The purpose of the current subsection is to given an infinite-dimensional analogue of this result. We begin with the following definition.

Definition 4.5. A sequence $\lambda_1, \lambda_2, \ldots$ of X-valued mappings on the metric space I is said to be *pointwise* Φ*-summable*, if for each $t \in I$ the multiset $\{\lambda_1(t), \lambda_2(t), \ldots\}^*$ is Φ-summable.

The question we would like to address is the following. Given a pointwise Φ-summable sequence $(\lambda_i(\cdot))_{i\in\mathbb{N}}$ of continuous mappings defined on I, is the induced mapping

$$I \ni t \longmapsto \{\lambda_1(t), \lambda_2(t), \ldots\}^* \in \mathcal{S}_\Phi(X, x_0), \tag{27}$$

which will be denoted by $\{\lambda_1, \lambda_2, \ldots\}^*$ from here on, continuous in general? The following counter example says otherwise:

Example 4.6. Let us consider the space $\mathcal{S}_1(\mathbb{R}_+, 0)$, where $\mathbb{R}_+$ is equipped with the standard metric $\rho(x, y) := |x - y|$. We define a mapping $g : \mathbb{R} \to \mathbb{R}_+$ by

$$g(t) = \begin{cases} \sin(\pi t), & \text{if } t \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Let $I = [0, 1]$ and consider the doubly-indexed sequence $(\lambda_{m,n}(\cdot))_{m,n\in\mathbb{N}}$ of continuous functions on I defined by $\lambda_{m,n}(t) = \frac{g(2^m t)}{2^n}$. Let us first prove that $(\lambda_{m,n}(\cdot))_{m,n\in\mathbb{N}}$ is pointwise 1-summable. Indeed, for any $t \in I$,

$$\sum_{m,n=1}^{\infty} |0 - \lambda_{m,n}(t)| = \sum_{m,n=1}^{\infty} \frac{g(2^m t)}{2^n} \le N_t \cdot \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \right) = N_t,$$

where N_t denotes the cardinality of the set $\{m \in \mathbb{N} \mid 0 \le 2^m t \le 1\}$. However, the mapping $S : I \to \mathcal{S}_1(\mathbb{R}_+, 0)$ induced by $(\lambda_{m,n}(\cdot))_{m,n\in\mathbb{N}}$ is not continuous at 0. This is because for any $t_0 \in (0, 1)$ there is a large enough index m_0 satisfying $2^{-m_0} < t_0$, and this gives

$$\rho_1\left(S(0), S(2^{-m_0})\right) = \rho_1\left(O_0, S(2^{-m_0})\right) = \sum_{m,n=1}^{\infty} \frac{g(2^{m-m_0})}{2^n} \ge \sum_{n=1}^{\infty} \frac{1}{2^n} = 1. \tag{28}$$

Nevertheless, we have the following criterion:

Theorem 4.7. *Let Φ be a regular symmetric norm, and let I be a compact metric space. Let $(\lambda_i(\cdot))_{i\in\mathbb{N}}$ be a sequence of pointwise Φ-summable sequence of continuous X-valued mappings on the metric space I. Then the following assertions are all equivalent:*

1. *$\Phi(d(x_0,\lambda_{n+1}(\cdot)),d(x_0,\lambda_{n+2}(\cdot)),\dots)\to 0$ uniformly as $n\to\infty$.*
2. *The mapping $\xi_0(\cdot):=(d(x_0,\lambda_i(\cdot)))_{i\in\mathbb{N}}:I\to\ell_\Phi$ is continuous.*
3. *The induced mapping $S(\cdot):=\{\lambda_1(\cdot),\lambda_2(\cdot),\dots\}^*$ is continuous.*

It is easy to observe that in Example 4.6 the mapping $(d(x_0,\lambda_{m,n}(\cdot)))_{m,n\in\mathbb{N}}$ fails to be continuous at 0 by (28).

Proof. We proceed as (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). As we shall see below, the assumption of I being compact is only used in the last implication (3) $\Rightarrow$ (1). Before taking up the proof, let us first introduce some notation. For each $n\in\mathbb{N}$, we define

$$S_n(\cdot):=\{\lambda_1(\cdot),\dots,\lambda_n(\cdot)\}^* \text{ and } \xi_n(\cdot):=(d(x_0,\lambda_1(\cdot)),\dots,d(x_0,\lambda_n(\cdot)),0,0,\dots).$$

Note that each ξ_n is continuous, because for all $t,t'\in I$ we have

$$\begin{aligned}\Phi(\xi_n(t)-\xi_n(t')) &= \Phi(|d(x_0,\lambda_1(t))-d(x_0,\lambda_1(t'))|,\dots,|d(x_0,\lambda_n(t))-d(x_0,\lambda_n(t'))|,0,0,\dots)\\ &\le \Phi(d(\lambda_1(t),\lambda_1(t'),\dots,d(\lambda_n(t),\lambda_n(t')),0,0,\dots)\\ &\le \sum_{i=1}^n d(x_0,\lambda_i).\end{aligned}$$

Since Φ is a regular symmetric norm, we have $\lim_{n\to\infty}\Phi(\xi_0(t)-\xi_n(t))=0$ for all $t\in I$. Let us first prove (1) $\Rightarrow$ (2). Suppose that the convergence

$$\lim_{n\to\infty}\Phi(d(x_0,\lambda_{n+1}(\cdot)),d(x_0,\lambda_{n+2}(\cdot),\dots)=\lim_{n\to\infty}\Phi(\xi_0(\cdot)-\xi_n(\cdot))=0$$

is uniform. Since ξ_0 is the uniform limit of the continuous mappings $\xi_1,\xi_2,\dots$, the mapping $\xi_0=(d(x_0,\lambda_i(\cdot)))_{i\in\mathbb{N}}$ is continuous. For (2) $\Rightarrow$ (3), we assume that ξ_0 is continuous. Observe that for each $N=0,1,2,\dots$ the "cut-off mapping"

$$\ell_\Phi\ni(\xi_1,\xi_2,\dots)\longmapsto(\xi_{N+1},\xi_{N+2},\dots)\in\ell_\Phi$$

is obviously (1-Lipschitz) continuous, and so $(d(x_0,\lambda_{N+1}(\cdot)),d(x_0,\lambda_{N+2}(\cdot)),\dots)$ is continuous. It follows from the continuity of the norm Φ that $d_\Phi(O_{x_0},(S-S_N)(\cdot))=\Phi(d(x_0,\lambda_{N+1}(\cdot)),d(x_0,\lambda_{N+2}(\cdot)),\dots)$ is continuous. To prove the continuity of S, we let $\epsilon>0$ and $t_0\in I$ be arbitrary. Since Φ is regular and $(d(x_0,\lambda_i(t_0)))_{i\in\mathbb{N}}\in\ell_\Phi$, there exists an index N (depending on both ϵ and t_0) such that

$$d_\Phi(O_{x_0},(S-S_N)(t_0))=\Phi(d(x_0,\lambda_{N+1}(t_0)),d(x_0,\lambda_{N+2}(t_0)),\dots)<\frac{\epsilon}{4}.$$

Since $d_\Phi(O_{x_0},(S-S_N)(\cdot))$ is continuous at t_0, there exists a neighborhood I_0 of t_0, s.t.

$$d_\Phi(O_{x_0},(S-S_N))(t)<\frac{\epsilon}{4}\qquad\forall t\in I_0. \tag{29}$$

Since S_N is continuous, we may shrink I_0 if necessary, to ensure that

$$d_\Phi(S_N(t_0),S_N(t))<\frac{\epsilon}{2}\qquad\forall t\in I_0. \tag{30}$$

It follows from (29), (30) that for all $t \in I_0$ we have

$$\begin{aligned} d_\Phi(S(t_0), S(t)) &= d_\Phi((S - S_N)(t_0) + S_N(t_0), (S - S_N)(t) + S_N(t)) \\ &\leq d_\Phi((S - S_N)(t_0), (S - S_N)(t)) + d_\Phi(S_N(t_0), S_N(t)) \\ &\leq d_\Phi((S - S_N)(t_0), O_{x_0}) + d_\Phi(O_{x_0}, (S - S_N)(t)) + d_\Phi(S_N(t_0), S_N(t)) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

thereby establishing the continuity of S. Finally, to prove (3) $\Rightarrow$ (1), we assume that S is continuous. Then $S - S_n$ is continuous for each $n \in \mathbb{N}$. It follows that each $f_n := d_\Phi(O_{x_0}, (S - S_n)(\cdot)) : I \to \mathbb{R}$ is continuous. By construction, $(f_n)_{n\in\mathbb{N}}$ is a pointwise decreasing sequence, and it converges to 0 pointwise. It follows from Dini's theorem (see [Rud, Theorem 7.13] for details) that $f_n \to 0$ uniformly. $\square$

Remark 4.8. Given a compact metric space I and a continuous X-valued mapping λ defined on I, we define the *radius* of λ to be

$$R(\lambda) := \sup_{t\in I} d(x_0, \lambda(t)).$$

If $(\lambda_i(\cdot))_{i\in\mathbb{N}}$ is a pointwise Φ-summable sequence of continuous X-valued mappings on I, then the following are immediate consequences of the first part of Theorem 4.7:

1. For any $\epsilon > 0$ there exists a large enough index N s.t. $\sup_{n>N} R(\lambda_n) < \epsilon$.
2. That is, no matter how small $\epsilon > 0$ may be, all but finitely many of $\lambda_1, \lambda_2, \dots$ have their images completely included in the open ϵ-neighborhood of x_0.

The following two assertions are immediate corollaries of Theorem 4.7:

Corollary 4.9. *Let Φ be a regular symmetric norm, and let I be a compact metric space. Let $(\lambda_i(\cdot))_{i\in\mathbb{N}}$ be a pointwise Φ-summable sequence of continuous X-valued mappings on I, s.t. $\{\lambda_1, \lambda_2, \dots\}^*$ is continuous. If $(\lambda'_i(\cdot))_{i\in\mathbb{N}}$ is a subsequence of $(\lambda_i(\cdot))_{i\in\mathbb{N}}$, then the induced mapping $\{\lambda'_1, \lambda'_2, \dots\}^*$ is continuous.*

Proof. Since $\{\lambda_1, \lambda_2, \dots\}^*$ is continuous, $\Phi(d(x_0, \lambda_{n+1}(\cdot)), d(x_0, \lambda_{n+2}(\cdot), \dots) \to 0$ uniformly as $n \to \infty$ by Theorem 4.7. It follows that the convergence

$$\Phi(d(x_0, \lambda'_{n+1}(\cdot)), d(x_0, \lambda'_{n+2}(\cdot)), \dots) = 0$$

is also uniform. By the same theorem, $\{\lambda'_1, \lambda'_2, \dots\}^*$ is continuous. $\square$

Corollary 4.10. *Let Φ be a regular symmetric norm, and let I be a compact metric space. If $(\lambda_i(\cdot))_{i\in\mathbb{N}}$ is a pointwise Φ-summable sequence of continuous X-valued mappings on I s.t. $\{\lambda_1, \lambda_2, \dots\}^*$ is continuous, then $(\lambda_i(\cdot))_{i\in\mathbb{N}}$ converges uniformly to x_0.*

Proof. Let $\epsilon > 0$ and $t_0 \in I$ be arbitrary. Since S is continuous, it follows from Theorem 4.7 that $\Phi(d(x_0, \lambda_{n+1}(\cdot)), d(x_0, \lambda_{n+2}(\cdot)), \dots) \to 0$ uniformly as $n \to \infty$. Then there exists an index $N \in \mathbb{N}$, s.t.

$$\frac{\epsilon}{2} > \Phi(d(x_0, \lambda_{N+1}(t)), d(x_0, \lambda_{N+2}(t), \dots) \geq \sup_{n>N} d(x_0, \lambda_n(t)) \quad \forall t \in I. \tag{31}$$

It follows that $\lambda_i(\cdot) \to x_0$ uniformly by triangle inequality. $\square$

5. Continuous Enumeration of Multiset-valued Mappings

Notation. We shall assume the following throughout the current section:

1. Let Φ be a symmetric norm.
2. Let (X,d) be a metric space with a fixed point $x_0 \in X$.

We stress that Φ is not necessarily regular.

5.1. **Introduction.** Let us recall that in §4.4, we have developed a criterion under which a given sequence $\lambda_1, \lambda_2, \dots$ of continuous X-valued mappings induces a continuous $\mathcal{S}_\Phi(X,x_0)$-valued mapping $\{\lambda_1, \lambda_2, \dots\}^*$. A different kind of continuity question is the following. Given a continuous $\mathcal{S}_\Phi(X,x_0)$-valued mapping S, does there exist a sequence $\lambda_1, \lambda_2, \dots$ of continuous X-valued mappings, s.t. $S = \{\lambda_1, \lambda_2, \dots\}^*$? Such a sequence $(\lambda_i(\cdot))_{i\in\mathbb{N}}$ is called a *continuous enumeration* of S. The ultimate purpose of the current section is to prove the following result:

Theorem 5.1 (existence of continuous enumeration)**.** *If $S : [0,1] \to \mathcal{S}_\Phi(X,x_0)$ is a continuous mapping, then it admits a continuous enumeration.*

This theorem is absolutely essential when we calculate the fundamental group of the metric space $\mathcal{S}_\Phi(X,x_0)$ in the next section. As we shall see below, the existence of a continuous enumeration is a simple corollary of the following two technical results:

Theorem 5.2 (finite-rank continuous enumeration)**.** *Let I be an interval in $\mathbb{R}$, and let $S : I \to \mathcal{S}_\Phi(X,x_0)$ be a continuous mapping of a finite rank n. Then there exist n continuous mappings $\lambda_1, \dots, \lambda_n : I \to X$ s.t. $S = \{\lambda_1, \dots, \lambda_n\}^*$ for all $t \in I$.*

This theorem is a multiset analogue of Theorem 1.1.

Theorem 5.3 (theorem of finite separation)**.** *Let $S : [0,1] \to \mathcal{S}_\Phi(X,x_0)$ be continuous, and let $\epsilon > 0$ be fixed. Then there exists a finite-rank continuous mapping $S_\epsilon : [0,1] \to \mathcal{S}_\Phi(X,x_0)$, s.t. for all $t \in [0,1]$ we have $S_\epsilon(t) \subseteq S(t)$ and $\operatorname{supp}(S - S_\epsilon)(t) \subseteq B_\epsilon(x_0)$.*

This theorem is motivated by the second part of Remark 4.8.

Proof of Theorem 5.1. We set $\epsilon_n := 1/n$ and proceed inductively. It follows from Theorem 5.3 that there exists a finite-rank continuous mapping $S_1 : [0,1] \to \mathcal{S}_\Phi(X,x_0)$, such that for all $t \in [0,1]$ we have $S_1(t) \subseteq S(t)$ and $\operatorname{supp}(S - S_1)(t) \subseteq B_\epsilon(x_0)$. We can then apply the same lemma with ϵ_2 to the continuous mapping $S - S_1$ and obtain another finite-rank continuous mapping $S_2 : [0,1] \to \mathcal{S}_\Phi(X,x_0)$ satisfying the desired properties. Proceeding this way, we can form a sequence $S_1, S_2, \dots : [0,1] \to \mathcal{S}_\Phi(X,x_0)$ of continuous mappings, s.t. each S_i admits a finite-rank continuous enumeration $\lambda_1^i, \dots, \lambda_{n_i}^i$. By construction, $S = \{\lambda_1^1, \dots, \lambda_{n_1}^1, \lambda_1^2, \dots, \lambda_{n_2}^2, \dots\}^*$, and so S admits a continuous enumeration. □

Before taking up proofs of Theorem 5.2 and Theorem 5.3, we introduce the notion of "simple continuous enumeration" which will be used in later sections:

Definition 5.4. A continuous mapping $\lambda : [0,1] \to X$ is said to be *simple* in (X,x_0), if its *support* given by $\operatorname{supp}\lambda := \{t \in [0,1] \mid \lambda(t) \neq x_0\}$ is an open sub-interval of $[0,1]$.

Theorem 5.5 (simple continuous enumeration)**.** *If $S : [0,1] \to \mathcal{S}_\Phi(X,x_0)$ is a continuous mapping, then there exists a continuous enumeration $(\lambda_i)_{i\in\mathbb{N}}$ of S with the property that all $\lambda_1, \lambda_2, \dots$ are simple continuous mappings in (X,x_0).*

Proof. Since S is continuous, it admits a continuous enumeration $(\lambda_i')_{i\in\mathbb{N}}$. Note that the support each λ_i' is an open subset of $[0,1]$, and so it admits an at most countable union of pairwise disjoint intervals $I_1, I_2, \ldots$ that are open in $[0,1]$. We can then define the continuous paths $\lambda_i^1, \lambda_i^2, \ldots$ in X by

$$\lambda_i^j(t) := \begin{cases} \lambda_i(t), & \text{if } t \in I_j, \\ x_0, & \text{otherwise.} \end{cases}$$

By construction, each λ_i^j is simple in (X, x_0). Now, the doubly-indexed sequence $(\lambda_i^j)_{i,j\in\mathbb{N}}$ is a continuous enumeration of S. The proof is complete. □

5.2. **A sketch of proofs.** The remaining part of the current section is devoted entirely to proving Theorem 5.2 (existence of finite-rank continuous enumeration) and Theorem 5.3 (theorem of finite separation). As was mentioned, Theorem 5.2 is nothing but a multiset analogue of Kato's selection theorem, and we will simply replicate his proof. As for the theorem of finite separation, the following "local version" is easy to obtain:

Lemma 5.6. *Let $S : [0,1] \to \mathcal{S}_\Phi(X, x_0)$ be continuous, and let $\epsilon > 0$ be fixed. Then for any $t_0 \in [0,1]$ there exists a neighborhood I_0 of t_0 and a finite-rank continuous mapping $S_\epsilon : I_0 \to \mathcal{S}_\Phi(X, x_0)$, such that for all $t \in I_0$ we have $S_\epsilon(t) \subseteq S(t)$ and $\operatorname{supp}(S - S_\epsilon)(t) \subseteq B_\epsilon(x_0)$.*

Proof. We can always shrink ϵ to ϵ_0, so that $S(t_0) \cap B_{\epsilon_0}(x_0) = S(t_0) \cap B_\epsilon(x_0)$ and for each $s \in \operatorname{supp} S(t_0)$ we have $d(x_0, s) \neq \epsilon_0$. If we set $U_0 := B_{\epsilon_0}(x_0)$, then there exists an open set U_1 in X s.t. $\operatorname{dist}(U_0, U_1) > 0$ and $\operatorname{supp}(S(t_0) \setminus U_0) \subseteq U_1$. It follows from Theorem 4.4 that there exists a neighborhood I_0 of t_0, s.t. $S \cap U_0, S \cap U_1$ are both continuous on I_0 and $S \cap U_1$ has a constant finite-rank on I_0. We set $S_\epsilon := (S \cap U_1)|_{I_0}$. Then for any $t \in I_0$, we have $S_\epsilon(t) \subseteq S(t)$ and $\operatorname{supp}(S - S_\epsilon)(t) \subseteq B_\epsilon(x_0)$. The claim follows. □

As we shall see, the theorem of finite separation is obtained by extending this local property to the global one by "patching" appropriately chosen neighborhoods finitely many times.

5.3. **A proof of Theorem 5.2 (finite-rank continuous enumeration).** Given a continuous $\mathcal{S}_\Phi(X, x_0)$-valued mapping $S(\cdot)$, there is no natural way to select a continuous enumeration even if S has a finite-rank. However, there are some trivial examples:

Example 5.7. Let $S(\cdot)$ be a continuous mapping of $\mathcal{S}_\Phi(X, x_0)$-valued mapping on a metric space I. Suppose that $S(\cdot)$ has a finite-rank n and that S can be written as

$$S(t) := \{\underbrace{\lambda(t), \ldots, \lambda(t)}_{n \text{ times}}\}^* \qquad \forall t \in I$$

for some $\lambda : I \to X$. In this case, the continuity of $\lambda : I \to X$ is an immediate consequence of Inequality (18). That is, S admits a continuous enumeration.

Proof of Theorem 5.2. For brevity, let us call the finite sequence $\lambda_1, \ldots, \lambda_n$ in the premise of Theorem 5.2 a *finite-rank continuous enumeration* of S.

(A) Let us develop one preliminary result beforehand. Let I_1, I_2 be two overlapping subintervals of I, such that I_1 is located to the left of I_2. Suppose that the two restrictions $S|_{I_1}, S|_{I_2}$ admit finite-rank continuous enumerations $(\lambda_1^1, \ldots, \lambda_n^1)$ and $(\lambda_1^2, \ldots, \lambda_n^2)$ respectively. For any $t_0 \in I_1 \cap I_2$, the two finite sequences $(\lambda_i^1(t_0))_{i=1}^n, (\lambda_i^2(t_0))_{i=1}^n$ are identical up to a permutation. It follows that a finite-rank continuous enumeration exists on $I_1 \cup I_2$. It follows that if J is a subinterval of I s.t. each point of J has a neighborhood on which a continuous enumeration exists, then a continuous enumeration exists on the whole interval J.

(B) Let us prove the claim by induction on n. The base step $n = 1$ is done in Example 5.7. Suppose that the claim is proved for n replaced by a smaller number and for any interval I. Let Γ be the set of all $t \in I$ for which $S(t)$ admits the representation $S(t) = \{x(t), \dots, x(t)\}^*$, where a point $x(t) \in X$ is repeated n times. It follows from Example 5.7 again that the $x(t)$ depends continuously on $t \in \Gamma$. Since Γ is a closed subset of I by Theorem 4.4, the open set $I \setminus \Gamma$ can be written as countable union of pairwise disjoint open subintervals $I_1, I_2, \dots$ of I. Given any such interval I_j and any point $t_j \in I_j$, since $\operatorname{supp} S(t_j)$ can be written as a union of two nonempty finite subsets of X, it follows from the induction hypothesis and Theorem 4.4 that t_j has a neighborhood on which a finite-rank continuous enumeration exists. It follows from (A) that a finite-rank continuous enumeration $(\lambda_1^j, \dots, \lambda_n^j)$ exists on each I_j. Then we define the mappings $\lambda_1, \dots, \lambda_n : I \to X$ by

$$\lambda_i(t) := \begin{cases} x(t), & \text{if } t \in \Gamma, \\ \lambda_i^j(t), & \text{if } t \in I_j,\ j = 1, 2, \dots. \end{cases} \tag{32}$$

It remains to prove the continuity of each λ_i. If $t_0 \notin \Gamma$, then the continuity of λ_i at t_0 follows by construction. If $t_0 \in \Gamma$, then Estimate (18) allows to establish

$$d(\lambda_i(t_0), \lambda_i(t)) = d(x(t_0), \lambda_i(t)) \le 2\, d_\Phi(S(t_0), S(t)) \quad \forall t \in I.$$

The continuity of λ_i at t_0 follows from that of S. □

5.4. **A proof of Theorem 5.3 (theorem of finite separation).** Let us first establish the following technical lemma:

Lemma 5.8. *Let I be an interval in $\mathbb{R}$, and let S be a continuous $\mathcal{S}_\Phi(X, x_0)$-valued mapping on I. Then for each $t_0 \in I$ and each $s \in \operatorname{supp} S(t_0)$, there exists a continuous mapping $\lambda : I \to X$ with the property that $\lambda(t_0) = s$ and $\lambda(t) \in \operatorname{supp} S(t)$ for all $t \in I$.*

Proof. Given any X-valued mapping λ, we denote its domain by I_λ throughout. We will proceed with Zorn's lemma and consider the non-trivial case $x \neq x_0$.

(A) Let $\mathcal{A}$ be the set of all those continuous X-valued mappings λ, s.t. I_λ is an open subinterval of I containing t_0 and $\lambda(t_0) = x$. We define a partial order $\preceq$ on $\mathcal{A}$ by

$$\lambda_1 \preceq \lambda_2 \iff I_{\lambda_1} \subseteq I_{\lambda_2} \text{ and } \lambda_2 \text{ restricted to } I_{\lambda_1} \text{ is } \lambda_1.$$

If $\mathcal{A}_0 = \{\lambda_\alpha\}_\alpha$ is a totally ordered subset of $\mathcal{A}$, then it has an upper bound $\lambda_0 : \bigcup_\alpha I_{\lambda_\alpha} \to X$ defined by $\lambda_0(t) = \lambda_\alpha(t) \iff t \in I_{\lambda_\alpha}$. Evidently, λ_0 is a well-defined continuous mapping.

(B) By Zorn's lemma, $\mathcal{A}$ contains a maximal element $\lambda : I_\lambda \to X$. It remains to prove $I_\lambda = I$. Assume the contrary that I_λ is a proper subset of I. Then there exists a boundary point b of I_λ with $b \notin I_\lambda$. Without loss of generality we assume that b is the supremum of I_λ. Let us extend the domain of λ by setting $\lambda(b) := x_0$. The extended mapping λ cannot be continuous at b, as this would contradict the maximality of I_λ. It follows that there exists an $\epsilon > 0$, s.t. any neighborhood I_b of b in I contains at least one point $t_b \in I_\lambda$ satisfying

$$\epsilon \le d(\lambda(b), \lambda(t_b)) = d(x_0, \lambda(t_b)). \tag{33}$$

It follows from Lemma 5.6 that there exist a neighborhood I_b of b and a finite-rank continuous mapping $S_\epsilon : I_b \to \mathcal{S}_\Phi(X, x_0)$, s.t. for all $t \in I_b$ we have $S_\epsilon(t) \subseteq S(t)$ and $\operatorname{supp}(S - S_\epsilon)(t) \subseteq B_\epsilon(x_0)$. As mentioned above, the neighborhood I_b contains a point t_b satisfying (33). Applying

Theorem 5.2 to S_b establishes the existence of a continuous mapping $\lambda_b : I_b \to X$ s.t. $\lambda_b(t_b) = \lambda(t_b)$. We define the mapping $\mu : I_\lambda \cup I_b \to X$ by

$$\mu(t) = \begin{cases} \lambda(t), & \text{if } t \leq t_b, \\ \lambda_b(t), & \text{if } t > t_b. \end{cases}$$

Since μ thus defined is continuous, this contradicts the maximality of λ. The proof is complete. □

We are now in a position to prove the theorem of finite separation:

Proof of Theorem 5.3. We say that a closed subinterval I of $[0,1]$ has *Property* $\mathcal{X}$, if there exists a continuous $\mathcal{S}_\Phi(X, x_0)$-valued mapping S' defined on I, such that for all $t \in I$ we have $S_\epsilon(t) \subseteq S(t)$ and $\operatorname{supp}(S - S_\epsilon)(t) \subseteq B_\epsilon(x_0)$. We have to prove that the closed interval $[0,1]$ itself has Property $\mathcal{X}$.

(A) We show that if two closed subintervals I_1, I_2 of $[0,1]$ have Property $\mathcal{X}$, then so does their union $I_1 \cup I_2$. As we shall see shortly, we may assume that I_1 is located to the left of I_2 and that $I_1 \cap I_2$ is a point-set $\{t_0\}$. Let S_1, S_2 be finite-rank continuous $\mathcal{S}_\Phi(X, x_0)$-valued mappings defined on I_1, I_2 respectively satisfying the required conditions so that I_1, I_2 both have Property $\mathcal{X}$. It follows from Theorem 5.2 that S_1, S_2 admit finite-rank continuous enumerations:

$$S_1 = \{\lambda_1^1, \ldots, \lambda_{n_1}^1\}^* \text{ and } S_2 = \{\lambda_1^2, \ldots, \lambda_{n_2}^2\}^*.$$

We will proceed to the following three steps:

1. After a suitable rearrangement of the finite sequence $(\lambda_i^2)_{i=1}^{n_2}$, we may assume that

$$(\lambda_1^1(t_0), \ldots, \lambda_n^1(t_0)) = (\lambda_1^2(t_0), \ldots, \lambda_n^2(t_0)) \tag{34}$$

   for some $n \in \mathbb{N}$. We may assume that n is the largest natural number s.t. (34) holds. For each $i = 1, \ldots, n$, define the continuous mappings $\lambda_1, \ldots, \lambda_n : I_1 \cup I_2 \to X$ by

$$\lambda_i(t) = \begin{cases} \lambda_i^1(t), & \text{if } t \leq t_0, \\ \lambda_i^2(t), & \text{if } t > t_0. \end{cases}$$

2. We define the continuous mapping $T_2 : I_2 \to \mathcal{S}_\Phi(X, x_0)$ by $T_2(t) := S(t) - S_2(t)$. It follows from Equality (34) that

$$\{\lambda_{n+1}^1(t_0), \ldots, \lambda_{N_1}^1(t_0)\}^* \subseteq S(t_0) - S_2(t_0) = T_2(t_0).$$

   It follows from Lemma 5.8 that there exist $n_1 - n$ continuous mappings $\mu_{n+1}^2, \ldots, \mu_{n_1}^2 : I_2 \to X$, such that $(\mu_{n+1}^2(t_0), \ldots, \mu_{n_1}^2(t_0)) = (\lambda_{n+1}^1(t_0), \ldots, \lambda_{n_1}^1(t_0))$ and

$$\{\mu_{n+1}^2(t), \ldots, \mu_{N_1}^2(t)\}^* \subseteq T_2(t) \quad \forall t \in I_2.$$

   We can then define the mappings $\lambda_{n+1}^{1'}, \ldots, \lambda_{n_1}^{1'} : I_1 \cup I_2 \to X$ by

$$\lambda_i^{1'}(t) = \begin{cases} \lambda_i^1(t), & \text{if } t \leq t_0, \\ \mu_i^2(t), & \text{if } t > t_0. \end{cases}$$

3. Repeat the previous step with 2 replaced by 1.

We can then define the finite-rank continuous mapping $S_\epsilon : I_1 \cup I_2 \to \mathcal{S}_\Phi(X, x_0)$ by

$$S_\epsilon = \{\lambda_1^1, \ldots, \lambda_n^1, \lambda_{n+1}^{1'}, \ldots, \lambda_{n_1}^{1'}, \lambda_{n+1}^{2'}, \ldots, \lambda_{n_2}^{2'}\}^*.$$

By construction, the mapping S_ϵ thus defined satisfies the desired properties in order for $I_1 \cup I_2$ to have Property $\mathfrak{X}$.

(B) By Lemma 5.6 each $t \in [0,1]$ has a neighborhood I_t, such that every closed subinterval of I_t has Property $\mathfrak{X}$. Let $L > 0$ be a Lebesgue number of the open cover $\mathcal{N} := \{I_t\}_{t\in[0,1]}$. We can then choose a large enough number $N \in \mathbb{N}$ such that $\frac{1}{N} < L$. It follows that each of

$$I_1 := \left[0, \frac{1}{N}\right], I_2 := \left[\frac{1}{N}, \frac{2}{N}\right], \ldots, I_N := \left[\frac{N-1}{N}, 1\right],$$

is contained entirely in some member of the cover $\mathcal{N}$. That is, the intervals $I_1, \ldots, I_N$ have Property $\mathfrak{X}$. It follows from (A) that $[0,1] = I_1 \cup \ldots \cup I_N$ has Property $\mathfrak{X}$. □

6. The Fundamental Group of $S_\Phi(X, x_0)$

The ultimate aim of the current section is to construct the following group-isomorphism:

Theorem 6.1. *Let Φ be a regular symmetric norm, and let X be a locally simply connected, path-connected metric space with a fixed point $x_0 \in X$. Then there exists a group isomorphism*

$$\Psi_\Phi : \pi_1(\mathcal{S}_\Phi(X, x_0), O_{x_0}) \to H_1(X),$$

where $\pi_1(\mathcal{S}_\Phi(X, x_0), O_{x_0})$ is the fundamental group of X and $H_1(X)$ is the first singular homology group of X.

6.1. **Preliminaries.**

Here, we recall standard facts in algebraic topology. The details can be found in any standard textbook in the subject. See, for example, [Hat, §1,2].

6.1.1. *Fundamental groups.* Let X be a topological space with fixed points $x_0, x_1, x_2 \in X$. A *path* in X from x_0 to x_1 is any continuous mapping $\lambda : [0,1] \to X$ with $\lambda(0) = x_0$ and $\lambda(1) = x_1$, where x_0, x_1 are referred to as the *end-points* of λ. The *inverse* of a path λ in X, denoted by λ^{-1}, is the path in X defined by $\lambda^{-1}(t) := \lambda(1-t)$. If λ is a path from x_0 to x_1 and λ' is a path from x_1 to x_2, we define their *product* $\lambda \circledast \lambda'$ by

$$(\lambda \circledast \lambda')(t) = \begin{cases} \lambda(2t), & \text{if } 0 \le t \le \frac{1}{2}, \\ \lambda'(2t-1), & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Paths λ, λ' from x_0 to x_1 are said to be *path-homotopic* or *homotopic* in short, if there exists a continuous mapping $H : [0,1] \times [0,1] \to X$ with $H(\cdot, 0) = \lambda(\cdot), H(\cdot, 1) = \lambda'(\cdot), H(0, \cdot) \equiv x_0, H(1, \cdot) \equiv x_1$. Such a mapping H is called a *homotopy* from λ to λ'.

A loop in (X, x_0) is any path λ in X satisfying $\lambda(0) = \lambda(1) = x_0$. Two loops λ, λ' in (X, x_0) are considered *equivalent*, if there exists a homotopy from λ to λ'. Equivalence of loops induces an equivalence relation on the set of loops in (X, x_0), and the set of all the equivalence classes, denoted by $\pi_1(X, x_0)$, forms a group under the operation

$$[\lambda]_{\pi_1} \circledast [\lambda']_{\pi_1} = [\lambda \circledast \lambda']_{\pi_1},$$

where $[\,\cdot\,]_{\pi_1}$ denotes equivalence classes in $\pi_1(X, x_0)$. The group $\pi_1(X, x_0)$ is called the *fundamental group* of (X, x_0). The identity element of the fundamental group $\pi_1(X, x_0)$ is the equivalence class represented by the constant loop $[0,1] \ni t \longmapsto x_0 \in X$, and any loops in this equivalence class are said to be *null-homotopic*. Let us recall the following basic terminology:

1. The topological space X is said to be *path-connected*, if any two points in X can be joined by some path in X. It is a well-known result that a path-connected space X has a unique fundamental group in the sense that for any two points $x_0, x_1 \in X$, the fundamental groups $\pi_1(X, x_0), \pi_1(X, x_1)$ are isomorphic to each other.
2. The topological space X is said to be *simply connected*, if X is path-connected and if X has the trivial fundamental group.
3. The topological space X is said to be *locally simply connected*, if every point of X has a local base of simply connected open subsets of X.

6.1.2. *Singular homology groups.* Let X be a topological space. The *standard n-simplex* is

$$\Delta^n := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \ \forall i = 0, \ldots, n \right\}.$$

A *(singular) n-simplex* in X is any continuous mapping of the form $\Delta^n \to X$. The free Abelian group generated by the set of all n-simplices in X is denoted by $C_n(X)$. A member of $C_n(X)$, known as an *n-chain* in X, is a formal finite sum of the form $\sum_i n_i \sigma_i$, where n_i are integers and σ_i are n-simplices. We define the mappings $d_0, \ldots, d_n : \Delta^{n-1} \to \Delta^n$ by $d_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$. Given an n-simplex σ of X, its *boundary* $\partial_n \sigma$ is the $(n-1)$-chain

$$\partial_n \sigma := \sum_{i=0}^{n} (-1)^i (\sigma \circ d_i) \in C_{n-1}(X).$$

We can then extend this formula in the obvious way to obtain a group homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$ known as the *n-th boundary homomorphism*. Members of $\ker \partial_n$ are referred to as *n-cycles*. It can be shown that $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$, and so we can form the quotient group $H_n(X) := \ker \partial_n / \operatorname{im} \partial_{n+1}$, known as the *$n$-th (singular) homology group*. From here on, by homology groups, we shall always mean singular homology groups.

6.1.3. *The Fundamental group and first homology group.* Let X be a topological space. We will make use of the following notations:

1. Given two paths λ, λ' in X having the same end-points, we write $\lambda \sim \lambda'$ if λ, λ' are path-homotopic to each other.
2. Given two 1-chains σ, σ' in X with $\sigma - \sigma' \in \ker \partial_1$, we write $\sigma \simeq \sigma'$ if $\sigma - \sigma' \in \operatorname{im} \partial_2$.
3. Let $[\,\cdot\,]_{\pi_1}, [\,\cdot\,]_{H_1}$ denote equivalence classes in $\pi_1(X, x_0), H_1(X)$ respectively.

Note that paths in X can be viewed as 1-chains in $C_1(X)$. With this convention in mind, we will freely use the following well-known result without any further comment:

Lemma 6.2. *Let λ, μ be two paths in X.*

1. $\lambda^{-1} \simeq -\lambda$.
2. *If $\lambda \sim \mu$, then $\lambda \simeq \mu$.*
3. *If $\lambda(1) = \mu(0)$, then $\lambda \circledast \mu \sim \lambda + \mu$.*

The following theorem shows that the first singular homology group $H_1(X)$ is the abelianisation of the fundamental group $\pi_1(X, x_0)$ provided that X is path-connected:

Theorem 6.3 ([Hat, Theorem 2A.1])**.** *The correspondence*

$$\pi_1(X, x_0) \ni [\lambda]_{\pi_1} \longmapsto [\lambda]_{H_1} \in H_1(X) \tag{35}$$

defines a group homomorphism from $\pi_1(X, x_0)$ into $H_1(X)$. Furthermore, if X is path-connected, then (35) is surjective and its kernel is the commutator subgroup $[\pi_1(X, x_0), \pi_1(X, x_0)]$.

Recall that the commutator subgroup $[\pi_1(X, x_0), \pi_1(X, x_0)]$ is the subgroup of $\pi_1(X, x_0)$ generated by elements of the form $[\lambda \ast \mu \ast \lambda^{-1} \ast \mu^{-1}]_{\pi_1}$ where λ, μ are loops in (X, x_0).

6.2. **Isomorphism $\Psi_\Phi : \pi_1(\mathcal{S}_\Phi(X, x_0), O_{x_0}) \to H_1(X)$.**

Notation. We will assume the following throughout the remaining part of the current section:

1. Let Φ be a regular symmetric norm.
2. Let (X, d) be a locally simply connected, path-connected metric space, and let $x_0 \in X$.
3. We identify the constant loops $t \to x_0$ and $t \to O_{x_0}$ with x_0 and O_{x_0} respectively.

If S is a loop in $(\mathcal{S}_\Phi(X, x_0), O_{x_0})$ admitting a continuous enumeration $(\lambda_i)_{i\in\mathbb{N}}$, then each λ_i is a loop in (X, x_0). Furthermore, since X has a simply connected neighborhood of x_0, it immediately follows from the second part of Remark 4.8 that all but finitely many λ_i's are null-homotopic. This allows us to understand the formal infinite sum

$$\sum_{i=1}^{\infty} \lambda_i = \lambda_1 + \lambda_2 + \lambda_3 + \dots \tag{36}$$

as the 1-cycle in X formed by summing up all those λ_i's that are not null-homotopic: if all λ_i's happen to be null-homotopic, then we set $\sum_{i=1}^{\infty} \lambda_i = x_0$. We will prove the following technical theorem in the next subsection:

Theorem 6.4. *If S, T are homotopic loops in $(\mathcal{S}_\Phi(X, x_0), O_{x_0})$ admitting continuous enumerations $(\lambda_i)_{i\in\mathbb{N}}, (\mu_i)_{i\in\mathbb{N}}$ respectively, then $\sum_{i=1}^{\infty} \lambda_i \simeq \sum_{i=1}^{\infty} \mu_i$.*

We are now in a position to introduce a mapping $\Psi_\Phi : \pi_1(\mathcal{S}_\Phi(X, x_0), O_{x_0}) \to H_1(X)$:

Definition 6.5. Given $\mathcal{S} \in \pi_1(\mathcal{S}_\Phi(X, x_0), O_{x_0})$, we select any loop $S \in \mathcal{S}$ and any continuous enumeration $(\lambda_i)_{i\in\mathbb{N}}$ of S. We define $\Psi_\Phi(\mathcal{S}) := [\sum_{i=1}^{\infty} \lambda_i]_{H_1}$.

We will prove that Ψ_Φ thus defined is a group isomorphism using the following two lemmas:

Lemma 6.6. *The following assertions hold true:*

1. *If S, S', T, T' are paths in $\mathcal{S}_\Phi(X, x_0)$ with $S \sim T, S' \sim T'$, then $S + S' \sim T + T'$.*
2. *If λ, λ' are homotopic paths in X, then $\{\lambda\}^*, \{\lambda'\}^*$ are homotopic paths.*
3. *If λ, λ' are homotopic loops in (X, x_0), then $\{\lambda, \lambda'\}^*, \{\lambda \ast \lambda'\}^*$ are homotopic loops.*

Proof. For the first part, if H, H' are homotopies from S to T and from S' to T' respectively, then $H + H'$ is a homotopy from $S + S'$ to $T + T$. For the second part, if h is a homotopy from λ to λ', then $\{h\}^*$ is a homotopy from $\{\lambda\}^*$ to $\{\lambda'\}^*$. For the last part, it is easy to observe that $\{\lambda \ast x_0, x_0 \ast \lambda'\}^* = \{\lambda \ast \lambda'\}^*$. It follows from the second part that $\{\lambda\}^* \sim \{\lambda \ast x_0\}^*$ and $\{\lambda'\}^* \sim \{x_0 \ast \lambda'\}^*$. The claim follows by the first part. $\square$

Lemma 6.7. *If S is a loop in $(\mathcal{S}_\Phi(X, x_0), O_{x_0})$ admitting a continuous enumeration $\lambda_1, \lambda_2, \dots$ all of which are null-homotopic loops, then S is also null-homotopic.*

We shall make use of the notation $R(\cdot)$ introduced in Remark 4.8.

Proof. (A) Since X is locally simply connected, for each $m \in \mathbb{N}$ there exists a pair of a simply connected neighborhood U_m of x_0 and a positive number $\delta_m < 1/m$ satisfying $B_{\delta_m}(x_0) \subseteq U_m \subseteq B_{1/m}(x_0)$. That is, if we have a loop λ in (X, x_0) satisfying $R(\lambda) < \delta_m$, then there exists a homotopy h_λ from x_0 to λ satisfying $R(h_\lambda) < 1/m$. We may assume $\sup_{i\in\mathbb{N}} R(\lambda_i) < \delta_1$ without loss of generality[5], and so for each loop λ_i there exists a unique $m_i \in \mathbb{N}$ s.t. $\delta_{m_i+1} \leq R(\lambda_i) < \delta_{m_i}$. It follows that there exists a homotopy h_{λ_i} from x_0 to λ_i s.t. $R(h_{\lambda_i}) < 1/m_i$. We renumber the sequence $(\lambda_i)_{i\in\mathbb{N}}$ so that $m_1 \leq m_2 \leq m_3 \leq \ldots$. As in the second part of Remark 4.8, the sequence $(m_i)_{i\in\mathbb{N}}$ thus defined is necessarily unbounded, and so $R(h_{\lambda_i}) \to 0$ as $i \to \infty$.

(B) Our idea is that instead of "continuously deforming" $\lambda_1, \lambda_2, \ldots$ at once, we do so one by one. More precisely, we try to construct a homotopy H from O_{x_0} to $S = \{\lambda_1, \lambda_2, \ldots\}$ in such a way that deformation of each λ_n takes place within the rectangular strip $[0,1] \times [\frac{1}{n}, \frac{1}{n-1}]$. For this purpose, we introduce the "reparametrisations" $\gamma_n : [\frac{1}{n}, \frac{1}{n-1}] \to [0,1]$, where $n \in \mathbb{N}$:

$$\gamma_n(s) := n(n-1)s - (n-1).$$

We can then define a homotopy $h_n : [0,1] \times [0,1] \to X$ from x_0 to λ_n by

$$h_n(t,s) = \begin{cases} x_0, & \text{if } s < \frac{1}{n}, \\ h_{\lambda_n}(t, \gamma_n(s)), & \text{if } s \in \left[\frac{1}{n}, \frac{1}{n-1}\right], \\ \lambda_n(t), & \text{if } s > \frac{1}{n-1}. \end{cases}$$

For each $(t,s) \in [0,1] \times [0,1]$, we set $H(t,s) := \{h_1(t,s), h_2(t,s), \ldots\}^*$. By construction, H restricted to the rectangular strip $[0,1] \times [\frac{1}{n}, \frac{1}{n-1}]$ is of the form

$$H(t,s) = \{h_{\lambda_n}(t, \gamma_n(s)), \lambda_{n+1}(t), \lambda_{n+2}(t), \ldots\}^* \qquad \forall (t,s) \in [0,1] \times \left[\frac{1}{n}, \frac{1}{n-1}\right],$$

and the Φ-summability of each $H(t,s)$ follows from that of $S(t)$. It follows that the sequence $h_1, h_2, \ldots : [0,1] \times [0,1] \to X$ is a pointwise Φ-summable sequence in X. It remains to prove the continuity of H by the first part of Theorem 4.7.

(C) Given arbitrary $\epsilon > 0$, there exists a large enough index N s.t. for all $n \geq N$

$$R(h_{\lambda_n}) < \frac{\epsilon}{2} \text{ and } \Phi(d(x_0, \lambda_{n+1}(\cdot)), d(x_0, \lambda_{n+2}(\cdot)), \ldots) < \frac{\epsilon}{2}.$$

To prove the continuity of H by Theorem 4.7, it remains to prove

$$\Phi(d(x_0, h_{N+1}(t,s)), d(x_0, h_{N+2}(t,s)), \ldots) < \epsilon \qquad \forall (t,s) \in [0,1] \times [0,1]. \tag{37}$$

Let (t,s) be an arbitrary point in $[0,1] \times [0,1]$, and suppose $s \in \left[\frac{1}{n}, \frac{1}{n-1}\right]$ for some $n \in \mathbb{N}$. If $n < N$, then (37) trivially holds as

$$\Phi(d(x_0, h_{N+1}(t,s)), d(x_0, h_{N+2}(t,s)), \ldots) = \Phi(d(x_0, \lambda_{N+1}(t)), d(x_0, \lambda_{N+2}(t)), \ldots) < \frac{\epsilon}{2} < \epsilon.$$

[5] Indeed, the first part of Remark 4.8 with $\epsilon := \delta_1$ asserts the existence of an index N s.t. $\sup_{n\geq N+1} M(\lambda_n) < \delta_1$, and we have $S = \{\lambda_1, \ldots, \lambda_N\}^* + \{\lambda_{N+1}, \ldots\}^* \sim O_{x_0} + \{\lambda_{N+1}, \ldots\}^* \sim \{\lambda_{N+1}, \ldots\}^*$ by Lemma 6.6.

It remains to prove (37) for the case $n \geq N$. Now,

$$
\begin{aligned}
&\Phi(d(x_0, h_{N+1}(t,s)), d(x_0, h_{N+2}(t,s)), \ldots) \\
&\quad = \Phi(d(x_0, h_{N+1}(t,s)), \ldots, d(x_0, h_{n-1}(t,s)), d(x_0, h_n(t,s)), d(x_0, h_{n+1}(t,s)) \ldots) \\
&\quad = \Phi(d(x_0, x_0), \ldots, d(x_0, x_0), d(x_0, h_{\lambda_n}(t, \gamma_n(s)), d(x_0, \lambda_{n+1}(t)), \ldots) \\
&\quad \leq d(x_0, h_{\lambda_n}(t, \gamma_n(s)) + \Phi(d(x_0, \lambda_{n+1}(t)), d(x_0, \lambda_{n+2}(t)), \ldots) \\
&\quad < R(h_{\lambda_n}) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}
$$

thereby establishing (37). The proof is complete. □

Proof of Theorem 6.1. Let us first show that Ψ_Φ is a group homomorphism. Let S, T be two homotopic loops in $(\mathcal{S}_\Phi(X, x_0), O_{x_0})$ admitting continuous enumerations $(\lambda_i)_{i\in\mathbb{N}}, (\mu_i)_{i\in\mathbb{N}}$ respectively. Since the sequence $(\lambda_i \divideontimes \mu_i)_{i\in\mathbb{N}}$ is a continuous enumeration of the loop $S \divideontimes T$,

$$
\begin{aligned}
\Psi_\Phi([S]_{\pi_1} \divideontimes [T]_{\pi_1}) &= \Psi_\Phi([S \divideontimes T]_{\pi_1}) \\
&= [\lambda_1 \divideontimes \mu_1 + \lambda_2 \divideontimes \mu_2 + \ldots]_{H_1} \\
&= [(\lambda_1 + \lambda_2 + \ldots) + (\mu_1 + \mu_2 + \ldots)]_{H_1} \\
&= \Psi_\Phi([S]_{\pi_1}) + \Psi_\Phi([T]_{\pi_1}),
\end{aligned}
$$

where the third equality follows from the fact that all but finitely many loops in $(\lambda_i \divideontimes \mu_i)_{i\in\mathbb{N}}$ are null-homotopic.

Since the surjectivity of Ψ_Φ follows immediately from Theorem 6.3, it suffices to prove the injectivity. Suppose that $\Psi_\Phi([S]) = 0$ and that $(\lambda_i)_{i\in\mathbb{N}}$ is a continuous enumeration of S. Then there exists a large enough index N s.t. λ_n is null-homotopic for all $n \geq N$. By Lemma 6.7 we have

$$
S = \{\lambda_1, \ldots, \lambda_n\}^* + \{\lambda_{n+1}, \lambda_{n+2}, \ldots\}^* \sim \{\lambda_1, \ldots, \lambda_n\}^* + O_{x_0} \sim \{\lambda_1, \ldots, \lambda_n\}^*.
$$

By assumption we have $x_0 \simeq \lambda_1 + \ldots + \lambda_n \simeq \lambda_1 \divideontimes \ldots \divideontimes \lambda_n$. Then Theorem 6.3 asserts that there exist $[\gamma_1], \ldots, [\gamma_m] \in [\pi_1(X, x_0), \pi_1(X, x_0)]$ s.t. $\lambda_1 \divideontimes \ldots \divideontimes \lambda_n \sim \gamma_1 \divideontimes \ldots \divideontimes \gamma_m$. Since $\{\gamma_i\}^* \sim O_{x_0}$ for each $i = 1, \ldots, m$, it follows from Lemma 6.6 that $\{\lambda_1, \ldots, \lambda_n\}^* \sim \{\gamma_1 \divideontimes \ldots \divideontimes \gamma_m\}^* \sim O_{x_0}$. Thus $S \sim O_{x_0}$. The injectivity of Ψ_Φ follows. □

6.3. **A proof of Theorem 6.4 (well-definedness of Ψ_Φ).** We now work towards proving Theorem 6.4. As we shall see, we will need to deal with loops in $\mathcal{S}_\Phi(X, x_0)$ that are not necessarily based at O_{x_0}. Note that members of a continuous enumeration $(\lambda_i)_{i\in\mathbb{N}}$ of such a loop S may not necessarily be loops, and so the formal sum (36) requires a certain modification. Let us begin our discussion with the following terminology:

Definition 6.8. Let U be a path-connected neighborhood of x_0, and let λ be a path in X. A path of the form $\lambda \divideontimes \theta$ is called a *U-right-extension* of λ, if θ satisfies the following:

1. If $\lambda(1) \in U$, then θ is a path in U s.t. $\theta(0) = \lambda(1)$ and $\theta(1) = x_0$.
2. If $\lambda(1) \notin U$, then θ is the constant path assuming $\lambda(1)$.

The notion of *U-left-extension* is defined analogously. A path of the form $\theta_l \divideontimes \lambda \divideontimes \theta_r$ is called a *U-extension*, if the two paths $\theta_l \divideontimes \lambda$ and $\lambda \divideontimes \theta_r$ are a U-left-extension and U-right-extension of λ respectively.

Let us consider basic properties of U-extension:

Lemma 6.9. *Let U be a simply connected neighborhood of x_0, and let λ, μ be paths in X:*

1. *If λ', λ'' are two U-extensions of λ, then $\lambda' \sim \lambda''$.*
2. *If λ' is a U-extension of λ, then $(\lambda')^{-1}$ is a U-extension of λ^{-1}.*
3. *If λ is a loop in X and if λ' is an U-extension of λ, then $\lambda \simeq \lambda'$.*
4. *If $\lambda(1) = \mu(0)$ and if λ', μ' are U-extensions of λ, μ respectively, then $\lambda' \divideontimes \mu'$ is homotopic to a U-extension of $\lambda \divideontimes \mu$.*

Proof. Suppose $\lambda', \lambda'', \mu'$ have the forms $\lambda' = \theta'_l \divideontimes \lambda \divideontimes \theta'_r$, $\lambda'' = \theta''_l \divideontimes \lambda \divideontimes \theta''_r$, and $\mu' = \theta'''_l \divideontimes \mu \divideontimes \theta'''_r$. For the first part, observe that $\theta'_l \sim \theta''_l$ and $\theta'_r \sim \theta''_r$. It follows that

$$\lambda' = \theta'_l \divideontimes \lambda \divideontimes \theta'_r \sim \theta''_l \divideontimes \lambda \divideontimes \theta''_r = \lambda''.$$

For the second part, we have $(\lambda')^{-1} = (\theta'_r)^{-1} \divideontimes \lambda^{-1} \divideontimes (\theta'_l)^{-1}$. Evidently, $(\lambda')^{-1}$ is a U-extension of λ^{-1}. The third part follows easily from the fact that U is simply connected. For the last part, since $\lambda(1) = \mu(0)$, the path $\theta'_r \divideontimes \theta'''_l$ is either a constant path or a null-homotopic loop. Thus, $\lambda' \divideontimes \mu' \sim \theta'_l \divideontimes (\lambda \divideontimes \mu) \divideontimes \theta'''_r$, where the right hand side is a U-extension of $\lambda \divideontimes \mu$. □

Let U be a simply connected neighborhood of x_0 and let S be a path in $\mathcal{S}_\Phi(X, x_0)$ admitting a continuous enumeration $(\lambda_i)_{i\in\mathbb{N}}$. A *$U$-extension* of $(\lambda_i)_{i\in\mathbb{N}}$ is any sequence $(\lambda'_i)_{i\in\mathbb{N}}$ of paths in X, s.t. $\lambda'_1, \lambda'_2, \ldots$ are U-extensions of $\lambda_1, \lambda_2, \ldots$ respectively. As before, it follows from Remark 4.8 that all but finitely many paths in $(\lambda_i)_{i\in\mathbb{N}}$ have their images entirely included in the neighborhood U. That is, all but finitely many paths in $(\lambda'_i)_{i\in\mathbb{N}}$ are null-homotopic loops in (X, x_0). This fact allows us to consider the infinite formal sum

$$\sum_{i=1}^{\infty} \lambda'_i = \lambda'_1 + \lambda'_2 + \lambda'_3 + \ldots$$

as the 1-chain in X formed by summing up all those paths in $(\lambda'_i)_{i\in\mathbb{N}}$ that are not null-homotopic loops in (X, x_0). It is always possible to choose a large enough index $N \in \mathbb{N}$, so that for each $n > N$ the image of the path λ_n is in U. In this case, $\sum_{i=1}^{\infty} \lambda'_i \simeq \lambda'_1 + \ldots + \lambda'_N$.

Lemma 6.10. *Let U be a simply connected neighborhood of x_0, and let S be a path in $\mathcal{S}_\Phi(X, x_0)$ admitting a continuous enumeration $(\lambda_i)_{i\in\mathbb{N}}$. Let $(\lambda'_i)_{i\in\mathbb{N}}, (\lambda''_i)_{i\in\mathbb{N}}$ be two U-extension of $(\lambda_i)_{i\in\mathbb{N}}$. Then:*

1. $\sum_{i=1}^{\infty} \lambda'_i \simeq \sum_{i=1}^{\infty} \lambda''_i$.
2. $\sum_{i=1}^{\infty} (\lambda'_i)^{-1} \simeq -\sum_{i=1}^{\infty} \lambda'_i$.
3. $\sum_{i=1}^{\infty} \lambda'_i = \sum_{i=1}^{\infty} \lambda'_{\pi_i}$ *for any permutation π.*
4. *If S is a loop in $\mathcal{S}_\Phi(X, x_0)$, then $\sum_{i=1}^{\infty} \lambda'_i$ is a 1-cycle in X.*

Proof. The first and second parts follow from Lemma 6.9. The third part is obvious. For the last part, suppose that S is a loop and that N is a large enough index s.t. for all $n > N$ the path λ'_n is a null-homotopic loop in (X, x_0). Since $S(0) = S(1)$, it is easy to see that the two sequences $(\lambda'_i(0))_{i\in\mathbb{N}}, (\lambda'_i(1))_{i\in\mathbb{N}}$ are identical up to a permutation. It follows that the two finite sequences $(\lambda'_i(0))_{i=1}^N, (\lambda'_i(1))_{i=1}^N$ are also identical up to a permutation. Now, since $\sum_{i=1}^{\infty} \lambda'_i \simeq \lambda'_1 + \ldots + \lambda'_N$, the sum $\sum_{i=1}^{\infty} \lambda'_i$ is a 1-cycle. □

This Lemma 6.10 allows us to introduce the following notation:

Definition 6.11. Let U be a simply connected neighborhood of x_0, and let S be a loop in $\mathcal{S}_\Phi(X, x_0)$ admitting a continuous enumeration $(\lambda_i)_{i\in\mathbb{N}}$. Then we define

$$\Psi_\Phi^U((\lambda_i)_{i\in\mathbb{N}}) := \left[\sum_{i=1}^{\infty} \lambda_i'\right]_{H_1},$$

where $(\lambda_i')_{i\in\mathbb{N}}$ is any U-extension of $(\lambda_i)_{i\in\mathbb{N}}$.

We are now in a position to state and prove the following generalisation of Theorem 6.4:

Theorem 6.12. *Let U be a simply connected neighborhood of x_0, and let $S_0 \in \mathcal{S}_\Phi(X, x_0)$ be fixed. If S, T are two homotopic loops in $(\mathcal{S}_\Phi(X, x_0), S_0)$ and if $(\lambda_i)_{i\in\mathbb{N}}, (\mu_i)_{i\in\mathbb{N}}$ are continuous enumerations of S, T respectively, then $\Psi_\Phi^U((\lambda_i)_{i\in\mathbb{N}}) = \Psi_\Phi^U((\mu_i)_{i\in\mathbb{N}})$.*

Evidently, Theorem 6.4 is an immediate corollary. Theorem 6.12 will be proved with the aid of the following two lemmas:

Lemma 6.13. *Let U be a simply connected neighborhood of x_0. Let $S_1, \ldots, S_n$ be paths in $\mathcal{S}_\Phi(X, x_0)$ s.t. $S_1 \circledast \ldots \circledast S_n$ is a loop. Suppose that $S_1, \ldots, S_n$ admit continuous enumerations $(\lambda_i^1)_{i\in\mathbb{N}}, \ldots, (\lambda_i^n)_{i\in\mathbb{N}}$ respectively, and that $\lambda_i^j(1) = \lambda_i^{j+1}(0)$ for each $i \in \mathbb{N}$ and each $j = 1, \ldots, n-1$. Let $(\lambda_i^{1'})_{i\in\mathbb{N}}, \ldots, (\lambda_i^{n'})_{i\in\mathbb{N}}$ be U-extensions of $(\lambda_i^1)_{i\in\mathbb{N}}, \ldots, (\lambda_i^n)_{i\in\mathbb{N}}$ respectively. Then $\sum_{i=1}^\infty \lambda_i^{1'} + \ldots + \sum_{i=1}^\infty \lambda_i^{n'}$ is a 1-cycle, and*

$$\Psi_\Phi^U((\lambda_i^1 \circledast \ldots \circledast \lambda_i^n)_{i\in\mathbb{N}}) = \left[\sum_{i=1}^{\infty} \lambda_i^{1'} + \ldots + \sum_{i=1}^{\infty} \lambda_i^{n'}\right]_{H_1}.$$

Note that the left hand side of the above expression makes sense, because $(\lambda_i^1 \circledast \ldots \circledast \lambda_i^n)_{i\in\mathbb{N}}$ is a continuous enumeration of the loop $S_1 \circledast \ldots \circledast S_n$.

Proof. For notational simplicity, we consider the case $n = 2$. Let N be a large enough index s.t. for all $n > N$ the two paths $\lambda_n^{1'}, \lambda_n^{2'}$ are both null-homotopic loops in (X, x_0). It follows that $\lambda_n^{1'} \circledast \lambda_n^{2'}$ is a null-homotopic loop in (X, x_0) for each $n > N$. Note also that each path $\lambda_i^{1'} \circledast \lambda_i^{2'}$ is homotopic to a U-extension of $\lambda_i^1 \circledast \lambda_i^2$ as in the last part of Lemma 6.9. That is, $\lambda_1^{1'} \circledast \lambda_1^{2'} + \ldots + \lambda_N^{1'} \circledast \lambda_N^{2'}$ is a 1-cycle, and

$$\Psi_\Phi^U((\lambda_i^1 \circledast \lambda_i^2)_{i\in\mathbb{N}}) = \left[\lambda_1^{1'} \circledast \lambda_1^{2'} + \ldots + \lambda_N^{1'} \circledast \lambda_N^{2'}\right]_{H_1}.$$

The claim follows by

$$\begin{aligned}
\lambda_1^{1'} \circledast \lambda_1^{2'} + \ldots + \lambda_N^{1'} \circledast \lambda_N^{2'} &\simeq \lambda_1^{1'} + \lambda_1^{2'} + \ldots + \lambda_N^{1'} + \lambda_N^{2'} \\
&\simeq (\lambda_1^{1'} + \ldots + \lambda_N^{1'}) + (\lambda_1^{2'} + \ldots + \lambda_N^{2'}) \\
&\simeq \sum_{i=1}^{\infty} \lambda_i^{1'} + \sum_{i=1}^{\infty} \lambda_i^{2'}.
\end{aligned}$$

□

Lemma 6.14. *Let U be a simply connected neighborhood of x_0, and let H be a continuous $\mathcal{S}_\Phi(X, x_0)$-valued mapping on a metric space I. Then for each $r \in I$ there exists a neighborhood $N(r)$ of r, s.t. for any loop γ in $N(r)$ and for any continuous enumeration $(\lambda_i)_{i\in\mathbb{N}}$ of the loop $H \circ \gamma$, we have $\Psi_\Phi^U((\lambda_i)_{i\in\mathbb{N}}) = 0$.*

Proof. Let $r \in I$ be fixed. We can choose an open ball $U_0 := B_\epsilon(x_0)$, s.t. $U_0 \subseteq U$ and for all $x \in \operatorname{supp} H(r)$ we have $d(x_0, x) \neq \epsilon$. Suppose that $\operatorname{supp}(H(r) \setminus U_0) = \{x_0, \ldots, x_n\}$, where $x_0, \ldots, x_n$ are distinct points in X. We can then choose neighborhoods $U'_1, \ldots, U'_n$ of $x_1, \ldots, x_n$ respectively, s.t. $(U_0, \ldots, U'_n)$ is positively separated. Since X is locally simply connected, we can choose simply connected neighborhoods $U_1, \ldots, U_n$ of $x_1, \ldots, x_n$ respectively, s.t. $U_i \subseteq U'_i$ for each $i = 1, \ldots, n$. By construction, $(U_0, \ldots, U_n)$ is also positively separated. It follows from Theorem 4.4 that there exists a neighborhood $N(r)$ of r, s.t. $H \cap U_0, \ldots, H \cap U_n$ are continuous and the following mapping is constant:

$$N(r) \ni t \longmapsto (\operatorname{rank}(H(t) \cap U_1), \ldots, \operatorname{rank}(H(t) \cap U_n)) \in \mathbb{Z}^n.$$

Suppose that γ is a loop in $N(r)$ and that $(\lambda_i)_{i\in\mathbb{N}}$ is a continuous enumeration of the loop $S_\gamma := H \circ \gamma$. By construction, S_γ admits a representation

$$S_\gamma = S_\gamma \cap U_0 + \ldots + S_\gamma \cap U_n.$$

It follows that the image of each path λ_i is included entirely in one of $U_0, \ldots, U_n$, since $(U_0, \ldots, U_n)$ was chosen to be positively-separated. Without loss of generality, we may assume that $\lambda_1, \ldots, \lambda_N$ are all those paths in $(\lambda_i)_{i\in\mathbb{N}}$ whose images are not in U_0. That is, $S_\gamma \setminus U_0 = \{\lambda_1, \ldots, \lambda_N\}^*$. We can then relabel $\lambda_1, \ldots, \lambda_N$ into $\lambda^1_1, \ldots, \lambda^1_{N_1}, \ldots, \lambda^n_1, \ldots, \lambda^n_{N_n}$, in such a way that for each $j = 1, \ldots, n$, we have $S_\gamma \cap U_j = \{\lambda^j_1, \ldots, \lambda^j_{N_j}\}^*$. Note that each sum $\lambda^j_1 + \ldots + \lambda^j_{N_j}$ is a 1-cycle, as the path $S \cap U_j$ is a loop. Since $U_1, \ldots, U_n$ are simply connected,

$$\lambda^j_1 + \ldots + \lambda^j_{N_j} \simeq x_0 \qquad \forall j = 1, \ldots, n.$$

Let $(\lambda'_i)_{i\in\mathbb{N}}$ be an U-extension of $(\lambda_i)_{i\in\mathbb{N}}$. Given each path λ^j_i, we let $\lambda^{j'}_i$ be the corresponding U-extension of λ^j_i taken from $(\lambda'_i)^N_{i=1}$. Since U is simply connected, we have $\lambda^{j'}_1 + \ldots + \lambda^{j'}_{N_j} \simeq \lambda^j_1 + \ldots + \lambda^j_{N_j}$ for each $j = 1, \ldots, n$. Now,

$$\begin{aligned}\Psi^U_p((\lambda_i)_{i\in\mathbb{N}}) &= [\lambda'_1 + \ldots + \lambda'_N]_{H_1} \\ &= [\lambda^{1'}_1 + \ldots + \lambda^{1'}_{N_1}]_{H_1} + \ldots + [\lambda^{n'}_1 + \ldots + \lambda^{n'}_{N_n}]_{H_1} \\ &= 0.\end{aligned}$$

□

The following proof is seemingly standard. See, for example, [Phi, Proposition 3].

Proof of Theorem 6.12. (A) Let U be a simply connected neighborhood of x_0, and let $S_0 \in \mathcal{S}_\Phi(X, x_0)$ be fixed. Let S, T be homotopic loops in $(\mathcal{S}_\Phi(X, x_0), S_0)$, and let H be a homotopy from S to T. Then the square $[0,1] \times [0,1]$ has an open cover $\mathcal{N} := \{N(r) \mid r \in [0,1] \times [0,1]\}$, where $N(r)$ is a neighborhood of r having the property specified in Lemma 6.14. Let $L > 0$ be a Lebesgue number of the open cover $\mathcal{N}$. Select a large enough positive number n satisfying $\frac{1}{n} < \frac{L}{\sqrt{2}}$, and form n closed intervals

$$I_1 := \left[0, \frac{1}{n}\right], I_2 := \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, I_n := \left[\frac{n-1}{n}, 1\right].$$

This allows us to partition the square $[0,1] \times [0,1]$ into a grid of $n \times n$ squares, $\{I_i \times I_j\}_{1 \le i,j \le n}$. By the choice of n, each square $I_i \times I_j$ has the diameter $< L$, and so each square $I_i \times I_j$ is entirely included in some member of the open cover $\mathcal{N}$.

(B) Let us first consider the n squares

$$R_1 := I_1 \times I_1,\ R_2 := I_2 \times I_1,\ \ldots,\ R_n := I_n \times I_1.$$

Let $a_1, \ldots, a_n, b_1, \ldots, b_n, c_0, \ldots, c_n$ be the paths in $[0,1] \times I_1$ as shown in the figure below:

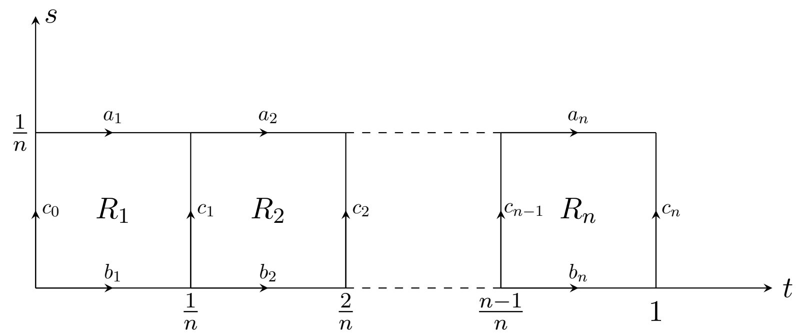

FIGURE 1. The boundaries of the squares are traced as above.

We will assume that all of the paths shown above trace the edges of the squares $R_1, \ldots, R_n$ at a "constant speed". Let us introduce some terminology:

1. Sets of the form $R_{i,j} := R_i \cup R_{i+1} \cup \ldots \cup R_{j-1} \cup R_j$ are referred to as *rectangles*.
2. The *boundary loop* of the rectangle $R_{i,j}$, denoted by $\Gamma_{i,j}$, is defined to be

$$\Gamma_{i,j} := (a_i \divideontimes a_{i+1} \divideontimes \ldots \divideontimes a_j) \divideontimes c_j^{-1} \divideontimes (b_i \divideontimes b_{i+1} \divideontimes \ldots \divideontimes b_j)^{-1} \divideontimes c_{i-1}.$$

3. The rectangle $R_{i,j}$ is said to have *Property* $\mathcal{Y}$, if for any continuous enumeration $(\lambda_i)_{i\in\mathbb{N}}$ of the loop $H \circ \Gamma_{i,j}$ we have $\Psi_\Phi^U((\lambda_i)_{i\in\mathbb{N}}) = 0$.

Note that each square $R_i = R_{i,i}$ has Property $\mathcal{Y}$, as it is included entirely in some member of the open cover $\mathcal{N}$ as in (A).

(C) We will show that if two rectangles $R_{i,j}, R_{j+1,k}$ have Property $\mathcal{Y}$, then so does their union $R_{i,k} = R_{i,j} \cup R_{j+1,k}$. For notational simplicity, we will consider the special case $R_{i,j} = R_1$ and $R_{j+1,k} = R_2$. Let $(\lambda_i)_{i\in\mathbb{N}}$ be an arbitrary continuous enumeration of the loop

$$H \circ \Gamma_{1,2} = (H \circ a_1) \divideontimes (H \circ a_2) \divideontimes (H \circ c_2^{-1}) \divideontimes (H \circ b_2^{-1}) \divideontimes (H \circ b_1^{-1}) \divideontimes (H \circ c_0).$$

We can then choose continuous enumerations $(\lambda_i^1)_{i\in\mathbb{N}}, \ldots, (\lambda_i^6)_{i\in\mathbb{N}}$ of the 6 paths $H \circ a_1$, $H \circ a_2$, $H \circ c_2^{-1}$,$H \circ b_2^{-1}$, $H \circ b_1^{-1}$, $H \circ c_0$ respectively, s.t. for each $i \in \mathbb{N}$ we have $\lambda_i = \lambda_i^1 \divideontimes \ldots \divideontimes \lambda_i^6$. Indeed, each λ_i^j is merely a "reparametrisation" of the restriction $\lambda_i|_{[\frac{j-1}{6}, \frac{j}{6}]}$[6]. Let $(\theta_i)_{i\in\mathbb{N}}$ be a continuous enumeration of $H \circ c_1$, and let $(\theta_i')_{i\in\mathbb{N}}$ be an U-extensions of (θ_i). Then $((\theta_i')^{-1})_{i\in\mathbb{N}}$

[6] More precisely, we define each path $\lambda_i^j : [0,1] \to X$ by

$$\lambda_i^j(t) := \lambda_i\left(\frac{(j-1)(1-t)}{6} + \frac{jt}{6}\right).$$

is a U-extension of $(\theta_i^{-1})_{i\in\mathbb{N}}$ by Lemma 6.9. By Lemma 6.13, we obtain

$$\begin{aligned}
\Psi_\Phi^U((\lambda_i)_{i\in\mathbb{N}}) &= \Psi_\Phi^U((\lambda_i^1 \circledast \ldots \circledast \lambda_i^6)_{i\in\mathbb{N}}) + 0 \\
&= \left[\sum_{i=1}^\infty \lambda_i^{1'} + \ldots + \sum_{i=1}^\infty \lambda_i^{6'}\right]_{H_1} + \left[\sum_{i=1}^\infty \theta_i' + \sum_{i=1}^\infty (\theta_i')^{-1}\right]_{H_1} \\
&= \left[\sum_{i=1}^\infty \lambda_i^{1'} + \sum_{i=1}^\infty (\theta_i')^{-1} + \sum_{i=1}^\infty \lambda_i^{5'} + \sum_{i=1}^\infty \lambda_i^{6'}\right]_{H_1} + \left[\sum_{i=1}^\infty \lambda_i^{2'} + \sum_{i=1}^\infty \lambda_i^{3'} + \sum_{i=1}^\infty \lambda_i^{4'} + \sum_{i=1}^\infty \theta_i'\right]_{H_1} \\
&= 0 + 0 = 0,
\end{aligned}$$

where the fourth equality follows from the fact that $(\lambda_i^{1'})_{i\in\mathbb{N}}, ((\theta_i')^{-1})_{i\in\mathbb{N}}, (\lambda_i^{5'})_{i\in\mathbb{N}}, (\lambda_i^{6'})_{i\in\mathbb{N}}$ and $(\lambda_i^{2'})_{i\in\mathbb{N}}, (\lambda_i^{3'})_{i\in\mathbb{N}}, (\lambda_i^{4'})_{i\in\mathbb{N}}, (\theta_i')_{i\in\mathbb{N}}$ can be renumbered to form continuous enumerations of $H \circ \Gamma_{1,1}, H \circ \Gamma_{1,2}$ respectively. That is, we have shown that if two rectangles $R_{i,j}, R_{j+1,k}$ have Property $\mathcal{Y}$, then so does $R_{i,k}$. Since each square has Property $\mathcal{Y}$, it follows that the rectangle $[0,1] \times I_1$ has Property $\mathcal{Y}$.

(D) Let $(\lambda_i)_{i\in\mathbb{N}}, (\mu_i)_{i\in\mathbb{N}}$ be continuous enumerations of $H(\cdot, 0), H(\cdot, \frac{1}{n})$ respectively. We show that $\Psi_\Phi^U((\lambda_i)_{i\in\mathbb{N}}) = \Psi_\Phi^U((\mu_i)_{i\in\mathbb{N}})$. Suppose that $S_0 = \{s_1, s_2, \ldots\}^*$ and that $\theta_1, \theta_2, \ldots$ are paths in X taking the constant values $s_1, s_2, \ldots$ respectively. Evidently, $\Psi_\Phi^U((\theta_i)_{i\in\mathbb{N}}) = 0$. If $(\lambda_i')_{i\in\mathbb{N}}, (\mu_i')_{i\in\mathbb{N}}, (\theta_i')_{i\in\mathbb{N}}$ are U-extensions of $(\lambda_i)_{i\in\mathbb{N}}, (\mu_i)_{i\in\mathbb{N}}, (\theta_i)_{i\in\mathbb{N}}$ respectively, then

$$\begin{aligned}
\Psi_\Phi^U((\lambda_i)_{i\in\mathbb{N}}) - \Psi_p^U((\mu_i)_{i\in\mathbb{N}}) &= \left[\sum_{i=1}^\infty \lambda_i' - \sum_{i=1}^\infty \mu_i'\right]_{H_1} \\
&= \left[\sum_{i=1}^\infty \lambda_i' + \sum_{i=1}^\infty \theta_i' + \sum_{i=1}^\infty (\mu_i')^{-1} + \sum_{i=1}^\infty \theta_i'\right]_{H_1} = 0,
\end{aligned}$$

where the last equality follows from the fact that $(\lambda_i)_{i\in\mathbb{N}}, (\theta_i)_{i\in\mathbb{N}}, (\mu_i^{-1})_{i\in\mathbb{N}}, (\theta_i)_{i\in\mathbb{N}}$ can be renumbered to form a continuous enumeration of the loop $H \circ \Gamma_{1,n}$. That is, $\Psi_\Phi^U((\lambda_i)_{i\in\mathbb{N}}) = \Psi_\Phi^U((\mu_i)_{i\in\mathbb{N}})$. The claim follows by applying the same argument to all of $H|_{[0,1]\times I_1}, \ldots, H|_{[0,1]\times I_n}$ finitely many times. ☐

Therefore, we obtain Theorem 6.4 as special case of this result.

7. Infinite-dimensional Analogues of T. Kato's Continuous Enumeration

Notation. Throughout this section we will assume the following:

1. Let $\mathcal{H}$ be a separable Hilbert space.
2. Let Φ be a symmetric norm.

The purpose of the current section is to give certain infinite-dimensional analogues of Kato's finite-dimensional continuous enumeration (Theorem 1.3). We will first discuss one preliminary concept, the notion of factor metric space.

7.1. **Factoring metric spaces by compact subsets.** Given a topological space $\mathbb{X}$ with a subset K, we denote by $\mathbb{X}/K$ the topological quotient space formed by the equivalence relation which identifies all points in K and leaves other points as they are. Equivalence classes in $\mathbb{X}/K$ shall be denoted by $[\,\cdot\,]_K$.

Theorem 7.1. *If $(\mathbb{X}, d)$ is a metric space having a compact subset K, then the factor space $\mathbb{X}/K$ is a metrizable topological space whose topology is given by the metric*

$$\operatorname{dist}([x]_K, [y]_K) := \min\left\{d(x,y),\ \inf_{k\in K} d(x,k) + \inf_{k\in K} d(k,y)\right\}, \tag{38}$$

Furthermore, if $\mathbb{X}$ is a path-connected, separable, complete metric space, then so is $\mathbb{X}/K$.

A proof of this theorem turns out to be technical, and so we will discuss it at the end of the current section. Given a metric space $\mathbb{X}$ and a compact subset K, the quotient space $\mathbb{X}/K$ contains the equivalence class $\mathcal{K}$ represented by points of K. Since $\mathbb{X}/K$ can be viewed as a metric space with a fixed point K, we may consider the multiset space

$$\mathcal{S}_\Phi(\mathbb{X}, K) := \mathcal{S}_\Phi(\mathbb{X}/K, \mathcal{K}).$$

If K happens to be a point-set $\{x_0\}$ for some $x_0 \in X$, then there is a canonical identification between $\mathcal{S}_\Phi(\mathbb{X}, \{x_0\})$ and $\mathcal{S}_\Phi(\mathbb{X}, x_0)$, as $\mathbb{X}/\{x_0\}$ and $\mathbb{X}$ are naturally isometric.

Lemma 7.2. *Let $\mathbb{X}$ be either $\mathbb{T}$ or $\mathbb{R}$, and let K be a compact subset of $\mathbb{X}$. If λ' is a simple continuous path in $(\mathbb{X}/K, \mathcal{K})$, then there exists a continuous path λ in $\mathbb{X}$ satisfying $\lambda'(\cdot) = [\lambda(\cdot)]_K$ with the property that λ assumes some constant value which is a boundary point of K on each connected component of $[0,1] \setminus \operatorname{supp} \lambda'$.*

Before taking up a proof, let us observe that this result does not seem to hold if $\mathbb{X} = \mathbb{C}$. Indeed, a continuous path in $\mathbb{C}/K$ can be "absorbed into the compact set K with increasing frequency". Such a path can easily be constructed using a topologist's sine curve for example.

Proof. Suppose $\operatorname{supp} \lambda' = (0,1]$ for notational simplicity. It follows that for each $t \in (0,1]$ there exists a unique point, denoted by $\lambda(t) \in \mathbb{X}$, satisfying $\lambda'(t) = [\lambda(t)]_K$. This uniquely defines a mapping $\lambda : (0,1] \to \mathbb{X}$ whose continuity follows from that of λ'. Since λ' is continuous at $t = 0$, we have

$$\lim_{t\to 0^+} \operatorname{dist}(\lambda'(t), \lambda'(0)) = \lim_{t\to 0^+} \operatorname{dist}(\lambda'(t), \mathcal{K}) = \lim_{t\to 0^+} \inf_{k\in K} d(\lambda(t), k) = 0.$$

Since $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = \mathbb{T}$, there exists a unique boundary point $k_0 \in \partial K$ s.t. $\inf_{k\in K} d(\lambda(t), k) = d(\lambda(t), k_0)$ for any t sufficiently close to 0. It follows that

$$\lim_{t\to 0^+} d(\lambda(t), k_0) = 0.$$

Setting $\lambda(0) := k_0$ continuously extends the domain of λ to the whole interval $[0,1]$. Evidently, $\lambda(\cdot) = [\lambda(\cdot)]_K$ still holds true on the whole interval $[0,1]$, and so the proof is complete. □

7.2. **A unitary analogue of Kato's continuous enumeration.** Given a fixed unitary operator U_0 on the separable Hilbert space $\mathcal{H}$, we define the set $\mathcal{U}_\Phi(\mathcal{H}, U_0)$ to be the collection of all unitary operators U on $\mathcal{H}$ with $U - U_0 \in \mathfrak{S}_\Phi(\mathcal{H})$. The collection $\mathcal{U}_\Phi(\mathcal{H}, U_0)$ forms a complete metric space with the metric

$$\operatorname{dist}(U, U') := \|U - U'\|_{\mathfrak{S}_\Phi} \qquad \forall U, U' \in \mathcal{U}_\Phi(\mathcal{H}, U_0).$$

We will make use of the following assumption throughout:

Assumption 7.3. *The unitary operator U_0 has the property that $\mathcal{U}_\Phi(\mathcal{H}, U_0)$ contains at least one unitary operator whose discrete spectrum is empty.*

Let us assume that the fixed unitary operator U_0 satisfies Assumption 7.3. In this case, the essential spectrum $K := \sigma_{\mathrm{ess}}(U_0)$ is nonempty. The spectrum of each unitary operator $U \in \mathfrak{U}_\Phi(\mathcal{H}, U_0)$ can then be identified with the following countable multiset in $(\mathbb{T}/K, \mathcal{K})$:

$$\sigma(U) := \{[z_1]_K, [z_2]_K, \dots\}^*, \tag{39}$$

where $(z_i)_{i\in\mathbb{N}}$ is any extended enumeration of $\sigma_{\mathrm{dis}}(U)$ in the sense of Definition 1.4. The following theorem holds true:

Theorem 7.4. *If U_0 is a unitary operator on the separable Hilbert space $\mathcal{H}$ satisfying Assumption 7.3, then the following mapping is a well-defined $\pi/2$-Lipschitz continuous mapping:*

$$\sigma : \mathfrak{U}_\Phi(\mathcal{H}, U_0) \ni U \longmapsto \sigma(U) \in \mathcal{S}_\Phi(\mathbb{T}, \sigma_{\mathrm{ess}}(U_0)).$$

Proof. Let $K := \sigma_{\mathrm{ess}}(U_0)$, and let d be the metric on $\mathbb{T}/K$ given by (38). We need to show

$$d_\Phi(\sigma(U), \sigma(U')) \leq \frac{\pi}{2}\|U - U'\|_{\mathfrak{S}_\Phi} \quad \forall U, U' \in \mathfrak{U}_\Phi(\mathcal{H}, U_0). \tag{40}$$

Let us first derive estimate (40). Theorem 1.7 asserts that there exist extended enumerations $(\lambda_i)_{i\in\mathbb{N}}, (\mu_i)_{i\in\mathbb{N}}$ of the discrete spectra of U, U' respectively, s.t.

$$\Phi(|\lambda_1 - \mu_1|, |\lambda_2 - \mu_2|, \dots) \leq \frac{\pi}{2}\|U - U'\|_{\mathfrak{S}_\Phi}.$$

Since $(\lambda_i)_{i\in\mathbb{N}}, (\mu_i)_{i\in\mathbb{N}}$ are enumerations of the multisets $\sigma(U), \sigma(U')$ respectively, we have

$$d_\Phi(\sigma(U), \sigma(U')) \leq \Phi(d(\lambda_1, \mu_1), d(\lambda_2, \mu_2), \dots) \leq \Phi(|\lambda_1 - \mu_1|, |\lambda_2 - \mu_2|, \dots) \leq \frac{\pi}{2}\|U - U'\|_{\mathfrak{S}_\Phi},$$

thereby establishing estimate (40). Since U_0 satisfies Assumption 7.3, there exists a unitary operator U' whose spectrum, viewed as the multiset, is $\mathcal{K}$. It follows that for each $U \in \mathfrak{U}_\Phi(\mathcal{H}, U_0)$, we have $\sigma(U) \in \mathcal{S}_\Phi(\mathbb{T}, K)$. It follows that $\sigma : \mathfrak{U}_\Phi(\mathcal{H}, U_0) \to \mathcal{S}_\Phi(\mathbb{T}, K)$ is a well-defined $\pi/2$-Lipschitz continuous mapping. The proof is now complete. □

We are now in a position to give the following unitary analogue of Kato's finite-dimensional continuous enumeration.

Theorem 7.5. *Let U_0 be a fixed unitary operator on the separable Hilbert space $\mathcal{H}$ satisfying Assumption 7.3, and let $K := \sigma_{\mathrm{ess}}(U_0)$. If U is a continuous path in $\mathfrak{U}_\Phi(\mathcal{H}, U_0)$, then there exists a sequence $(\lambda_i)_{i\in\mathbb{N}}$ of continuous paths in $\mathbb{T}$, s.t.*

1. $\sigma(U(\cdot)) = \{[\lambda_1(\cdot)]_K, [\lambda_2(\cdot)]_K, \dots\}^*$.
2. $(\lambda_i(\cdot))_{i\in\mathbb{N}}$ *is an extended enumeration of* $\sigma_{\mathrm{dis}}(U(\cdot))$ *pointwise.*

Proof. This is an immediate consequence of Theorem 5.5, Lemma 7.2, and Theorems 7.4. □

7.3. **A self-adjoint analogue of Kato's continuous enumeration.** Given a fixed self-adjoint operator H_0 on $\mathcal{H}$, we define the set $\mathcal{H}_\Phi(\mathcal{H}, H_0)$ to be the collection of all self-adjoint operators H on $\mathcal{H}$ with $H - H_0 \in \mathfrak{S}_\Phi(\mathcal{H})$. The collection $\mathcal{H}_\Phi(\mathcal{H}, H_0)$ forms a complete metric space with the metric

$$\mathrm{dist}\,(H, H') := \|H - H'\|_{\mathfrak{S}_\Phi} \qquad \forall H, H' \in \mathcal{H}_\Phi(\mathcal{H}, H_0).$$

Assumption 7.6. *The self-adjoint operator H_0 has the property that $\mathcal{H}_\Phi(\mathcal{H}, H_0)$ contains at least one self-adjoint operator whose discrete spectrum is empty.*

As before, we identify the spectrum of any self-adjoint operator $H \in \mathcal{H}_\Phi(\mathcal{H}, H_0)$ to be the multiset in the quotient space $\mathbb{R}/\sigma_{\mathrm{ess}}(H_0)$. Since the following two theorems require the obvious modifications, we omit their proofs:

Theorem 7.7. *If H_0 is a fixed self-adjoint operator on the separable Hilbert space $\mathcal{H}$ satisfying Assumption 7.6, then the following mapping is a well-defined* 1*-Lipschitz continuous mapping:*

$$\sigma : \mathcal{H}_\Phi(\mathcal{H}, H_0) \ni H \longmapsto \sigma(H) \in \mathcal{S}_\Phi(\mathbb{R}, \sigma_{\mathrm{ess}}(H_0)).$$

Theorem 7.8. *Let H_0 be a fixed self-adjoint operator on the separable Hilbert space $\mathcal{H}$ satisfying Assumption 7.6, and let $K := \sigma_{\mathrm{ess}}(H_0)$. If H is a continuous path in $\mathcal{H}_\Phi(\mathcal{H}, H_0)$, then there exists a sequence $(\lambda_i)_{i\in\mathbb{N}}$ of continuous paths in $\mathbb{R}$, s.t.*

1. $\sigma(H(\cdot)) = \{[\lambda_1(\cdot)]_K, [\lambda_2(\cdot)]_K, \dots\}^*$.
2. $(\lambda_i(\cdot))_{i\in\mathbb{N}}$ *is an extended enumeration of* $\sigma_{\mathrm{dis}}(H(\cdot))$ *pointwise.*

7.4. **A proof of Theorem 7.1 (metrizability of quotient space).** Before taking up the proof, we recall the following well-known result first. If $(\mathbb{X}, d)$ is a metric space and if $\sim$ is any equivalence class on $\mathbb{X}$, then the quotient set $\mathbb{X}/\sim = \{[x] \mid x \in X\}$ admits a pseudo-metric

$$d_\sim([x], [y]) := \inf \sum_{i=1}^{n} d(p_i, q_i),$$

where the infimum is taken over all pairs of finite sequences $(p_i)_{i=1}^n, (q_i)_{i=1}^n$ of points in X with the property that $[p_1] = [x], [q_n] = [y], [q_i] = [p_{i+1}]$ for each $i = 1, \dots, n-1$. See [BBI, Section 3.1.2] for details.

Proof of the first part of Theorem 7.1. Let d_K be the metric (38), and let $\sim$ be the equivalence class used to form the factor space $\mathbb{X}/K$. We will first prove first

$$d_\sim([x]_K, [y]_K) = d_K([x]_K, [y]_K) \qquad \forall x, y \in X. \tag{41}$$

Firstly, $d_\sim \le d_K$ follows from the following two obvious inequalities:

$$d_\sim([x]_K, [y]_K) \le d(x, y) \text{ and } d_\sim([x]_K, [y]_K) \le \inf_{k\in K} d(x, k) + \inf_{k\in K} d(k, y).$$

To prove $d_K \le d_\sim$, we let $(p_1, \dots, p_n), (q_1, \dots, q_n)$ be an arbitrary pair of finite sequences of points in X with the property that $p_1 \sim x, q_n \sim y$ and $q_i \sim p_{i+1}$ for each $i = 1, \dots, n-1$. Note that $q_i \sim p_{i+1}$ happens if and only if either q_i, p_{i+1} both belong to K or $q_i = p_{i+1}$. In the latter case, we have $d(p_i, q_{i+1}) \le d(p_i, q_i) + d(p_{i+1}, q_{i+1})$ by triangle inequality. That is, without loss of generality, we may assume that for each $i = 1, \dots, n-1$, the points p_i, q_{i+1} both belong to K. This leads to

$$\sum_{i=1}^{n} d(p_i, q_i) \ge d_K([x]_K, [y]_K)$$

Taking the infimum over $(p_i)_{i=1}^n, (q_i)_{i=1}^n$ gives $d_K \le d_\sim$. It follows that d_K is a pseudo-metric. Note also that since K is a closed set, d_K is non-degenerate. That is, d_K is a genuine metric on the quotient set $\mathbb{X}/\sim$.

It remains to show that the quotient topology τ_q on $\mathbb{X}/\sim$ agrees with the metric topology τ_m induced by the metric d_K. Recall that τ_q is defined to be the finest topology, s.t. the quotient map $q : \mathbb{X} \to \mathbb{X}/\sim$ is continuous. Since q is continuous with respect to τ_m, we have $\tau_m \subseteq \tau_q$. It remains to prove $\tau_q \subseteq \tau_m$. Let $U \in \tau_q$, and let $[x]_K \in U$. Since $q^{-1}(U)$ is a neighborhood of x, there exists a small enough $\epsilon > 0$ s.t. $B_\epsilon(x) \subseteq q^{-1}(U)$. Let $B^q_\epsilon([x]_K)$ be the open ϵ-ball centred at $[x]_K$ with respect to d_K. We show that by shrinking ϵ appropriately $B^q_\epsilon([x]_K) \subseteq U$ holds. We will consider the following two cases separately:

1. Suppose $x \notin K$. Shrink ϵ further, if necessary, to make sure that $\epsilon < \inf_{k\in K} d(x,k)$ holds. This implies $B_\epsilon(x) \cap K = \emptyset$. If $[y]_K \in B_\epsilon^q([x]_K)$, then

$$\begin{aligned}\epsilon &> d_K([x]_K, [y]_K)\\ &= \min\left\{d(x,y),\ \inf_{k\in K} d(x,k) + \inf_{k\in K} d(k,y)\right\}\\ &> \min\left\{d(x,y),\ \epsilon + \inf_{k\in K} d(k,y)\right\} = d(x,y).\end{aligned}$$

   It follows that $y \in B_\epsilon(x)$, and so $B_\epsilon^q([x]_K) \subseteq q(B_\epsilon(x)) \subseteq q(q^{-1}(U)) \subseteq U$.
2. Suppose $x \in K$ now. In this case, $K \subseteq q^{-1}(U)$. Set $F := X \setminus q^{-1}(U)$. Since K, F are disjoint, $R := \operatorname{dist}(K,F) > 0$. We shrink ϵ so that $\epsilon < R$ holds. If $[y]_K \in B_\epsilon^q([x]_K)$, then

$$d_K([x]_K, [y]_K) = \inf_{k\in K} d(k,y) < \operatorname{dist}(K,F) = \inf_{(k,f)\in K\times F} d(k,f).$$

   This immediately implies $y \in q^{-1}(U)$. It follows that $B_\epsilon^q([x]_K) \subseteq U$.

□

Proof of the second part of Theorem 7.1. Let $\mathbb{X}$ be a path-connected, separable, complete metric space. The path-connectedness and separability of the quotient $\mathbb{X}/K$ easily follows from that of $\mathbb{X}$. For the completeness, let d_K be the metric (38), and let $([x_n]_K)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{X} \setminus K$. We consider the case where the sequence $([x_n]_K)_{n\in\mathbb{N}}$ does not converge to the equivalence class $\mathcal{K}$. In this case, we can choose an $\epsilon_0 > 0$ and a subsequence $([x'_n]_K)_{n\in\mathbb{N}}$ with $d_K([x'_n]_K, K) \geq \epsilon_0$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$ be arbitrary. Without loss of generality, we may assume that $\epsilon < \epsilon_0$. Then there exists an index N s.t. for all $m, n \geq \mathbb{N}$

$$\begin{aligned}\epsilon_0 &> d_K([x'_m]_K, [x'_n]_K)\\ &= \min\left\{d(x'_m, x'_n),\ \inf_{k\in K} d(x'_m,k) + \inf_{k\in K} d(x'_n,k)\right\}\\ &\geq \min\left\{d(x'_m, x'_n),\ 2\epsilon_0\right\} = d(x'_m, x'_n).\end{aligned}$$

It follows that $(x'_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and so it has a limit $x_0 \in \mathbb{X}$. It follows from the continuity of the quotient mapping that $[x'_n]_K \to [x_0]_K$ as $n \to \infty$. Since $([x_n]_K)_{n\in\mathbb{N}}$ is a Cauchy sequence having a convergent subsequence $([x'_n]_K)_{n\in\mathbb{N}}$, it converges. □

8. Pushnitski's Unitary Spectral Flow

Notation. We shall assume the following throughout:

1. Let Φ be a regular symmetric norm.
2. Let $\mathcal{H}$ be a separable Hilbert space.

8.1. **The flow of paths in $\mathcal{S}_\Phi(\mathbb{T},1)$.** Let S be an arbitrary multiset in $\mathcal{S}_\Phi(\mathbb{T},1)$ admitting a representation $S = \{e^{i\theta_1}, e^{i\theta_2}, \dots\}^*$, where $\theta_1, \theta_2, \dots$ are in $[0, 2\pi]$. Given $\theta \in (0, 2\pi)$, we define the paths $\gamma_1(-;\theta), \gamma_2(-;\theta), \dots : [0,1] \to [0,2\pi]$ by

$$\gamma_j(-;\theta) := \begin{cases}[0,1] \ni t \longmapsto \theta_j(1-t) \in [0,2\pi], & \text{if } \theta_j \leq \theta,\\ [0,1] \ni t \longmapsto \theta_j(1-t) + 2\pi t \in [0,2\pi], & \text{if } \theta_j > \theta.\end{cases}$$

The *canonical θ-contruction* of S, denoted by $\Gamma_\theta(S)$, is the mapping

$$[0,1] \ni t \longmapsto \{e^{i\gamma_1(t;\theta)}, e^{i\gamma_2(t;\theta)}, \dots\}^* \in \mathcal{S}_\Phi(\mathbb{T}, 1). \tag{42}$$

Proposition 8.1. *$\Gamma_\theta(S)$ is continuous for any $S \in \mathcal{S}_\Phi(\mathbb{T}, 1)$ and any $\theta \in (0, 2\pi)$.*

Proof. Without loss of generality, we may assume that $\theta < \pi$. Since $e^{i\theta_j} \to 1$ as $j \to \infty$, there exists a large enough index $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\theta_n \in [0, \theta] \cup [2\pi - \theta, 2\pi]$. It is geometrically obvious that

$$\left|e^{i\gamma_n(t;\theta)} - 1\right| \leq \left|e^{i\theta_n} - 1\right| \quad \forall t \in [0,1]\ \forall n \geq N.$$

It follows that

$$\Phi(|e^{i\gamma_N(t;\theta)} - 1|, |e^{i\gamma_{N+1}(t;\theta)} - 1|, \dots) \leq \Phi(|e^{i\theta_N} - 1|, |e^{i\theta_{N+1}} - 1|, \dots) \qquad \forall t \in [0,1].$$

Now, the claim follows from Theorem 4.7. □

Recall that we have defined the following group isomorphism in §6:

$$\Psi_\Phi : \pi_1(\mathcal{S}_\Phi(\mathbb{T}, 1), O_1) \to H_1(\mathbb{T}) \cong \mathbb{Z}.$$

Definition 8.2. Given a path $S(\cdot)$ in $\mathcal{S}_\Phi(\mathbb{T}, 1)$, we define the *flow* of S to be a function $\mu(-; S) : (0, 2\pi) \to \mathbb{Z}$ given by

$$\mu(\theta; S) := \Psi_\Phi\left([\Gamma_\theta(S(0))^{-1} \circledast S \circledast \Gamma_\theta(S(1))]_{\pi_1}\right)$$

Theorem 8.3. *Let S, T be two paths in $\mathcal{S}_\Phi(\mathbb{T}, 1)$:*

1. *If S, T are homotopic paths, then $\mu(-; S) = \mu(-; T)$.*
2. *If $S(1) = T(0)$, then $\mu(-; S \circledast T) = \mu(-; S) + \mu(-; T)$.*

Proof. Let us fix $\theta \in (0, 2\pi)$. For notational simplicity, we write

$$\Gamma_S(t) := \Gamma_\theta(S(t)) \text{ and } \Gamma_T(t) := \Gamma_\theta(T(t)) \quad \forall t \in [0,1].$$

For the first part, let us assume that S, T are homotopic. Since S, T have the same end-points, $\Gamma_S(0) = \Gamma_T(0)$ and $\Gamma_S(1) = \Gamma_T(1)$. It is now easy to observe that

$$\mu(\theta; S) = \Psi_\Phi([\Gamma_S(0)^{-1} \circledast S \circledast \Gamma_S(1)]_{\pi_1}) = \Psi_\Phi([\Gamma_T(0)^{-1} \circledast T \circledast \Gamma_T(1)]_{\pi_1}) = \mu(\theta; T).$$

For the second part, we assume $S(1) = T(0)$. That is, $\Gamma_S(1) = \Gamma_T(0)$. Now,

$$\begin{aligned}
\mu(\theta; S \circledast T) &= \Psi_\Phi([\Gamma_{S \circledast T}(0)^{-1} \circledast (S \circledast T) \circledast \Gamma_{S \circledast T}(1)]_{\pi_1}) \\
&= \Psi_\Phi([\Gamma_S(0)^{-1} \circledast (S \circledast T) \circledast \Gamma_T(1)]_{\pi_1}) \\
&= \Psi_\Phi([\Gamma_S(0)^{-1} \circledast S \circledast \Gamma_S(1) \circledast \Gamma_T(0)^{-1} \circledast T \circledast \Gamma_T(1)]_{\pi_1}) \\
&= \Psi_\Phi([\Gamma_S(0)^{-1} \circledast S \circledast \Gamma_S(1)]_{\pi_1}) + \Psi_p([\Gamma_T(0)^{-1} \circledast T \circledast \Gamma_T(1)]_{\pi_1}) \\
&= \mu(\theta; S) + \mu(\theta; T).
\end{aligned}$$

□

8.2. **Unitary spectral flow.**

Lemma 8.4. *If U, V are two homotopic paths in $\mathcal{U}_\Phi(\mathcal{H}, I)$, then $\sigma(U), \sigma(U')$ are homotopic paths in $\mathcal{S}_\Phi(\mathbb{T}, 1)$.*

In particular, this establishes a well-defined homomorphism (homotopy functor)

$$\pi_1(\mathcal{U}_\Phi(\mathcal{H}, I), I) \ni [U]_{\pi_1} \longmapsto [\sigma(U)]_{\pi_1} \in \pi_1(\mathcal{S}_\Phi(\mathbb{T}, 1), O_1).$$

Proof. If H is a homotopy from U to V, then spec $\circ H$ is the required homotpy. □

Definition 8.5. Given a path $U(\cdot)$ in $\mathcal{U}_\Phi(\mathcal{H}, I)$, we define the *spectral flow* of U to be a function $\mathrm{sf}\,(-; U) : (0, 2\pi) \to \mathbb{Z}$ given by $\mathrm{sf}\,(-; U) := \mu(-; \sigma(U))$.

Theorem 8.6. *Let U, V be two paths in $\mathcal{U}_p(I)$:*

1. *If U, V are homotopic, then* $\mathrm{sf}\,(-; U) = \mathrm{sf}\,(-; V)$.
2. *If $U(1) = V(0)$, then* $\mathrm{sf}\,(-; U \circledast V) = \mathrm{sf}\,(-; U) + \mathrm{sf}\,(-; V)$.

Proof. The assertions follow by Theorem 8.3 and Lemma 8.4. □

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