Orbits and Khinchine-type inequalities in symmetric spaces

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A Thesis presented for the Degree of Doctor of Philosophy

in

School of Computer Science, Engineering and Mathematics
Faculty of Science and Engineering
Flinders University

April 8, 2011

Acknowledgement

It is pleasure for me to thank my supervisor Professor Peter Dodds and my co-supervisor Professor Fyodor Sukochev for their help and inspiration. I would like to thank Flinders University and School of Computer Science, Engineering and Mathematics for the financial support during my PhD candidature.

Contents

Chapter 1

Introduction & preliminaries

1.1 Introduction

The primary aim of this thesis is the study of various geometric and probabilistic properties of symmetric Banach and quasi-Banach spaces.

In Chapter 1, we gather the necessary background material and technical preliminary information.

In Chapter 2, we study the action of some important semi-groups in symmetric (quasi-)Banach spaces. Our aim is to determine the geometric structure of their orbits and to give simple and constructive criteria which characterise the orbits in terms of their extreme points.

In Chapter 3, we study various generalizations of Khinchine and Johnson-Schechtman inequalities. These important inequalities are shown to be useful tools for studying connections between the geometric and probabilistic structures of symmetric spaces. We prove the most general possible form of the Johnson-Schechtman inequalities. This allows us to prove the Khinchine inequality in very general form. As a bonus, our proof, which is based on an inequality of Prokhorov, is radically simpler than any currently available in the literature.

A further important topic covered in Chapter 3 is the connection between the Kruglov operator (see Section 1.2 below) and random permutations of matrices. An important estimate due to Montgomery-Smith and Semenov is proved to be valid if and only if the space satisfies the Kruglov property.

The last sections of the thesis deal with various analogs of the Banach-Saks index. We introduce an operator estimate which is equivalent to the latter index being non-trivial. In particular, this allows us to completely characterize Lorentz spaces with non-trivial (modified) Banach-Saks index.

1.1.1 Orbits and their importance

The most important object in the theory of interpolation of two symmetric (quasi-)Banach spaces is the semigroup of operators which are simultaneously contractions in both spaces.

Historically, interpolation spaces between L_1 and L_∞ were studied first. Orbits of the interpolation semigroup in this case have been precisely characterised via the Calderon-Mityagin theorem in terms of submajorization in the sense of Hardy, Littlewood and Polya.

We are also interested in the other semigroups such as the positive part of the interpolation semigroup and the bistochastic semigroup. The former consists of all positive operators from the interpolation semi-group. The latter consists of all bistochastic operators and is, therefore, a subset of the interpolation semigroup associated with L_1 and L_∞. Arguing as in the Calderon-Mityagin theorem, one can obtain a precise description for the orbits of these two semigroups.

Let E be a symmetric (quasi-)Banach function space which is an interpolation space for the Banach couple (L_1, L_∞). This thesis will study the following question.

Question 1.1.1. *Which conditions guarantee that the orbits of the element $x \in E$ (corresponding to the interpolation semigroup, the positive part of the interpolation semigroup and the bistochastic semigroup, respectively) coincide with the closed convex hull of their extreme points?*

The answer to this question depends strongly on the topology in which the closure is taken.

If $E = L_1(0,1)$, then it has been shown by Ryff (see [49]) that the bistochastic orbit of every element is weakly compact. It follows now from the Krein-Milman theorem that the bistochastic orbit is the weak (and hence norm)-closed convex hull of its extreme points. A generalisation of this result can be found in [22]. According to [22], the bistochastic orbit of every element is weakly compact in any separable symmetric Banach space on the interval $(0,1)$. Thus, in any such space, the bistochastic orbit is the weak (and hence norm)-closed convex hull of its extreme points.

The situation is very different for non-separable spaces. First of all, orbits are not weakly compact anymore. For example, if $E = L_\infty$, then the interpolation orbit of a constant is a ball. Clearly, a ball in L_∞ is not a weakly-compact set because L_∞ is not a reflexive space. Hence, the proofs given in [49] and [22] are not valid for non-separable spaces.

We wish to determine whether the orbits of a given element are the closed convex hulls of their extreme points in the natural topology of a space induced by the (quasi-)norm. Such studies were pioneered by Braverman and Mekler (see [11]) for symmetric Banach spaces on the interval $(0,1)$. They proved that, for every fully symmetric space E on $(0,1)$ (i.e. exact interpolation space for the couple (L_1, L_∞)) with non-trivial upper Boyd index, the interpolation orbit of every element coincides with the norm-closed convex hull of the set of its extreme points.

They also proved the converse assertion for Marcinkiewicz spaces. In general, however, this converse assertion is false. As shown above, any separable space (such as L_1) would be a counter-example.

We show (Theorem 2.7.1) that, in every symmetric quasi-Banach space E which is an $(L_1, L_\infty)-$interpolation space, the interpolation orbit of an element $x \in E$ is the norm-closed convex hull of its extreme points if and only if

$$\varphi(x) := \lim_{s\to\infty} \frac{1}{s} \|\sigma_s(x^*)\|_E = 0. \tag{1.1}$$

This result trivially implies the result of Braverman and Mekler mentioned above. Here, σ_s denotes the dilation operator (see Subsection 1.2.8 below).

The important class of Orlicz spaces is considered in section 2.10. We demonstrate that the condition (1.1) is always valid in these spaces. Thus, for Orlicz spaces, the answer to Question 1.1.1 is always positive. Note, that the results of [22] and [11] are insufficient to cover this result in such generality. Indeed, the results of [22] are only applicable to separable Orlicz space, that is, those with non-trivial lower Boyd index. The results from [11] are only applicable to Orlicz spaces with non-trivial upper Boyd index. However, one can easily construct (see Appendix B) an Orlicz space with both Boyd indices being trivial.

As an application, we study the notion of symmetric and fully symmetric functionals in Section 2.10. The latter are a "commutative" counterpart of the Dixmier traces which appear in non-commutative geometry (see e.g. [16]). Symmetric and fully symmetric functionals are extensively studied in [23], [30] (see also [16] and the references therein). Note, however, that our terminology differs from that used in the articles just cited. These classes of symmetric and fully symmetric functionals are different in general. For example, the Marcinkiewicz space $M_{1,\infty}$ admits symmetric functionals which fail to be fully symmetric (see paper for details [29]). It follows from Theorem 2.7.1 that any symmetric functional on a fully symmetric space satisfying (1.1) is automatically fully symmetric. In particular, this implies that an Orlicz space does not possess any singular symmetric functionals (see Proposition 2.10.6). This latter result strengthens Theorem 3.1 from [23] which states that an Orlicz space does not possess any singular fully symmetric functionals.

The main results of Chapter 2 are contained in Sections 2.7 and 2.8 which deal with function spaces. In Section 2.9, we derive similar results for sequence spaces. Section 2.1 treats various properties of the functional φ and the modifications needed in later sections. In the section 2.4, we obtain some results about expectation operators. Section 2.5 is devoted to a theorem of Mekler (see [39, 40]). This Theorem (see Theorem 2.5.8) is an important ingredient in the original proof of Braverman and Mekler but is also of interest in its own right and and can be treated as a generalization of Birkhoff theorem.

The precise description of the extreme points of the orbits is heavily used in the chapter. This description is due to Ryff (see [48]) for the bistochastic semi-group. Descriptions for the other 2 semi-groups are less well-known. We present them in an Appendix for the convenience of the reader, together with details of proof.

1.1.2 The Kruglov operator

The Khinchine inequality

$$\mathrm{const} \cdot \|\{a_n\}\|_2 \leq \|\sum_n a_n r_n\|_p \leq \mathrm{const} \cdot p^{1/2} \|\{a_n\}\|_2$$

is one of the most important inequalities in analysis. In this classical setting, the proof of the left hand side inequality is almost trivial. In this thesis, we will be concerned only with the generalisation of the right hand side inequality.

The proof of the Khinchine inequality heavily uses the fact that the Rademacher functions are independent. It seems natural to extend the Khinchine inequality so that it is valid for arbitrary sequences $\{a_n\}_{n=1}^{\infty}$ of independent mean zero functions.

Most attempts at such a generalisation have proved ineffective because their unnatural formulation prevented any interesting applications. Rosenthal [46] was probably the first who found a useful general inequality of Khinchine type. The best constants for the Rosenthal inequality may be found in the paper [27].

All these papers, however, are generalisations of the Khinchine inequality in the classic setting of L_p-spaces. In 1989, Johnson and Schechtman introduced a new inequality and proved it for all symmetric Banach spaces E such that $E \supset L_p$ for $1 \leq p < \infty$.

Braverman (see [12]) applied some earlier ideas of Kruglov to the Johnson-Schechtman inequalities. He was able to generalize them to a significantly wider class of symmetric spaces under an additional assumption. More precisely, he required that the supports $\mathrm{supp}(x_n)$, $n \in \mathbb{N}$, of the independent functions x_n, $n \in \mathbb{N}$ should be such that

$$\sum_{n=1}^{\infty} m(\mathrm{supp}(x_n)) \leq 1. \tag{1.2}$$

An equivalent characterisation of the spaces considered by Braverman was discovered by Astashkin & Sukochev (see [6]). They observed that a complicated non-linear condition used by Braverman may be reformulated in terms of the boundedness of a linear operator introduced by Kruglov. Hence, the powerful machinery of linear operator theory could now be applied. In this way, Astashkin & Sukochev (see [3]) managed to prove the Johnson-Schechtman inequality for all spaces considered by Braverman. They also showed that the rather technical assumption (1.2) is superfluous.

The Kruglov operator K maps a symmetric quasi-Banach space $E(0,1)$ into the space $E((0,1)^\infty)$ according to the formula

$$(Kx)(\omega) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} x(\omega^{(k)}) \chi_{A_n}(\omega^{(0)}), \quad \omega = \{\omega^{(k)}\}_{k=0}^{\infty} \in (0,1)^\infty.$$

Here, $\{A_n\}_{n=1}^\infty$ is a fixed collection of disjoint subsets of the interval $(0,1)$ such that $m(A_n) = 1/en!$.

In order to emphasize the contribution of Kruglov to probability theory, this class of spaces considered by Astashkin & Sukochev is called the Kruglov class or $\mathbb{K}$ in [6]. We usually refer to its members as spaces with the Kruglov property.

The proof of the Johnson-Schechtman inequality in [3] is quite complicated. In this thesis, proof is considerably simplified and generalised to the quasi-Banach setting. This is presented in Chapter 3 (see Theorems 3.1.15 and 3.2.4).

It is well-known (see [34],[12]) that the Orlicz space $\exp(L_1)$ defined by the function $e^t - 1$ satisfies the Kruglov property. The latter property also holds for the separable part $(\exp(L_1))_0$.

All previously known symmetric spaces E with the Kruglov property satisfy the inclusion $E \supset (\exp(L_1))_0$. This, together with Theorem 7.2 of [6] suggests that $(\exp(L_1))_0$ is the minimal space with the Kruglov property.

In section 4.1, we show that this hypothesis fails. Moreover, for every given symmetric space $E \in \mathbb{K}$, there exists a Marcinkiewicz space satisfying the Kruglov property such that $M_\psi \subset E$ and $M_\psi \neq E$.

The situation is quite different in the subclass of Lorentz spaces. Indeed, every Lorentz space satisfying the Kruglov property necessarily contains $\exp(L_1)$ (see Theorem 4.2.6).

1.1.3 The operators T_n

In [35], S. Kwapien and C. Schutt considered random permutations and applied their results to the geometry of Banach spaces. The results of [35] were further strengthened in [54] and [41] via an operator approach. The following family of operators was introduced there.

Given a symmetric norm $\|\cdot\|$ on $L_\infty(0,1)$, one may define a symmetric norm on $\mathbb{R}^n$ by the formula

$$\|x\| = \|\sum_{k=1}^{n} x_k \chi_{((k-1)/n,k/n)}\|, \quad x = \{x_k\}_{k=1}^{n} \in \mathbb{R}^n.$$

Let $n \in \mathbb{N}$ and let S_n be the set of all permutations of the set $1, 2, \ldots, n$. Let M_n be the algebra of all $n \times n$ matrices. Consider the operator $A_n : M_n \to \mathbb{R}^n$ defined by the following formula

$$(A_n x)(\pi) = \sum_{i=1}^{n} x_{i,\pi(i)} \quad \pi \in S_n. \tag{1.3}$$

The uniform boundedness of the operators A_n, $n \in \mathbb{N}$, is essential for applications in the geometry of symmetric Banach spaces. One of the major results of [41] (see Corollary 8 there) says that if the sequence of operators $\{A_n\}_{n\geq 1}$ is uniformly bounded on the set of diagonal matrices, then it is uniformly bounded on the set of all matrices.

For every $x \in L_1(0,1)$, we define the vector $B_n x \in \mathbb{R}^n$ by the formula

$$(B_n x)_i = n \int_{(i-1)/n}^{i/n} x(t)dt, \quad i = 1, 2, \cdots, n.$$

For every $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we define the function $C_n x \in L_\infty(0,1)$ by the formula

$$C_n x = \sum_{k=1}^{n} x_k \chi_{((k-1)/n,k/n)}.$$

We now define the operator $T_n : L_1(0,1) \to L_\infty(0,1)$ by setting

$$T_n = C_{n!} A_n B_n, \quad n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, T_n is a positive operator. Sometimes, we also use the notation T_n for the operator $C_{n!}A_n$, defined on $\mathbb{R}^n$ (this does not cause any ambiguity).

Example 1.1.2. *Evidently,*

$$\|T_n x\|_{L_1} = \|x\|_{L_1}$$

for every positive $x \in L_1(0,1)$. It follows that the sequence of operators $A_n : L_1 \to L_1$, $n \in \mathbb{N}$, is uniformly bounded.

There is not any immediately obvious connection between the operator K and the sequence T_n, $n \geq 1$. Nevertheless, the following interesting fact follows from comparison of results in [41] and [6]: the criterion for boundedness of the operator K in any Lorentz space Λ_ψ coincides with that for the uniform boundedness of the family of operators $\{T_n\}_{n\geq 1}$ in Λ_ψ.

Theorem 1.1.3. *The operator K maps the Lorentz space Λ_ψ into itself if and only if*

$$\sup_{0<t\leq 1} \frac{1}{\psi(t)} \sum_{k=1}^{\infty} \psi(\frac{t^k}{k!}) < \infty.$$

Theorem 1.1.4. *The family of operators $\{T_n\}_{n\geq 1}$ is uniformly bounded in the Lorentz space Λ_ψ if and only if*

$$\sup_{0<t\leq 1} \frac{1}{\psi(t)} \sum_{k=1}^{\infty} \psi(\frac{t^k}{k!}) < \infty.$$

It is now natural to ask whether the boundedness of the operator K in an arbitrary symmetric space E is equivalent to the uniform boundedness of the family of operators $\{T_n\}_{n\geq 1}$ in E. In Chapter 3, we establish that it is indeed the case. The proof is based on combinatorial estimates for the corresponding distribution functions. The equivalence that we establish implies some new corollaries for the operator K and operators T_n, $n \geq 1$. In particular, Corollary 4.5.2 strengthens Theorem 19 from [41] by showing that the uniform boundedness of the family of operators $\{T_n\}_{n\geq 1}$ in the Orlicz space $\exp(L_p)$ is equivalent to the condition $p \leq 1$.

1.1.4 The Banach-Saks indices

The Banach-Saks theorem says that if a sequence in L_p is weakly null, then there exists a subsequence which converges to 0 in the sense of Cesaro.

A sequence $\{x_k\}_{k=1}^{\infty}$ in the Banach space is called a p-Banach-Saks-sequence if, for every subsequence $\{y_k\}_{k=1}^{\infty} \subset \{x_k\}_{k=1}^{\infty}$, we have

$$\|\sum_{k=1}^{n} y_k\|_E = O(n^{1/p}).$$

A symmetric Banach space is said to have the p-Banach-Saks property if every weakly null sequence $\{x_n\}_{n=1}^{\infty} \subset E$ contains a p-Banach-Saks-subsequence. The infimum of all such p is called the Banach-Saks index of E. It was proved in [4] that the Banach-Saks index is non-trivial (i.e. is not equal to 1) if and only if $0 < \alpha_E \leq \beta_E < 1$.

In section 1.2, we introduce modified versions of the Banach-Saks index. In our setting, we require the weakly null sequence to be independent (respectively, independent and identically distributed; respectively, disjoint). It is not clear, a priori, how to compute these modified indices.

It is shown in this thesis that the modified Banach-Saks index for independent sequences is the minimum of that for disjoint sequences and that for independent identically distributed sequences. The former is usually easier to compute. As for the latter, we show how to compute it in terms of the estimates on the norms of certain sequences of operators.

This sequence of operators is a semigroup. This allows us to characterise precisely those spaces for which the modified Banach-Saks index (for independent identically distributed functions) is trivial.

As an application, we establish a criterion for the triviality of those indices for Lorentz spaces. Such questions were considered earlier in the literature by Carothers & Dilworth (see [18, 17]) in the setting of $L_{p,q}-$spaces.