Orbits and Khinchine-type inequalities in symmetric spaces

Dmitriy Zanin

A Thesis presented for the Degree of Doctor of Philosophy

in

School of Computer Science, Engineering and Mathematics
Faculty of Science and Engineering
Flinders University

April 8, 2011

Contents

Chapter 1

Introduction & preliminaries

1.1 Introduction

The primary aim of this thesis is the study of various geometric and probabilistic properties of symmetric Banach and quasi-Banach spaces.

In Chapter 1, we gather the necessary background material and technical preliminary information.

In Chapter 2, we study the action of some important semi-groups in symmetric (quasi-)Banach spaces. Our aim is to determine the geometric structure of their orbits and to give simple and constructive criteria which characterise the orbits in terms of their extreme points.

In Chapter 3, we study various generalizations of Khinchine and Johnson-Schechtman inequalities. These important inequalities are shown to be useful tools for studying connections between the geometric and probabilistic structures of symmetric spaces. We prove the most general possible form of the Johnson-Schechtman inequalities. This allows us to prove the Khinchine inequality in very general form. As a bonus, our proof, which is based on an inequality of Prokhorov, is radically simpler than any currently available in the literature.

A further important topic covered in Chapter 3 is the connection between the Kruglov operator (see Section 1.2 below) and random permutations of matrices. An important estimate due to Montgomery-Smith and Semenov is proved to be valid if and only if the space satisfies the Kruglov property.

The last sections of the thesis deal with various analogs of the Banach-Saks index. We introduce an operator estimate which is equivalent to the latter index being non-trivial. In particular, this allows us to completely characterize Lorentz spaces with non-trivial (modified) Banach-Saks index.

1.1.1 Orbits and their importance

The most important object in the theory of interpolation of two symmetric (quasi-)Banach spaces is the semigroup of operators which are simultaneously contractions in both spaces.

Historically, interpolation spaces between L_1 and L_∞ were studied first. Orbits of the interpolation semigroup in this case have been precisely characterised via the Calderon-Mityagin theorem in terms of submajorization in the sense of Hardy, Littlewood and Polya.

We are also interested in the other semigroups such as the positive part of the interpolation semigroup and the bistochastic semigroup. The former consists of all positive operators from the interpolation semi-group. The latter consists of all bistochastic operators and is, therefore, a subset of the interpolation semigroup associated with L_1 and L_∞. Arguing as in the Calderon-Mityagin theorem, one can obtain a precise description for the orbits of these two semigroups.

Let E be a symmetric (quasi-)Banach function space which is an interpolation space for the Banach couple (L_1, L_∞). This thesis will study the following question.

Question 1.1.1. *Which conditions guarantee that the orbits of the element $x \in E$ (corresponding to the interpolation semigroup, the positive part of the interpolation semigroup and the bistochastic semigroup, respectively) coincide with the closed convex hull of their extreme points?*

The answer to this question depends strongly on the topology in which the closure is taken.

If $E = L_1(0,1)$, then it has been shown by Ryff (see [49]) that the bistochastic orbit of every element is weakly compact. It follows now from the Krein-Milman theorem that the bistochastic orbit is the weak (and hence norm)-closed convex hull of its extreme points. A generalisation of this result can be found in [22]. According to [22], the bistochastic orbit of every element is weakly compact in any separable symmetric Banach space on the interval $(0,1)$. Thus, in any such space, the bistochastic orbit is the weak (and hence norm)-closed convex hull of its extreme points.

The situation is very different for non-separable spaces. First of all, orbits are not weakly compact anymore. For example, if $E = L_\infty$, then the interpolation orbit of a constant is a ball. Clearly, a ball in L_∞ is not a weakly-compact set because L_∞ is not a reflexive space. Hence, the proofs given in [49] and [22] are not valid for non-separable spaces.

We wish to determine whether the orbits of a given element are the closed convex hulls of their extreme points in the natural topology of a space induced by the (quasi-)norm. Such studies were pioneered by Braverman and Mekler (see [11]) for symmetric Banach spaces on the interval $(0,1)$. They proved that, for every fully symmetric space E on $(0,1)$ (i.e. exact interpolation space for the couple (L_1, L_∞)) with non-trivial upper Boyd index, the interpolation orbit of every element coincides with the norm-closed convex hull of the set of its extreme points.

They also proved the converse assertion for Marcinkiewicz spaces. In general, however, this converse assertion is false. As shown above, any separable space (such as L_1) would be a counter-example.

We show (Theorem 2.7.1) that, in every symmetric quasi-Banach space E which is an $(L_1, L_\infty)-$interpolation space, the interpolation orbit of an element $x \in E$ is the norm-closed convex hull of its extreme points if and only if

$$\varphi(x) := \lim_{s\to\infty} \frac{1}{s}\|\sigma_s(x^*)\|_E = 0. \tag{1.1}$$

This result trivially implies the result of Braverman and Mekler mentioned above. Here, σ_s denotes the dilation operator (see Subsection 1.2.8 below).

The important class of Orlicz spaces is considered in section 2.10. We demonstrate that the condition (1.1) is always valid in these spaces. Thus, for Orlicz spaces, the answer to Question 1.1.1 is always positive. Note, that the results of [22] and [11] are insufficient to cover this result in such generality. Indeed, the results of [22] are only applicable to separable Orlicz space, that is, those with non-trivial lower Boyd index. The results from [11] are only applicable to Orlicz spaces with non-trivial upper Boyd index. However, one can easily construct (see Appendix B) an Orlicz space with both Boyd indices being trivial.

As an application, we study the notion of symmetric and fully symmetric functionals in Section 2.10. The latter are a "commutative" counterpart of the Dixmier traces which appear in non-commutative geometry (see e.g. [16]). Symmetric and fully symmetric functionals are extensively studied in [23], [30] (see also [16] and the references therein). Note, however, that our terminology differs from that used in the articles just cited. These classes of symmetric and fully symmetric functionals are different in general. For example, the Marcinkiewicz space $M_{1,\infty}$ admits symmetric functionals which fail to be fully symmetric (see paper for details [29]). It follows from Theorem 2.7.1 that any symmetric functional on a fully symmetric space satisfying (1.1) is automatically fully symmetric. In particular, this implies that an Orlicz space does not possess any singular symmetric functionals (see Proposition 2.10.6). This latter result strengthens Theorem 3.1 from [23] which states that an Orlicz space does not possess any singular fully symmetric functionals.

The main results of Chapter 2 are contained in Sections 2.7 and 2.8 which deal with function spaces. In Section 2.9, we derive similar results for sequence spaces. Section 2.1 treats various properties of the functional φ and the modifications needed in later sections. In the section 2.4, we obtain some results about expectation operators. Section 2.5 is devoted to a theorem of Mekler (see [39, 40]). This Theorem (see Theorem 2.5.8) is an important ingredient in the original proof of Braverman and Mekler but is also of interest in its own right and and can be treated as a generalization of Birkhoff theorem.

The precise description of the extreme points of the orbits is heavily used in the chapter. This description is due to Ryff (see [48]) for the bistochastic semi-group. Descriptions for the other 2 semi-groups are less well-known. We present them in an Appendix for the convenience of the reader, together with details of proof.

1.1.2 The Kruglov operator

The Khinchine inequality

$$\text{const} \cdot \|\{a_n\}\|_2 \leq \|\sum_n a_n r_n\|_p \leq \text{const} \cdot p^{1/2} \|\{a_n\}\|_2$$

is one of the most important inequalities in analysis. In this classical setting, the proof of the left hand side inequality is almost trivial. In this thesis, we will be concerned only with the generalisation of the right hand side inequality.

The proof of the Khinchine inequality heavily uses the fact that the Rademacher functions are independent. It seems natural to extend the Khinchine inequality so that it is valid for arbitrary sequences $\{a_n\}_{n=1}^{\infty}$ of independent mean zero functions.

Most attempts at such a generalisation have proved ineffective because their unnatural formulation prevented any interesting applications. Rosenthal [46] was probably the first who found a useful general inequality of Khinchine type. The best constants for the Rosenthal inequality may be found in the paper [27].

All these papers, however, are generalisations of the Khinchine inequality in the classic setting of L_p-spaces. In 1989, Johnson and Schechtman introduced a new inequality and proved it for all symmetric Banach spaces E such that $E \supset L_p$ for $1 \leq p < \infty$.

Braverman (see [12]) applied some earlier ideas of Kruglov to the Johnson-Schechtman inequalities. He was able to generalize them to a significantly wider class of symmetric spaces under an additional assumption. More precisely, he required that the supports $\text{supp}(x_n)$, $n \in \mathbb{N}$, of the independent functions x_n, $n \in \mathbb{N}$ should be such that

$$\sum_{n=1}^{\infty} m(\text{supp}(x_n)) \leq 1. \tag{1.2}$$

An equivalent characterisation of the spaces considered by Braverman was discovered by Astashkin & Sukochev (see [6]). They observed that a complicated non-linear condition used by Braverman may be reformulated in terms of the boundedness of a linear operator introduced by Kruglov. Hence, the powerful machinery of linear operator theory could now be applied. In this way, Astashkin & Sukochev (see [3]) managed to prove the Johnson-Schechtman inequality for all spaces considered by Braverman. They also showed that the rather technical assumption (1.2) is superfluous.

The Kruglov operator K maps a symmetric quasi-Banach space $E(0,1)$ into the space $E((0,1)^{\infty})$ according to the formula

$$(Kx)(\omega) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} x(\omega^{(k)}) \chi_{A_n}(\omega^{(0)}), \quad \omega = \{\omega^{(k)}\}_{k=0}^{\infty} \in (0,1)^{\infty}.$$

Here, $\{A_n\}_{n=1}^{\infty}$ is a fixed collection of disjoint subsets of the interval $(0,1)$ such that $m(A_n) = 1/en!$.

In order to emphasize the contribution of Kruglov to probability theory, this class of spaces considered by Astashkin & Sukochev is called the Kruglov class or $\mathbb{K}$ in [6]. We usually refer to its members as spaces with the Kruglov property.

The proof of the Johnson-Schechtman inequality in [3] is quite complicated. In this thesis, proof is considerably simplified and generalised to the quasi-Banach setting. This is presented in Chapter 3 (see Theorems 3.1.15 and 3.2.4).

It is well-known (see [34],[12]) that the Orlicz space $\exp(L_1)$ defined by the function $e^t - 1$ satisfies the Kruglov property. The latter property also holds for the separable part $(\exp(L_1))_0$.

All previously known symmetric spaces E with the Kruglov property satisfy the inclusion $E \supset (\exp(L_1))_0$. This, together with Theorem 7.2 of [6] suggests that $(\exp(L_1))_0$ is the minimal space with the Kruglov property.

In section 4.1, we show that this hypothesis fails. Moreover, for every given symmetric space $E \in \mathbb{K}$, there exists a Marcinkiewicz space satisfying the Kruglov property such that $M_{\psi} \subset E$ and $M_{\psi} \neq E$.

The situation is quite different in the subclass of Lorentz spaces. Indeed, every Lorentz space satisfying the Kruglov property necessarily contains $\exp(L_1)$ (see Theorem 4.2.6).

1.1.3 The operators T_n

In [35], S. Kwapien and C. Schutt considered random permutations and applied their results to the geometry of Banach spaces. The results of [35] were further strengthened in [54] and [41] via an operator approach. The following family of operators was introduced there.

Given a symmetric norm $\|\cdot\|$ on $L_\infty(0,1)$, one may define a symmetric norm on $\mathbb{R}^n$ by the formula

$$\|x\| = \|\sum_{k=1}^n x_k \chi_{((k-1)/n,k/n)}\|, \quad x = \{x_k\}_{k=1}^n \in \mathbb{R}^n.$$

Let $n \in \mathbb{N}$ and let S_n be the set of all permutations of the set $1, 2, \ldots, n$. Let M_n be the algebra of all $n \times n$ matrices. Consider the operator $A_n : M_n \to \mathbb{R}^n$ defined by the following formula

$$(A_n x)(\pi) = \sum_{i=1}^n x_{i,\pi(i)} \quad \pi \in S_n. \tag{1.3}$$

The uniform boundedness of the operators A_n, $n \in \mathbb{N}$, is essential for applications in the geometry of symmetric Banach spaces. One of the major results of [41] (see Corollary 8 there) says that if the sequence of operators $\{A_n\}_{n\geq 1}$ is uniformly bounded on the set of diagonal matrices, then it is uniformly bounded on the set of all matrices.

For every $x \in L_1(0,1)$, we define the vector $B_n x \in \mathbb{R}^n$ by the formula

$$(B_n x)_i = n \int_{(i-1)/n}^{i/n} x(t)dt, \quad i = 1, 2, \cdots, n.$$

For every $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we define the function $C_n x \in L_\infty(0,1)$ by the formula

$$C_n x = \sum_{k=1}^n x_k \chi_{((k-1)/n,k/n)}.$$

We now define the operator $T_n : L_1(0,1) \to L_\infty(0,1)$ by setting

$$T_n = C_{n!} A_n B_n, \quad n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, T_n is a positive operator. Sometimes, we also use the notation T_n for the operator $C_{n!}A_n$, defined on $\mathbb{R}^n$ (this does not cause any ambiguity).

Example 1.1.2. *Evidently,*

$$\|T_n x\|_{L_1} = \|x\|_{L_1}$$

for every positive $x \in L_1(0,1)$. *It follows that the sequence of operators* $A_n : L_1 \to L_1$, $n \in \mathbb{N}$, *is uniformly bounded.*

There is not any immediately obvious connection between the operator K and the sequence T_n, $n \geq 1$. Nevertheless, the following interesting fact follows from comparison of results in [41] and [6]: the criterion for boundedness of the operator K in any Lorentz space Λ_ψ coincides with that for the uniform boundedness of the family of operators $\{T_n\}_{n\geq 1}$ in Λ_ψ.

Theorem 1.1.3. *The operator K maps the Lorentz space Λ_ψ into itself if and only if*

$$\sup_{0<t\leq 1} \frac{1}{\psi(t)} \sum_{k=1}^{\infty} \psi(\frac{t^k}{k!}) < \infty.$$

Theorem 1.1.4. *The family of operators $\{T_n\}_{n\geq 1}$ is uniformly bounded in the Lorentz space Λ_ψ if and only if*

$$\sup_{0<t\leq 1} \frac{1}{\psi(t)} \sum_{k=1}^{\infty} \psi(\frac{t^k}{k!}) < \infty.$$

It is now natural to ask whether the boundedness of the operator K in an arbitrary symmetric space E is equivalent to the uniform boundedness of the family of operators $\{T_n\}_{n\geq 1}$ in E. In Chapter 3, we establish that it is indeed the case. The proof is based on combinatorial estimates for the corresponding distribution functions. The equivalence that we establish implies some new corollaries for the operator K and operators T_n, $n \geq 1$. In particular, Corollary 4.5.2 strengthens Theorem 19 from [41] by showing that the uniform boundedness of the family of operators $\{T_n\}_{n\geq 1}$ in the Orlicz space $\exp(L_p)$ is equivalent to the condition $p \leq 1$.

1.1.4 The Banach-Saks indices

The Banach-Saks theorem says that if a sequence in L_p is weakly null, then there exists a subsequence which converges to 0 in the sense of Cesaro.

A sequence $\{x_k\}_{k=1}^\infty$ in the Banach space is called a p-Banach-Saks-sequence if, for every subsequence $\{y_k\}_{k=1}^\infty \subset \{x_k\}_{k=1}^\infty$, we have

$$\|\sum_{k=1}^{n} y_k\|_E = O(n^{1/p}).$$

A symmetric Banach space is said to have the p-Banach-Saks property if every weakly null sequence $\{x_n\}_{n=1}^\infty \subset E$ contains a p-Banach-Saks-subsequence. The infimum of all such p is called the Banach-Saks index of E. It was proved in [4] that the Banach-Saks index is non-trivial (i.e. is not equal to 1) if and only if $0 < \alpha_E \leq \beta_E < 1$.

In section 1.2, we introduce modified versions of the Banach-Saks index. In our setting, we require the weakly null sequence to be independent (respectively, independent and identically distributed; respectively, disjoint). It is not clear, a priori, how to compute these modified indices.

It is shown in this thesis that the modified Banach-Saks index for independent sequences is the minimum of that for disjoint sequences and that for independent identically distributed sequences. The former is usually easier to compute. As for the latter, we show how to compute it in terms of the estimates on the norms of certain sequences of operators.

This sequence of operators is a semigroup. This allows us to characterise precisely those spaces for which the modified Banach-Saks index (for independent identically distributed functions) is trivial.

As an application, we establish a criterion for the triviality of those indices for Lorentz spaces. Such questions were considered earlier in the literature by Carothers & Dilworth (see [18, 17]) in the setting of $L_{p,q}-$spaces.

1.2 Preliminaries

1.2.1 Rearrangements & their properties

Let L_0 be the space of Lebesgue measurable functions either on $(0,1)$ or on $(0,\infty)$ which are finite almost everywhere (with identification $m-$a.e.). Here m is Lebesgue measure. Define S_0 to be the subset of L_0 which consists of all functions x such that $m(\{t : |x(t)| > s\})$ is finite for some $s > 0$.

Definition 1.2.1. *Let $x \in S_0$. The function d_x defined by the formula*

$$d_x(s) = m(\{t : |x(t)| > s\}), \quad s > 0$$

is called the **distribution function** *of x.*

Definition 1.2.2. *Two functions x and y are called* **equimeasurable** *if their distribution functions coincide, that is $d_x = d_y$.*

Equimeasurability is an binary relation. Clearly, this relation is reflexive, symmetric and transitive. Therefore, it is an equivalence relation.

Definition 1.2.3. *Let $x \in S_0$. We define the* **right-continuous rearrangement** *of x by the formula*

$$x^*(t) = \inf\{s \geq 0 : \ m(\{|x| > s\}) \leq t\}.$$

Lemma 1.2.4. *For every $x \in S_0$, the function x^* is equimeasurable with x.*

Thus, x^* is the unique monotone representative of the equivalence class of functions equimeasurable with x. The term "rearrangement" we apply to x^* is widely used in a literature. The following theorem (see [50]) clarifies the naming convention.

Theorem 1.2.5. *Let $x \in S_0$ be a function on the interval $(0,1)$. Let x^* be the right-continuous rearrangement of x. Then there exists a measure-preserving transformation from $(0,1)$ to itself such that $|x| = x^* \circ \gamma$.*

Note that the converse assertion is false, as is shown by the following example.

Example 1.2.6. *Let*

$$x(t) = \begin{cases} 2t, & 0 \leq t < 1/2 \\ 2t-1, & 1/2 \leq t \leq 1 \end{cases}$$

Here, we have that $x^*(t) = 1-t$, $t > 0$. However, there is no measure-preserving transform from $(0,1)$ to itself such that $x^* = x \circ \gamma$.

In the case of the semi-axis, the preceding theorem is not valid and the situation is more complicated. The following example is worth noting.

Example 1.2.7. *If* $x(t) = \frac{2}{\pi}\mathrm{arctg}(t)$, $t > 0$, *then,* $x^*(t) = 1$ *for all* $t > 0$. *Therefore,* $x^* \circ \gamma = 1$ *for every measure preserving transform* γ.

However, under mild additional restriction, then preceding theorem is valid on the semi-axis.

Theorem 1.2.8. *Let* $x \in S_0$ *be a function on the semi-axis. If* $x \geq x^*(\infty)$, *then there exists a measure-preserving transformation from the semi-axis into itself such that* $x = x^* \circ \gamma$.

Example 1.2.9. *Let* $x(t) = t$ *and* $y(t) = 1 - t$ *for every* $t \in (0, 1)$. *We have* $x^* = y^* = y$ *and* $(x + y)^* = 1$. *Thus, the inequality*

$$(x + y)^*(t) \leq x^*(t) + y^*(t)$$

fails for every $t \in (0, 1/2)$.

However, the following weaker inequality is valid (see [33])

Lemma 1.2.10. *Let* $x, y \in S_0$. *For every* $t_1, t_2 > 0$ *we have*

$$(x + y)^*(t_1 + t_2) \leq x^*(t_1) + y^*(t_2). \tag{1.4}$$

The following proposition follows directly from the definition of rearrangement (see [33, II.2.2]).

Proposition 1.2.11. *Let* $x \in S_0$. *For every* $t > 0$,

$$\int_0^t x^*(s)ds = \sup_{m(A)=t} \int_A |x(s)|ds.$$

In general, one cannot replace sup in the proposition above with a max (see Example 1.2.7). However, if $x \in S_0(0, 1)$ or $x \geq x^*(\infty)$, then sup may be replaced by max in Proposition 1.2.11 (see [33, II.2.2]).

The following semi-orderings play an important role in the theory of symmetric spaces.

Definition 1.2.12. *Let* $x, y \in S_0$. *We say that* y *is* **submajorized** *by* x **in the sense of Hardy-Littlewood-Polya** *if*

$$\int_0^t y^*(s)ds \leq \int_0^t x^*(s)ds, \quad \forall t > 0.$$

In this case, we write $y \prec\prec x$.

Definition 1.2.13. *Let* $0 \leq x, y \in L_1$. *We say that* y *is* **majorized** *by* x **in the sense of Hardy-Littlewood-Polya** *if* $y \prec\prec x$ *and* $\|y\|_1 = \|x\|_1$. *In this case, we write* $y \prec x$.

Lemma 1.2.14. *Let $\{x_k\}_{k=1}^{\infty} \subset S_0$ and $\{y_k\}_{k=1}^{\infty} \subset S_0$ be sequences of mutually disjoint functions. If $y_k \prec\prec x_k$ for every k, then*

$$y = \sum_k y_k \prec\prec \sum_k x_k = x.$$

Proof. Fix $\varepsilon > 0$. There exists a set A such that $m(A) = t$ and

$$\int_0^t y^*(s)ds \leq \varepsilon + \int_A |y(s)|ds = \varepsilon + \sum_k \int_{A\cap \mathrm{supp}(y_k)} |y_k(s)|ds.$$

However,

$$\int_{A\cap \mathrm{supp}(y_k)} |y_k(s)|ds \leq \int_0^{m(A\cap \mathrm{supp}(y_k))} y_k^*(s)ds \leq \int_0^{m(A\cap \mathrm{supp}(y_k))} x_k^*(s)ds.$$

Again, there exist sets $B_k \subset \mathrm{supp}(x_k)$ such that $m(B_k) = m(A \cap \mathrm{supp}(y_k))$ and

$$\int_0^{m(A\cap \mathrm{supp}(y_k))} x_k^*(s)ds \leq \varepsilon \cdot 2^{-k}\varepsilon + \int_{B_k} |x_k(s)|ds.$$

Set $B = \cup_k B_k$. It follows that

$$\int_0^t y^*(s)ds \leq 2\varepsilon + \int_B |x(s)|ds \leq 2\varepsilon + \int_0^t x^*(s)ds.$$

Since ε is arbitrary, we are done. □

The following properties of rearrangement are well-known and can be found in Chapter II of [33] (see Equation (2.17), Theorem 3.1 and Section 6.1 there).

Proposition 1.2.15. *If $x, y \in S_0$, then*

$$(x + y)^* \prec\prec x^* + y^* \tag{1.5}$$

and

$$(x^* - y^*) \prec\prec (x - y)^*. \tag{1.6}$$

In fact, even stronger version of (1.5) is valid. If $x_k \in S_0$, $k \in \mathbb{N}$, then

$$\sum_{k=1}^{\infty} x_k \prec\prec \sum_{k=1}^{\infty} x_k^*, \tag{1.7}$$

provided that the latter series converges pointwise.

1.2.2 Convergence almost everywhere, in measure and in distribution

The following definitions of convergence are well-known.

Definition 1.2.16. *Let $x_n \in S_0$ be a sequence of functions and let $x \in S_0$. We say that x_n converges to x*

1. **almost everywhere** *if the set of non-convergence has measure* 0, *that is*

$$m(\{t : \quad x_n(t) \not\to x(t)\}) = 0.$$

2. **in measure** *if for any fixed $\varepsilon > 0$,*

$$m(\{t : \quad |x_n(t) - x(t)| > \varepsilon\}) \to 0.$$

3. **in distribution** *if $d_{x_n}(t)$ converges to $d_x(t)$ for all $t > 0$.*

The following result gathers several well-known properties which will be needed in sequel.

Lemma 1.2.17. *Let $\{x_n\}_{n\in\mathbb{N}} \subset S_0$ be a sequence of functions on the interval $(0, 1)$ and let $x \in S_0$.*

1. *If $x_n \to x$ almost everywhere, then $x_n \to x$ in measure.*
2. *If $x_n \to x$ in measure, then $x_n \to x$ in distribution.*
3. *Let $x_n = x_n^*$ for all $n \in \mathbb{N}$ and let $x = x^*$. If $x_n \to x$ in distribution, then $x_n \to x$ almost everywhere.*
4. *If $x_n \to x$ in measure, then there exists a subsequence $\{y_k\}_{k\in\mathbb{N}} \subset \{x_n\}_{n\in\mathbb{N}}$ which converges to x almost everywhere.*

1.2.3 Quasi-Banach spaces

We recall the definition of a quasi-Banach space (see [43]).

Definition 1.2.18. *Let E be a linear space over $\mathbb{R}$. A function $\|\cdot\| : E \to \mathbb{R}$ is called* **quasi-norm** *if the following conditions are satisfied.*

1. *There exists a constant $C(E)$ (which depends only on E) such that $\|x + y\| \leq C(E)(\|x\| + \|y\|)$ for every $x, y \in E$.*
2. *For every $x \in E$ and $c \in \mathbb{R}$, we have $\|cx\| = |c| \cdot \|x\|$*
3. *For every $x \in E$, we have $\|x\| \geq 0$. Moreover, if $\|x\| = 0$, then $x = 0$.*

We refer to the constant $C(E)$ as the concavity modulus of E.

Definition 1.2.19. *If E is a linear space over $\mathbb{R}$ and if $\|\cdot\|_E : E \to \mathbb{R}$ is a quasi-norm, then the pair $(E, \|\cdot\|_E)$ is called a* **quasi-normed space**.

For brevity, we will say that E is a quasi-normed space since this will not cause any confusion in the current text.

As in the case of normed spaces, we have the following definition.

Definition 1.2.20. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a quasi-normed space E. We say that the sequence x_n, $n \in \mathbb{N}$, is a* **Cauchy sequence** *if $\|x_n - x_m\|_E \to 0$ provided that $m, n \to \infty$.*

The usual definition of completeness follows.

Definition 1.2.21. *If E is a quasi-normed space such that every Cauchy sequence in E converges, then E is called* **quasi-Banach** *space.*

Example 1.2.22. *If $E = L_p(0,1)$ or $E = L_p(0,\infty)$ with $0 < p < 1$, then E is a quasi-Banach space with concavity modulus $C(L_p) = 2^{1/p-1}$.*

In many cases, the study of quasi-Banach spaces is significantly more difficult than that of Banach spaces. The reason is that many basic principles of functional analysis fail in quasi-Banach spaces. The most common example is that Hahn-Banach theorem fails even for such simple quasi-normed spaces as L_p, $0 < p < 1$.

Lemma 1.2.23. *The quasi-Banach space L_p, $0 < p < 1$, does not admit any continuous linear functional (see Section 1.47 of [47]). Moreover, there are no convex open subsets in this space (except the trivial ones).*

1.2.4 The Aoki-Rolewicz theorem

The very worst property of a quasi-Banach space is that the quasi-norm is not necessarily continuous in the topology induced by the quasi-norm itself. In order to somehow deal with such spaces, we need the Aoki-Rolewicz theorem. For completeness, we include the details of proof and will follow that given by Gustavsson (see [25]).

Lemma 1.2.24. *Let E be a linear space and let $f : E \to \mathbb{R}$ be such that*

$$f(x+y) \leq 2\max\{f(x), f(y)\}, \quad \forall x, y \in E.$$

If $i_j \geq 0$ are such that $\sum_{j=1}^{n} 2^{-i_j} \leq 1$, then

$$f(\sum_{j=1}^{n} x_j) \leq \max_{1\leq j\leq n} 2^{i_j} f(x_j).$$

Proof. We use induction on n. The assertion is valid for $n = 1$. Assume it is valid for $n < k$ and let us prove it for $n = k$. After permutation (if necessary), one can find $1 \leq l < k$ such that

$$\sum_{j=1}^{l} 2^{-i_j} \leq \frac{1}{2}, \quad \sum_{j=l+1}^{k} 2^{-i_j} \leq \frac{1}{2}.$$

By induction,

$$f(\sum_{j=1}^{l} x_j) \leq \max_{1\leq j\leq l} 2^{i_j-1} f(x_j), \quad f(\sum_{j=l+1}^{k} x_j) \leq \max_{l+1\leq j\leq k} 2^{i_j-1} f(x_j).$$

Hence,

$$f(\sum_{j=1}^{k} x_j) \leq 2\max\{f(\sum_{j=1}^{l} x_j), f(\sum_{j=1+1}^{k} x_j)\} \leq \max_{1\leq j\leq k} 2^{i_j} f(x_j).$$

□

Definition 1.2.25. *Two quasi-norms* $\|\cdot\|_{1,2}$ *on* E *are said to be* **equivalent** *if there exist constants* c_1, c_2 *such that*

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1, \quad \forall x \in E.$$

Theorem 1.2.26. *Every quasi-Banach space* $(E, \|\cdot\|)$ *admits an equivalent quasi-norm* $\|\cdot\|_{new}$ *such that*

$$\|x+y\|_{E,new}^p \leq \|x\|_{E,new}^p + \|y\|_{E,new}^p, \quad \forall x, y \in E \tag{1.8}$$

for some $p < 1$.

Proof. Let $p = \log_2^{-1}(2C(E))$. It is clear that

$$\|x+y\|_E^p \leq C(E)^p(\|x\|_E + \|y\|_E)^p \leq 2\max\{\|x\|_E^p, \|y\|_E^p\}, \quad \forall x, y \in E.$$

Define $f : E \to \mathbb{R}$ by the formula $f(x) = \|x\|_E^p$. Clearly, f satisfies the assumption of Lemma 1.2.24.

Let $x_1, \cdots, x_n \in E$. For every $1 \leq j \leq n$, find i_j such that

$$2^{-i_j} \leq \frac{\|x_j\|_E^p}{\sum_{j=1}^n \|x_j\|_E^p} \leq 2^{1-i_j}.$$

It follows that

$$\|\sum_{j=1}^{n} x_j\|_E^p \leq \max_{1\leq j\leq n} 2^{i_j}\|x_j\|_E^p \leq 2\sum_{j=1}^{n} \|x_j\|_E^p.$$

Define the new quasi-norm by the formula

$$\|x\|_{E,new}^p = \inf\{\sum_i \|x_i\|_E^p : \sum_i x_i = x\}.$$

It follows from the above that

$$\|x\|_{E,new} \leq \|x\|_E \leq 2^{1/p}\|x\|_{E,new} = 2C(E)\|x\|_{E,new}.$$

It is clear that the new quasi-norm satisfies the condition (1.8). □

Corollary 1.2.27. *Every quasi-Banach space admits an equivalent continuous quasi-norm.*

Proof. By the Aoki-Rolewicz theorem, we may assume that the inequality (1.8) holds for our quasi-norm. It follows that E is a metric space with a distance given by the formula

$$\mathrm{dist}(x, y) = \|x - y\|_E^p.$$

Since distance is continuous in any metric space, we are done. □

Note, that if the original quasi-norm takes the same value on equimeasurable functions, then so does the quasi-norm given by the Aoki-Rolewicz theorem. From now on, we assume that quasi-norm is continuous.

1.2.5 Symmetric spaces & their properties

Definition 1.2.28. *Let E be a quasi-Banach space of real-valued Lebesgue measurable functions either on $(0,1)$ or $(0,\infty)$ (with identification $m-$a.e.). E is said to be* **ideal lattice** *if $x \in E$ and $|y| \leq |x|$ implies that $y \in E$ and $\|y\|_E \leq \|x\|_E$.*

Definition 1.2.29. *The ideal lattice $E \subseteq S_0$ is said to be a* **symmetric** *quasi-Banach space if for every $x \in E$ and every $y \in S_0$ the assumption $y^* = x^*$ implies that $y \in E$ and $\|y\|_E = \|x\|_E$.*

In particular, if E is a Banach space, the following assertion is valid.

Lemma 1.2.30. *Let E be a symmetric Banach space either on the interval $(0,1)$ or on the semi-axis.*

1. *If $E = E(0,1)$ is a symmetric Banach space on $(0,1)$, then*

$$L_\infty \subseteq E \subseteq L_1.$$

These inclusions are continuous. Moreover, there exist absolute constants c_1 and c_2 such that

$$c_1\|x\|_{L_1} \leq \|x\|_E \leq c_2\|x\|_{L_\infty}$$

for every $x \in E$.

2. *If $E = E(0,\infty)$ is a symmetric Banach space on $(0,\infty)$, then*

$$L_1 \cap L_\infty \subseteq E \subseteq L_1 + L_\infty.$$

These inclusions are continuous. Moreover, there exist absolute constants c_1 and c_2 such that

$$c_1\|x\|_{L_1+L_\infty} \leq \|x\|_E \leq c_2\|x\|_{L_1\cap L_\infty}$$

for every $x \in E$.

Definition 1.2.31. *Let E be a symmetric quasi-Banach space. The space E is said to have* **order-continuous quasi-norm** *if $\|x_n\|_E \to 0$ for every sequence $\{x_n\}_{n=1}^{\infty} \subset E$ such that $x_n \downarrow 0$ almost everywhere.*

The following theorem can be found in [8] (see Theorem 5.5 of Chapter II there).

Theorem 1.2.32. *If E is a symmetric Banach space, then E is separable if and only if the norm in E is order-continuous.*

Definition 1.2.33. *The symmetric quasi-Banach space E is said to have the* **Fatou property** *if, whenever $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in E such that $x_n \to x$ almost everywhere for some $x \in S_0$, it follows that $x \in E$ and*

$$\|x\|_E \leq \liminf_{n\to\infty} \|x_n\|_E.$$

The following notion is somewhat weaker.

Definition 1.2.34. *Let E be a symmetric space. If unit ball of E is closed in E with respect to almost everywhere convergence, then the quasi-norm on E is said to be a* **Fatou quasi-norm**.

Example 1.2.35. *If the quasi-norm on E is order-continuous, then it is a Fatou quasi-norm.*

Note that if E is a symmetric Banach space, then the Banach dual of E is not necessarily a symmetric space. In this setting, the appropriate notion is that of Köthe duality. The necessary definition now follows.

Definition 1.2.36. *The Köthe dual space $E^{\times}$ of a symmetric Banach space E consists of all functions $x \in S_0$ for which the norm*

$$\|x\|_{E^{\times}} = \sup_{\|y\|_E \leq 1} \int_0^1 x(t)y(t)dt$$

is finite.

If E is a symmetric Banach space, then $E^{\times}$ is also a symmetric Banach space which is isometrically embedded into the Banach dual E^* of the space E.

The following theorem is proved in Zaanen [58] (see Chapter 15 there).

Theorem 1.2.37. *Let E be a symmetric Banach space.*

1. *The Köthe dual $E^{\times}$ satisfies the Fatou property.*
2. *The norm on E is order-continuous if and only if its Köthe dual $E^{\times}$ coincides with its Banach dual E^*.*
3. *E is equipped with a Fatou norm if and only if it can be isometrically embedded into its second Köthe dual $E^{\times\times}$.*

4. *E satisfies the Fatou property if and only if the natural embedding of E into $E^{\times\times}$ is a surjective isometry.*

Definition 1.2.38. *Let E be a symmetric quasi-Banach space. E is said to be* **fully symmetric** *if and only if $x \in E$, $y \in L_1 + L_\infty$, $y \prec\prec x$ implies that $y \in E$ and $\|y\|_E \leq \|x\|_E$.*

The following assertion can be found in Chapter II of [33] (see Theorem 4.9 and 4.10 there).

Theorem 1.2.39. *Let E be a symmetric Banach space.*

1. *If E is separable, then E is fully symmetric.*
2. *If E satisfies the Fatou property, then E is fully symmetric.*

For symmetric quasi-Banach space E, define E_0 to be closure of the set of simple functions with finite support in E.

Lemma 1.2.40. *If $E = E(0,1)$ is a symmetric quasi-Banach space on the interval $(0,1)$ and if $E \neq L_\infty(0,1)$, then E_0 is separable.*

Proof. For every $x \in E$,

$$\|x\|_E \geq x^*(u)\|\chi_{(0,u)}\|_E.$$

If x is unbounded, then $\|\chi_{(0,u)}\|_E \to 0$ as $u \to 0$. The assertion follows now from Theorem 4.8 of [33]. □

If E is as in the lemma above, then E_0 is called the **separable part** of E.

1.2.6 Interpolation

Let E_0 and E_1 be quasi-Banach function spaces. The intersection $E_0 \cap E_1$ equipped with the quasi-norm

$$\|x\|_{E_0 \cap E_1} = \max\{\|x\|_{E_0}, \|x\|_{E_1}\}, \quad x \in E_0 \cap E_1$$

is a quasi-Banach space. The sum $E_0 + E_1$ equipped with the norm

$$\|x\|_{E_0+E_1} = \inf\{\|x_0\|_{E_0} + \|x_1\|_{E_1} : x = x_0 + x_1,\ x_i \in E_i,\ i = 0,1\}, \quad \forall x \in E_0 + E_1$$

is a quasi-Banach space.

Definition 1.2.41. *The quasi-Banach space F with $E_0 \cap E_1 \subset F \subset E_0 + E_1$ is called an* **interpolation space** *with respect to E_0 and E_1 if every linear operator T bounded in E_0 and in E_1 is also bounded in F.*

It follows from the closed graph theorem that there exists a constant $C > 0$ such that

$$\|T\|_{F\to F} \leq C \max\{\|T\|_{E_0\to E_0}, \|T\|_{E_1\to E_1}\}.$$

If $C = 1$, the space F is called an **exact interpolation space** with respect to E_0 and E_1.

The following theorem due to Calderon and Mityagin can be found in Chapter II of [33] (see Theorem 4.3 there).

Theorem 1.2.42. *A symmetric quasi-Banach space E is an exact interpolation space with respect to L_1 and L_∞ if and only if E is fully symmetric.*

In fact, every interpolation space with respect to L_1 and L_∞ can be made fully symmetric by equivalent renorming.

1.2.7 Symmetric sequence spaces

Symmetric sequence spaces are the natural couterpart of the symmetric function spaces.

Let $x = \{x_n\}_{n\in\mathbb{N}} \in l_\infty$ and let $x_n \to 0$ as $n \to \infty$. The sequence x^* is a rearrangement of the sequence $|x| = \{x_n\}_{n\in\mathbb{N}}$ in decreasing order.

Definition 1.2.43. *The quasi-Banach space $E \subset c_0$ is called a* **symmetric quasi-Banach sequence space** *if*

1. *$x \in E$ and $|y| \leq |x|$ implies that $y \in E$ and $\|y\|_E \leq \|x\|_E$,*
2. *for every $x \in E$ and every $y \in S_0$ the assumption $y^* = x^*$ implies that $y \in E$ and $\|y\|_E = \|x\|_E$.*

It is also convenient to call l_∞ a Banach sequence space.

Definition 1.2.44. *Let $x, y \in c_0$. We say that y is* **submajorized** *by x* **in the sense of Hardy-Littlewood-Polya** *if*

$$\sum_{k=1}^{n} y_k^* \leq \sum_{k=1}^{n} x_k^*, \quad \forall n \in \mathbb{N}.$$

In this case, we write $y \prec\prec x$.

Definition 1.2.45. *Let $0 \leq x, y \in l_1$. We say that y is* **majorized** *by x* **in the sense of Hardy-Littlewood-Polya** *if $y \prec\prec x$ and $\|y\|_1 = \|x\|_1$. In this case, we write $y \prec x$.*

Definition 1.2.46. *Let E be a symmetric quasi-Banach space. E is said to be* **fully symmetric** *if and only if $x \in E$, $y \in l_\infty$, $y \prec\prec x$ implies that $y \in E$ and $\|y\|_E \leq \|x\|_E$.*

Theorem 1.2.47. *A symmetric quasi-Banach space E is an exact interpolation space with respect to l_1 and l_∞ if and only if E is fully symmetric.*

In fact, every interpolation space with respect to l_1 and l_∞ can be made fully symmetric by equivalent renorming.

1.2.8 Dilation operators & Boyd indices

If $\tau > 0$, the dilation operator σ_τ is defined by setting

$$(\sigma_\tau(x))(s) = x(\frac{s}{\tau}), \quad s > 0$$

in the case of the semi-axis. In the case of the interval $(0,1)$, the operator σ_τ is defined by

$$(\sigma_\tau x)(s) = \begin{cases} x(s/\tau), & s \le \min\{1,\tau\} \\ 0, & \tau < s \le 1. \end{cases}$$

Lemma 1.2.48. *If $x, y \in L_1 + L_\infty$ and $y \prec\prec x$, then,*

$$(\sigma_\tau(y))^* \le \sigma_\tau(y^*) \prec\prec \sigma_\tau(x^*).$$

Proof. In the case of the semi-axis, $d_{\sigma_\tau y} = \tau d_y = d_{\sigma_\tau(y^*)}$. In the case of the interval $(0,1)$, $d_{\sigma_\tau y} \le \tau d_y$ and $d_{\sigma_\tau(y^*)} = \min\{1, \tau d_y\}$. Hence, $d_{\sigma_\tau y} \le d_{\sigma_\tau(y^*)}$ and so $(\sigma_\tau(y))^* \le \sigma_\tau(y^*)$. Finally,

$$\int_0^t \sigma_\tau(y^*)(s)ds = \tau \int_0^{\frac{t}{\tau}} y^*(s)ds \le \tau \int_0^{\frac{t}{\tau}} x^*(s)ds = \int_0^t \sigma_\tau(x^*)(s)ds.$$

□

The following assertion is widely used in the literature. However, no direct reference seems to be available. We include the proof for convenience of the reader.

Lemma 1.2.49. *If $0 \le x, y \in L_1 + L_\infty$, then*

$$x^* + y^* \prec\prec 2\sigma_{\frac{1}{2}}((x+y)^*). \tag{1.9}$$

Proof. Fix $\varepsilon > 0$. It follows from Proposition 1.2.11 that, for each $t > 0$,

$$\int_0^t x^*(s)ds \le \varepsilon + \int_{e_1} x(s)ds, \quad \int_0^t y^*(s)ds \le \varepsilon + \int_{e_2} y(s)ds$$

for some e_1 and e_2 with $m(e_i) = t$. However,

$$\int_{e_1} x(s)ds + \int_{e_2} y(s)ds \le \int_{e_1 \cup e_2} (x+y)(s)ds \le$$

$$\le \sup_{m(e)=2t} \int_e (x+y)(s)ds = \int_0^{2t} (x+y)^*(s)ds,$$

again using Lemma 1.2.11. Since $\varepsilon > 0$ is arbitrary, it follows that

$$\int_0^t (x^* + y^*)(s)ds \le \int_0^{2t} (x+y)^*(s)ds.$$

Observing that

$$\int_0^{2t} u(s)ds = \int_0^t (2\sigma_{\frac{1}{2}} u)(s)ds,$$

the assertion follows immediately. □

If E is a symmetric quasi-Banach space and if $\tau > 0$, then the dilation operator σ_τ is a bounded operator on E (see [33], Chapter II.4, Theorem 4.4). If, in addition, E is a Banach space, then

$$\|\sigma_\tau\|_{E\to E} \le \max\{1, \tau\}, \quad \tau > 0.$$

As is easily seen, the operators σ_τ $(\tau \ge 1)$ satisfy the semi-group property $\sigma_{\tau_1}\sigma_{\tau_2} = \sigma_{\tau_1\tau_2}$.

Theorem 1.2.50. *Let E be a symmetric quasi-Banach space. The following two limits exist.*

$$\alpha_E = \lim_{\tau\to 0} \frac{1}{\log(\tau)} \log(\|\sigma_\tau\|_{E\to E})$$

and

$$\beta_E = \lim_{\tau\to\infty} \frac{1}{\log(\tau)} \log(\|\sigma_\tau\|_{E\to E}).$$

These two numbers α_E and β_E are called the **Boyd indices** of the symmetric space E.

Lemma 1.2.51. *For every symmetric Banach space E, $0 \le \alpha_E \le \beta_E \le 1$.*

Thus, α_E is called the **lower Boyd index** and β_E is called the **upper Boyd index**.

In a some sense, if $\alpha_E = 0$, then the symmetric Banach space E is close to L_∞. Similarly, if $\beta_E = 1$, then the symmetric Banach E is close to L_1. For any other case, the following theorem is valid

Theorem 1.2.52. *Let E be a symmetric Banach space either on the interval $(0,1)$ or on the semi-axis. If $0 < \alpha_E$ and $\beta_E < 1$, then E is an interpolation space between L_p and L_q for some $1 < p < q < \infty$.*

1.2.9 Convex and concave functions

Definition 1.2.53. *A function $f : \mathbb{R} \to \mathbb{R}$ is called* **convex** *if*

$$f(\lambda_1 t_1 + \lambda_2 t_2) \le \lambda_1 f(t_1) + \lambda_2 f(t_2) \quad \forall t_1, t_2 \in \mathbb{R}$$

provided that $0 \le \lambda_1, \lambda_2$ and $\lambda_1 + \lambda_2 = 1$.

Definition 1.2.54. *A function $f : \mathbb{R} \to \mathbb{R}$ is called* **concave** *if*

$$f(\lambda_1 t_1 + \lambda_2 t_2) \ge \lambda_1 f(t_1) + \lambda_2 f(t_2) \quad \forall t_1, t_2 \in \mathbb{R}$$

provided that $0 \le \lambda_1, \lambda_2$ and $\lambda_1 + \lambda_2 = 1$.

The following theorem is well-known.

Theorem 1.2.55. *If a function f is convex (respectively, concave), then it is continuous. Moreover, it is right-differentiable and left-differentiable at every point and the derivative function f' is increasing (respectively, decreasing).*

Definition 1.2.56. *The function $f : \mathbb{R} \to \mathbb{R}$ is called* **quasi-concave** *if there exists a concave function g and constant $c > 0$ such that $c^{-1}f \leq g \leq cf$.*

The following theorem can be found in Chapter II of [33] (see Theorem 1.1 there).

Theorem 1.2.57. *The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is quasi-concave if and only if the following inequalities are valid*

$$f(t_1) \leq Cf(t_2), \quad g(t_1) \leq Cg(t_2), \quad \forall 0 < t_1 < t_2.$$

Here, the function g is defined by setting $g(t) = t/f(t)$ for all $t > 0$.

Definition 1.2.58. *The fundamental function φ_E of a symmetric space E is defined by setting $\varphi(t) = \|\chi_{[0,t]}\|_E$ for every $t > 0$.*

Example 1.2.59. *The fundamental function φ_E of a symmetric Banach space E is always quasi-concave.*

1.2.10 Examples of symmetric spaces

We list below some of the most important examples of symmetric space.

Definition 1.2.60. *A function $\Phi : \mathbb{R} \to \mathbb{R}$ is called an* **Orlicz function** *if it satisfies the following conditions*

1. *$\Phi(t)$ is positive for every $t \in \mathbb{R}$.*
2. *Φ is even function, that is $\Phi(-t) = \Phi(t)$ for every $t \in \mathbb{R}$*
3. *Φ is convex function*
4. *$\Phi(0) = 0$*

Example 1.2.61. *If $p > 1$, then the function Φ_p defined by*

$$\Phi_p(t) = e^{|t|^p} - 1, \quad t \in \mathbb{R}$$

is an Orlicz function. If $0 < p \leq 1$, then the function Φ_p defined by

$$\Phi_p(t) = e^{|t|^p} - \sum_{k=0}^{[1/p]} \frac{|t|^{kp}}{k!}, \quad t \in \mathbb{R}$$

is an Orlicz function.

Definition 1.2.62. *The* **Orlicz space** *L_Φ consists of all $x \in S_0$ such that*

$$\|x\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_0^1 \Phi\left(\frac{x(t)}{\lambda}\right) dt \leq 1\right\} < \infty.$$

If Φ_p is the Orlicz function defined in the Example 1.2.61, then the Orlicz space L_{Φ_p} is called the **exponential Orlicz space** and denoted by $\exp(L_p)$.

Let ψ be an increasing concave continuous function either on $(0,1)$ or on the semi-axis such that $\psi(+0)=0$. The following definitions can be found in [33] (see Chapter II, Section 5 there).

Definition 1.2.63. *The* **Lorentz space** Λ_ψ *is the space of all measurable functions on the interval* $(0,1)$ *such that*

$$\|x\|_{\Lambda_\psi} = \int_0^1 x^*(t)d\psi(t) < \infty.$$

Definition 1.2.64. *The* **Marcinkiewicz space** M_ψ *is the space of all measurable functions on* $(0,1)$ *such that*

$$\|x\|_{M_\psi} = \sup_{0<t\le 1} \frac{1}{\psi(t)} \int_0^t x^*(s)ds < \infty.$$

Lorentz and Marcinkiewicz spaces on the semi-axis can be defined in similar manner. We now gather some of the most important properties of Lorentz and Marcinkiewicz spaces (see [33]).

Theorem 1.2.65. *Let* ψ *be an increasing concave continuous function either on* $(0,1)$ *or on the semi-axis such that* $\psi(0)=0$.

1. *The Lorentz space* Λ_ψ *is always separable.*
2. *The Köthe dual of the Lorentz space* Λ_ψ *is the Marcinkiewicz space* M_ψ.
3. *The Köthe dual of the Marcinkiewicz space* M_ψ *is the Lorentz space* Λ_ψ.
4. *Both Lorentz and Marcinkiewicz spaces satisfy the Fatou property.*
5. *If* E *is a symmetric Banach space with fundamental function* φ_E, *then*

$$\Lambda_{\varphi_E} \subset E \subset M_{\psi_E}.$$

Here, the function ψ_E *is defined by setting* $\psi_E(t) = t/\varphi_E(t)$ *for* $t>0$.

We need the following description of the Boyd indices for Lorentz and Marcinkiewicz spaces.

Lemma 1.2.66. *Let* Λ_ψ *be a Lorentz space on the interval* $(0,1)$.

1.
$$\liminf_{t\to 0} \frac{\psi(2t)}{\psi(t)} = 1 \iff \alpha_{\Lambda_\psi} = 0.$$

2.
$$\limsup_{t\to 0} \frac{\psi(2t)}{\psi(t)} = 2 \iff \beta_{\Lambda_\psi} = 0.$$

If Λ_ψ is a Lorentz space on the semi-axis, one should replace "$t \to 0$"with "either $t \to 0$ or $t \to \infty$".

Lemma 1.2.67. *Let M_ψ be a Marcinkiewicz space on the interval $(0,1)$.*

1. $$\liminf_{t\to 0} \frac{\psi(2t)}{\psi(t)} = 1 \iff \beta_{M_\psi} = 1.$$

2. $$\limsup_{t\to 0} \frac{\psi(2t)}{\psi(t)} = 2 \iff \alpha_{M_\psi} = 0.$$

If M_ψ is a Marcinkiewicz space on the semi-axis, one should replace "$t \to 0$"with "either $t \to 0$ or $t \to \infty$".

One usually refers to the following class of spaces as "Lorentz spaces". We will not use this name in order to avoid confusion with Lorentz spaces defined above.

Definition 1.2.68. *If $p, q > 0$, then the space $L_{p,q}(0,1)$ is the space of all measurable functions x on the interval $(0,1)$ such that*

$$\|x\|_{p,q} = (\int_0^1 (x^*(t)))^q dt^{q/p})^{1/q} < \infty.$$

$L_{p,q}(0,\infty)$ is defined in a similar manner.

Definition 1.2.69. *If $p > 0$, then the space $L_{p,\infty}(0,1)$ is the space of all measurable functions x on the interval $(0,1)$ such that*

$$\|x\|_{p,\infty} = \sup_{0<t<1} t^{1/p} x^*(t) < \infty.$$

$L_{p,\infty}(0,\infty)$ is defined in a similar manner.

It is well-known (see [8]) that, for $p, q \geq 1$, the quasi-norm $\|\cdot\|_{p,q}$ is equivalent to a symmetric norm.

For further properties of Lorentz, Marcinkiewicz and Orlicz spaces, we refer to [33, 36] and [44].

1.2.11 Some special spaces.

Definition 1.2.70. *A* **Poisson random variable** *with a parameter $a > 0$ is the random variable which takes values $n \in \mathbb{Z}_+$ with probability $e^{-a}\frac{a^n}{n!}$.*

Lemma 1.2.71. *If N is Poisson random variable, then*

$$\exp(-1 - 2t \cdot \operatorname{arcsinh}(2t)) \leq m(\{|N| > t\}) \leq \exp(1 - \frac{t}{3e} \operatorname{arcsinh}(\frac{t}{3e})).$$

Proof. Let us prove the left inequality. Assume first that $t \geq 1$. There exists $n \in \mathbb{N}$ such that $t \in [n, n+1)$. Clearly,

$$m(\{|N| > t\}) \geq m(\{N = n+1\}) = \frac{1}{e \cdot (n+1)!} \geq \frac{1}{e \cdot (n+1)^{n+1}} =$$

$$= \exp(-1 - (n+1)\log(n+1)) \geq \exp(-1 - (n+1)\text{arcsinh}(n+1)).$$

However, $n + 1 \leq 2n \leq 2t$ and, therefore,

$$m(\{N > t\}) \geq \exp(-1 - 2t \cdot \text{arcsinh}(2t)).$$

If $t \in [0, 1)$, then

$$m(\{|N| > t\}) \geq m(\{|N| = 1\}) = \frac{1}{e} \geq \exp(-2t\text{arcsinh}(2t)).$$

Let us now prove the right inequality. Assume first that $t \geq 5$. There exists $n \in \mathbb{N}$ such that $t \in [n, n+1)$ and $n \geq 3$. Clearly,

$$m(\{|N| > t\}) = m(\{N \geq n+1\}) = \sum_{k=n+1}^{\infty} \frac{1}{e \cdot k!} =$$

$$= \frac{1}{e \cdot (n+1)!}(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots) \leq$$

$$\leq \frac{1}{e \cdot (n+1)!} \cdot \sum_{k=0}^{\infty} \frac{1}{(n+2)^k} = \frac{1}{e(n+1)(n+1)!} \leq \frac{1}{(n+1)!}.$$

Note that by the Stirling formula,

$$n! \geq (\frac{n}{e})^n.$$

Therefore,

$$m(\{|N| > t\}) \leq (\frac{n+1}{e})^{-(n+1)} = \exp(-(n+1)\log(\frac{n+1}{e})).$$

However,

$$\log(\frac{n+1}{e}) \geq \log(\frac{t}{e}) \geq \text{arcsinh}(\frac{t}{3e}).$$

The latter inequality is valid since $t \geq 5$. Hence,

$$m(\{|N| > t\}) \leq \exp(-\frac{t}{3e}\text{arcsinh}(\frac{t}{3e})).$$

If $t \in [0, 5)$, then

$$m(\{|N| > t\}) \leq 1 \leq \exp(1 - \frac{t}{3e}\text{arcsinh}(\frac{t}{3e})).$$

$\square$

Corollary 1.2.72. *If N is a Poisson random variable, then the smallest symmetric space containing N is M_ψ where $\psi(t) = t\log(e/t)/\log(\log(e^e/t))$.*

Proof. It follows from Corollary 3.1.8 that the smallest symmetric space containing N coincides with enveloping Marcinkiewicz space. The computation above shows that the decreasing rearrangement of N is equivalent to ψ' at 0. This suffices to conclude the statement. □

Definition 1.2.73. *A* **Gaussian random variable** *is any random variable ξ such that*

$$m(\{t:\ \xi(t)<s\}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s} e^{-u^2/2}du.$$

Lemma 1.2.74. *If ξ is a Gaussian random variable, then the smallest symmetric space containing ξ is $\exp(L_2)$.*

Proof. Let ψ be a concave increasing function such that $\psi' = \xi^*$. It is easy to show that $\psi(2t)/\psi(t) \to 2$ as $t \to 0$. Thus, the smallest symmetric space containing ξ coincides with enveloping Marcinkiewicz space, which is $\exp(L_2)$. □

1.2.12 Expectation operators

Let E be a fully symmetric quasi-Banach space either on the interval $(0,1)$ or on the semi-axis. We need the notion of an averaging operator (see [11]) and that of an expectation operator.

Let $\mathcal{A} = \{A_k\}$ be a (finite or infinite) sequence of disjoint sets of finite measure and denote by $\mathfrak{A}$ the collection of all such sequences. Denote by A_∞ the complement of $\cup_k A_k$.

The partial averaging operator $P(\cdot|\mathcal{A}) : L_1 + L_\infty \to L_1 + L_\infty$ is defined by setting

$$P(x|\mathcal{A}) = \sum_k \frac{1}{m(A_k)}(\int_{A_k} x(s)ds)\chi_{A_k} + x\chi_{A_\infty}.$$

Note, that we do not require A_∞ to have finite measure.

The averaging operator $E(\cdot|\mathcal{A}) : L_1 + L_\infty \to L_1 + L_\infty$ is defined by setting

$$E(x|\mathcal{A}) = \sum_k \frac{1}{m(A_k)}(\int_{A_k} x(s)ds)\chi_{A_k}.$$

Note, that we do not require A_∞ to have finite measure.

Every partial averaging operator is a contraction both in L_1 and L_∞. Hence, $P(\cdot|\mathcal{A})$ is also contraction in E. Moreover, $P(\cdot|\mathcal{A})$ is a doubly stochastic operator in the sense of [49]. Every averaging operator is a contraction both in L_1 and L_∞. Hence, $E(\cdot|\mathcal{A})$ is also contraction in E.

Since $P(\cdot|\mathcal{A}) \in \Sigma$, it follows that $P(x|\mathcal{A}) \in \Omega(x)$ (respectively, $P(x|\mathcal{A}) \in \Omega'(x)$ if $x \in L_1$) for every $\mathcal{A} \in \mathfrak{A}$. Since $E(\cdot|\mathcal{A}) \in \Sigma$, it follows that $E(x|\mathcal{A}) \in \Omega(x)$ for every $\mathcal{A} \in \mathfrak{A}$. As will be seen, elements of the form $P(x|\mathcal{A})$ and $E(x|\mathcal{A})$ play a central role.

1.2.13 The orbits & their properties

Define the semigroups Σ and Σ^+ by setting

$$\Sigma = \{A : L_1 + L_\infty \to L_1 + L_\infty, \quad \|A\|_{L_1 \to L_1}, \|A\|_{L_\infty \to L_\infty} \leq 1\},$$
$$\Sigma^+ = \{A \geq 0, \quad A \in \Sigma\}.$$

The positive operator $A : L_1+L_\infty \to L_1+L_\infty$ is called bistochastic if $A1 = 1$ and

$$\int_0^\infty (Ax)(s)ds = \int_0^\infty x(s)ds, \quad x \in L_1 + L_\infty.$$

The semigroup of bistochastic operators is denoted by Σ'.

The orbits of the element x with respect to the semi-groups Σ, Σ^+ and Σ' will be denoted by $\Omega(x)$, $\Omega^+(x)$ and $\Omega'(x)$ respectively. That is,

$$\Omega(x) = \{Ax, \quad A \in \Sigma\}, \quad \Omega^+(x) = \{Ax, \quad A \in \Sigma^+\}, \quad \Omega'(x) = \{Ax, \quad A \in \Sigma'\}.$$

The following assertion is known as Calderon-Mityagin theorem (see [33]).

Theorem 1.2.75. *If $x \in L_1 + L_\infty$, then*

$$\Omega(x) = \{y \in L_1 + L_\infty : \; y \prec\prec x\},$$
$$\Omega^+(x) = \{0 \leq y \in L_1 + L_\infty : \; y \prec\prec x\},$$
$$\Omega'(x) = \{0 \leq y \in L_1 : \; y \prec x\}.$$

It is clear that for all equimeasurable functions x, y from $L_1 + L_\infty$ we have

$$\Omega(x) = \Omega(y), \, \Omega^+(x) = \Omega^+(y), \, \Omega'(x) = \Omega'(y).$$

In particular,

$$\Omega(x) = \Omega(x^*), \, \Omega^+(x) = \Omega^+(x^*), \, \Omega'(x) = \Omega'(x^*).$$

The following question was initially investigated by Braverman & Mekler (see [11]). They only considered symmetric Banach spaces on the interval $(0, 1)$.

Question 1.2.76. *Find conditions which guarantee that the orbit $\Omega(x)$ coincides with the norm-closed convex hull of its extreme points.*

In this thesis, we study this question also for the sets $\Omega^+(x)$, $\Omega'(x)$ in the more general setting of symmetric quasi-Banach spaces on the interval $(0, 1)$ or on the semi-axis. Necessary and sufficient conditions are given in Chapter 2.

Fo the convenience of the reader, we give here the classification of the extreme points of the sets $\Omega(x)$, $\Omega^+(x)$ and $\Omega'(x)$ (see Theorem A.0.14).

1. For every $x \in L_1(0, 1)$, we have

$$\text{extr}(\Omega(x)) = \{y : \; y^* = x^*\},$$
$$\text{extr}(\Omega^+(x)) = \{y \geq 0 : \; y^* = x^* \chi_{[0,\beta]}\},$$
$$\text{extr}(\Omega'(x)) = \{y \geq 0 : \; y^* = x^*\}.$$

2. For every $x \in (L_1 + L_\infty)(0, \infty)$, we have
$$\mathrm{extr}(\Omega(x)) = \{y : \ y^* = x^*, \ |y| \geq y^*(\infty)\},$$
$$\mathrm{extr}(\Omega^+(x)) = \{y : \ y^* = x^* \chi_{[0,\beta]}, \ |y| \geq y^*(\infty)\}.$$
3. For every $x \in L_1(0, \infty)$, we have
$$\mathrm{extr}(\Omega'(x)) = \{y \geq 0 : \ y^* = x^*\}.$$

We suppose that E is a fully symmetric quasi-Banach space. Note that since E is fully symmetric, it follows that each of the orbits $\Omega(x)$, $\Omega^+(x)$ and $\Omega'(x)$ is a subset of E.

Here, we use the notation of [31] as opposed to that of [55]. More precisely, the following notation is employed.
$$\mathcal{Q}(x) = \mathrm{Conv}(\mathrm{extr}(\Omega(x))), \quad \mathcal{Q}_E(x) = \overline{\mathcal{Q}(x)},$$
$$\mathcal{Q}^+(x) = \mathrm{Conv}(\mathrm{extr}(\Omega^+(x))), \quad \mathcal{Q}_E^+(x) = \overline{\mathcal{Q}^+(x)},$$
$$\mathcal{Q}'(x) = \mathrm{Conv}(\mathrm{extr}(\Omega'(x))), \quad \mathcal{Q}'_E(x) = \overline{\mathcal{Q}'(x)}.$$
Here, closure is taken in the natural topology in E (that is the one generated by the quasi-norm).

In addition, if $x \in (L_1 + L_\infty)(0, \infty)$ but $x \notin L_1(0, \infty)$, we set
$$\mathcal{Q}'(x) = \mathrm{Conv}(\{y \geq 0 : \ y^* = x^*, y \geq y^*(\infty)\})$$
and
$$\mathcal{Q}'_E(x) = \overline{\mathcal{Q}'(x)}.$$

A partial answer to Question 1.2.76 was first given by Braverman and Mekler in [11]. They showed that if
$$\lim_{\tau \to \infty} \frac{1}{\tau} \|\sigma_\tau\|_{E \to E} = 0, \tag{1.10}$$
then $\Omega(x) = \mathcal{Q}_E(x)$ for symmetric Banach spaces on the interval $(0, 1)$. In contrast, we will show that
$$\mathcal{Q}_E(x) = \Omega(x) \Longleftrightarrow \lim_{\tau \to \infty} \frac{1}{\tau} \|\sigma_\tau(x^*)\|_E = 0. \tag{1.11}$$
Note that our condition is a localised version of the global condition (1.10) and goes much further as it permits us to characterise those elements $x \in E$ for which the equality $\Omega(x) = \mathcal{Q}_E(x)$ is valid.

The implication $\Longleftarrow$ in the equivalence (1.11) was proved by Sukochev and Zanin in [55]. Also, the implication $\Longleftarrow$ was proved in [55] for every symmetric Banach space E on the semi-axis such that $E \not\subset L_1$. If $E \subset L_1$ is a space on the semi-axis, then the implication $\Longleftarrow$ in (1.11) must be replaced with the stronger statement
$$\mathcal{Q}_E(x) = \Omega(x) \Longleftarrow \lim_{\tau \to \infty} \frac{1}{\tau} \|\sigma_\tau(x^*) \chi_{(0,1)}\|_E = 0.$$
The implication $\Longrightarrow$ was not proved in [55] in full generality. A complete proof of the implication $\Longrightarrow$ was given by Kalton, Sukochev and Zanin in [31] using very different methods.

1.2.14 Characteristic function of a random variable

In probability theory, a measurable function on the interval $(0,1)$ is called a random variable. We refer the reader to [24] for the following assertions.

Definition 1.2.77. *Let x be a random variable. The function $\varphi_x : \mathbb{R} \to \mathbb{C}$ defined by the following formula*

$$\varphi_x(t) = \int_0^1 e^{itx(s)} ds$$

is called the **characteristic function** *of a random variable x.*

We need the following basic facts from probability theory.

Lemma 1.2.78. *Let x_1, x_2 be random variables. If $\varphi_{x_1} = \varphi_{x_2}$, then x_1 and x_2 are equidistributed.*

Lemma 1.2.79. *If the random variables x_k, $1 \leq k \leq n$ are independent, then*

$$\varphi_{\sum_{k=1}^n x_k} = \prod_{k=1}^n \varphi_{x_k}.$$

Conversely, if, for every real sequence λ_k, $1 \leq k \leq n$, we have

$$\varphi_{\sum_{k=1}^n \lambda_k x_k} = \prod_{k=1}^n \varphi_{\lambda_k x_k},$$

then the random variables x_k, $1 \leq k \leq n$ are independent.

Lemma 1.2.80. *Let x and x_n, $n \in \mathbb{N}$, be random variables. If $\varphi_{x_n}(t) \to \varphi_x(t)$ for every $t \in \mathbb{R}$, then $x_n \to x$ in distribution.*

1.2.15 Basic properties of the operator K

In [12], Braverman introduced a new approach to the Johnson-Schechtman inequality based on earlier ideas of Kruglov and depending on a certain nonlinear construction. To simplify the approach of Braverman, Astashkin & Sukochev introduced a linear operator, called the Kruglov operator, which we now describe.

To do so, it is necessary to change the underlying measure space from the interval to the space Ω defined by

$$\Omega = \prod_{n=0}^{\infty} ((0,1), m).$$

It is a well-known fact that the measure space Ω equipped with product measure is isomorphic to the interval $(0,1)$ equipped with Lebesgue measure.

Definition 1.2.81. *Let x be a random variable (measurable function) on the interval $(0,1)$. Let $\{B_n\}_{n=0}^{\infty}$ be a fixed sequence of mutually disjoint measurable subsets of $(0,1)$ such that $m(B_n) = \dfrac{1}{en!}$. The* **Kruglov operator** *$K : S_0(0,1) \to S_0(\Omega)$ is defined by setting*

$$Kx(\omega) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} x(\omega_k)\chi_{B_n}(\omega_0), \quad x \in S_0(0,1).$$

Here, $\omega = (\omega_0, \omega_1, \cdots)$ is an element of Ω.

Remark 1.2.82. *The operator K is well-defined.*

Proof. Suppose that x_1, x_2 are measurable functions such that $x_1 = x_2$ almost everywhere. The set $A = \{\omega : x_1(\omega) = x_2(\omega)\}$ is a set of full measure in $(0,1)$. If $\omega_k \in A$, then $(Kx_1)(\omega) = (Kx_2)(\omega)$. However, $(0,1) \times A^{\infty}$ is a set of full measure in Ω. Thus, $Kx_1 = Kx_2$ almost everywhere. □

It is clear that if $0 \le x \in S_0(0,1)$, then $0 \le Kx \in S_0(\Omega)$, so that K is a positive linear operator. The following assertion was proved in [6] for the case of symmetric Banach spaces.

Lemma 1.2.83. *Let E and F be symmetric quasi-Banach spaces on the interval $(0,1)$. If K has the property that $Kx \in F$ for each $x \in E$, then $K : E \to F$ is a bounded operator.*

Proof. Let $C(E)$ be the concavity modulus of E. If K is not bounded, one can select a sequence $\{x_n\}_{n=1}^{\infty} \subset E$ of positive elements such that $\|x_n\|_E = 1$ and $\|Kx_n\|_F \ge n^3 C(E)^n$.

We claim that the series $x = \sum_{n=1}^{\infty} n^{-2} C(E)^{-n} x_n$ converges in E. Indeed, for every $M > N$, we have

$$\|\sum_{n=N+1}^{M} \frac{1}{n^2} C(E)^{-n} x_n\|_E \le C(E)^{-N} \sum_{n=N+1}^{M} \frac{1}{n^2}\|x_n\|_E \le \frac{1}{N} C(E)^{-N}.$$

For every $n \in \mathbb{N}$, $x \ge n^{-2}C(E)^{-n}x_n$. Since the operator K is positive, we have $Kx \ge n^{-2}C(E)^{-n}Kx_n \ge 0$. It follows that $\|Kx\|_F \ge n$ for every $n \in \mathbb{N}$. □

The following lemma is similar to Lemma 1 of [12].

Lemma 1.2.84. *Let E be an arbitrary symmetric quasi-Banach space on the interval $(0,1)$. For every $x \in S_0$ such that $Kx \in E$ we also have $x \in E$.*

Proof. We have
$$|Kx(\omega)| \ge |x(\omega_1)|\chi_{B_1}(\omega_0) \sim \sigma_{1/e}(x).$$
Therefore, $\sigma_{1/e}x \in E$ and so $x \in E$. □

It is now possible to define the Kruglov property for a symmetric quasi-Banach space E.

Definition 1.2.85. *A symmetric quasi-Banach space E on the interval $(0,1)$ is said to have the* **Kruglov property** *($E \in \mathbb{K}$) if and only if $x \in E$ implies that $Kx \in E$.*

The following crucial property of the operator K may be found in [6]. This property was taken by Braverman as a definition of the Kruglov operator.

Lemma 1.2.86. *For every $x \in S_0$, we have $\varphi_{Kx} = \exp(\varphi_x - 1)$.*

Proof. It is clear that

$$\varphi_{Kx}(t) = \int_\Omega e^{it(Kx)(\omega)} \prod_{n=0}^{\infty} d\omega_n = \sum_{n=0}^{\infty} \frac{1}{en!} \int_{(0,1)^n} \exp(it \sum_{k=1}^{n} x(\omega_k)) \prod_{k=1}^{n} d\omega_k =$$

$$= \sum_{n=0}^{\infty} \frac{1}{en!} \prod_{k=1}^{n} \int_{(0,1)} \exp(itx(\omega_k) d\omega_k = \sum_{n=0}^{\infty} \frac{1}{en!} \varphi_x^n(t) = e^{\varphi_x(t)-1}.$$

□

1.2.16 Banach-Saks indices

The following theorem is due to Banach and Saks.

Theorem 1.2.87. *Let H be a Hilbert space. If a sequence $\{x_n\} \subset H$ converges weakly, then there exists a subsequence $\{y_k\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ and there exists $x \in E$ such that*

$$\frac{1}{n} \sum_{k=1}^{n} y_k \to x.$$

This leads to the following definition.

Definition 1.2.88. *Let E be a Banach space. If any weakly-convergent sequence $\{x_n\}_{n=1}^{\infty} \subset E$ contains a subsequence $\{y_k\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ such that Cesaro means $n^{-1} \sum_{k=1}^{n} y_k$ are convergent in E, then E is said to have* **Banach-Saks property**.

Definition 1.2.89. *Let E be a Banach space and let $p > 1$. The bounded sequence $\{x_n\}_{n=1}^{\infty} \subset E$ is called a* **p-BS-sequence** *if for all subsequences $\{y_k\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$,*

$$\sup_{m \in \mathbb{N}} m^{-\frac{1}{p}} \| \sum_{k=1}^{m} y_k \|_E < \infty.$$

Definition 1.2.90. *Let E be a Banach space and let $p > 1$. We say that E has the* **p-BS-property** *if each weakly null sequence contains a p-BS-subsequence.*

Consider the set

$$\Gamma(E) = \{p : p \geq 1, E \in BS(p)\}.$$

Clearly, either $\Gamma(E) = [1, \gamma]$, or $\Gamma(E) = [1, \gamma)$ for some $\gamma \geq 1$.

If, in the preceding definition, we replace all weakly null sequences by weakly null sequences of independent random variables (respectively, by weakly null sequences of pairwise disjoint elements; by weakly null sequences of independent identically distributed random variables), we obtain the set $\Gamma_i(E)$ (respectively, $\Gamma_d(E)$, $\Gamma_{iid}(E)$).

The general problem of describing and comparing the sets $\Gamma(E)$, $\Gamma_i(E)$, $\Gamma_{iid}(E)$ and $\Gamma_d(E)$ in various classes of symmetric spaces was addressed in [51, 21, 53, 4, 52, 5]. In particular, it follows directly from the definition that

$$1 \in \Gamma(E) \subset \Gamma_i(E) \subset \Gamma_{iid}(E) \subseteq [1, 2].$$

It follows from Lemma 3 of [28] that

$$\Gamma_i(E) \subset \Gamma_d(E)$$

for any symmetric space E.

Moreover, the sets $\Gamma(E)$ and $\Gamma_i(E)$ often coincide. For example, it follows from Corollary 4.4 and Theorem 4.5 of [52] that

$$\Gamma(L_p) = \Gamma_i(L_p) = \Gamma_{iid}(L_p), \quad \forall 1 < p < \infty.$$

If $E = \Lambda_{t^{1/2}}$ is Lorentz space generated by the function $\psi(t) = t^{1/2}$, $t > 0$, then it follows from Theorem 5.9 of [52] and Proposition 4.12 of [4] that

$$\Gamma(\Lambda_{t^{1/2}}) = [1, 2), \quad \Gamma_i(\Lambda_{t^{1/2}}) = [1, 2].$$

The following theorem (see [53, Theorem 9] for the proof) states that these two situations are typical

Theorem 1.2.91. *Let E be a symmetric Banach space on the interval $(0, 1)$. Assume that $\Gamma(E) \neq \{1\}$. One of the following possibilities occur*

1. $\Gamma_i(E) = \Gamma(E)$
2. $\Gamma_i(E) = [1, 2]$ *and* $\Gamma(E) = [1, 2)$.

In Theorem 3.5.4, we show the connection between the class of all symmetric spaces with Kruglov property and the estimates on Γ_{iid}. We will prove the general theorem (see Theorem 4.6.3 below) that

$$\Gamma_i(E) = \Gamma_{iid}(E) \cap \Gamma_d(E). \tag{1.12}$$

Since every Lorentz space Λ_ψ satisfies the condition

$$\Gamma_d(\Lambda_\psi) = [1, \infty),$$

the equality (1.12) then shows that $\Gamma_i(\Lambda_\psi) = \Gamma_{iid}(\Lambda_\psi)$ an this permits us to describe all Lorentz spaces Λ_ψ for which $\Gamma_i(\Lambda_\psi)$ is non-trivial.

Examples of symmetric spaces E such that $\Gamma(E) = \{1\}$ but $\Gamma_i(E) \neq \{1\}$ have been produced in [5] under the assumption that E has the Kruglov property. We present examples of Lorentz and Marcinkiewicz spaces E such that $\Gamma_i(E) = \Gamma_{iid}(E) \neq \{1\}$ and which do not possess the Kruglov property (see Example 3.6.8).

Chapter 2

Orbits

The results of this chapter were mostly published in [55] and [31].

2.1 The dilation functional and its properties

The following lemma introduces dilation functionals φ, φ_{fin} and φ_{cut} on E, which are a priori non-linear. The behavior of these functionals on the positive part E_+ of E provides the key to our main question on orbits.

Lemma 2.1.1. *Let E be a fully symmetric quasi-Banach space either on the interval $(0,1)$ or on the semi-axis. For every $x \in E$, the following limit exists and is finite.*

$$\varphi(x) = \lim_{s\to\infty} \frac{1}{s}\|\sigma_s(x^*)\|_E,\ x \in E. \tag{2.1}$$

Proof. We prove that the function

$$s \to \frac{1}{s}\|\sigma_s x^*\|_E$$

is decreasing. Let $s_2 > s_1$. We have $s_2 = s_3 s_1$ and $s_3 > 1$. Note that according to the semi-group property of the operators σ_τ,

$$\sigma_{s_2}(x^*) = \sigma_{s_3 s_1}(x^*) = \sigma_{s_3}(\sigma_{s_1}(x^*)).$$

Therefore,

$$\frac{1}{s_2}\|\sigma_{s_3}(\sigma_{s_1}(x^*))\|_E \leq \frac{||\sigma_{s_3}\|_{E\to E}}{s_2}\|\sigma_{s_1}(x^*)\|_E \leq \frac{1}{s_1}\|\sigma_{s_1}(x^*)\|_E,$$

since

$$\|\sigma_{s_3}\|_{E\to E} \leq s_3.$$

It follows immediately that the limit in (2.1) exists. □

Lemma 2.1.2. *Let E be a fully symmetric quasi-Banach space on the semi-axis. The following limits exist and are finite.*

$$\varphi_{fin}(x) = \lim_{s\to\infty} \frac{1}{s}\|\sigma_s(x^*)\chi_{[0,1]}\|_E,\ x \in E, \tag{2.2}$$

$$\varphi_{cut}(x) = \lim_{s\to\infty} \frac{1}{s}\|\sigma_s(x^*)\chi_{[0,s]}\|_E,\ x \in E. \tag{2.3}$$

Proof. It is trivial to see that

$$\sigma_s(x^*\chi_{[0,1]}) = \sigma_s(x^*)\chi_{[0,s]}.$$

Therefore,

$$\varphi_{cut}(x) = \varphi(x^*\chi_{[0,1]}).$$

Hence, existence of the limit in (2.3) follows from Lemma 2.1.1.

It is trivial to see that the operators ζ_τ defined as

$$\zeta_s(x) = \sigma_s(x)\chi_{[0,1]}, \quad s \geq 1$$

also satisfy the semi-group property. Thus, existence of the limit in (2.2) follows *mutatis mutandi*. □

Note that we assumed continuity of the quasi-norm in the introduction. According to the Corollary 1.2.27, this is not a restriction.

Lemma 2.1.3. *If E is a fully symmetric quasi-Banach space either on the interval $(0,1)$ or on the semi-axis, then, the functional φ defined by (2.1) is continuous. If $E = E(0,\infty)$ then the functionals φ_{fin} and φ_{cut} are also continuous.*

Proof. According to the Aoki-Rolewicz theorem 1.2.26, we may assume that our quasi-norm satisfies inequality (1.8) with some $p < 1$. It follows that

$$|\,\|x_1\|_E^p - \|y_1\|_E^p\,| \leq \|x_1 - y_1\|_E^p.$$

Note the elementary inequality

$$|a^{1/p} - b^{1/p}| \leq \frac{1}{p}|a-b| \cdot (\max\{a,b\})^{1/p-1}.$$

Substitute $a = \|x_1\|_E^p$ and $b = \|y_1\|_E^p$. It follows that

$$|\,\|x_1\|_E - \|y_1\|_E\,| \leq \frac{1}{p}\|x_1 - y_1\|_E^p \cdot (\max\{\|x_1\|_E, \|y_1\|_E\})^{1-p}.$$

Substitute $x_1 = s^{-1}\sigma_s(x^*)$ and $y_1 = s^{-1}\sigma_s(y^*)$. It follows that

$$|\,\frac{1}{s}\|\sigma_s(x^*)\|_E - \frac{1}{s}\|\sigma_s(y^*)\|_E\,| \leq \frac{1}{p}\|\frac{1}{s}\sigma_s(x^*-y^*)\|_E^p \max\{\frac{1}{s}\|\sigma_s(x^*)\|_E, \frac{1}{s}\|\sigma_s(y^*)\|_E\}^{1-p}.$$

Letting $s \to \infty$, it is clear that

$$|\varphi(x) - \varphi(y)| = \lim_{s\to\infty} |\,\|\frac{1}{s}\sigma_s(x^*)\|_E - \|\frac{1}{s}\sigma_s(y^*)\|_E\,|.$$

Evidently,

$$\|\frac{1}{s}\sigma_s(x^*)\|_E \leq \|x\|_E, \ \|\frac{1}{s}\sigma_s(y^*)\|_E \leq \|y\|_E$$

and

$$\frac{1}{s}\|\sigma_s(x^* - y^*)\|_E \leq \|x^* - y^*\|_E \leq \|x - y\|_E.$$

Therefore,

$$|\varphi(x) - \varphi(y)| \leq \frac{1}{p}\|x-y\|_E^p \cdot (\max\{\|x\|_E, \|y\|_E\})^{1-p}.$$

The proof for φ_{fin} and φ_{cut} follows *mutatis mutandi*. □

The assertion of Lemma 2.1.3 can be significantly improved.

Lemma 2.1.4. *Let E be a fully symmetric quasi-Banach space either on the interval $(0,1)$ or on the semi-axis. If $x, y \in E$, then*

$$|\varphi(x) - \varphi(y)| \leq \frac{1}{p}\varphi(x-y)^p(\max\{\varphi(x), \varphi(y)\})^{1-p}.$$

The proof follows that of Lemma 2.1.3 *mutatis mutandi.*

Corollary 2.1.5. *Let E be a fully symmetric quasi-Banach space either on the interval $(0,1)$ or on the semi-axis. If $x, y \in E$ and $\varphi(x-y) = 0$, then $\varphi(x) = \varphi(y)$.*

Lemma 2.1.6. *Let E be a fully symmetric quasi-Banach space.*

i) If $x, y \in E$ are equimeasurable, then $\varphi(y) = \varphi(x)$.

ii) If $x, y \in E$ satisfy $|y| \leq |x|$, then $\varphi(y) \leq \varphi(x)$.

iii) $\varphi(x) \leq \|x\|_E$ for every $x \in E$.

iv) If $x, y \in E$ satisfy $y \prec\prec x$, then $\varphi(y) \leq \varphi(x)$.

If, in addition, $E = E(0,\infty)$, then φ_{fin} and φ_{cut} also satisfy the same properties.

Proof. (i) If x and y are equimeasurable, then $x^* = y^*$. Therefore,

$$\varphi(x) = \lim_{s\to\infty}\frac{1}{s}\|\sigma_s(x^*)\|_E = \lim_{s\to\infty}\frac{1}{s}\|\sigma_s(y^*)\|_E = \varphi(y).$$

(ii) If $|y| \leq |x|$, then $y^* \leq x^*$. Therefore,

$$\varphi(y) = \lim_{s\to\infty}\frac{1}{s}\|\sigma_s(y^*)\|_E \leq \lim_{s\to\infty}\frac{1}{s}\|\sigma_s(x^*)\|_E = \varphi(x).$$

(iii) This follows from the fact that $\|\sigma_s(x^*)\|_E \leq s\|x\|_E$.

(iv) According to Lemma 1.2.48, we have

$$y \prec\prec x \Longrightarrow \sigma_s(y^*) \prec\prec \sigma_s(x^*)$$

for every $s > 0$. Since E is fully symmetric, it now follows that

$$\|\sigma_s(y^*)\|_E \leq \|\sigma_s(x^*)\|_E$$

for every $s > 0$. Therefore,

$$\varphi(y) = \lim_{s\to\infty}\frac{1}{s}\|\sigma_s(y^*)\|_E \leq \lim_{s\to\infty}\frac{1}{s}\|\sigma_s(x^*)\|_E = \varphi(x).$$

□

Lemma 2.1.7. *Let E be a fully symmetric quasi-Banach space.*

1. *If $E = E(0,1)$ is a space on the interval $(0,1)$ and if $x \in L_\infty$, then $\varphi(x) = 0$.*
2. *If $E = E(0,\infty)$ is a space on the semi-axis and if $x \in L_\infty \cap E$, then $\varphi_{fin}(x) = 0$.*
3. *If $E = E(0,\infty) \not\subseteq L_1$ and if $x \in E \cap L_\infty$, then $\varphi_{cut}(x) = 0$.*
4. *The functional φ vanishes on every separable space $E = E(0,1)$.*

Proof. The proofs are straightforward, but are included for completeness.

1. If $x \in L_\infty(0,1)$, then
$$\varphi(x) \leq \|x\|_\infty \varphi(1) = 0.$$
2. If $x \in L_\infty(0,\infty) \cap E(0,\infty)$, then
$$\varphi_{fin}(x) \leq \|x\|_\infty \varphi_{fin}(\chi_{(0,1)}) = 0.$$
3. Note that $E(0,\infty) \not\subset L_1(0,\infty)$ implies that $\|\chi_{(0,s)}\|_E = o(s)$. Thus, if $x \in L_\infty(0,\infty) \cap E(0,\infty)$, then
$$\varphi_{fin}(x) \leq \|x\|_\infty \varphi_{fin}(\chi_{(0,1)}) = \|x\|_\infty \lim_{s\to\infty} \frac{1}{s} \|\chi_{(0,s)}\|_E = 0.$$
4. If E is separable, then the bounded functions are dense in E. Therefore, it is sufficient to prove that φ vanishes on such functions. The proof now follows as in (3).

□

Lemma 2.1.8. *Let E be a fully symmetric quasi-Banach space either on the interval $(0,1)$ or on the semi-axis. If $x \in E$, then*
$$\varphi(\sigma_\tau(x^*)) = \tau\varphi(x), \quad \forall \tau > 0. \tag{2.4}$$
If $E = E(0,\infty)$, then φ_{fin} satisfies (2.4). If $E = E(0,\infty) \not\subset L_1(0,\infty)$, then φ_{cut} also satisfies (2.4).

Proof. Applying the semigroup property of the dilation operators σ_τ, we obtain that
$$\lim_{s\to\infty} \frac{1}{s} ||\sigma_s(\sigma_\tau(x^*))||_E = \tau \lim_{\tau\to\infty} \frac{1}{s\tau} ||\sigma_{s\tau}(x^*)||_E = \tau\varphi(x).$$

The proof for φ_{fin} follows *mutatis mutandi.* Assume now that $E = E(0,\infty) \not\subset L_1(0,\infty)$. It follows from the above that
$$\varphi_{cut}(\sigma_\tau(x^*)) = \varphi(\sigma_\tau(x^*)\chi_{(0,1)}) = \varphi(\sigma_\tau(x^*\chi_{(0,\tau^{-1})})) = \tau\varphi(x^*\chi_{(0,\tau^{-1})}).$$
Note that $x^*\chi_{(\tau^{-1},1)}$ is a bounded function. Therefore, $\varphi(x^*\chi_{(\tau^{-1},1)}) = 0$. It now follows from the Corollary 2.1.5 that
$$\varphi(x^*\chi_{(0,\tau^{-1})}) = \varphi(x^*\chi_{(0,1)}) = \varphi_{cut}(x).$$
and the proof is complete. □

Lemma 2.1.9. *Let E be a fully symmetric quasi-Banach space. For all $0 \leq x_1, \dots, x_k \in E$ and numbers $\lambda_1, \dots, \lambda_k \geq 0$*

$$\varphi(\sum_{i=1}^{k} \lambda_i x_i) = \varphi(\sum_{i=1}^{k} \lambda_i x_i^*).$$

If $E = E(0,\infty)$, then the same statement is valid for φ_{fin}. If, in addition, $E \not\subseteq L_1$, then the same assertion is valid for φ_{cut}.

Proof. Applying the inequality (1.9) n times, we have for positive functions $x_1, \dots, x_{2^n}$

$$(x_1^* + \ldots + x_{2^n}^*) \prec\prec 2^n \sigma_{2^{-n}}(x_1 + \ldots + x_{2^n}).$$

Therefore, by Lemma 2.1.6(iv),

$$\varphi(x_1^* + \ldots + x_{2^n}^*) \leq \varphi(2^n \sigma_{2^{-n}}(x_1 + \ldots + x_{2^n})^*).$$

By Lemma 2.1.8, the equality (2.4) holds. Therefore,

$$\varphi(2^n \sigma_{2^{-n}}(x_1 + \ldots + x_{2^n})^*) = \varphi(x_1 + \ldots + x_{2^n}).$$

Therefore,

$$\varphi(x_1^* + \ldots + x_{2^n}^*) \leq \varphi(x_1 + \ldots + x_{2^n}).$$

The converse inequality follows trivially from (1.5) and Lemma 2.1.6(iv).

The assertion of Lemma follows now from the continuity of the functional φ (see Lemma 2.1.3). □

2.2 Linearity and non-linearity of φ and related functionals

The following proposition proves that the functional φ cannot be linear on the positive cone of E unless it is 0. However, as will be shown in Proposition 2.2.4, it is indeed the case that φ is additively homogeneous on the positive cone generated by $\mathcal{Q}_E^+(x)$ for all $x \in E$. Note, that y and z in the proposition below are arbitrary, that is y, z do not necessary belong to $\mathcal{Q}_E^+(x)$.

Proposition 2.2.1. *Let E be a fully symmetric quasi-Banach space equipped with a Fatou quasi-norm. If $x \geq 0 \in E$, then, in each of the following cases, there exists a decomposition $x = y + z$, such that $y, z \geq 0$ and such that the following assertions hold.*

i) If $E = E(0,1)$, then $\varphi(x) = \varphi(y) = \varphi(z)$.

ii) If $E = E(0,\infty)$ and $\varphi_{cut}(x) = 0$, then $\varphi(x) = \varphi(y) = \varphi(z)$.

iii) If $E = E(0,\infty)$, then $\varphi_{fin}(x) = \varphi_{fin}(y) = \varphi_{fin}(z)$.

iv) If $E = E(0,\infty)$, then $\varphi_{cut}(x) = \varphi_{cut}(y) = \varphi_{cut}(z)$.

Proof. We will prove only the first assertion. The proofs of the third and fourth assertions are exactly the same. The proof of the second assertion requires replacement of the interval $[\frac{1}{m}, \frac{1}{n}]$ with the interval $[n, m]$.

We may assume that $x = x^*$. Fix $n \in N$. If $m \to \infty$, then we obtain

$$\sigma_n(x\chi_{[\frac{1}{m},\frac{1}{n}]}) \uparrow \sigma_n(x\chi_{[0,\frac{1}{n}]})$$

almost everywhere. By the definition of a Fatou quasi-norm, it follows that

$$\|\sigma_n(x\chi_{[\frac{1}{m},\frac{1}{n}]})\|_E \to_m \|\sigma_n(x\chi_{[0,\frac{1}{n}]})\|_E.$$

For each $n \in N$, one can select $f(n) > n$, such that

$$\|\sigma_n(x\chi_{[\frac{1}{f(n)},\frac{1}{n}]})\|_E \geq (1 - \frac{1}{n})\|\sigma_n(x\chi_{[0,\frac{1}{n}]})\|_E. \tag{2.5}$$

Fix some n_0 and set $n_k = f^k(n_0)$, $k \in N$. Here, $f^k = f \circ \ldots \circ f$ (k times). Define

$$y = \sum_{k=0}^{\infty} x\chi_{[\frac{1}{n_{2k+1}},\frac{1}{n_{2k}}]},$$

$$z = \sum_{k=1}^{\infty} x\chi_{[\frac{1}{n_{2k}},\frac{1}{n_{2k-1}}]}.$$

It is clear that

$$\frac{1}{n_{2k}}\|\sigma_{n_{2k}}(y^*)\|_E \geq \frac{1}{n_{2k}}\|\sigma_{n_{2k}}(y)\|_E.$$

On the other hand,

$$\sigma_{n_{2k}}y \geq \sigma_{n_{2k}}(x\chi_{[\frac{1}{n_{2k+1}},\frac{1}{n_{2k}}]}).$$

It follows that

$$\frac{1}{n_{2k}}\|\sigma_{n_{2k}}(y^*)\|_E \geq \frac{1}{n_{2k}}\|\sigma_{n_{2k}}(x\chi_{[\frac{1}{n_{2k+1}},\frac{1}{n_{2k}}]})\|_E. \tag{2.6}$$

By definition of the sequence n_k, we have $n_{2k+1} = f(n_{2k})$. Applying the inequality 2.5, we obtain that

$$\|\sigma_{n_{2k}}(x\chi_{[\frac{1}{n_{2k+1}},\frac{1}{n_{2k}}]})\|_E \geq (1 - \frac{1}{n_{2k}})\|\sigma_{n_{2k}}(x\chi_{[0,\frac{1}{n_{2k}}]})\|_E. \tag{2.7}$$

Note the inequality

$$\frac{1}{n_{2k}}\|\sigma_{n_{2k}}(x\chi_{[0,\frac{1}{n_{2k}}]})\|_E \geq \varphi(x\chi_{[0,\frac{1}{n_{2k}}]}). \tag{2.8}$$

It follows from (2.6), (2.7) and (2.8) that

$$\frac{1}{n_{2k}}\|\sigma_{n_{2k}}(y^*)\|_E \geq (1 - \frac{1}{n_{2k}})\varphi(x\chi_{[0,\frac{1}{n_{2k}}]}). \tag{2.9}$$

By Lemma 2.1.7, $\varphi(x\chi_{[\frac{1}{n_{2k}},1]}) = 0$. It follows from Corollary 2.1.5 that

$$\varphi(x\chi_{[0,\frac{1}{n_{2k}}]}) = \varphi(x). \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\frac{1}{n_{2k}}\|\sigma_{n_{2k}}(y^*)\|_E \geq (1 - \frac{1}{n_{2k}})\varphi(x).$$

Passing to the limit, we obtain $\varphi(y) \geq \varphi(x)$. The converse inequality is obvious.

Hence, $\varphi(y) = \varphi(x) = \varphi(z)$, and this completes proof of the proposition. □

The following assertion says that the functionals φ, φ_{fin}, φ_{cut} take only one value on the whole set $\mathcal{Q}'_E(x)$

Lemma 2.2.2. *Let E be a fully symmetric quasi-Banach space and let $x \in E$.*

1. *For every $y \in \mathcal{Q}'_E(x)$, we have*

$$\varphi(y) = \varphi(x).$$

2. *If, in addition, $E = E(0,\infty)$, then*

$$\varphi_{fin}(y) = \varphi_{fin}(x)$$

for every $y \in \mathcal{Q}'_E(x)$.

3. *If $E = E(0,\infty) \not\subseteq L_1(0,\infty)$, then*

$$\varphi_{cut}(y) = \varphi_{cut}(x)$$

for every $y \in \mathcal{Q}'_E(x)$.

Proof. We will only prove the first assertion because the second and third statements are proved in exactly the same way.

Since φ is a continuous functional (see Lemma 2.1.3), it follows that we only need to prove the assertion for $y \in \mathcal{Q}'(x)$. Every $y \in \mathcal{Q}'(x)$ can be written as

$$y = \sum_{i=1}^{s} \lambda_i x_i,$$

where $\lambda_i \geq 0$, $\sum_{i=1}^{s} \lambda_i = 1$, $x_i \geq 0$ and $x_i^* = x$. By Lemma 2.1.9, we obtain

$$\varphi(y) = \varphi(\sum_{i=1}^{s} \lambda_i x_i) = \varphi(\sum_{i=1}^{s} \lambda_i x_i^*).$$

However,

$$\sum_{i=1}^{s} \lambda_i x_i^* = x^*.$$

Therefore, $\varphi(y) = \varphi(x)$.

□

The following definition is a weak form of linearity.

Definition 2.2.3. *If A is a convex set, then real function θ defined on the cone generated by A is called* **additively homogeneous** *if and only if*

$$\theta(\alpha y_1 + \beta y_2)) = \alpha\theta(y_1) + \beta\theta(y_2), \quad y_1, y_2 \in A, \quad \alpha, \beta \in \mathbb{R}_+.$$

Proposition 2.2.4. *If E is a fully symmetric quasi-Banach space and if $x \in E$, then the following assertions hold.*

i) If $E = E(0,1)$, then φ is additively homogeneous on $\mathcal{Q}_E^+(x)$.

ii) If $E = E(0,\infty)$, then φ_{fin} is additively homogeneous on $\mathcal{Q}_E^+(x)$.

iii) If $E = E(0,\infty) \not\subseteq L_1(0,\infty)$, then φ_{cut} is additively homogeneous on $\mathcal{Q}_E^+(x)$.

Proof. We will only prove the first assertion. The proofs of the other two assertions are exactly the same.

Since the functional φ is continuous (see Lemma 2.1.3), it follows that we only need to prove that φ is additive homogeneous on $\mathcal{Q}^+(x)$. We may assume that $\alpha + \beta = 1$. Fix $y_1, y_2, y_3 \in \mathcal{Q}^+(x)$ such that

$$y_3 = \alpha y_1 + \beta y_2.$$

These three elements can be written as

$$y_j = \sum_{i=1}^{m} \lambda_{ij} x_i.$$

Here, $\lambda_{ij} \geq 0$,

$$\sum_{i=1}^{m} \lambda_{ij} = 1, \ \lambda_{i3} = \alpha\lambda_{i1} + \beta\lambda_{i2}.$$

$x_i \geq 0$ and $x_i^* = x^*\chi_{[0,\beta_i]}$. Define functions z_j and u_j by the formulae.

$$z_j = \sum_{i=1}^{m} \lambda_{ij} x_i^*, \ u_j = \sum_{\beta_i > 0} \lambda_{ij} x^*.$$

It follows from Lemma 2.1.9 that $\varphi(y_j) = \varphi(z_j)$ for $j = 1, 2, 3$.

If $\beta_i > 0$, then $x_i^* - x^*$ is a bounded function. Hence, $u_j - z_j$ is also bounded and, therefore, $\varphi(u_j - z_j) = 0$ for $j = 1, 2, 3$. By Corollary 2.1.5, we obtain $\varphi(z_j) = \varphi(u_j)$. It follows that

$$\varphi(y_j) = \varphi(u_1) = \sum_{\beta_i > 0} \lambda_{ij} \varphi(x).$$

Therefore,

$$\alpha\varphi(y_1) + \beta\varphi(y_2) = \varphi(y_3)$$

and this concludes the proof. □

Proposition 2.2.5. *Let $E = E(0,\infty)$ be a fully symmetric quasi-Banach space on the semi-axis equipped with a Fatou quasi-norm. Suppose that $E = E(0,\infty) \not\subseteq L_1(0,\infty)$ and $x \in E$. If $\Omega^+(x) = \mathcal{Q}_E^+(x)$, then φ is additively homogeneous on $\Omega^+(x)$.*

Proof. It follows from Proposition 2.2.4 that φ_{cut} is additively homogeneous on $\mathcal{Q}_E^+(x)$. By assumption, $\Omega^+(x) = \mathcal{Q}_E^+(x)$. Hence, φ_{cut} is additively homogeneous on $\Omega^+(x)$. It follows now from Proposition 2.2.1(iv) that $\varphi_{cut}(x) = 0$. Hence, $\varphi(x^*\chi_{[0,1]}) = 0$. Since $x^*\chi_{[1,\beta]}$ is bounded, it follows that $\varphi(x^*\chi_{[1,\beta]}) = 0$. According to the Corollary 2.1.5, we obtain $\varphi(x^*\chi_{[0,\beta]}) = 0$ for every finite β.

Since the functional φ is continuous (see Lemma 2.1.3), it follows that we only need to prove that φ is additively homogeneous on $\mathcal{Q}^+(x)$.

We may assume that $\alpha + \beta = 1$. Fix $y_1, y_2, y_3 \in \mathcal{Q}^+(x)$ such that

$$y_3 = \alpha y_1 + \beta y_2.$$

These three elements can be written as

$$y_j = \sum_{i=1}^m \lambda_{ij} x_i.$$

Here, $\lambda_i \geq 0$,

$$\sum_{i=1}^m \lambda_{ij} = 1,\ \lambda_{i3} = \alpha\lambda_{i1} + \beta\lambda_{i2}.$$

$x_i \geq 0$ and $x_i^* = x^*\chi_{[0,\beta_i]}$. Define functions z_j and u_j by the formulae.

$$z_j = \sum_{i=1}^m \lambda_{ij} x_i^*, u_j = \sum_{\beta_i<\infty} \lambda_{ij} x_i^*,$$

$$v_j = \sum_{\beta_i=\infty} \lambda_{ij} x_i^* = \sum_{\beta_i=\infty} \lambda_{ij} x^*.$$

It follows from Lemma 2.1.9 that $\varphi(y_j) = \varphi(z_j)$ for $j = 1, 2, 3$.

Clearly, $z_j = u_j + v_j$. If $\beta = \max_{\beta_i<\infty} \beta_i$, then

$$u_j \leq x^*\chi_{[0,\beta]} \Longrightarrow \varphi(u_j) \leq \varphi(x^*\chi_{[0,\beta]}) = 0.$$

Since $\varphi(u_j) = 0$, it follows from Corollary 2.1.5 that $\varphi(u_j + v_j) = \varphi(v_j)$.

Hence,

$$\varphi(y_j) = \varphi(z_j) = \varphi(u_j + v_j) = \varphi(v_j) = \sum_{\beta_i=\infty} \lambda_{ij}\varphi(x)$$

for $j = 1, 2, 3$. Therefore,

$$\alpha\varphi(y_1) + \beta\varphi(y_2) = \varphi(y_3)$$

and the proof is complete. □

2.3 Further properties of the sets $\mathcal{Q}_E(x)$

We now summarize some properties of the sets $\mathcal{Q}_E(x)$.

Lemma 2.3.1. *The closure of a convex set is a closed convex set.*

Proof. Let A be a convex set and let $x_1, x_2 \in \overline{A}$. Let $\lambda \in (0,1)$. Our aim is to prove that $\lambda x_1 + (1-\lambda)x_2 \in \overline{A}$.

Fix $\varepsilon > 0$. There exist $y_1, y_2 \in A$ such that $\|y_i - x_i\|_E \le \varepsilon$ for $i = 1, 2$. It follows that

$$\|(\lambda x_1 + (1-\lambda)x_2) - (\lambda y_1 + (1-\lambda y_2))\|_E = \|\lambda(x_1 - y_1) + (1-\lambda)(x_2 - y_2)\|_E \le$$

$$\le C(E)(\lambda\|x_1 - y_1\|_E + (1-\lambda)\|x_2 - y_2\|_E) \le C(E)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

Corollary 2.3.2. *The sets $\mathcal{Q}_E(x)$, $\mathcal{Q}_E^+(x)$ and $\mathcal{Q}'_E(x)$ are closed convex sets.*

Lemma 2.3.3. *Let E be a symmetric quasi-Banach space on the interval $(0,1)$ and let $x \in E$. If $0 \le z \in \mathcal{Q}_E(x)$ and $|y| \le z$, then $y \in \mathcal{Q}_E(x)$.*

Proof. Fix $n \in \mathbb{N}$. Define sets e_i, f_{1i} and f_{2i} $i = 1-n, \dots, n$ by the formulae

$$e_i = \{t: \ \frac{i-1}{n} z(t) \le y(t) < \frac{i}{n} z(t)\},$$

$$f_{1i} = \{t: \ y(t) \ge \frac{k}{n} z(t)\}, \ f_{2i} = \{t: \ y(t) < \frac{k}{n} z(t)\}.$$

Define functions z_k, $k = 1-n, \dots, n$ by the formula

$$z_k = z\chi_{f_{1k}} - z\chi_{f_{2k}}.$$

It is clear that for $t \in e_i$, $z_k(t) = -z(t)$ if $k \ge i$ and $z_k(t) = z(t)$ if $k < i$. Therefore, for $t \in e_i$,

$$\sum_{k=1-n}^{n} z_k(t) = z(t)(\sum_{k=1-n}^{i-1} 1 - \sum_{k=i}^{n} 1) = 2z(t)(i-1).$$

Thus,

$$|\frac{1}{2n} \sum_{k=1-n}^{n} z_k - y| \le \frac{1}{n} z. \tag{2.11}$$

Note that z_k is equimeasurable with z and, therefore, $z_k \in \mathcal{Q}_E(x)$. However, $\mathcal{Q}_E(x)$ is a convex set. Therefore,

$$\frac{1}{2n} \sum_{k=1-n}^{n} z_k \in \mathcal{Q}_E(x).$$

It follows from the inequality (2.11) that

$$\|\frac{1}{2n}\sum_{k=1-n}^{n} z_k - y\|_E \le \frac{1}{n}\|z\|_E.$$

Thus,

$$\operatorname{dist}(y, \mathcal{Q}_E(x)) \le \frac{1}{n}\|z\|_E.$$

Since n is arbitrarily large and $\mathcal{Q}_E(x)$ is a closed set, this suffices to complete the proof of the lemma. □

A stronger version of Lemma 2.3.3 is given below in Lemma 2.3.8.

The following assertion seems to be known. We include details of proof for lack of a convenient reference.

Lemma 2.3.4. *Let E be a symmetric quasi-Banach space either on the interval $(0,1)$ or on the semi-axis and let $x \in E$. If $y \in \mathcal{Q}'_E(z)$ and $z \in \mathcal{Q}'_E(x)$, then $y \in \mathcal{Q}'_E(x)$.*

Proof. Without loss of generality, $y = y^*$, $z = z^*$ and $x = x^*$. Let $y \in \mathcal{Q}'_E(z)$. Hence, for every $\varepsilon > 0$, one can find $n \in \mathbb{N}$, $\lambda_i \in \mathbb{R}_+$ and positive measurable functions $z_i \sim z$, $i = 1, \dots, n$, such that $\sum_{i=1}^n \lambda_i = 1$ and

$$\|y - \sum_{i=1}^{n} \lambda_i z_i\|_E \le \varepsilon.$$

By assumption, $z \in \mathcal{Q}'_E(x)$. Since z_i are equimeasurable with z, it follows that $z_i \in \mathcal{Q}'_E(x)$ for all $1 \le i \le n$. However, the set $\mathcal{Q}'_E(x)$ is convex by Lemma 2.3.2. Therefore,

$$\sum_{i=1}^{n} \lambda_i z_i \in \mathcal{Q}'_E(x).$$

Thus,

$$\operatorname{dist}(y, \mathcal{Q}'_E(x)) \le \varepsilon.$$

Since $\mathcal{Q}'_E(x)$ is a closed set, it follows that $y \in \mathcal{Q}'_E(x)$. □

Remark 2.3.5. *The collection of sets $\{\mathcal{Q}_E(x),\ x \in E\}$ also satisfies the transitivity property expressed in Lemma 2.3.4. We do not know whether this is always the case for the collection $\{\mathcal{Q}^+_E(x),\ x \in E\}$.*

Proposition 2.3.6. *Let $E = E(0,\infty)$ be a fully symmetric quasi-Banach space on the semi-axis and let $x \in E$. If $\varphi(x) = 0$, then, $x\chi_A \in \mathcal{Q}'_E(x)$ for every Lebesgue measurable subset $A \subseteq (0,\infty)$.*

Proof. Fix $n \in \mathbb{N}$. Our first claim is that we can split the semi-axis into the union of two disjoint sets B and C such that $m(B) = m(A)$ and $m(C) = nm(\mathbb{R}_+ \backslash A)$.

Note that sets A and $\mathbb{R}_+ \backslash A$ cannot simultaneously have finite measure. If $m(A) < \infty$, then $B = (0, m(A))$ and $C = (m(A), \infty)$. If $m(\mathbb{R}_+ \backslash A) < \infty$, then $C = (0, nm(\mathbb{R}_+\backslash A))$ and $B = (nm(\mathbb{R}_+\backslash A), \infty)$. If $m(\mathbb{R}_+\backslash A) = \infty$ and $m(A) = \infty$, then $B = \cup_{n=0}^{\infty}(2n, 2n+1)$ and $C = \cup_{n=1}^{\infty}(2n-1, 2n)$. Clearly, in every particular case, sets B and C satisfy the required assumptions.

Since $m(C) = nm(\mathbb{R}_+\backslash A)$, the set C can be written as a union of disjoint sets $C = \cup_{i=1}^{n} C_i$ such that $m(C_i) = m(\mathbb{R}_+\backslash A)$. Fix measure-preserving bijections $\gamma : B \to A$ and $\gamma_i : C_i \to \mathbb{R}_+\backslash A$.

Define functions x_i, $1 \le i \le n$, in the following manner. If $t \in B$, then $x_i(t) = x(\gamma(t))$. If $t \in C_i$, then $x_i(t) = x(\gamma_i(t))$ and $x_j(t) = 0$ for $j \ne i$. It is clear that functions x_i are equimeasurable with x. Clearly, the functions $x_i\chi_C$ are disjoint and equimeasurable. Since $(x_i\chi_C)^* \le x^*$, it follows that

$$(\frac{1}{n}\sum_{i=1}^{n} x_i\chi_C)^* \le \frac{1}{n}\sigma_n(x^*). \tag{2.12}$$

Note, however, that

$$\frac{1}{n}\sum_{i=1}^{n} x_i\chi_C = \frac{1}{n}\sum_{i=1}^{n} x_i - (x \circ \gamma)\chi_B. \tag{2.13}$$

It follows from (2.13) and (2.12) that

$$\text{dist}(x\chi_A, \mathcal{Q}'_E(x)) = \text{dist}((x \circ \gamma)\chi_B, \mathcal{Q}'_E(x)) \le \frac{1}{n}\|\sigma_n(x^*)\|_E.$$

Since the latter expression tends to 0 as $n \to \infty$ and $\mathcal{Q}'_E(x)$ is a closed set, this suffices to complete the proof of the proposition. □

Corollary 2.3.7. *Let $E = E(0, \infty)$ be a fully symmetric quasi-Banach space on the semi-axis and let $x \in E$. If $\varphi(x) = 0$, then $y\chi_A \in \mathcal{Q}'_E(x)$ for every $y \in \mathcal{Q}'_E(x)$.*

Proof. It follows from the assumption and Lemma 2.2.2 that $\varphi(y) = \varphi(x) = 0$. Lemma 2.3.6 implies that $y\chi_A \in \mathcal{Q}'_E(y)$. Since $y\chi_A \in \mathcal{Q}'_E(y)$ and $y \in \mathcal{Q}'_E(x)$, it follows from Lemma 2.3.4 that $y\chi_A \in \mathcal{Q}'_E(x)$. □

An assertion somewhat similar to the lemma below is contained in [11, Lemma 1.3].

Lemma 2.3.8. *Let $E = E(0, \infty)$ be a fully symmetric quasi-Banach space on the semi-axis and let $x \in E$. Suppose that $\varphi(x) = 0$. If $y \in \mathcal{Q}'_E(x)$ and $0 \le z \le y$, then, $z \in \mathcal{Q}'_E(x)$.*

Proof. Fix $n \in \mathbb{N}$. Define sets e_i and f_i, $i = 1, \dots, n$ by

$$e_i = \{t : \frac{i-1}{n}y(t) \le z(t) < \frac{i}{n}y(t)\},$$

$$f_i = \bigcup_{i \le (n+j)/2} e_j.$$

Define functions y_k, $k = 1, \dots, n$ by the formula

$$y_k = y \sum_{k \le (i+n)/2} \chi_{e_i} = y \chi_{f_k}.$$

By Corollary 2.3.7, $y_k \in \mathcal{Q}'_E(x)$. Since the set $\mathcal{Q}'_E(x)$ is convex, it follows that

$$\frac{1}{n} \sum_{k=1}^{n} y_k \in \mathcal{Q}'_E(x).$$

On the other hand, if $t \in e_i$, then $y_k(t) = y(t)$ if $k \le (i+n)/2$ and $y_k(t) = 0$ otherwise. Therefore,

$$\sum_{k=1}^{n} y_k(t) = y(t) \sum_{k \le (i+n)/2} 1 = y(t)[\frac{i+n}{2}], \ \forall t \in e_i,$$

and so

$$|\frac{1}{n} \sum_{k=1}^{n} y_k - \frac{1}{2}(y+z)| \le \frac{2y}{n}.$$

Thus,

$$\mathrm{dist}(\frac{y+z}{2}, \mathcal{Q}'_E(x)) \le \frac{2}{n} \|y\|_E.$$

Letting $n \to \infty$, it follows that $(y+z)/2 \in \mathcal{Q}'_E(x)$.

Repeat this process m times and obtain

$$2^{-m}((2^m - 1)z + y) \in \mathcal{Q}'_E(x).$$

Hence,

$$\mathrm{dist}(z, \mathcal{Q}'_E(x)) \le 2^{-m} \|y - z\|_E.$$

Letting $m \to \infty$, it follows that $z \in \mathcal{Q}'_E(x)$. □

2.4 Elements of the form $P(x|\mathcal{A})$.

Lemma 2.4.1. *Let $E = E(0,\infty)$ be a fully symmetric quasi-Banach space on the semi-axis and let $x \in E$. Suppose that $E = E(0,\infty) \not\subseteq L_1(0,\infty)$. If $P(x|\mathcal{A}) \in \mathcal{Q}'_E(x)$ for every $\mathcal{A} \in \mathfrak{A}$, then $\varphi_{cut}(x) = 0$.*

Proof. Suppose that $x = x^*$. Set $\mathcal{A} = \{[0,1]\}$ and $y = P(x|\mathcal{A}) \in E \cap L_\infty$. By assumption, $y \in \mathcal{Q}'_E(x)$. From Lemma 2.2.2 and Lemma 2.1.7, it follows that $\varphi_{cut}(x) = \varphi_{cut}(y) = 0$. □

Lemma 2.4.2. *Let E and x be as in Lemma 2.4.1. If $L_\infty \subseteq E$, then $\varphi(x) = 0$.*

Proof. Due to the choice of E, we have $1 \in E$. However, $\sigma_\tau(1) = 1$ implies $\varphi(1) = 0$. Thus, for every $z \in E \cap L_\infty$, we have $\varphi(z) = 0$. However, for every $x \in E$, we have $\varphi(x^*\chi_{[0,1]}) = 0$ due to Lemma 2.4.1. Hence,

$$0 \le \varphi(x) = \varphi(x^*) \le C(E)(\varphi(x^*\chi_{[0,1]}) + \varphi(x^*\chi_{[1,\infty)})) = 0 + 0 = 0.$$

□

Define $\omega(x,y)$, $x, y \in E$, by setting

$$\omega(x,y) := \limsup_{t\to\infty} \frac{\int_0^t y^*(s)ds}{\int_0^t x^*(s)ds}.$$

Clearly, $\omega(x,y)$ is either 0, or a finite non-zero number or ∞.

Lemma 2.4.3. *Let E and x be as in Lemma 2.4.1. If $y \in E \cap L_\infty$, then $\varphi(y) \le \omega(x,y)\varphi(x)$.*

Proof. Fix $\varepsilon > 0$. There exists $T > 0$, such that for every $t > T$,

$$\int_0^t y^*(s) \le (\omega(x,y) + \varepsilon) \int_0^t x^*(s)ds.$$

On the other hand, for every $t < T$, we have

$$\int_0^t y^*(s)ds \le \int_0^t \|y\|_\infty \chi_{[0,T]}(s)ds.$$

One can unify theses inequalitites and obtain

$$\int_0^t y^*(s)ds \le \int_0^t ((\omega(x,y) + \varepsilon)x^* + \|y\|_\infty \chi_{[0,T]})(s)ds.$$

Therefore,

$$y \prec\prec (\omega(x,y) + \varepsilon)x^* + \|y\|_\infty \chi_{[0,T]}$$

and

$$\varphi(y) \le \varphi((\omega(x,y) + \varepsilon)x^* + \|y\|_\infty \chi_{[0,T]}).$$

Note that $\varphi(\chi_{[0,T]}) = 0$ since $E \not\subset L_1$. Therefore, by Remark 2.1.4,

$$\varphi((\omega(x,y)+\varepsilon)x^* + \|y\|_\infty \chi_{[0,T]}) = (\omega(x,y)+\varepsilon)\varphi(x).$$

Hence,

$$\varphi(y) \leq (\omega(x,y)+\varepsilon)\varphi(x).$$

Since $\varepsilon > 0$ is arbitrary, the assertion of the lemma follows. □

Lemma 2.4.4. *Let E and x be as in Lemma 2.4.1. If $y \in E \cap L_\infty$, then $\varphi(y) \geq \omega(x,y)\varphi(x)$. In particular, if in addition $\varphi(x) > 0$, then $\omega(x,y) < \infty$.*

Proof. Without loss of generality, $y = y^*$ and $x = x^*$.

Now, fix $\omega < \omega(x,y)$. There exists a sequence $t_k \uparrow \infty$, such that

$$\int_0^{t_k} y^*(s)ds \geq \omega \int_0^{t_k} x^*(s)ds.$$

Without loss of generality, $t_0 = 0$.

Let $\mathcal{A}$ be the partition of $(0,\infty)$ given by $\mathcal{A} = \{[t_k, t_{k+1})\}_{k=1}^\infty$.

It then follows that

$$\omega \int_0^t P(x|\mathcal{A})^*(s)ds \leq \int_0^t y^*(s)ds$$

for arbitrary $t > 0$. Therefore,

$$\omega P(x|\mathcal{A}) \prec\prec y$$

and

$$\omega\varphi(P(x|\mathcal{A})) \leq \varphi(y).$$

However, $P(x|\mathcal{A}) \in \mathcal{Q}'_E(x)$ by assumption. It follows from Lemma 2.2.2 that

$$\omega\varphi(x) = \omega\varphi(P(x|\mathcal{A})) \leq \varphi(y).$$

Since ω can be arbitrarily close to $\omega(x,y)$, the lemma is established. □

Corollary 2.4.5. *Let E and x be as in Lemma 2.4.1. If $y \in E \cap L_\infty$, then $\varphi(y) = \omega(x,y)\varphi(x)$. In particular, if in addition $\varphi(x) > 0$, then $\omega(x,y) < \infty$.*

Lemma 2.4.6. *Let E and x be as in Lemma 2.4.1. If $\varphi(x) = 0$, then $E \cap L_\infty$ is a Marcinkiewicz space M_ψ such that*

$$\lim_{t\to\infty} \frac{\psi(2t)}{\psi(t)} = 1. \tag{2.14}$$

Proof. Without loss of generality, $x = x^*$. Set $\mathcal{A} = \{[0,1]\}$. It follows from the assumption that $P(x|\mathcal{A}) \in \mathcal{Q}'_E(x)$. Therefore, by Lemma 2.2.2, we have

$$\varphi(P(x|\mathcal{A})) = \varphi(x).$$

Set

$$\psi(t) = \int_0^t P(x|\mathcal{A})^*(s)ds.$$

It follows from Corollary 2.4.5 that $\omega(x, y)$ is finite and, therefore,

$$\limsup_{t\to\infty} \frac{1}{\psi(t)} \int_0^t y^*(s)ds < \infty$$

for every $y \in E \cap L_\infty$. Since both ψ' and y are bounded functions, it follows that

$$\limsup_{t\to 0} \frac{1}{\psi(t)} \int_0^t y^*(s)ds = \frac{\|y\|_\infty}{\|\psi'\|_\infty} < \infty.$$

Therefore, for every $y \in E \cap L_\infty$, we have

$$\sup_t \frac{1}{\psi(t)} \int_0^t y^*(s)ds < \infty$$

and, therefore, $y \in M_\psi$. Hence, $E \cap L_\infty \subset M_\psi$. However, $\psi' = x \in E \cap L_\infty$. Therefore, $M_\psi \subset E \cap L_\infty$, and so $E \cap L_\infty = M_\psi$.

In order to obtain 2.14, consider the function $2\sigma_{\frac{1}{2}}\psi'$. It follows that $\varphi(2\sigma_{1/2}\psi') = \varphi(\psi')$ by Lemma 2.1.8. Hence

$$\omega(\psi', 2\sigma_{1/2}\psi')\varphi(\psi') = \varphi(2\sigma_{1/2}\psi')$$

and $\omega(\psi', 2\sigma_{1/2}\psi') = 1$. However,

$$\omega(\psi', 2\sigma_{1/2}\psi') = \limsup_{t\to\infty} \frac{\int_0^t 2x(2s)ds}{\int_0^t x(s)ds} = \limsup_{t\to\infty} \frac{\psi(2t)}{\psi(t)}.$$

Therefore,

$$\limsup_{t\to\infty} \frac{\psi(2t)}{\psi(t)} = 1$$

and the proof is complete. □

Lemma 2.4.7. *Let E be a symmetric quasi-Banach space. The set G defined by the formula*

$$G = \{y \in E : \ \exists C \ \sup_{t\geq 1} \frac{y^*(t)}{\psi'(Ct)} < \infty\}$$

is a linear space.

Proof. If $y_1, y_2 \in G$, then $y_i^*(t) \leq C_1\psi'(C_2t)$. Let $y = ay_1 + by_2$.

$$y^*(t) \leq |a|y_1^*(t/2) + |b|y_2^*(t/2).$$

Hence, for $t \geq 2$ we have

$$y^*(t) \leq C_1(a + b)\psi'(C_2t/2).$$

If $t \in (1,2)$, then

$$y^*(t) \leq \lambda y_1^*(1/2) + (1-\lambda) y_2^*(1/2) = C_3\psi'(C_2) \leq C_3\psi'(C_2 t/2).$$

If $C_4 = \max\{C_1(a+b), C_3\}$, then $y^*(t) \leq C_4\psi'(C_2 t/2)$ for every $t \geq 1$. □

Corollary 2.4.8. *Let E and x be as in Lemma 2.4.6 and let ψ be as given in the statement of Lemma 2.4.6. It follows that $\mathcal{Q}'_E(x) \subset \overline{G}$, where G is the linear space defined in Lemma 2.4.7*

Proof. It is sufficient to prove that $\mathcal{Q}'(x) \subset G$. Since $x^*(t) = \psi'(t)$ for $t \geq 1$, it follows that $x \in G$. So is any y equimeasurable with x. However, the set of all convex combinations of such functions is exactly $\mathcal{Q}'(x)$. The assertion follows from the fact that G is a linear space. □

Theorem 2.4.9. *Let E be a fully symmetric quasi-Banach space either on the interval $(0,1)$ or on the semi-axis and let $x \in E$. Suppose that $P(x|\mathcal{A}) \in \mathcal{Q}'_E(x)$ for every partition $\mathcal{A} \in \mathfrak{A}$.*

1. *If $E = E(0,1)$, then $\varphi(x) = 0$.*
2. *If $E = E(0,\infty) \not\subset L_1(0,\infty)$, then $\varphi(x) = 0$.*
3. *If $E = E(0,\infty) \subset L_1(0,\infty)$, then $\varphi_{fin}(x) = 0$.*

Proof. 1. Let $E = E(0,1)$ and $x = x^*$. Set $\mathcal{A} = \{[0,1]\}$ and $y = P(x|\mathcal{A})$. By assumption, $y \in \mathcal{Q}'_E(x)$. By Lemma 2.2.2 and Lemma 2.1.7, $\varphi(x) = \varphi(y) = 0$.

2. Let $E = E(0,\infty)$ and $L_\infty \subseteq E \not\subseteq L_1$. In this case, the assertion is proved in Lemma 2.4.2.

   Let $E = E(0,\infty)$ and suppose now that $L_\infty \not\subseteq E \not\subseteq L_1$. Note that $E \cap L_\infty = M_\psi$ according to Lemma 2.4.6. It follows from (2.14) that there exists a sequence $t_k \uparrow \infty$, such that $t_0 = 0$, $t_1 = 1$ and for every $k \in \mathbb{N}$

$$\frac{\psi(t_{k+1}) - \psi(t_k)}{t_{k+1} - t_k} \geq \frac{2}{3}\frac{\psi(\frac{1}{2}t_{k+1})}{t_{k+1}}.$$

   Set $\mathcal{A} = \{[t_k, t_{k+1}]\}$ and $z = P(x|\mathcal{A})$. It follows from the construction given in [33] that

$$||(z-y)\chi_{[\frac{1}{2}t_k, t_k]}||_{M_\psi} \geq \frac{1}{4}$$

   for every $y \in G$ and every sufficiently large k. However,

$$||(y-z)\chi_{[\frac{1}{2}t_k, t_k]}||_{L_\infty} \to 0.$$

   Since $M_\psi = E \cap L_\infty$, it follows that

$$||(z-y)\chi_{[\frac{1}{2}t_k, t_k]}||_E \geq \frac{1}{4}$$

   for sufficiently large k. In particular, $||y-z||_E \geq \frac{1}{4}$. Hence, $\mathrm{dist}_E(z, G) \geq \frac{1}{4}$ and $\mathrm{dist}_E(z, \mathcal{Q}'(x)) \geq \frac{1}{4}$. This contradicts the assumption that $P(x|\mathcal{A}) \in \mathcal{Q}'_E(x)$.

3. Let $E = E(0,\infty)$ and suppose that $E \subseteq L_1$. Set $\mathcal{A} = \{[0,1]\}$ and $y = P(x|\mathcal{A})$. By assumption, $y \in \mathcal{Q}'_E(x)$. By Lemma 2.2.2 and Lemma 2.1.7, it follows that $\varphi_{fin}(x) = \varphi_{fin}(y) = 0$.

□

Lemma 2.4.10. *Let $x \in L_1(0,a)$ be arbitary. If, for every fixed $n \in \mathbb{N}$, $x_0, \cdots, x_{n-1}$ are defined on the interval $(0,a)$ by the formula*

$$x_i(t) = x((t + \frac{ia}{n})\mathrm{mod}a),$$

then

$$\frac{1}{a}\int_0^a x(s)ds - \frac{1}{n}\sum_{i=0}^{n-1} x_i \prec\prec \frac{2}{n}\sigma_n(x^*)\chi_{(0,a)}.$$

Proof. Without loss of generality, $a = 1$ and $x = x^*$ on the interval $(0,1)$.

Set

$$z = x(t - \frac{i}{n}),\ if\ \frac{i}{n} \leq t \leq \frac{i+1}{n},\ 0 \leq i \leq n-1.$$

Clearly, z is equimeasurable with $\sigma_n(x^*)\chi_{(0,1)}$.

We will show that

$$\int_0^1 x(s)ds - \frac{1}{n}\sum_{i=0}^{n-1} x((t+\frac{i}{n})(\mathrm{mod}1)) \leq \int_0^{\frac{1}{n}} x(s)ds$$

and

$$\int_0^1 x(s)ds - \frac{1}{n}\sum_{i=0}^{n-1} x((t+\frac{i}{n})(\mathrm{mod}1)) \geq -\frac{1}{n}z(t).$$

We will prove only the first inequality. The proof of the second one is identical.

Without loss of generality, $t \in [0, \frac{1}{n}]$. Clearly,

$$\frac{1}{n}x(t+\frac{i}{n}) \geq \int_{\frac{i+1}{n}}^{\frac{i+2}{n}} x(s)ds$$

for $i = 0, \ldots, n-2$. Hence,

$$\int_0^1 x(s)ds - \frac{1}{n}\sum_{i=0}^{n-1} x(t+\frac{i}{n}) = \int_0^{\frac{1}{n}} x(s)ds - \frac{1}{n}x(t+\frac{n-1}{n}) -$$

$$-\sum_{i=0}^{n-2}(\frac{1}{n}x(t+\frac{i}{n}) - \int_{\frac{i+1}{n}}^{\frac{i+2}{n}} x(s)ds) \leq \int_0^{\frac{1}{n}} x(s)ds.$$

Therefore,

$$|\int_0^1 x(s)ds - \frac{1}{n}\sum_{i=0}^{n-1} x_i(t)| \leq \frac{1}{n}(z(t) + \int_0^1 z(s)ds) \quad t \in [0,1].$$

However,

$$z + \int_0^1 z(s)ds \prec\prec 2z.$$

Hence,

$$|\int_0^1 x(s)ds - \frac{1}{n}\sum_{i=0}^{n-1} x_i| \prec\prec \frac{2}{n}z.$$

Since z is equimeasurable with $\sigma_n(x^*)$, it follows that

$$|\int_0^1 x(s)ds - \frac{1}{n}\sum_{i=0}^{n-1} x_i| \prec\prec \frac{2}{n}\sigma_n(x^*)\chi_{(0,1)}.$$

□

Theorem 2.4.11. *Let E be a fully symmetric quasi-Banach space either on the interval $(0,1)$ or on the semi-axis. If $x \in E$, then $P(x|\mathcal{A}) \in \mathcal{Q}'_E(x)$ for every partition $\mathcal{A} \in \mathfrak{A}$, provided that one of the following conditions is satisfied.*

1. $E = E(0,1)$ *and* $\varphi(x) = 0$.
2. $E = E(0,\infty)$ *is such that* $E \not\subseteq L_1$ *and* $\varphi(x) = 0$.
3. $E = E(0,\infty)$ *is such that* $E \subseteq L_1$ *and* $\varphi_{fin}(x) = 0$.

Proof. 1. Consider some partition $\mathcal{A}$. By the definition of the operator $P(\cdot|\mathcal{A})$, we have for every $t \in A_k$, $k \in \mathbb{N}$,

$$P(x|\mathcal{A})(t) = \frac{1}{m(A_k)}\int_{A_k} x(s)ds.$$

Fix $n \in \mathbb{N}$. By Lemma 2.4.10, there exist functions x_{ik}, $0 \le i \le n-1$, on the set A_k equimeasurable with $x\chi_{A_k}$ such that

$$\frac{1}{m(A_k)}\int_{A_k} x(s)ds - \frac{1}{n}\sum_{i=0}^{n-1} x_{ik} \prec\prec \frac{2}{n}\sigma_n((x\chi_{A_k})^*)\chi_{(0,m(A_k))}, \quad k \in \mathbb{N}. \tag{2.15}$$

Let now

$$x_i = \sum_k x_{ik}\chi_{A_k} + x\chi_{(0,\infty)\setminus \cup_k A_k}, \quad 0 \le i \le n-1.$$

Clearly, x_i are equimeasurable with x.

Let z_k be a function on A_k such that

$$z_k^* = \sigma_n((x\chi_{A_k})^*)\chi_{(0,m(A_k))}, \quad k \in \mathbb{N}$$

and let $z = \sum_k z_k$.

It follows from the (2.15) and Lemma 1.2.14 that

$$P(x|\mathcal{A}) - \frac{1}{n}\sum_{i=0}^{n-1} x_i \prec\prec \frac{2}{n}z.$$

However,

$$\{t : \ z(t) > s\} = \cup_k \{t \in A_k : \ z_k(t) > s\}.$$

It follows that

$$m(\{t \in A_k : \ z_k(t) > s\}) \leq nm(\{t \in A_k : \ x(t) > s\}).$$

Therefore,

$$m(\{t : \ z(t) > s\}) \leq n\sum_k m(\{t \in A_k : \ x(t) > s\}) = nm(\{t : \ x(t) > s\}),$$

and so $z^* \leq \sigma_n(x^*)$ and

$$P(x|\mathcal{A}) - \frac{1}{n}\sum_{i=0}^{n-1} x_i \prec\prec \frac{2}{n}\sigma_n(x^*).$$

2. Repeat the previous argument *mutatis mutandi.*
3. The assertion follows from Theorem 2.7.9.

□

2.5 The Mekler theorem

The Mekler theorem (see Theorem 2.5.8 below) is a remarkable result which connects arbitrary elements of the orbit $\Omega(x)$ with those generated by the expectation operators.

The principle ideas of the proofs are due to Mekler, but our approach is simpler and more general. For example, the original Mekler result was only available for functions on the interval $(0,1)$, while our result is valid also on the semi-axis.

The original proof of Mekler required very complicated machinery. We eliminate any need for this complexity and use a theorem due to Birkhoff instead.

We set the notion of

$$y \lhd x \Longleftrightarrow \int_0^t y^*(s)ds < \int_0^t x^*(s)ds \quad \forall t.$$

Lemma 2.5.1. *Let $x = x^* \in L_1(0,1)$ and $y = y^* \prec\prec x$. If $q \in (0,1)$ is a constant, then there exists a sequence $t_n \to 0$ such that $qy\chi_{[t_n,1]} \lhd x\chi_{[t_n,1]}$.*

Proof. Fix a sequence $t_n \downarrow 0$. If there exists a subsequence $\{s_k\}_{k=1}^\infty \subset \{t_n\}_{n=1}^\infty$ such that $qy\chi_{(s_k,1)} \lhd x\chi_{(s_k,1)}$, then the proof is finished. Otherwise, $qy\chi_{(t_n,1)} \not\lhd x\chi_{(t_n,1)}$ for all sufficiently large n. Without loss of generality, $qy\chi_{(t_n,1)} \not\lhd x\chi_{(t_n,1)}$ for all $n \in \mathbb{N}$. Let B_n, $n \geq 1$, be the set of all $t \geq t_n$ such that

$$\int_{t_n}^t qy^*(s)ds \geq \int_{t_n}^t x^*(s)ds.$$

Clearly, B_n is a closed set. Therefore, $u_n = \sup B_n \in B_n$. In particular,

$$\int_{t_n}^t qy^*(s)ds < \int_{t_n}^t x^*(s)ds \quad \forall t > u_n. \tag{2.16}$$

Assume first that there is a subsequence of the sequence $\{u_n\}_{n=1}^\infty$ which converges to 0. Passing to this subsequence, it may be assumed that u_n converges to 0. If there exists a subsequence $\{v_k\}_{k=1}^\infty \subset \{u_n\}_{n=1}^\infty$ such that $qy\chi_{(v_k,1)} \lhd x\chi_{(v_k,1)}$, then the proof is finished. Otherwise, $qy\chi_{(u_n,1)} \not\lhd x\chi_{(u_n,1)}$ for all sufficiently large n. Without loss of generality, $qy\chi_{(u_n,1)} \not\lhd x\chi_{(u_n,1)}$ for all $n \in \mathbb{N}$. Thus, there exists $w_n > u_n$ such that

$$\int_{u_n}^{w_n} qy^*(s)ds \geq \int_{u_n}^{w_n} x^*(s)ds.$$

It follows that

$$\int_{t_n}^{w_n} qy^*(s)ds \geq \int_{w_n}^{v_n} x^*(s)ds,$$

which contradicts (2.16).

Let now the sequence $\{u_n\}_{n=1}^{\infty}$ be bounded away from 0. Without loss of generality, $u_n \to u > 0$. Therefore,

$$\int_0^u qy^*(s)ds = \lim_{n\to\infty} \int_{t_n}^{u_n} qy^*(s)ds \geq \lim_{n\to\infty} \int_{t_n}^{u_n} x^*(s)ds = \int_0^u x^*(s)ds.$$

The latter is impossible because $y \prec\prec x$ and $0 < q < 1$. $\square$

This lemma cannot be extended to the case of the semi-axis, as the following example shows.

Example 2.5.2. *Let $x = x^* = \chi_{[0,1]}$. If $y = y^*$ is such that $\|y\|_{L_1 \cap L_\infty} \leq 1$, then $y \prec\prec x$. If, in addition, the support of y has infinite measure, then $qy\chi_{(t,\infty)} \ntriangleleft x\chi_{(t,\infty)}$ for every $t > 1$.*

However, if x is non-integrable, then an extension of Lemma 2.5.1 to the case of the semi-axis does exist.

Lemma 2.5.3. *Let $x = x^* \in (L_1 + L_\infty)(0,\infty)$ and $x \notin L_1(0,\infty)$. Let $y = y^* \prec\prec x$ and let $q \in (0,1)$ be a constant. Then there exists a sequence $t_n \to \infty$ such that $qy\chi_{[t_n,\infty)} \triangleleft x\chi_{[t_n,\infty)}$.*

Proof. Fix a sequence $t_n \uparrow \infty$. If there exists a subsequence $\{s_k\}_{k=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$ such that $qy\chi_{(s_k,\infty)} \triangleleft x\chi_{(s_k,\infty)}$, then the proof is finished. Otherwise, $qy\chi_{(t_n,\infty)} \ntriangleleft x\chi_{(t_n,\infty)}$ for all sufficiently large n. Without loss of generality, $qy\chi_{(t_n,\infty)} \ntriangleleft x\chi_{(t_n,\infty)}$ for all $n \in \mathbb{N}$. Let B_n, $n \geq 1$, be the set of all $t \geq t_n$ such that

$$\int_{t_n}^{t} qy^*(s)ds \geq \int_{t_n}^{t} x^*(s)ds.$$

Clearly, B_n is a closed set. We claim that B_n is bounded.

Indeed, if this were not the case, there would be a sequence $t_{n,k} \to \infty$ such that

$$\int_{t_n}^{t_{n,k}} qy^*(s)ds \geq \int_{t_n}^{t_{n,k}} x^*(s)ds. \tag{2.17}$$

However,

$$\int_{t_n}^{t_{n,k}} (qy^*(s) - x^*(s))ds = -\int_0^{t_n} (qy^*(s) - x^*(s))ds +$$

$$+q\int_0^{t_{n,k}} (y^*(s) - x^*(s))ds - (1-q)\int_0^{t_{n,k}} x^*(s)ds.$$

The first term does not depend on k. The second term is always negative since $y \prec\prec x$. The third term tends to $-\infty$ since $x \notin L_1(0,\infty)$. Therefore, (2.17) is impossible for large k.

Let now $u_n = \sup B_n$, $n \in \mathbb{N}$. Evidently, $u_n \in B_n$ and

$$\int_{t_n}^{t} qy^*(s)ds < \int_{t_n}^{t} x^*(s)ds, \quad \forall t > u_n. \tag{2.18}$$

Clearly, $u_n \uparrow \infty$. If there exists a subsequence $\{v_k\}_{k=1}^{\infty} \subset \{u_n\}_{n=1}^{\infty}$ such that $qy\chi_{(v_k,\infty)} \lhd x\chi_{(v_k,\infty)}$, then the proof is finished. Otherwise, $qy\chi_{(u_n,\infty)} \not\lhd x\chi_{(u_n,\infty)}$ for all sufficiently large n. Without loss of generality, $qy\chi_{(u_n,\infty)} \not\lhd x\chi_{(u_n,\infty)}$ for all $n \in \mathbb{N}$. Thus, there exists $w_n > u_n$ such that

$$\int_{u_n}^{w_n} qy^*(s)ds \geq \int_{u_n}^{w_n} x^*(s)ds.$$

It follows that

$$\int_{t_n}^{w_n} qy^*(s)ds \geq \int_{t_n}^{w_n} x^*(s)ds,$$

which contradicts (2.18). □

Lemma 2.5.4. *If $z = z^* \in L_\infty(0,1)$, then*

$$\sup_{1\leq m\leq n} \frac{z(0) - z(\frac{m}{n})}{m} \to 0.$$

Proof. Assume the contrary. There exists $\varepsilon > 0$ and sequences $1 \leq m_k \leq n_k \to \infty$ such that

$$z(0) - z(\frac{m_k}{n_k}) \geq \varepsilon m_k.$$

It follows that $m_k \leq \varepsilon^{-1} z(0)$ is a bounded sequence. Since $z = z^*$ is right-continuous, we have $z(0) - z(m_k/n_k) \to 0$. Thus, $\varepsilon m_k \to 0$, which is impossible. □

Lemma 2.5.5. *Let $x = x^* \in L_\infty(0,1)$ and $y = y^* \lhd x$. If $y(0) < x(0)$, then there exists $n \in \mathbb{N}$ and functions*

$$u = \sum_{k=1}^{n} x(\frac{k}{n})\chi_{[(k-1)/n,k/n]}, \quad v = \sum_{k=1}^{n} y(\frac{k-1}{n})\chi_{[(k-1)/n,k)/n]}$$

such that $y \leq v \prec\prec u \leq x$.

Proof. It follows from $y(0) < x(0)$ that

$$\varepsilon = \inf_{t>0} \frac{1}{t} \int_0^t (x-y)(s)ds > 0.$$

By Lemma 2.5.4, one can select n with

$$(x+y)(0) - (x+y)(\frac{m}{n}) \leq \varepsilon m, \quad 0 \leq m \leq n.$$

Fix $1 \leq m \leq n$. We have

$$\int_0^{m/n} (u-v)(s)ds = \frac{1}{n} \sum_{k=1}^{m} (x(\frac{k}{n}) - y(\frac{k-1}{n})) =$$

$$= \frac{1}{n}\sum_{k=1}^{m}(x(\frac{k-1}{n}) - y(\frac{k}{n})) - \frac{1}{n}((x+y)(0) - (x+y)(\frac{m}{n})) \geq$$

$$\geq \int_0^{m/n}(x-y)(s)ds - \frac{1}{n}((x+y)(0) - (x+y)(\frac{m}{n})) \geq$$

$$\geq \varepsilon\frac{m}{n} - \frac{1}{n}((x+y)(0) - (x+y)(\frac{m}{n})) \geq 0.$$

Therefore,

$$\int_0^t (u-v)(s)ds \geq 0$$

for every $t > 0$. □

Recall that a matrix $C = (c_{ij})_{i,j=1}^n \in M_n(\mathbb{R})$ is called bistochastic if $c_{ij} \geq 0$, $\sum_{i=1}^n c_{ij} = 1$ and $\sum_{j=1}^n c_{ij} = 1$ for every $1 \leq i, j \leq n$. The following result is due to Birkhoff (see [9]).

Theorem 2.5.6. *Let $a, b \in \mathbb{R}^n$ be positive vectors. If $b \prec a$, then there exists a bistochastic matrix $C \in M_n(\mathbb{R})$ such that $b = Ca$.*

Corollary 2.5.7. *Let $a, b \in \mathbb{R}^n$ be positive vectors. If $b \prec\prec a$, then there exists a bistochastic matrix $C \in M_n(\mathbb{R})$ such that $b \leq Ca$.*

Proof. Without loss of generality, $a = a^*$ and $b = b^*$. We prove the assertion by induction on n. Fix

$$\varepsilon_0 = \min_{1\leq k\leq n} \frac{1}{k}\sum_{i=1}^{k}(a_i - b_i).$$

Set $d = b + \varepsilon_0 \cdot \mathbf{1}$. Clearly, $b \leq d$ and $d \prec\prec a$. Moreover, there exists k such that

$$\sum_{i=1}^{k} d_i = \sum_{i=1}^{k} a_i.$$

If $k = n$, then by the Birkhoff theorem, $d = Ca$ for some bistochastic matrix C and we are done. If $k < n$, set $a^1 = (a_1, \cdots, a_k)$ and $a^2 = (a_{k+1}, \cdots, a_n)$. Similarly, $d^1 = (d_1, \cdots, d_k)$ and $d^2 = (d_{k+1}, \cdots, d_n)$. By induction, there exist bistochastic matrices C_1 and C_2 such that $d^1 = C_1a^1$ and $d^2 \leq C_2a^2$. The assertion follows for the matrix $C = C_1 \oplus C_2$. □

The theorem which follows was proved by Mekler (see [39, 40]) in the case of the interval $(0, 1)$. Our approach considerably simplifies that of Mekler and permits extension to functions on the semi-axis.

Theorem 2.5.8. *Let $x \in L_1(0, 1)$ (or $x \in (L_1 + L_\infty)(0, \infty)$ such that $x \notin L_1(0, \infty)$) and $y \prec\prec x$. It follows that for every fixed $q \in (0, 1)$ there exists a positive function z such that $z^* = x^*$ and $qy \leq P(z|\mathcal{A})$. Here, $\mathcal{A}$ is a some partition of the interval (or that of the semi-axis).*

Proof. Without loss of generality, it may be assumed that $x = x^*$ and $y = y^*$.

Let t_n be as in Lemma 2.5.1 (respectively, in Lemma 2.5.3). It follows that

$$qy\chi_{(t_n,t_{n+1})} \lhd x\chi_{(t_n,t_{n+1})}.$$

Define functions u, v on the interval (t_n, t_{n+1}) according to Lemma 2.5.5. It is sufficient to prove the assertion for the functions u and v.

Therefore, we may assume without loss of generality that our interval is $(0, 1)$ and

$$x = \sum_{i=1}^{n} x_i\chi_{[(i-1)/n,i/n]}, \quad y = \sum_{i=1}^{n} y_i\chi_{[(i-1)/n,i/n]}.$$

Let $a = \{x_i\}_{i=1}^n$ and $b = \{y_i\}_{i=1}^n$. Clearly, $b \prec\prec a$. According to Corollary 2.5.7, there exists a bistochastic matrix C such that $b \leq Ca$. Set

$$z(t) = x_j \quad \forall t \in [\frac{i-1}{n} + \frac{1}{n}\sum_{k=1}^{j-1} c_{ik}, \frac{i-1}{n} + \frac{1}{n}\sum_{k=1}^{j} c_{ik}].$$

It is clear that

$$m(\{t : \ z(t) = x_j\}) = \frac{1}{n}\sum_{i=1}^{n} c_{ij} = \frac{1}{n} = m(\{t : \ x(t) = x_j\}).$$

Therefore, z is equimeasurable with x. On the other hand,

$$\int_{(i-1)/n}^{i/n} z(s)ds = \frac{1}{n}\sum_{k=1}^{n} c_{ik}x_k = \frac{1}{n}(Ca)_i \geq \frac{1}{n}y_i = y|_{((i-1)/n,i/n)}.$$

If $\mathcal{A} = \{(i-1)/n, i/n\}_{i=1}^n$, then $y \leq P(z|\mathcal{A})$. □

2.6 Example of a fully symmetric quasi-Banach space

We are going to discuss the properties of quasi-Banach symmetric spaces which are interpolation spaces between L_1 and L_∞. Clearly, every fully symmetric quasi-Banach space is an interpolation space between L_1 and L_∞. Indeed, an inspection of Theorem 4.3 in [33] shows that the proof in the Banach setting carries verbatim to the quasi-Banach case.

However, one can ask for an example with a quasi-norm not equivalent to a norm. Even the existence of such spaces is a non-trivial fact. It is possible to find such an example among the spaces $L_{p,q}$. In fact, if $0 < q < 1$ and $p > 1$, then $L_{p,q}$ is a fully symmetric quasi-Banach space. On the other hand, its quasi-norm $\|\cdot\|_{p,q}$ is not equivalent to a Banach norm.

Lemma 2.6.1. *Let $0 < q < 1$. We have $L_{p,q}(0,1) \subset L_1(0,1)$ if and only if $p \geq 1$.*

Proof. Let $p \geq 1$. Clearly, $x \in L_{p,q}$ if and only if $x^q \in \Lambda_{t^{q/p}} \subset L_{p/q}$. Therefore, $x \in L_p \subset L_1$. Now let $p \in (0,1)$. Fix $\alpha \in (1, 1/p)$. It is clear that $t^{-\alpha}$ belongs to $L_{p,q}$ but not to L_1. □

Lemma 2.6.2. *Let $0 < q < 1$ and $p > 1$. There exists a constant $C_{p,q}$ such that for every $u \in \Lambda_{t^{q/p}}$ and every partition $\mathcal{A}$ we have*

$$L(u) = \int_0^1 P(u^{1/q}|\mathcal{A})^q dt^{q/p} \leq C_{p,q}\|u\|_{\Lambda_{t^{q/p}}}. \tag{2.19}$$

Proof. The Lorentz space $\Lambda_{t^{q/p}}$ is an interpolation space $[L_\infty, L_{1/q}]_{1/p,1}$ by Theorem 2.g.18 from [36]. Therefore, by Theorem 2.g.14 from [36],

$$\|\lambda\|_{\Lambda_{t^{q/p}}} \leq C_{p,q}\|\lambda\|_\infty^{1-1/p}\|\lambda\|_{1/q}^{1/p}.$$

In particular, for every $0 \leq \lambda \leq 1$, we have

$$\int_0^1 \lambda(t) dt^{q/p} \leq \int_0^1 \lambda^*(t) dt^{q/p} \leq C_{p,q}(\int_0^1 \lambda^{1/q}(t)dt)^{q/p}.$$

Set

$$\lambda = P(\chi_A|\mathcal{A})^q \leq 1.$$

We have

$$L(\chi_A) = \int_0^1 \lambda(t)dt^{q/p} \leq C_{p,q}(\int_0^1 \lambda^{1/q}(t)dt)^{q/p} = C_{p,q}\|\chi_A\|_{\Lambda_{t^{q/p}}}.$$

Since $q < 1$, the transformation

$$u \to P((u \circ \gamma)^{1/q}|\mathcal{A})^q, \quad u \in \Lambda_{t^{q/p}}$$

is a convex mapping. Therefore, $L(\cdot)$ is a convex functional.

The assertion (2.19) holds for functions u taking finitely many values by Lemma 5.2 from [33].

For every $u \in \Lambda_{t^{q/p}}$, there exist a sequence $\{u_n\}_{n=1}^{\infty}$ of finitely-valued functions such that $u_n \uparrow u$. By the Levi theorem,

$$P((u_n \circ \gamma)^{1/q}|\mathcal{A})^q \uparrow P((u \circ \gamma)^{1/q}|\mathcal{A})^q.$$

Applying the Levi theorem again, we obtain $L(u_n) \uparrow L(u)$. The assertion follows immediately. □

Lemma 2.6.3. *If* $0 < q < 1$ *and* $p > 1$, *then* $L_{p,q}$ *is fully symmetric.*

Proof. For every $x \in L_{p,q}$ and every measure preserving transform γ, we have $(x \circ \gamma)^q \in \Lambda_{t^{q/p}}$. It follows from Lemma 2.6.2 that

$$\int_0^1 P(x \circ \gamma|\mathcal{A})^q dt^{q/p} \leq C_{p,q} \int_0^1 (x^*)^q dt^{q/p}.$$

Therefore,

$$\int_0^1 P(x^* \circ \gamma|\mathcal{A})^q dt^{q/p} \leq C_{p,q} \int_0^1 (x^*)^q dt^{q/p}$$

for every $x \in L_{p,q}$.

Note that $P(x^* \circ \gamma|\mathcal{A})$ is a step function. Hence, there exists a measure-preserving bijection γ_0 such that

$$P(x^* \circ \gamma|\mathcal{A})^* = P(x^* \circ \gamma|\mathcal{A}) \circ \gamma_0 = P(x^* \circ \gamma \circ \gamma_0|\gamma_0^{-1}\mathcal{A}).$$

It follows that

$$\int_0^1 (P(x^* \circ \gamma|\mathcal{A})^*)^q dt^{q/p} \leq C_{p,q} \int_0^1 (x^*)^q dt^{q/p}$$

for every $x \in L_{p,q}$.

The latter statement means that

$$\|P(x^* \circ \gamma|\mathcal{A})\|_{p,q} \leq C_{p,q}^{1/q} \|x\|_{p,q}.$$

According to Theorem 2.5.8, if $y \prec\prec x$, then $y \leq 2P(z|\mathcal{A})$ with $z^* = x^*$. According to Theorem 1.2.5, one can represent z as $z = x^* \circ \gamma$ with γ being a measure-preserving transformation. It follows immediately that $\|y\|_E \leq \text{const} \cdot \|x\|_E$. Hence, there exists an equivalent quasi-norm on E, which turns it into a fully symmetric space. □

Now we prove that the quasi-norm in $L_{p,q}$ (for $q < 1 < p$) is not equivalent to a norm. In order to prove it, we establish an inclusion of l_q into $L_{p,q}$. Such results are available in e.g. [32]. We provide the proof here for the sake of completeness.

Lemma 2.6.4. *Let Λ_ψ be an arbitrary Lorentz space on the interval $(0,1)$ and let $\{x_n\}_{n=1}^\infty$ be a normalised disjoint sequence in Λ_ψ. There exists a subsequence $\{y_k\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ equivalent to the unit vector basis of l_1.*

Proof. Without loss of generality, each x_n is piecewise-constant. Set $A_n = m(\mathrm{supp}(x_n))$. Select B_n such that

$$\int_{B_n}^{A_n} x_n^* d\psi(t) = \frac{1}{2}.$$

Let $n_1 = 1$ and let n_k be the least possible natural number such that $A_{n_k} \leq B_{n_{k-1}}$. Set $y_k = x_{n_k}$.

Since $\|\cdot\|_{\Lambda_\psi}$ is convex and $\|y_n\|_{\Lambda_\psi} = 1$, it follows that

$$\|\sum_k a_k y_k\|_{\Lambda_\psi} \leq \sum_k |a_k|.$$

Let $y_k^* = |y_k| \circ \gamma_k$, where γ_k is a measure-preserving transform. Let γ be a measure-preserving transform such that $\gamma = \gamma_k$ on $[A_{n_{k+1}}, A_{n_k}]$. It follows that

$$\|\sum_k a_k y_k\|_{\Lambda_\psi} = \int_0^1 (\sum_k |a_k| \cdot |y_k|)^*(t) d\psi(t) \geq \int_0^1 (\sum_k |a_k| \cdot |y_k|)(\gamma(t)) d\psi(t).$$

However,

$$\sum_k |a_k| \cdot |y_k|(\gamma(t)) \geq |a_k| \cdot y_k^*(t), \quad \forall t \in [A_{n_{k+1}}, A_{n_k}].$$

Since $A_{n_{k+1}} \leq B_{n_k}$, it follows that

$$\|\sum_k a_k y_k\|_{\Lambda_\psi} \geq \sum_k |a_k| \int_{B_{n_k}}^{A_{n_k}} x_{n_k}^*(t) d\psi(t) \geq \frac{1}{2} \sum_k |a_k|.$$

□

Corollary 2.6.5. *Let $\{x_n\}_{n=1}^\infty$ be a normalised disjoint sequence in $L_{p,q}$ $(q < p)$. There exists a subsequence $\{y_k\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ equivalent to the unit vector basis of l_q.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be such a sequence. It follows that $\{x_n^q\}_{n=1}^\infty$ forms a disjoint sequence in $\Lambda_{t^{q/p}}$. Clearly, $\|x_n^q\|_{\Lambda_{t^{q/p}}} = 1$. Let $\{y_k^q\}_{k=1}^\infty \subset \{x_n^q\}_{n=1}^\infty$ be a subsequence defined in Lemma 2.6.4. It is clear that $\{y_k\}_{k=1}^\infty$ satisfies our requirements. □

Corollary 2.6.6. *The quasi-norm in the space $L_{p,q}$ (for $0 < q < 1 < p$) is not equivalent to any norm.*

Proof. Assume the contrary. Let $\{x_n\} \subset L_{p,q}$ be any normalised disjoint sequence and let $\{y_n\} \subset \{x_n\}$ be a sequence defined in Corollary 2.6.5. The $L_{p,q}-$norm on the linear span of $\{y_n\}$ is equivalent to l_q-quasinorm, which is impossible. □

Example 2.6.7. *If $0 < q < 1 < p$, then space $L_{p,q}$ is a fully symmetric quasi-Banach space on the interval $(0, 1)$ whose quasi-norm is not equivalent to any norm.*

2.7 Sufficiency

2.7.1 The Mekler approach

The Mekler approach allows us to consider the orbits $\Omega(x)$ and $\Omega^+(x)$ in the case that $E = E(0,1)$ or that $E = E(0,\infty)$, provided that $E(0,\infty) \not\subset L_1(0,\infty)$.

Theorem 2.7.1. *Let $E = E(0,1)$ be a fully symmetric quasi-Banach space on the interval $(0,1)$ and let $x \in E$. If $\varphi(x) = 0$, then $\Omega(x) = \mathcal{Q}_E(x)$.*

Proof. Let $x = x^*$ and let y satisfy $y \prec\prec x$. There exists a measure-preserving transform $\gamma : (0,1) \to (0,1)$ such that $y = \mathrm{sgn}(y) \cdot y^* \circ \gamma$. Hence, we may assume without loss of generality that $y = y^*$.

Fix $\varepsilon > 0$ and $q \in (1-\varepsilon, 1)$. According to Theorem 2.5.8, there exists a positive function z such that $z^* = x^*$ and a partition $\mathcal{A}$ such that

$$0 \le qy \le P(z|\mathcal{A}).$$

By Theorem 2.4.11,

$$P(z|\mathcal{A}) \in \mathcal{Q}_E(x).$$

By Lemma 2.3.3, $qy \in \mathcal{Q}_E(x)$. Therefore,

$$y = qy + (1-q)y \in \mathcal{Q}_E(x) + \varepsilon B_E(0,1).$$

Since ε is arbitrary and $\mathcal{Q}_E(x)$ is a closed set, it follows that $y \in \mathcal{Q}_E(x)$ □

If the space E is not a subset of $L_1(0,\infty)$, we are able to prove a significantly stronger assertion.

Theorem 2.7.2. *Let $E = E(0,\infty)$ be a fully symmetric quasi-Banach space on the semi-axis such that $E(0,\infty) \not\subset L_1(0,\infty)$. If $\varphi(x) = 0$, then $\Omega^+(x) = \mathcal{Q}'_E(x)$.*

Proof. Let $x = x^* \in E \backslash L_1$ and let $y = y^*$ be such that $y \prec\prec x$. Fix $\varepsilon > 0$ and $q \in (1-\varepsilon, 1)$. According to Theorem 2.5.8, there exists a positive function z such that $z^* = x^*$ and a partition $\mathcal{A}$ such that

$$0 \le qy \le P(z|\mathcal{A}).$$

By Theorem 2.4.11,

$$P(z|\mathcal{A}) \in \mathcal{Q}'_E(x).$$

By Lemma 2.3.8, $qy \in \mathcal{Q}'_E(x)$. Therefore,

$$y = qy + (1-q)y \in \mathcal{Q}'_E(x) + \varepsilon B_E(0,1).$$

Since ε is arbitrary and $\mathcal{Q}'_E(x)$ is a closed set, it follows that $y \in \mathcal{Q}'_E(x)$.

Let $x = x^* \in E \backslash L_1$ and let y satisfy $y \prec\prec x$. For every fixed $\varepsilon > 0$, there exists a measure-preserving transform $\gamma : (0,\infty) \to (0,\infty)$ such that

$$\| \max\{y, y^*(\infty)\} - y^* \circ \gamma\|_{L_1 \cap L_\infty} \le \varepsilon.$$

By the argument above, $y^* \in \mathcal{Q}'_E(x)$. Therefore,

$$\max\{y, y^*(\infty)\} \in \mathcal{Q}'_E(x) + \varepsilon B_E(0,1).$$

Since ε is arbitrary and $\mathcal{Q}'_E(x)$ is a closed set, it follows that $\max\{y, y^*(\infty)\} \in \mathcal{Q}'_E(x)$. By Lemma 2.3.8, $y \in \mathcal{Q}'_E(x)$.

Let $x = x^* \in E \cap L_1$ and let y be such that $y \prec\prec x$. Since $y^*(\infty) = 0$, there exists a measure-preserving transform $\gamma : (0,\infty) \to (0,\infty)$ such that $y = y^* \circ \gamma$. Hence, we may assume without loss of generality that $y = y^*$.

Fix $\varepsilon > 0$ and q such that $0 < (1-q)\|y\|_E \le \varepsilon$. There exists $T > 0$ such that

$$\|x^*\chi_{(T,\infty)}\|_{L_1\cap L_\infty} \le \varepsilon, \quad \|y^*\chi_{(T,\infty)}\|_{L_1\cap L_\infty} \le \varepsilon.$$

Clearly, $y^*\chi_{[0,T]} \prec\prec x^*\chi_{[0,T]}$. According to Theorem 2.5.8, there exists a positive function z such that $z^* = x^*\chi_{[0,T]}$ and a partition $\mathcal{A}$ such that

$$0 \le qy^*\chi_{[0,T]} \le P(z|\mathcal{A}).$$

By Theorem 2.4.11,

$$P(z|\mathcal{A}) \in \mathcal{Q}'_E(x^*\chi_{[0,T]}).$$

By Lemma 2.3.8, $qy^*\chi_{[0,T]} \in \mathcal{Q}'_E(x^*\chi_{[0,T]})$. By Lemma 2.3.4, $qy^*\chi_{[0,T]} \in \mathcal{Q}'_E(x)$. Therefore,

$$y = qy^*\chi_{[0,T]} + y^*\chi_{(T,\infty)} + (1-q)y \in \mathcal{Q}'_E(x) + \varepsilon B_E(0,1) + (1-q)y \subset$$
$$\subset \mathcal{Q}'_E(x) + \varepsilon B_E(0,1) + \varepsilon B_E(0,1) \subset \mathcal{Q}'_E(x) + 2C(E)\varepsilon B_E(0,1).$$

Since ε is arbitrary and $\mathcal{Q}'_E(x)$ is a closed set, it follows that $y \in \mathcal{Q}'_E(x)$. □

The corresponding result for the case of the full orbit $\Omega(x)$ now follows as a direct consequence.

Theorem 2.7.3. *Let $E = E(0,\infty)$ be a fully symmetric quasi-Banach space on the semi-axis such that $E(0,\infty) \not\subset L_1(0,\infty)$. If $\varphi(x) = 0$, then $\Omega(x) = \mathcal{Q}_E(x)$.*

Proof. If $y \in \Omega(x)$, then $|y| \in \Omega(x)$. By Theorem 2.7.2, $|y| \in \mathcal{Q}'_E(x)$. Hence, for every fixed $\varepsilon > 0$,

$$|y| \in \mathcal{Q}'(x) + \varepsilon B_E(0,1)$$

and

$$y \in \operatorname{sgn}(y)\mathcal{Q}'(x) + \varepsilon B_E(0,1) \subset \mathcal{Q}_E(x) + \varepsilon B_E(0,1).$$

Since $\varepsilon > 0$ is arbitrary and $\mathcal{Q}_E(x)$ is a convex set, it follows that $y \in \mathcal{Q}_E(x)$. □

2.7.2 The Braverman approach

The Mekler approach of the preceding section cannot deal with the case of the orbits $\Omega'(x)$ either on the interval $(0,1)$ or on the semi-axis. In the following sections, we follow the approach of Braverman (see [11]). In addition, the Braverman approach permits us to consider the orbits $\Omega^+(x)$ in the case that $E = E(0,1)$ or $E = E(0,\infty) \subset L_1(0,\infty)$.

Preliminary results

The following proposition is the core of the Braverman approach. In the case of the interval $(0,1)$ it can be found in [11, Lemma 3.2]. However, our proof is more general, simpler and shorter.

We consider functions of the form

$$x = \sum_{i \in Z} x_i \chi_{[a_{i-1}, a_i]}, \quad y = \sum_{i \in Z} y_i \chi_{[a_{i-1}, a_i]}, \tag{2.20}$$

where $\{a_i\}_{i \in Z}$ is an increasing sequence (possibly finite or one-sidedly infinite).

Proposition 2.7.4. *Let $y = y^*$ and $x = x^*$ be functions of the form (2.20) either on the interval $(0,1)$ or on the semi-axis. If $y \prec x$, then there exists a countable collection $\{\Delta_k\}_{k \in \mathcal{K}}$ of disjoint sets, such that*

i) Every set Δ_k can be represented as

$$\Delta_k = I_k \cup J_k,$$

where I_k and J_k are intervals of finite measure. The interval I_k lies to the left of J_k for every $k \in \mathcal{K}$.

ii) The functions x and y are constant on the intervals I_k and J_k.

iii) For every $k \in \mathcal{K}$,

$$y|_{\Delta_k} \prec x|_{\Delta_k}.$$

iv) $y(t) = x(t)$ if $t \notin \cup_{k \in \mathcal{K}} \Delta_k$.

Proof. There exists a subsequence $\{a_{m_i}\}_{i \in \mathcal{I}}$ (possibly finite or one-sidedly infinite) such that

$$\{x < y\} = \cup_{i \in \mathcal{I}} [a_{m_i - 1}, a_{m_i}].$$

Since $y \prec x$, we have

$$\int_0^t (x-y)_+(s)ds - \int_0^t (y-x)_+(s)ds = \int_0^t x(s)ds - \int_0^t y(s)ds \geq 0.$$

For each $i \in \mathcal{I}$, denote by b_i the minimal $t > 0$, such that

$$\int_0^t (x-y)_+(s)ds = \int_0^{a_{m_i}} (y-x)_+(s)ds.$$

Clearly, for every $i \in \mathcal{I}$,

$$\int_0^{a_{m_i - 1}} (x-y)_+(s)ds = \int_0^{a_{m_i}} (x-y)_+(s)ds \geq \int_0^{a_{m_i}} (y-x)_+(s)ds.$$

Hence, $b_i \leq a_{m_i - 1}$.

For each $i \in \mathcal{I}$, the set

$$[b_{i-1}, b_i] \cap \{x > y\} = \cup_{j=1}^{n_i} I_i^j.$$

is a finite union of disjoint intervals on which each of x and y is finite.

By the definition of b_i, we have

$$\int_{a_{m_i-1}}^{a_{m_i}} (y-x)_+(s)ds = \int_{b_{i-1}}^{b_i} (x-y)_+(s)ds = \sum_{j=1}^{n_i} \int_{I_i^j} (x-y)_+(s)ds.$$

Set

$$\mathcal{K} = \{(i,j) : 1 \le j \le n_i, i \in \mathcal{I}\}.$$

If $k = (i,j) \in \mathcal{K}$, set $I_k = I_i^j$ and

$$J_k = J_i^j = [a_{m_i-1} + (y_{m_i} - x_{m_i})^{-1} c_i^{j-1}, a_{m_i-1} + (y_{m_i} - x_{m_i})^{-1} c_i^j],$$

where

$$c_i^j = \sum_{l=1}^{j} \int_{I_i^l} (x-y)_+(s)ds, \quad i \in \mathcal{I}, 0 \le j \le n_i.$$

Using the fact that x and y are constant on the interval $[a_{m_i-1}, a_{m_i}]$, we obtain

$$J_k = J_i^j \subset [a_{m_i-1}, a_{m_i}]$$

and

$$\cup_{j=1}^{n_i} J_i^j = [a_{m_i-1}, a_{m_i}].$$

Since

$$I_i^j \subset [b_{i-1}, b_i] \subset [0, b_i] \subset [0, a_{m_i-1}]$$

and

$$J_i^j \subset [a_{m_i-1}, a_{m_i}],$$

it follows that every interval I_k lies to the left of J_k.

(ii) By the definition of I_i^j, the functions x and y are constant on it. Since J_i^j is a subset of the interval $[a_{m_i-1}, a_{m_i}]$, it follows that functions x and y are constant on J_i^j.

(iii) It is clear that

$$\int_{I_k} (x-y)_+(s)ds = \int_{J_k} (y-x)_+(s)ds, \quad k \in \mathcal{K} \tag{2.21}$$

and

$$\int_{I_k \cup J_k} x(s)ds = \int_{I_k \cup J_k} y(s)ds.$$

Note that

$$x(t) \ge y(t) \ \forall t \in I_k, \quad y(t) \ge x(t) \ \forall t \in J_k$$

for all $k \in \mathcal{K}$. The assertion follows immediately.

(iv) By the definition of the set J_i^j,

$$\{y > x\} = \cup_{i \in \mathcal{I}} \cup_{j=1}^{n_i} J_i^j \subseteq \cup_{k \in \mathcal{K}} \Delta_k.$$

Therefore, $y(t) \leq x(t)$ for every $t \notin \cup_{k \in \mathcal{K}} \Delta_k$. However,

$$\int_{\Delta_k} y(s)ds = \int_{\Delta_k} x(s)ds \quad \forall k \in \mathcal{K}.$$

Since $y \prec x$, it follows that

$$\int_{s \notin \cup_{k \in \mathcal{K}} \Delta_k} y(s)ds = \int_{s \notin \cup_{k \in \mathcal{K}} \Delta_k} x(s)ds.$$

Therefore, $y(t) = x(t)$ for every $t \notin \cup_{k \in \mathcal{K}} \Delta_k$. □

Corollary 2.7.5. *Let E be a fully symmetric quasi-Banach space either on the interval $(0,1)$ or on the semi-axis. If x, y and $\mathcal{B} = \{\Delta_k\}_{k \in \mathcal{K}}$ are as in Proposition 2.7.4, then y can be arbitrarily well approximated by convex combinations of functions of the form $P(x|\mathcal{A})$, $\mathcal{A} \in \mathfrak{A}$. Here, approximation is in the topology induced by the quasi-norm of E.*

Proof. Set

$$\lambda_k = (y|_{I_k} - y|_{J_k})/(x|_{I_k} - x|_{J_k})$$

for every $k \in \mathcal{K}$. According to Proposition 2.7.4,

$$y|_{\Delta_k} \prec x|_{\Delta_k}$$

and, therefore, it is not difficult to verify that $\lambda_k \in [0,1]$ for every $k \in \mathcal{K}$.

Further, a simple calculation shows that

$$y = (1 - \lambda_k)P(x|\mathcal{B}) + \lambda_k x$$

on the set Δ_k, $k \in \mathcal{K}$.

As is well-known, every $[0,1]-$valued sequence can be uniformly approximated by convex combinations of $\{0,1\}-$valued sequences.

Fix $\varepsilon > 0$. There exists $\mu \in l_\infty(\mathcal{K})$ such that

$$\mu = \sum_{i=1}^n \theta_i \chi_{D_i}, \quad \|\lambda - \mu\|_\infty \leq \varepsilon.$$

Here, $n \in \mathbb{N}$, $\theta_i \in \mathbb{R}_+$ are such that $\sum_{i=1}^n \theta_i = 1$ and $D_i \subseteq \mathcal{K}$.

Set

$$z(t) = (1 - \mu_k)P(x|\mathcal{B})(t) + \mu_k x(t), \quad \forall t \in \Delta_k$$

for every $k \in \mathcal{K}$ and

$$z(t) = x(t), \quad \forall t \notin \cup_{k \in \mathcal{K}} \Delta_k.$$

It is clear that

$$|y - z|\chi_{\Delta_k} = |\lambda_k - \mu_k|\, |x - P(x|\mathcal{B})|\chi_{\Delta_k}$$

for every $k \in \mathcal{K}$. Hence,

$$|y - z| = \sum_{k \in \mathcal{K}} |y - z|\chi_{\Delta_k} \leq 2\varepsilon(x + P(x|\mathcal{B})).$$

Since E is fully symmetric, it follows that

$$\|y - z\|_E \leq 2\varepsilon C(E)\|x\|_E.$$

Set $F_i = \cup_{k \in D_i} \Delta_k$ and $\mathcal{A}_i = \{\Delta_k\}_{k \notin D_i} \in \mathfrak{A}$, $1 \leq i \leq n$. It is then clear that

$$z = \sum_{i=1}^n \theta_i((1 - \chi_{F_i})P(x|\mathcal{B}) + \chi_{F_i} x) = \sum_{i=1}^n \theta_i P(x|\mathcal{A}_i).$$

□

Lemma 2.7.6. *Let E be a fully symmetric quasi-Banach space on the interval $(0, 1)$. If x and y are as in Proposition 2.7.4 and $\varphi(x) = 0$, then $y \in \mathcal{Q}'_E(x)$.*

Proof. Fix $\varepsilon > 0$. According to Corollary 2.7.5, there exists $z \in E$ such that

$$z = \sum_{i=1}^n \theta_i P(x|\mathcal{A}_i), \quad \|y - z\|_E \leq \varepsilon.$$

Here, $n \in \mathbb{N}$ and $0 \leq \theta_i \in \mathbb{R}$ are such that $\sum_{i=1}^n \theta_i = 1$. According to Theorem 2.4.11, there exist $z_i \in \mathcal{Q}'(x)$ such that

$$P(x|\mathcal{A}_i) \in z_i + \varepsilon C(E)^{-n} B_E(0, 1).$$

It follows that

$$z = \sum_{i=1}^n \theta_i P(x|\mathcal{A}_i) \in \sum_{i=1}^n \theta_i z_i + \varepsilon C(E)^{-n} \sum_{i=1}^n \theta_i B_E(0, 1).$$

By the definition of a quasi-norm,

$$\sum_{i=1}^n \theta_i B_E(0, 1) \subset C(E)^n B_E(0, 1).$$

Since $\mathcal{Q}'(x)$ is convex, it follows that

$$\sum_{i=1}^n \theta_i z_i \in \mathcal{Q}'(x).$$

Therefore,

$$z \in \mathcal{Q}'(x) + \varepsilon B_E(0, 1)$$

and

$$y \in z + \varepsilon B_E(0, 1) \subset \mathcal{Q}'(x) + \varepsilon B_E(0, 1) + \varepsilon B_E(0, 1) \subset \mathcal{Q}'(x) + 2C(E)\varepsilon B_E(0, 1).$$

Since ε is arbitrarily small, it follows that

$$y \in \mathcal{Q}'_E(x)$$

and we are done. □

The case that $E \subseteq L_1$

Theorem 2.7.7. *Let $E = E(0,1)$ be a fully symmetric quasi-Banach space on the interval $(0,1)$. If $x \in E$ is such that $\varphi(x) = 0$, then $\Omega'(x) = \mathcal{Q}'_E(x)$.*

Proof. Let $x = x^*$ and $0 leq y \in \Omega'(x)$. In this case, $y = y^* \circ \gamma$ for some measure-preserving transformation γ (see [50] or [8, Theorem 7.5,p.82]). Without loss of generality, we may assume that $y = y^*$.

Fix $\varepsilon > 0$. Set

$$s_n(\varepsilon) = \inf\{s : y(s) \le y(1) + n\varepsilon\}, \quad n \in \mathbb{N}.$$

Let $\mathcal{A}_\varepsilon$ be the partition, determined by the points $s_n(\varepsilon)$, $n \in \mathbb{N}$. Set $u = P(y|\mathcal{A}_\varepsilon)$ and $z = P(x|\mathcal{A}_\varepsilon)$.

Clearly, the functions u and z are of the form given in (2.20) and $u \prec z$. Therefore, one can apply Lemma 2.7.6.

We obtain

$$u \in \mathcal{Q}'(z) + \varepsilon B_E(0,1).$$

On the other hand,

$$\|x - z\|_E \le \|x - z\|_\infty \le \varepsilon, \quad \|y - u\|_E \le \|y - u\|_\infty \le \varepsilon.$$

Clearly,

$$\mathcal{Q}'(z) \subset \mathcal{Q}'(x) + \mathcal{Q}'(x - z) \subset \mathcal{Q}'(x) + \Omega(x - z) \subset \mathcal{Q}'(x) + \varepsilon B_E(0,1)$$

and

$$y = u + (y - u) \in u + \varepsilon B_E(0,1).$$

Therefore,

$$y \in u + \varepsilon B_E(0,1) \subset \mathcal{Q}'(z) + \varepsilon B_E(0,1) + \varepsilon B_E(0,1) \subset$$

$$\subset \mathcal{Q}'(x) + \varepsilon B_E(0,1) + \varepsilon B_E(0,1) + \varepsilon B_E(0,1) \subset \mathcal{Q}'(x) + 3C^2(E)\varepsilon B_E(0,1).$$

Since ε is arbitrarily small, it follows that

$$y \in \mathcal{Q}'_E(x)$$

and the proof is complete. □

Theorem 2.7.8. *Let $E = E(0,1)$ be a fully symmetric quasi-Banach space on the interval $(0,1)$. If $x \in E$ is such that $\varphi(x) = 0$, then $\Omega^+(x) = \mathcal{Q}^+_E(x)$.*

Proof. Suppose that $\varphi(x) = 0$ and let $0 \le y \in \Omega^+(x)$. There exists $s_0 \in [0,1]$, such that

$$\int_0^{s_0} x^*(s)ds = \int_0^1 y^*(s)ds$$

and this implies $y \prec x^*\chi_{[0,s_0]}$.

It is clear that

$$0 \leq \varphi(x^* \chi_{[0,s_0]}) \leq \varphi(x) = 0.$$

It follows now from Theorem 2.7.7 that

$$y \in \mathcal{Q}'_E(x^* \chi_{[0,s_0]}) \subset \mathcal{Q}^+_E(x).$$

Hence, $y \in \mathcal{Q}^+_E(x)$ and this completes the proof. □

We now consider the case that $E = E(0,\infty)$.

Theorem 2.7.9. *Let $E = E(0,\infty)$ be a fully symmetric quasi-Banach space on the semi-axis. Suppose that $E(0,\infty) \subset L_1(0,\infty)$. If $x \in E$ is such that $\varphi_{fin}(x) = 0$, then $\Omega'(x) = \mathcal{Q}'_E(x)$.*

Proof. Let $x = x^*$ and $0 \leq y \in \Omega'(x)$. It follows from [33, Lemma II.2.1] that there exists a measure-preserving transformation γ such that $y = y^* \circ \gamma$. Without loss of generality, we may assume that $y = y^*$.

Initially, we consider the case when $\mathrm{supp}(x) = (0,\infty)$.

Fix $\varepsilon > 0$. It is clear that there exists T such that

$$\|x\chi_{[T,\infty)}\|_E \leq \|x\chi_{T,\infty)}\|_{L_1 \cap L_\infty} \leq \varepsilon, \quad \|y\chi_{[T,\infty)}\|_E \leq \|y\chi_{T,\infty)}\|_{L_1 \cap L_\infty} \leq \varepsilon.$$

Since $\mathrm{supp}(x) = (0,\infty)$, it follows that

$$\int_0^T x(s)ds < \int_0^\infty x(s)ds = \int_0^\infty y(s)ds.$$

Hence, there exists $S \geq T$ such that

$$\int_0^T x(s)ds = \int_0^S y(s)ds.$$

Therefore, $y\chi_{[0,S]} \prec x\chi_{[0,T]}$.

Consider the fully symmetric quasi-Banach space F on the interval $[0,S]$ defined by the formula

$$F = \{x \in E : \ \mathrm{supp}(x) \subset [0,S]\}, \quad \|x\|_F = \|x\|_E \ \forall x \in F.$$

It is clear that $x\chi_{[0,T]}, y\chi_{[0,S]} \in F$.

However,

$$\frac{1}{s}\|(\sigma_s x)\chi_{[0,S]}\|_E \leq \frac{S}{s}\|(\sigma_s x)\chi_{[0,1]}\|_E \to 0$$

when $s \to \infty$. Therefore, the assumption $\varphi_{fin,E}(x) = 0$ implies that $\varphi_F(x\chi_{[0,T]}) = 0$. Hence, by Theorem 2.7.7,

$$y\chi_{[0,S]} \in \mathcal{Q}'_F(x\chi_{[0,T]}) \subset \mathcal{Q}'(x\chi_{[0,T]}) + \varepsilon B_F(0,1) \subset \mathcal{Q}'(x\chi_{[0,T]}) + \varepsilon B_E(0,1).$$

On the other hand,

$$\mathcal{Q}'(x\chi_{[0,T]}) \subset \mathcal{Q}'(x\chi) + \mathcal{Q}'(x\chi_{[T,\infty)}) \subset \mathcal{Q}'(x) + \Omega(x\chi_{[T,\infty)}) \subset \mathcal{Q}'(x) + \varepsilon B_E(0,1)$$

and

$$y = y\chi_{[0,S]} + y\chi_{[S,\infty)} \in y\chi_{[0,S]} + \varepsilon B_E(0,1).$$

Hence,

$$y \in y\chi_{[0,S]} + \varepsilon B_E(0,1) \subset \mathcal{Q}'(x\chi_{[0,T]}) + \varepsilon B_E(0,1) + \varepsilon B_E(0,1) \subset$$

$$\subset \mathcal{Q}'(x) + \varepsilon B_E(0,1) + \varepsilon B_E(0,1) + \varepsilon B_E(0,1) \subset \mathcal{Q}'(x) + 3C^2(E)\varepsilon B_E(0,1).$$

Since ε is arbitrarily small, it follows that

$$y \in \mathcal{Q}'_E(x).$$

Let us now consider the case that $m(\text{supp}(x)) < \infty$.

Fix $z = z^* \in L_1 \cap L_\infty$ with $m(\text{supp}(z)) = \infty$. It is clear that for every $\varepsilon > 0$ we have

$$(y + \varepsilon z) \in \Omega'(x + \varepsilon z).$$

Clearly,

$$0 \le \varphi_{fin}(x + \varepsilon z) \le C(E)(\varphi_{fin}(x) + \varepsilon\varphi_{fin}(z)) = 0.$$

Here, the last inequality follows from the assumption and Lemma 2.1.7. Thus, $\varphi_{fin}(x + \varepsilon z) = 0$.

It follows from above that

$$(y + \varepsilon z) \in \mathcal{Q}'_E(x + \varepsilon z) \subset \mathcal{Q}'(x + \varepsilon z) + \varepsilon B_E(0,1) \subset$$

$$\subset \mathcal{Q}'(x) + \varepsilon\mathcal{Q}'(z) + \varepsilon B_E(0,1) \subset \mathcal{Q}'(x) + \varepsilon\Omega(z) + \varepsilon B_E(0,1) \subset$$

$$\subset \mathcal{Q}'(x) + \varepsilon B_E(0,1) + \varepsilon\|z\|_E B_E(0,1) \subset \mathcal{Q}'(x) + C(E)(1 + \|z\|_E)\varepsilon B_E(0,1).$$

Since ε is arbitrarily small, it follows that

$$y \in \mathcal{Q}'_E(x)$$

and this suffices to complete the proof. □

Theorem 2.7.10. *Let $E = E(0,\infty)$ be a fully symmetric quasi-Banach space on the semi-axis such that $E(0,\infty) \subseteq L_1(0,\infty)$. If $x \in E$ is such that $\varphi_{fin}(x) = 0$, then $\Omega^+(x) = \mathcal{Q}^+_E(x)$.*

Proof. Suppose that $\varphi(x) = 0$ and let $0 \le y \in \Omega^+(x)$. Hence, there exists $s_0 \in [0,\infty]$, such that

$$\int_0^{s_0} x^*(s)ds = \int_0^\infty y^*(s)ds.$$

Therefore, $y \prec x^*\chi_{[0,s_0]}$.

It is clear that

$$0 \le \varphi(x^*\chi_{[0,s_0]}) \le \varphi(x) = 0.$$

It follows now from Theorem 2.7.9 that

$$y \in \mathcal{Q}'_E(x^*\chi_{[0,s_0]}) \subset \mathcal{Q}^+_E(x).$$

Hence, $y \in \mathcal{Q}^+_E(x)$. □

The corresponding result for the case of the full orbit $\Omega(x)$ follows from Theorem 2.7.10.

Theorem 2.7.11. *Let $E = E(0,\infty)$ be a fully symmetric quasi-Banach space on the semi-axis such that $E(0,\infty) \subseteq L_1(0,\infty)$. If $x \in E$ is such that $\varphi_{fin}(x) = 0$, then $\Omega(x) = \mathcal{Q}_E(x)$.*

Proof. Let $x = x^*$ and $y \in \Omega(x)$. It follows from [33, Lemma II.2.1] that there exists measure-preserving transformation γ such that $y = \mathrm{sgn}(y)y^* \circ \gamma$. Without loss of generality, we may assume that $y = y^*$.

Fix $\varepsilon > 0$. By Theorem 2.7.8, there exist $n \in \mathbb{N}$, scalars $\lambda_i, \beta_i \in [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$ and positive functions x_i such that

$$x_i^* = x\chi_{[0,\beta_i]}$$

and

$$\| |y| - \sum_{i=1}^n \lambda_i x_i \|_E \le \varepsilon.$$

For every $1 \le i \le n$ there exist measure-preserving transformations $\gamma_i : (0,\infty) \to (0,\infty)$ (see [50]) such that

$$x_i = (x\chi_{[0,\beta_i]}) \circ \gamma_i.$$

Set

$$x_i^1 = x \circ \gamma_i, \quad x_i^2 = (x\chi_{[0,\beta_i]} - x\chi_{[\beta_i,1]}) \circ \gamma_i$$

for every $1 \le i \le n$.

It is clear that

$$(x_i^1)^* = x, \quad (x_i^2)^* = x \quad \forall 1 \le i \le n$$

and

$$x_i = \frac{1}{2}(x_i^1 + x_i^2).$$

Therefore,

$$\|y - \frac{1}{2}\sum_{i=1}^n \lambda_i x_i^1 - \frac{1}{2}\sum_{i=1}^n \lambda_i x_i^2\|_E \le \varepsilon.$$

□

2.8 Necessity

In this section, we will show that the sufficient conditions given in each of the Theorems of preceding section are, in fact, necessary.

We start from the relatively simple proof of the necessity results in the case of positive orbits and then proceed with the case of full orbits.

2.8.1 The case of positive orbits

Theorem 2.8.1. *Let $E = E(0,1)$ be a fully symmetric quasi-Banach space on the interval $(0,1)$. If $x \in E$ is such that $\Omega'(x) = \mathcal{Q}'_E(x)$, then $\varphi(x) = 0$.*

Proof. Suppose that $\mathcal{Q}'_E(x) = \Omega'(x)$. Set $\mathcal{A} = \{[0,1]\}$ and $y = P(x|\mathcal{A})$. Clearly, $y \in \Omega'(x) = \mathcal{Q}'_E(x)$. Lemma 2.2.2 implies that $\varphi(x) = \varphi(y)$. Lemma 2.1.7 implies $\varphi(y) = 0$. The assertion is proved. □

Theorem 2.8.2. *Let $E = E(0,1)$ be a fully symmetric quasi-Banach space on the interval $(0,1)$. Suppose that quasi-norm on E is a Fatou quasi-norm. If $x \in E$ is such that $\Omega^+(x) = \mathcal{Q}^+_E(x)$, then $\varphi(x) = 0$.*

Proof. By Proposition 2.2.1(iv), there exist $0 \leq y_1, z_1 \in E$, such that $x = y_1 + z_1$ and

$$\varphi(x) = \varphi(y_1) = \varphi(z_1).$$

By assumption, $\Omega^+(x) = \mathcal{Q}^+_E(x)$ and so $y_1, z_1 \in \mathcal{Q}^+_E(x)$. By Proposition 2.2.4,

$$\varphi(x) = \varphi(y_1) + \varphi(z_1).$$

Consequently, $\varphi(x) = 0$. □

Theorem 2.8.3. *Let $E = E(0,\infty)$ be a fully symmetric quasi-Banach space on the semi-axis. Suppose that $E(0,\infty) \subset L_1(0,\infty)$. If $x \in E$ is such that $\Omega'(x) = \mathcal{Q}'_E(x)$, then $\varphi_{fin}(x) = 0$.*

Proof. Let $x = x^*$ and suppose that $\mathcal{Q}'_E(x) = \Omega'(x)$. Set $\mathcal{A} = \{[0,1]\}$ and $y = P(x|\mathcal{A})$. Clearly, $y \in \Omega'(x) = \mathcal{Q}'_E(x)$. Lemma 2.2.2 implies that $\varphi_{fin}(x) = \varphi_{fin}(y)$. Lemma 2.1.7 implies $\varphi_{fin}(y) = 0$. The assertion is proved. □

Theorem 2.8.4. *Let $E = E(0,\infty)$ be a fully symmetric quasi-Banach space on the semi-axis. Suppose that $E(0,\infty) \subset L_1(0,\infty)$ and that the quasi-norm on E is a Fatou quasi-norm. If $x \in E$ is such that $\Omega^+(x) = \mathcal{Q}^+_E(x)$, then $\varphi_{fin}(x) = 0$.*

Proof. By Proposition 2.2.1(iv), there exist $0 \leq y_1, z_1 \in E$, such that $x = y_1 + z_1$ and

$$\varphi_{fin}(x) = \varphi_{fin}(y_1) = \varphi_{fin}(z_1).$$

By assumption, $\Omega^+(x) = \mathcal{Q}^+_E(x)$ and so $y_1, z_1 \in \mathcal{Q}^+_E(x)$. By Proposition 2.2.4,

$$\varphi_{fin}(x) = \varphi_{fin}(y_1) + \varphi_{fin}(z_1).$$

Consequently, $\varphi_{fin}(x) = 0$. □

Theorem 2.8.5. *Let $E = E(0,\infty)$ be a fully symmetric quasi-Banach space on the semi-axis. Suppose that $E(0,\infty) \not\subset L_1(0,\infty)$ and that the quasi-norm on E is a Fatou quasi-norm. If $x \in E$ is such that $\Omega^+(x) = \mathcal{Q}_E^+(x)$, then $\varphi(x) = 0$.*

Proof. By Proposition 2.2.1(iv), there exist $0 \le y_1, z_1 \in E$, such that $x = y_1 + z_1$ and

$$\varphi_{cut}(x) = \varphi_{cut}(y_1) = \varphi_{cut}(z_1).$$

By assumption, $\Omega^+(x) = \mathcal{Q}_E^+(x)$ and so $y_1, z_1 \in \mathcal{Q}_E^+(x)$. By Proposition 2.2.4,

$$\varphi_{cut}(x) = \varphi_{cut}(y_1) + \varphi_{cut}(z_1).$$

Consequently, $\varphi_{cut}(x) = 0$. By Proposition 2.2.1(ii), there exist $0 \le y_2, z_2 \in E$, such that $x = y_2 + z_2$ and

$$\varphi(x) = \varphi(y_2) = \varphi(z_2).$$

Again, by the assumption, we have $y_2, z_2 \in \mathcal{Q}_E^+(x)$ and therefore, by Proposition 2.2.5, we have

$$\varphi(x) = \varphi(y_2) + \varphi(z_2).$$

Consequently, $\varphi(x) = 0$. □

Theorem 2.8.6. *Let $E = E(0,\infty)$ be a fully symmetric quasi-Banach space on the semi-axis. Suppose that $E(0,\infty) \not\subset L_1(0,\infty)$. If $x \in E \cap L_1$ is such that $\Omega'(x) \subset \mathcal{Q}'_E(x)$, then $\varphi(x) = 0$.*

Proof. The assertion follows from Theorem 2.4.9. □

2.8.2 The case of full orbits

The arguments in this subsection considerably simplify those given by Kalton, Sukochev and Zanin in [31].

Throughout, we assume that $x = x^*$. Let $X(t) = \int_0^t x(s)ds$. By assumption, X is a concave increasing function.

Let $\{a_n\}_{n\in\mathbb{Z}}$, $\{a_n(\theta)\}_{n\in\mathbb{Z}}$ be such that $X(a_n) = (3/2)^n$ and $X(a_n(\theta)) = (3/2)^n\theta$. We define

$$\mathcal{A}_m = \{ma_n :\ ma_n \le a_{n+1},\ n \in \mathbb{Z}\}.$$

If $\{\kappa_n\}_{n\in\mathbb{Z}}$ is an arbitrary sequence such that $\kappa_n > 1$, then we define

$$\mathcal{B}_{\kappa,\theta} = \{\kappa_n a_{3n} :\ \kappa_n a_{3n}(\theta) \le a_{3n+1}(\theta),\ n \in \mathbb{Z}\}.$$

It will be convenient to introduce the following notation. If $\mathcal{A}$ is a discrete subset of the semi-axis, then the elements of $\mathcal{A} \cup \{0\}$ partition the semi-axis. This partition consists of a (finite or infinite) sequence of sets of finite measure. We identify this partition with the set $\mathcal{A}$. Elements of $\mathcal{A}$ will be called nodes of the partition $\mathcal{A}$. The corresponding averaging operator will be denoted by $E(\cdot|\mathcal{A})$.

Lemma 2.8.7. *If $\mathcal{C}_i$, $1 \leq i \leq k$, are discrete sets, then*

$$E(x| \cup_{i=1}^k \mathcal{C}_i) \prec\prec \sum_{i=1}^k E(x|\mathcal{C}_i).$$

Proof. It is sufficient to verify

$$\int_0^t E(x| \cup_{i=1}^k \mathcal{C}_i)(s)ds \leq \sum_{i=1}^k \int_0^t E(x|\mathcal{C}_i)(s)ds$$

only at the nodes of $E(x| \cup_{i=1}^k \mathcal{C}_i)$, that is at the nodes of $E(x|\mathcal{C}_i)$ for every i. However, if $t \in \mathcal{C}_i$ for some i, then

$$\int_0^t E(x| \cup_{i=1}^k \mathcal{C}_i)(s)ds = \int_0^t E(x|\mathcal{C}_i)(s)ds$$

and we are done. □

Clearly,

$$\mathcal{B}_{m,1} \cup \mathcal{B}_{m,3/2} \cup \mathcal{B}_{m,(3/2)^2} = \mathcal{A}_m.$$

Therefore, by Lemma 2.8.7, we have

$$E(x|\mathcal{A}_m) \prec\prec E(x|\mathcal{B}_{m,1}) + E(x|\mathcal{B}_{m,3/2}) + E(x|\mathcal{B}_{m,(3/2)^2}). \tag{2.22}$$

In what follows, we will only work with a fixed value of θ. It will therefore be convenient, for simplicity of notation, to drop the explicit dependence on θ and write $\mathcal{B}_\kappa, a_n$ rather that $\mathcal{B}_{\kappa,\theta}, a_n(\theta)$.

We will need the following lemma.

Lemma 2.8.8. *If $\kappa \geq \kappa'$ (that is $\kappa_n \geq \kappa'_n$ for every n), then*

$$\int_0^t E(x|\mathcal{B}_\kappa)(s)ds \leq 3/2 \int_0^t E(x|\mathcal{B}_{\kappa'})(s)ds. \tag{2.23}$$

Proof. Since $E(x|\mathcal{B}_\kappa)$ is piecewise-constant, it is sufficient to prove the assertion for the nodes of $E(f|\mathcal{B}_\kappa)$. If $\kappa_n a_{3n} \leq a_{3n+1}$, then $\kappa'_n a_{3n} \leq a_{3n+1}$ and

$$\int_0^{\kappa_n a_{3n}} E(x|\mathcal{B}_\kappa)(s)ds \leq \int_0^{a_{3n+1}} f(t)dt = 3/2(3/2)^{3n}\theta \leq$$

$$\leq 3/2 \int_0^{\kappa'_n a_{3n}} x(s)ds = 3/2 \int_0^{\kappa'_n a_{3n}} E(f|\mathcal{B}_{\kappa'})(s)ds.$$

Since $\kappa'_n \leq \kappa_n$, it follows that the assertion is proved for $s = \kappa_n a_{3n}$. □

Remark 2.8.9. *The inequality* (2.23) *holds if $\kappa_n \leq \kappa'_n$ only for such n that $\kappa_n a_{3n} \leq a_{3n+1}$ or $\kappa'_n a_{3n} \leq a_{3n+1}$.*

We use the following remarkable characterization of the set $\mathcal{Q}(x)$ given in [30].

Theorem 2.8.10. *Let x be a measurable function either on the interval $(0,1)$ or on the semi-axis. The following assertions are valid.*

1. [30, Lemma 4.4]. *If $y \in \mathcal{Q}(x)$, then there exists $p \in \mathbb{N}$ such that*

$$\int_{pa}^{b} y^*(s)ds \le \int_{a}^{b} x^*(s)ds, \quad \forall 0 \le pa \le b. \tag{2.24}$$

2. [30, Theorem 6.3]. *If y is a measurable function satisfying (2.24), then, for every $\varepsilon > 0$, there exist z such that $|y| \le z$ and $z \in (1+\varepsilon)\mathcal{Q}(x)$.*

For each sequence κ and $\lambda > 0$, we define the sequence κ^λ by

$$\kappa_n^\lambda = \begin{cases} \kappa_n, & \kappa_n \ge \lambda \\ \infty, & \kappa_n < \lambda. \end{cases}$$

Proposition 2.8.11. *If y satisfies (2.24), then*

$$\int_0^t E(x|\mathcal{B}_{\kappa^{100p}})(s)ds \le 30 \int_0^t |E(x|\mathcal{B}_\kappa) - y^*|^*(s)ds. \tag{2.25}$$

Proof. It is sufficient to prove (2.25) only at $t = \kappa_n a_{3n}$, where $100pa_{3n} \le \kappa_n a_{3n} \le a_{3n+1}$. These are the only nodes of $E(x|\mathcal{B}_{\kappa^{100p}})$.

Clearly,

$$\int_0^{\kappa_n a_{3n}} |E(x|\mathcal{B}_\kappa) - y^*|^*(s)ds \ge \int_{pa_{3n}}^{\kappa_n a_{3n}} E(x|\mathcal{B}_\kappa)(s)ds - \int_{pa_{3n}}^{\kappa_n a_{3n}} y^*(s)ds \ge$$

$$\ge (1 - \frac{p}{\kappa_n})\kappa_n a_{3n} E(x|\mathcal{B}_\kappa)(\kappa_n a_{3n} - 0) - \int_{a_{3n}}^{\kappa_n a_{3n}} x(s)ds \ge$$

$$\ge (1 - \frac{p}{\kappa_n})\frac{\kappa_n a_{3n}}{\kappa_n a_{3n} - \kappa_m a_{3m}}(X(\kappa_n a_{3n}) - X(\kappa_m a_{3m})) - \int_{a_{3n}}^{a_{3n+1}} x(s)ds,$$

where m is the largest integer number such that $m < n$ and $\kappa_m a_{3m} \le a_{3m+1}$. It then follows that

$$X(\kappa_n a_{3n}) - X(\kappa_m a_{3m}) \ge X(a_{3n}) - X(a_{3n-2}) = 5/9(3/2)^{3n}\theta.$$

By definition,

$$\int_{a_{3n}}^{a_{3n+1}} x(s)ds = \frac{1}{2}(3/2)^{3n}\theta.$$

Hence,

$$\int_0^{\kappa_n a_{3n}} |E(x|\mathcal{B}_\kappa) - y^*|^*(s)ds \ge (1 - \frac{1}{100}) \cdot \frac{5}{9}(3/2)^{3n}\theta - \frac{1}{2}(3/2)^{3n}\theta =$$

$$= \frac{1}{30}\int_0^{a_{3n+1}} x(s)ds \ge \frac{1}{30}\int_0^{\kappa_n a_{3n}} E(x|\mathcal{B}_{\kappa^{100p}})(s)ds.$$

□

Corollary 2.8.12. *Let E be a fully symmetric quasi-Banach space either on the interval $(0,1)$ or on the semi-axis and let $x = x^* \in E$. If $\Omega(x) = \mathcal{Q}_E(x)$, then*

$$E(x|\mathcal{B}_{\kappa^\lambda}) \to 0 \tag{2.26}$$

as λ approaches ∞.

Proof. It follows from the assumption that

$$P(x|\mathcal{B}_\kappa) \in \Omega(x) = \mathcal{Q}_E(x).$$

Hence, for every $\varepsilon > 0$, there exists $y \in \mathcal{Q}(x)$ such that

$$\|P(x|\mathcal{B}_\kappa) - y\|_E \le \varepsilon.$$

By Theorem 2.8.10, there exists $p \in \mathbb{N}$ such that (2.24) is valid. By Proposition 2.8.11,

$$P(x|\mathcal{B}_{\kappa^{100p}}) \prec\prec 30(P(x|\mathcal{B}_\kappa) - y).$$

It follows from Lemma 2.8.8 that for every $\lambda > 100p$

$$\|P(x|\mathcal{B}_{\kappa^\lambda})\|_E \le 45\varepsilon.$$

Since ε is arbitrarily small, this completes the proof. □

Proposition 2.8.13. *Let E be a fully symmetric quasi-Banach space either on the interval $(0,1)$ or on the semi-axis equipped with a Fatou quasi-norm. If $x \in E$ is such that $\Omega(x) = \mathcal{Q}_E(x)$, then $E(x|\mathcal{B}_m) \to 0$.*

Proof. Assume the contrary. If there exists a sequence $m_s \to \infty$ such that $E(x|\mathcal{B}_{m_s}) \to 0$, then, by Lemma 2.8.8 we have $E(x|\mathcal{B}_m) \to 0$. Therefore, there exists an ε such that $\|E(x|\mathcal{B}_m)\|_E > \varepsilon$ for all m. Set

$$\kappa_n^{m,r} = \begin{cases} m & 0 \le |n| < r \\ \infty & r \le |n|. \end{cases}$$

Clearly, $E(x|\mathcal{B}_{\kappa^{m,r}}) \to E(x|\mathcal{B}_m)$ almost everywhere. Since $E(x|\mathcal{B}_{\kappa^{m,r}}) \uparrow_r$, it follows from the definition of Fatou quasi-norm that

$$\lim_{r\to\infty} \|E(x|\mathcal{B}_{\kappa^{m,r}})\|_E = \|E(x|\mathcal{B}_m)\|_E.$$

Hence, for each m, there exists r_m such that $\|E(x|\mathcal{B}_{\kappa^{m,r_m}})\|_E > \varepsilon$.

Now define

$$\kappa_n = \inf_{m\ge 1} \kappa_n^{m,r_m} = \inf_{r_m > |n|} m, \quad n \in \mathbb{N}.$$

Clearly, $\kappa_n \uparrow \infty$ as $|n| \uparrow \infty$. By the Corollary 2.8.12, there exists λ such that

$$\|E(x|\mathcal{B}_{\kappa^\lambda})\|_E < \frac{2}{3}\varepsilon.$$

Now the set $\{n : \kappa_n < \lambda\}$ is finite. Fix an integer m large enough so that $ma_{3n} > a_{3n+1}$ whenever $\kappa_n < \lambda$. This means that $\kappa_n^\lambda \le \kappa_n^{m,r_m}$ except at points where both are bigger than a_{3n+1}/a_{3n}. According to the Remark 2.8.9, it follows that

$$E(x|\mathcal{B}_{\kappa^{m,r_m}}) \prec\prec \frac{3}{2}E(x|\mathcal{B}_{\kappa^\lambda}).$$

This implies $\|E(x|\mathcal{B}_{\kappa^{m,r_m}})\|_E < \varepsilon$, giving a contradiction. □

Lemma 2.8.14. *Let $x \in L_1 + L_\infty$ be a function on the semi-axis. If $x \notin L_1$, then, for every $t > 0$,*

$$X(t) \le \frac{2}{3}X(m^2t) + \frac{3}{2}\int_0^{m^2t} E(x|\mathcal{A}_m)(s)ds. \tag{2.27}$$

Proof. Let $t \in [a_n, a_{n+1}]$. If $a_{n+1} > ma_n$, then

$$\int_0^{m^2t} E(x|\mathcal{A}_m)(s)ds \ge \int_0^{ma_n} E(x|\mathcal{A}_m)(t)dt = X(ma_n) \ge \frac{2}{3}X(t).$$

If $a_{n+1} \le ma_n$ and $a_{n+2} > ma_{n+1}$, then

$$\int_0^{m^2t} E(x|\mathcal{A}_m)(s)ds \ge \int_0^{ma_{n+1}} E(x|\mathcal{A}_m)(s)ds = X(ma_{n+1}) \ge X(t).$$

If $a_{n+2} \le ma_{n+1}$ and $a_{n+1} \le ma_n$, then

$$X(m^2t) \ge X(a_{n+2}) = \frac{3}{2}X(a_{n+1}) \ge \frac{3}{2}X(t)$$

and the inequality follows. □

Corollary 2.8.15. *Let $x \in L_1 + L_\infty$ be a function on the semi-axis. If $x \notin L_1$, then for every $t > 0$*

$$X(m^{-2l}t) \le (\frac{2}{3})^l X(t) + \frac{9}{2}\int_0^t E(x|\mathcal{A}_m)(s)ds, \quad m, l \in \mathbb{N}. \tag{2.28}$$

Proof. Denote, for brevity,

$$Z(t) = \int_0^t E(x|\mathcal{A}_m)(s)ds, \quad t > 0.$$

We will use induction on l to prove that

$$X(t) \le (\frac{2}{3})^l X(m^{2l}t) + \frac{9}{2}(1 - (\frac{2}{3})^l)Z(m^{2l}t), \quad l \in \mathbb{N}. \tag{2.29}$$

Indeed, (2.29) is valid for $l = 1$. Assume that (2.29) is valid for $l = k$. Let us prove (2.29) for $l = k + 1$. We obtain

$$X(t) \le (\frac{2}{3})^l X(m^{2l}t) + \frac{9}{2}(1 - (\frac{2}{3})^l)Z(m^{2l}t) \le$$

$$\leq (\frac{2}{3})^l(\frac{2}{3}X(m^{2(l+1)}t) + \frac{3}{2}Z(m^{2(l+1)}t)) + \frac{9}{2}(1-(\frac{2}{3})^l)Z(m^{2l}t) \leq$$
$$\leq (\frac{2}{3})^{l+1}X(m^{2(l+1)}t) + ((\frac{2}{3})^{l-1} + \frac{9}{2}(1-(\frac{2}{3})^l))Z(m^{2(l+1)}t)) =$$
$$= (\frac{2}{3})^{l+1}X(m^{2(l+1)}t) + \frac{9}{2}(1-(\frac{2}{3})^{l+1})Z(m^{2(l+1)}t)).$$

The assertion follows immediately from (2.29). □

The situation in the case that $x \in L_1$ is slightly more complicated.

Lemma 2.8.16. *If $x \in L_1(0,1)$ or $x \in L_1(0,\infty)$, then there exists constant C such that for every $t > 0$*

$$X(t) \leq \frac{2}{3}X(m^2t) + \frac{3}{2}\int_0^{m^2t} E(x|\mathcal{A}_m)(s)ds + \frac{3}{2}C\int_0^{m^2t}\chi_{[0,1]}(s)ds. \quad (2.30)$$

Proof. Consider first the case of the semi-axis. Fix n_0 such that $X(a_{n_0}) \leq 4/9X(\infty)$. Then the argument in Lemma 2.8.14 applies *mutatis mutandi* for $0 \leq t \leq a_{n_0}$. For every $t \geq a_{n_0}$, we have

$$X(t) \leq \frac{X(\infty)}{\min\{a_{n_0},1\}}\min\{m^2t,1\}$$

and the inequality follows in this case.

The same argument applies in the case of the interval $(0,1)$ by replacing $X(\infty)$ by $X(1)$. □

Corollary 2.8.17. *If x and C are as in Lemma 2.8.16, then for every $t > 0$,*

$$X(m^{-2l}t) \leq (\frac{2}{3})^lX(t) + \frac{9}{2}\int_0^t E(x|\mathcal{A}_m)(s)ds + \frac{9}{2}C\min\{m^2t,1\}, \quad m,l \in \mathbb{N}. \quad (2.31)$$

Proof. If

$$Z(t) = \int_0^t E(x|\mathcal{A}_m)(s)ds + C\min\{m^2t,1\},$$

then the proof of Corollary 2.8.15 applies *mutatis mutandi.* □

Theorem 2.8.18. *Let E be a fully symmetric quasi-Banach space either on the interval $(0,1)$ or on the semi-axis and let $x \in E$. Suppose that the quasi-norm on E is a Fatou quasi-norm. If $\Omega(x) = \mathcal{Q}_E(x)$, then $\varphi(x) = 0$ provided that one of the following conditions is satisfied*

1. *$E = E(0,1)$ is a space on the interval $(0,1)$.*
2. *$E = E(0,\infty)$ is a space on the semi-axis and $E(0,\infty) \not\subset L_1(0,\infty)$.*

Proof. It follows from Proposition 2.8.13 that $E(x|\mathcal{B}_{m,\theta}) \to 0$ for every θ. It follows from the inequality (2.22) that $E(x|\mathcal{A}_m) \to 0$.

Fix $l \in \mathbb{N}$. If $x \notin L_1$, then by Corollary 2.8.15,

$$\frac{1}{m^{2l}}\sigma_{m^{2l}}x \prec\prec (\frac{2}{3})^l x + \frac{9}{2}E(x|\mathcal{A}_m), \quad m \in \mathbb{N}$$

and, therefore,

$$\frac{1}{sm^{2l}}\sigma_{sm^{2l}}x \prec\prec (\frac{2}{3})^l \frac{1}{s}\sigma_s x + \frac{9}{2}E(x|\mathcal{A}_m), \quad m, s \in \mathbb{N}.$$

Let $m, s \to \infty$. It follows that

$$\varphi(x) \le C_0(\frac{2}{3})^l \varphi(x) + 0.$$

Let $l \to \infty$. It follows that $\varphi(x) = 0$.

Fix $l \in \mathbb{N}$. If $x \in L_1$ and C are as in Lemma 2.8.16, then it follows from Corollary 2.8.17 that

$$\frac{1}{m^{2l}}\sigma_{m^{2l}}x \prec\prec (\frac{2}{3})^l x + \frac{9}{2}E(x|\mathcal{A}_m) + \frac{9}{2}C\chi_{(0,1)}, \quad m \in \mathbb{N}$$

and, therefore,

$$\frac{1}{sm^{2l}}\sigma_{sm^{2l}}x \prec\prec (\frac{2}{3})^l \frac{1}{s}\sigma_s x + \frac{9}{2}E(x|\mathcal{A}_m) + \frac{9}{2}Cs^{-1}\sigma_s\chi_{(0,1)}, \quad m, s \in \mathbb{N}.$$

Let $m, s \to \infty$. It follows that

$$\varphi(x) \le C_0(\frac{2}{3})^l \varphi(x) + 0 + 0.$$

Let $l \to \infty$. It now follows that $\varphi(x) = 0$. □

Theorem 2.8.19. *Let $E = E(0,\infty)$ be a fully symmetric quasi-Banach space on the semi-axis equipped with a Fatou quasi-norm such that $E(0,\infty) \subset L_1(0,\infty)$. If $x \in E$ is such that $\Omega(x) = \mathcal{Q}_E(x)$, then $\tau^{-1}\sigma_\tau(x^*)\chi_{[0,1]} \to 0$.*

Proof. Set $F = E + L_\infty$. Clearly,

$$\Omega(x) = \mathcal{Q}_E(x) \subset \mathcal{Q}_F(x) \subset \Omega(x).$$

Hence, $\mathcal{Q}_F(x) = \Omega(x)$. Applying Theorem 2.8.18, we obtain $\varphi_F(x) = 0$, which proves the assertion. □

2.9 The sequence space case

In this section, we will follow Kalton, Sukochev and Zanin [31].

Suppose now that E is a symmetric quasi-Banach sequence space. We associate it with a symmetric quasi-Banach function space on the semi-axis as follows.

Consider the partition of the semi-axis $\mathcal{A} = \{[k-1,k]\}_{k\in\mathbb{N}}$. It is clear that the operator $P(\cdot|\mathcal{A})$ maps $(L_1+L_\infty)(0,\infty)$ into the set of step functions. Elements of the latter can be readily identified with bounded sequences.

More precisely, for any bounded sequence ξ we set

$$x_\xi = \sum_{k=1}^{\infty} \xi_k \chi_{(k-1,k]}$$

and identify ξ and x_ξ.

Definition 2.9.1. *Let E be a symmetric quasi-Banach sequence space. We define the function space F_E on the semi-axis as the set of all $x \in L_\infty(0,\infty)$ such that $P(x^*|\mathcal{A}) \in E$, and set*

$$\|x\|_{F_E} = \|x\|_\infty + \|P(x^*|\mathcal{A})\|_E.$$

It is easy to see that F_E is a linear space and that $\|\cdot\|_{F_E}$ is a quasi-norm on F_E. Further, equipped with the quasi-norm $\|\cdot\|_{F_E}$, the space F_E is a quasi-Banach symmetric space on the semi-axis. It is not difficult to see that the space E is fully symmetric if and only if F_E is fully symmetric.

If E is a fully symmetric sequence space and if $\xi \in E$, then the sets $\Omega(\xi)$, $\mathcal{Q}(\xi)$, $\mathcal{Q}_E(\xi)$ are defined in the same way as in the function space setting.

The set $\mathcal{Q}(\xi)$ admits a characterization fully analogous to that in the the function space setting given in Theorem 2.8.10

Theorem 2.9.2. *Let ξ be a bounded sequence. The following assertions are valid.*

1. [30, Lemma 4.4]. *If $\eta \in \mathcal{Q}(\xi)$, then there exists $p \in \mathbb{N}$ such that*

$$\sum_{k=pm+1}^{n} \eta_k^* \le \sum_{k=m+1}^{n} \xi_k^*, \quad \forall n,m \in \mathbb{N} : pm+1 \le n. \tag{2.32}$$

2. [30, Theorem 5.4]. *If η is a sequence satisfying (2.32), then, for every $\varepsilon > 0$, there exist ζ such that $|\eta| \le \zeta$ and $\zeta \in (1+\varepsilon)\mathcal{Q}(\xi)$.*

Lemma 2.9.3. *Let $\xi = \xi^* \notin l_1$ be a sequence and let $y = y^* \in L_\infty(0,\infty)$. If $y = y^* \prec\prec x_\xi$ and $0 < q < 1$, then there exist sequences $n_k \to \infty$ and $\varepsilon_k \to 0$ such that*

$$x_\xi \chi_{[0,n_k]} + q y \chi_{(n_k,\infty)} \prec\prec (1+\varepsilon_k) x_\xi.$$

Proof. By Lemma 2.5.3, there exists a sequence $t_k \to \infty$ such that

$$qy\chi_{(t_k,\infty)} \prec\prec x_\xi \chi_{(t_k,\infty)}.$$

Set $n_k = [t_k]$. Clearly,

$$\int_0^t (x_\xi \chi_{(0,n_k)} + qy\chi_{n_k,\infty})^*(s)ds = \sup_{u+v=t, u\le n_k} \int_0^u x_\xi(s)ds + \int_{n_k}^{n_k+v} qy(s)ds.$$

Evidently,

$$\int_{n_k}^{n_k+v} qy(s)ds \le y(n_k)\min\{v,1\} + \int_{t_k}^{t_k+v} x_\xi(s)ds.$$

Therefore,

$$\int_0^t (x_\xi \chi_{(0,n_k)} + qy\chi_{n_k,\infty})^*(s)ds \le y(n_k)\min\{t,1\} + \int_0^t x_\xi(s)ds$$

and the assertion follows. □

Lemma 2.9.4. *If $E \neq l_\infty, l_1$ is a fully symmetric quasi-Banach sequence space and if $\xi \in E \backslash l_1$ is such that $\Omega(\xi) = \mathcal{Q}_E(\xi)$, then $\Omega(x_\xi) = \mathcal{Q}_{F_E}(x_\xi)$.*

Proof. Fix $y \in \Omega(x_\xi)$. Since $E \neq l_\infty$, it follows that $x_\xi^*(\infty) = 0$. Thus, $y^*(\infty) = 0$. It follows that $|y| = y^* \circ \gamma$, where γ is a measure-preserving transformation. Hence, we assume that $y = y^*$.

Fix $\varepsilon > 0$ and $0 < q < 1$. By Lemma 2.9.3, there exists n such that $y(n) < \varepsilon$ and

$$x_\xi \chi_{[0,n]} + qy\chi_{(n,\infty)} \prec\prec (1+\varepsilon)x_\xi. \tag{2.33}$$

Since $qy\chi_{[0,n]} \prec\prec x_\xi\chi_{[0,n]}$, there exist functions x_i, $1 \le i \le m$, on the interval $[0,n]$ such that $x_i^* = x_\xi\chi_{[0,n]}$ and

$$\|\frac{1}{m}\sum_{i=1}^m x_i - qy\|_{L_\infty(0,n)} \le \|\chi_{[0,n]}\|_{F_E}^{-1}\varepsilon.$$

Define x_i to be qy on (n,∞) for $1 \le i \le m$. It follows that

$$\|\frac{1}{m}\sum_{i=1}^m x_i - qy\|_{F_E} \le \|\frac{1}{m}\sum_{i=1}^m x_i - qy\|_{L_\infty(0,n)} \cdot \|\chi_{[0,n]}\|_{F_E} \le \varepsilon.$$

Set $\mathcal{A}_n = \{[k,k+1]\}_{k\ge n}$ and $z_i = P(x_i|\mathcal{A}_n)$. It is clear that

$$\frac{1}{m}\sum_{i=1}^m x_i - \frac{1}{m}\sum_{i=1}^n z_i = q(y - P(y|\mathcal{A}_n))$$

and

$$\|y - P(y|\mathcal{A}_n)\|_{L_1\cap L_\infty} \le \sum_{k=n}^\infty y(k) - y(k+1) = y(n).$$

Hence,

$$\|\frac{1}{m}\sum_{i=1}^{m}x_i - \frac{1}{m}\sum_{i=1}^{m}z_i\|_{F_E} \le \|y - P(y|\mathcal{A}_n)\|_{L_1\cap L_\infty} \le y(n) \le \varepsilon.$$

Clearly, z_i is equimeasurable with $x_\xi\chi_{[0,n]} + qP(y|\mathcal{A}_n)\chi_{(n,\infty)} = x_\eta$ for some $\eta \in E$. It follows from (2.33) that $\eta \prec\prec (1+\varepsilon)\xi$. By assumption, $\eta \in (1+\varepsilon)\mathcal{Q}_E(\xi)$. Therefore, $z_i \in (1+\varepsilon)\mathcal{Q}_{F_E}(x_\xi)$, $1 \le i \le m$.

Since $\mathcal{Q}_{F_E}(x_\xi)$ is a convex set, it follows that

$$\frac{1}{m}\sum_{i=1}^{m}z_i \in \mathcal{Q}_{F_E}(x_\xi) \subset \mathcal{Q}_{F_E}(x_\xi) + \varepsilon B_{F_E}(0,1).$$

Hence,

$$qy \in \frac{1}{m}\sum_{i=1}^{m}x_i + \varepsilon B_{F_E}(0,1) \subset \frac{1}{m}\sum_{i=1}^{m}z_i + \varepsilon B_{F_E}(0,1) + \varepsilon B_{F_E}(0,1) \subset$$

$$\subset \mathcal{Q}_{F_E}(x_\xi) + \varepsilon B_{F_E}(0,1) + \varepsilon B_{F_E}(0,1) + \varepsilon B_{F_E}(0,1).$$

Since ε is arbitrarily small and $\mathcal{Q}_{F_E}(x_\xi)$ is a closed set, the assertion of the lemma follows. □

Theorem 2.9.5. *Let E be a fully symmetric quasi-Banach sequence space and let $\xi \in E$. If $\Omega(\xi) = \mathcal{Q}_E(\xi)$, then $\Omega(x_\xi) = \mathcal{Q}_{F_E}(x_\xi)$.*

Proof. If $\xi \in l_1$, then $\varphi_{fin}(x_\xi) = 0$. It follows from Theorem 2.7.11 that

$$\Omega(x_\xi) = \mathcal{Q}_{L_1\cap L_\infty}(x_\xi) \subset \mathcal{Q}_{F_E}(x_\xi)$$

and we are done. Assume now that $\xi \notin l_1$. If $E \neq l_1, l_\infty$, then the assertion is proved in Lemma 2.9.4. If $E = l_\infty$, then $\varphi(x_\xi) = 0$ and our assertion follows from Theorem 2.7.2. □

Lemma 2.9.6. *Let $E \neq l_\infty$ be a fully symmetric quasi-Banach sequence space. If $\Omega(x_\xi) = \mathcal{Q}_{F_E}(x_\xi)$, then $\Omega(\xi) = \mathcal{Q}_E(\xi)$.*

Proof. Let $\xi \in E$ and let $\Omega(x_\xi) = \mathcal{Q}_{F_E}(x_\xi)$. Assume for simplicity that $\|\xi\|_E = 1$. Let $\eta \in \Omega(\xi)$. Since $E \neq l_\infty$, it follows that $\xi^*(\infty) = 0$ and $\eta^*(\infty) = 0$. Thus, we may assume without loss of generality, that $\xi = \xi^*$ and $\eta = \eta^*$.

Evidently, $x_\eta \in \Omega(x_\xi)$. Fix $\varepsilon > 0$. There exists $z \in \mathcal{Q}(x_\xi)$ such that $\|z - x_\eta\|_{F_E} \le \varepsilon$. Since x_η decreases, it follows that

$$\|z^* - x_\eta\|_{F_E} \le \|z - x_\eta\|_{F_E} \le \varepsilon.$$

Define the sequence ζ by the formula $x_\zeta = P(z^*|\mathcal{A})$. Since $z \in \mathcal{Q}(x_\xi)$, it follows from Theorem 2.8.10 that there exists $p \in \mathbb{N}$ such that

$$\int_{pa}^{b} z^*(s)ds \le \int_{a}^{b} x_\xi^*(s)ds, \quad \forall pa < b.$$

If $m, n \in \mathbb{N}$, then

$$\sum_{k=pm+1}^{n} \zeta_k^* = \int_{pm}^{n} z^*(s)ds \leq \int_{m}^{n} x_\xi^*(s)ds = \sum_{k=m+1}^{n} \xi_k^*, \quad \forall pm+1 \leq n.$$

Therefore, by Theorem 2.9.2, there exists ζ' such that $|\zeta| \leq \zeta'$ and

$$\zeta' \in (1+\varepsilon)\mathcal{Q}(\xi).$$

By Lemma 2.3.3,

$$\zeta \in (1+\varepsilon)\mathcal{Q}_E(\xi) \subset \mathcal{Q}_E(\xi) + \varepsilon B_E(0,1) \subset \mathcal{Q}(\xi) + \varepsilon B_E(0,1) + \varepsilon B_E(0,1).$$

On the other hand,

$$\|\zeta - \eta\|_E \leq \|x_\zeta - x_\eta\|_{F_E} = \|P(z^* - x_\eta|\mathcal{A})\|_{F_E} \leq \|z^* - x_\eta\|_{F_E} \leq \varepsilon,$$

and, therefore,

$$\eta \in \zeta + \varepsilon B_E(0,1) \subset \mathcal{Q}(\xi) + \varepsilon B_E(0,1) + \varepsilon B_E(0,1) + \varepsilon B_E(0,1).$$

Since ε is arbitrarily small and $\mathcal{Q}_E(\xi)$ is closed, this suffices to prove the lemma. □

Theorem 2.9.7. *Let E be a fully symmetric quasi-Banach sequence space. Suppose that $E \neq l_1$. If $\xi \in E$ and if $\varphi(\xi) = 0$, then $\Omega(\xi) = \mathcal{Q}_E(\xi)$.*

Proof. If $E = l_\infty$, the assertion is well-known. Let $E \neq l_\infty$. Let $\xi \in E$ be such that $\varphi_E(\xi) = 0$. Clearly, this implies $\varphi_{F_E}(x_\xi) = 0$. It follows from Theorem 2.7.3 that $\Omega(x_\xi) = \mathcal{Q}_{F_E}(x_\xi)$. By Lemma 2.9.6, $\Omega(\xi) = \mathcal{Q}_E(\xi)$. □

The following result is a corollary of the preceding results.

Theorem 2.9.8. *Let E be a fully symmetric quasi-Banach sequence space equipped with a Fatou quasi-norm and let $\xi \in E$. The following conditions are equivalent.*

1. $\Omega(\xi) = \mathcal{Q}_E(\xi)$.
2. $\Omega^+(\xi) = \mathcal{Q}'_E(\xi)$.
3. $\varphi(\xi) = 0$.

2.10 Applications & examples

2.10.1 Orlicz spaces are always "good"

The following proposition shows that Orlicz spaces always satisfy the condition (1.1).

Lemma 2.10.1. *Let Φ be an Orlicz function and L_Φ be the corresponding Orlicz space on the semi-axis. Suppose that $\Phi'(0) = 0$. If $x \in L_\Phi$, then*

$$n \int_0^\infty \Phi(\frac{x(s)}{n})ds \to 0.$$

The same assertion is valid for Orlicz spaces on the interval $(0, 1)$.

Proof. Without loss of generality, it may be assumed that $\|x\|_{L_\Phi} = 1$, that is $\int_0^\infty \Phi(x(s))ds = 1$. It follows from the condition $\Phi'(0) = 0$ that $n\Phi(x/n) \to 0$ almost everywhere. It is clear that $n\Phi(x/n) \leq \Phi(x)$. The assertion of the lemma follows now from the Lebesgue dominated convergence principle. □

Proposition 2.10.2. *Let Φ be an Orlicz function and L_Φ be the corresponding Orlicz space on the semi-axis. If $x \in L_\Phi$, then $\varphi(x) = 0$ for every $x \in L_\Phi$. The same assertion is valid for Orlicz spaces on the interval $(0, 1)$.*

Proof. Assume the contrary. Suppose first that $\Phi'(0) = 0$. It follows that $L_\Phi \not\subset L_1(0, \infty)$. Let

$$\|\sigma_n x\|_{L_\Phi} \geq n\alpha$$

for some $0 \leq x \in L_\Phi$, some $\alpha > 0$ and every $n \geq 1$. By the definition of the norm $\|\cdot\|_{L_\Phi}$, we have

$$\int_0^\infty \Phi(\frac{1}{n\alpha}\sigma_n x(s))ds \geq 1.$$

Hence,

$$n \int_0^\infty \Phi(\frac{1}{n}y(s))ds \geq 1$$

with $y = \alpha^{-1}x \in L_\Phi$. A contradiction.

Suppose first that $\Phi'(0) = 0$. It follows that $L_\Phi \not\subset L_1(0, \infty)$. Let

$$\|(\sigma_n x^*)\chi_{(0,1)}\|_{L_\Phi} \geq n\alpha$$

for some $0 \leq x \in L_\Phi$, some $\alpha > 0$ and every $n \geq 1$. By the definition of the norm $\|\cdot\|_{L_\Phi}$, we have

$$\int_0^1 \Phi(\frac{1}{n\alpha}(\sigma_n x^*)(s))ds \geq 1.$$

Hence,

$$n \int_0^{1/n} \Phi(\frac{1}{n}y(s))ds \geq 1$$

with $y = \alpha^{-1}x^* \in L_\Phi$. Define Orlicz function Φ_0 by setting $\Phi_0(z) = \Phi(z) - \Phi'(+0)|z|$ for every $z \in \mathbb{R}$. It follows that

$$n\int_0^\infty \Phi_0(\frac{1}{n}y(s))ds \to 0.$$

Therefore,

$$n\int_0^{1/n} \Phi(\frac{1}{n}y(s))ds \le \Phi'(+0)\int_0^{1/n} y(s)ds + n\int_0^\infty \Phi_0(\frac{1}{n}y(s))ds \to 0.$$

A contradiction. □

Corollary 2.10.3. *If L_Φ is an Orlicz space and if $x \in L_\Phi$, then $\Omega(x) = \mathcal{Q}_{L_\Phi}(x)$.*

It follows from Theorem 15.3 of [38] that every separable symmetric Banach space satisfies the condition $\Omega(x) = \mathcal{Q}_E(x)$. In [11], Braverman & Mekler showed that $\Omega(x) = \mathcal{Q}_E(x)$ provided that $\beta_E < 1$. For the subclass of Orlicz spaces, Corollary 2.10.3 substantially improves each of these results. Indeed, it is quite easy to construct (see Appendix B) a non-separable Orlicz space L_Φ such that $\beta_{L_\Phi} = 1$.

2.10.2 Symmetric functionals

Let E be a fully symmetric quasi-Banach space.

Definition 2.10.4. *A positive functional $\omega \in E^*$ is said to be* **symmetric** *if $\omega(y) = \omega(x)$ for all $0 \le x, y \in E$ such that $y^* = x^*$.*

Definition 2.10.5. *A positive functional $\omega \in E^*$ is said to be* **fully symmetric** *if $\omega(y) \le \omega(x)$ for all $0 \le x, y \in E$ such that $y \prec\prec x$.*

We refer to [23, 16] and references therein for the exposition of the theory of singular fully symmetric functionals and their applications. Recently, symmetric functionals which fail to be fully symmetric were constructed in [30] on some Marcinkiewicz spaces. However, for Orlicz spaces situation is different. The following proposition shows that a symmetric functional on an Orlicz space is necessary fully symmetric.

Proposition 2.10.6. *Any symmetric functional on the Orlicz space L_Φ is fully symmetric.*

Proof. Let $\omega \in E^*$ be symmetric. It is clear, that $\omega(x^*\chi_{[0,\beta]}) \le \omega(x)$ for $x \ge 0$. Therefore, $\omega(y) \le \omega(x)$ for $y \in \mathcal{Q}^+(x)$. Since ω is continuous, we have $\omega(y) \le \omega(x)$ for $y \in \mathcal{Q}^+_{L_\Phi}(x)$. By Theorem 2.7.8 and Proposition 2.10.2, we have $\mathcal{Q}^+_{L_\Phi}(x) = \Omega^+(x)$, and so ω is a fully symmetric functional on L_Φ. □

Corollary 2.10.7. *There are no non-zero symmetric functionals on the Orlicz space L_Φ.*

Proof. Indeed, there are no non-zero fully symmetric singular functionals on L_Φ (see [23, Theorem 3.1]). □

It was shown (in an unpublished paper of Kalton & Sukochev) that a similar assertion is valid for every fully symmetric space E. That is, space E admits non-zero symmetric functionals if and only if E admits non-zero fully symmetric functionals.

2.10.3 Marcinkiewicz spaces with trivial functional φ

Lemma 2.10.8. *Let M_ψ be a Marcinkiewicz space either on the interval $(0,1)$ or on the semi-axis. Then $\varphi(x) = 0$ for every $x \in M_\psi$ if and only if $\beta_{M_\psi} < 1$.*

Proof. Note that $\varphi(x) \leq \|x\|_{M_\psi} \varphi(\psi')$ for $x \in M_\psi$. Clearly, for the Marcinkiewicz space on the interval $(0,1)$, the condition $\varphi(\psi') = 0$ holds if and only if

$$\liminf_{t\to 0} \frac{\psi(2t)}{\psi(t)} > 1.$$

If M_ψ is a space on the semi-axis, then one should replace $\to 0$ with $\to 0, \infty$. □

Chapter 3

Khinchine-type inequalities

The result of sections 3.5, 3.6 and 3.7 were published in [56]. The results of sections 3.1, 3.2, 3.3 and 3.4 were submitted for publication (see [57]).

3.1 The Johnson-Schechtman inequality for positive functions

In this section, we extend the results of Astashkin & Sukochev (see [6]) concerning the Johnson-Schechtman inequality for positive functions to the quasi-normed setting. Our proofs are significantly shorter and easier to understand than those of Astashkin & Sukochev, which do not extend to the case of quasi-Banach symmetric spaces.

We begin by recalling the definition of the Kruglov operator for the convenience of the reader. The measure space $\Omega = \prod_{n=0}^{\infty}(0,1)$ equipped with the product measure is isomorphic to the interval $(0,1)$ equipped with a Lebesgue measure. Let x be a random variable (measurable function) on the interval $(0,1)$. Let $\{B_n\}_{n=0}^{\infty}$ be a fixed sequence of mutually disjoint measurable subsets of $(0,1)$ such that $m(B_n) = \dfrac{1}{en!}$. The Kruglov operator $K : S_0(0,1) \to S_0(\Omega)$ is defined by setting

$$Kx(\omega) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} x(\omega_k)\chi_{B_n}(\omega_0), \quad x \in S_0(0,1).$$

Here, $\omega = (\omega_0, \omega_1, \cdots)$ is an element of Ω.

Let x_n, $1 \le k \le n$, be a (finite) sequence of random variables. In what follows, we will denote by $\bar{x}_k$, $1 \le k \le n$, the sequence of their disjoint copies. If $\sum_{k=1}^{n} m(\text{supp}(x_n)) \le 1$, then it will be assumed that $\text{supp}(x_k) \subset (0,1)$, $1 \le k \le n$.

We will use the following approximation to Kx, where x is an arbitrary measurable function on the interval $(0,1)$.

Define the operator $H_n : S_0(0,1) \to S_0(\Omega)$ by the formula

$$(H_n x)(\omega) = \sum_{k=1}^{n} \sigma_{\frac{1}{n}} x(\omega_k), \quad x \in S_0(0,1). \tag{3.1}$$

Here, $\omega = (\omega_0, \omega_1, \cdots)$ is an element of Ω.

Lemma 3.1.1. *The sequence of functions $\{H_n x\}_{n=1}^{\infty}$ converges to Kx in distribution.*

Proof. It is clear that

$$\varphi_{H_n x} = \varphi^n_{\sigma_{1/n} x}.$$

However,

$$\varphi_{\sigma_{1/n} x}(t) = \int e^{it\sigma_{1/n} x} dm = (1 - \frac{1}{n}) + \frac{1}{n}\varphi_x(t).$$

Therefore,

$$\varphi_{H_n x} = (1 + \frac{\varphi_x - 1}{n})^n \to \exp(\varphi_x - 1) = \varphi_{Kx}.$$

Convergence in distributions now follows from Lemma 1.2.80. □

Theorem 3.1.2. *Let E and F be symmetric quasi-Banach spaces. Assume that, for any sequence of independent functions x_k, $1 \le k \le n$, such that $\sum_{k=1}^n m(\mathrm{supp}(x_k)) \le 1$ we have*

$$\|\sum_{k=1}^n x_k\|_F \le C \cdot \|\sum_{k=1}^n \bar{x}_k\|_E. \tag{3.2}$$

If F has the Fatou property, then K maps E into F and $\|K\|_{E\to F} \le C$.

Proof. Let $x \in E$. Define $x_k \in E(\Omega)$ by setting $x_k(\omega) = (\sigma_{1/n}x)(\omega_k)$ for every $\omega \in \Omega$. It is easily seen that we may take

$$\bar{x}_k(t) = \sigma_{1/n}x(t - \frac{k-1}{n}\mathrm{mod}1), \quad 1 \le k \le n.$$

It is clear that

$$H_n x \sim \sum_{k=1}^n x_k, \quad x \sim \sum_{k=1}^n \bar{x}_k.$$

It follows from the inequality (3.2) that $\|H_n x\|_F \le C\|x\|_E$. It follows from Lemma 3.1.1 that the sequence $H_n x$, $n \in \mathbb{N}$, converges to Kx in distribution and hence $(H_n x)^* \to Kx$ almost everywhere. Since F has the Fatou property, it follows that $Kx \in F$ and $\|Kx\|_F \le C\|x\|_E$. □

The crucial property of the Kruglov operator, which is stated in the proposition below, strengthens the property used by Astashkin and Sukochev (see [6]). Essentially, the proposition says that the operator K maps disjoint functions into independent ones.

Proposition 3.1.3. *If the functions x_k, $1 \le k \le n$, are disjoint, then the functions Kx_k, $1 \le k \le n$, are independent.*

Proof. Let $\lambda_k \in \mathbb{R}$, $1 \le k \le n$. Since the functions x_k are disjoint, it follows that

$$\exp(it\sum_{k=1}^n \lambda_k x_k) - 1 = \sum_{k=1}^n (\exp(it\lambda_k x_k) - 1).$$

This implies immediately the following relations for the characteristic functions.

$$\varphi_{\sum_{k=1}^n \lambda_k x_k} - 1 = \sum_{k=1}^n (\varphi_{\lambda_k x_k} - 1).$$

Since $\varphi_{Kx} = \exp(\varphi_x - 1)$, it follows that

$$\varphi_{\sum_{k=1}^n \lambda_k Kx_k} = \exp(\sum_{k=1}^n (\varphi_{\lambda_k x_k} - 1)) = \prod_{k=1}^n \exp(\varphi_{\lambda_k x_k} - 1) = \prod_{k=1}^n \varphi_{\lambda_k Kx_k}.$$

The assertion now follows from Lemma 1.2.79 □

Lemma 3.1.4. *For every positive $x \in S_0$, we have $\sigma_{1/2}x^* \leq (Kx)^*$.*

Proof. Let $B \subset \cup_{n\geq 1} B_n$ be such that $m(B) = 1/2$. It is clear that $(Kx)(\omega) \geq x(\omega_1)\chi_B(\omega_0)$ for every $\omega \in \Omega$. However, the mapping $\omega \to x(\omega_1)\chi_B(\omega_0)$ is equimeasurable with $\sigma_{1/2}x^*$. The assertion follows immediately. □

Proposition 3.1.5. *Let E, F be symmetric quasi-Banach spaces. If x_k, $1 \leq k \leq n$, are independent and if $\sum_{k=1}^n m(\mathrm{supp}(x_k)) \leq 1$, then*

$$\|\sum_{k=1}^n x_k\|_F \leq 2C(F)\|K\|_{E\to F}\|\sum_{k=1}^n \bar{x}_k\|_E.$$

Proof. Clearly, $|x_k|$, $1 \leq k \leq n$, are independent.

$$|\sum_{k=1}^n x_k| \leq \sum_{k=1}^n |x_k|, \quad |\sum_{k=1}^n \bar{x}_k| = \sum_{k=1}^n |\bar{x}_k|.$$

If one proves the assertion for $|x_k|$, then the assertion for x_k follows immediately. Without loss of generality, it may be assumed that $0 \leq x_k$, $1 \leq k \leq n$.

It follows from Proposition 3.1.3 that $K(\sum_{k=1}^n \bar{x}_k)$ is equimeasurable with the mapping

$$\omega \to \sum_{k=1}^n (Kx_k)^*(\omega_k).$$

Therefore,

$$\|\sum_{k=1}^n (Kx_k)^*(\omega_k)\|_{F(\Omega)} = \|K(\sum_{k=1}^n \bar{x}_k)\|_{F(\Omega)} \leq \|K\|_{E\to F}\|\sum_{k=1}^n \bar{x}_k\|_E.$$

It follows from Lemma 3.1.4 that

$$\sum_{k=1}^n (\sigma_{1/2}x_k^*)(\omega_k) \leq \sum_{k=1}^n (Kx_k)^*(\omega_k)$$

and, therefore,

$$\|\sum_{k=1}^n (\sigma_{1/2}x_k^*)(\omega_k)\|_{F(\Omega)} \leq \|K\|_{E\to F}\|\sum_{k=1}^n \bar{x}_k\|_E. \tag{3.3}$$

The function $\sum_{k=1}^n x_k$ is equimeasurable with the function $y_1 + y_2$ defined by

$$y_i(\omega) = \sum_{k=1}^n (\sigma_{1/2}x_k^*)((\omega_k - \frac{i}{2})\mathrm{mod}1), \quad \omega \in \Omega.$$

Since

$$\|y_1 + y_2\|_{F(\Omega)} \leq C(F)(\|y_1\|_{F(\Omega)} + \|y_2\|_{F(\Omega)}),$$

then

$$\|\sum_{k=1}^{n} x_k\|_F \leq 2C(F)\|\sum_{k=1}^{n}(\sigma_{1/2}x_k^*)(\omega_k)\|_{F(\Omega)}. \tag{3.4}$$

The assertion follows now from (3.3) and (3.4). □

Remark 3.1.6. *By the definition of operator K, the function $K1$ has a Poisson distribution with parameter 1. Let ψ be a piecewise-constant concave function such that $\psi' = (K1)^*$. It is clear that $K : L_\infty \to M_\psi$ and $\|K\|_{L_\infty \to M_\psi} = 1$.*

Lemma 3.1.7. *Let*

$$s_k = \sum_{n=k}^{\infty} \frac{1}{e \cdot n!}.$$

It follows that for every $k \in \mathbb{N}$,

$$4ks_{k+1} \geq s_k.$$

Proof. Clearly,

$$4ks_{k+1} \geq \frac{(k+1)^2}{k}s_{k+1} \geq \frac{(k+1)^2}{k} \cdot \frac{1}{e \cdot (k+1)!} = \frac{k+1}{k} \cdot \frac{1}{e \cdot k!}.$$

On the other hand,

$$\frac{k+1}{k} \cdot \frac{1}{e \cdot k!} = \frac{1}{e \cdot k!} \cdot \frac{1}{1 - \frac{1}{k+1}} = \frac{1}{e \cdot k!}(1 + \frac{1}{k+1} + \frac{1}{(k+1)^2} + \cdots).$$

It is clear that $k! \cdot (k+1)^n \leq (k+n)!$. Therefore,

$$\frac{1}{e \cdot k!}(1 + \frac{1}{k+1} + \frac{1}{(k+1)^2} + \cdots) \geq \sum_{n=k}^{\infty} \frac{1}{e \cdot n!}.$$

□

Corollary 3.1.8. *If ψ is as in the Remark 3.1.6, then*

$$\inf_{0<t<1-1/e} \frac{t\psi'(t)}{\psi(t)} \geq \frac{1}{4}.$$

Proof. Let s_k be as in Lemma 3.1.7. Since ψ' is a Poisson random variable, it follows that

$$\psi'(t) = k, \quad \forall t \in (s_{k+1}, s_k), \quad k \in \mathbb{N}.$$

Therefore,

$$\psi(s_{k+1}) = \sum_{n=k+1}^{\infty} \frac{k+1}{e \cdot (k+1)!} = s_k, \quad k \in \mathbb{N}.$$

If $0 < t < 1 - 1/e$, then $t \in [s_{k+1}, s_k]$ for some $k \geq 1$. Clearly, $\psi'(t) = k$ on this interval. Since ψ is concave, it follows that the function $t/\psi(t)$ increases. Therefore,

$$\frac{t\psi'(t)}{\psi(t)} = \frac{kt}{\psi(t)} \geq \frac{ks_{k+1}}{\psi(s_{k+1})} = \frac{ks_{k+1}}{s_k} \geq \frac{1}{4}.$$

The last inequality is valid by Lemma 3.1.7. □

Corollary 3.1.9. *If F be an arbitrary symmetric quasi-Banach space and if $x \in F$, then*

$$\|x\|_F \leq 8C(F)\|x\|_{M_\psi} \cdot \|K\|_{L_\infty \to F}.$$

Proof. It is clear that

$$\|x\|_{M_\psi} = \sup_{t>0} \frac{1}{\psi(t)} \int_0^t x^*(s)ds \geq \sup_{0<t<1/2} \frac{1}{\psi(t)} \int_0^t x^*(s)ds \geq$$

$$\geq \sup_{0<t<1/2} \frac{tx^*(t)}{\psi(t)} = \sup_{0<t<1/2} \frac{t\psi'(t)}{\psi(t)} \cdot \frac{x^*(t)}{\psi'(t)} \geq$$

$$\geq \inf_{0<t<1/2} \frac{t\psi'(t)}{\psi(t)} \cdot \sup_{0<t<1/2} \frac{x^*(t)}{\psi'(t)} \geq \frac{1}{4} \sup_{0<t<1/2} \frac{x^*(t)}{\psi'(t)}.$$

Here, the last inequality follows from Corollary 3.1.8. Therefore,

$$x^*(t) \leq 4\|x\|_{M_\psi}\psi'(t)$$

for every $t \in (0, 1/2)$. In particular,

$$x^* \leq \sigma_2 x^* \leq 4\|x\|_{M_\psi}\sigma_2\psi'.$$

Therefore,

$$\|x\|_F \leq \|\sigma_2 x^*\|_F \leq 4\|x\|_{M_\psi}\|\sigma_2\psi'\|_F \leq 8C(F)\|x\|_{M_\psi}\|K1\|_F.$$

The assertion follows now from the obvious equality

$$\|K\|_{L_\infty \to F} = \|K1\|_F.$$

□

Lemma 3.1.10. *Let $x_k, y_k \in L_1(0,1)$, $1 \leq k \leq n$, be positive and independent. If $y_k \prec x_k$ for each k, then*

$$\sum_{k=1}^n y_k \prec \sum_{k=1}^n x_k.$$

Proof. Define functions $x, y \in L_1((0,1)^n)$ by setting

$$x(\omega) = \sum_{k=1}^{n} x_k(\omega_k), \quad y(\omega) = \sum_{k=1}^{n} y_k(\omega_k), \quad \omega = (\omega_1, \cdots, \omega_n) \in (0,1)^n.$$

By assumption, for every $1 \le k \le n$, there exists a bistochastic operator A_k such that $A_k x_k = y_k$. The operator $A = \otimes_{k=1}^{n} A_k$ is a bounded operator on $L_1((0,1)^n)$ (see Appendix C). Since each of the operators A_k is bistochastic, so is the operator A. Evidently,

$$y = \sum_{k=1}^{n} 1 \otimes \cdots \otimes y_k \otimes \cdots \otimes 1 = \sum_{k=1}^{n} A_1 1 \otimes \cdots \otimes A_k x_k \otimes \cdots \otimes A_n 1 = Ax \prec x.$$

Since $\sum_{k=1}^{n} x_k$ (respectively, $\sum_{k=1}^{n} y_k$) is equimeasurable with x (respectively, y), the assertion of the lemma follows. □

The above proof does not work if we replace $\prec$ with $\prec\prec$. However, we are able to derive the following corollary.

Lemma 3.1.11. *Let $x_k, y_k \in L_1(0,1)$, $1 \le k \le n$, be positive and independent. If $y_k \prec\prec x_k$ for each k, then*

$$\sum_{k=1}^{n} y_k \prec\prec \sum_{k=1}^{n} x_k.$$

Proof. For $1 \le k \le n$, select $s_k \in (0,1)$ such that $y_k \prec x_k^* \chi_{(0,s_k)}$. Define functions $x, y, z \in L_1((0,1)^n)$ by setting

$$x(\omega) = \sum_{k=1}^{n} x_k^*(\omega_k), \quad y(\omega) = \sum_{k=1}^{n} y_k(\omega_k), \quad z(\omega) = \sum_{k=1}^{n} (x_k^* \chi_{(0,s_k)})(\omega_k)$$

for every $\omega = (\omega_1, \cdots, \omega_n) \in (0,1)^n$. It follows from Lemma 3.1.10 that $y \prec z \le x$. Since $\sum_{k=1}^{n} x_k$ (respectively, $\sum_{k=1}^{n} y_k$) is equimeasurable with x (respectively, y), this suffices to conclude the lemma. □

Proposition 3.1.12. *If x_k, $1 \le k \le n$, are bounded and independent, then*

$$\| \sum_{k=1}^{n} x_k \|_{M_\psi} \le 2 \| \sum_{k=1}^{n} \bar{x}_k \|_{L_1 \cap L_\infty}.$$

Proof. Without loss of generality, $x_k \ge 0$ for $1 \le k \le n$. Suppose that

$$\| \sum_{k=1}^{n} \bar{x}_k \|_\infty = 1, \quad \|x_k\|_1 = \alpha_k.$$

If $\alpha = \sum_{k=1}^{n} \alpha_k > 1$, then $x_k \prec \alpha \chi_{[0, \alpha^{-1}\alpha_k]}$ for $1 \le k \le n$. It follows from Lemma 3.1.10 that

$$\sum_{k=1}^{n} x_k \prec \alpha \sum_{k=1}^{n} \chi_{[0,\alpha^{-1}\alpha_k]}(\omega_k).$$

Therefore, by Proposition 3.1.5,

$$\|\sum_{k=1}^{n} x_k\|_{M_\psi} \leq \alpha\|\sum_{k=1}^{n} \chi_{[0,\alpha^{-1}\alpha_k]}(\omega_k)\|_{M_\psi} \leq 2\alpha\|\sum_{k=1}^{n} \overline{\chi_{[0,\alpha^{-1}\alpha_k]}}\|_\infty = 2\alpha.$$

If $\alpha = \sum_{k=1}^{n} \alpha_k < 1$, then $x_k \prec \chi_{[0,\alpha_k]}$ for $1 \leq k \leq n$. It follows from Lemma 3.1.10 that

$$\sum_{k=1}^{n} x_k \prec \sum_{k=1}^{n} \chi_{[0,\alpha_k]}(\omega_k).$$

Therefore, by Proposition 3.1.5,

$$\|\sum_{k=1}^{n} x_k\|_{M_\psi} \leq \|\sum_{k=1}^{n} \chi_{[0,\alpha_k]}(\omega_k)\|_{M_\psi} \leq 2\|\sum_{k=1}^{n} \overline{\chi_{[0,\alpha_k]}}\|_\infty = 2.$$

□

Proposition 3.1.13. *Let F be a symmetric quasi-Banach space. If x_k, $1 \leq k \leq n$, are bounded and independent, then*

$$\|\sum_{k=1}^{n} x_k\|_F \leq 16C(F)\|K\|_{L_\infty \to F}\|\sum_{k=1}^{n} \bar{x}_k\|_{L_1 \cap L_\infty}.$$

Here, $\bar{x}_k$ are disjoint copies of x_k.

Proof. The assertion follows directly from Proposition 3.1.12 and Corollary 3.1.9 applied to the function $x = \sum_{k=1}^{n} x_k \in M_\psi$. □

Lemma 3.1.14. *Let E, F be symmetric quasi-Banach spaces. If $K : E \to F$, then $K : L_\infty \to F$ and*

$$\|K\|_{L_\infty \to F} \leq \|K\|_{E \to F}.$$

Proof. Since $\|x\|_E \leq \|x\|_\infty$ for every $x \in L_\infty$, it follows that

$$\|K\|_{E \to F} = \sup_{x \in E} \frac{\|Kx\|_F}{\|x\|_E} \geq \sup_{x \in L_\infty} \frac{\|Kx\|_F}{\|x\|_E} \geq \sup_{x \in L_\infty} \frac{\|Kx\|_F}{\|x\|_\infty} = \|K\|_{L_\infty \to F}.$$

□

According to Lemma 3.1.14, one can replace $\|K\|_{L_\infty \to F}$ with $\|K\|_{E \to F}$ in Proposition 3.1.13.

Theorem 3.1.15. *Let E, F be symmetric quasi-Banach spaces. If x_k, $1 \leq k \leq n$, are independent and if $X = \sum_{k=1}^{n} \bar{x}_k$, then*

$$\|\sum_{k=1}^{n} x_k\|_F \leq 32C^2(F)\|K\|_{E \to F}(\|X^*\chi_{[0,1]}\|_E + \|X^*\chi_{(1,\infty)}\|_1). \tag{3.5}$$

Proof. Define the function g and the constant c by the formulae

$$g(s) = m(\{t : \ |X(t)| > s\}) = \sum_{k=1}^{n} m(\{t : \ |x_k(t)| \geq s\}),$$

$$c = X^*(1) = \inf\{s : \ g(s) < 1\}.$$

If g is discontinuous at c, then some of the sets $|x_k|^{-1}\{c\}$ have positive measure. For those k, select sets $A_k \subset |x_k|^{-1}\{c\}$ such that $\sum_{k=1}^{n} m(A_k) = 1 - g(c+0)$. Such a selection is possible since $g(c) \geq 1$.

Define functions

$$x_{1k} = x_k \chi_{\{|x_k|>c\} \cup A_k}, \quad x_{2k} = x_k - x_{1k}, \quad 1 \leq k \leq n.$$

The functions x_{1k}, $1 \leq k \leq n$ are independent. So are the functions x_{2k}, $1 \leq k \leq n$.

It is clear that

$$\sum_{k=1}^{n} \bar{x}_{1k} \sim X^* \chi_{[0,1]}, \quad \sum_{k=1}^{n} \bar{x}_{2k} \sim X^* \chi_{(1,\infty)}.$$

Therefore, applying Propositions 3.1.5 and 3.1.13, we obtain

$$\|\sum_{k=1}^{n} x_k\|_F \leq C(F)(\|\sum_{k=1}^{n} x_{1k}\|_F + \|\sum_{k=1}^{n} x_{2k}\|_F) \leq$$

$$\leq 16C^2(F)\|K\|_{E\to F}(\|\sum_{k=1}^{n} \bar{x}_{1k}\|_E + \|\sum_{k=1}^{n} \bar{x}_{2k}\|_{L_1\cap L_\infty}) =$$

$$= 16C^2(F)\|K\|_{E\to F}(\|X^*\chi_{[0,1]}\|_E + \|X^*\chi_{(1,\infty)}\|_{L_1\cap L_\infty}).$$

If

$$\|X^*\chi_{(1,\infty)}\|_\infty \leq \|X^*\chi_{(1,\infty)}\|_1,$$

then

$$\|\sum_{k=1}^{n} x_k\|_F \leq 16C^2(F)\|K\|_{E\to F}(\|X^*\chi_{[0,1]}\|_E + \|X^*\chi_{(1,\infty)}\|_1)$$

and we are done.

Otherwise, note that

$$\|X^*\chi_{(1,\infty)}\|_\infty = X^*(1) \leq \|X^*\chi_{[0,1]}\|_E.$$

Hence,

$$\|\sum_{k=1}^{n} x_k\|_F \leq 32C^2(F)\|K\|_{E\to F}\|X^*\chi_{[0,1]}\|_E$$

and this suffices to complete the proof. □

3.2 The Johnson-Schechtman inequality for symmetrically distributed & mean zero functions

Recall that the random variable x is said to be symmetrically distributed if x^+ and x^- have the same distribution.

If we assume that the independent random variables x_k, $1 \le k \le n$, in the statement of Theorem 3.1.15 are, in addition, symmetrically distributed (or are mean zero), then the Johnson-Schechtman inequality given in (3.5) can be significantly improved. In this section we extend estimates due to Astashkin & Sukochev (see [3]) to the quasi-Banach setting. Our proofs significantly simplify those of [3].

The following remarkable inequality was proved by Prokhorov. We include the following simple proof from [27].

Lemma 3.2.1. *If x_k, $1 \le k \le n$, is a sequence of uniformly bounded independent symmetrically distributed random variables, then*

$$m(\{\sum_{k=1}^n x_k > t\}) \le \exp(-\frac{t}{2\|\sum_{k=1}^n \bar{x}_k\|_\infty}\mathrm{arcsinh}(\frac{t\|\sum_{k=1}^n \bar{x}_k\|_\infty}{2\|\sum_{k=1}^n \bar{x}_k\|_2^2})). \qquad (3.6)$$

Proof. Recall the inequality $e^u - u - 1 \le |u|\sinh(|u|)$. For every symmetrically distributed random variable x and every $\lambda > 0$, we have

$$\int_{(0,1)} e^{\lambda x(\omega)} d\omega = 1 + \int_{(0,1)} (e^{\lambda x(\omega)} - \lambda x(\omega) - 1) d\omega \le$$

$$\le 1 + \int_{(0,1)} (\lambda^2 x^2(\omega)) \frac{\sinh(|\lambda x(\omega)|)}{|\lambda x(\omega)|} \le 1 + \frac{\lambda\|x\|_2^2}{\|x\|_\infty}\sinh(\lambda\|x\|_\infty).$$

Recall that $1 + u \le \exp(u)$ for every $u > 0$. It follows that

$$\int_{(0,1)^n} \exp(\lambda \sum_{k=1}^n x_k(\omega_k)) \prod_{k=1}^n d\omega_k \le \exp(\sum_{k=1}^n \frac{\lambda\|x_k\|_2^2}{\|x_k\|_\infty}\sinh(\lambda\|x_k\|_\infty)).$$

Therefore,

$$\int_{(0,1)} \exp(\lambda \sum_{k=1}^n x_k) \le \exp(\frac{\lambda\|\sum_{k=1}^n \bar{x}_k\|_2^2}{\|\sum_{k=1}^n \bar{x}_k\|_\infty}\sinh(\lambda\|\sum_{k=1}^n \bar{x}_k\|_\infty)).$$

It is clear that

$$m(\{\sum_{k=1}^n x_k > t\}) \le e^{-\lambda t}\int_{(0,1)} \exp(\lambda \sum_{k=1}^n x_k).$$

Setting

$$\lambda = \frac{1}{\|\sum_{k=1}^n \bar{x}_k\|_\infty}\mathrm{arcsinh}(\frac{\|\sum_{k=1}^n \bar{x}_k\|_\infty t}{2\|\sum_{k=1}^n \bar{x}_k\|_2^2}),$$

the assertion follows. $\square$

Proposition 3.2.2. *If x_k, $1 \le k \le n$, are bounded, symmetrically distributed and independent, then*

$$\|\sum_{k=1}^{n} x_k\|_{M_\psi} \le C_{abs} \|\sum_{k=1}^{n} \bar{x}_k\|_{L_2 \cap L_\infty}.$$

Proof. Define the set of operators

$$A_n : L_2 \cap L_\infty \to M_\psi(\Omega), \quad A_n x(\omega) = \sum_{k=0}^{n-1} x(k + \omega_{2k-1}) r(\omega_{2k}), \quad \omega \in \Omega.$$

It is clear that

$$\|A_n\|_{L_2 \cap L_\infty \to M_\psi} \le n.$$

On the other hand, for any fixed x set

$$\alpha(x) = \|x\|_\infty + \sup_n \frac{\|x\chi_{[0,n]}\|_2^2}{\|x\chi_{[0,n]}\|_\infty}.$$

Here, $0/0$ is set to be 0. Clearly, $\alpha(x)$ is always finite. It follows from the Prokhorov inequality that

$$m(\{|A_n x| > t\alpha(x)\}) \le 2 \cdot \exp(-\frac{t}{2} \text{arcsinh}(\frac{t}{2})), \quad n \in \mathbb{N}.$$

By Lemma 1.2.71,

$$m(\{4\psi' > t\}) \ge \exp(-1 - \frac{t}{2} \text{arcsinh}(\frac{t}{2})).$$

Therefore,

$$m(\{|A_n x| > t\alpha(x)\}) \le 2e \cdot m(\{4\psi' > t\}), \quad n \in \mathbb{N}.$$

Now it is clear that

$$\|A_n x\|_{M_\psi(\Omega)} \le 8e \cdot \alpha(x).$$

Therefore, the norms $\|A_n x\|_{M_\psi(\Omega)}$, $n \in \mathbb{N}$, are uniformly bounded. It follows from the uniform boundedness principle that the norms $\|A_n\|_{L_2 \cap L_\infty \to M_\psi(\Omega)}$ are uniformly bounded. □

Corollary 3.2.3. *Let E, F be symmetric quasi-Banach spaces. If x_k, $1 \le k \le n$, are bounded, symmetrically distributed and independent, then*

$$\|\sum_{k=1}^{n} x_k\|_F \le C_{abs} C(F) \|K\|_{L_\infty \to F} \|\sum_{k=1}^{n} \bar{x}_k\|_{L_2 \cap L_\infty}.$$

Proof. The assertion follows directly from Proposition 3.2.2 and Corollary 3.1.9 applied to the function $x = \sum_{k=1}^{n} x_k \in M_\psi$. □

According to Lemma 3.1.14, one can replace $\|K\|_{L_\infty\to F}$ with $\|K\|_{E\to F}$ in Proposition 3.2.3.

Theorem 3.2.4. *Let E, F be symmetric quasi-Banach spaces. If x_k, $1\le k\le n$, are independent and symmetrically distributed and if $X=\sum_{k=1}^n \bar{x}_k$, then*

$$\|\sum_{k=1}^n x_k\|_F \le C_{abs}C^2(F)\|K\|_{E\to F}(\|X^*\chi_{[0,1]}\|_E+\|X^*\chi_{(1,\infty)}\|_2). \tag{3.7}$$

Proof. Let the function g and the constant c be defined as in the proof of Theorem 3.1.15. Select sets A_k as in the proof of Theorem 3.1.15 such that in addition $x_k|_{A_k}$, $1\le k\le n$, are symmetrically distributed. The proof follows *mutatis mutandi.* □

From now on, we restrict ourselves to the case $F\subset L_1$. Theorem 3.2.4 can be extended to the case when the random variables x_k, $1\le k\le n$, are not symmetrically distributed but just mean zero.

We need the following assertion proved by Braverman (see [12]) in the Banach setting. The proof in the quasi-Banach setting is identical.

Lemma 3.2.5. *If symmetric quasi-Banach space E is such that $E\subset L_1$, then there exists a constant $C_0(E)$ such that*

$$\|x\|_E \le C_0(E)\|x(\omega_1)-x(\omega_2)\|_E$$

for every mean zero random variable x.

Theorem 3.2.6. *Let E, F be symmetric quasi-Banach spaces and let $F\subset L_1$. If x_k, $1\le k\le n$, are independent and mean zero and if $X=\sum_{k=1}^n \bar{x}_k$, then*

$$\|\sum_{k=1}^n x_k\|_F \le C_{abs}C_0(F)C^2(F)C(E)(\|X^*\chi_{[0,1]}\|_E+\|X\|_{L_1+L_2}). \tag{3.8}$$

Proof. Define functions $z_1, z_2\in E(\omega)$ and functions $y_k\in E(\Omega)$, $1\le k\le n$ by setting

$$z_1(\omega)=\sum_{k=1}^n x(\omega_{2k-1}),\quad z_2(\omega)=\sum_{k=1}^n x_k(\omega_{2k-1})-x_k(\omega_{2k}),\quad \omega\in\Omega,$$

$$y_k(\omega)=x_k(\omega_{2k-1})-x_k(\omega_{2k}),\quad \omega\in\Omega.$$

By Lemma 3.2.5,

$$\|\sum_{k=1}^n x_k\|_F=\|z_1\|_{F(\Omega)}\le C_0(F)\|z_2\|_{F(\Omega)}=C_0(F)\|\sum_{k=1}^n y_k\|_{F(\Omega)}.$$

Evidently, y_k, $1\le k\le n$, are independent and symmetrically distributed. Therefore, by Theorem 3.2.4,

$$\|\sum_{k=1}^n y_k\|_{F(\Omega)}\le C_{abs}C^2(F)\|K\|_{E\to F}(\|Y^*\chi_{(0,1)})\|_E+\|Y^*\chi_{(1,\infty)}\|_2.$$

Here, $Y = \sum_{k=1}^n \bar{y}_k$ is the sum of disjoint copies of y_k.

It follows from the inequality (1.4) that $Y^* \leq 2\sigma_2 X^*$. Thus,

$$\|Y^*\chi_{(1,\infty)}\|_2 \leq 4\|X^*\chi_{(1/2,\infty)}\|_2 \leq 100\|X\|_{L_1+L_2}.$$

Similarly,

$$\|Y^*\chi_{(0,1)}\|_E \leq 2\|(\sigma_2 X^*)\chi_{[0,1]}\|_E \leq 4C(E)\|X^*\chi_{(0,1)}\|_E.$$

□

3.3 The reverse Johnson-Schechtman inequality

The reverse Johnson-Schechtman inequality was first proved in [28]. We reproduce it here for several reasons. First, it is not said in [28] that the reverse inequality is valid in the quasi-Banach setting. Second, we need precise values of various constants in the subsequent section devoted to the Khinchine inequality. Finally, our proof is somewhat different from that of [28].

Proposition 3.3.1. *Let E be a symmetric quasi-Banach space on the interval $(0,1)$. Let $x_k \in E$, $1 \leq k \leq n$, be positive and independent random variables. If $\sum_{k=1}^n m(\mathrm{supp}(x_k)) \leq 1$, then*

$$\|\sum_{k=1}^n \bar{x}_k\|_E \leq 2C(E)\|\sum_{k=1}^n x_k\|_E.$$

Proof. The following elementary inequalities are valid for every $t \in [0,1]$.

$$e^{-t} \geq 1 - t, \quad 1 - e^{-t} \geq \frac{1}{2}t.$$

It follows that

$$m(\{t : \max_{1\leq k\leq n} x_k(t) > s\}) = 1 - \prod_{k=1}^n (1 - m(\{t : x_k(t) > s\})) \geq$$

$$\geq 1 - \exp(-\sum_{k=1}^n m(\{t : x_k(t) > s\})) \geq \frac{1}{2}\sum_{k=1}^n m(\{t : x_k(t) > s\}).$$

Therefore,

$$\sum_{k=1}^n m(\{t : x_k(t) > s\}) \leq 2m(\{t : \max_{1\leq k\leq n} x_k(t) > s\}) \leq 2m(\{t : \sum_{k=1}^n x_k(t) > s\}).$$

Hence,

$$(\sum_{k=1}^n \bar{x}_k)^* \leq \sigma_2(\sum_{k=1}^n x_k)^*.$$

The assertion follows immediately. □

3.4 The Khinchine inequality

In this section, we provide the most natural extension of the classical Khinchine inequality (see Theorem 3.4.6).

Lemma 3.4.1. *Let E be a symmetric quasi-Banach space on the interval* $(0,1)$. *If* $p = 1/2 \cdot \log_2^{-1}(2C(E))$, *then* $E \subset L_p$ *and*

$$\|x\|_p \leq 8C^3(E)\|x\|_E.$$

Proof. Define the increasing function ψ by the formula $\psi(u) = \|\chi_{[0,u]}\|_E$, $0 < u < 1$. It follows from the definition of a quasi-norm that

$$\psi(2u) \leq 2C(E)\psi(u), \quad u > 0.$$

In particular,

$$\psi(2^{-n}) \geq (2C(E))^{-n}, \quad n \geq 0.$$

If $u \in (0,1)$ is arbitrary, then $u \in [2^{-n-1}, 2^{-n}]$. Hence,

$$\psi(u) \geq \psi(2^{-n-1}) \geq 2^{-(n+1)\log_2(2C(E))} \geq \frac{1}{2C(E)} u^{\log_2(2C(E))}.$$

If $x \in E$, then

$$\|x\|_E \geq \|x^*(t)\chi_{[0,t]}\|_E \geq x^*(t)\frac{1}{2C(E)} t^{\log_2(2C(E))}.$$

Hence,

$$x^*(t) \leq 2\|x\|_E C(E) t^{-\log_2(2C(E))}, \quad t > 0.$$

The assertion follows immediately. □

Lemma 3.4.2. *If* $x, y \in L_1(0,1)$ *are positive and* $y \prec x$, *then* $\|y\|_p \geq \|x\|_p$ *provided that* $0 < p < 1$.

Proof. Fix $\varepsilon > 0$. According to Theorem 2.7.7, there exists $z \in \mathcal{Q}'(x)$ such that $\|y - z\|_1 \leq \varepsilon$. In particular,

$$z = \sum_{k=1}^{n} \lambda_k x_k, \quad x_k \geq 0, \ x_k^* = x^,$$

where

$$\sum_{k=1}^{n} \lambda_k = 1, \quad \lambda_k \geq 0.$$

Therefore,

$$\|z\|_p = \|\sum_{k=1}^{n} \lambda_k x_k\|_p \geq \sum_{k=1}^{n} \lambda_k \|x_k\|_p = \|x\|_p.$$

Since $\varepsilon > 0$ is arbitrarily small and the quasi-norm in L_p is continuous with respect to L_1-convergence, this suffices to complete the proof of the lemma. □

Lemma 3.4.3. *Let $0 < p < 1$ and let y_k, $1 \le k \le n$, be independent, positive and bounded random variables. It follows that*

$$\|\sum_{k=1}^{n} \bar{y}_k\|_1 \le 2^{1/p} \max\{\sup_{1\le k\le n} \|y_k\|_\infty, \|\sum_{k=1}^{n} y_k\|_p\}. \tag{3.9}$$

Proof. Without loss of generality,

$$\sup_{1\le k\le n} \|y_k\|_\infty = 1, \quad \|y_k\|_1 = \alpha_k, \quad 1 \le k \le n.$$

Let $\alpha = \sum_{k=1}^{n} \alpha_k$. If $\alpha \le 1$, then the assertion is evident. If $\alpha \ge 1$, then

$$y_k \prec \alpha \chi_{[0,\alpha^{-1}\alpha_k]}, \quad 1 \le k \le n.$$

It follows from Lemma 3.1.10 that

$$\sum_{k=1}^{n} y_k \prec \alpha \sum_{k=1}^{n} \chi_{[0,\alpha^{-1}\alpha_k]}(\omega_k).$$

According to Lemma 3.4.2,

$$\|\sum_{k=1}^{n} y_k\|_p \ge \alpha \|\sum_{k=1}^{n} \chi_{[0,\alpha^{-1}\alpha_k]}(\omega_k)\|_p.$$

It follows now from Lemma 3.3.1 that

$$2C(L_p)\|\sum_{k=1}^{n} y_k\|_p \ge \alpha \|\sum_{k=1}^{n} \overline{\chi_{[0,\alpha^{-1}\alpha_k]}}\|_p = \alpha.$$

Since the concavity modulus of L_p for $0 < p < 1$ can be estimated as

$$C(L_p) \le 2^{1/p-1},$$

we are done. □

Lemma 3.4.4. *Let E be a symmetric quasi-Banach space. If $x_k \in E$, $1 \le k \le n$, are bounded independent random variables, then*

$$\|\sum_{k=1}^{n} \bar{x}_k\|_2 \le 32C^5(E) \max\{\sup_{1\le k\le n} \|x_k\|_\infty, \|(\sum_{k=1}^{n} x_k^2)^{1/2})\|_E\}. \tag{3.10}$$

Proof. Let $p = 1/2 \log_2^{-1}(2C(E))$. It clearly follows from Lemma 3.4.1 that

$$8C^3(E)\|(\sum_{k=1}^{n} x_k^2)^{1/2})\|_E \ge \|(\sum_{k=1}^{n} x_k^2)^{1/2}\|_p = \|\sum_{k=1}^{n} x_k^2\|_{p/2}^{1/2}.$$

Clearly,

$$\|\sum_{k=1}^{n} \bar{x}_k\|_2 = \|\sum_{k=1}^{n} \bar{x}_k^2\|_1^{1/2}, \quad \|x_k\|_\infty = \|x_k^2\|_\infty^{1/2}.$$

Set $y_k = x_k^2$, $1 \le k \le n$. The assertion follows immediately from Lemma 3.4.3. □

Lemma 3.4.5. *Let E be a symmetric quasi-Banach space. If $x_k \in E$, $1 \le k \le n$, are independent random variables and if $X = \sum_{k=1}^n \bar{x}_k$, then*

$$2C(E)\|(\sum_{k=1}^n x_k^2)^{1/2}\|_E \ge X^*(1).$$

Proof. Without loss of generality,

$$\sum_{k=1}^n m(\mathrm{supp}(x_k)) = 1.$$

It follows that

$$\sum_{k=1}^n x_k^2 \ge (X^*(1))^2 \sum_{k=1}^n \chi_{\mathrm{supp}(x_k)}.$$

The support of the latter function has measure

$$1 - \prod_{k=1}^n (1 - m(\mathrm{supp}(x_k))) \ge 1/2.$$

Therefore,

$$\|(\sum_{k=1}^n x_k^2)^{1/2}\|_E \ge X^*(1)\|\chi_{[0,1/2]}\|_E.$$

The assertion follows immediately. □

Theorem 3.4.6. *Let E and F be symmetric quasi-Banach spaces on the interval $(0,1)$. If $x_k \in E$, $1 \le k \le n$, are independent symmetrically distributed random variables, then*

$$\|\sum_{k=1}^n x_k\|_F \le C_{abs}C^6(E)C^2(F)\|K\|_{E\to F}\|(\sum_{k=1}^n x_k^2)^{1/2}\|_E. \quad (3.11)$$

Proof. Recall the assertion of Theorem 3.2.4: if x_k, $1 \le k \le n$, are independent and symmetrically distributed, then

$$\|\sum_{k=1}^n x_k\|_F \le C_{abs}C^2(F)\|K\|_{E\to F}(\|X^*\chi_{[0,1]}\|_E + \|X^*\chi_{(1,\infty)}\|_2).$$

Here, $X = \sum_{k=1}^n \bar{x}_k$ is a sum of disjoint copies of x_k.

Define the function g and the constant c by the formulae

$$g(s) = m(\{t: \ |X(t)| > s\}) = \sum_{k=1}^n m(\{t: \ |x_k(t)| \ge s\}),$$

$$c = X^*(1) = \inf\{s: \ g(s) < 1\}.$$

If g is discontinuous at c, then some of the sets $|x_k|^{-1}\{c\}$ have positive measure. For those k, select sets $A_k \subset |x_k|^{-1}\{c\}$ such that $\sum_{k=1}^n m(A_k) = 1-g(c+0)$ and $x_k|_{A_k}$ is symmetrically distributed. Such a selection is possible since $g(c) \geq 1$.

Define functions

$$x_{1k} = x_k \chi_{\{|x_k|>c\}\cup A_k}, \quad x_{2k} = x_k - x_{1k}, \quad 1 \leq k \leq n.$$

The functions x_{1k}, $1 \leq k \leq n$ are independent. So are the functions x_{2k}, $1 \leq k \leq n$. It is clear that

$$\sum_{k=1}^n \bar{x}_{1k} \sim X^* \chi_{[0,1]}, \quad \sum_{k=1}^n \bar{x}_{2k} \sim X^* \chi_{(1,\infty)}.$$

Apply Lemma 3.4.4 to the functions x_{2k}. It follows that

$$\|X^* \chi_{(1,\infty)}\|_2 \leq 32C^5(E) \max\{X^*(1), \|(\sum_{k=1}^n x_{2k}^2)^{1/2}\|_E\}.$$

By Lemma 3.4.5,

$$\|X^* \chi_{(1,\infty)}\|_2 \leq 64C^6(E) \|(\sum_{k=1}^n x_k^2)^{1/2}\|_E. \tag{3.12}$$

On the other hand,

$$\|X^* \chi_{[0,1]}\|_E = \|\sum_{k=1}^n \bar{x}_{1k}\|_E = \|\sum_{k=1}^n \bar{x}_{1k}^2\|_{E^{1/2}}^{1/2}$$

and

$$\|(\sum_{k=1}^n x_k^2)^{1/2}\|_E \geq \|(\sum_{k=1}^n x_{1k}^2)^{1/2}\|_E = \|\sum_{k=1}^n x_{1k}^2\|_{E^{1/2}}^{1/2}.$$

Apply Lemma 3.3.1 to the space $E^{1/2}$ and functions x_{1k}^2. It follows that

$$\|X^* \chi_{[0,1]}\|_E \leq (2C(E^{1/2}))^{1/2} \|(\sum_{k=1}^n x_k^2)^{1/2}\|_E. \tag{3.13}$$

Since $C(E^{1/2}) \leq 4C^2(E)$, the assertion follows from (3.13) and (3.12). □

Note that setting $x_k = a_k r_k$ and $E = F = L_p$ in Theorem 3.4.6, we obtain the classical Khinchine inequality.

3.5 The operators A_n, $n \geq 0$

For every $n \geq 1$, we consider the operator $A_n : E(0,1) \to E(\Omega)$ given by

$$(A_n x)(\omega) = \sum_{k=1}^{n} x(\omega_{2k-1}) r(\omega_{2k}), \quad \omega \in \Omega.$$

where r is a centered Bernoulli random variable. We set $A_0 = 0$.

Norm-estimates for these operators can yield a Khinchine-type inequality (see Theorem 3.5.3 below).

The following theorem is the main result of the present section.

Theorem 3.5.1. *If E is a fully symmetric quasi-Banach space on the interval $(0,1)$, then one of the following assertions is valid.*

1. *$\|A_n\|_{E \to E} = n$ for every $n \in \mathbb{N}$.*
2. *There exists a constant $\frac{1}{2} \leq q < 1$, such that $\|A_n\|_{E\to E} \leq \mathrm{const} \cdot n^q$ for all $n \in \mathbb{N}$.*

Proof. Observing that $A_{mn}x$ and $A_m(A_n x)$ are identically distributed, we have

$$\|A_{mn}x\|_E = \|A_m(A_n x)\|_E, \quad x \in E(0,1).$$

Hence,

$$\|A_{mn}\|_{E\to E} \leq \|A_m\|_{E\to E} \cdot \|A_n\|_{E\to E}. \tag{3.14}$$

Thus, we have the following alternative:

1. $\|A_n\|_{E\to E} = n$ for every natural n.
2. There exists $n_0 \geq 2$, such that $\|A_{n_0}\|_{E\to E} < n_0$.

To finish the proof of Theorem 3.5.1, we need only to consider the second case. Suppose there exists a constant $\frac{1}{2} \leq q < 1$, such that $\|A_{n_0}\|_{E\to E} \leq n_0^q$. By (3.14) we have

$$\|A_{n_0^m}\|_{E\to E} \leq \|A_{n_0}\|_{E\to E}^m \leq n_0^{qm}, \ \forall m \in \mathbb{N}.$$

The map $P_{n,m} : L_1(\Omega) \to L_1(\Omega)$ defined by the formula

$$(P_{n,m}x)(\omega) = \int x(\omega) \prod_{n<k\leq m} d\omega_{2k}$$

is a contraction in $L_1(\Omega)$ and in $L_\infty(\Omega)$.

For every $n \leq m$, we have $A_n x = P_{n,m}(A_m x) \prec\prec A_m x$. Since the space E is fully symmetric, the sequence of norms $\|A_n x\|_E$ increases. Therefore,

$$\|A_n x\|_E \leq \|A_{n_0^m} x\|_E, \quad \forall n \leq n_0^m.$$

Hence,

$$\|A_n\|_{E\to E} \le \|A_{n_0^m}\|_{E\to E}, \quad \forall n \le n_0^m.$$

For every $n \in \mathbb{N}$ fix $m \in \mathbb{N}$ such that $n_0^{m-1} \le n \le n_0^m$. It follows that

$$\|A_n\|_{E\to E} \le \|A_{n_0^m}\|_{E\to E} \le n_0^{qm} \le n_0^q n^q.$$

This proves the theorem. □

Recall that the Lorentz sequence space $l_{1/q,1}$, $0 < q \le 1$, is the space of all sequences $\{a_k\}_{k=1}^{\infty}$ such that

$$\|\{a_k\}_{k=1}^{\infty}\|_{1/q,1} = \sum_{k=1}^{\infty} a_k^*(k^q - (k-1)^q).$$

The proof of the following lemma is identical to that of [33, Lemma II.5.2].

Lemma 3.5.2. *If a convex functional Φ is bounded on the indicator sequences of all finite sets, then it is bounded on all sequences with finite support.*

Theorem 3.5.3. *Let E be a symmetric Banach space on the interval $(0,1)$ such that $\|A_n\|_{E\to E} \le cn^q$, $n \in \mathbb{N}$, with some $0 < q < 1$. If $x_k \in E$, $k \in \mathbb{N}$ are independent, symmetrically distributed and equimeasurable with x, then*

$$\|\sum_{k=1}^{\infty} a_k x_k\|_E \le c\|\{a_k\}_{k=1}^{\infty}\|_{1/q,1} \cdot \|x\|_E. \tag{3.15}$$

Proof. Let $A \subset \mathbb{N}$ be a finite set and let $\{a_k\}_{k=1}^{\infty} = \chi_A$. If $|A| = n$, then

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{k\in A} x_k \sim \sum_{k=1}^{n} x(\omega_{2k-1}) r(\omega_{2k}) = (A_n x)(\omega).$$

Therefore,

$$\|\sum_{k=1}^{n} a_k x_k\|_E = \|A_n x\|_E \le cn^q \cdot \|x\|_E = c\|\{a_k\}_{k=1}^{\infty}\|_{1/q,1} \cdot \|x\|_E.$$

Consider the convex functional

$$\Phi : \{a_k\}_{k=1}^{\infty} \longrightarrow \|\sum_{k=1}^{\infty} a_k x_k\|_E.$$

It follows from Lemma 3.5.2 that (3.15) holds for every finitely supported sequence.

Let now $\{a_k\}_{k=1}^{\infty} \in l_{1/q,1}$ be an arbitrary sequence. It follows from above that

$$\|\sum_{k=n+1}^{m} a_k x_k\|_E \le \|\{a_k\}_{k=n+1}^{m}\|_{1/q,1} \cdot \|x\|_E.$$

Therefore, the sequence of functions $\sum_{k=1}^{n} a_k x_k$, $n \in \mathbb{N}$, is a Cauchy sequence in E. Its limit is $\sum_{k=1}^{\infty} a_k x_k$. The assertion follows immediately. □

We complete this section with an estimate of $\|A_n\|_{E\to E}$, $n \geq 1$ in general symmetric spaces with the Kruglov property.

Theorem 3.5.4. *Let $E = E(0,1)$ be a quasi-Banach symmetric space on the interval $(0,1)$. Suppose that E satisfies the Kruglov property and that $\beta_E < 1$. Then there exists $q < 1$ such that $\|A_n\|_{E\to E} \leq \mathrm{const}\cdot n^q$ for all sufficiently large $n \geq 1$.*

Proof. Set $x_k = x(\omega_{2k-1})r(\omega_{2k})$ and $X = \sum_{k=1}^n \bar{x}_k$. Clearly, $X^*\chi_{[0,1]}$ is equimeasurable with $\sigma_n(x^*)\chi_{[0,1]}$ while $X^*\chi_{(1,\infty)}$ is equimeasurable with $\sigma_n(x^*\chi_{(1/n,1)})$.

By Theorem 3.2.4,

$$\|A_n x\|_E \leq \mathrm{const}(\|\sigma_n(x^*)\chi_{[0,1]}\|_E + n^{1/2}\|x^*\chi_{(1/n,1)}\|_2).$$

Note that

$$\|\sigma_n(x^*)\chi_{[0,1]}\|_E \leq \|\sigma_n\|_{E\to E}\|x\|_E \leq c(\varepsilon)n^{\beta_E+\varepsilon}\|x\|_E.$$

On the other hand,

$$x^*(\frac{1}{n})\chi_{[0,1]} \leq \sigma_n(x^*)\chi_{[0,1]}.$$

Therefore,

$$x^*(\frac{1}{n}) = \|x^*(\frac{1}{n})\chi_{[0,1]}\|_E \leq \|\sigma_n(x^*)\|_E \leq \|\sigma_n\|_{E\to E}\|x\|_E.$$

For every $t \in [0,1]$, fix $n \in \mathbb{N}$ such that $nt \leq 1 \leq (n+1)t$. It follows that

$$x^*(t) \leq c(\varepsilon)(n+1)^{\beta_E+\varepsilon}\|x\|_E \leq c(\varepsilon)2^{\beta_E+\varepsilon}t^{-(\beta_E+\varepsilon)}\|x\|_E.$$

If $\beta_E < 1/2$, then $x \in L_2(0,1)$. If $\beta_E \geq 1/2$, then

$$\|x^*\chi_{[1/n,1]}\|_2 \leq c(\varepsilon)2^{\beta_E+\varepsilon}\|x\|_E(2(\beta_E+\varepsilon)-1)^{-1/2}n^{(\beta_E+\varepsilon)-1/2}.$$

Thus,

$$\|A_n x\|_E \leq \mathrm{const}\cdot n^{\max\{\beta_E+\varepsilon,1/2\}}\|x\|_E.$$

This proves the theorem. □

Remark 3.5.5. *The assumption $\beta_E < 1$ in Theorem 3.5.4 is necessary (see [4, Theorem 4.2]). For example, the space $E = L_1$ satisfies the Kruglov property and $\beta_E = 1$. However, $\|A_n\|_{E\to E} = n$, $n \in \mathbb{N}$. On the other hand, the condition that E satisfies the Kruglov property is not optimal (see Example 3.6.8).*

3.6 The operators A_n, $n \geq 1$ in Lorentz spaces.

We need the following technical facts. Some of these facts are elementary but we present a proof for convenience of the reader.

Lemma 3.6.1. *If random variable ξ takes values $0, 1, \cdots, n$, then*

$$\|\xi\|_{\Lambda_\psi} \leq n\psi(\frac{1}{n}\|\xi\|_1).$$

Proof. Indeed,

$$\xi^*(s) = k \Longleftrightarrow s \in (m(\{\xi \geq k+1\}), m(\{\xi \geq k\})).$$

Therefore,

$$\|\xi\|_{\Lambda_\psi} = n\psi(m(\{\xi = n\})) + \sum_{k=1}^{n-1} k(\psi(m(\{\xi \geq k\})) - \psi(m(\{\xi \geq k+1\}))) =$$

$$= \sum_{k=1}^{n} \psi(m(\{\xi \geq k\})) \leq n\psi(\frac{1}{n}\sum_{k=1}^{n} m(\{\xi \geq k\})).$$

However,

$$\|\xi\|_1 = nm(\{\xi = n\}) + \sum_{k=1}^{n-1} k(m(\{\xi \geq k\}) - m(\{\xi \geq k+1\})) = \sum_{k=1}^{n} m(\{\xi \geq k\}).$$

Hence,

$$\|x\|_{\Lambda_\psi} \leq n\psi(\frac{1}{n}\|\xi\|_1).$$

□

Lemma 3.6.2. *If ψ is strictly monotone, then*

$$\|A_n\chi_{[0,u]}\|_{\Lambda_\psi} < n\|\chi_{[0,u]}\|_{\Lambda_\psi}$$

for every $0 < u < 1$.

Proof. Set

$$x(\omega) = \sum_{k=1}^{n} \chi_{[0,u]}(\omega_{2k-1}), \quad \omega \in \Omega.$$

Clearly, $\|x\|_1 = nu$ and $|A_n\chi_{[0,u]}| \leq x$. Therefore,

$$\|x\|_1 = \|A_n\chi_{[0,u]}\|_1 + \|x - |A_n\chi_{[0,u]}|\|_1.$$

Since $|A_n\chi_{[0,u]}| \neq x$, it follows that

$$\|A_n\chi_{[0,u]}\|_1 < \|x\|_1 = nu.$$

Note that random variable $\xi = |A_n\chi_{[0,u]}|$ satisfies the conditions of Lemma 3.6.1. Therefore,

$$\|A_n\chi_{[0,u]}\|_{\Lambda_\psi} \leq n\psi(\frac{1}{n}\|A_n\chi_{[0,u]}\|_1) < n\psi(u), \quad n \in \mathbb{N}$$

since ψ is strictly monotone. □

Lemma 3.6.3. *The following upper limits are equal.*

$$\limsup_{u\to 0} \frac{1}{n\psi(u)} \|A_n\chi_{[0,u]}\|_{\Lambda_\psi} = \limsup_{u\to 0} \frac{1}{n\psi(u)} \sum_{s=1}^{n} \psi(2^{1-s}\binom{n}{s}u^s).$$

Proof. For every $s \geq 1$, using a well-known formula for conditional probabilities, we have

$$m(|\sum_{k=1}^{n} \chi_{[0,u]}(\omega_{2k-1})r(\omega_{2k})| \geq s) = \sum_{k=1}^{n} \binom{n}{s} u^k(1-u)^{n-k} m(|r_1 + \cdots + r_k| \geq s).$$

Actually, the summation above is taken from $k = s$ to n, since

$$m(|r_1 + \cdots + r_k| \geq s) = 0, \quad \forall k < s.$$

If now $u \to 0$, then, for every $s \geq 1$ and $k > s$, we have

$$\binom{n}{k} u^k(1-u)^{n-k} = o(u^s).$$

Therefore,

$$m(|\sum_{k=1}^{n} \chi_{[0,u]}(\omega_{2k-1})r(\omega_{2k})| \geq s) = 2^{1-s}\binom{n}{s}u^s(1 + o(1)). \tag{3.16}$$

Let $\xi = |A_n\chi_{[0,u]}|$. It is clear that

$$\|\xi\|_{\Lambda_\psi} = n\psi(m(\{\xi = n\})) + \sum_{k=1}^{n-1} k(\psi(m(\{\xi \geq k\})) - \psi(m(\{\xi \geq k+1\}))) =$$

$$= \sum_{k=1}^{n} \psi(m(\{\xi \geq k\})).$$

Therefore,

$$\|A_n\chi_{[0,u]}\|_{\Lambda_\psi} = \sum_{k=1}^{n} \psi(2^{1-s}\binom{n}{s}u^s(1 + o(1))).$$

Since ψ is concave, it follows that

$$\lim_{u\to 0} \frac{\psi(u(1 + o(1)))}{\psi(u)} = 1. \tag{3.17}$$

Hence,

$$\|A_n\chi_{[0,u]}\|_{\Lambda_\psi} = (1 + o(1)) \sum_{k=1}^{n} \psi(2^{1-s}\binom{n}{s}u^s)$$

and

$$\limsup_{u\to 0} \frac{1}{n\psi(u)} \|A_n\chi_{[0,u]}\|_{\Lambda_\psi} = \limsup_{u\to 0} \frac{1}{n\psi(u)} \sum_{s=1}^{n} \psi(2^{1-s}\binom{n}{s}u^s), \quad n \in \mathbb{N}.$$

□

We need to consider the following properties of the function ψ.

$$a_\psi := \limsup_{u\to 0} \frac{\psi(ku)}{\psi(u)} < k. \tag{3.18}$$

$$c_\psi := \limsup_{u\to 0} \frac{\psi(u^l)}{\psi(u)} < 1. \tag{3.19}$$

$$\limsup_{u\to 0} \frac{1}{\psi(u)} \sum_{s=1}^{n} \psi(2^{1-s}\binom{n}{s}u^s) < n. \tag{3.20}$$

Proposition 3.6.4. *Suppose, there exist $k \geq 2$ such that* (3.18) *holds and $l \geq 2$ such that* (3.19) *holds. Then,* (3.20) *holds for all sufficiently large $n \in \mathbb{N}$.*

Proof. Consider the sum

$$\sum_{s=1}^{n} \psi(\binom{n}{s}2^{1-s}u^s).$$

For any sufficiently large n, we write

$$\sum_{s=1}^{n} = \sum_{s=1}^{1+[\frac{n}{k}]} + \sum_{s=2+[\frac{n}{k}]}^{n}.$$

Consequently, the upper limit in (3.20) can be estimated as

$$\begin{aligned}\limsup_{u\to 0} \frac{1}{\psi(u)} \sum_{s=1}^{n} \psi(\binom{n}{s}2^{1-s}u^s) \leq \limsup_{u\to 0} \frac{1}{\psi(u)} \sum_{s=1}^{1+[\frac{n}{k}]} \psi(\binom{n}{s}2^{1-s}u^s)+ \\ + \limsup_{u\to 0} \frac{1}{\psi(u)} \sum_{s=2+[\frac{n}{k}]}^{n} \psi(\binom{n}{s}2^{1-s}u^s)\end{aligned} \tag{3.21}$$

Consider the first upper limit in (3.21). Since ψ is concave, we have

$$\sum_{s=1}^{1+[\frac{n}{k}]} \psi(\binom{n}{s}2^{1-s}u^s) \leq (1+[\frac{n}{k}])\psi(\frac{1}{1+[\frac{n}{k}]} \sum_{s=1}^{1+[\frac{n}{k}]} \binom{n}{s}2^{1-s}u^s) =$$

$$= (1+[\frac{n}{k}])\psi(\frac{nu(1+o(1))}{1}1+[\frac{n}{k}]) \leq (1+[\frac{n}{k}])\psi(ku(1+o(1))).$$

Therefore,

$$\limsup_{u\to 0} \frac{1}{\psi(u)} \sum_{s=1}^{1+[\frac{n}{k}]} \psi(\binom{n}{s}2^{1-s}u^s) \leq (1+[\frac{n}{k}]) \limsup_{u\to 0} \frac{\psi(ku(1+o(1)))}{\psi(u)} \leq (1+\frac{n}{k})a_\psi.$$

Consider the second upper limit in (3.21). It is clear that for all $\frac{1}{k}n \leq s \leq n$

$$\binom{n}{s} \cdot 2^{1-s} \leq 2^n$$

and

$$\binom{n}{s}2^{1-s}u^s \leq 2^n u^{\frac{1}{k}n} = (2^k u)^{\frac{1}{k}n}.$$

Thus, the second upper limit in (3.21) can be estimated as

$$\limsup_{u\to 0} \frac{1}{\psi(u)} \sum_{s=2+[\frac{n}{k}]}^{n} \psi(\binom{n}{s}2^{1-s}u^s) \leq n(1-\frac{1}{k}) \limsup_{u\to 0} \frac{\psi((2^k u)^{\frac{n}{k}})}{\psi(u)}.$$

Substituting the variable $w = 2^k u$ on the right hand side, we have

$$n(1-\frac{1}{k}) \limsup_{w\to 0} \frac{\psi(w^{\frac{n}{k}})}{\psi(2^{-k}w)}.$$

By the concavity of ψ, we have $\psi(2^{-k}w) \geq 2^{-k}\psi(w)$. Therefore, the second upper limit in (3.21) is bounded from above by

$$n(1-\frac{1}{k})2^k \limsup_{w\to 0} \frac{\psi(w^{\frac{n}{k}})}{\psi(w)}.$$

Now, we observe that

$$\limsup_{w\to 0} \frac{\psi(w^m)}{\psi(w)} \leq c_\psi^{\frac{\log(m)}{\log(l)}-1}. \tag{3.22}$$

Indeed, let $l^r \leq m \leq l^{r+1}$,

$$\frac{\psi(w^m)}{\psi(w)} \leq \frac{\psi(w^{l^r})}{\psi(w)} = \frac{\psi(w^{l^r})}{\psi(w^{l^{r-1}})} \cdots \frac{\psi(w^l)}{\psi(w)}$$

and

$$\limsup_{w\to 0} \frac{\psi(w^m)}{\psi(w)} \leq c_\psi^r \leq c_\psi^{\frac{\log(m)}{\log(l)}-1}.$$

If n tends to infinity, then, thanks to the assumption $c_\psi < 1$, we have

$$n(1-\frac{1}{k})2^k \limsup_{w\to 0} \frac{\psi(w^{\frac{n}{k}})}{\psi(w)} = o(n).$$

Therefore, the upper limit in (3.20) (see also (3.21)) is bounded from above by

$$\frac{a_\psi}{k}n + o(n).$$

Thus, the upper limit in (3.20) is strictly less than n for every sufficiently large n. □

Lemma 3.6.5. *Suppose, that* (3.20) *holds for some n. Then, there exist $k \geq 2$ such that* (3.18) *holds and $l \geq 2$ such that* (3.19) *holds.*

Proof. Clearly,

$$\limsup_{u\to 0}\frac{\psi(nu)}{n\psi(u)} = \limsup_{u\to 0}\frac{\psi(2^{1-1}\binom{n}{1}u^1)}{n\psi(u)} \le \limsup_{u\to 0}\frac{1}{n\psi(u)}\sum_{s=1}^{n}\psi(2^{1-s}\binom{n}{s}u^s) < 1.$$

Thus, (3.18) holds for $k = n$.

Since $\binom{n}{s}2^{1-s}u^s \ge u^{n+1}$ for every $s = 1, 2, \cdots, n$ and every sufficiently small u, we have

$$1 > \limsup_{u\to 0}\frac{1}{n\psi(u)}\sum_{s=1}^{n}\psi(\binom{n}{s}2^{1-s}u^s) \ge \limsup_{u\to 0}\frac{n\psi(u^{n+1})}{n\psi(u)}.$$

Thus, (3.19) holds for $l = n+1$. □

The following theorem is the main result in this section.

Theorem 3.6.6. *Let ψ be increasing concave function. The following conditions are equivalent.*

1. $\|A_n\|_{\Lambda_\psi\to\Lambda_\psi} \le \mathrm{const}\cdot n^q$ *for some* $0 < q < 1$ *and for all* $n \in \mathbb{N}$.
2. *Estimates* (3.18) *and* (3.19) *hold for some* $k \ge 2$ *and* $l \ge 2$.

Proof. [Sufficiency] Without loss of generality, ψ is a strictly monotone function. It follows from Lemma 3.6.2, Lemma 3.6.3 and Proposition 3.6.4 that there exists $n \in \mathbb{N}$ such that

$$\sup_{0<u\le 1}\frac{1}{n\psi(u)}\|A_n\chi_{[0,u]}\|_{\Lambda_\psi} < 1.$$

Hence, there exists $n \in \mathbb{N}$ and $c < n$ such that

$$\|A_n\chi_A\|_{\Lambda_\psi} \le c\|\chi_A\|_{\Lambda_\psi}$$

for every measurable set $A \subset [0,1]$.

It follows from [33, Lemma II.5.2] that

$$\|A_n x\|_{\Lambda_\psi} \le c\|x\|_{\Lambda_\psi}$$

for every $x \in \Lambda_\psi$.

The assertion follows from Theorem 3.5.1.

[Necessity] Fix n such that $\|A_n\|_{\Lambda_\psi\to\Lambda_\psi} = c < n$. In particular, $\|A_n\chi_{[0,u]}\|_{\Lambda_\psi} \le c\psi(u)$ for every $u \in (0,1]$. It follows now from Lemma 3.6.3 that

$$\limsup_{u\to 0}\frac{1}{n\psi(u)}\sum_{s=1}^{n}\psi(2^{1-s}\binom{n}{s}u^s) < 1.$$

The assertion follows now from Lemma 3.6.5. □

Remark 3.6.7. *The condition* (3.18) *is equivalent to the assumption* $\beta_{\Lambda_\psi} < 1$. *The condition* (3.19) *follows from (but is not equivalent to) the condition* $\alpha_{\Lambda_\psi} > 0$.

Example 3.6.8. *Define the concave increasing function* ψ *by the formula* $\psi(t) = \log^{-1/2}(e^2/t)$, $t \in [0,1]$. *It follows that*

$$\|A_n\|_{\Lambda_\psi \to \Lambda_\psi} \leq \text{const} \cdot n^q, \quad 0 < q < 1, \forall n \in \mathbb{N}.$$

However, Λ_ψ *does not have the Kruglov property.*

Proof. For every $k, l > 1$ we have

$$\lim_{u \to 0} \frac{\psi(ku)}{\psi(u)} = \lim_{u \to 0} (\frac{\log(u)}{\log(ku)})^{\frac{1}{2}} = 1 < k$$

and

$$\lim_{u \to 0} \frac{\psi(u^l)}{\psi(u)} = \lim_{u \to 0} (\frac{\log(u)}{\log(u^l)})^{\frac{1}{2}} = \frac{1}{l^{\frac{1}{2}}} < 1.$$

It follows from Theorem 3.6.6 that

$$\|A_n\|_{\Lambda_\psi \to \Lambda_\psi} \leq \text{const} \cdot n^q, \quad 0 < q < 1, \forall n \in \mathbb{N}.$$

It is clear that

$$\Lambda_{\log^{-1/2}(1/t)} \subset M_{t \log^{1/2}(1/t)} \subset M_{t \log(e/t)} = \exp(L_1).$$

However, Theorem 4.2.6 says that Λ_ψ must contain $\exp(L_1)$ if $\Lambda_\psi \in \mathbb{K}$. This implies that $\Lambda_\psi \notin \mathbb{K}$. □

3.7 The operators A_n, $n \geq 1$ in the Orlicz spaces $\exp(L_p)$

Theorem 3.7.1. *The following norm estimates of the operators* A_n, $n \in \mathbb{N}$, *are valid in the spaces* $\exp(L_p)$.

1. *For every* $0 \leq p \leq 2$, *we have* $\|A_n\|_{\exp(L_p) \to \exp(L_p)} \leq \text{const} \cdot n^{1/2}, \quad n \in \mathbb{N}$.
2. *For every* $2 \leq p \leq \infty$, *we have* $\|A_n\|_{\exp(L_p) \to \exp(L_p)} \leq \text{const} \cdot n^{1-1/p}, \quad n \in \mathbb{N}$.

Proof. By Lemma 1.2.74, $\exp(L_2) = M_\psi$, where $\psi' \otimes r$ is Gaussian. The only extreme points of the unit ball in M_ψ are the functions equimeasurable with ψ'. Therefore,

$$\|A_n \xi\|_{M_\psi} = \|A_n \psi'\|_{M_\psi} = \|A_n\|_{M_\psi \to M_\psi}, \quad \forall n \in \mathbb{N}.$$

However, $A_n \xi$ is equimeasurable with $n^{1/2}\xi$. Hence,

$$\|A_n\|_{\exp(L_2) \to \exp(L_2)} \leq \text{const} \cdot n^{1/2}. \tag{3.23}$$

If $0 < p \leq 1$, then the space $\exp(L_p)$ satisfies the Kruglov property. It follows from Theorem 3.2.4 that

$$\|A_n x\|_{\exp(L_p)} \leq \mathrm{const}(\|\sigma_n(x^*)\|_{\exp(L_p)} + n^{1/2}\|x\|_2).$$

Therefore,

$$\|A_n\|_{\exp(L_p)\to\exp(L_p)} \leq \mathrm{const} \cdot n^{1/2}, \quad 0 < p \leq 1, \quad n \in \mathbb{N}. \tag{3.24}$$

If $1 < p < 2$, then

$$[\exp(L_1), \exp(L_2)]_{\theta,\infty} = \exp(L_p)$$

with $\frac{1}{p} = 1 - \theta/2$ (see, for example [14]). Here, the notion $[\cdot,\cdot]_{\theta,\infty}$ denotes the real interpolation method. If $2 < p < \infty$, then

$$[\exp(L_2), L_\infty]_{\theta,\infty} = \exp(L_p)$$

with $\frac{1}{p} = (1-\theta)/2$ (see, for example [14]).

In both cases the assertion follows immediately by interpolation from (3.23) and (3.24). □

Chapter 4

Complementary results

The results of this section were mostly published in [7] (see also [56]).

4.1 No minimal space in the class $\mathbb{K}$

We need the following lemma proved in [6].

Lemma 4.1.1. *Let K be the Kruglov operator and let ψ_n be a piecewise-linear concave function such that $\psi_n' = (K^n 1)^*$. Then*

1. $M_{\psi_n} \subset M_{\psi_{n+1}}$.
2. $M_{\psi_{n+1}} \neq M_{\psi_n}$.

It follows from Lemma 4.1.1 that

$$L_\infty = M_{\psi_0} \subset M_{\psi_1} \subset \cdots \subset (\exp(L_1))_0.$$

In a certain sense the spaces M_{ψ_n}, $n \geq 1$ can be treated as "approximations" of the space $(\exp(L_1))_0$. By [6, Theorem 7.2], we have $M_{\psi_n} \subset E$ for every symmetric space $E \in \mathbb{K}$ and every $n = 1, 2, \ldots$ This suggests a rather natural hypothesis that the space $(\exp(L_1))_0$ is the minimal symmetric space among the class of all symmetric spaces with the Kruglov property. However, the latter class has no minimal element (see Theorem 4.1.8).

Lemma 4.1.2. *For every $x \in L_1[0,1]$*

$$\lim_{n\to\infty} m(\operatorname{supp} K^n x) = 0.$$

Proof. Without loss of generality, we may assume that $x > 0$. Set

$$a_n := m(\{t : \ (K^n x)(t) = 0\}), \quad n \in \mathbb{N}.$$

It follows from the definition of the operator K that

$$a_{n+1} = \frac{1}{e} + \frac{1}{e}\sum_{k=1}^{\infty} \frac{a_n^k}{k!} = e^{a_n - 1}, \quad \forall n \in \mathbb{N}.$$

Evidently, the sequence $\{a_n\}$ increases and $a_n \in [0,1]$. Hence, $\{a_n\}$ converges to the fixed point of the mapping

$$t \longrightarrow e^{t-1}, \quad t > 0.$$

However, the only fixed point of the above mapping is 1. Therefore,

$$\lim_{n\to\infty} a_n = 1,$$

which proves the lemma. □

Construction 4.1.3. *Let* ψ_n, $n \in \mathbb{N}$, *be as in Lemma 4.1.1. For every fixed* $\varepsilon > 0$, *define the concave function* ψ_ε *by the formula*

$$\psi_\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \psi_n. \tag{4.1}$$

Lemma 4.1.4. *The function* ψ_ε *defined in* (4.1) *is concave and piecewise-linear. Moreover,*

$$\psi'_\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \psi'_n. \tag{4.2}$$

Proof. Since $\psi'_n = (K^n 1)^*$, it follows that $\operatorname{supp}(\psi'_n) = [0, a_n]$ with $a_n \to 0$. Therefore, for any fixed value of t, the series in (4.2) is nothing more than a finite sum. In particular, it converges almost everywhere. By the Levi theorem,

$$\int_0^t \sum_{n=0}^{\infty} \varepsilon^n \psi'_n(s)ds = \sum_{n=0}^{\infty} \varepsilon^n \int_0^t \psi'_n(s)ds = \psi_\varepsilon.$$

This implies the assertion of (4.2).

On any fixed interval $[\delta, 1]$ the function ψ'_ε is piecewise-constant because of (4.2). Thus, ψ'_ε is piecewise-constant and ψ_ε is piecewise-linear. □

Lemma 4.1.5. *Let* E *be a fully symmetric quasi-Banach space. If* $K : E \to E$, *then* $M_{\psi_\varepsilon} \subset E$ *for every sufficiently small* ε.

Proof. Indeed, if $K : E \to E$, then

$$\|\psi'_n\|_E = \|K^n 1\|_E \leq \|K\|^n_{E\to E} \|1\|_E = \|K\|^n_{E\to E}.$$

Therefore, if $\varepsilon \cdot \|K\|_{E\to E} < 1$, then the series (4.2) converges in E.

Hence, $\psi'_\varepsilon \in E$ and $M_{\psi_\varepsilon} \subset E$. □

Lemma 4.1.6. *For every* $\varepsilon < 1$, *the operator* K *maps* M_{ψ_ε} *to* M_{ψ_ε}.

Proof. Let us prove that the operator K is bounded in M_{ψ_ε}. The extreme points of the unit ball in this space are equimeasurable with ψ'_ε [48]. Therefore, it is sufficient to show that $K\psi'_\varepsilon \in M_{\psi_\varepsilon}$. Since K is bounded in L_1, it follows that

$$K\psi'_\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n K\psi'_n \prec \sum_{n=0}^{\infty} \varepsilon^n \psi'_{n+1} \leq \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n \psi'_n = \frac{1}{\varepsilon} \psi'_\varepsilon.$$

Here, the first inequality follows from $K\psi'_n \sim \psi'_{n+1}$ and (1.7). Thus, $K\psi'_\varepsilon \in M_{\psi_\varepsilon}$. □

Lemma 4.1.7. *Functions* ψ_ε, $\varepsilon > 0$, *are not equivalent. More precisely,*

$$\lim_{t\to 0} \frac{\psi_\varepsilon(t)}{\psi_\delta(t)} = 0 \tag{4.3}$$

if $0 < \varepsilon < \delta < 1$.

Proof. Arguing as in the proof of Theorem 7.2 in [6], one can obtain

$$\lim_{t\to 0} \frac{\psi_m(t)}{\psi_{m+1}(t)} = 0$$

for every $m = 1, 2, \ldots$. It follows that

$$\lim_{t\to 0} \frac{1}{\psi_\varepsilon(t)} (\sum_{n=m}^{\infty} \varepsilon^n \psi_n) = 1.$$

Hence,

$$\limsup_{t\to 0} \frac{\psi_\varepsilon(t)}{\psi_\delta(t)} = \limsup_{t\to 0} \frac{\sum_{n=m}^{\infty} \varepsilon^n \psi_n}{\sum_{n=m}^{\infty} \delta^n \psi_n}.$$

However, for $n \geq m$, we have $\varepsilon^n \leq \delta^n (\varepsilon/\delta)^m$. Therefore,

$$\limsup_{t\to 0} \frac{\psi_\varepsilon(t)}{\psi_\delta(t)} \leq (\varepsilon/\delta)^m.$$

Since m is arbitrary, it follows that

$$\limsup_{t\to 0} \frac{\psi_\varepsilon(t)}{\psi_\delta(t)} = 0.$$

The assertion follows immediately. □

Theorem 4.1.8. *The class $\mathbb{K}$ of symmetric quasi-Banach spaces does not contain a minimal element. That is, there is no symmetric space F such that $K : E \to E$ implies $F \subset E$.*

Proof. Assume the contrary. It follows from Lemma 4.1.5 that

$$M_{\psi_\varepsilon} \subset F$$

for $0 < \varepsilon < \varepsilon_0$.

However, the space $M_{\psi_\varepsilon} \in \mathbb{K}$ by Lemma 4.1.6. By assumption,

$$M_{\psi_\varepsilon} \supset F.$$

Therefore, all the spaces M_{ψ_ε} for $0 < \varepsilon < \varepsilon_0$ coincide with F.

On the other hand, these spaces cannot coincide because of Lemma 4.1.7. □

4.2 Lorentz spaces from the class $\mathbb{K}$

Despite Theorem 4.1.8, one can restrict the question of minimality to Lorentz spaces only. We will prove that all Lorentz spaces with the Kruglov property are "on one side" of the space $\exp(L_1)$.

Let us prove the following preliminary results.

Lemma 4.2.1. *Let* a_n, $n \in \mathbb{N}$ *be decreasing positive sequence and let* b_n, c_n, $n \in \mathbb{N}$ *be such that* $\sum_{k=1}^{n} b_k \leq \sum_{k=1}^{n} c_k$ *for every* $n \in \mathbb{N}$. *It follows that*

$$\sum_{k=1}^{n} a_k b_k \leq \sum_{k=1}^{n} a_k c_k$$

for every $n \in \mathbb{N}$.

Proof. Set $B_n = \sum_{k=1}^{n} b_n$ and $C_n = \sum_{k=1}^{n} c_n$. Also, $B_0 = C_0 = 0$. It follows that

$$\sum_{k=1}^{n} a_k b_k = a_n B_n + \sum_{k=1}^{n} (a_{k-1} - a_k) B_{k-1} \leq$$

$$\leq a_n C_n + \sum_{k=1}^{n} (a_{k-1} - a_k) C_{k-1} = \sum_{k=1}^{n} a_k c_k.$$

□

We need the following number-theoretic estimate.

Lemma 4.2.2. *For every* $n \in \mathbb{N}$ *we have*

$$\sum_{k=1}^{n} \tau(k) \geq \log(n!). \tag{4.4}$$

Proof. It is clear that

$$\sum_{k=1}^{n} \tau(k) = \sum_{k=1}^{n} [\frac{n}{k}] = n \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \{\frac{n}{k}\}.$$

If $k \geq 3n/4$, then $1 \leq n/k \leq 4/3$ and $\{n/k\} \leq 1/3$. Therefore,

$$\sum_{k=1}^{n} \{\frac{n}{k}\} \leq \frac{1}{3} \cdot (\frac{n}{4} + 1) + \frac{3n}{4} = \frac{5n}{6} + \frac{1}{3}.$$

Note that

$$\sum_{k=1}^{n} \frac{1}{k} \geq \log(n).$$

Therefore,

$$\sum_{k=1}^{n} \tau(k) \geq n \log(n) - \frac{5}{6} n - \frac{1}{3}.$$

On the other hand, by Stirling formula,

$$\log(n!) \leq n \log(n) - n + \frac{1}{2} \log(2\pi n) + \frac{1}{12n}.$$

Therefore, for every $n \geq 2$,

$$\sum_{k=1}^{n} \tau(k) - \log(n!) \geq \frac{1}{6}(n - 3\log(n) - 8).$$

It is clear that the latter expression is positive for $n \geq 17$. If $n \leq 16$, the assertion can be verified directly. ☐

Lemma 4.2.3. *Let φ be an increasing function on the interval $(0,1)$. If*

$$\sum_{j=1}^{\infty} \varphi(\frac{t^j}{j!}) \leq M \cdot \varphi(t) \tag{4.5}$$

for every $t \in (0,1)$, then for every $l \in \mathbb{N}$

$$2M \sum_{n=1}^{l} \varphi(e^{-n}) \geq \sum_{n=1}^{l} \log(n)\varphi(e^{-2n}). \tag{4.6}$$

Proof. Note, that $e^{-2ij} \leq e^{-ij}/j!$ if $j \leq i$. Substitute $t = e^{-i}$ in the inequality (4.5). We obtain that

$$\sum_{j=1}^{i} \varphi(e^{-2ij}) \leq \sum_{j=1}^{i} \varphi(\frac{e^{-ij}}{j!}) \leq M\varphi(e^{-i}).$$

Therefore,

$$M \sum_{i=1}^{l} \varphi(e^{-i}) \geq \sum_{1 \leq j \leq i \leq l} \varphi(e^{-2ij}) \geq \frac{1}{2} \sum_{n=1}^{l} \tau(n)\varphi(e^{-2n}).$$

Here, $\tau(n)$ is the number of divisors of n. It follows from Lemma 4.2.1 and (4.4) that

$$\sum_{n=1}^{l} \tau(n)\varphi(e^{-2n}) \geq \sum_{n=1}^{l} \log(n)\varphi(e^{-2n}).$$

The assertion follows immediately. ☐

Lemma 4.2.4. *Let φ be an increasing function on the interval $(0,1)$ such that* (4.6) *is valid. It follows that*

$$\sum_{n=0}^{\infty} \varphi(e^{-n}) \leq 20\exp(5M)\varphi(1). \tag{4.7}$$

Proof. It is clear that

$$\sum_{n=1}^{l} \varphi(e^{-n}) = \varphi(e^{-1}) + \sum_{n=1}^{[l/2]} \varphi(e^{-2n}) + \sum_{n=1}^{[(l-1)/2]} \varphi(e^{-2n}) \leq 2\sum_{n=0}^{l} \varphi(e^{-2n}).$$

It follows now from (4.6) that

$$4M\sum_{n=0}^{l}\varphi(e^{-2n}) \geq 2M\sum_{n=1}^{l}\varphi(e^{-n}) \geq \sum_{n=1}^{l}\log(n)\varphi(e^{-2n}).$$

Fix minimal $n_0 > \exp(5M)$ so that $\log(k) \geq 5M$ for $k \geq n_0$. Therefore,

$$4M\sum_{n=0}^{l}\varphi(e^{-2n}) \geq 5M\sum_{n=n_0}^{l}\varphi(e^{-2k})$$

and

$$4\sum_{n=0}^{n_0-1}\varphi(e^{-2n}) \geq \sum_{n=n_0}^{l}\varphi(e^{-2n}).$$

Since l is arbitrary, it follows that

$$\sum_{n=0}^{\infty}\varphi(e^{-2n}) \leq 5\sum_{n=0}^{n_0-1}\varphi(e^{-2n}) \leq 5n_0\varphi(1)$$

and so

$$\sum_{n=0}^{\infty}\varphi(e^{-n}) \leq 2\sum_{n=0}^{\infty}\varphi(e^{-2n}) \leq 10n_0\varphi(1).$$

The assertion follows immediately. □

Lemma 4.2.5. *In every Lorentz space* Λ_φ,

$$\|\log(1/t)\|_{\Lambda_\varphi} \leq \sum_{n=0}^{\infty}\varphi(e^{-n}).$$

Proof. The inequality is a consequence of the following estimates:

$$\|\log(1/t)\|_{\Lambda_\varphi} = \int_0^1 \log(1/t)\,d\varphi(t) = \sum_{n=0}^{\infty}\int_{e^{-n}}^{e^{-n-1}}\log(1/t)\,d\varphi(t) \leq$$

$$\leq \sum_{n=0}^{\infty}(n+1)(\varphi(e^{-n}) - \varphi(e^{-n-1})) = \sum_{n=0}^{\infty}\varphi(e^{-n}) < \infty.$$

□

Theorem 4.2.6. *Let* φ *be an increasing concave function on the interval* $[0,1]$ *such that* $\varphi(0) = 0$. *If* $\Lambda_\varphi \in \mathbb{K}$, *then* $\Lambda_\varphi \supset \exp L_1$.

Proof. According to Theorem 1.1.3, the condition (4.5) is satisfied if $\Lambda_\varphi \in \mathbb{K}$. Lemma 4.2.3 and Lemma 4.2.4 imply that the condition (4.7) is also satisfied. According to Lemma 4.2.5, $\log(1/t) \in \Lambda_\varphi$. Since Lorentz spaces are fully symmetric, it follows that the Marcinkiewicz space M_ψ with $\psi(t) = t\log(e/t)$ is a subset of Λ_φ. However, M_ψ coincides with $\exp(L_1)$. □

4.3 Uniform boundedness of the sequence $\{T_n\}_{n\in\mathbb{N}}$ implies boundedness of the Kruglov operator

For every $m \in \mathbb{N}$, let H_m be the operator defined in (3.1). For any fixed $n \in \mathbb{N}$ and $a = (a_1, \cdots, a_n) \in \mathbb{R}_+^n$, we set

$$H_m a = H_m(\sum_{k=1}^{n} a_k \chi_{((k-1)/n,k/n)}).$$

For any fixed $n \in \mathbb{N}$ and $a \in \mathbb{R}_+^n$, we define the sequence $\sigma_m a \in \mathbb{R}_+^{mn}$ by the formula

$$\sigma_m a = (\underbrace{a_1, a_1, \dots, a_1}_{m}, \underbrace{a_2, a_2, \dots, a_2}_{m}, \dots, \underbrace{a_n, a_n, \dots, a_n}_{m}).$$

It is clear that $C_{mn}(\sigma_m a) = C_n a$.

Definition 4.3.1. *Let $Ch(n)$ be the number of permutations π of the set $\{1, 2, \dots, n\}$, such that $\pi(i) \neq i$ for every $i = 1, 2, \dots, n$.*

It is well-known (see [26, c. 20]) that

$$Ch(n) = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}).$$

Therefore, $Ch(n)$ is a closest natural number to $e^{-1}n!$ and

$$\frac{1}{3}n! \leq Ch(n) \leq n!, \quad \forall n \in \mathbb{N}. \tag{4.8}$$

We are going to estimate the rearrangement of $H_m a$ by that of $T_{nm}(\sigma_m a)$.

Sometimes it is useful to require numbers a_k, $1 \leq k \leq n$ to be independent over $\mathbb{Z}$. That is, we require any $\mathbb{Z}-$linear combination of a_i to be non-zero if coefficients are not all equal to 0. In this case,

$$\sum_{i=1}^{n} k_i a_i \neq \sum_{i=1}^{n} k_i' a_i \quad \text{if} \quad (k_1, k_2, \dots, k_n) \neq (k_1', k_2', \dots, k_n').$$

The function $H_m a$ only take values of the form $\sum_{i=1}^n k_i a_i$, where $k_i \in \mathbb{Z}$, $k_i \geq 0$ for every $i = 1, 2, \dots, n$ and $\sum_{i=1}^n k_i \leq m$.

Lemma 4.3.2. *If numbers a_i, $1 \leq i \leq n$, are independent over $\mathbb{Z}$, then*

$$m(\{\omega : \ (H_m a)(\omega) = \sum_{i=1}^{n} k_i a_i\}) \leq C_m^{m-q,k_1,\cdots,k_n} \left(\frac{1}{mn}\right)^q.$$

Here, $q = \sum_{i=1}^n k_i$.

Proof. It is clear that the function $H_m a$ equals $\sum_{i=1}^n k_i a_i$ if and only if exactly k_i (respectively, $m-q$) of the independent copies of the function

$$\sigma_{\frac{1}{m}} \sum_{k=1}^{n} a_k \chi_{((k-1)/n,k/n)}$$

take the value a_i (respectively, 0). Note, that

$$m(\{\omega_s : \ (\sigma_{1/m}(\sum_{k=1}^{n} a_k \chi_{((k-1)/n,k/n)}))(\omega_s) = a_i\}) = \frac{1}{mn}$$

and

$$m(\{\omega_s : \ (\sigma_{1/m}(\sum_{k=1}^{n} a_k \chi_{((k-1)/n,k/n)}))(\omega_s) = 0\}) = 1 - \frac{1}{m}.$$

Therefore, we obtain

$$m(\{\omega : \ (H_m a)(\omega) = \sum_{i=1}^{n} k_i a_i\}) = C_m^{m-q,k_1,\cdots,k_n}(1-\frac{1}{m})^{m-q}(\frac{1}{mn})^{k_1+\cdots+k_n} \leq$$

$$\leq C_m^{m-q,k_1,\cdots,k_n}(\frac{1}{mn})^q.$$

□

On the other hand, the function $T_{nm}(\sigma_m a)$ only takes values of the form $\sum_{i=1}^n k_i a_i$, where $k_i \in \mathbb{Z}$, $k_i \geq 0$ for every $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n k_i \leq mn$.

Lemma 4.3.3. *If the numbers a_i, $1 \leq i \leq n$, are independent over $\mathbb{Z}$, then*

$$m(\{t : \ (T_{mn}(\sigma_m a))(t) = \sum_{i=1}^{n} k_i a_i\}) \geq \frac{1}{3} C_m^{m-q,k_1,\cdots,k_n} \left(\frac{1}{mn}\right)^q.$$

Here, $q = \sum_{i=1}^n k_i$.

Proof. It follows from (1.3) and (4.8) that

$$m(\{t : \ (T_{mn}(\sigma_m a))(t) = \sum_{i=1}^{n} k_i a_i\}) = C_m^{k_1} C_m^{k_2} \ldots C_m^{k_n} Ch(mn-q)\frac{1}{(mn)!} \geq$$

$$\geq \frac{(m!)^n (mn-q)!}{3(m-k_1)!\cdots(m-k_n)!k_1!\cdots k_n!(mn)!}.$$

Since

$$(m-k_1)!\cdots(m-k_n)! \leq (m!)^{n-1}(m-q)!$$

and

$$\frac{(mn-q)!}{(mn)!} \geq \frac{1}{(mn)^q},$$

we have

$$m(\{t:\,(T_{mn}(\sigma_m a))(t)=\sum_{i=1}^{n}k_i a_i\})\geq\frac{m!(mn-q)!}{3k_1!\cdots k_n!(m-q)!(mn)!}\geq$$

$$\geq\frac{1}{3}C_m^{m-q,k_1,\cdots,k_n}\left(\frac{1}{mn}\right)^q.$$

□

Lemma 4.3.4. *For any $n,m\in\mathbb{N}$ and any $a\in\mathbb{R}^n$, we have*

$$(H_m a)^*\leq\sigma_3(T_{mn}(\sigma_m a))^*.$$

Proof. Without loss of generality, the numbers a_k, $1\leq k\leq n$, are independent over $\mathbb{Z}$.

It follows from Lemma 4.3.2 and Lemma 4.3.3 that

$$m(\{t:\,H_m a(t)=\sum_{i=1}^{n}k_i a_i\})\leq 3m(\{t:\,(T_{mn}(\sigma_m a))(t)=\sum_{i=1}^{n}k_i a_i\}).$$

Thus, for $\tau>0$, we have

$$m(\{t:\,H_m a(t)>\tau\})\leq 3m(\{t:\,(T_{mn}(\sigma_m a))(t)>\tau\}).$$

The required estimate follows immediately. □

Lemma 4.3.5. *Let E be a symmetric quasi-Banach space. If $K:E\to E$, then $K:(E)_0\to(E)_0$.*

Proof. Let ψ_n be piecewise-linear concave functions such that $\psi_n'=(K^n 1)^*$. Since $K:E\to E$, it follows that $\psi_2\in E$ and

$$M_{\psi_1}\subset(M_{\psi_2})_0\subset(E)_0.$$

Thus, $K:L_\infty\to(E)_0$.

Let $x\in(E)_0$. Since E is separable, there exists a sequence $x_n\in L_\infty$ such that $\|x_n-x\|_E\to 0$. Since $K:E\to E$, it follows that $\|Kx_n-Kx\|_E\to 0$. However, $Kx_n\in(E)_0$ and, therefore, $Kx\in(E)_0$. □

Theorem 4.3.6. *Let E be a symmetric quasi-Banach with Fatou property. If the operators $T_n:E\to E$ are uniformly bounded, then K maps E into E. Moreover,*

$$\|K\|_{E\to E}\leq 3\sup_{k\in\mathbb{N}}\|T_k\|_{E\to E}.$$

Proof. Let $a\in\mathbb{R}^n$. It follows that $C_n a\in L_\infty$ and, therefore, $C_n a\in E$. It follows from Lemma 4.3.4 that

$$\|H_m(C_n a)\|_E\leq 3\sup_{k\in\mathbb{N}}\|T_n\|_{E\to E}\|C_n a\|_E. \tag{4.9}$$

Since $H_m(C_n a)$ converges to $K(C_n a)$ in distribution, it follows that $(H_m(C_n a))^* \to (K(C_n a))^*$ almost everywhere. It follows from (4.9) and the fact that E has the Fatou property that

$$\|K(C_n a)\|_E \leq 3 \sup_{k\in\mathbb{N}} \|T_k\|_{E\to E} \|C_n a\|_E. \tag{4.10}$$

Let now $x = x^* \in E$ be arbitrary. For every $n \in \mathbb{N}$, define the function $x_n \in E$ by the formula

$$x_n = \sum_{k=1}^{2^n} x(k2^{-n})\chi_{((k-1)2^{-n}, k2^{-n})}.$$

It is clear that $x_n \uparrow x$ almost everywhere. Therefore, x_n converges to x in distribution. Hence, φ_{x_n} converges to φ_x. Thus,

$$\varphi_{Kx_n} = \exp(\varphi_{x_n} - 1) \to \exp(\varphi_x - 1) = \varphi_{Kx}.$$

It follows Kx_n converges to Kx in distribution and, therefore, $(Kx_n)^*$ converges to $(Kx)^*$ almost everywhere. Since inequality (4.10) is valid for x_n, it follows that

$$\|Kx\|_E \leq 3 \sup_{k\in\mathbb{N}} \|T_k\|_{E\to E} \|x\|_E.$$

□

Theorem 4.3.7. *Let E be a separable symmetric Banach space. If the operators $T_n : E \to E$ are uniformly bounded, then K maps E into E. Moreover,*

$$\|K\|_{E\to E} \leq 3 \sup_n \|T_n\|_{E\to E}.$$

Proof. If E is separable, then the natural inclusion $E \to E^{\times\times}$ is an isometry. Repeating the previous argument, we obtain $K : E^{\times\times} \to E^{\times\times}$. The assertion follows now from Lemma 4.3.5. □

4.4 Boundedness of the Kruglov operator implies uniform boundedness of the sequence $\{T_n\}_{n\in\mathbb{N}}$

Lemma 4.4.1. *If $n, k \in \mathbb{N}$, $1 \leq k \leq n$, then*

$$\frac{(n-k)!}{n!(k-1)!} \leq \frac{3}{n^k}.$$

Proof. Assume that $n \geq 4$. Since $j(n-j) \geq n$ for $2 \leq j \leq n-2$, we have

$$\frac{n^k(n-k)!}{n!(k-1)!} = \prod_{j=1}^{k-1} \frac{n}{j(n-j)} \leq \left(\frac{n}{n-1}\right)^2 < 3.$$

If $n \leq 3$, the assertion is evident. □

Let $A = \{1, 2, \cdots, n\}$ and let 2^A be the collection of all subsets of the set A. Denote

$$S(U) := \sum_{j\in U} a_j$$

for every $U \in 2^A$. Note that if numbers a_k, $1 \le k \le n$, are independent over $\mathbb{Z}$, then $S(U_1) \neq S(U_2)$ if $U_1 \neq U_2$.

For every $U \in 2^A$ denote

$$\mathcal{A}_U = \{V \in 2^A,\ |V| = 2|U|,\ V \supset U,\ S(V\backslash U) \le S(U)\}$$

and

$$\mathcal{B}_U = \{V \in 2^A,\ |V| = 2|U| - 1,\ V \supset U,\ S(V\backslash U) \le S(U)\}.$$

It clearly follows that

$$|\mathcal{A}_U| \le C_{n-|U|}^{|U|} = \frac{(n-|U|)!}{|U|!(n-2|U|)!} \tag{4.11}$$

and similarly

$$|\mathcal{B}_U| \le C_{n-|U|}^{|U|-1} = \frac{(n-|U|)!}{(|U|-1)!(n-2|U|+1)!}. \tag{4.12}$$

On the other hand,

$$\{V \in 2^A :\ |V| \text{ is even}\} = \bigcup_U \mathcal{A}_U$$

and

$$\{V \in 2^A :\ |V| \text{ is odd}\} = \bigcup_U \mathcal{B}_U.$$

Therefore,

$$2^A = \bigcup_U (\mathcal{A}_U \cup \mathcal{B}_U). \tag{4.13}$$

It follows from the definition of $\mathcal{A}_U$ and $\mathcal{B}_U$ that for every $V \in \mathcal{A}_U \cup \mathcal{B}_U$

$$S(U) \le S(V) \le 2S(U). \tag{4.14}$$

Lemma 4.4.2. *Let $n \in \mathbb{N}$ and $a = (a_1, \cdots, a_n) \in \mathbb{R}_+^n$ be such that the coordinates a_k, $1 \le k \le n$, are independent over $\mathbb{Z}$. It follows that, for every $\tau > 0$,*

$$m(\{t :\ T_n a(t) > \tau\}) \le 6 \sum_{S(U)>\tau/2} \frac{1}{n^{|U|}}.$$

Proof. It is clear that

$$m(\{t :\ T_n a(t) > \tau\}) = \sum_{V\in 2^A, S(V)>\tau} m(\{t :\ T_n a(t) = S(V)\}).$$

It follows from the equation (4.13) that

$$\sum_{V\in 2^A} \leq \sum_U \Big(\sum_{V\in\mathcal{A}_U} + \sum_{V\in\mathcal{B}_U} \Big).$$

If $V \in \mathcal{A}_U$ or $V \in \mathcal{B}_U$ and $S(V) > \tau$, then $S(U) > \tau/2$. Therefore,

$$m(\{t:\ T_na(t) > \tau\}) \leq \sum_{S(u)>\tau/2} \Big(\sum_{V\in\mathcal{A}_U} m(\{t:\ T_na(t) = S(V)\}) +$$

$$+ \sum_{V\in\mathcal{A}_U} m(\{t:\ T_na(t) = S(V)\}) \Big).$$

If $V \in \mathcal{A}_U$, then

$$m(\{t:\ T_na(t) = S(V)\}) = \frac{Ch(n-|V|)}{n!} \leq \frac{(n-2|U|)!}{n!}.$$

If $V \in \mathcal{B}_U$, then

$$m(\{t:\ T_na(t) = S(V)\}) = \frac{Ch(n-|V|)}{n!} \leq \frac{(n-2|U|+1)!}{n!}.$$

It follows from (4.11) and Lemma 4.4.1 that

$$\sum_{V\in\mathcal{A}_U} m(\{t:\ T_na(t) = S(V)\}) \leq \frac{(n-2|U|)!}{n!} \cdot |\mathcal{A}_U| \leq \frac{(n-|U|)!}{|U|!n!} \leq \frac{3}{n^{|U|}}.$$

It follows from (4.12) and Lemma 4.4.1 that

$$\sum_{V\in\mathcal{B}_U} m(\{t:\ T_na(t) = S(V)\}) \leq \frac{(n-2|U|+1)!}{n!} \cdot |\mathcal{B}_U| \leq \frac{(n-|U|)!}{(|U|-1)!n!} \leq \frac{3}{n^{|U|}}.$$

The assertion follows immediately. □

Lemma 4.4.3. *Let $n \in \mathbb{N}$ and $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ be such that the coordinates a_k, $1 \leq k \leq n$, are distinct and strictly positive. It follows that for every $m \geq 2n^2$ and for all $\tau > 0$*

$$m(\{\omega : H_ma(\omega) > \tau\}) \geq \frac{1}{8} \sum_{S(U)>\tau} \frac{1}{n^{|U|}}.$$

Proof. Let $U \in 2^A$. Let $W \subset \{1, 2, \cdots, m\}$ be such that $|W| = |U|$ and let θ be a bijection $\theta : W \to U$. Let $F_{U,W,\theta}$ be the set of all ω such that

$$(\sigma_{1/m}(C_na))(\omega^{(j)}) = a_{\theta(j)}, \quad j \in W$$

and

$$(\sigma_{1/m}(C_na))(\omega^{(j)}) = 0, \quad j \notin W.$$

Let $\omega \in F_{U_1,W_1,\theta_1} \cap F_{U_2,W_2,\theta_2}$. If $j \notin W_1$ or $j \notin W_2$, then $(\sigma_{1/m}(C_n a))(\omega^{(j)}) = 0$. It follows that $W_1 = W_1 \cap W_2 = W_2$. Therefore, $a_{\theta_1(j)} = a_{\theta_2(j)}$ for every $j \in W_1 = W_2$. Therefore, $\theta_1 = \theta_2$ and $U_1 = U_2$. Hence, the sets $F_{U,W,\theta}$ are disjoint.

Let F_U be the union of all sets $F_{U,W,\theta}$. It is clear that the sets F_U are disjoint.

Let $\omega \in F_U$. Hence, $\omega \in F_{U,W,\theta}$ and

$$H_m a(\omega) = \sum_{j \in W} a_{\theta(j)} = S(U).$$

Therefore,

$$\{\omega : H_m a(\omega) > \tau\} \supset \bigcup_{S(U) > \tau} F_U. \tag{4.15}$$

We proceed with the estimate of $m(F_U)$ from below. It is clear that

$$m(F_{U,W,\theta}) = (\frac{1}{mn})^{|U|}(1 - \frac{1}{m})^{m-|U|}.$$

There exist exactly $C_m^{|U|}$ relevant sets W. For fixed U and W, there exist exactly $|U|!$ relevant bijections θ. Therefore,

$$m(F_U) = C_m^{|U|}|U|!\frac{1}{m^{|U|}} \cdot (1 - \frac{1}{m})^{m-|U|} \cdot \frac{1}{n^{|U|}}.$$

It is clear that

$$C_m^{|U|}|U|!m^{-|U|} = \frac{m!}{m^{|U|}(m-|U|)!} \geq \frac{(m-|U|)^{|U|}(m-|U|)!}{m^{|U|}(m-|U|)!} = (1 - \frac{|U|}{m})^{|U|}.$$

If now $m \geq 2n^2$, then $m \geq 2|U|^2$ and

$$C_m^{|U|}|U|!m^{-|U|} \geq (1 - \frac{1}{2|U|})^{|U|} \geq \frac{1}{2}.$$

Since $m \geq 2$, it follows that $(1 - 1/m)^{m-|U|} \geq (1 - 1/m)^m \geq 1/4$. Hence,

$$m(F_U) \geq \frac{1}{8} \cdot \frac{1}{n^{|U|}}.$$

Since the sets F_U are disjoint, the assertion follows immediately from (4.15). □

Lemma 4.4.4. *Let $n \in \mathbb{N}$ and $a = (a_1, \cdots, a_n) \in \mathbb{R}_+^n$. For every $m \geq 2n^2$, the following inequality is valid:*

$$(T_n a)^* \leq 2\sigma_{48}(H_m a)^*.$$

Proof. Without loss of generality, the coordinates a_k, $1 \leq k \leq n$, are independent over $\mathbb{Z}$. It follows from Lemma 4.4.2 and Lemma 4.4.3 that

$$m(\{t : T_n a(t) > \tau\}) \leq 6 \sum_{S(U) > \tau/2} n^{-|U|} \leq 48 m(\{\omega : H_m a(\omega) > \tau/2\}).$$

The assertion follows immediately. □

Remark 4.4.5. *The estimate*

$$m(\{t : T_n a(t) > \tau\}) \leq C m(\{\omega : H_n a(\omega) > \tau\}) \quad \tau > 0$$

fails for any constant C. Indeed, if $a_1 = a_2 = \ldots = a_n = 1$, then

$$m(\{t : T_n a(t) = n\}) = \frac{1}{n!},$$

while

$$m(\{\omega : H_n a(\omega) = n\}) = \frac{1}{n^n}.$$

Theorem 4.4.6. *Let E be a fully symmetric quasi-Banach space. If $K : E \to E$, then the operators T_n, $n \in \mathbb{N}$, are uniformly bounded in E. Moreover,*

$$\|T_n\|_{E \to E} \leq 96 \|K\|_{E \to E}, \quad n \in \mathbb{N}.$$

Proof. It follows from Lemma 4.4.4 that for arbitrary $n \in \mathbb{N}$, $a \in \mathbb{R}^n$, $\tau > 0$ and every $m \geq 2n^2$, we have

$$(T_n a)^* \leq 2\sigma_{48} (H_m a)^*.$$

By Lemma 3.1.1, $H_m a$ converges to $K(C_n a)$ in distribution. Therefore, $(H_m(C_n a))^*$ converges to $(K(C_n a))^*$ almost everywhere.

It follows that

$$(T_n a)^* \leq 2\sigma_{48} (K(C_n a))^*.$$

This implies that

$$\|T_n a\|_E \leq 96 \|K(C_n a)\|_E.$$

Since E is fully symmetric, we have

$$\|T_n x\|_E = \|T_n(B_n x)\|_E \leq 96 \|K(C_n B_n x)\|_E \leq 96 \|K\|_{E \to E} \|x\|_E.$$

□

4.5 The Kruglov property and random permutations

We are going to infer some corollaries from Theorems 4.4.6 and 4.3.6.

Corollary 4.5.1. *Let E be a fully symmetric quasi-Banach space. For every $n \in \mathbb{N}$ and every $x = (x_{i,j})_{1 \le i,j \le n}$, we have*

$$\|A_n x\|_E \le C(\|\sum_{k=1}^{n} x_k^* \chi_{((k-1)/n,k/n)}\|_E + \frac{1}{n}\sum_{k=n+1}^{n^2} x_k^*).$$

Here, $(x_k^*)_{k=1}^{n^2}$ is the decreasing rearrangement of the sequence $(|x_{i,j}|)_{i,j=1}^n$ and $C > 0$ does not depend either on n or x.

Corollary 4.5.2. *The operators T_n, $n \ge 1$, are uniformly bounded on the Orlicz space $\exp(L_p)$ if and only if $p \le 1$.*

Indeed, the Orlicz space $\exp(L_p)$ has the Kruglov property if and only if $p \le 1$ (see [12, 2.4, p. 42]). The assertion follows immediately from Theorems 4.4.6 and 4.3.6.

Theorem 4.1.8 in conjunction with Theorems 4.4.6 and 4.3.6 implies the following corollary.

Corollary 4.5.3. *If the symmetric quasi-Banach space E has the Fatou property and if $\sup_n \|T_n\|_E < \infty$, then there exists a symmetric space $F \subsetneq E$, such that $\sup_n \|T_n\|_F < \infty$.*

If E is a symmetric space and $p \ge 1$, then $E(p)$ denotes the space of all measurable functions x on the interval $[0,1]$ such that $|x|^p \in E$. We equip $E(p)$ with the norm

$$\|x\|_{E(p)} = \|\,|x|^p\,\|_E^{1/p}.$$

It is well-known that $E(p) \subset E$ and $\|x\|_E \le \|x\|_{E(p)}$ for all $x \in E(p)$ (see [36, 1.d]).

Let E and F be symmetric spaces such that $E \subset F$ and $K : E \to E$. This does not necessary imply that $K : F \to F$ (see [6, Corollaries 5.6 and 5.7]). However,

Corollary 4.5.4. *Let E be a symmetric Banach space such that either E is separable or $E = E^{\times\times}$. If the Kruglov operator K is bounded in $E(p)$, then it is bounded in E.*

Proof. Assume that the operator K is bounded in $E(p)$. According to Theorem 4.4.6, the operators T_n, $n \in \mathbb{N}$, are uniformly bounded in $E(p)$.

Let $a \in \mathbb{R}_+^n$ and let

$$\|T_n(C_n a)\|_{E(p)} \le C\|C_n a\|_{E(p)},$$

so that

$$\|(T_n(C_n a))^p\|_E^{1/p} \le C\|(C_n a)^p\|_E^{1/p}.$$

If $x = (C_n a)^p$, then

$$\|(T_n x^{1/p})^p\|_E \le C^p\|x\|_E.$$

It follows from the definition of the operator T_n, $n \geq 1$ that

$$(T_n x^{1/p})^p \geq T_n x.$$

Hence, $\|T_n x\|_E \leq C^p \|x\|_E$, $n \in \mathbb{N}$. Thus, the operators T_n, $n \geq 1$ are uniformly bounded in E. According to Theorem 4.3.6, the Kruglov operator K is bounded in E. $\square$

4.6 Applications to Banach-Saks index sets

Let E be a Banach space and let $p > 1$. The bounded sequence $\{x_n\}_{n=1}^{\infty} \subset E$ is called a p-BS-sequence if for all subsequences $\{y_k\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$,

$$\sup_{m\in\mathbb{N}} m^{-\frac{1}{p}} \|\sum_{k=1}^{m} y_k\|_E < \infty.$$

Define the set $\Gamma_i(E)$ (respectively, $\Gamma_{iid}(E)$; respectively, $\Gamma_d(E)$) as the set of those p such that every independent (respectively, independent and identically distributed; respectively, disjoint) weakly null sequence contains a p-BS-subsequence.

The first main result of this section characterizing a subclass of the class of all symmetric spaces E such that $\Gamma_{iid}(E) = \Gamma_i(E)$ is given in Theorem 4.6.3 below. We first need a modification of the subsequence splitting result from [52, Theorem 3.2].

Lemma 4.6.1. *Let E be a symmetric separable Banach space on the interval $(0,1)$. Suppose that E is separable. If the sequence $\{x_n\} \subset E$ converges to 0 in measure, then there exists a subsequence $\{z_n\} \subset \{x_n\}$ such that $z_n = v_n + w_n$, where the functions v_n, $n \in \mathbb{N}$, are mutually disjoint and $\|w_n\|_E \to 0$.*

Proof. Since $x_n \to 0$ by measure, it follows that $m(\{t : |x_n(t)| \leq \varepsilon\}) \to 1$. Passing to a subsequence, if needed, we obtain $m(\{t : |x_n(t)| \leq 1/n) \geq 1 - 1/n$.

Since E is separable, it follows that there exists a sequence $\varepsilon_n > 0$ such that $\|x_n^* \chi_{[0,\varepsilon_n]}\|_E \leq 1/n$.

Set

$$v_{n,1} = x_n \chi_{1/n<|x_n|<x^*(\varepsilon_n)}, \quad w_{n1} = x_n \chi_{|x_n|<1/n} + x_n \chi_{|x_n|>x^*(\varepsilon_n)}.$$

Clearly, $w_{n1} \to 0$ in norm and $m(\mathrm{supp}(v_{n,1})) \to 0$ and $v_{n,1} \in L_\infty(0,1)$. Without loss of generality, $x_n = v_{n,1}$, $n \in \mathbb{N}$.

Passing to a subsequence, if needed, we obtain $m(\mathrm{supp}(x_{n+1})) \leq \frac{1}{2} m(\mathrm{supp}(x_n))$. Passing to a further subsequence, if needed, we obtain

$$\|x_n\|_\infty \cdot \|\chi_{\mathrm{supp}(x_{n+1})}\|_E \leq \frac{1}{n}.$$

Therefore,

$$\|x_n \chi_{\cup_{k>n} \mathrm{supp}(x_k)}\|_E \leq \|x_n\|_\infty \|\chi_{\cup_{k>n} \mathrm{supp}(x_k)}\|_E \leq \|x_n\|_\infty \|\chi_{\mathrm{supp}(x_{n+1})}\|_E \leq \frac{2}{n}.$$

Clearly, the sequence $v_n = x_n - x_n \chi_{\cup_{k>n} \mathrm{supp}(x_k)}$, $n \in \mathbb{N}$, is disjoint and the assertion of the lemma follows. □

Theorem 4.6.2. *Let E be a symmetric Banach space on the interval $(0,1)$. Suppose that E is separable and satisfies Fatou property. Let $\{x_n\}_{n\geq 1} \subset E$ be a* *sequence of independent random variables. If $x_n \to 0$ weakly, then there exists a subsequence $\{y_n\}_{n\ge1} \subset \{x_n\}_{n\ge1}$, such that*

$$y_n = u_n + v_n + w_n, n \geq 1,$$

where

1. *$\{u_n\}_{n\ge1}$ is a sequence of independent identically distributed functions.*
2. *$\{v_n\}_{n\ge1}$ is a sequence of functions with pairwise disjoint support.*
3. *$\{w_n\}_{n\ge1}$ is a sequence of functions such that $\|w_n\|_E \to 0$.*
4. *$u_n \to 0$ and $v_n \to 0$ weakly.*

Proof. Since sequence $\{x_n\}$ converges weakly, it is bounded. Clearly,

$$\|x_n\|_E \geq \|x_n^* \chi_{[0,s]}\|_E \geq x_n^*(s)\|\chi_{[0,s]}\|_E, \quad s \in [0,1].$$

By the Helly Selection theorem, there exists a sequence $\{x_{n,1}\} \subset \{x_n\}$ such that $(x_{n,1}^+)^* \to (x^+)^*$ and $(x_{n,1}^-)^* \to (x^-)^*$ uniformly on the interval $[1/2, 1]$. Without loss of generality, $\|(x_{n,1}^+)^* - (x^+)^*\|_{L_\infty(1/2,1)} \leq 1$ and $\|(x_{n,1}^-)^* - (x^-)^*\|_{L_\infty(1/2,1)} \leq 1$. Repeating the argument, we get a subsequence $\{x_{n,2}\} \subset \{x_{n,1}\}$ such that $(x_{n,1}^+)^* \to (x^+)^*$ and $(x_{n,1}^-)^* \to (x^-)^*$ uniformly on the interval $[1/3, 1]$. Without loss of generality, $\|(x_{n,2}^+)^* - (x^+)^*\|_{L_\infty(1/3,1)} \leq 1/2$ and $\|(x_{n,2}^-)^* - (x^-)^*\|_{L_\infty(1/3,1)} \leq 1/2$.

Repeat this process, consider diagonal subsequence $x_{n,n}$, $n \in \mathbb{N}$. It is clear that

$$m(\{t : |(x_{n,n}^+)^*(t) - (x^+)^*(t)| \geq \frac{1}{n}\}) \leq m([0, \frac{1}{n+1}]) = \frac{1}{n+1}$$

and

$$m(\{t : |(x_{n,n}^-)^*(t) - (x^-)^*(t)| \geq \frac{1}{n}\}) \leq m([0, \frac{1}{n+1}]) = \frac{1}{n+1}.$$

Hence, $(x_{n,n}^+)^* \to (x^+)^*$ and $(x_{n,n}^-)^* \to (x^-)^*$ in measure. Since E has the Fatou property and $(x_{n,n}^+)^*, (x_{n,n}^-)^* \in E$, $n \in \mathbb{N}$, it follows that $(x^+)^*, (x^-)^* \in E$.

There exist an isomorphism $\gamma : \Omega \to (0,1)$ and measure-preserving transforms $\gamma_n : (0,1) \to (0,1)$, $n \in \mathbb{N}$, such that

$$(x_{n,n} \circ \gamma)(\omega) = (((x_{n,n}^+)^* +^d (-(x_{n,n}^-)^*)) \circ \gamma_n)(\omega_n), \quad \omega \in \Omega.$$

Here, $+^d$ denotes the disjoint sum. Set

$$x = (x^+)^* +^d (x^-)^*$$

and

$$(u_n \circ \gamma)(\omega) = (x \circ \gamma_n)(\omega_n).$$

It follows immediately that u_n, $n \in \mathbb{N}$, are independent and identically distributed and that $x_{n,n} - u_n \to 0$ in measure.

Apply Lemma 4.6.1 to the sequence $\{x_{n,n} - u_n\}$. Passing to a subsequence if needed, we obtain $x_{n,n} = u_n + v_n + w_n$, where u_n are identically distributed, v_n are disjoint and $\|w_n\|_E \to 0$.

Since $E^{\times\times} = E$, we can apply Proposition 2.1 from [22]. It follows that the sequence $\{u_n\}$ is weakly compact. Without loss of generality, $u_n \to u$ weakly. Hence, $v_n = (x_{n,n} - u_n) - w_n \to -u$ weakly.

If $E = L_1$, then $v_n \to 0$ weakly, which proves theorem for this case.

Suppose now that $E \neq L_1$. It follows that $E^\times \neq L_\infty$ and

$$|\int_0^1 v_n(s)\mathrm{sgn}(u)(s)ds| = |\int_0^1 v_n(s)(\mathrm{sgn}(u)\chi_{\mathrm{supp}(v_n)})(s)ds| \leq$$

$$\leq \|v_n\|_E \cdot \|\chi_{\mathrm{supp}(v_n)}\|_{E^\times} \to 0.$$

However, $v_n \to -u$ weakly. Therefore,

$$|\int_0^1 v_n(s)\mathrm{sgn}(u)(s)ds| \to \int_0^1 |u(s)|ds.$$

Hence, $u = 0$. Thus, $u_n \to 0$ and $v_n \to 0$ weakly. □

Theorem 4.6.3. *Let E be a symmetric Banach space on the interval* $(0,1)$. *Suppose that E is separable and satisfies the Fatou property. Then*

$$\Gamma_i(E) = \Gamma_{iid}(E) \cap \Gamma_d(E).$$

Proof. It is clear that

$$\Gamma_i(E) \subset \Gamma_{iid}(E).$$

It follows from Lemma 3.3.1 that

$$\Gamma_i(E) \subset \Gamma_d(E).$$

Therefore,

$$\Gamma_i(E) \subset \Gamma_{iid}(E) \cap \Gamma_d(E).$$

Let us prove the reverse inclusion. Let q be such that $1/q \in \Gamma_{iid}(E)$ and $1/q \in \Gamma_d(E)$. Let x_n be a sequence weakly convergent to 0. Let u_n, v_n, w_n be as in Theorem 4.6.2. Passing to a subsequence if needed, we obtain $\|w_n\|_E \leq 2^{-n}$, $n \in \mathbb{N}$. Without loss of generality, $w_n = 0$ and $x_n = u_n + v_n$, $n \in \mathbb{N}$.

Since $1/q \in \Gamma_{iid}(E)$, it follows that

$$\|\sum_{k=1}^n u_k\|_E \leq \mathrm{const} \cdot n^{1/q}, \quad n \in \mathbb{N}.$$

Since $1/q \in \Gamma_d(E)$, it follows that

$$\|\sum_{k=1}^n v_k\|_E \leq \mathrm{const} \cdot n^{1/q}, \quad n \in \mathbb{N}.$$

It is now clear that

$$\|\sum_{k=1}^{n} x_k\|_E \leq \text{const} \cdot n^{1/q}, \quad n \in \mathbb{N}.$$

Hence, $q \in \Gamma_i(E)$ and we are done. □

Lemma 4.6.4. *Let E be a fully symmetric Banach space on the interval $(0,1)$. Then $\Gamma_{iid}(E) = \{1\}$ if and only if $\|A_n\|_{E\to E} = n$ for every $n \in \mathbb{N}$.*

Proof. Let $1 < q \in \Gamma_{iid}(E)$. Define the sequence $x_k \in E(\Omega)$, $k \in \mathbb{N}$, by the formula $x_k(\omega) = x(\omega_{2k-1})r(\omega_{2k})$. It converges weakly to 0 (see Lemma 3.4. in [53]). Hence,

$$\|\sum_{k=1}^{n} x_k\|_E \leq C(x)n^{1/q}.$$

It follows that $n^{-1/q}\|A_n x\|_E \leq C(x)$. By the uniform boundedness principle, $\|A_n\|_{E\to E} \leq Cn^{1/q}$.

Let us prove the converse assertion. Let $\|A_n\|_{E\to E} \leq Cn^{1/q}$, $n \in \mathbb{N}$. Suppose that $\{x_k\} \subset E$ is a sequence of independent identically distributed functions which converges weakly to 0. Without loss of generality,

$$x_k(\omega) = x(\omega_{2k-1}), \quad \omega \in \Omega.$$

Since $x_k \to 0$ weakly, it follows that

$$\int x(\omega)d\omega = \int x_k(\omega)d\omega \to 0.$$

Hence, x is mean zero and, therefore, x_k, $k \in \mathbb{N}$, are mean zero.

By Lemma 3.2.5,

$$\|\sum_{k=1}^{n} x_k\|_E \leq \text{const} \cdot \|\sum_{k=1}^{n}(x(\omega_{2k-1}) - x(\omega_{2k}))\|_E =$$

$$= \text{const}\|A_n(x(\omega_1) - x(\omega_2))\|_E \leq \text{const} \cdot n^{1/q}\|x(\omega_1) - x(\omega_2)\|_E.$$

Hence,

$$\|\sum_{k=1}^{n} x_k\|_E \leq \text{const} \cdot n^{1/q}\|x\|_E$$

and we are done. □

Remark 4.6.5. *Let E be a symmetric Banach space on the interval $(0,1)$. Suppose that E has Fatou property and is separable. If E satisfies an upper 2-estimate, then $\Gamma_{iid}(E) = \Gamma_i(E)$.*

Indeed, in this case $2 \in \Gamma_d(E)$.

Our second main result in this section completely characterizes the subclass of all Lorentz spaces whose Banach-Saks index set $\Gamma_i(\Lambda_\psi)$ is non-trivial.

Corollary 4.6.6. $\Gamma_i(\Lambda_\psi) \neq \{1\}$ *if and only if the function* ψ *satisfies the conditions* (3.18) *and* (3.19) *for some* $k, l \geq 2$.

Proof. We have

$$\Gamma_d(\Lambda_\psi) = [1, \infty)$$

(see e.g. the proof of [4, Corollary 4.8]). Therefore,

$$\Gamma_i(\Lambda_\psi) = \Gamma_{iid}(\Lambda_\psi).$$

The assertion follows now from Lemma 4.6.4 and Theorem 3.6.6. □

We complete this section with the description of $\Gamma_i(\exp(L_p)_0)$, $0 < p < \infty$.

Theorem 4.6.7. *For every* $0 < p \leq 2$,

$$\Gamma_{iid}(\exp(L_p)_0) = \Gamma_i(\exp(L_p)_0) = [1, 2].$$

For every $2 \leq p < \infty$,

$$\Gamma_{iid}(\exp(L_p)_0) = \Gamma_i(\exp(L_p)_0) = [1, \frac{p}{p-1}].$$

Proof. Note that $\exp(L_p)$ coincides with the Marcinkiewicz space $M_{t \log^{1/p}(1/t)}$. We have

$$\Gamma_d((M_\psi)_0) = [1, \infty)$$

(see e.g. [4, p.897]). It follows from Lemma 4.6.4 and Theorem 3.7.1 that

$$\Gamma_{iid}(\exp(L_p)_0) = [1, 2]$$

for $0 < p \leq 2$ and

$$\Gamma_{iid}(\exp(L_p)_0) = [1, \frac{p}{p-1}]$$

for $2 < p < \infty$. The assertion follows from Theorem 4.6.3. □

Remark 4.6.8. *The preceding theorem shows that the set* $\Gamma_i(\exp(L_p)_0)$ *is non-trivial for all* $0 < p < \infty$, *whereas* $\exp(L_p)_0$ *has the Kruglov property if and only if* $0 < p \leq 1$. *Therefore, the Kruglov property is not necessary for the condition* $\Gamma_i(E) \neq \{1\}$.

Example 4.6.9. *If* $E = L_{p,q}$ *for* $1 < q < p < 2$, *then* $\Gamma_i(E) \neq \Gamma_{iid}(E)$.

Proof. For every normalized sequence $\{v_n\}_{n\geq 1} \subset L_{p,q}$ of functions with disjoint support, there exists a subsequence spanning the space l_q (see Lemma 2.6.5). Hence, $\Gamma_d(L_{p,q}) \subset \Gamma(l_q) = [1, q]$. Therefore, by Theorem 4.6.3, we have $\Gamma_i(L_{p,q}) \subseteq [1, q]$.

Next, it is proved in [18, Corollary 3.7] (see also [13, Corollary 5.2]) that if $p < 2$ then for every sequence of identically distributed independent random variables we have

$$\|\sum_{k=1}^{n} x_k\|_{L_{p,q}} = o(n^{\frac{1}{p}}).$$

Hence, $[1, p] \subseteq \Gamma_{iid}(L_{p,q})$. □

Appendix A

Classification of extreme points

The following theorem is due to Ryff (see [48]).

Theorem A.0.10. *If* $0 \le x \in L_1(0,1)$, *then* $y \in \mathrm{extr}(\Omega'(x))$ *if and only if* $y^* = x^*$.

In this appendix, we prove similar results for the sets $\Omega(x)$ and $\Omega^+(x)$ and extend them to the semi-axis.

Lemma A.0.11. *Let* $x_k = x_k^* \in S_0$ *and let* $0 \le y_k \prec\prec x_k$, $k \in \mathbb{N}$, *be mutually disjoint functions. If*

$$\sum_{k=1}^{\infty} y_k \in \mathrm{extr}(\Omega(\sum_{k=1}^{\infty} x_k)),$$

then $y_k \in \mathrm{extr}(\Omega(x_k))$, $k \in \mathbb{N}$. *The same assertion is valid for the sets* Ω^+ *and* Ω'.

Proof. Assume the contrary. Let $y_n = \frac{1}{2}(y_{1n} + y_{2n})$ with $y_{1n}, y_{2n} \prec\prec x_n$ and $\mathrm{supp}(y_{1n}), \mathrm{supp}(y_{2n}) \subset \mathrm{supp}(y_n)$. It follows from Lemma 1.2.14 that

$$y_{1n} + \sum_{k \ne n} y_k \prec\prec \sum_{k=1}^{\infty} x_k, \quad y_{2n} + \sum_{k \ne n} y_k \prec\prec \sum_{k=1}^{\infty} x_k.$$

The assertion follows immediately. □

Lemma A.0.12. *Let* $x = x^* \in L_1(0,1)$ *(respectively,* $x = x^* \in (L_1+L_\infty)(0,\infty)$*). Assume that* $y = y^*$ *is not a step function and that the inequality*

$$\int_0^t y(s)ds < \int_0^t x(s)ds$$

holds for all $t \in (0,1)$ *(respectively, for all* $t > 0$*). It follows that* $y \notin \mathrm{extr}(\Omega(x))$, $y \notin \mathrm{extr}(\Omega'(x))$ *and* $y \notin \mathrm{extr}(\Omega^+(x))$.

Proof. There exists a point $t_0 \in (0,1)$ (respectively, $t_0 > 0$) such that $y|_{(t_0,t_0+\varepsilon)}$ takes infinitely many values for arbitrary $\varepsilon > 0$. In particular, $y(t_0) > 0$. Select $t_4 > t_0$ such that

$$\int_0^{t_4} y(s)ds \leq \int_0^{t_0} x(s)ds.$$

Fix $t_1, t_3 \in (t_0, t_4)$ such that $y(t_0) > y(t_1) > y(t_3) > y(t_4)$. Set $t_2 = (t_1+t_3)/2$ and

$$y_\pm = y \pm \delta(\chi_{(t_1,t_2)} - \chi_{(t_2,t_3)}), \quad \delta = \min\{y(t_0) - y(t_1), y(t_3) - y(t_4)\}.$$

It is clear that $y_\pm = y$ on the complement of the interval (t_0, t_4) and $y(t_4) \leq y_+, y_- \leq y(t_0)$ on the interval (t_0, t_4). Therefore,

$$\int_0^t y_\pm^*(s)ds = \int_0^t y(s)ds \leq \int_0^t x(s)ds, \quad t \notin (t_0, t_4)$$

and

$$\int_0^t y_\pm^*(s)ds \leq \int_0^{t_4} y(s)ds \leq \int_0^{t_0} x(s)ds \leq \int_0^t x(s)ds, \quad t \in (t_0, t_4).$$

Therefore, $y = (y_+ + y_-)/2$ with $y_+, y_- \prec\prec x$. □

Lemma A.0.13. *Let $x = x^* \in L_1(0,1)$ (respectively, $x = x^* \in (L_1+L_\infty)(0,\infty)$). Assume that $y = y^*$ is a step function and that the inequality*

$$\int_0^t y(s)ds < \int_0^t x(s)ds$$

holds for all $t \in (0,1)$ (respectively, for all $t > 0$). It follows that $y \notin \mathrm{extr}(\Omega(x))$ and $y \notin \mathrm{extr}(\Omega'(x))$. If $y \in \mathrm{extr}(\Omega^+(x))$, then $y = 0$.

Proof. Suppose that y has at least three maximal intervals of constancy. Let (a,b) be the maximal interval of constancy for y such that $a > 0$ and $b < 1$ (respectively, $b < \infty$). Set

$$y_\pm = y \pm \delta(\chi_{(a,(a+b)/2)} - \chi_{((a+b)/2,b)}).$$

Clearly, $y_\pm \prec\prec x$ for sufficiently small δ and $y = (y_+ + y_-)/2$. Therefore, $y \notin \mathrm{extr}(\Omega(x))$.

The case that y has one or two maximal intervals of constancy can be treated similarly. □

Theorem A.0.14. *Let $x \in L_1(0,1)$ (or $x \in (L_1+L_\infty)(0,\infty)$) and let $y \prec\prec x$.*

1. *If $x \in L_1(0,1)$ or $x \in L_1(0,\infty)$, then $y \in \mathrm{extr}(\Omega'(x))$ if and only if $y^* = x^*$ and $y \geq 0$.*
2. *If $x \in L_1(0,1)$ or $x \in (L_1+L_\infty)(0,\infty)$, then $y \in \mathrm{extr}(\Omega(x))$ if and only if $y^* = x^*$ and $|y| \geq y^*(\infty)$.*

3. *If* $x \in L_1(0,1)$ *or* $x \in (L_1 + L_\infty)(0,\infty)$, *then* $y \in \mathrm{extr}(\Omega^+(x))$ *if and only if* $y^* = x^*\chi_{(0,\beta)}$ *and* $|y| \geq y^*(\infty)$.

Proof. Let y be an extreme point of the set $\Omega(x)$. Without loss of generality, $x = x^*$ and $y = y^*$. Define the set

$$A = \{t : \int_0^t y^*(s)ds < \int_0^t x^*(s)ds\}.$$

It is clear that A is an open set. If $A \neq \emptyset$, then $A = \cup_{i\geq 1} I_i$ — a union of disjoint intervals.

In the second case, $y|_{I_i} \prec\prec x|_{I_i}$. By the Lemma A.0.11, we have $y|_{I_i} \in \mathrm{extr}(\Omega(x|_{I_i}))$, which contradicts Lemma A.0.12 or Lemma A.0.13. Thus, $A = \emptyset$ and $y = x$.

The same construction works in case of $\Omega'(x)$.

Consider now the case of $\Omega^+(x)$. It follows from the Lemma A.0.12 that $y = 0$ on A. Since $y = y^*$, it follows that $A = (\beta, \infty)$ and $y = x\chi_{(0,\beta)}$. □

Appendix B

A pathological Orlicz space

Example B.0.15. *There exists a non-separable Orlicz space L_Φ such that $\beta_{L_\Phi} = 1$.*

Proof. Let $a_0 = 1$ and $a_{n+1} = e^{a_n}$. Set $\Phi(t) = t^2$ on $(0, 1)$,

$$\Phi(t) = e^t + \Phi(a_{2n}) - e^{a_{2n}}, \quad \forall t \in [a_{2n}, a_{2n+1}],$$

$$\Phi(t) = \Phi(a_{2n-1}) + e^{a_{2n-1}}(t - a_{2n-1}), \quad \forall t \in [a_{2n-1}, a_{2n}]$$

for all $t \in \mathbb{N}$. Clearly, $\Phi'(t) = e^t$ on $[a_{2n}, a_{2n+1}]$ and $\Phi'(t) = e^{a_{2n-1}}$ on $[a_{2n-1}, a_{2n}]$. Hence, $\Phi'(t) \le e^t$ and $\Phi(t) \le e^t - 1$ for all $t > 0$.

If $\alpha_{L_\Phi} > 0$, then (see [36, 2.b.5]) there exists $q > 0$ such that

$$\sup_{\lambda, t \ge 1} \frac{\Phi(\lambda t)}{\Phi(\lambda) t^q} < \infty.$$

In particular, $\Phi(t) \le \mathrm{const} \cdot t^q$ for $t \ge 1$. However,

$$\Phi(a_{2n+1}) \ge e^{a_{2n+1}} - e^{a_{2n}} = e^{a_{2n+1}}(1 + o(1)), \quad \forall n \in \mathbb{N}.$$

Therefore, $\alpha_{L_\Phi} = 0$ and L_Φ is non-separable.

If $\beta_{L_\Phi} < 1$, then (see [36, 2.b.5]) there exists $p > 1$ such that

$$\inf_{\lambda, t \ge 1} \frac{\Phi(\lambda t)}{\Phi(\lambda) t^p} > 0.$$

For every $n \in \mathbb{N}$, set $t_n = n$ and $\lambda_n = a_{2n}/n$. Hence, $\lambda_n t_n = a_{2n}$ and

$$\Phi(\lambda_n t_n) = \Phi(a_{2n-1}) + e^{a_{2n-1}}(a_{2n} - a_{2n-1}) = a_{2n}(1 + o(1)) + a_{2n}^2(1 + o(1)).$$

Since $a_{2n-1} = \frac{1}{n} o(a_{2n})$, it follows that

$$\Phi(\lambda_n) = \Phi(a_{2n-1}) + e^{a_{2n-1}}(\frac{1}{n} a_{2n} - a_{2n-1}) = a_{2n}(1 + o(1)) + \frac{1}{n} a_{2n}^2(1 + o(1)).$$

Therefore,

$$\frac{\Phi(\lambda_n t_n)}{\Phi(\lambda_n) t_n^p} = (1 + o(1)) \frac{a_{2n}^2}{\frac{1}{n} a_{2n}^2 \cdot n^p} = (1 + o(1)) n^{1-p} = o(1)$$

and we conclude that $\beta_{L_\Phi} = 1$. $\square$

Appendix C

An operator tensor product

If $z_1, \cdots, z_k \in L_1(0,1)$, we denote by $z_1 \otimes \cdots \otimes z_n \in L_1((0,1)^n)$ the function

$$\omega = (\omega_1, \cdots, \omega_n) \to z_1(\omega_1) \cdots z_n(\omega_n).$$

The following assertion is well-known. We provide a proof due to the lack of a convenient reference.

Lemma C.0.16. *Let $A_k : L_1(0,1) \to L_1(0,1)$, $1 \le k \le n$, be bounded operators. There exists a bounded linear operator $A_1 \otimes \cdots \otimes A_n : L_1((0,1)^n) \to L_1((0,1)^n)$ such that*

$$(A_1 \otimes \cdots \otimes A_n)(z_1 \otimes \cdots \otimes z_n) = A_1 z_1 \otimes \cdots \otimes A_n z_n, \quad z_1, \cdots, z_n \in L_1(0,1).$$

Proof. For simplicity of notation, we will assume that $n = 2$.

If z_1, z_2 are simple functions (finite linear combinations of indicator functions), define

$$(A_1 \otimes A_2)(z_1 \otimes z_2) = (A_1 z_1) \otimes (A_2 z_2).$$

The operator $A_1 \otimes A_2$ is extended to the linear hull $\mathcal{L}$ of such functions by linearity. We now show that $A_1 \otimes A_2$ is well-defined on $\mathcal{L}$.

Indeed, let

$$\sum_{k=1}^{n} z_{1k} \otimes z_{2k} = \sum_{k=1}^{m} z_{3k} \otimes z_{4k}.$$

One can represent $z_{ik} = \sum_{j=1}^{l} a_{ijk} \chi_{B_j}$ with B_j, $1 \le j \le l$ being disjoint sets. Therefore,

$$\sum_{k=1}^{n} z_{1k} \otimes z_{2k} = \sum_{j_1, j_2 = 1}^{l} \sum_{k=1}^{n} a_{1 j_1 k} a_{2 j_2 k} \chi_{B_{j_1}} \otimes \chi_{B_{j_2}} =$$

$$= \sum_{j_1, j_2 = 1}^{l} \sum_{k=1}^{m} a_{3 j_1 k} a_{4 j_2 k} \chi_{B_{j_1}} \otimes \chi_{B_{j_2}} = \sum_{k=1}^{m} z_{3k} \otimes z_{4k}.$$

It follows that

$$(A_1 \otimes A_2)(\sum_{k=1}^{n} z_{1k} \otimes z_{2k}) = \sum_{j_1,j_2=1}^{l} \sum_{k=1}^{n} a_{1j_1k} a_{2j_2k} (A_1 \chi_{B_{j_1}}) \otimes (A_2 \chi_{B_{j_2}}) =$$

$$= \sum_{j_1,j_2=1}^{l} \sum_{k=1}^{m} a_{3j_1k} a_{4j_2k} (A_1 \chi_{B_{j_1}}) \otimes (A_2 \chi_{B_{j_2}}) = (A_1 \otimes A_2)(\sum_{k=1}^{m} z_{3k} \otimes z_{4k})$$

and this implies that $A_1 \otimes A_2$ is well-defined on $\mathcal{L}$.

If $z = \sum_{k=1}^{n} a_k \chi_{B_k} \otimes \chi_{C_k}$ with the sets B_k, $1 \leq k \leq n$ and C_k, $1 \leq k \leq n$, being disjoint, then

$$\|(A_1 \otimes A_2)z\|_1 \leq \sum_{k=1}^{n} |a_k| \cdot \|A_1 \chi_{B_k}\|_1 \cdot \|A_2 \chi_{C_k}\|_1 \leq$$

$$\leq \|A_1\|_{L_1 \to L_1} \|A_2\|_{L_1 \to L_1} \sum_{k=1}^{n} |a_k| m(B_k) m(C_k) = \|A_1\|_{L_1 \to L_1} \|A_2\|_{L_1 \to L_1} \|z\|_1.$$

Therefore, $A_1 \otimes A_2$ is a bounded operator on a dense subset of $L_1((0,1)^2)$. The assertion follows immediately. □

Lemma C.0.17. *Let $A_k : L_1(0,1) \to L_1(0,1)$, $1 \leq k \leq n$, be bounded operators.*

1. *If $A_1, \cdots, A_n$ are positive, then so is $A_1 \otimes \cdots \otimes A_n$.*
2. *If $A_1, \cdots, A_n$ preserve integral, then so does $A_1 \otimes \cdots \otimes A_n$.*
3. *If $A_k 1 = 1$, $1 \leq k \leq n$, then $(A_1 \otimes \cdots \otimes A_n)1 = 1$.*
4. *If $A_1, \cdots, A_n$ are bistochastic, the so is $A_1 \otimes \cdots \otimes A_n$.*

Proof. The fourth assertion is an immediate corollary of the first, second and third assertions.

1. Let $z = \sum_{k=1}^{m} z_{1k} \otimes \cdots \otimes z_{nk}$ with z_{ik}, $1 \leq i \leq n$, $1 \leq k \leq m$, being simple functions. One can write $z = \sum_{k=1}^{l} u_{1k} \otimes \cdots \otimes u_{nk}$ with u_{ik}, $1 \leq i \leq n$, $1 \leq k \leq l$, being positive simple functions. It follows that $Az \geq 0$. Since functions of the above form are dense in $L_1((0,1)^n)$, the assertion follows.
2. Let $z = \sum_{k=1}^{m} z_{1k} \otimes \cdots \otimes z_{nk}$ with z_{ik}, $1 \leq i \leq n$, $1 \leq k \leq m$, being simple functions. It is clear that

   $$\int_{(0,1)^n} (A_1 \otimes \cdots \otimes A_n)(z) = \int_{(0,1)} z.$$

   Since functions of the above form are dense in $L_1((0,1)^n)$, the assertion follows.
3. Clear.

□

Bibliography

[1] Ando T. *Weakly compact sets in Orlicz spaces,* Canad.J.Math **14** (1962), 170-176.

[2] Astashkin S.V., Sukochev F.A. *Comparison of the sums of independent and disjoint functions in symmetric spaces.* Mathematical Notes 2004. V.76, No.4. p.483-489. (in Russian). English translation at Math.Notes, **80**, no.3-4 (2006), 593-598.

[3] Astashkin S.V., Sukochev F.A. *Series of independent, mean zero random variables in rearrangement-invariant spaces having the Kruglov property* Zapiski Nauchnyh Seminarov POMI **345** (2007), 25-50 (in Russian). English translation at Journal of Mathematical Sciences, 2008 V.148, No.6

[4] Astashkin S.V., Semenov E.M. and Sukochev F.A. *The Banach-Saks p-property*, Math. Ann., **332**, no. 4 (2005), 879–900.

[5] Astashkin S.V., Semenov E.M. and Sukochev F.A. *Banach-Saks type properties in rearrangement- invariant spaces with Kruglov property*, Houston J.Math (to appear).

[6] Astashkin S.V. and Sukochev F.A. *Series of independent random variables in rearrangement invariant spaces: an operator approach*, Israel J. Math., **145** (2005), 125–156.

[7] Astashkin S., Semenov E., Sukochev F. and Zanin D. *The Kruglov operator and operators defined by random permutations* (Russian) Funktsional. Anal. i Prilozhen. **43** (2009), no. 2, 318.

[8] Bennett C. and Sharpley R. *Interpolation of operators,* Pure and Applied Mathematics, **129**, Academic Press, Inc., Boston, MA, 1988.

[9] Birkhoff G. *Three observations on linear algebra* (Spanish) Univ. Nac. Tucumn. Revista A. **5** (1946). 147151.

[10] Borovkov A.A. *Probability theory.* Gordon and Breach Science Publishers, Amsterdam, 1998.

[11] Braverman M.S. and Mekler A.A. *The Hardy-Littlewood property for symmetric spaces,* Siberian Math. J. **18** (1977), 371-385.

[12] Braverman M.S. *Independent random variables and rearrangement invariant spaces*, London Mathematical Society, Lecture Note Series, vol. 194, Cambridge University Press, Cambridge, 1994.

[13] Braverman M.S. *Independent random variables in Lorentz spaces,* Bull. London Math. Soc. **28**, no. 1 (1996), 79–87.

[14] Brudnyi Yu.A and Kruglyak N.Ya. *Interpolation functors and interpolation spaces,* V.1.North Holland, (1991)

[15] Calderón A.P. *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math., **26** (1966), 273–299.

[16] Carey A.L. and Sukochev F.A. *Dixmier traces and their applications in noncommutative geometry*, Russian Math. Surveys **61** (2006), 45–110.

[17] Carothers N.L. and Dilworth S.J. *Subspaces of $L_{p,q}$*, Proc. Amer. Math. Soc. **104** (1988), 537–545.

[18] Carothers N.L. and Dilworth S.J. *Equidistributed random variables in $L_{p,q}$* J.Funct.Anal. **84** (1989), 146-159.

[19] Chilin V.I., Krygin A.V. and Sukochev F.A. *Extreme points of convex fully symmetric sets of measurable operators,* Integral Equations Operator Theory **15** (1992), 186-226.

[20] Dilworth S.J. *Special Banach lattices and their applications*, Handbook of the geometry of Banach spaces, vol.1, 497–532, North-Holland, Amsterdam, (2001).

[21] Dodds P.G., Semenov E.M. and Sukochev F.A. *The Banach-Saks property in rearrangement invariant spaces*, Studia Math., **162**, no. 3 (2004), 263–294.

[22] Dodds P.G., Sukochev F.A. and Schlüchtermann G. *Weak compactness criteria in symmetric spaces of measurable operators*, Math.Proc.Camb.Phil.Soc. (2001), **131**, 363.

[23] Dodds P.G., de Pagter B., Semenov E.M. and Sukochev F.A. *Symmetric functionals and singular traces*, Positivity **2** (1998), 47–75.

[24] Gnedenko B. Theory of probability Gordon and Breach, 1997

[25] Gustavsson J. *Metrization of quasi-metric spaces* Math. Scand. **35** (1974), 5660.

[26] Hall M. *Combinatorial theory.* John Wiley and Sons, Inc., New York, 1986.

[27] Johnson W.B., Schechtman G., Zinn J. *Best constants in moment inequalities for linear combinations of independent and exchangeable random variables.* Ann. Probab. 1985. V.13. No.1 p.234-253.

[28] Johnson W.B. and Schechtman G. *Sums of independent random variables in rearrangement invariant function spaces*, Ann.Probab., **17**, no. 2 (1989), 789–808.

[29] Kalton N. and Sukochev F. *Rearrangement-invariant functionals with applications to traces on symmetrically normed ideals,* Canad. Math. Bull. **51** (2008), 67–80.

[30] Kalton N. and Sukochev F. *Symmetric norms and spaces of operators* J.reine angew. Math. (2008), 1-41

[31] Kalton N., Sukochev F. and Zanin D. *Orbits in symmetric spaces. II* Studia Math. **197** (2010), no. 3, 257274

[32] Kaminska A., Maligranda L. *Order convexity and concavity of Lorentz spaces* $\Lambda_{p,w}$, $0 < p < \infty$. Studia Math. **160** (2004) no. 3, 267–286.

[33] Krein S.G., Petunin Ju.I. and Semenov E.M. *Interpolation of linear operators*, Nauka, Moscow, 1978 (in Russian); English translation in Translations of Math. Monographs, Vol. **54**, Amer. Math. Soc., Providence, RI, 1982.

[34] Kruglov V.M. *A remark on the theory of infinitely divisible laws*, Teor. Veroyatnost. i Primenen., **15** (1970), 331–336 (in Russian).

[35] Kwapien S., Schütt C. *Some combinatorial and probabilistic inequalities and their applications to Banach space theory* Studia Math. 1985. V.82. P.91-106.

[36] Lindenstrauss J. and Tzafriri L. *Classical Banach Spaces I and II: Sequence Spaces; Function Spaces*, Springer, 1996.

[37] Lorentz G.G. *Relations between function spaces* Proc. Amer. Math. Soc. 1961. V.12. P.127-132.

[38] Luxemburg W. *Rearrangement invariant Banach function spaces* Queen's Papers in Pure & Appl. Math. **10** (1967) 83-144

[39] Mekler A. *About averaged majorisation of functions using permutations* Trudy LIAP **87** 1974

[40] Mekler A. *Intermediate spaces, and bistochastic projections* Mat. Issled. **10** (1975), no. 1 (35), 270275, 302303.

[41] Montgomery-Smith S., Semenov E.M. *Random rearrangements and operators* Amer. Math. Soc. Transl. 1998. V.184(2). P.157-183.

[42] Montgomery-Smith S. *Rearrangement invariant norms of symmetric sequence norms of independent sequences of random variables*, Israel J. Math., **131** (2002), 51–60.

[43] Nikolskii, N. *Functional Analysis I: Linear Functional Analysis* (1992) Encyclopaedia of Mathematical Sciences. **19**.

[44] Rao M.M. and Ren Z.D. *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146 (Marcel Dekker Inc., New York, 1991).

[45] Rodin V.A. and Semenov E.M. *Rademacher series in symmetric spaces,* Anal. Math. **1**, no.4, (1975), 207-222.

[46] Rosenthal H.P. *On the subspaces of* L^p $(p > 2)$ *spanned by sequences of independent random variables* Israel J. Math. **8** 1970 273303.

[47] Rudin W. *Functional analysis* International series in pure and applied mathematics. (1991)

[48] Ryff J.V. *Extreme points of some convex subsets of* $L_1(0,1)$, Proc. Amer. Math. Soc. **18** (1967), 1026-1034.

[49] Ryff J.V. *Orbits of* L_1-*functions under doubly stochastic transformations,* Trans. Amer. Math. Soc. **117** (1965), 92-100.

[50] Ryff J.V. *Measure preserving transformations and rearrangements,* J. Math. Anal. Appl. **31** (1970), 449-458.

[51] Semenov E.M. and Sukochev F.A. *The Banach–Saks index of rearrangement invariant spaces on* $[0,1]$, C.R. Acad. Sci. Paris, ser. *I* **337** (2003), 397-401.

[52] Semenov E.M. and Sukochev F.A. *Banach–Saks index,* Sbornik: Mathematics, **195** (2004), no. 5, 263–285.

[53] Semenov E.M. and Sukochev F.A. *Estimates for a quadratic function and the Banach-Saks p-property*, Algebra and Analysis **18**, no. 4 (2006), 185–197 (in Russian).

[54] Semenov E.M. *Operator properties of random permutations* Funct. Anal. Appl. 1994, V.28, No. 3, p. 215–217.

[55] Sukochev F. and Zanin D. *Orbits in symmetric spaces* J.Funct.Anal. **257** (2009), no.1, 194-218.

[56] Sukochev F. and Zanin D. *Khinchin inequality and Banach-Saks type properties in rearrangement-invariant spaces* Studia Math. **191** (2009), no. 2, 101122.

[57] Sukochev F. and Zanin D. *Johnson-Schechtman and Khinchine inequalities in quasi-Banach spaces* submitted for publication

[58] Zaanen A. *Integration* 1967 North Holland.

Index