An Information Geometric Approach to Sensor Management

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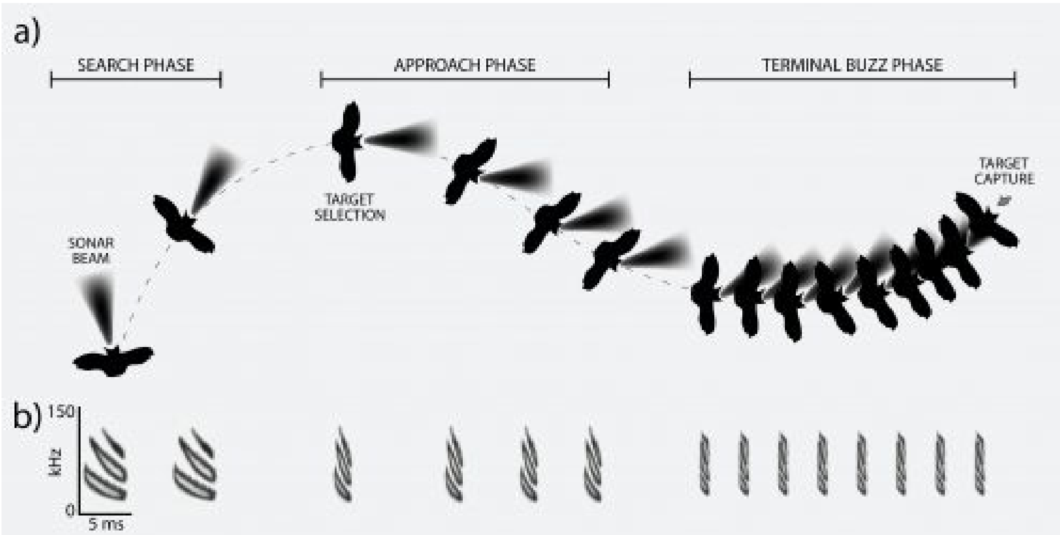
Contents

Summary

This work examines optimal trajectories through the space of target tracking system parameters. In particular, the focus is on pulsed signals with linear frequency modulation. The signal configuration parameters are the pulse width, T, the inter-pulse spacing, T_p, and chirp rate, b. The study is motivated by the observation that optimal values for these parameters, from the point of view of maximising the information gathered regarding the target, depend on time varying target parameters.



`https://wind-energy-wildlife.unl.edu/researchbat.asp#echolocation`

Figure 1: A bat searching for food transmits a broad signal with low chirp rate, large pulse width and large inter-pulse spacing. Once a target is detected, the bat approaches the target and adjusts the transmitted sonar signal so as to increase the chirp rate, decrease the pulse width and decrease the inter-pulse spacing. This adjustment continues until the target is captured.

The view taken in this study is that the configuration parameters should change in such a way as to maintain optimality from the point of view of information gathering. Thus the trajectory of the curve
$\gamma(t) = (T(t), T_p(t), b(t))$ through the parameter space should be a geodesic on a suitable manifold. To this end, mathematical aspects of a pulse-Doppler radar transmitter/receiver system are considered, beginning with the radar ambiguity function.

This function measures the disparity between transmitted and returning signals and is obtained by taking the square of a radar auto-correlation function.

The ambiguity function is also referred to as the likelihood, describing the probability of locating the target given a set of measurements and forming the basis for construction of the Fisher information matrix which, in turn, allows measurement of the information content of a given location for a specific target as a function of the configuration parameters.

The auto-correlation function leads to a collection of integrals containing the products of shifted sinc functions. These integrals did not appear in the literature and so a novel method for solving such integrals is developed.

Once the calculation of the Fisher information metric for a given sensor configuration is achieved, a family of such metrics is required in order that the optimal selection of configuration parameters may be undertaken. This family of metrics is accounted for by construction of the Gil Medrano metric, which provides a metric for the manifold of all Riemannian metrics corresponding to sensor configurations.

The calculation of geodesics on this manifold allows an optimal choice of sensor configuration to be made, facilitating efficient selection of sensor parameters and increasing the information content of received signals. Such configurations are those that possess an optimal Time-Bandwidth product, an important feature in signal processing that describes how efficiently the available bandwidth is being utilised while simultaneously describing the inverse relationship between the range and frequency resolution of the system.

The main contributions of the thesis are:

- Computing an ambiguity function for an infinite Gaussian modulated pulse with linear frequency modulation.
- Using notions from information geometry to determine optimal trajectories for configuration parameters.
- A method for computing a class of definite integrals comprising products of sinc function shifted by integer multiples of pi and derivatives of such functions.

This thesis explores the mathematics needed, in principle, to determine geodesics for configuration parameters and the mathematics represent the sole focus of this thesis, ignoring all aspects of physical implementation via real world sensor systems.

Declaration

I certify that this thesis:

1. does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university; and

2. to the best of my knowledge and belief, does not contain any material previously published or written by another person except where due reference is made in the text.

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Chapter 1

Introduction

The methods developed in this thesis seek to establish a mathematical framework for exploring the optimisation of target tracking sensor systems. This approach is a collaborative effort between applied mathematics and radar engineering, with a strong focus on construction of the mathematical aspects of the problem rather than the practical execution of an engineering task. A typical radar system consists of a transmitter and a receiver, with complex systems of multiple transmitter/receivers being common. The detection of a target is performed by transmitting a signal pulse, waiting for the pulse to strike a target, be reflected and return to the receiver. The radar ambiguity function is employed to determine how much the returned signal differs from the transmitted signal. The ambiguity function is a function of the round-trip time delay of the signal and the Doppler shift due to target motion. Integrating the square of the ambiguity function is a sufficient statistic for the detection problem.

The optimisation of target tracking sensor systems has been well researched and practically demonstrated from a wide variety of approaches [12], [29], [14]...etc. The application of information geometric methods to the optimisation of target tracking sensor systems represents a relatively new option for optimisation of such a system, and represents one of the main contributions of this thesis. Information geometry provides a mechanism by which to view the space inhabited by a collection of sensors and targets as a coordinate space in which points contain probabilistic information about the inhabitants. This is known as a statistical manifold. In this thesis, the method by which such information may be used to optimally configure a sensor system is established for two scenarios.

The first scenario describes an infinite Gaussian pulse train with a linear chirp and the second scenario describes a single Doppler bin model with a Gaussian modulation and linear chirp. The motivation for developing calculations for two independent cases is simply to demonstrate the method for these extreme cases. The techniques required vary greatly between the two cases and provide insight into the technical challenges of this method. The infinite Gaussian pulse train is addressed in Chapter 4 and the single Doppler bin case in Chapter 5.

The Fisher information is a Riemannian metric that provides a means of measuring the instantaneous information gain, pertaining to the target. In the case of a radar system, this information consists of the target's range and relative speed, represented using the time delay and Doppler shift of the returned signal. By taking this view, one effectively constructs an information landscape in which all points contain distributions of target information.

A good example of optimising the information content of a signal is described by the hunting strategies of bats. In figure 1 the phases of a bats hunting process are shown. The bat echolocation signal incorporates distinct patterns depending on the information required for a specific phase. When the bat is searching for its target, the chirp rate is relatively low, the pulse width is wide and the space between pulses is large. This large inter-pulse spacing allows pulses time to return, where shorter inter-pulse spacing would result in ambiguous range estimates. In phase two, the bat acquires its target and begins its approach. Now there is a need for higher accuracy and the bat adjusts its signal appropriately. The chirp rate increases, the pulse width decreases and the inter-pulse spacing decreases. This trend continues and indeed accelerates as the bat prepares to catch its prey. The chirp rate of phase three is very high, the pulse width is very short and the bat produces many chirps in rapid succession. The modification of the signal parameters by the bat represents a clear case where the optimisation of the signal is being utilised to maintain high information content under changing physical conditions.

The result of these changes of parameter is that in order to construct a method for selecting an optimal configuration a family of Fisher metrics is required. Here each Fisher metric is associated with a specific configuration which may themselves be described as inhabiting a single manifold.

This manifold of Fisher metrics is a submanifold of the manifold of all Riemannian metrics and is, in this thesis, referred to as the configuration manifold, G, as defined by Gil Medrano [13]. On this manifold points represent individual sensor configuration manifolds, each with their own associated Fisher metric. The advantage of viewing the entire family of Fisher metrics for sensor configurations in this way is that, once G is obtained it is possible to determine geodesics on G. These geodesics are determined by solving a system of ordinary differential equations (ODE), the coefficients of which are partial derivatives of the Gil Medrano metric. The objective is to provide formulae for calculating these geodesics without the necessity of using numerical differentiation to set the coefficients of the ODE specifying the geodesics.

The context of the problem and related literature are discussed in Chapter 2. In computing the components of the Fisher matrix for the infinite pulse train scenario, a class of integrals of products of shifted sinc functions were encountered for which formulae were not available in the literature. A method was developed for evaluating these integrals analytically. This method and formulae for the required integrals are presented in Chapter 3. Chapter 4 develops formulae for the Fisher metric and configuration metric for the case of the infinite pulse train scenario and Chapter 5 develops the same for the single Doppler bin case. A short summary and suggestions for future work is appears in Chapter 6 and some extended calculation may be found in the appendix 7.

Chapter 2

Literature Review

Presented here is an extensive, but by no means exhaustive, cross section of the work performed in fields relevant to the information geometric approach to the optimisation of target tracking sensor systems. This chapter focuses on forming a cohesive picture of the advances made across the range of mathematical disciplines required in the execution of this research.

Commencing with a brief outline of the basics of Radar systems in Section 2.1 and followed by an examination of statistics in Section 2.2. Information geometry is discussed in Section 2.3, where Subsection 2.4.1 describes work on exponential families and how such families are connected to the field of information geometry by the related works of Amari. Finally, Section 2.5 is concerned with more recent work relating directly with ideas upon which this thesis is based.

2.1 Radar Basics

The operation of a typical target tracking sensor system, such as radar, involves the transmission of a signal, which travels until it is incident with an object and is reflected back to be picked up by the receivers. Any object in the signal's field of view that is sufficiently reflective can return a signal. Typically only a fraction of the returning data is of interest. In general, the data associated with an object of interest is referred to as target data, where as the undesirable data is referred to as noise. There is a large body of literature describing the operation of such systems. Providing a broad scope of research, these works address topics ranging from the basics of radar system operation [6][34], to information on more specific topics such as sensor scheduling [24] or Riemannian metrics on parameter space [25]. A brief examination of some of the literature relevant to the intersection of radar systems and information geometry is presented here.

2.1.1 Sensor Systems

Beginning with some foundational information, Moran [24] provides a thorough breakdown of sensor management, describing features of signal transmission, construction, range accuracy and Doppler shift. Moran defines a signal $s(t)$ with wave form $w(t)$ and carrier frequency f_c as

$$s(t) = w(t)\cos\left(2\pi f_c t\right). \tag{2.1}$$

If the wave form is assumed to be complex then equation (2.1) becomes

$$s(t) = (\Re w(t))\cos\left(2\pi f_c t\right) - (\Im w(t))\sin\left(2\pi f_c t\right). \tag{2.2}$$

The complex signal can then be considered as consisting of two components, where Moran suggests that the 'in-phase' component, denoted $\mathbf{I}$, may be separated from the 'quadrature', denoted $\mathbf{Q}$, by demodulation against $\sin\left(2\pi f_c t\right)$ and $\cos\left(2\pi f_c t\right)$ respectively. The complex version of the signal in equation (2.2) is

$$s_c(t) = w(t)e^{2\pi i f_c t}. \tag{2.3}$$

The signal voltage at the receiver antenna is described by

$$s_u(t) = As\left(t - 2r\right), \tag{2.4}$$

where A represents the reduction in the amplitude of the received signal (the attenuation) and r is the distance from the transmitter/receiver and the target. The receiver generates heat, which produces 'thermal noise', that to a high degree can be regarded as a white Gaussian process (stochastic process)

$$s_r(t) = s_u(t) + N(t), \tag{2.5}$$

where $N(t)$ represents the noise.

The next feature of the signal that needs to be accounted for is the effect of a moving target on the frequency of the return pulse; the Doppler shift. If the target has a radial velocity, v, relative to the radar then the received signal is

$$s_u(t) = As\left(\alpha t - 2r\right), \tag{2.6}$$

with

$$\alpha = \frac{\left(1 - \frac{v}{c}\right)}{\left(1 + \frac{v}{c}\right)}. \tag{2.7}$$

Here c is the speed of light in the medium and v is the speed of the receiver relative to the medium. Typically $v << c$, in which case

$$\alpha \approx \left(1 - \frac{2v}{c}\right). \tag{2.8}$$

As long as the signal does not significantly exceed the channels coherence bandwidth, "narrow band", then the range $(f_c - \delta, f_c + \delta)$ approximately accommodates the Fourier spectrum, in the case where δ is small compared to f_c. Given a return pulse from a stationary target at the same range as the moving target and shifting it by $f_d = \left(\frac{2v}{c}\right) f_c$, called the Doppler frequency, the complex signal is adjusted to account for this shift as follows

$$s_u(t) = \Re\left(w\left(t - \frac{2R}{c}\right) e^{2\pi i f_c\left(1-\frac{2v}{c}\right)\left(t-\frac{2R}{c}\right)}\right). \tag{2.9}$$

Demodulation of the return signal is performed to remove the carrier frequency, f_c, from equation (2.9). This demodulation is achieved by first 'mixing' the signal, by multiplying it with $\cos(2\pi f t)$, the mixed signal is then low-pass filtered. The demodulated signal is then typically match filtered against a second signal $v(t)$, where v is generally selected to be the same as the fluctuating wave form w, resulting in

$$A_{w,v}(x, f) = \int_R v(t)^\star w(t - x) e^{2\pi i f_c t} dt. \tag{2.10}$$

Moran [24] describes a 'scene' as a function of both the range and the Doppler and assigning a corresponding 'reflectivity' $\rho(t, f)$ to each range and Doppler. The superposition of these signals show that the range and Doppler of the scene, convolved with the ambiguity, produce the return signal.

$$R(\tau, f) = \int\int_{R^2} \rho(\tau', f') A_{w,v}(\tau - \tau', f - f') d\tau' df'. \tag{2.11}$$

Here the ambiguity refers to a two dimensional function of the signal round-trip time delay and the Doppler frequency. Van Trees [35] deals with the ambiguity function from the perspective of hypothesis testing, for the detection of a slowly fluctuating point target. The signal, $s(t)$ is given by

$$s(t) = \sqrt{2} Re\left[r(t) \times e^{j\omega_c t}\right], \tag{2.12}$$

where

$$r(t) = b\sqrt{E_1} f(1 - \tau) \times e^{j\omega_D t} + n(t), \tag{2.13}$$

and ω_c is the carrier frequency, ω_D is the Doppler shift, τ is the time delay, E_1 is the energy in the signal. Additive white noise is modelled by $n(t)$. The null hypothesis, H_0 is when only noise is received, $n(t)$ and H_1 is when the full signal is $s(t)$ is received. To generate a set of sufficient statistics for this hypothesis test we can integrate against any complete orthonormal basis. Van Trees [36] uses the transmitted signal, $s_0(t)$ (i.e. $s(t)$ where $\tau = \omega_D = 0$) as the first element of the orthonormal basis, leading to the sufficient statistic r_1

$$r_1 = \int s_0(t)s(t)dt, \tag{2.14}$$

$$= 2E_1 \int f(t)f(t-\tau) \times e^{j\omega_D t}dt, \tag{2.15}$$

where r_1 is the ambiguity function. The likelihood ratio is

$$\Lambda(r_1) = \frac{p\left(R_1 = r_1|H_1\right)}{p\left(R_1 = r_1|H_0\right)}, \tag{2.16}$$

and the white noise, $n(t)$, with variance σ gives a Gaussian distribution for R_1

$$p\left(R_1 = r_1\right) = \frac{1}{2\pi\sigma^2}\exp\{-\frac{|r_1|^2}{2\sigma^2}\}. \tag{2.17}$$

So the log likelihood, $\log\Lambda$ is

$$\log\Lambda = C|r_1|^2 = C|\phi(\tau,\omega)|^2, \tag{2.18}$$

where $\phi(\tau,\omega)$ is the radar autocorrelation function. The practicality of using multiple sensor systems relies on the effective use of sensor scheduling, which arises as a result of restrictions on which members of a sensor array, some subset of the total number of sensors, are able to utilise measurement data at any given time. Thus, an optimised schedule allows for a more efficient generation of estimates.

Kershaw and Evans [16] describe a typical tracking system explaining that such a system consists of a sensor and a tracker, where the sensor's job is to perform signal processing tasks and obtain estimates of the targets position and velocity. An example of a tracker is the Kalman filter [28], which takes a series of measurements represented by the results of the sensor signal processing, and produces a joint probability distribution which represents a likelihood function. This method enables predictions of the targets future position and velocity, with much greater accuracy than a similar estimate using only a single measurement. This method has a familiar feel, being quite similar to the process of "Dead Reckoning" used in navigation, where current position estimates are made using past positional measurements in combination with estimates of direction and velocity. The sensor and tracker elements of the system must work in tandem while performing their tasks over two separate time frames; the sensor operates over the duration of a received pulse and the tracker on the interval between pulses. Kershaw and Evans [16] determine a tracking filter structure and waveform parameter selection that minimises the variance of the target state vector. The process of determining this structure returns to the methods already described, calculating the Fisher information metric ([16] denotes the FIM: $\mathbf{J}$), which is done for a restricted case, showing that the elements of the FIM are second derivatives of the ambiguity function. The ambiguity function describes the degree of distortion in a returning pulse, as compared to the signal template at the receiver, the matched filter. Provided with the appropriate conditions, the FIM may be calculated, thus enabling determination of the Cramer-Rao lower bound . In [16] the Cramer-Rao lower bound is obtained by taking the inverse of the FIM, an option that was discussed earlier. The Cramer-Rao lower bound is employed in [16] to provide a lower bound for the error covariance matrix and is achieved when the mean squared error of an unbiased estimator is minimised. In general the Cramer-Rao lower bound is found to have an inverse dependence on several features of the pulse train [15]. These are the pulse 'on-time', the number of pulses K and the pulse time variance $\tilde{R}_2$. The argument is made in [16] that, for signal-to-noise ratios allowing the neglect of unwanted signal noise from the ambiguity function, represented by the sensor sidelobes, it is possible to select an optimal receiver. Combining this with the fact that the maximum likelihood estimates are jointly Gaussian with the covariance matrix g^{-1} suggests that the covariance matrix is a reasonable interpretation of an optimal waveform selection receiver. The FIM is expressed in the notation of [16] as $g = \eta\mathbf{U}(\theta)$, where η as the signal-to-noise ratio, $\mathbf{U}(\theta)$ a symmetric matrix with real valued constants for any specific waveform, combined with a transform matrix $\mathbf{T} = \text{diag}(c/2, c/2\omega_c)$, where c is the propagation speed of the wave form and w_c is the transmission frequency. The form of the measurement covariance matrix $\mathbf{N}(\theta)$ is

$$\mathbf{N}(\theta) = \frac{1}{\eta}\mathbf{T}\mathbf{U}^{-1}\mathbf{T}. \tag{2.19}$$

This form of the covariance matrix is valid only for transmission of constant energy waveforms.

2.2 The Geometry of Statistics

The construction of a target tracking sensor optimisation method by information geometric means is a relatively unexplored application of the theory and comes as a new instalment in a long chain of advances from disparate research topics and fields. The development of these methods begins with the work of Fisher, who in his 1921 paper [11], laid the foundations for the analysis of data by application of efficient statistics, where efficiency in this context refers to minimising the number of observations required to achieve accurate results. This work was later developed by Rao who, in 1945 [17], first introduced the idea of a Riemannian metric to the field of statistics, applying differential geometric methods to probability distributions in order to analyse statistical data.

In the 1970s, the statistician Efron, [10][9], expanded on this idea, with an examination of 'exponential families', in which he showed that the second-order information loss and second-order variance of a maximum likelihood estimator are related to the statistical curvature.

Efron introduced a notion of curvature in the context of a space of density functions [9]. More specifically, he considered one parameter families of densities of the form

$$F = \{f_\theta(x) : \theta \in \Theta\}, \tag{2.20}$$

where $\theta \in \Theta$ is a possibly infinite interval of the real line. The functions

$$L_\theta(x) = \log f_\theta(x),\ \dot{L}_\theta(x) = \frac{\partial}{\partial\theta} L_\theta(x),\ \ddot{L}_\theta(x) = \frac{\partial}{\partial\theta} \dot{L}_\theta(x), \tag{2.21}$$

were defined and smoothness and uniform growth conditions were assumed for $f \in F$ so that the expectations exist for these functions. Then $E_\theta L_\theta = 0$ and $E_\theta L_\theta = -E_\theta \ddot{L}$. Efron defines $i_\theta = -E_\theta \ddot{L}$ and so the covariance matrix of $\left(L_\theta,\ \ddot{L}_\theta\right)$ is

$$M_\theta = \begin{bmatrix} E_\theta L_\theta^2 & E_\theta L_\theta \ddot{L}_\theta \\ E_\theta L_\theta \ddot{L}_\theta & E_\theta \ddot{L}_\theta^2 - i_\theta^2 \end{bmatrix}. \tag{2.22}$$

The statistical curvature of the family F is defined as

$$\gamma_\theta = \left(\frac{|M_\theta|}{i_\theta^3}\right)^{\frac{1}{2}}. \tag{2.23}$$

The quantity i_θ is the Fisher information. Efron goes on to show that for the special case of one parameter exponential families, the curvature is zero and that the formula for γ_θ is entirely analogous to the usual formula for the curvature of a function defined on the real line. Additionally Efron defines

$$U_\theta(x) \equiv \frac{\dot{l}_\theta(x)}{\dot{i}_\theta} + \theta, \tag{2.24}$$

which is an unbiased estimator for θ near θ_0. The statistical curvature can be described as the derivative, at $\theta = \theta_0$, of the unexplained fraction of the standard deviation of U_θ given U_{θ_0}. In the event that the exponential families discussed by Efron possess curvature, they may be described as representing geodesics through the space of probability distributions [9]. The ability to consider these curved exponentials as geodesics on a probability space is a feature that will be explored further later on.

It is important to note that the square of the statistical curvature, γ_θ^2 describes, approximately, the amount by which the variance exceeds the Cramer-Rao lower bound. Here the Cramer-Rao lower bound [17] defines the minimum variance of an unbiased estimator and Efron's proposed unbiased estimator U_{θ_0} turns out to be a local formulation of the Fisher score, which achieves this bound.

2.3 Information Geometry

In 1982 the pioneer of information geometry, Shun-ichi Amari, explored exponential families from the perspective of differential geometry [1]. Amari uses the typical notation for probability functions, $p(x|\theta)$ and suggests that, if one considers points, θ, in the space of distributions, S^n, then these points can be thought of as carrying the function $\ell(x|\theta) = log(p(x|\theta))$ of x. Letting T_θ be the tangent space of S^n with a basis, $e_i(i = 1, \cdot \cdot \cdot, n)$, Amari states that since each point θ has an associated $\ell(x|\theta)$, the basis for the tangent space at θ, $e_i(\theta)$, can be represented as

$$e_i(\theta) = \partial_i \ell(x|\theta). \tag{2.25}$$

This representation is commonly used when constructing the Fisher information metric (FIM) which, with a small amount of additional effort, may be defined for θ. Following the method of [1] the FIM is

$$g_{ij}(\theta) = \mathbf{E}_\theta\{\partial_i \ell(x|\theta)\partial_j \ell(x|\theta)\}, \tag{2.26}$$

noting that the Fisher score used by Efron and the FIM of equation (2.26) are directly related. The Fisher metric, rather than the score $l(\theta)$, is of interest and is obtained by the additional calculation of the expectation $g(\theta) = E\left[l(\theta)\right]$. Amari [1] describes the metric tensor for the probability space, S^n, which provides a means of determining the length, ds, of the line element, $d\theta$.

$$ds^2 = d\theta \cdot d\theta = \sum_{i,j} g_{ij} d\theta^i d\theta^j. \tag{2.27}$$

However, the Cramer-Rao theorem mentioned in Section 2.2, states that *the covariance matrix of any unbiased estimator $\hat{\theta}$ of θ cannot be smaller than the inverse of the Fisher information matrix*

$$E_\theta\left\{\left(\hat{\theta}^i - \theta^i\right)\left(\hat{\theta}^j - \theta^j\right)\right\} - g^{ij} \geq 0. \tag{2.28}$$

According to Amari, viewing equation (2.27) with the Cramer-Rao theorem in mind shows that, for some Riemannian space, the length ds is a measure of the difference between the distributions $p(x|\theta)$ and $p(x|\theta + d\theta)$. Taking the integral of ds over all curves connecting a pair of points in some Riemannian space, provides a curve of minimum length with respect to the metric, a geodesic. These observations cement the idea that a Riemannian metric, such as the FIM, is an appropriate object to apply to probability spaces.

2.4 A Manifold of Metrics

In a 1991 paper, Gil Medrano and Michor [13] explored the manifold of all Riemannian metrics, following from earlier work by Michor [23]. The authors consider a Riemannian space $(\mathcal{M}, G)$, consisting of the space of all Riemannian metrics $\mathcal{M} = \mathcal{M}(\mathrm{M})$ on a manifold M. Here $\mathcal{M}$ is itself an infinite dimensional manifold and G is smooth Riemannian metric defined as

$$G_g(h, k) = \int_M tr(g^{-1}hg^{-1}k)vol(g), \tag{2.29}$$

with $\mathrm{vol}(g) = \sqrt{\det(g)}dx$. The curvature of M may be determined by first developing the covariant derivative of a vector field along geodesics. These geodesics on $\mathcal{M}$ represent a stable state, where the energy functional has been minimised. Gil Medrano and Michor [13] use the covariant derivative to facilitate the construction of the Jacobi equation, as presented in Do Carmo [21], in terms of the covariant derivative D, the Riemann curvature tensor R, $\dot{\gamma} = d\gamma(t)/dt$ the tangent vector field and the Jacobi field J.

$$\frac{D^2}{dt^2}J(t) + R\left(J(t), \dot{\gamma}(t)\right)\dot{\gamma}(t) = 0, \tag{2.30}$$

where the solution of this equation defines a Jacobi field. Such a field represents a tangent space to a geodesic on M, inhabited by a family of infinitesimal variations of the geodesic. Do Carmo [21] expands on the idea of Jacobi fields, by first developing the basic outline of Gauss' Lemma which states that, given a Riemannian manifold M, a sphere of sufficient size, centred at a point p on M is perpendicular to any geodesic through p. Indeed, given the point p and a velocity vector $v \in T_pM$, then a geodesic γ_v is defined, with the properties

$$\gamma(0) = p \;\text{ and }\; \dot{\gamma}(0) = v \in T_pM. \tag{2.31}$$

Following [27], there exists an exponential mapping from the tangent space T_pM at a point p on the manifold M, given by

$$e : T_pM \to M, \tag{2.32}$$

where $t \mapsto \exp(tv)$ represents a unique geodesic through the point p with tangent v. A feature directly related to the Jacobi field is that of conjugate points, where according to [30] a pair of points, $p, q \in \gamma$, are called conjugate if there exists a non-zero Jacobi field that vanishes at p and q. This is equivalent to stating that, given a second geodesic infinitesimally close to γ and passing through the points p and q, then p and q are conjugate. Importantly, the simple existence of any second geodesic through p and q does not mean that they are conjugate. The Jacobi field is a requirement for this condition to be true. The feature of conjugate points most relevant in the examination of the configuration metric is that, for Riemannian geometry, the conjugate points represent a localised failure of γ to be a curve of minimum length through p and q [30], meaning that the curve through p and q may be the shortest path locally, but not necessarily globally. An example of local failure to be path length minimising is on the sphere, where, tracing a curve originating at the north pole to some point on the sphere, it is clear that this curve cannot be of minimal length if it first passes through the south pole before reaching the point. The path length between these points depends on in which direction the geodesic is followed. The existence of conjugate points can be used as a test to establish that the curve under investigation is reliably a shortest path, where failure to possess a conjugate could confirm a shortest path. The Jacobi field describes the space of vectors tangent to a geodesic curve and according to [30] the Jacobi equation shows that in the event that the covariant derivative $\frac{D^2}{dt^2}J(t)$ is the Levi-Civita connection [33], which is an affine torsion free metric connection, then the geodesics of the connection are the geodesics of the metric. It is worth noting that the Levi-Civita connection is able to differentiate sections of the vector bundle along tangents of the underlying manifold. The concept of a vector bundle (or fiber bundle) is discussed in detail by [33] and is used by Gil Medrano [13], who defines the fiber bundle $S^2T^{\star}M$ on M, additionally proposing a connector, K, which is related to the Christoffel symbol in a chart by the equation

$$K(g, h; k, \ell) = (g, \ell - \Gamma_g(h, k)) \,. \tag{2.33}$$

Here, h and k represent modified elements of the Fisher matrix, for instance the partial derivatives $\partial_{\alpha,\beta}g$. The connector, K, is directly related to the process of parallel transport on the bundle, connecting the local geometry by identifying fibers over nearby points. Making use of this relationship, the curvature for the canonical Riemannian metric on M is determined. This method for constructing the manifold of all Riemannian metrics allows for the calculation of a metric describing the 'distance' between two arbitrary Riemannian manifolds. Again, this is a feature that will be of significant use in the construction of a completed target tracking sensor optimisation method.

2.4.1 Exponential Families

Returning in greater depth to exponential families, both Murray [27] and Amari [2] investigate the idea that exponential families possess an affine structure. Murray claims that utilisation of the log-likelihood can be considered as treating sets of probability distributions as affine geometries. The benefit of viewing these distributions as affine emerges from the fact that exponential families are affine spaces and as such are geometrically flat. This concept is of interest with regards to the target tracking sensor optimisation method as the log-likelihood is a common feature in radar calculations.

Affine connections are discussed in [27], [13], [33], where the connection of primary interest in this case is the Levi-Civita connection. This connection is a Riemannian metric connection which is both affine and torsion free, meaning that it is invariant under transformation (symmetrical). As its name implies, the Levi-Civita connection, connects nearby tangent spaces and does so in such a way that tangent vectors are differentiable as though they were functions on the manifold, rather than the tangent space. This characterises the idea of parallel transport, which may be summarised as connecting the geometry of nearby points by transporting vectors along curves on the manifold such that the vectors remain parallel with respect to the connection. Amari proposes that it is necessary to understand three types of metric connection, defined in [2]. However, here it is only necessary to examine the α-connection, which is defined as

$$\Gamma^{(\alpha)}_{ij,k} = \langle \nabla^{(\alpha)}_{\partial_i} \partial_j, \partial_k \rangle. \tag{2.34}$$

The α-connection should be understood alongside an investigation of dual connections, which Amari regards as vital to the subject of information geometry, but is not further investigated here. Another relationship of importance is the link between the FIM and an object called the Kullback-Leibler divergence [19] which is also considered by both Murray [27] and Amari [2]. The Kullback-Leibler divergence is typically expressed as

$$D_{KL}\left(P||Q\right) = \int_{\chi} \log\left(\frac{P(x)}{Q(x)}\right) dP, \tag{2.35}$$

where P is a continuous probability measure with respect to the measure Q, describing the amount by which the distribution P differs from Q and for this reason it is also known as the relative entropy. Considering, for instance, a pair of distributions P_x and Q_x. Taking the Hessian of the Kullback-Leibler divergence and evaluating at $\theta = \theta_0$ results in the expression

$$g_{ij}\left(\theta\right) = \frac{\partial^2}{\partial\theta^i \partial\theta^j} D_{KL}\left(P\left(\theta\right)||P\left(\theta_0\right)\right), \tag{2.36}$$

where the evaluation at $\theta = \theta_0$ indicates that the two distributions are close together. The Hessian of the Kullback-Leibler divergence is shown to be the Fisher information metric which, for practical purposes such as formulation of the manifold of metrics, is more appropriate, since the Kullback-Leibler divergence does not represent a true metric where as the FIM does. The Kullback-Leibler divergence is also related to another information measure, the Shannon information, sometimes referred to as the Shannon entropy and provides a different view to that of the Fisher information, instead focusing on the communication of information. If a higher measurement for the Shannon entropy is obtained then the message being sent is regarded as containing a higher information content. This is due to the idea that learning an unlikely event has occurred provides more information than learning a likely event has occurred. Frieden [4] shows that the relationship between the Shannon entropy, $H[X]$ and the Kullback-Leibler divergence is expressed as

$$H[X] = \log(n) - D(P_x || Q_x), \tag{2.37}$$

where n is the number of values X can take. The standard form of the Shannon entropy $H[X]$ is

$$H[X] = -\sum_{i=1}^{n} p(x_i) \log(p(x_i)). \tag{2.38}$$

Frieden [4] suggests the Fisher information is more sensitive than the Shannon information. This is because the Fisher information is a local measure of the system with a dependency on the probability distribution. The Shannon information, on the other hand, is a global measure and behaves as a probability distribution function itself. Amari's work forms a foundation for the field of information geometry, a field that is concerned with probability distributions and the mechanisms by which they may be translated into the language of differential geometry. For example, Amari describes probability densities in terms of objects such as a metric or an affine connections such as the Levi-Civita connection [33]. Another piece of investigative machinery described by Cheng et al [5] is the relationship between the Ricci-curvature tensor field and the amount of information it is possible for a given sensor network to obtain. This relationship begins with the definition of a statistical manifold, as follows.

Given a state of interest, θ, in the parameter space $\Theta \in \mathbb{R}^n$. The measurement, x, in the sample space $X \in \mathbb{R}^m$ is an example of a probability distribution $p(x|\theta)$.

The points, $S(\theta)$, of the manifold S each describe a probability distribution, $p(x|\theta)$. A parametrised family of probability distributions, denoted $S = \{P(x|\theta)\}$, form an n-dimensional statistical manifold where θ acts as a coordinate system. For such a parametrised family of probability distributions the Fisher information matrix plays the role of a Riemannian metric tensor. For continuity the FIM is reiterated here in a modified form, denoted $G(\theta) = [g_{ij}(\theta)]$ and defined

$$g_{ij}(\theta) = E\left(\frac{\partial \ln(p(x|\theta))}{\partial \theta_i} \cdot \frac{\partial \ln(p(x|\theta))}{\partial \theta_j}\right), \tag{2.39}$$

where again E denotes the expectation. The Fisher information measures the ability of the random variable x to discriminate the values of the parameter θ' from θ, for θ close to θ' and, as mentioned earlier, is used to define the metric of a statistical manifold S, a smooth Riemannian manifold, providing a sense of distance between distributions. Three methods are proposed in [5] to achieve such a calculation, the integrated Fisher information distance, the Kullback-Leibler divergence and the Energy difference. Recall the definition of the infinitesimal squared distance seen in equation (2.27)

$$ds^2 = \sum_{ij} g_{ij} d\theta_i d\theta_j = d\theta^T G(\theta) d\theta, \tag{2.40}$$

where the alternate form, the final factor of the equality, is of use. Consider a curve $\theta(t) \in \Theta$ joining distributions $\theta_1 = \theta(t_1)$ and $\theta_2 = \theta(t_2)$ for $t_1 \leq t \leq t_2$. Equation (2.41) determines the distance between these distributions

$$\mathcal{D}\left(\theta_1, \theta_2\right) = \int_{t_1}^{t_2} \left(\sqrt{\left(\frac{d\theta}{dt}\right)^T G(\theta) \left(\frac{d\theta}{dt}\right)} \right) dt. \tag{2.41}$$

The integrated Fisher information distance is the integral along the curve $\theta(t)$ that minimises equation (2.41).

$$\mathcal{D}_F\left(\theta_1, \theta_2\right) = \min_{\{\theta(t):\theta(t_1)=\theta_1,\theta(t_2)=\theta_2\}} \int_{t_1}^{t_2} \left(\sqrt{\left(\frac{d\theta}{dt}\right)^T G(\theta) \left(\frac{d\theta}{dt}\right)} \right) dt. \tag{2.42}$$

The integrated Fisher information distance can be difficult to calculate and in such cases an alternative to the integrated Fisher information distance is to utilise the Kullback-Leibler divergence [18]. Despite the Kullback-Leibler divergence allowing the information distance to be approximated without statistical manifold geometry, the fact that it does not constitute a genuine metric limits its value. It is still possible to take advantage of the property discussed previously, that the Hessian of the KLD once calculated is in fact the FIM. The Hessian, as discussed in [27],[33] is a matrix of second order partial derivatives, the definition of which is expressed here as per [33].

Consider a function $f : V \to \mathbb{R}$, where $v_1 ... v_n$ is a basis for a vector space V. Then for any two vectors, v, $w \in V$, the second derivative is

$$f_{\star\star}(v)(w) = \frac{d^2}{dt^2}\Big|_{t=0} f(v + tw). \tag{2.43}$$

Then given $\phi : \mathbb{R}^n \to \mathbb{R}$, defined by

$$\phi(a^1 ... a^n) = f\left(\sum_{i=1}^{n} a^i v_i\right), \tag{2.44}$$

then

$$f_{\star\star}\left(\sum_{i=1}^{n} b^i v_i\right)\left(\sum_{i=1}^{n} c^i v_i\right) = \sum_{i,\ j} \frac{\partial^2 \phi}{\partial x^i \partial x^j}(b) \cdot c^i c^j, \tag{2.45}$$

where, according to Spivak [33], the map $f_{\star\star}(v) : V \to \mathbb{R}$ is the Hessian of f at $v \in V$.

There are two final pieces of machinery worth noting here, the Riemann curvature tensor and the Ricci curvature tensor. The Riemann curvature tensor, $\mathbf{R}^l_{ijk}$, is defined by

$$\mathbf{R}^l_{ijk} = \frac{\partial}{\partial \theta^j}\Gamma^l_{ik} - \frac{\partial}{\partial \theta^k}\Gamma^l_{ij} + \sum_s \left(\Gamma^l_{js}\Gamma^s_{ik} - \Gamma^l_{ks}\Gamma^s_{ij}\right), \tag{2.46}$$

where

$$\Gamma^k_{ij} = \sum_{l=1}^{n} g^{kl}\frac{1}{2}\left(\frac{\partial g_{il}}{\partial y^j} + \frac{\partial g_{jl}}{\partial y^i} - \frac{\partial g_{ij}}{\partial y^l}\right), \tag{2.47}$$

defines the Christoffel symbol [33], [7]. The Ricci curvature tensor, $\mathbf{R}_{ij}$, is defined by

$$\mathbf{R}_{ij} = \sum_l \mathbf{R}^l_{ilj} = \sum_l \left(\frac{\partial}{\partial \theta^l}\Gamma^l_{ij} - \frac{\partial}{\partial \theta^j}\Gamma^l_{il}\right) + \sum_{l\ m} \left(\Gamma^l_{ij}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{jm}\right). \tag{2.48}$$

The main feature of interest here is the Ricci scalar (scalar curvature) which is described by [5] as the 'simplest curvature invariant of a Riemannian manifold'. The Ricci curvature tensor is a contraction of the Riemann curvature tensor and represents the amount by which a section of a curved Riemannian metric deviates from standard Euclidean space.

The scalar curvature is defined as the trace of the Ricci curvature

$$\mathbf{R} = \sum_{ij} g^{ij}\mathbf{R}_{ij}. \tag{2.49}$$

The Ricci curvature tensor controls the the growth rate of the volume of metric balls in a Riemannian manifold.

2.5 Current Parallel Work

This thesis builds on the information geometric approach to sensor configuration in [32] and this construction is of such importance it is worth reiterating some of the main ideas. The general construction of the manifold of metrics and the corresponding sensor manifold is discussed, developing all the familiar features incorporated in the radar problem under investigation. The framework for calculation is reiterated here though purely for additional familiarisation with the notation.

The collected set of measurements from individual sensors is expressed as

$$x = \{x_i\}_{i=1}^N. \tag{2.50}$$

Assuming independent measurements the probability of this set of measurements x, given the target can be found at location θ is

$$p(x|\theta) = \prod_{i=1}^{N} p_i(x_i|\theta). \tag{2.51}$$

Equation (2.52) calculates the likelihood ratio of obtaining the same measurement, x, from a target located at θ' rather than at θ

$$L\left(\theta, \theta'\right) = \log \frac{p(x|\theta)}{p(x|\theta')}. \tag{2.52}$$

The average over all measurements is the Kullback-Leibler divergence

$$D\left(\theta || \theta'\right) = E_x\left[L\left(\theta, \theta'\right)\right] = \int p(x|\theta) \log \frac{p(x|\theta)}{p(x|\theta')}, \tag{2.53}$$

here it is reiterated that the Kullback-Leibler divergence does not satisfy the triangle inequality and is not a true metric for the space [18]. However, its Hessian, the Fisher metric, is. The total information when traversing a curve $\gamma(t) \subset M$ is

$$I(T) = \int_0^T g\left(\gamma'(t), \gamma'(t)\right) dt, \tag{2.54}$$

and is equivalent to the energy functional [8].

There is a conceptual framework in development, principally by Williams [32], in which the information space is endowed with an *information flow*, that determines the rate at which the geodesics must be traversed in order to optimise the magnitude of the information exchanged between the target and the sensors. Traversing the geodesic at a rate slower than $\sqrt{g\left(\gamma'(t), \gamma'(t)\right)}$ results in the 'flow' pressing against the target from behind. The flow pressure concept and the mechanism by which interaction with the target leads to an information gain by the sensors are not subjects that have, at this stage, been investigated in any meaningful way.

Traversing the geodesic at a rate faster than $\sqrt{g\left(\gamma'(t), \gamma'(t)\right)}$ is currently being interpreted as producing a larger change in angle between sweeps of a bearing only radar and as such provides fewer but better quality position estimates at the sensor. Better position estimates obtained by the sensor is quite clearly not a desirable outcome for the target.

The arclength of the path is

$$l_g(\gamma) = \int_0^T \sqrt{g\left(\gamma'(t), \gamma'(t)\right)} dt, \tag{2.55}$$

where the extrema of the information functional and the arc length are the same as sets. The geodesics minimise the information functional if traversed at velocity

$$\frac{dl_g}{dt} = +\sqrt{g\left(\gamma'(t), \gamma'(t)\right)}, \tag{2.56}$$

this is equivalent to the arc-length parametrisation of the geodesic.

A set, $\gamma = \{\lambda_i\}_{i=1}^{N}$, of sensor locations defines a particular choice of sensor configurations. The Fisher-Rao metric g can be viewed as a function of both the sensor locations, Γ, and the target location, θ. Consider a normal distribution, with mean θ' and covariance g^{-1}. The inverse of the Fisher metric, g, represents the Cramer-Rao lower bound which defines a lower bound on the variance of an unbiased estimator.

Taking the ideas proposed in [32], an exploration of a simple target tracking sensor system can be undertaken. This involves setting up a system in which a configuration consisting of one or more fixed sensors are positioned to perform a 'sweep', and on each sweep detects the same target, only the bearing to the target is obtained by this means. Examining [32], the configuration manifold for this system is discussed. The integrals for the configuration metric require some consideration as the comparison between the 'bearings' configuration manifold and the Gaussian pulse configuration manifold should be of great interest and enable a deeper understanding of how the curvature of these manifolds effects detection.

An investigation of the bearing-only case begins with the notation for a set of N landmarks, λ_n $(n = 1, ..., N)$. The establishment of position estimations for an unknown point θ using these landmarks requires the use of a special case of a probability distribution function known as the von Mises-Fisher distribution. This is a distribution defined on the $(p-1)$-dimensional sphere in $\mathbb{R}^p$. The special case in question, $p = 2$, is called the von Mises distribution and gives the probability distribution on the 1-dimensional circle. The von Mises distribution

$$\tau_n \sim p_n(\cdot|\theta) = \frac{e^{\kappa \cos(\cdot - \arg(\theta - \lambda_n))}}{2\pi \boldsymbol{I}_0(\kappa)}, \tag{2.57}$$

provides angle measurements (bearings) from the landmarks with respect to θ. Here κ is known as the concentration parameter, a fixed parameter representing the degree of concentration of points distributed about the mean (usually denoted μ,). At this stage κ is regarded as being independent of n. The Fisher information for this von Mises distribution is calculated, observing first that the full likelihood is expressed as

$$p\left(\tau = (\tau_n)_{n=1}^{N} | \theta\right) = \prod_{n=1}^{N} p(\tau_n|\theta) = \frac{1}{\left(2\pi \boldsymbol{I}_0(\kappa)\right)^N} e^{\kappa \sum_{n=1}^{N} \cos(\tau_n - \arg(\theta - \lambda_n))}. \tag{2.58}$$

Here the distinction between a probability and a likelihood should be noted. The calculation of a probability is one that is performed prior to data being available to adequately estimate an outcome. A likelihood, on the other hand, is calculated after the required data has been collected. Additional note should be made of the log-likelihood, which appears as a useful tool here since the log and the actual function achieve their respective maximums at the same point.

The log-likelihood may then be used when calculating a maximum likelihood estimation (MLE).

$$\ell(\theta|\tau) = \kappa \sum_{n=1}^{N} \cos\left(\tau_n - \arg(\theta - \lambda_n)\right). \tag{2.59}$$

Considering equation (2.59), some further advantages of taking the log-likelihood are the conversion of the product, in equation (2.58), into a sum and the elimination of arbitrary constants. Further development requires a differentiation of the log-likelihood with respect to θ

$$d_\theta \ell = \kappa \sum_{n=1}^{N} \sin\left(\tau_n - \arg(\theta - \lambda_n)\right) d_\theta \arg(\theta - \lambda_n). \tag{2.60}$$

The form of the Fisher information equation, as defined in [2], is written

$$\mathbf{FI}(\theta) = \mathbb{E}_{\tau|\theta}\left[d_\theta \ell \otimes d_\theta \ell\right], \tag{2.61}$$

$$= \mathbb{E}_{\tau|\theta}\left[\kappa \sum_{n=1}^{N} \sin\left(\tau_n - \arg(\theta - \lambda_n)\right) d_\theta \arg(\theta - \lambda_n) \otimes \kappa \sum_{m=1}^{N} \sin\left(\tau_m - \arg(\theta - \lambda_m)\right) d_\theta \arg(\theta - \lambda_m)\right]. \tag{2.62}$$

Taking into account that the sine function is odd, some additional simplification is made, reducing the Fisher information to

$$\mathbf{FI}(\theta) = \mathbb{E}_{\tau|\theta}\left[d_\theta \ell \otimes d_\theta \ell\right] = \kappa\left(1 - \frac{I_2(\kappa)}{2I_0(\kappa)}\right) \sum_{n=1}^{N} \sin\left(\tau_n - \arg(\theta - \lambda_n)\right) d_\theta \arg(\theta - \lambda_n). \tag{2.63}$$

Here the factors $I_0(\kappa)$, $I_2(\kappa)$ are Bessel functions of the zeroth and second order respectively. Further calculation is certainly possible, however, there is little utility following this example further. The formulation of this problem represents a basis from which the construction of a target tracking sensor optimisation method has been pursued. Other methods of calculating, as well as additional motivation from van Trees [37] have resulted in a structurally reminiscent but content modified formulation of this calculation. In order to examine a bearings only system in a different scenario the work of Moreno-Salinas [26] is considered, in which bearings only target localisation methods are used in estimating the position of targets located under water. A slightly modified expression describing the relationship between the Cramer-Rao lower bound and the FIM is included here, illustrating how this relationship describes the variance of an unbiased estimator

$$\text{var}\left\{\hat{q}\right\} := \text{tr}(\text{Cov}\left\{\hat{q}\right\}) = \text{tr}\left(E\left\{(\hat{q} - q)(\hat{q} - q)^T\right\}\right) \geq \text{tr}(CRLB(q)). \tag{2.64}$$

This work, again, follows the well trodden path to acquisition of the FIM and, as discussed in previous cases, the optimisation is achieved by minimisation of the Cramer-Rao lower bound. It is important to note that, as in other papers (citation needed), the D-optimality criterion is considered a perfectly valid method by which to achieve optimality rather than via the Cramer-Rao lower bound, where D-optimality instead maximises the determinant of the FIM. According to [26], the maximisation of the FIM determinant has the effect of minimising the volume of the uncertainty ellipsoid of the target estimate. Where this type of uncertainty ellipsoid can typically be seen as below

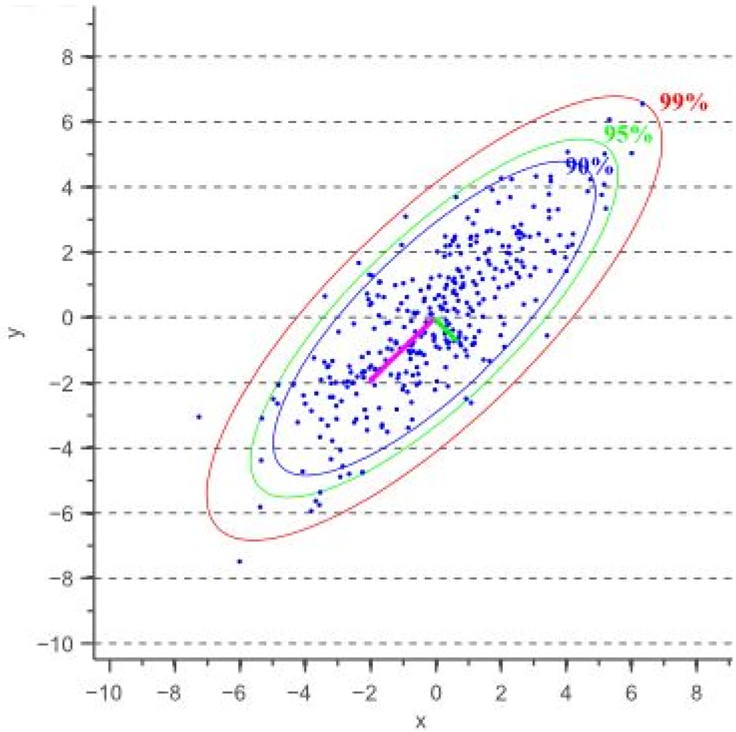

Figure 2.1: An uncertainty ellipsoid, or confidence region arises from repeated measurement and calculation of uncertainty for a set of data. Comparing these sets of measurements results in the 'true' value being contained within a particular confidence region some percentage of the time. In the image above, the red ellipse captures the true value in 99% of cases, the green in 95% and the blue in 90% .

http://www.visiondummy.com/2014/04/draw-error-ellipse-representing-covariance-matrix/

and is also commonly referred to as the confidence region. An interesting observation here is that the D-optimality can lead to errors due to rapid improvement in information in only one dimension, giving a very large FIM determinant, while other dimensions contain little-to-no information. It is also possible to make use of A-optimality, which minimises the trace of the Cramer-Rao lower bound, or another alternative method E-optimality, which takes the minimum eigenvalue of the FIM and maximises it. According to [26], using A or E optimality can eliminate the type of error to which D-Optimality is subject, though these methods are subject to variation under scalar changes to parameters or linear transformation of output, which D-optimality is not. Nonetheless [26] elects to utilise A-optimality.

Much of this work outlines the method by which optimal sensor locations are expressed and selected. This is obviously a valuable notion, but since their method does not utilise information geometry it is of interest but not of use with respect to the calculation under way. It is also noted that an adequate model for system noise is imperative if the sensor configuration is to be optimally selected; while this seems obvious it is well worth keeping in mind. In practical scenarios, the location of the target is typically known only within some error region. The geographical relocation of sensors is therefore performed with the express purpose of minimising the trace of the Cramer-Rao lower bound, thus minimising the error region. Obviously, since the Cramer-Rao lower bound is the inverse of the FIM this describes the same type of requirement seen elsewhere, where the maximisation of the FIM is desired, thereby maximising the information content of points within the confidence region.

This concludes the background material supporting the analysis of target tracking sensor systems by the methods of information geometry.

Chapter 3

Integrals of Products of Shifted Sinc Functions

The calculation of the Fisher metric for an infinite pulse-train, performed in Chapter 4, requires the values of definite integrals of the forms in equations (3.1) and (3.2).

For $z \in \mathbb{Z}$, let ψ_z denote the unnormalised sinc

$$\psi_z = \frac{\sin(x - z\pi)}{x - z\pi}. \tag{3.1}$$

The integrals of interest are of the form

$$I = \int_{-\infty}^{\infty} \prod_{k=1}^{K} \prod_{l=1}^{L} \psi_{z_k} \psi'_{w_l} dx, \tag{3.2}$$

where $\{z_k, w_l : k = 1, 2, ..., K,\ l = 1, 2, ...L\}$ is a collection of (not necessarily distinct) integers. To the author's knowledge, values for integrals in equation (3.2) have not appeared in the literature. A method for evaluating these integrals in general, for all combination of K and L, is presented in Section 3.1. In particular, each of these integrals will be reduced to an integral of linear combinations of terms of the form $\sin^{n_1} x / \sin^{n_2} x$ in order that closed form formulae may be found. The specific versions of the integrals in Section 3.2 needed in Chapter 4 are those with $K + L = 4$ and $L \leq 2$. There are 21 such integrals and since these play a crucial role in this work, closed-form solutions are listed in Section 3.2. Full details for the proofs for these 21 formulae appear in Section 3.3.

3.1 A General Method

Let $a > M = \max\{|z_k\pi|, |w_l\pi| : k = 1, 2, ..., K, l = 1, 2, ...L\}$, where $z,\ z_k,\ w$, and w_l are integers. Define

$$I_a = \int_{-\infty}^{\infty} \prod_{k=1}^{K} \prod_{l=1}^{L} \frac{\sin(x - z_k\pi)}{(x - z_k\pi)} \frac{(x - w_l\pi)\cos(x - w_l\pi) - \sin(x - w_l\pi)}{(x - w_l\pi)^2} dx. \tag{3.3}$$

Making use of the identities $\sin(x - z\pi) = (-1)^z \sin(x)$ and $\cos(x - z\pi) = (-1)^z \cos(x)$,

$$I_a = \gamma \int_{-a}^{a} \prod_{k=1}^{K} \frac{\sin(x)}{x - z_k\pi} \prod_{l=1}^{L} \left(\frac{\cos(x)}{x - w_l\pi} - \frac{\sin(x)}{x - w_l\pi} \right) dx, \tag{3.4}$$

where $\gamma = (-1)^{\sum_{k=1}^{K} z_k + \sum_{l=1}^{L} w_l}$.

The Product of binomials may be written conveniently as

$$\prod_{l=1}^{L}(A_l - B_l) = \sum_{v\in\mathbb{Z}_2^L} (-1)^{L-\eta_v} = \prod_{l=1}^{L} A_l^{vl} B_l^{1-vl}, \tag{3.5}$$

where $v = (v_1, v_2, ..., v_L)$ and $\eta_v = \sum_{l=1}^{L} v_l$. Applying this expansion to the product over l in equation (3.4),

$$I_a = \gamma \int_{-a}^{a} \sum_{v\in\mathbb{Z}_2^L} (-1)^{L-\eta_v} \frac{\sin^{K+L-\eta_v}(x) \cos^{\eta_v(x)}}{\prod_{k=1}^{K} \prod_{l=1}^{L} (x - z_k\pi)(x - w_l\pi)^{2-\eta_l}} dx. \tag{3.6}$$

The next step is to expand the product in equation (3.6) by partial fractions. To do so, the shift parameters z_k and w_l are re-indexed as a list of distinct integers of two classes; those appearing among the derivatives ψ'_w and those appearing only among the shifted sinc functions ψ_z. Set,

$$Q = \{q \in \mathbb{Z} : \exists l \in 1, 2, ..., L \text{ with } q = w_l\},$$
$$P = \{p \in \mathbb{Z} : \exists k \in 1, 2, ..., K \text{ with } P = z_k, p \notin Q\}.$$

Set $R = |Q|, S = |P|$, let m_r denote the multiplicity of $x - p_r\pi$ in $\prod_{k=1}^{K}(x - z_k\pi)$ and for $v \in \mathbb{Z}_2^L$, let $n_{v,s}$ denote the multiplicity of $x - q_s\pi$ in $\prod_{k=1}^{K} \prod_{l=1}^{L} (x - z_k\pi)(x - w_k\pi)^{2-\eta_v}$. With this notation equation (3.6) is

$$I_a = \gamma \int_{-a}^{a} \sum_{v\in\mathbb{Z}_2^L} (-1)^{L-\eta_v} \frac{\sin^{K+L-\eta_v}(x) \cos^{\eta_v(x)}}{\prod_{r=1}^{R} \prod_{s=1}^{S} (x - p_r\pi)^{m_r} (x - q_s\pi)^{n_{v,s}}}, \tag{3.7}$$

where the shift parameters $\{p_1, p_1, ..., p_R, q_1, q_2, ..., q_S, \}$ form a collection of distinct integers. This allows the decomposition of each term into partial fractions,

$$I_a = \gamma \int_{-a}^{a} \sum_{v\in\mathbb{Z}} (-1)^{L-\eta_v} \left\{ \sum_{r=1}^{R} \sum_{m=1}^{m_r} \frac{A_{v,r,m} \sin^{K+L-\eta_v(x)} \cos^{\eta_v}(x)}{(x - p_r\pi)^m} \right.$$
$$\left. + \sum_{s=1}^{S} \sum_{n=1}^{n_{v,s}} \frac{B_{v,s,n} \sin^{K+L-\eta_v}(x) \cos^{\eta_v}(x)}{(x - q_s\pi)^n} \right\} dx, \tag{3.8}$$

where the constants $A_{v,r,m}$ and $B_{v,s,n}$ satisfy the 2^L (one for each $v \in \mathbb{Z}_2^L$) polynomial equations

$$1 = \sum_{r=1}^{R} \sum_{m=1}^{m_r} \frac{A_{v,r,m} W_v}{(x - p_r\pi)^m} + \sum_{s=1}^{S} \sum_{n=1}^{n_{v,s}} \frac{B_{v,s,n} W_v}{(x - q_s\pi)^n}, \tag{3.9}$$

where

$$W_v = \prod_{r=1}^{R}\prod_{m=1}^{S}(x-p_r\pi)^{m_r}(x-q_s\pi)^{n_{v,s}}. \tag{3.10}$$

Not all individual terms of equation (3.8) are necessarily integrable, but the sum of terms comprising a single shift are integrable since they arise from shifted sinc functions, derivatives of shifted functions or products thereof. Hence

$$I_a = \gamma\sum_{v\in\mathbb{Z}}(-1)^{L-\eta_v}\Bigg\{\sum_{r=1}^{R}\int_{-a}^{a}\sum_{m=1}^{m_r}\frac{A_{v,r,m}\sin^{K+L-\eta_v}(x)\cos^{\eta_v}(x)}{(x-p_r\pi)^m} + \sum_{s=1}^{S}\int_{-a}^{a}\sum_{n=1}^{n_{v,s}}\frac{B_{v,s,n}\sin^{K+L-\eta_v}(x)\cos^{\eta_v}(x)}{(x-q_s\pi)^n}\Bigg\}dx. \tag{3.11}$$

With changes of variable $t = x - p_r\pi$ and $t = x - q_s\pi$ in the respective terms of equation (3.11) and using identities $\sin(t + j\pi) = (-1)^j \sin(t)$ and $\cos(t + j\pi) = (-1)^j \cos(t)$ for integers j,

$$I_a = \gamma\sum_{v\in\mathbb{Z}}(-1)^{L-\eta_v}\Bigg\{\sum_{r=1}^{R}(-1)^{(K+L)_{p_r}}\int_{-a-p_r\pi}^{a-p_r\pi}\sum_{m=1}^{m_r}\frac{A_{v,r,m}\sin^{K+L-\eta_v}(x)\cos^{\eta_v}(x)}{(x-p_r\pi)^m} + \sum_{s=1}^{S}(-1)^{(K+L)_{q_s}}\int_{-a-q_s\pi}^{a-q_s\pi}\sum_{n=1}^{n_{v,s}}\frac{B_{v,s,n}\sin^{K+L-\eta_v}(x)\cos^{\eta_v}(x)}{(x-q_s\pi)^n}\Bigg\}dx. \tag{3.12}$$

The integrals in equation (3.12) are over different intervals but since $a > M$ and $M = \max\{|p_r\pi|, |q_s\pi| : r = 1, 2, ..., R, s = 1, 2, ..., S.\}$, each interval may be broken into three subintervals as,

$$[-a - p_r\pi, a - p_r\pi] = [-a - p_r\pi, -a + M] \cup [-a + M, a - M] \cup [a - M, a - p_r\pi]. \tag{3.13}$$

This means that I_a may be written as an integral over the common subinterval $[-a+M, a-M]$ plus separate integrals over the remaining intervals. Thus

$$I_a = \gamma\sum_{v\in\mathbb{Z}}(-1)^{L-\eta_v}\Bigg\{\sum_{r=1}^{R}(-1)^{(K+L)_{p_r}}\int_{-a+M}^{a-M}\sum_{m=1}^{m_r}\frac{A_{v,r,m}\sin^{K+L-\eta_v}(x)\cos^{\eta_v}(x)}{(x-p_r\pi)^m} + \sum_{s=1}^{S}(-1)^{(K+L)_{q_s}}\int_{-a+M}^{a-M}\sum_{n=1}^{n_{v,s}}\frac{B_{v,s,n}\sin^{K+L-\eta_v}(x)\cos^{\eta_v}(x)}{(x-q_s\pi)^n}\Bigg\}dx + J_a, \tag{3.14}$$

where J_a is the sum of the integrals in equation (3.12) over intervals $[-a - p_r\pi, -a + M], [a - M, a - p_r\pi], [-a - q_s\pi, -a + M]$ and $[a - M, a - q_s\pi]$.

Lemma

$$\lim_{a\to\infty} J_a = 0. \tag{3.15}$$

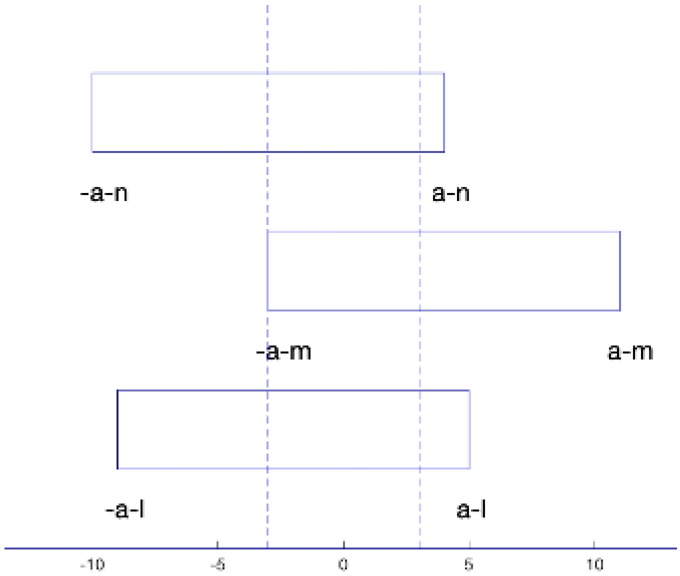

Figure 3.1: Intervals of Integration: The example intervals represented here are constructed by setting $l = 2$, $m = -4$, $n = 3$, thus $M = \max\{|l|, |m|, |n|\} = 4$, with $a = 7$. The three rectangles show the alignment of the intervals and the vertical lines show the bounds for the common interval $[-a + M, a - M]$ symmetric about zero. As a increases, the width of the common interval increases but the lengths of the rectangles not within the bounds of the common interval remain the same.

Proof

J_a is the sum of integrals of the form

$$J^-_{v,p,m} = \int_{-a-p_r\pi}^{-a+M} \frac{A_{v,r,m}\sin^{K+L-S_v}(t)\cos^{S_v}(t)}{t^m}dt, \tag{3.16}$$

$$J^+_{v,p,m} = \int_{a-M}^{-a-p_r\pi} \frac{A_{v,r,m}\sin^{K+L-S_v}(t)\cos^{S_v}(t)}{t^m}dt, \tag{3.17}$$

$$J^-_{v,p,m} = \int_{-a-q_s\pi}^{-a+M} \frac{B_{v,s,n}\sin^{K+L-S_v}(t)\cos^{S_v}(t)}{t^n}dt, \tag{3.18}$$

$$J^+_{v,p,m} = \int_{a-M}^{-a-q_s\pi} \frac{B_{v,s,n}\sin^{K+L-S_v}(t)\cos^{S_v}(t)}{t^n}dt, \tag{3.19}$$

All these individual integrals exist since none of the intervals contain zero. In the first case,

$$|J^-_{v,p,m}| \leq \frac{|A_{v,r,m}|(M + p_r\pi)}{|-a+M|^m}, \tag{3.20}$$

and so

$$\lim_{a\to\infty} J^-_{v,p,m} = 0. \tag{3.21}$$

The other cases follow similarly. ■

Taking $a \to \infty$ in equation (3.14) shows that the integral in equation (3.2) is

$$I = \gamma \sum_{v \in \mathbb{Z}} (-1)^{L-\eta_v} \Bigg\{ \sum_{r=1}^{R} (-1)^{(K+L)_{p_r}} \int_{-\infty}^{\infty} \sum_{m=1}^{m_r} \frac{A_{v,r,m} \sin^{K+L-\eta_v}(x) \cos^{\eta_v}(x)}{(x - p_r\pi)^m} + \sum_{s=1}^{S} (-1)^{(K+L)_{q_s}} \int_{-\infty}^{\infty} \sum_{n=1}^{n_{v,s}} \frac{B_{v,s,n} \sin^{K+L-\eta_v}(x) \cos^{\eta_v}(x)}{(x - q_s\pi)^n} \Bigg\} dx. \tag{3.22}$$

In many cases, writing equation (3.22) as

$$I = \gamma \int_{-\infty}^{\infty} \sum_{v \in \mathbb{Z}} (-1)^{L-\eta_v} \Bigg\{ \sum_{r=1}^{R} (-1)^{(K+L)_{p_r}} \sum_{m=1}^{m_r} \frac{A_{v,r,m} \sin^{K+L-\eta_v}(x) \cos^{\eta_v}(x)}{(x - p_r\pi)^m} + \sum_{s=1}^{S} (-1)^{(K+L)_{q_s}} \sum_{n=1}^{n_{v,s}} \frac{B_{v,s,n} \sin^{K+L-\eta_v}(x) \cos^{\eta_v}(x)}{(x - q_s\pi)^n} \Bigg\} dx, \tag{3.23}$$

is more practical since some combinations of terms may add to zero or reduce to simpler forms.

3.2 Integral Formulas Required for the Fisher Metric Calculation

Formulae for the 21 definite integrals required in Chapter 4 are presented here. Section 3.2.1 lists all five integrals in the form of equation (3.2) with $K = 4$ and $L = 0$, Section 3.2.2 lists the seven integrals with $K = 3$ and $L = 1$ and the nine integrals with $K = 2$ and $L = 2$ appear in Section 3.2.3. Proofs appear in Sections 3.3.1, 3.3.2 and 3.3.3 respectively. The formulae for equations (3.28), (3.35) and (3.44) are well known and do not require the general method presented in Section 3.1 but are included here for completeness. Throughout, k, l, m and n represent distinct integers.

3.2.1 Four Shifted Sinc Functions

$$\int_{-\infty}^{\infty} \psi_k \psi_l \psi_m \psi_n dx = 0, \tag{3.24}$$

$$\int_{-\infty}^{\infty} \psi_l^2 \psi_m \psi_n dx = \frac{(-1)^{m+n}}{2(m-l)(n-l)\pi}, \tag{3.25}$$

$$\int_{-\infty}^{\infty} \psi_m^2 \psi_n^2 dx = \frac{1}{(n-m)^2\pi}, \tag{3.26}$$

$$\int_{-\infty}^{\infty} \psi_m^3 \psi_n dx = \frac{(-1)^{m+n+1}}{2(m-n)^2\pi}, \tag{3.27}$$

$$\int_{-\infty}^{\infty} \psi_n^4 dx = \frac{2\pi}{3}, \tag{3.28}$$

3.2.2 Three Shifted Sinc Functions and One Derivative

$$\int_{-\infty}^{\infty} \psi_k \psi_l \psi_m \psi_n' dx = \frac{(-1)^{k+l+m+n}}{2(k-n)(l-n)(m-n)\pi^2}, \tag{3.29}$$

$$\int_{-\infty}^{\infty} \psi_l^2 \psi_m \psi_n' dx = \frac{(-1)^{m+n}(n+l-2m)}{2(l-n)^2(l-m)(m-n)\pi^2}, \tag{3.30}$$

$$\int_{-\infty}^{\infty} \psi_l \psi_m \psi_n \psi_n' dx = \frac{(-1)^{l+m}(2n-l-m)}{2(l-n)^2(m-n)^2\pi^2}, \tag{3.31}$$

$$\int_{-\infty}^{\infty} \psi_m^3 \psi_n' dx = (-1)^{m+n}\left(\frac{1}{2(m-n)} + \frac{3}{2(m-n)^3\pi^2}\right), \tag{3.32}$$

$$\int_{-\infty}^{\infty} \psi_m^2 \psi_n \psi_n' dx = \frac{-3}{2(m-n)^3\pi^2}, \tag{3.33}$$

$$\int_{-\infty}^{\infty} \psi_m \psi_n^2 \psi_n' dx = (-1)^{m+n}\left(\frac{1}{2(m-n)^3\pi^2} + \frac{1}{6(m-n)}\right), \tag{3.34}$$

$$\int_{-\infty}^{\infty} \psi_n^3 \psi_n' dx = 0, \tag{3.35}$$

3.2.3 Two Shifted Sinc Functions and Two Derivatives

$$\int_{-\infty}^{\infty} \psi_k \psi_l \psi_m' \psi_n' dx = \frac{(-1)^{k+l+m+n}}{2(m-n)^2\pi^3}\left(\frac{1}{(k-m)(l-m)} + \frac{1}{(k-n)(l-n)}\right), \tag{3.36}$$

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_l^2 \psi_m' \psi_n' dx = (-1)^{m+n} \Bigg(&\frac{1}{2(l-m)(l-n)\pi} + \frac{1}{2(l-m)^2(l-n)^2\pi^3} \\ &+ \frac{1}{2(l-m)^2(m-n)^2\pi^3} + \frac{1}{2(l-n)^2(m-n)^2\pi^3}\Bigg),\end{aligned} \tag{3.37}$$

$$\int_{-\infty}^{\infty} \psi_l \psi_m \psi_n'^2 dx = (-1)^{l+m}\left(\frac{1}{6(l-n)(m-n)\pi} - \frac{-l^2 - lm + 3ln - m^2 + 3mn - 3n^2}{2(l-n)^3(m-n)^3\pi^3}\right), \tag{3.38}$$

$$\int_{-\infty}^{\infty} \psi_l \psi_m \psi_m' \psi_n' dx = \frac{(-1)^{l+n}}{2\pi^3(m-n)^3}\left(\frac{2l-3m+n}{(l-m)^2} + \frac{1}{l-n}\right), \tag{3.39}$$

$$\int_{-\infty}^{\infty} \psi_m \psi_n \psi_m' \psi_n' dx = -\frac{3}{(m-n)^4\pi^3}, \tag{3.40}$$

$$\int_{-\infty}^{\infty} \psi_m^2 \psi_m' \psi_n' dx = (-1)^{m+n}\left(\frac{1}{6(m-n)^2\pi} + \frac{2}{(m-n)^4\pi^3}\right), \tag{3.41}$$

$$\int_{-\infty}^{\infty} \psi_m^2 \psi_n'^2 = \frac{2}{(m-n)^4\pi^3} + \frac{2}{3(m-n)^2\pi}, \tag{3.42}$$

$$\int_{-\infty}^{\infty} \psi_m \psi_n \psi_n'^2 dx = (-1)^{m+n+1}\left(\frac{1}{6(m-n)^2\pi} + \frac{1}{2(m-n)^4\pi^3}\right), \tag{3.43}$$

$$\int_{-\infty}^{\infty} \psi_n^2 \psi_n'^2 dx = \frac{\pi}{15}. \tag{3.44}$$

3.3 Proofs of the Integral Formulae

The proofs that follow show how the integral forms in Section 3.2 are obtained.

3.3.1 Four Shifted Sinc Functions

Proof of equation (3.24): $I = \int \psi_k \psi_l \psi_m \psi_n dx$

From equation (3.24) the integral is

$$I = \gamma \int_{-\infty}^{\infty} \left(\frac{A}{x - k\pi} + \frac{B}{x - l\pi} + \frac{C}{x - m\pi} + \frac{D}{x - n\pi} \right) \sin^4(x) dx, \tag{3.45}$$

where $\gamma = (-1)^{k+l+m+n}$ and the constants satisfy the polynomial equation

$$\begin{aligned} 1 =& A(x - l\pi)(x - m\pi)(x - n\pi) + B(x - k\pi)(x - m\pi)(x - n\pi) + C(x - k\pi)(x - l\pi)(x - n\pi) \\ &+ D(x - k\pi)(x - l\pi)(x - m\pi). \end{aligned} \tag{3.46}$$

Comparing the coefficients of x^3 on both sides of equation (3.46) shows that $A+B+C+D = 0$. From equation (3.23),

$$I = \gamma \int_{-\infty}^{\infty} \left(\frac{A}{t} + \frac{B}{t} + \frac{C}{t} + \frac{D}{t} \right) \sin^4(t) dt, \tag{3.47}$$

Hence $I = 0$.

Proof of equation (3.25): $I = \int \psi_l^2 \psi_m \psi_n dx$

From equation (3.25) the integral is

$$I = \gamma \int_{-\infty}^{\infty} \left(\frac{A}{x - l\pi} + \frac{B}{x - m\pi} + \frac{C}{x - n\pi} + \frac{D}{(x - l\pi)^2} \right) \sin^4(x) dx, \tag{3.48}$$

where $\gamma = (-1)^{m+n}$ and the constants satisfy the polynomial equation

$$\begin{aligned} 1 =& A(x - l\pi)(x - m\pi)(x - n\pi) + B(x - l\pi)^2(x - n\pi) + C(x - l\pi)^2(x - m\pi) \\ &+ D(x - m\pi)(x - n\pi). \end{aligned} \tag{3.49}$$

Comparing the coefficients of x^3 on both sides of equation (3.49) shows that $A+B+C+D = 0$. From equation (3.23),

$$I = \gamma \int_{-\infty}^{\infty} \left(\frac{A}{t} + \frac{B}{t} + \frac{C}{t} + \frac{D}{t^2} \right) \sin^4(t) dt = \gamma \int_{-\infty}^{\infty} \frac{D \sin^4(t)}{t^2} dt = \frac{\gamma D \pi}{2}. \tag{3.50}$$

By solving equation (3.49) for the coefficients,

$$D = \frac{1}{(m - l)(n - l)\pi^2}. \tag{3.51}$$

Proof of equation (3.26): $\int \psi_m^2 \psi_n^2 dx$

From equation (3.26) the integral is

$$I = \gamma \int_{-\infty}^{\infty} \left(\frac{A}{x - l\pi} + \frac{B}{x - m\pi} + \frac{C}{x - n\pi} + \frac{D}{(x - l\pi)^2} \right) \sin^4(x) dx, \tag{3.52}$$

where $\gamma = 1$ in this case and the constants satisfy the polynomial equation

$$1 = A(x - m\pi)(x - n\pi)^2 + B(x - m\pi)^2(x - n\pi) + C(x - n\pi)^2 + D(x - m\pi)^2. \tag{3.53}$$

Comparing the coefficients of x^3 on both sides of equation (3.53) shows that $A + B = 0$. From equation (3.23),

$$I = \int_{-\infty}^{\infty} \left(\frac{A}{t} + \frac{B}{t} + \frac{C}{t^2} + \frac{D}{t^2} + \right) \sin^4(t) dt = (C + D) \int_{-\infty}^{\infty} \frac{\sin^4(t)}{t^2} dt = \frac{(C + D)\pi}{2}. \tag{3.54}$$

By solving equation (3.53) for the coefficients,

$$C = D = \frac{1}{(m - n)^2 \pi^2}. \tag{3.55}$$

Proof of equation (3.27): $I = \int \psi_m^3 \psi_n dx$

From equation (3.27) the integral is

$$I = \gamma \int_{-\infty}^{\infty} \left(\frac{A}{x - n\pi} + \frac{B}{x - m\pi} + \frac{C}{(x - m\pi)^2} + \frac{D}{(x - m\pi)^3} \right) \sin^4(x) dx, \tag{3.56}$$

where $\gamma = (-1)^{m+n}$ and the constants satisfy the polynomial equation

$$1 = A(x - m\pi)^3 + B(x - m\pi)^2(x - n\pi) + C(x - m\pi)(x - n\pi) + D(x - n\pi). \tag{3.57}$$

Comparing the coefficients of x^3 on both sides of equation (3.53) shows that $A + B = 0$. From equation (3.23),

$$I = \gamma \int_{-\infty}^{\infty} \left(\frac{A}{t} + \frac{B}{t} + \frac{C}{t^2} + \frac{D}{t^3} + \right) \sin^4(t) dt = \gamma C \int_{-\infty}^{\infty} \frac{\sin^4(t)}{t^2} dt = \frac{\gamma C \pi}{2}. \tag{3.58}$$

By solving equation (3.57) for the coefficients,

$$C = -\frac{1}{(m - n)^2 \pi^2}. \tag{3.59}$$

Proof of equation (3.28): $I = \int \psi_n^4 dx$

With change of variable, $t = x - n\pi$, equation (3.28) is

$$I = \int_{-\infty}^{\infty} \frac{\sin^4(t)}{t^4} dt = \frac{2\pi}{3}. \tag{3.60}$$

3.3.2 Three Shifted Sinc Functions And One Derivative

Proof of equation (3.29): $I = \int \psi_k \psi_l \psi_m \psi_n' dx$

From equation (3.29) the integral is

$$I = \gamma \int_{-\infty}^{\infty} \left\{ \left(\frac{A_1}{x - l\pi} + \frac{B_1}{x - m\pi} + \frac{C_1}{x - n\pi} + \frac{D_1}{(x - n\pi)^2} \right) \sin^3(x) \cos(x) \right. \tag{3.61}$$

$$\left. - \left(\frac{A_2}{x - l\pi} + \frac{B_2}{x - m\pi} + \frac{C_2}{x - n\pi} + \frac{D_2}{(x - n\pi)^2} + \frac{E_2}{(x - n\pi)^3} \right) \sin^4(x) \right\} dx, \tag{3.62}$$

where $\gamma = (-1)^{l+m}$ and the constants satisfy the polynomial equations

$$\begin{aligned} 1 =& A_1(x-m\pi)(x-n\pi)^2 + B_1(x-l\pi)(x-n\pi)^2 + C_1(x-l\pi)(x-m\pi)(x-n\pi) \\ &+ D_1(x-l\pi)(x-m\pi), \end{aligned} \tag{3.63}$$

and

$$\begin{aligned} 1 =& A_2(x-m\pi)(x-n\pi)^3 + B_2(x-l\pi)(x-n\pi)^3 + C_2(x-l\pi)(x-m\pi)(x-n\pi)^2 \\ &+ D_2(x-l\pi)(x-m\pi)(x-n\pi) + E_2(x-l\pi)(x-m\pi). \end{aligned} \tag{3.64}$$

Comparing the coefficients of x^3 in equation (3.63) and x^4 in equation (3.64) shows that $A_1 + B_1 + C_1 = 0$ and $A_1 + B_1 + C_1 = 0$. From equation (3.23),

$$\begin{aligned} I =& \gamma \int_{-\infty}^{\infty} \left\{ \left(\frac{A_1}{t} + \frac{B_1}{t} + \frac{C_1}{t} + \frac{D_1}{t} \right) \sin^3(t)\cos(t) \right. \\ & \left. - \left(\frac{A_2}{t} + \frac{B_2}{t} + \frac{C_2}{t} + \frac{D_2}{t} + \frac{E_2}{t^2} \right) \sin^4(t) \right\} dt, \end{aligned} \tag{3.65}$$

$$= -\gamma E_2 \int_{-\infty}^{\infty} \frac{\sin^4(t)}{t^2} dt, \tag{3.66}$$

$$= -\frac{\gamma E_2 \pi}{2}, \tag{3.67}$$

By solving the polynomial equation (3.64) for the coefficients

$$E_2 = \frac{-1}{(k-n)(l-n)(m-n)\pi^3}. \tag{3.68}$$

The integrals containing one or more derivative factors become slightly more complicated to evaluate and so a more detailed description of the method is provided in the following example (3.69). The cases, thereafter, do not contain the additional explanation but will follow this method or explicitly show where additional steps are required.

Proof of equation (3.30): $I = \int \psi_l^2 \psi_m \psi_n' dx$

$$I = \int_{-\infty}^{\infty} \frac{\sin^2(x-l\pi)}{(x-l\pi)^2} \frac{\sin(x-m\pi)}{(x-m\pi)} \frac{P\left(\cos(x-n\pi)(x-n\pi) - \sin(x-n\pi)\right)}{(x-n\pi)^2} dx. \tag{3.69}$$

Since $\sin(x-n\pi) = (-1)^j \sin(x)$ and setting $\gamma = (-1)^{m+n}$,

$$\begin{aligned} &= P\gamma \int_{-a}^{a} \sin^3(x)\cos(x)(x-n\pi) \left[\frac{1}{(x-l\pi)^2(x-m\pi)(x-n\pi)^2} \right] dx \\ &- P\gamma \int_{-a}^{a} \sin^4(x) \left[\frac{1}{(x-l\pi)^2(x-m\pi)(x-n\pi)^2} \right] dx, \end{aligned} \tag{3.70}$$

$$\begin{aligned} &= P\gamma \int_{-a}^{a} \sin^3(x)\cos(x) \left[\frac{A_1}{(x-l\pi)^2} + \frac{B_1}{(x-l\pi)} + \frac{C_1}{(x-m\pi)} + \frac{D_1}{(x-n\pi)} \right] dx \\ &- P\gamma \int_{-a}^{a} \sin^4(x) \left[\frac{A_2}{(x-l\pi)^2} + \frac{B_2}{(x-l\pi)} + \frac{C_2}{(x-m\pi)} + \frac{D_2}{(x-n\pi)^2} + \frac{E_2}{(x-n\pi)} \right] dx, \end{aligned} \tag{3.71}$$

where A,B,C,D in equation (3.70) satisfy the polynomial equation

$$1 = A_1(x - m\pi)(x - n\pi) + B_1(x - l\pi)(x - m\pi)(x - n\pi) + C_1(x - l\pi)^2(x - n\pi) \tag{3.72}$$

$$+ D_1(x - l\pi)^2(x - m\pi). \tag{3.73}$$

Comparing the leading order coefficients of x^3 on both sides of equation (3.73) shows that $B_1 + C_1 + D_1 = 0$. Solving for the coefficients gives the result

$$A_1 = \frac{1}{\pi^2(l - m)(l - n)}, \tag{3.74}$$

$$B_1 = \frac{-2l + m + n}{\pi^3(l - m)^2(l - n)^2}, \tag{3.75}$$

$$C_1 = \frac{1}{\pi^3(m - l)^2(m - n)}, \tag{3.76}$$

$$D_1 = \frac{1}{\pi^3(n - l)^2(n - m)}. \tag{3.77}$$

Recalling the discussion of intervals in Section 3.1 and illustrated by figure 3.1, evaluation of equation (3.70) is considered over the common interval $[-a + M,\ a - M]$, where the limit "a" is chosen large enough that all these intervals include zero. In particular, "a" must be chosen so that $a > M = max|l\pi|, |m\pi|, |n\pi|$. For this reason, each of the intervals $[-a - j\pi, a - j\pi], j = l, m, n$ may be decomposed into three intervals $[-a - j\pi, a - j\pi] = [-a - j\pi, -a + M] \cup [-a + M, a - M] \cup [a - M, a - j\pi]$. The integrals in equations (3.70) and (3.71) will be denoted I_1 and I_2 respectively, in order to reduce clutter. Substitute $s = x - (l, m, n)\pi$ and apply $\sin(s + n\pi) = (-1)^j \sin(s)$, with $\gamma = (-1)^{m+n}$

$$I_1 = P\gamma \int_{-a+M}^{a-M} \frac{A_1 \sin^3(s + l\pi)\cos(s + l\pi)}{(s)^2} ds + P\gamma \int_{-a+M}^{a-M} \frac{B_1 \sin^3(s + l\pi)\cos(s + l\pi)}{(s)} ds$$
$$+ P\gamma \int_{-a+M}^{a-M} \frac{C_1 \sin^3(s + m\pi)\cos(s + m\pi)}{(s)} ds + P\gamma \int_{-a+M}^{a-M} \frac{D_1 \sin^3(s + n\pi)\cos(s + n\pi)}{(s)} ds, \tag{3.78}$$

$$= P\gamma \int_{-a+M}^{a-M} \frac{A_1 \sin^3(s)\cos(s)}{(s)^2} ds + P\gamma \int_{-a+M}^{a-M} \frac{(B_1 + C_1 + D_1)\sin^3(s)\cos(s)}{(s)} ds, \tag{3.79}$$

$$= P\gamma \int_{-a+M}^{a-M} \frac{A_1 \sin^3(s)\cos(s)}{(s)^2} ds. \tag{3.80}$$

Since the integrand of equation (3.80) is odd, this integral equals zero.

$$P\gamma \int_{-a+M}^{a-M} \frac{A_1 \sin^3(s)\cos(s)}{(s)^2} ds = 0. \tag{3.81}$$

Examination of equation (3.71) where $\gamma = (-1)^m$

$$I_2 = -P\gamma \int_{-a}^{a} \sin^4(x) \left[\frac{A_2}{(x-l\pi)^2} + \frac{B_2}{(x-l\pi)} + \frac{C_2}{(x-m\pi)} + \frac{D_2}{(x-n\pi)^2} + \frac{E_2}{(x-n\pi)} \right] dx, \quad (3.82)$$

for A,B,C,D,E satisfying the polynomial equation

$$\begin{aligned} 1 =& A_2(x-m\pi)(x-n\pi)^2 + B_2(x-l\pi)(x-m\pi)(x-n\pi)^2 + C_2(x-l\pi)^2(x-m\pi)^2 \\ &+ D_2(x-l\pi)^2(x-m\pi) + E_2(x-l\pi)^2(x-m\pi)(x-n\pi). \end{aligned} \quad (3.83)$$

Comparing the leading order coefficients of x^4 on both sides of equation (3.83) shows that $B_2 + C_2 + E_2 = 0$. Solving for the coefficients gives the result

$$A_2 = \frac{1}{\pi^3(l-m)(l-n)^2}, \quad (3.84)$$

$$B_2 = \frac{-3l+2m+n}{\pi^4(l-m)^2(l-n)^3}, \quad (3.85)$$

$$C_2 = \frac{1}{\pi^4(m-l)^2(m-n)^2}, \quad (3.86)$$

$$D_2 = \frac{1}{\pi^3(n-l)^2(n-m)}, \quad (3.87)$$

$$E_2 = \frac{-l-2m+3n}{\pi^4(n-l)^3(n-m)^2}. \quad (3.88)$$

Continuing on with equation (3.71), over the common interval $[-a+M,\ a-M]$ and again substituting $s = x-(l,m,n)\pi$ and applying $\sin(s+n\pi) = (-1)^j \sin(s)$ and $\gamma = (-1)^{m+n}$ as before.

$$\begin{aligned} I_2 =& -P\gamma \int_{-a+M}^{a-M} \frac{A_2 \sin^4(s+l\pi)}{(s)^2} ds - P\gamma \int_{-a+M}^{a-M} \frac{B_2 \sin^4(s+l\pi)}{(s)} ds \\ &- P\gamma \int_{-a+M}^{a-M} \frac{C_2 \sin^4(s+m\pi)}{(s)} ds - P\gamma_2 \int_{-a+M}^{a-M} \frac{D_2 \sin^4(s+n\pi)}{(s)^2} ds - P\gamma \int_{-a+M}^{a-M} \frac{E_2 \sin^4(s+n\pi)}{(s)} ds, \end{aligned} \quad (3.89)$$

$$= -P\gamma \int_{-a+M}^{a-M} \frac{(A_2+D_2)\sin^4(s)}{(s)^2} ds - P\gamma \int_{-a+M}^{a-M} \frac{(B_2+C_2+D_2)\sin^4(s)}{(s)} ds, \quad (3.90)$$

$$= -P\gamma \int_{-a+M}^{a-M} \frac{(A_2+D_2)\sin^4(s)}{(s)^2} ds. \quad (3.91)$$

Since

$$\int_{-\infty}^{\infty} \frac{\sin^4(x)}{(x)^x} dx = \frac{\pi}{2}, \quad (3.92)$$

equation (3.91) becomes

$$= \frac{\pi}{2} P\gamma \left(\frac{1}{\pi^3(l-m)(l-n)^2} + \frac{1}{\pi^3(l-n)^2(n-m)} \right). \quad (3.93)$$

So equation (3.93) becomes

$$= \frac{\pi}{2} P\gamma \left(\frac{1}{\pi^3(l-m)(l-n)^2} + \frac{1}{\pi^3(l-n)^2(n-m)} \right), \tag{3.94}$$

$$I = \frac{l-2m+n}{2\pi^2(l-m)(l-n)^2(m-n)} P\gamma. \tag{3.95}$$

Proof of equation (3.31): $I = \int \psi_l \psi_m \psi_n \psi_n' dx$

$$I = \int_{-\infty}^{\infty} \frac{\sin(x-l\pi)}{(x-l\pi)} \frac{\sin(x-m\pi)}{(x-m\pi)} \frac{\sin(x-n\pi)}{(x-n\pi)} \frac{P\left(\cos(x-n\pi)x_n - \sin(x-n\pi)\right)}{(x-n\pi)^2} dx. \tag{3.96}$$

Since $\sin(x-n\pi) = (-1)^j \sin(x)$ and $\gamma = (-1)^{l+m}$,

$$= \gamma \int_{-a}^{a} P\sin^3(x) \left(\cos(x)(x-n\pi) - \sin(x)\right) \left[\frac{1}{(x-l\pi)(x-m\pi)(x-n\pi)^3} \right] dx,$$

$$= P\gamma \int_{-a}^{a} \sin^3(x)\cos(x) \left[\frac{A}{(x-l\pi)} + \frac{B}{(x-m\pi)} + \frac{C}{(x-n\pi)^2} + \frac{D}{(x-n\pi)} \right] dx \tag{3.97}$$

$$- P\gamma \int_{-a}^{a} \sin^4(x) \left[\frac{A}{(x-l\pi)} + \frac{B}{(x-m\pi)} + \frac{C}{(x-n\pi)^3} + \frac{D}{(x-n\pi)^2} + \frac{E}{(x-n\pi)} \right] dx, \tag{3.98}$$

where A,B,C,D in equation (3.97) satisfy the polynomial equation

$$\begin{aligned} 1 =& A_1(x-m\pi)(x-n\pi)^3 + B_1(x-l\pi)(x-n\pi)^3 + C_1(x-l\pi)(x-m\pi)(x-n\pi) \\ &+ D_1(x-l\pi)(x-m\pi)(x-n\pi)^2. \end{aligned} \tag{3.99}$$

Comparing the leading order coefficients of x^4 on both sides of equation (3.99) shows that $A_1 + B_1 + D_1 = 0$. Solving for the coefficients gives the result

$$A_1 = \frac{1}{\pi^4(l-m)(l-n)^3}, \tag{3.100}$$

$$B_1 = -\frac{1}{\pi^4(m-l)(m-n)^3}, \tag{3.101}$$

$$C_1 = -\frac{1}{\pi^2(n-l)(n-m)}, \tag{3.102}$$

$$D_1 = \frac{l+m-2n}{\pi^3(n-l)^2(n-m)^2}. \tag{3.103}$$

Continuing on with equation (3.97), over the common interval $[-a+M,\ a-M]$ as discussed in Section 3.1, with $\gamma = (-1)^{l+m+n}$.

$$I_1 = P\gamma \int_{-a+M}^{a-M} \frac{A_1 \sin^3(s+l\pi)\cos(s+l\pi)}{s} ds + P\gamma \int_{-a+M}^{a-M} \frac{B_1 \sin^3(s+m\pi)\cos(s+m\pi)}{s} ds$$
$$+ P\gamma \int_{-a+M}^{a-M} \frac{C_1 \sin^3(s+n\pi)\cos(s+n\pi)}{s^2} ds + P\gamma \int_{-a+M}^{a-M} \frac{D_1 \sin^3(s+n\pi)\cos(s+n\pi)}{s} ds, \tag{3.104}$$

$$= P\gamma \int_{-a+M}^{a-M} \frac{(A_1+B_1+D_1)\sin^3(s+l\pi)\cos(s+l\pi)}{s} ds$$
$$+ P\gamma \int_{-a+M}^{a-M} \frac{C_1 \sin^3(s+n\pi)\cos(s+n\pi)}{s^2} ds, \tag{3.105}$$
$$= P\gamma \int_{-a+M}^{a-M} \frac{C_1 \sin^3(s+n\pi)\cos(s+n\pi)}{s^2} ds. \tag{3.106}$$

Since the integrand of equation (3.106) is odd, this integral equals zero.

$$P\gamma \int_{-a+M}^{a-M} \frac{C_1 \sin^3(s)\cos(s)}{(s)^2} ds = 0. \tag{3.107}$$

Examining equation (3.98) where $\gamma = (-1)^{l+m+n}$

$$I_2 = -P\gamma \int_{-a}^{a} \sin^4(x) \left[\frac{A}{(x-l\pi)} + \frac{B}{(x-m\pi)} + \frac{C}{(x-n\pi)^3} + \frac{D}{(x-n\pi)^2} + \frac{E}{(x-n\pi)} \right] dx, \tag{3.108}$$

for A,B,C,D,E satisfying the polynomial equation

$$1 = A(x-m\pi)(x-n\pi)^3 + B(x-l\pi)(x-n\pi)^3 + C(x-l\pi)(x-m\pi)$$
$$+ D(x-l\pi)(x-m\pi)(x-n\pi) + E(x-l\pi)(x-m\pi)(x-n\pi)^2. \tag{3.109}$$

Comparing the leading order coefficients of x^4 on both sides of equation (3.109) shows that $A+B+E=0$. Solving for the coefficients gives the result

$$A = \frac{1}{\pi^4(l-m)(l-n)^3}, \tag{3.110}$$
$$B = -\frac{1}{\pi^4(m-l)(m-n)^3}, \tag{3.111}$$
$$C = -\frac{1}{\pi^2(n-l)(n-m)}, \tag{3.112}$$
$$D = \frac{l+m-2n}{\pi^3(n-l)^2(n-m)^2}, \tag{3.113}$$
$$E = \frac{-l^2-lm+3ln-m^2+3mn-3n^2}{\pi^4(n-l)^3(n-m)^3}, \tag{3.114}$$

then

$$I_2 = -P\gamma \int_{-a-l\pi}^{a-l\pi} \frac{A\sin^4(s+l\pi)}{(s)} d\tau - P\gamma \int_{-a-m\pi}^{a-m\pi} \frac{B\sin^4(s+m\pi)}{(s)} d\tau$$
$$- P\gamma \int_{-a-n\pi}^{a-n\pi} \frac{C\sin^4(s+n\pi)}{(s)^3} d\tau - P\gamma \int_{-a-n\pi}^{a-n\pi} \frac{D\sin^4(s+n\pi)}{(s)^2} dx - P\gamma \int_{-a-n\pi}^{a-n\pi} \frac{E\sin^4(s+n\pi)}{(s)} dx, \tag{3.115}$$

$$= -P\gamma \int_{-a-n\pi}^{a-n\pi} \frac{C\sin^4(s+n\pi)}{(s)^3} ds - P\gamma \int_{-a-n\pi}^{a-n\pi} \frac{D\sin^4(s+n\pi)}{(s)^2} ds$$
$$- P\gamma \int_{-a-l\pi}^{a-l\pi} \frac{(A+B+E)\sin^4(s+l\pi)}{(s)} ds, \tag{3.116}$$
$$= -P\gamma \int_{-a-n\pi}^{a-n\pi} \frac{C\sin^4(s+n\pi)}{(s)^3} ds - P\gamma \int_{-a-n\pi}^{a-n\pi} \frac{D\sin^4(s+n\pi)}{(s)^2} ds. \tag{3.117}$$

Since

$$\int_{-\infty}^{\infty} \frac{\sin^4(x)}{x^3} dx = 0, \text{ and } \int_{\infty}^{\infty} \frac{\sin^4(x)}{x^3} dx = \frac{\pi}{2}, \tag{3.118}$$

equation (3.117) becomes

$$I_2 = -\frac{\pi}{2} P\gamma \left(\frac{l+m-2n}{\pi^3 (l-n)^2 (n-m)^2} \right). \tag{3.119}$$

As $I_1 = 0$, the I_2 equation represents the final expression of I.

Proof of equation (3.32): $I = \int \psi_m^3 \psi_n' dx$

$$I = \int_{-\infty}^{\infty} \frac{\sin^3(x-m\pi)}{(x-m\pi)^3} \left(\frac{P\left(\cos(x-n\pi)(x-n\pi) - \sin(x-n\pi)\right)}{(x-n\pi)^2} \right) dx. \tag{3.120}$$

Since $\sin(x-n\pi) = (-1)^j \sin(x)$ and $\gamma = (-1)^m$

$$= P\gamma \int_{-a}^{a} \frac{\sin^3(x)\cos(x)(x-n\pi)}{(x-m\pi)^3 (x-n\pi)^2} d\tau - P\gamma_2 \int_{-\infty}^{\infty} \frac{\sin^4(x)}{(x-m\pi)^3 (x-n\pi)^2} dx, \tag{3.121}$$
$$= P\gamma \int_{-a}^{a} \sin^3(x)\cos(x) \left(\frac{A}{(x-m\pi)^3} + \frac{B}{(x-m\pi)^2} + \frac{C}{(x-m\pi)} + \frac{D}{(x-n\pi)} \right) dx \tag{3.122}$$
$$- P\gamma \int_{-a}^{a} \sin^4(x) \left(\frac{A}{(x-m\pi)^3} + \frac{B}{(x-m\pi)^2} + \frac{C}{(x-m\pi)} + \frac{D}{(x-n\pi)^2} + \frac{E}{(x-n\pi)} \right) dx, \tag{3.123}$$

where A,B,C,D in equation (3.122) satisfy the polynomial equation

$$1 = A_1(x-n\pi) + B_1(x-m\pi)(x-n\pi) + C_1(x-m\pi)^2(x-n\pi) + D_1(x-m\pi)^3. \tag{3.124}$$

Comparing the leading order coefficients of x^3 on both sides of equation (3.124) shows that

$C + D = 0$. Solving for the coefficients give the result

$$A_1 = \frac{1}{\pi(m-n)}, \quad (3.125)$$

$$B_1 = -\frac{1}{\pi^2(m-n)^2}, \quad (3.126)$$

$$C_1 = \frac{1}{\pi^3(m-n)^3}, \quad (3.127)$$

$$D_1 = -\frac{1}{\pi^3(m-n)^3}. \quad (3.128)$$

Continuing on with equation (3.122), over the common interval $[-a+M,\ a-M]$ as discussed in Section 3.1, with $\gamma = (-1)^m$.

$$I_1 = P\gamma \int_{-a+M}^{a-M} \frac{A_1 \sin^3(s+m\pi)\cos(s+m\pi)}{(s)^3} ds + P\gamma \int_{-a+M}^{a-M} \frac{B_1 \sin^3(s+m\pi)\cos(s+m\pi)}{(s)^2} ds$$
$$+ P\gamma \int_{-a+M}^{a-M} \frac{C_1 \sin^3(s+m\pi)\cos(s+m\pi)}{(s)} ds + P\gamma \int_{-a+M}^{a-M} \frac{D_1 \sin^3(s+n\pi)\cos(s+n\pi)}{(s)} ds, \quad (3.129)$$

$$= P\gamma \int_{-a+M}^{a-M} \frac{A_1 \sin^3(s+m\pi)\cos(s+m\pi)}{(s)^3} ds + P\gamma \int_{-a+M}^{a-M} \frac{B_1 \sin^3(s+m\pi)\cos(s+m\pi)}{(s)^2} ds$$
$$+ P\gamma \int_{-a+M}^{a-M} \frac{(C_1 + D_1)\sin^3(s+m\pi)\cos(s+m\pi)}{(s)} ds, \quad (3.130)$$
$$= P\gamma \int_{-a+M}^{a-M} \frac{A_1 \sin^3(s+m\pi)\cos(s+m\pi)}{(s)^3} ds + P\gamma \int_{-a+M}^{a-M} \frac{B_1 \sin^3(s+m\pi)\cos(s+m\pi)}{(s)^2} ds. \quad (3.131)$$

Since the two integrands of equation (3.131) are,

$$\int_{-a+M}^{a-M} \frac{\sin^3(s)\cos(s)}{(s)^3} ds = \frac{\pi}{2}, \quad \int_{-a+M}^{a-M} \frac{\sin^3(s)\cos(s)}{(s)^2} ds = 0, \quad (3.132)$$

equation (3.131) becomes

$$= P\gamma \frac{\pi}{2} A_1, \quad (3.133)$$

$$I_1 = \frac{P\gamma}{2(m-n)}. \quad (3.134)$$

Examining equation (3.123), where $\gamma = (-1)^m$

$$I_2 = -P\gamma \int_{-a}^{a} \sin^4(x) \left(\frac{A}{(x-m\pi)^3} + \frac{B}{(x-m\pi)^2} + \frac{C}{(x-m\pi)} + \frac{D}{(x-n\pi)^2} + \frac{E}{(x-n\pi)} \right) dx, \quad (3.135)$$

where A,B,C,D equation (3.122) satisfy the polynomial equation

$$1 = A_2(x-n\pi)^2 + B_2(x-m\pi)(x-n\pi)^2 + C_2(x-m\pi)^2(x-n\pi)^2 + D_2(x-m\pi)^3 + E_2(x-m\pi)^3(x-n\pi). \quad (3.136)$$

Comparing the leading order coefficients of x^3 on both sides of equation (3.136) shows that $C_2 + E_2 = 0$. Solving for the coefficients

$$\begin{aligned} A_2 &= \frac{1}{\pi^2(m-n)^2}, \\ B_2 &= -\frac{2}{\pi^3(m-n)^3}, \\ C_2 &= +\frac{3}{\pi^4(m-n)^4}, \\ D_2 &= -\frac{1}{\pi^3(m-n)^3}, \\ E_2 &= -\frac{3}{\pi^4(m-n)^4}, \end{aligned} \tag{3.137}$$

then

$$I_2 = -P\gamma \int_{-a+M}^{a-M} \frac{A_2 \sin^4(s+m\pi)}{(s)^3} ds - P\gamma \int_{-a+M}^{a-M} \frac{B_2 \sin^4(s+m\pi)}{(s)^2} ds$$
$$- P\gamma \int_{-a+M}^{a-M} \frac{C_2 \sin^4(s+m\pi)}{(s)} ds - P\gamma \int_{-a+M}^{a-M} \frac{D_2 \sin^4(s+n\pi)}{(s)^2} ds - P\gamma \int_{-a+M}^{a-M} \frac{E_2 \sin^4(s+n\pi)}{(s)} ds. \tag{3.138}$$

$$= -P\gamma \int_{-a+M}^{a-M} \frac{A_2 \sin^4(s+m\pi)}{(s)^3} ds - P\gamma \int_{-a+M}^{a-M} \frac{(B_2+D_2) \sin^4(s+m\pi)}{(s)^2} ds$$
$$- P\gamma \int_{-a+M}^{a-M} \frac{(C_2+E_2) \sin^4(s+n\pi)}{(s)} ds, \tag{3.139}$$
$$= -P\gamma \int_{-a+M}^{a-M} \frac{A_2 \sin^4(s+m\pi)}{(s)^3} ds - P\gamma \int_{-a+M}^{a-M} \frac{(B_2+D_2) \sin^4(s+m\pi)}{(s)^2} ds. \tag{3.140}$$

Since the two integrands of equation (3.140) are,

$$\int_{-a+M}^{a-M} \frac{\sin^4(s)}{(s)^3} ds = 0, \quad \int_{-a+M}^{a-M} \frac{\sin^4(s)}{(s)^2} ds = \frac{\pi}{2}, \tag{3.141}$$

equation (3.140) becomes

$$I_2 = -P\gamma \frac{\pi}{2}(B_2 + D_2), \tag{3.142}$$
$$= P\gamma \frac{\pi}{2} \left(\frac{2}{\pi^3(m-n)^3} + \frac{1}{\pi^3(m-n)^3} \right), \tag{3.143}$$
$$= P\gamma \frac{\pi}{2} \left(\frac{3}{\pi^3(m-n)^3} \right). \tag{3.144}$$

Recombining, $I_1 + I_2$, results in the full expression of I

$$I = \frac{(-1)^m \left((-1)^m (m-n)^2 + \frac{3}{\pi^2} \right)}{2(m-n)^3}. \tag{3.145}$$

Proof of equation (3.33): $I = \int \psi_m^2 \psi_n \psi_n' dx$

$$I = \int_{-\infty}^{\infty} \frac{\sin^2(x - m\pi)}{(x - m\pi)^2} \frac{\sin(x - n\pi)}{(x - n\pi)} \frac{P\left(\cos(x - n\pi)(x - n\pi) - \sin(x - n\pi)\right)}{(x - n\pi)^2} dx. \quad (3.146)$$

Since $\sin(x - n\pi) = (-1)^j \sin(x)$ and $\gamma = (-1)^n$

$$= P\gamma_1 \int_{-a}^{a} \sin^3(x)\cos(x) \left(\frac{A}{(x - m\pi)^2} + \frac{B}{(x - m\pi)} + \frac{C}{(x - n\pi)^2} + \frac{D}{(x - n\pi)} \right) dx, \quad (3.147)$$

$$- P\gamma_2 \int_{-a}^{a} \sin^4(x) \left(\frac{A}{(x - m\pi)^2} + \frac{B}{(x - m\pi)} + \frac{C}{(x - n\pi)^3} + \frac{D}{(x - n\pi)^2} + \frac{E}{(x - n\pi)} \right) dx, \quad (3.148)$$

where A,B,C,D equation (3.147) satisfy the polynomial equation

$$1 = A_1(x - n\pi)^2 + B_1(x - m\pi)(x - n\pi)^2 + C_1(x - m\pi)^2 + D_1(x - m\pi)^2(x - n\pi). \quad (3.149)$$

Comparing the leading order coefficients of x^3 on both sides of equation (3.136) shows that $B_1 + D_1 = 0$. Solving for the coefficients in the bracket give the result

$$A = \frac{1}{\pi^2(m - n)^2(x - \pi m)^2}, \quad (3.150)$$

$$B = -\frac{2}{\pi^3(m - n)^3(x - \pi m)}, \quad (3.151)$$

$$C = \frac{1}{\pi^2(m - n)^2(x - \pi n)^2}, \quad (3.152)$$

$$D = \frac{2}{\pi^3(m - n)^3(x - \pi n)}. \quad (3.153)$$

Continuing on with equation (3.147), over the common interval $[-a + M,\ a - M]$ as discussed in Section 3.1, with $\gamma = (-1)^n$.

$$I_1 = P\gamma \int_{-a+M}^{a-M} \frac{A_1 \sin^3(s + m\pi)\cos(s + m\pi)}{(s)^2} ds + P\gamma \int_{-a+M}^{a-M} \frac{B_1 \sin^3(s + m\pi)\cos(s + m\pi)}{(s)} ds$$

$$+ P\gamma \int_{-a+M}^{a-M} \frac{C_1 \sin^3(s + n\pi)\cos(s + n\pi)}{(s)^2} ds + P\gamma \int_{-a+M}^{a-M} \frac{D_1 \sin^3(s + n\pi)\cos(s + n\pi)}{(s)} ds,$$

$$= P\gamma \int_{-a+M}^{a-M} \frac{(A_1 + C_1)\sin^3(s + m\pi)\cos(s + m\pi)}{(s)^2} ds$$

$$+ P\gamma \int_{-a+M}^{a-M} \frac{(B_1 + D_1)\sin^3(s + m\pi)\cos(s + m\pi)}{(s)} ds, \quad (3.154)$$

$$= P\gamma \int_{-a+M}^{a-M} \frac{(A_1 + C_1)\sin^3(s + m\pi)\cos(s + m\pi)}{(s)^2} ds. \quad (3.155)$$

Since

$$\int_{-a+M}^{a-M} \frac{\sin^3(s)\cos(s)}{(s)^2} ds = 0, \quad (3.156)$$

then $I_1 = 0$.

Examining equation (3.148), where $\gamma = 1$

$$I_2 = -P\gamma \int_{-a}^{a} \sin^4(x) \left(\frac{A}{(x-m\pi)^3} + \frac{B}{(x-m\pi)^2} + \frac{C}{(x-m\pi)} + \frac{D}{(x-n\pi)^2} + \frac{E}{(x-n\pi)} \right) dx, \tag{3.157}$$

where A,B,C,D,E equation (3.148) satisfy the polynomial equation

$$1 = A_2(x-n\pi)^3 + B_2(x-m\pi)(x-n\pi)^3 + C_2(x-m\pi)^2 + D_2(x-m\pi)^2(x-n\pi) + E_2(x-m\pi)^2(x-n\pi)^2. \tag{3.158}$$

Comparing the leading order coefficients of x on both sides of equation (3.158) shows that $B_2 + E_2 = 0$. Solving for the coefficients gives

$$A_2 = \frac{1}{\pi^3(m-n)^3}, \tag{3.159}$$
$$B_2 = -\frac{3}{\pi^4(m-n)^4},$$
$$C_2 = \frac{1}{\pi^2(m-n)^2},$$
$$D_2 = \frac{2}{\pi^3(m-n)^3},$$
$$E_2 = \frac{3}{\pi^4(m-n)^4}.$$

Then,

$$I_2 = -P\gamma \int_{-a+M}^{a-m} \frac{A\sin^4(s+m\pi)}{(s)^2} ds - P\gamma \int_{-a+M\pi}^{a-m} \frac{B\sin^4(s+m\pi)}{(s)} ds$$
$$- P\gamma \int_{-a+M}^{a-m} \frac{C\sin^4(s+n\pi)}{(s)^3} ds - P\gamma \int_{-a+M}^{a-m} \frac{D\sin^4(s+n\pi)}{(s)^2} ds - P\gamma \int_{-a+M}^{a-m} \frac{E\sin^4(s+n\pi)}{(s)} ds, \tag{3.160}$$

$$= -P\gamma \int_{-a+M}^{a-m} \frac{C\sin^4(s+n\pi)}{(s)^3} ds - P\gamma \int_{-a+M}^{a-m} \frac{(A+D)\sin^4(s+m\pi)}{(s)^2} ds$$
$$- P\gamma \int_{-a+M}^{a-m} \frac{(B+E)\sin^4(s+n\pi)}{(s)} ds, \tag{3.161}$$
$$= -P\gamma \int_{-a+M}^{a-m} \frac{C\sin^4(s+n\pi)}{(s)^3} ds - P\gamma \int_{-a+M}^{a-m} \frac{(A+D)\sin^4(s+m\pi)}{(s)^2} ds. \tag{3.162}$$

Since the two integrands of equation (3.162) are,

$$\int_{-a+M}^{a-M} \frac{\sin^4(s)}{(s)^3} ds = 0, \quad \int_{-a+M}^{a-M} \frac{\sin^4(s)}{(s)^2} ds = \frac{\pi}{2}, \tag{3.163}$$

then I_2 is

$$= -P\gamma\frac{\pi}{2}(A+D), \tag{3.164}$$
$$= -P\gamma\frac{\pi}{2}\left(\frac{1}{\pi^3(m-n)^3} + \frac{2}{\pi^3(m-n)^3} \right), \tag{3.165}$$
$$I_2 = -\frac{(-1)^m 3P}{2\pi^2(m-n)^3}. \tag{3.166}$$

Since $I_1 = 0$ the full expression is $I = I_2$.

Proof of equation (3.34): $I = \int \psi_m \psi_n^2 \psi_n' dx$

$$I = \int_{-\infty}^{\infty} \frac{\sin(x - m\pi)}{(x - m\pi)} \frac{\sin^2(x - n\pi)}{(x - n\pi)^2} \frac{P\left(\cos(x - n\pi)(x - n\pi) - \sin(x - n\pi)\right)}{(x - n\pi)^2} dx. \tag{3.167}$$

Since $\sin(x - n\pi) = (-1)^j \sin(x)$ and $\gamma = (-1)^m$

$$= P\gamma \int_{-a}^{a} \sin^3(x)\cos(x)(x - n\pi) \left(\frac{1}{(x - m\pi)(x - n\pi)^4} \right) dx$$

$$- P\gamma \int_{-a}^{a} \sin^4(x) \left(\frac{1}{(x - m\pi)(x - n\pi)^4} \right) dx, \tag{3.168}$$

$$= P\gamma \int_{-a}^{a} \sin^3(x)\cos(x) \left(\frac{A}{(x - m\pi)} + \frac{B}{(x - n\pi)^3} + \frac{C}{(x - n\pi)^2} + \frac{D}{(x - n\pi)} \right) dx \tag{3.169}$$

$$- P\gamma \int_{-a}^{a} \sin^4(x) \left(\frac{A}{(x - m\pi)} + \frac{B}{(x - n\pi)^4} + \frac{C}{(x - n\pi)^3} + \frac{D}{(x - n\pi)^2} + \frac{E}{(x - n\pi)} \right) dx, \tag{3.170}$$

where A,B,C,D equation (3.169) satisfy the polynomial equation

$$1 = A_1(x - n\pi)^3 + B_1(x - m\pi) + C_1(x - m\pi)(x - n\pi) + D_1(x - m\pi)(x - n\pi)^2. \tag{3.171}$$

Comparing the leading order coefficients of x^3 on both sides of equation (3.171) shows that $A_1 + D_1 = 0$. Solving for the coefficients in the bracket give the result

$$A = \frac{1}{\pi^3(m - n)^3}, \tag{3.172}$$

$$B = -\frac{1}{\pi(m - n)}, \tag{3.173}$$

$$C + -\frac{1}{\pi^2(m - n)^2}, \tag{3.174}$$

$$D = -\frac{1}{\pi^3(m - n)^3}. \tag{3.175}$$

Continuing on with equation (3.169), over the common interval $[-a + M,\ a - M]$ as discussed in Section 3.1, with $\gamma = (-1)^m$.

$$I_1 = P\gamma \int_{-a+M}^{a-M} \frac{A_1 \sin^3(s + m\pi)\cos(s + m\pi)}{(s)} ds + P\gamma \int_{-a+M}^{a-M} \frac{B_1 \sin^3(s + n\pi)\cos(s + n\pi)}{(s)^3} ds$$

$$+ P\gamma \int_{-a+M}^{a-M} \frac{C_1 \sin^3(s + n\pi)\cos(s + n\pi)}{(s)^2} ds + P\gamma \int_{-a+M}^{a-M} \frac{D_1 \sin^3(s + n\pi)\cos(s + n\pi)}{(s)} ds, \tag{3.176}$$

$$= P\gamma \int_{-a+M}^{a-M} \frac{B_1 \sin^3(s + n\pi)\cos(s + n\pi)}{(s)^3} + P\gamma \int_{-a+M}^{a-M} \frac{C_1 \sin^3(s + n\pi)\cos(s + n\pi)}{(s)^2} ds \tag{3.177}$$

$$+ P\gamma \int_{-a+M}^{a-M} \frac{(A_1 + D_1)\sin^3(s + m\pi)\cos(s + m\pi)}{(s)} ds, \tag{3.178}$$

$$= P\gamma \int_{-a+M}^{a-M} \frac{B_1 \sin^3(s + n\pi)\cos(s + n\pi)}{(s)^3} ds + P\gamma \int_{-a+M}^{a-M} \frac{C_1 \sin^3(s + n\pi)\cos(s + n\pi)}{(s)^2} ds. \tag{3.179}$$

Since the two integrands of equation (3.179) are,

$$\int_{-\infty}^{\infty} \frac{\sin^3(x)\cos(x)}{(x)^3} dx = \frac{\pi}{2}, \quad \int_{-\infty}^{\infty} \frac{\sin^3(x)\cos(x)}{(x)^2} dx = 0, \tag{3.180}$$

then

$$= P\gamma \frac{\pi}{2} B_1, \tag{3.181}$$

$$I_1 = -\frac{(-1)^{m+n} P}{2(m-n)}. \tag{3.182}$$

Examining equation (3.170), where $\gamma = (-1)^m$

$$I_2 = -P\gamma \int_{-a}^{a} \sin^4(x) \left(\frac{A}{(x - m\pi)} + \frac{B}{(x - n\pi)^4} + \frac{C}{(x - n\pi)^3} + \frac{D}{(x - n\pi)^2} + \frac{E}{(x - n\pi)} \right) dx, \tag{3.183}$$

where A,B,C,D,E equation (3.170) satisfy the polynomial equation

$$1 = A_2(x-n\pi)^3 + B_2(x-m\pi)(x-n\pi)^3 + C_2(x-m\pi)^2 + D_2(x-m\pi)^2(x-n\pi) + E_2(x-m\pi)^2(x-n\pi)^2. \tag{3.184}$$

Comparing the leading order coefficients of x^3 on both sides of equation (3.184) shows that $A_2 + E_2 = 0$. Solving for the coefficients gives

$$A_2 = \frac{1}{\pi^4(m-n)^4}, \tag{3.185}$$

$$B_2 = -\frac{1}{\pi(m-n)},$$

$$C_2 = -\frac{1}{\pi^2(m-n)^2},$$

$$D_2 = -\frac{1}{\pi^3(m-n)^3},$$

$$E_2 = -\frac{1}{\pi^4(m-n)^4},$$

then

$$= -P\gamma \int_{-a+M}^{a-M} \frac{A_2 \sin^4(s+m\pi)}{(s)} ds - P\gamma \int_{-a+M}^{a-M} \frac{B_2 \sin^4(s+n\pi)}{(s)^4} ds$$
$$- P\gamma \int_{-a+M}^{a-M} \frac{C_2 \sin^4(s+n\pi)}{(s)^3} ds - P\gamma \int_{-a+M}^{a-M} \frac{D_2 \sin^4(s+n\pi)}{(s)^2} ds - P\gamma \int_{-a+M}^{a-M} \frac{E_2 \sin^4(s+n\pi)}{(s)} ds,$$
$$= -P\gamma \int_{-a+M}^{a-M} \frac{B_2 \sin^4(s+n\pi)}{(s)^4} ds - P\gamma \int_{-a+M}^{a-M} \frac{C_2 \sin^4(s+n\pi)}{(s)^3} ds$$
$$- P\gamma \int_{-a+M}^{a-M} \frac{D_2 \sin^4(s+n\pi)}{(s)^2} ds - P\gamma \int_{-a+M}^{a-M} \frac{(A_2+E_2) \sin^4(s+m\pi)}{(s)} ds, \quad (3.186)$$
$$= -P\gamma \int_{-a+M}^{a-M} \frac{B_2 \sin^4(s+n\pi)}{(s)^4} ds - P\gamma \int_{-a+M}^{a-M} \frac{C_2 \sin^4(s+n\pi)}{(s)^3} ds$$
$$- P\gamma \int_{-a+M}^{a-M} \frac{D_2 \sin^4(s+n\pi)}{(s)^2} ds. \quad (3.187)$$

Since the three integrands of equation (3.187) are,

$$\int_{-a+M}^{a-M} \frac{\sin^4(x)}{(x)^4} dx = \frac{2\pi}{3}, \quad \int_{-a+M}^{a-M} \frac{\sin^4(x)}{(x)^3} dx = 0, \quad \int_{-a+M}^{a-M} \frac{\sin^4(x)}{(x)^2} dx = \frac{\pi}{2}, \quad (3.188)$$

then

$$= -P\gamma \frac{2\pi}{3} B_2 - P\gamma \frac{\pi}{2} D_2,$$
$$I_2 = \frac{2P(-1)^m}{3(m-n)} + \frac{P(-1)^m}{2\pi^2(m-n)^3}. \quad (3.189)$$

Recombining $I_1 + I_2$ gives

$$= -\frac{(-1)^m P}{2(m-n)} + \frac{2P(-1)^m}{3(m-n)} + \frac{P(-1)^m}{2\pi^2(m-n)^3}, \quad (3.190)$$
$$I = (-1)^m P \left(\frac{1}{6(m-n)} + \frac{1}{2\pi^2(m-n)^3} \right). \quad (3.191)$$

<u>Proof of equation (3.35): $I = \int \psi_n^3 \psi_n' dx$</u>

$$I = \int_{-\infty}^{\infty} \frac{\sin^3(x-n\pi)}{(x-n\pi)^3} \frac{P\left(\cos(x-n\pi)(x-n\pi) - \sin(x-n\pi)\right)}{(x-n\pi)^2} dx. \quad (3.192)$$

Since $\sin(x-n\pi) = (-1)^j \sin(x)$ and $\gamma = (-1)^n$

$$= P\gamma \int_{-a}^{a} \sin^3(x)\cos(x) \left(\frac{A}{(x-n\pi)^4} + \frac{B}{(x-n\pi)^3} + \frac{C}{(x-n\pi)^2} + \frac{D}{(x-n\pi)} \right) dx \quad (3.193)$$
$$- P\gamma \int_{-a}^{a} \sin^4(x) \left(\frac{A}{(x-n\pi)^5} + \frac{B}{(x-n\pi)^4} + \frac{C}{(x-n\pi)^3} + \frac{D}{(x-n\pi)^2} + \frac{E}{(x-n\pi)} \right) dx. \quad (3.194)$$

Thus, $I = 0$.

3.3.3 Two Shifted Sinc Functions And Two Derivatives

Proof of equation (3.36): $\psi_k\psi_l\psi'_m\psi'_n$

As before $\sin(x-n\pi)=(-1)^j\sin(x)$ and $\gamma=(-1)^{k+l}$

$$I=\int_{-\infty}^{\infty}\frac{\sin(x-k\pi)}{(x-k\pi)}\frac{\sin(x-l\pi)}{(x-l\pi)}\frac{P\left(\cos(x-m\pi)(x-m\pi)-\sin(x-m\pi)\right)}{(x-m\pi)^2} \tag{3.195}$$

$$\cdot\frac{P\left(\cos(x-n\pi)(x-n\pi)-\sin(x-n\pi)\right)}{(x-n\pi)^2}dx, \tag{3.196}$$

$$=P^2\gamma\int_{-a}^{a}\sin^2(x)\left(\cos(x)(x-m\pi)-\sin(x)\right)\left(\cos(x)(x-n\pi)-\sin(x)\right)$$

$$\left(\frac{1}{(x-k\pi)(x-l\pi)(x-m\pi)^2(x-n\pi)^2}\right)dx, \tag{3.197}$$

$$I_1=P^2\gamma\int_{-a}^{a}\sin^2(x)\cos^2(x)\left(\frac{1}{(x-k\pi)(x-l\pi)(x-m\pi)(x-n\pi)}\right)dx,$$

$$=P^2\gamma\int_{-a}^{a}\sin^2(x)\cos^2(x)\left(\frac{A}{(x-k\pi)}+\frac{B}{(x-l\pi)}+\frac{C}{(x-m\pi)}+\frac{D}{(x-n\pi)}\right), \tag{3.198}$$

$$I_2=-P^2\gamma\int_{-a}^{a}\sin^3(x)\cos(x)\left(\frac{1}{(x-k\pi)(x-l\pi)(x-m\pi)(x-n\pi)^2}\right)dx,$$

$$=-P^2\gamma\int_{-a}^{a}\sin^3(x)\cos(x)\left(\frac{A}{(x-k\pi)}+\frac{B}{(x-l\pi)}+\frac{C}{(x-m\pi)}+\frac{D}{(x-n\pi)^2}+\frac{E}{(x-n\pi)}\right)dx, \tag{3.199}$$

$$I_3=-P^2\gamma\int_{-a}^{a}\sin^3(x)\cos(x)\left(\frac{1}{(x-k\pi)(x-l\pi)(x-m\pi)^2(x-n\pi)}\right)dx,$$

$$=-P^2\gamma\int_{-a}^{a}\sin^3(x)\cos(x)\left(\frac{A}{(x-k\pi)}+\frac{B}{(x-l\pi)}+\frac{C}{(x-m\pi)^2}+\frac{D}{(x-m\pi)}+\frac{E}{(x-n\pi)}\right)dx, \tag{3.200}$$

$$I_4=P^2\gamma\int_{-a}^{a}\sin^4(x)\left(\frac{1}{(x-k\pi)(x-l\pi)(x-m\pi)^2(x-n\pi)^2}\right)dx,$$

$$=P^2\gamma\int_{-a}^{a}\sin^4(x)$$

$$\cdot\left(\frac{A}{(x-k\pi)}+\frac{B}{(x-l\pi)}+\frac{C}{(x-m\pi)^2}+\frac{D}{(x-m\pi)}+\frac{E}{(x-n\pi)^2}+\frac{F}{(x-n\pi)}\right)dx, \tag{3.201}$$

where A,B,C,D equation (3.198) satisfy the polynomial equation

$$1=A_1(x-l\pi)(x-m\pi)(x-n\pi)+B_1(x-k\pi)(x-m\pi)(x-n\pi)+C_1(x-k\pi)(x-l\pi)(x-n\pi)$$

$$+D_1(x-k\pi)(x-l\pi)(x-m\pi). \tag{3.202}$$

Comparing the leading order coefficients of x^3 on both sides of equation (3.202) shows that

$A_1 + B_1 + C_1 + D_1 = 0$. Solving for the coefficients

$$A = +\frac{1}{\pi^3(k-l)(k-m)(k-n)}, \tag{3.203}$$

$$B = -\frac{1}{\pi^3(l-k)(l-m)(l-n)}, \tag{3.204}$$

$$C = -\frac{1}{\pi^3(m-k)(m-l)(m-n)}, \tag{3.205}$$

$$D = -\frac{1}{\pi^3(n-k)(n-l)(n-m)}, \tag{3.206}$$

Continuing with equation (3.198), over the common interval $[-a+M,\ a-M]$ as discussed in Section 3.1, with $\gamma = (-1)^{k+l}$.

then

$$= P^2\gamma \int_{-a+M}^{a-M} \frac{A\sin^2(s+k\pi)\cos^2(s+k\pi)}{(s)}ds + P^2\gamma_1 \int_{-a+M}^{a-M} \frac{B\sin^2(s+l\pi)\cos^2(s+l\pi)}{(s)}ds,$$
$$+ P^2\gamma \int_{-a+M}^{a-M} \frac{C\sin^2(s+m\pi)\cos^2(s+m\pi)}{(s)}ds + P^2\gamma_1 \int_{-a+M}^{a-M} \frac{D\sin^2(s+n\pi)\cos^2(s+n\pi)}{(s)}ds \tag{3.207}$$

$$= P^2\gamma \int_{-a+M}^{a-M} \frac{(A+B+C+D)\sin^2(s+k\pi)\cos^2(s+k\pi)}{(s)}ds, \tag{3.208}$$

$$I_1 = 0. \tag{3.209}$$

As seen in equation (3.203), $(A+B+C+D) = 0$.

Next, A,B,C,D,E in I_2, equation (3.199), satisfy the polynomial equation

$$\begin{aligned} 1 &= A_2(x-l\pi)(x-m\pi)(x-n\pi)^2 + B_2(x-k\pi)(x-m\pi)(x-n\pi)^2 \\ &+ C_2(x-k\pi)(x-l\pi)(x-n\pi)^2 + D_2(x-k\pi)(x-l\pi)(x-m\pi) \\ &+ E_2(x-k\pi)(x-l\pi)(x-m\pi)(x-n\pi). \end{aligned} \tag{3.210}$$

Comparing the leading order coefficients of x^4 on both sides of equation (3.210) shows that $A_2 + B_2 + C_2 + E_2 = 0$. Solving for the coefficients give the result

$$A_2 = \frac{1}{(k-l)(k-m)(k-n)^2\pi^4}, \tag{3.211}$$

$$B_2 = \frac{1}{(l-k)(l-m)(l-n)^2\pi^4}, \tag{3.212}$$

$$C_2 = \frac{1}{(m-k)(m-l)(m-n)^2\pi^4}, \tag{3.213}$$

$$D_2 = \frac{1}{(n-k)(n-l)(n-m)\pi^3}, \tag{3.214}$$

$$E_2 = \frac{kl + km - 2kn + lm - 2ln - 2mn + 3n^2}{\pi^4(k-n)^2(n-l)^2(n-m)^2}. \tag{3.215}$$

Calculation of I_2 is performed, where the substitution $s = x - (k, l, m, n)\pi$ is made and applying $\sin(s + n\pi) = (-1)^j \sin(s)$ and $\gamma = (-1)^{k+l}$

$$\begin{aligned}
I_2 &= -P^2\gamma \int_{-a+M}^{a-M} \frac{A\sin^3(s+k\pi)\cos(s+k\pi)}{(s)}ds - P^2\gamma \int_{-a+M}^{a-M} \frac{B\sin^3(s+l\pi)\cos(s+l\pi)}{(s)}ds \\
&- P^2\gamma \int_{-a+M}^{a-M} \frac{C\sin^3(s+m\pi)\cos(s+m\pi)}{(s)}ds - P^2\gamma \int_{-a+M}^{a-M} \frac{D\sin^3(s+n\pi)\cos(s+n\pi)}{(s)^2}ds \\
&- P^2\gamma \int_{-a+M}^{a-M} \frac{E\sin^3(s+n\pi)\cos(s+n\pi)}{(s)}ds, \qquad (3.216) \\
&= -P^2\gamma \int_{-a+M}^{a-M} \frac{(A+B+C+E)\sin^3(s+k\pi)\cos(s+k\pi)}{(s+n\pi)}ds \\
&- P^2\gamma \int_{-a+M}^{a-M} \frac{D\sin^3(s+n\pi)\cos(s+n\pi)}{(s)^2}ds, \\
&= -P^2\gamma \int_{-a+M}^{a-M} \frac{D\sin^3(s+n\pi)\cos(s+n\pi)}{(s)^2}ds, \qquad (3.217) \\
I_2 &= 0, \qquad (3.218)
\end{aligned}$$

as

$$\int_{-a+M}^{a-M} \frac{D\sin^3(x)\cos(x)}{(x)^2}dx = 0. \qquad (3.219)$$

Examining I_3,

$$I_3 = -P^2\gamma \int_{-a}^{a} \sin^3(x)\cos(x)\left(\frac{A}{(x-k\pi)} + \frac{B}{(x-l\pi)} + \frac{C}{(x-m\pi)^2} + \frac{D}{(x-m\pi)} + \frac{E}{(x-n\pi)}\right)dx. \qquad (3.220)$$

where A,B,C,D equation (3.200) satisfy the polynomial equation

$$\begin{aligned}
1 &= A_3(x-l\pi)(x-m\pi)^2(x-n\pi) + B_3(x-k\pi)(x-m\pi)^2(x-n\pi) \\
&+ C_3(x-k\pi)(x-l\pi)(x-n\pi) + D_3(x-k\pi)(x-l\pi)(x-m\pi)(x-n\pi) \\
&+ E_3(x-k\pi)(x-l\pi)(x-m\pi)^2. \qquad (3.221)
\end{aligned}$$

Comparing the leading order coefficients of x^4 on both sides of equation (3.221) shows that $A_3 + B_3 + D_3 + E_3 = 0$. Solving for the coefficients give the result

$$\begin{aligned}
A &= \frac{1}{(k-l)(k-m)^2(k-n)\pi^4}, \qquad (3.222) \\
B &= \frac{1}{(l-k)(l-m)^2(l-n)\pi^4}, \\
C &= \frac{1}{(m-k)(m-l)(m-n)\pi^3}, \\
D &= -\frac{-kl+2km-kn+2lm-ln-3m^2+2mn}{\pi^4(k-m)^2(m-l)^2(m-n)^2}, \\
E &= \frac{1}{(n-k)(n-l)(n-m)^2\pi^4}.
\end{aligned}$$

Calculation of I_3 is now possible, where the substitution $s = x - (k, l, m, n)\pi$ is made and applying $\sin(s + n\pi) = (-1)^j \sin(s)$ and $\gamma = (-1)^{k+l}$

$$I_3 = -P^2\gamma \int_{-a+M}^{a-M} \frac{A_3 \sin^3(s + k\pi)\cos(s + k\pi)}{(s)} ds - P^2\gamma \int_{-a+M}^{a-M} \frac{B_3 \sin^3(s + l\pi)\cos(s + l\pi)}{(s)} ds$$

$$- P^2\gamma \int_{-a+M}^{a-M} \frac{C_3 \sin^3(s + m\pi)\cos(s + m\pi)}{(s)^2} ds - P^2\gamma \int_{-a+M}^{a-M} \frac{D_3 \sin^3(s + m\pi)\cos(s + m\pi)}{(s)} ds$$

$$- P^2\gamma \int_{-a+M}^{a-M} \frac{E_3 \sin^3(s + n\pi)\cos(s + n\pi)}{(s)} ds, \tag{3.223}$$

$$= -P^2\gamma \int_{-a+M}^{a-M} \frac{(A_3 + B_3 + D_3 + E_3)\sin^3(s + k\pi)\cos(s + k\pi)}{(s)} ds \tag{3.224}$$

$$- P^2\gamma \int_{-a+M}^{a-M} \frac{C_3 \sin^3(s + m\pi)\cos(s + m\pi)}{(s)^2} ds. \tag{3.225}$$

Again, the equation

$$\int_{-a+M}^{a-M} \frac{\sin^3(x)\cos(x)}{(x)^2} dx = 0, \tag{3.226}$$

and so,

$$I_3 = 0. \tag{3.227}$$

Considering I_4,

$$I_4 = P^2\gamma \int_{-a}^{a} \sin^4(x) \left(\frac{A}{(x - k\pi)} + \frac{B}{(x - l\pi)} + \frac{C}{(x - m\pi)^2} + \frac{D}{(x - m\pi)} + \frac{E}{(x - n\pi)^2} + \frac{F}{(x - n\pi)} \right) dx, \tag{3.228}$$

where A,B,C,D equation (3.201) satisfy the polynomial equation

$$1 = A_4(x - l\pi)(x - m\pi)^2(x - n\pi)^2 + B_4(x - k\pi)(x - m\pi)^2(x - n\pi)^2$$

$$+ C_4(x - k\pi)(x - l\pi)(x - n\pi)^2 + D_4(x - k\pi)(x - l\pi)(x - m\pi)(x - n\pi)^2$$

$$+ E_4(x - k\pi)(x - l\pi)(x - m\pi)^2 + F_4(x - k\pi)(x - l\pi)(x - m\pi)^2(x - n\pi). \tag{3.229}$$

Comparing the leading order coefficients of x^3 on both sides of equation (3.229) shows that $A_4 + B_4 + D_4 + F_4 = 0$. Solving for the coefficients give the result

$$A_4 = \frac{1}{(k - l)(k - m)^2(k - n)\pi^4}, \tag{3.230}$$

$$B_4 = \frac{1}{(l - k)(l - m)^2(l - n)\pi^4},$$

$$C_4 = \frac{1}{(m - k)(m - l)(m - n)\pi^3},$$

$$D_4 = -\frac{-kl + 2km - kn + 2lm - ln - 3m^2 + 2mn}{\pi^4(m - k)^2(m - l)^2(m - n)^2},$$

$$E_4 = \frac{1}{(n - k)(n - l)(n - m)^2\pi^4},$$

$$F_4 = \frac{-2kl - km + 3kn - lm + 3ln + 2mn - 4n^2}{\pi^5(n - k)^2(n - l)^2(n - m)^3}. \tag{3.231}$$

The final integral, I_4 is calculated, substituting $s = x - (k, l, m, n)\pi$ and applying $\sin(s + n\pi) = (-1)^j \sin(s)$ and $\gamma = (-1)^{k+l}$

$$I_4 = P^2\gamma \int_{-a+M}^{a-M} \frac{A\sin^4(s+k\pi)}{(s)} ds + P^2\gamma \int_{-a+M}^{a-M} \frac{B\sin^4(s+l\pi)}{(s)} ds + P^2\gamma \int_{-a+M}^{a-M} \frac{C\sin^4(s+m\pi)}{(s)^2} ds \tag{3.232}$$

$$+ P^2\gamma \int_{-a+M}^{a-M} \frac{D\sin^4(s+m\pi)}{(s)} ds + P^2\gamma \int_{-a+M}^{a-M} \frac{E\sin^4(s+n\pi)}{(s)^2} ds + P^2\gamma \int_{-a+M}^{a-M} \frac{F\sin^4(s+n\pi)}{(s)} ds, \tag{3.233}$$

$$= P^2\gamma \int_{-a+M}^{a-M} \frac{(A_4+B_4+D_4+F_4)\sin^4(s+k\pi)}{(s)} ds + P^2\gamma \int_{-a+M}^{a-M} \frac{C\sin^4(s+m\pi)}{(s)^2} ds$$
$$+ P^2\gamma \int_{-a+M}^{a-M} \frac{E\sin^4(s+n\pi)}{(s)^2} ds, \tag{3.234}$$
$$= P^2\gamma \int_{-a+M}^{a-M} \frac{(C+E)\sin^4(s+m\pi)}{(s)^2} ds. \tag{3.235}$$

Since

$$\int_{-a+M}^{a-M} \frac{\sin^4(x)}{(x)^2} dx = \frac{\pi}{2}, \tag{3.236}$$

so,

$$= P^2\gamma \frac{\pi}{2}(C_4 + E_4), \tag{3.237}$$
$$I_4 = P^2\gamma \frac{\pi}{2}\left(\frac{1}{(m-k)(m-l)(m-n)\pi^3} + \frac{1}{(n-k)(n-l)(n-m)^2\pi^4}\right). \tag{3.238}$$

Since $I_1 + I_2 + I_3 = 0$ the final result for I is

$$I = P^2\gamma \frac{k\left(l(\pi m - \pi n + 1) - m(\pi n + 1) + \pi n^2\right) - l\left(m(\pi n + 1) - \pi n^2\right) + m^2 + \pi m n^2 - \pi n^3}{\pi^4(k-m)(k-n)(l-m)(l-n)(m-n)^2}. \tag{3.239}$$

<u>Proof of equation (3.37): $I = \int \psi_l^2 \psi'_m \psi'_n dx$</u>

Again, $\sin(x - n\pi) = (-1)^j \sin(x)$ and $\gamma = 1$

$$I = \int_{-a}^{a} \frac{\sin^2(x - l\pi)}{(x - l\pi)^2}$$
$$\cdot \frac{P\left(\cos(x-m\pi)(x-m\pi) - \sin(x-m\pi)\right)}{(x-m\pi)^2} \frac{P\left(\cos(x-n\pi)(x-n\pi) - \sin(x-n\pi)\right)}{(x-n\pi)^2} dx, \tag{3.240}$$

$$= P^2\gamma \int_{-a}^{a} \frac{\sin^2(x)\cos^2(x)(x-m\pi)(x-n\pi)}{(x-l\pi)^2(x-m\pi)^2(x-n\pi)^2} dx - P^2\gamma \int_{-a}^{a} \frac{\sin^3(x)\cos(x)(x-m\pi)}{(x-l\pi)^2(x-m\pi)^2(x-n\pi)^2} dx$$
$$- P^2\gamma \int_{-a}^{a} \frac{\sin^3(x)\cos(x)(x-n\pi)}{(x-l\pi)^2(x-m\pi)^2(x-n\pi)^2} ds - P^2\gamma \int_{-a}^{a} \frac{\sin^4(x)}{(x-l\pi)^2(x-m\pi)^2(x-n\pi)^2} dx, \tag{3.241}$$

$$I_1 = P^2\gamma \int_{-a}^{a} \sin^2(x)\cos^2(x)\left(\frac{A}{(x-l\pi)^2}+\frac{B}{(x-l\pi)}+\frac{C}{(x-m\pi)}+\frac{D}{(x-n\pi)}\right)dx, \tag{3.242}$$

$$I_2 = -P^2\gamma \int_{-a}^{a} \sin^3(x)\cos(x)\left(\frac{A}{(x-l\pi)^2}+\frac{B}{(x-l\pi)}+\frac{C}{(x-m\pi)}+\frac{D}{(x-n\pi)^2}+\frac{E}{(x-n\pi)}\right)dx, \tag{3.243}$$

$$I_3 = -P^2\gamma \int_{-a}^{a} \sin^3(x)\cos(x)\left(\frac{A}{(x-l\pi)^2}+\frac{B}{(x-l\pi)}+\frac{C}{(x-m\pi)^2}+\frac{D}{(x-m\pi)}+\frac{E}{(x-n\pi)}\right)dx, \tag{3.244}$$

$$I_4 = -P^2\gamma \int_{-a}^{a} \sin^4(x)\left(\frac{A}{(x-l\pi)^2}+\frac{B}{(x-l\pi)}+\frac{C}{(x-m\pi)^2}+\frac{D}{(x-m\pi)}+\frac{E}{(x-n\pi)^2}\right.$$
$$\left.+\frac{F}{(x-n\pi)}\right)dx, \tag{3.245}$$

where A,B,C,D in equation (3.242) satisfy the polynomial equation

$$1 = A_1(x-m\pi)(x-n\pi)+B_1(x-l\pi)(x-m\pi)(x-n\pi)+C_1(x-l\pi)^2(x-n\pi)$$
$$+D_1(x-l\pi)^2(x-m\pi). \tag{3.246}$$

Comparing the leading order coefficients of x^3 on both sides of equation (3.246) shows that $B_1+C_1+D_1=0$. Solving for the coefficients give the result

$$A = \frac{1}{(l-m)(l-n)\pi^2}, \tag{3.247}$$

$$B = \frac{-2l+m+n}{(l-m)^2(l-n)^2\pi^3}, \tag{3.248}$$

$$C = \frac{1}{(m-l)^2(m-n)\pi^3}, \tag{3.249}$$

$$D = \frac{1}{(n-l)^2(n-m)\pi^3}. \tag{3.250}$$

Continuing with equation (3.242), over the common interval $[-a+M,\ a-M]$ as discussed in Section 3.1, with $\gamma=1$.

$$I_1 = P^2\gamma\int_{-a+M}^{a-M}\frac{A\sin^2(s+l\pi)\cos^2(s+l\pi)}{(s)^2}ds + P^2\gamma\int_{-a+M}^{a-M}\frac{B\sin^2(s+l\pi)\cos^2(s+l\pi)}{(s)}ds$$
$$+P^2\gamma\int_{-a+M}^{a-M}\frac{C\sin^2(s+m\pi)\cos^2(s+m\pi)}{(s)}ds + P^2\gamma\int_{-a+M}^{a-M}\frac{D\sin^2(s+n\pi)\cos^2(s+n\pi)}{(s)}ds, \tag{3.251}$$

$$= P^2\gamma\int_{-a+M}^{a-M}\frac{A\sin^2(s+l\pi)\cos^2(s+l\pi)}{(s)^2}ds$$
$$+P^2\gamma\int_{-a+M}^{a-M}\frac{(B+C+D)\sin^2(s+m\pi)\cos^2(s+m\pi)}{(s)}ds, \tag{3.252}$$

$$= P^2\gamma\int_{-a+M}^{a-M}\frac{A\sin^2(s+l\pi)\cos^2(s+l\pi)}{(s)^2}ds. \tag{3.253}$$

Since,

$$\int_{-a+M}^{a-M} \frac{\sin^2(x)\cos^2(x)}{(x)^2} ds = \frac{\pi}{2}, \tag{3.254}$$

then

$$= \frac{\pi}{2} P^2 \gamma A, \tag{3.255}$$

$$I_1 = \gamma \frac{P^2 \gamma}{2\pi(l-m)(l-n)}. \tag{3.256}$$

Examining I_2, A,B,C,D,E equation (3.243) satisfy the polynomial equation

$$\begin{aligned}1 &= A(x-m\pi)(x-n\pi)^2 + B(x-l\pi)(x-m\pi)(x-n\pi)^2 + C(x-l\pi)^2(x-n\pi)^2\\ &+ D(x-l\pi)^2(x-m\pi) + E(x-l\pi)^2(x-m\pi)(x-n\pi).\end{aligned} \tag{3.257}$$

Comparing the leading order coefficients of x^3 on both sides of equation (3.257) shows that $B_2 + C_2 + D_2 = 0$. Solving for the coefficients give the result

$$A_2 = \frac{1}{\pi^3(l-m)(l-n)^2}, \tag{3.258}$$

$$B_2 = \frac{-3l+2m+n}{\pi^4(l-m)^2(l-n)^3}, \tag{3.259}$$

$$C_2 = \frac{1}{\pi^4(m-l)^2(m-n)^2}, \tag{3.260}$$

$$D_2 = \frac{1}{\pi^3(n-l)^2(n-m)}, \tag{3.261}$$

$$E_2 = \frac{-l-2m+3n}{\pi^4(n-l)^3(n-m)^2}. \tag{3.262}$$

Calculation of I_2 is now possible, where the substitution $s = x - (k,l,m,n)\pi$ is made and applying $\sin(s+n\pi) = (-1)^j \sin(s)$ and $\gamma = 1$

$$\begin{aligned}I_2 = &-P^2\gamma \int_{-a+M}^{a-M} \frac{A_2 \sin^3(s+l\pi)\cos(s+l\pi)}{(s)^2} ds - P^2\gamma \int_{-a+M}^{a-M} \frac{B_2 \sin^3(s+l\pi)\cos(s+l\pi)}{(s)} ds\\ &- P^2\gamma \int_{-a+M}^{a-M} \frac{C_2 \sin^3(s+m\pi)\cos(s+m\pi)}{(s)} ds - P^2\gamma \int_{-a+M}^{a-M} \frac{D_2 \sin^3(s+n\pi)\cos(s+n\pi)}{(s)^2} ds\\ &- P^2\gamma \int_{-a+M}^{a-M} \frac{E_2 \sin^3(s+l\pi)\cos(s+l\pi)}{(s)} ds,\end{aligned} \tag{3.263}$$

$$\begin{aligned}&= -P^2\gamma \int_{-a+M}^{a-M} \frac{(A_2+D_2)\sin^3(s+l\pi)\cos(s+l\pi)}{(s)^2} ds\\ &- P^2\gamma \int_{-a+M}^{a-M} \frac{(B_2+C_2+D_2)\sin^3(s+l\pi)\cos(s+l\pi)}{(s)} ds,\end{aligned} \tag{3.264}$$

$$= -P^2\gamma \int_{-a+M}^{a-M} \frac{(A_2+D_2)\sin^3(s+l\pi)\cos(s+l\pi)}{(s)^2} ds. \tag{3.265}$$

Since,

$$\int_{-a+M}^{a-M} \frac{\sin^3(x)\cos(x)}{(x)^2} dx = 0, \tag{3.266}$$

so

$$I_2 = 0. \tag{3.267}$$

Next examining I_3, A,B,C,D,E equation (3.244) satisfy the polynomial equation

$$\begin{aligned} 1 &= A(x-m\pi)^2(x-n\pi) + B(x-l\pi)(x-m\pi)^2(x-n\pi) + C(x-l\pi)^2(x-m\pi)(x-n\pi) \\ &+ D(x-l\pi)^2(x-m\pi)(x-n\pi) + E(x-l\pi)^2(x-m\pi)^2. \end{aligned} \tag{3.268}$$

Comparing the leading order coefficients of x^3 on both sides of equation (3.268) shows that $B_3 + D_3 + E_3 = 0$. Solving for the coefficients give the result

$$A_3 = \frac{1}{\pi^3(l-m)^2(l-n)}, \tag{3.269}$$

$$B_3 = \frac{-3l+m+2n}{\pi^4(l-m)^3(l-n)^2}, \tag{3.270}$$

$$C_3 = \frac{1}{\pi^3(m-l)^2(m-n)}, \tag{3.271}$$

$$D_3 = \frac{-l+3m-2n}{\pi^4(l-m)^3(m-n)^2}, \tag{3.272}$$

$$E_3 = \frac{1}{\pi^4(n-l)^2(n-m)^2}. \tag{3.273}$$

Calculation of I_3 is undertaken, where the substitution $s = x-(k,l,m,n)\pi$ is made and applying $\sin(s+n\pi) = (-1)^j \sin(s)$ and $\gamma = 1$

$$\begin{aligned} I_3 &= -P^2\gamma \int_{-a+M}^{a-M} \frac{A_3 \sin^3(s+l\pi)\cos(s+l\pi)}{(s)^2} ds - P^2\gamma \int_{-a+M}^{a-M} \frac{B_3 \sin^3(s+l\pi)\cos(s+l\pi)}{(s)} ds \\ &- P^2\gamma \int_{-a+M}^{a-M} \frac{C_3 \sin^3(s+m\pi)\cos(s+m\pi)}{(s)^2} ds - P^2\gamma \int_{-a+M}^{a-M} \frac{D_3 \sin^3(s+m\pi)\cos(s+m\pi)}{(s)} ds \\ &- P^2\gamma \int_{-a+M}^{a-M} \frac{E_3 \sin^3(s+n\pi)\cos(s+n\pi)}{(s)} ds, \end{aligned} \tag{3.274}$$

$$\begin{aligned} &= -P^2\gamma \int_{-a+M}^{a-M} \frac{(A_3+C_3)\sin^3(s+l\pi)\cos(s+l\pi)}{(s)^2} ds \\ &- P^2\gamma \int_{-a+M}^{a-M} \frac{(B_3+D_3+E_3)\sin^3(s+n\pi)\cos(s+n\pi)}{(s)} ds, \end{aligned}$$

$$= -P^2\gamma \int_{-a+M}^{a-M} \frac{(A_3+C_3)\sin^3(s+l\pi)\cos(s+l\pi)}{(s)^2} ds, \tag{3.275}$$

$$= 0, \tag{3.276}$$

since,

$$\int_{-a+M}^{a-M} \frac{\sin^3(x)\cos(x)}{(x)^2} dx = 0. \tag{3.277}$$

Finally, calculating I_4, where A,B,C,D,E,F equation (3.245) satisfy the polynomial equation

$$\begin{aligned}1 &= A_4(x-m\pi)^2(x-n\pi)^2 + B_4(x-l\pi)(x-m\pi)^2(x-n\pi)^2 + C_4(x-l\pi)^2(x-n\pi)^2 \\ &+ D_4(x-l\pi)^2(x-m\pi)(x-n\pi)^2 + E_4(x-l\pi)^2(x-m\pi)^2 + F(x-l\pi)^2(x-m\pi)^2(x-n\pi).\end{aligned} \tag{3.278}$$

Comparing the leading order coefficients of x^5 on both sides of equation (3.278) shows that $B_4 + D_4 + F_4 = 0$. Solving for the coefficients give the result

$$A = \frac{1}{(l-m)^2(l-n)^2\pi^4}, \tag{3.279}$$

$$B = -\frac{2(2l-m-n)}{(l-m)^3(l-n)^3\pi^5}, \tag{3.280}$$

$$C = \frac{1}{(m-l)^2(m-n)^2\pi^4}, \tag{3.281}$$

$$D = -\frac{2(l-2m+n)}{(l-m)^3(m-n)^3\pi^5}, \tag{3.282}$$

$$E = \frac{1}{(n-l)^2(n-m)^2\pi^4}, \tag{3.283}$$

$$F = -\frac{2(l+m-2n)}{(l-n)^3(n-m)^3\pi^5}. \tag{3.284}$$

The final integral, I_4 is calculated, substituting $s = x - (k,l,m,n)\pi$ and applying $\sin(s+n\pi) = (-1)^j \sin(s)$ and $\gamma = 1$,

$$\begin{aligned}I_4 &= -P^2\gamma \int_{-a+M}^{a-M} \frac{A_4 \sin^4(s+l\pi)}{(s)^2} ds - P^2\gamma \frac{B_4 \sin^4(s+l\pi)}{(s)} ds - P^2\gamma \frac{C_4 \sin^4(s+m\pi)}{(s)^2} ds \\ &- P^2\gamma \frac{D_4 \sin^4(s+m\pi)}{(s)} ds - P^2\gamma \frac{E_4 \sin^4(s+n\pi)}{(s)^2} ds - P^2\gamma \frac{F_4 \sin^4(s+n\pi)}{(s)} ds, \end{aligned} \tag{3.285}$$

$$= -P^2\gamma \int_{-a+M}^{a-M} \frac{(A_4+C_4+E_4)\sin^4(s+l\pi)}{(s)^2} ds - P^2\gamma \frac{(B_4+D_4+F_4)\sin^4(s+n\pi)}{(s)} ds, \tag{3.286}$$

$$= -P^2\gamma \int_{-a+M}^{a-M} \frac{(A_4+C_4+E_4)\sin^4(s+l\pi)}{(s)^2} ds. \tag{3.287}$$

Since

$$\int_{-\infty}^{\infty} \frac{\sin^4(x)}{x^2} dx = \frac{\pi}{2}, \tag{3.288}$$

then,

$$= -P^2\gamma \frac{\pi}{2} \left(\frac{1}{\pi^4(l-m)^2(l-n)^2} + \frac{1}{\pi^4(m-l)^2(m-n)^2} + \frac{1}{\pi^4(n-l)^2(n-m)^2} \right), \tag{3.289}$$

$$I_4 = -P^2\gamma \frac{2\left(l^2 + l(-m-n) + m^2 - mn + n^2\right)}{\pi^4(l-m)^2(l-n)^2(m-n)^2}. \tag{3.290}$$

Finally, the full expression for $I = I_1 + I_2 + I_3 + I_4$ is

$$= \gamma\frac{2P^2\gamma}{\pi(l-m)(l-n)} - P^2\gamma\frac{2\left(l^2 + l(-m-n) + m^2 - mn + n^2\right)}{\pi^4(l-m)^2(l-n)^2(m-n)^2}, \tag{3.291}$$

$$I = \frac{2P^2\gamma\left(\pi^3(l-m)(l-n) - \frac{l^2 - l(m+n) + m^2 - mn + n^2}{(m-n)^2}\right)}{\pi^4(l-m)^2(l-n)^2}. \tag{3.292}$$

Proof of equation (3.38): $I = \int \psi_l\psi_m\psi_n'^2 dx$

Again, $\sin(x - n\pi) = (-1)^j \sin(x)$ and $\gamma = (-1)^{l+m}$,

$$I = \int_{-\infty}^{\infty} \frac{\sin(x-l\pi)}{(x-l\pi)}\frac{\sin(x-m\pi)}{(x-m\pi)}\frac{P^2\left(\cos(x-n\pi)(x-n\pi) - \sin(x-n\pi)\right)^2}{(x-n\pi)^4}dx, \tag{3.293}$$

$$I_1 = P^2\gamma\int_{-a}^{a} \sin^2(x)\cos^2(x)(x-n\pi)^2\left(\frac{1}{(x-l\pi)(x-m\pi)(x-n\pi)^4}\right)dx, \tag{3.294}$$

$$= P^2\gamma\int_{-a}^{a} \sin^2(x)\cos^2(x)\left(\frac{A}{(x-l\pi)} + \frac{B}{(x-m\pi)} + \frac{C}{(x-n\pi)^2} + \frac{D}{(x-n\pi)}\right)dx, \tag{3.295}$$

$$I_2 = -2P^2\gamma\int_{-a}^{a} \sin^3(x)\cos(x)(x-n\pi)\left(\frac{1}{(x-l\pi)(x-m\pi)(x-n\pi)^4}\right)dx, \tag{3.296}$$

$$= -2P^2\gamma\int_{-a}^{a} \sin^3(x)\cos(x)\left(\frac{A}{(x-l\pi)} + \frac{B}{(x-m\pi)} + \frac{C}{(x-n\pi)^3} + \frac{D}{(x-n\pi)^2}\right.$$
$$\left. + \frac{E}{(x-n\pi)}\right)dx, \tag{3.297}$$

$$I_3 = P^2\gamma\int_{-a}^{a} \sin^4(x)\left(\frac{1}{(x-l\pi)(x-m\pi)(x-n\pi)^4}\right)dx, \tag{3.298}$$

$$= P^2\gamma\int_{-a}^{a} \sin^4(x)\left(\frac{A}{(x-l\pi)} + \frac{B}{(x-m\pi)} + \frac{C}{(x-n\pi)^4} + \frac{D}{(x-n\pi)^3} + \frac{E}{(x-n\pi)^2}\right.$$
$$\left. + \frac{F}{(x-n\pi)}\right)dx, \tag{3.299}$$

where A,B,C,D equation (3.294) satisfy the polynomial equation

$$1 = A_1(x-m\pi)(x-n\pi)^2 + B_1(x-l\pi)(x-n\pi)^2 + C_1(x-l\pi)(x-m\pi) + D_1(x-l\pi)(x-m\pi)(x-n\pi).$$

Comparing the leading order coefficients of x^3 on both sides of equation (3.300) shows that $A_1 + B_1 + D_1 = 0$. Solving for the coefficients give the result

$$A_1 = \frac{1}{(l-m)(l-n)^2\pi^3}, \tag{3.300}$$

$$B_1 = -\frac{1}{(l-m)(m-n)^2\pi^3}, \tag{3.301}$$

$$C_1 = -\frac{1}{(l-n)(n-m)\pi^2}, \tag{3.302}$$

$$D_1 = \frac{l+m-2n}{(l-n)^2(n-m)^2\pi^3}. \tag{3.303}$$

Continuing with equation (3.294), over the common interval $[-a+M,\ a-M]$ as discussed in Section 3.1, with $\gamma = (-1)^{l+m}$.

$$
\begin{aligned}
I_1 &= P^2\gamma \int_{-a+M}^{a-M} \frac{A_1 \sin^2(s)\cos^2(s)}{(s)} ds + P^2\gamma \int_{-a+M}^{a-M} \frac{B_1 \sin^2(s)\cos^2(s)}{(s)} ds \\
&+ P^2\gamma \int_{-a+M}^{a-M} \frac{C_1 \sin^2(s)\cos^2(s)}{(s)^2} ds + P^2\gamma \int_{-a+M}^{a-M} \frac{D_1 \sin^2(s)\cos^2(s)}{(s)} ds, \quad (3.304) \\
&= P^2\gamma \int_{-a+M}^{a-M} \frac{(A_1+B_1+D_1) \sin^2(s)\cos^2(s)}{(s)} ds + P^2\gamma \int_{-a+M}^{a-M} \frac{C_1 \sin^2(s)\cos^2(s)}{(s)^2} ds, \quad (3.305) \\
&= P^2\gamma \int_{-a+M}^{a-M} \frac{C_1 \sin^2(s)\cos^2(s)}{(s)^2} ds. \quad (3.306)
\end{aligned}
$$

Since

$$
\int_{-\infty}^{\infty} \frac{\sin^2(x)\cos^2(x)}{(x)^2} dx = \frac{\pi}{2}, \quad (3.307)
$$

then

$$
\begin{aligned}
&= P^2\gamma \frac{\pi}{2} C_1, \quad (3.308) \\
I_1 &= -\frac{P^2\gamma}{2(l-n)(n-m)\pi}. \quad (3.309)
\end{aligned}
$$

Solving for I_2, considering A,B,C,D,E equation (3.296) satisfy the polynomial equation

$$
\begin{aligned}
1 &= A_2(x-m\pi)(x-n\pi)^3 + B_2(x-l\pi)(x-n\pi)^3 + C_2(x-l\pi)(x-m\pi) \\
&+ D_2(x-l\pi)(x-m\pi)(x-n\pi) + E_2(x-l\pi)(x-m\pi)(x-n\pi)^2. \quad (3.310)
\end{aligned}
$$

Comparing the leading order coefficients of x^4 on both sides of equation (3.310) shows that $A_2 + B_2 + E_2 = 0$. Solving for the coefficients give the result

$$
\begin{aligned}
A_2 &= \frac{1}{(l-m)(l-n)^3\pi^4}, \quad (3.311) \\
B_2 &= -\frac{1}{(l-m)(m-n)^3\pi^4}, \quad (3.312) \\
C_2 &= -\frac{1}{(l-n)(n-m)\pi^2}, \quad (3.313) \\
D_2 &= \frac{l+m-2n}{(l-n)^2(n-m)^2\pi^3}, \quad (3.314) \\
E_2 &= \frac{-l^2-lm+3ln-m^2+3mn-3n^2}{(l-n)^3(n-m)^3\pi^4}. \quad (3.315)
\end{aligned}
$$

Evaluation of I_2 is undertaken, where $\sin(x-n\pi)=(-1)^j\sin(x)$ and $\gamma=(-1)^{l+m}$

$$I_2 = -2P^2\gamma\int_{-a+M}^{a-M}\frac{A_2\sin^3(s)\cos(s)}{(s)}ds - 2P^2\gamma\int_{-a+M}^{a-M}\frac{B_2\sin^3(s)\cos(s)}{(s)}ds$$
$$- 2P^2\gamma\int_{-a+M}^{a-M}\frac{C_2\sin^3(s)\cos(s)}{(s)^3}ds - 2P^2\gamma\int_{-a+M}^{a-M}\frac{D_2\sin^3(s)\cos(s)}{(s)^2}ds$$
$$- 2P^2\gamma\int_{-a+M}^{a-M}\frac{E_2\sin^3(s)\cos(s)}{(s)}ds, \tag{3.316}$$
$$= -2P^2\gamma\int_{-a+M}^{a-M}\frac{(A_2+B_2+E_2)\sin^3(s)\cos(s)}{(s)}ds - 2P^2\gamma\int_{-a+M}^{a-M}\frac{C_2\sin^3(s)\cos(s)}{(s)^3}ds \tag{3.317}$$
$$- 2P^2\gamma\int_{-a+M}^{a-M}\frac{D_2\sin^3(s)\cos(s)}{(s)^2}ds, \tag{3.318}$$
$$= -2P^2\gamma\int_{-a+M}^{a-M}\frac{C_2\sin^3(s)\cos(s)}{(s)^3}ds - 2P^2\gamma\int_{-a+M}^{a-M}\frac{D_2\sin^3(s)\cos(s)}{(s)^2}ds, \tag{3.319}$$

since

$$\int_{-\infty}^{\infty}\frac{\sin^3(x)\cos(x)}{(x)^2}dx = 0 \text{ and } \int_{-\infty}^{\infty}\frac{\sin^3(x)\cos(x)}{(x)^3}dx = \frac{\pi}{2}, \tag{3.320}$$

then

$$= -2P^2\gamma\frac{\pi}{2}C_2, \tag{3.321}$$
$$I_2 = \frac{P^2\gamma}{(l-n)(n-m)\pi}. \tag{3.322}$$

Solving for I_3, considering A,B,C,D,E,F equation (3.298) satisfy the polynomial equation

$$1 = A_3(x-m\pi)(x-n\pi)^4 + B_3(x-l\pi)(x-n\pi)^4 + C_3(x-l\pi)(x-m\pi)$$
$$+ D_3(x-l\pi)(x-m\pi)(x-n\pi) + E_3(x-l\pi)(x-m\pi)(x-n\pi)^2$$
$$+ F_3(x-l\pi)(x-m\pi)(x-n\pi)^3. \tag{3.323}$$

Comparing the leading order coefficients of x^5 on both sides of equation (3.323) shows that $A_3+B_3+F_3=0$. Solving for the coefficients give the result

$$A_3 = \frac{1}{(l-m)(l-n)^4\pi^5}, \tag{3.324}$$
$$B_3 = -\frac{1}{(l-m)(m-n)^4\pi^5}, \tag{3.325}$$
$$C_3 = -\frac{1}{(l-n)(n-m)\pi^2}, \tag{3.326}$$
$$D_3 = \frac{l+m-2n}{(l-n)^2(n-m)^2\pi^3}, \tag{3.327}$$
$$E_3 = \frac{-l^2-lm+3ln-m^2+3mn-3n^2}{(l-n)^3(n-m)^3\pi^4}, \tag{3.328}$$
$$F_3 = \frac{l^3+l^2m-4l^2n+lm^2-4lmn+6ln^2+m^3-4m^2n+6mn^2-4n^3}{(l-n)^4(n-m)^4\pi^5}. \tag{3.329}$$

Evaluation of I_3 may be undertaken, where $\sin(x-n\pi)=(-1)^j\sin(x)$ and $\gamma=(-1)^{l+m}$

$$I_3 = P^2\gamma\int_{-a+M}^{a-M}\frac{A_3\sin^4(s)}{(s)}ds + P^2\gamma\int_{-a+M}^{a-M}\frac{B_3\sin^4(s)}{(s)}ds + P^2\gamma\int_{-a+M}^{a-M}\frac{C_3\sin^4(s)}{(s)^4}ds \quad (3.330)$$

$$+ P^2\gamma\int_{-a+M}^{a-M}\frac{D_3\sin^4(s)}{(s)^3}ds + P^2\gamma\int_{-a+M}^{a-M}\frac{E_3\sin^4(s)}{(s)^2}ds + P^2\gamma\int_{-a+M}^{a-M}\frac{F_3\sin^4(s)}{(s)}ds, \quad (3.331)$$

$$= P^2\gamma\int_{-a+M}^{a-M}\frac{(A_3+B_3+F_3)\sin^4(s)}{(s)}ds + P^2\gamma\int_{-a+M}^{a-M}\frac{C_3\sin^4(s)}{(s)^4}ds \quad (3.332)$$

$$+ P^2\gamma\int_{-a+M}^{a-M}\frac{D_3\sin^4(s)}{(s)^3}ds + P^2\gamma\int_{-a+M}^{a-M}\frac{E_3\sin^4(s)}{(s)^2}ds, \quad (3.333)$$

$$= P^2\gamma\int_{-a+M}^{a-M}\frac{C_3\sin^4(s)}{(s)^4}ds + P^2\gamma\int_{-a+M}^{a-M}\frac{D_3\sin^4(s)}{(s)^3}ds + P^2\gamma\int_{-a+M}^{a-M}\frac{E_3\sin^4(s)}{(s)^2}ds. \quad (3.334)$$

Since

$$\int_{-\infty}^{\infty}\frac{\sin^4(x)}{(x)^4}dx = \frac{2\pi}{3},\ \int_{-\infty}^{\infty}\frac{\sin^4(x)}{(x)^3}dx = 0,\ \int_{-\infty}^{\infty}\frac{\sin^4(x)}{(x)^2}dx = \frac{\pi}{2}, \quad (3.335)$$

then

$$= P^2\gamma\left(\frac{2\pi}{3}C_3 + \frac{\pi}{2}E_3\right), \quad (3.336)$$

$$I_3 = P^2\gamma\left(-\frac{2}{3\pi(l-n)(n-m)} + \frac{-l^2-lm+3ln-m^2+3mn-3n^2}{2(l-n)^3(n-m)^3\pi^3}\right). \quad (3.337)$$

The full expression for I is

$$I = P^2\gamma\left(-\frac{1}{6\pi(l-n)(n-m)} + \frac{-l^2-lm+3ln-m^2+3mn-3n^2}{2(l-n)^3(n-m)^3\pi^3}\right). \quad (3.338)$$

Proof of equation (3.39): $I=\int\psi_l\psi_m\psi'_m\psi'_n dx$

Again, $\sin(x-n\pi)=(-1)^j\sin(x)$ and $\gamma=(-1)^{l+m}$,

$$I = \int_{-\infty}^{\infty}\frac{\sin(x-l\pi)}{(x-l\pi)}\frac{\sin(x-m\pi)}{(x-m\pi)}\frac{P\left(\cos(x-m\pi)(x-m\pi)-\sin(x-m\pi)\right)}{(x-m\pi)^2}$$
$$\cdot\frac{P\left(\cos(x-n\pi)(x-n\pi)-\sin(x-n\pi)\right)}{(x-n\pi)^2}dx, \quad (3.339)$$

$$= P\gamma\int_{-a}^{a}\Bigg\{\sin^2(x)\cos^2(x)\left(\frac{A_1}{(x-l\pi)}+\frac{B_1}{(x-m\pi)^2}+\frac{C_1}{(x-m\pi)}+\frac{D_1}{(x-n\pi)}\right) \quad (3.340)$$

$$-\sin^3(x)\cos(x)\left(\frac{A_2}{(x-l\pi)}+\frac{B_2}{(x-m\pi)^2}+\frac{C_2}{(x-m\pi)}+\frac{D_2}{(x-n\pi)^2}+\frac{E_2}{(x-n\pi)}\right) \quad (3.341)$$

$$-\sin^3(x)\cos(x)\left(\frac{A_3}{(x-l\pi)}+\frac{B_3}{(x-m\pi)^3}+\frac{C_3}{(x-m\pi)^2}+\frac{D_3}{(x-m\pi)}+\frac{E_3}{(x-n\pi)}\right) \quad (3.342)$$

$$+\sin^4(x)\Bigg(\frac{A_4}{(x-l\pi)}+\frac{B_4}{(x-m\pi)^3}+\frac{C_4}{(x-m\pi)^2}+\frac{D_4}{(x-m\pi)}+\frac{E_4}{(x-n\pi)^2} \quad (3.343)$$

$$+\frac{F_4}{(x-n\pi)}\Bigg)\Bigg\}dx. \quad (3.344)$$

where equation (3.344) the coefficients satisfy the polynomial equations

$$1 = A_1(x - m\pi)^2(x - n\pi)^2 + B_1(x - l\pi)(x - n\pi)^2 + C_1(x - l\pi)(x - m\pi)(x - n\pi) + D_1(x - l\pi)(x - m\pi)^2, \tag{3.345}$$

$$1 = A_2(x - m\pi)^2(x - n\pi) + B_2(x - l\pi)(x - n\pi) + C_2(x - l\pi)(x - m\pi)(x - n\pi)^2 + D_2(x - l\pi)(x - m\pi)^2 + E_2(x - l\pi)(x - m\pi)^2(x - n\pi), \tag{3.346}$$

$$1 = A_3(x - m\pi)^3(x - n\pi) + B_3(x - l\pi)(x - n\pi) + C_3(x - l\pi)(x - m\pi)(x - n\pi) + D_3(x - l\pi)(x - m\pi)^2(x - n\pi) + E_3(x - l\pi)(x - m\pi)^3, \tag{3.347}$$

$$1 = A_4(x - m\pi)^3(x - n\pi)^2 + B_4(x - l\pi)(x - n\pi)^2 + C_4(x - l\pi)(x - m\pi)(x - n\pi)^2 + D_4(x - l\pi)(x - m\pi)^2(x - n\pi)^2 + E_4(x - l\pi)(x - m\pi)^3 + F_4(x - l\pi)(x - m\pi)^3(x - n\pi). \tag{3.348}$$

Comparing the leading order coefficients of x on both sides of equations (3.345), (3.345), (3.345) and (3.345). This comparison shows that $A_1 + C_1 + D_1 = 0$, for (3.345), $A_2 + C_2 + E_2 = 0$, for (3.346), $A_3 + D_3 + E_3 = 0$, for (3.347) and $A_3 + D_3 + F_3 = 0$, for (3.348).
Solving for the coefficients give the result

$$A_1 = \frac{1}{(l - m)^2(l - n)\pi^3}, \tag{3.349}$$

$$B_1 = -\frac{1}{(l - m)(m - n)\pi^2}, \tag{3.350}$$

$$C_1 = \frac{l - 2m + n}{(l - m)^2(m - n)^2\pi^3}, \tag{3.351}$$

$$D_1 = -\frac{1}{(l - n)(n - m)^2\pi^3}, \tag{3.352}$$

$$A_2 = \frac{1}{\pi^4(l - m)^2(l - n)^2}, \tag{3.353}$$

$$B_2 = -\frac{1}{\pi^3(l - m)(m - n)^2}, \tag{3.354}$$

$$C_2 = \frac{2l - 3m + n}{\pi^4(l - m)^2(m - n)^3}, \tag{3.355}$$

$$D_2 = -\frac{1}{\pi^3(l - n)(n - m)^2}, \tag{3.356}$$

$$E_2 = -\frac{-2l - m + 3n}{\pi^4(l - n)^2(n - m)^3}, \tag{3.357}$$

$$A_3 = \frac{1}{(l - m)^3(l - n)\pi^4}, \tag{3.358}$$

$$B_3 = -\frac{1}{(l - m)(m - n)\pi^2}, \tag{3.359}$$

$$C_3 = \frac{l - 2m + n}{(l - m)^2(m - n)^2\pi^3}, \tag{3.360}$$

$$D_3 = \frac{-l^2 + 3lm - ln - 3m^2 + 3mn - n^2}{(l - m)^3(m - n)^3\pi^4}, \tag{3.361}$$

$$E_3 = -\frac{1}{(l-n)(n-m)^3\pi^4}, \tag{3.362}$$
$$A_4 = \frac{1}{\pi^5(l-m)^3(l-n)^2}, \tag{3.363}$$
$$B_4 = -\frac{1}{\pi^3(l-m)(m-n)^2}, \tag{3.364}$$
$$C_4 = \frac{2l-3m+n}{\pi^4(l-m)^2(m-n)^3}, \tag{3.365}$$
$$D_4 = \frac{-3l^2+8lm-2ln-6m^2+4mn-n^2}{\pi^5(l-m)^3(m-n)^4}, \tag{3.366}$$
$$E_4 = -\frac{1}{\pi^4(l-n)(n-m)^3}, \tag{3.367}$$
$$F_4 = \frac{3l+m-4n}{\pi^5(l-n)^2(n-m)^4}. \tag{3.368}$$

Continuing with equation (3.344), over the common interval $[-a+M,\ a-M]$ as discussed in Section 3.1, substitute $s = x-(k,l,m,n)\pi$, apply $\sin(s+n\pi) = (-1)^j \sin(s)$ and $\gamma = (-1)^{l+m}$

$$I = P\gamma \int_{-a+M}^{a-M} \left\{ \sin^2(s)\cos^2(s)\left(\frac{A_1}{(s)} + \frac{B_1}{(s)^2} + \frac{C_1}{(s)} + \frac{D_1}{(s)}\right) \right.$$
$$- \sin^3(s)\cos(s)\left(\frac{A_2}{(s)} + \frac{B_2}{(s)^2} + \frac{C_2}{(s)} + \frac{D_2}{(s)^2} + \frac{E_2}{(s)}\right)$$
$$- \sin^3(x)\cos(x)\left(\frac{A_3}{(s)} + \frac{B_3}{(s)^3} + \frac{C_3}{(s)^2} + \frac{D_3}{(s)} + \frac{E_3}{(s)}\right)$$
$$\left. + \sin^4(s)\left(\frac{A_3}{(s)} + \frac{B_3}{(s)^3} + \frac{C_3}{(s)^2} + \frac{D_3}{(s)} + \frac{E_3}{(s)^2} + \frac{F_3}{(s)}\right)\right\} ds, \tag{3.369}$$
$$= P\gamma \int_{-a+M}^{a-M} \left\{ \left(\frac{B_1\sin^2(s)\cos^2(s)}{(s)^2}\right) - \left(\frac{B_2\sin^3(s)\cos(s)}{(s)^2} + \frac{D_2\sin^3(s)\cos(s)}{(s)^2}\right) \right.$$
$$\left. - \left(\frac{B_3\sin^3(s)\cos(s)}{(s)^3} + \frac{C_3\sin^3(s)\cos(s)}{(s)^2}\right) + \left(\frac{B_4\sin^4(s)}{(s)^3} + \frac{C_4\sin^4(s)}{(s)^2} + \frac{E_4\sin^4(s)}{(s)^2}\right)\right\} ds. \tag{3.370}$$

since

$$\int_{-\infty}^{\infty} \frac{\sin^2(s)\cos^2(s)}{(s)^2} ds = \frac{\pi}{2}, \int_{-\infty}^{\infty} \frac{\sin^3(s)\cos(s)}{(s)^2} ds = 0, \int_{-\infty}^{\infty} \frac{\sin^4(s)}{(s)^3} ds = 0, \int_{-\infty}^{\infty} \frac{\sin^4(s)}{(s)^2} ds = \frac{\pi}{2},$$
$$\int_{-\infty}^{\infty} \frac{\sin^3(s)\cos(s)}{(s)^3} ds = \frac{\pi}{2},$$

then

$$= P\gamma \int_{-a+M}^{a-M} \left\{ \frac{B_1\sin^2(s)\cos^2(s)}{(s)^2} - \frac{B_3\sin^3(s)\cos(s)}{(s)^3} + \frac{C_3\sin^4(s)}{(s)^2} + \frac{E_3\sin^4(s)}{(s)^2} \right\} ds, \tag{3.371}$$
$$I = P\gamma\frac{\pi}{2}\left(B_1 - B_3 + C_3 + E_3\right). \tag{3.372}$$

Proof of equation (3.40): $I = \int \psi_m \psi_n \psi'_m \psi'_n dx$

$$\begin{aligned}I = &\int_{-\infty}^{\infty} \frac{\sin(x-m\pi)}{(x-m\pi)} \frac{\sin(x-n\pi)}{(x-n\pi)} \frac{P\left(\cos(x-m\pi)(x-m\pi)-\sin(x-m\pi)\right)}{(x-m\pi)^2} \\ &\cdot \frac{P\left(\cos(x-n\pi)(x-n\pi)-\sin(x-n\pi)\right)}{(x-n\pi)^2} dx, \qquad (3.373)\\ &= P\gamma \int_{-a}^{a} \left\{ \sin^2(x)\cos^2(x) \left(\frac{A_1}{(x-m\pi)^2} + \frac{B_1}{(x-m\pi)} + \frac{C_1}{(x-n)^2} + \frac{D_1}{(x-n\pi)} \right) \right. \\ &- \sin^3(x)\cos(x) \left(\frac{A_2}{(x-m\pi)^2} + \frac{B_2}{(x-m\pi)} + \frac{C_2}{(x-n\pi)^3} + \frac{D_2}{(x-n\pi)^2} + \frac{E_2}{(x-n\pi)} \right) \\ &- \sin^2(x)\cos(x) \left(\frac{A_3}{(x-m\pi)^3} + \frac{B_3}{(x-m\pi)^2} + \frac{C_3}{(x-m\pi)} + \frac{D_3}{(x-n\pi)^2} + \frac{E_3}{(x-n\pi)} \right) \\ &+ \sin^4(x) \left(\frac{A_4}{(x-m\pi)^3} + \frac{B_4}{(x-m\pi)^2} + \frac{C_4}{(x-m\pi)} + \frac{D_4}{(x-n\pi)^3} + \frac{E_4}{(x-n\pi)^2} \right. \\ &\left.\left. + \frac{F_4}{(x-n\pi)} \right) \right\} dx, \qquad (3.374)\end{aligned}$$

where the coefficients satisfy the following polynomial equations

$$1 = A_1(x-n\pi)^2 + B_1(x-m\pi)(x-n\pi)^2 + C_1(x-m\pi)^2 + D_1(x-m\pi)^2(x-n\pi), \qquad (3.375)$$

$$\begin{aligned}1 = &A_2(x-m\pi)(x-n\pi)^2 + B_2(x-l\pi)(x-m\pi)(x-n\pi)^2 + C_2(x-l\pi)^2(x-m\pi)(x-n\pi)^2 \\ &+ D_2(x-m\pi)^(3)(x-n\pi)^(2) + E_2(x-l\pi)^(3)(x-m\pi) + F_2(x-l\pi)^3(x-m\pi)(x-n\pi),\end{aligned} \qquad (3.376)$$

$$\begin{aligned}1 = &A_3(x-m\pi)(x-n\pi)^2 + B_3(x-l\pi)(x-m\pi)(x-n\pi)^2 + C_3(x-l\pi)^2(x-m\pi)(x-n\pi)^2 \\ &+ D_3(x-m\pi)^(3)(x-n\pi)^(2) + E_3(x-l\pi)^(3)(x-m\pi) + F_4(x-l\pi)^3(x-m\pi)(x-n\pi),\end{aligned} \qquad (3.377)$$

$$\begin{aligned}1 = &A_4(x-m\pi)(x-n\pi)^2 + B_4(x-l\pi)(x-m\pi)(x-n\pi)^2 + C_4(x-l\pi)^2(x-m\pi)(x-n\pi)^2 \\ &+ D_4(x-m\pi)^(3)(x-n\pi)^(2) + E_4(x-l\pi)^(3)(x-m\pi) + F_4(x-l\pi)^3(x-m\pi)(x-n\pi).\end{aligned} \qquad (3.378)$$

Comparing the coefficients of x^3 on both sides of equations (3.375),(3.376),(3.377) and (3.378) shows that $B_1 + D_1 = 0$ for (3.375), $B_2 + E_2 = 0$ for (3.376), $C_3 + E_3 = 0$ for (3.377) and $C_4 + F_4 = 0$ for (3.378).

Solving for the coefficients give the result

$$A_1 = \frac{1}{(m-n)^2\pi^2}, \qquad (3.379)$$

$$B_1 = \frac{2}{(n-m)^3\pi^3}, \qquad (3.380)$$

$$C_1 = \frac{1}{(n-m)^2\pi^2}, \qquad (3.381)$$

$$D_1 = \frac{2}{(m-n)^3\pi^3}, \tag{3.382}$$

$$A_2 = \frac{1}{(m-n)^3\pi^3}, \tag{3.383}$$

$$B_2 = -\frac{3}{(m-n)^4\pi^4}, \tag{3.384}$$

$$C_2 = \frac{1}{(n-m)^2\pi^2}, \tag{3.385}$$

$$D_2 = \frac{2}{(m-n)^3\pi^3}, \tag{3.386}$$

$$E_2 = \frac{3}{(m-n)^4\pi^4}, \tag{3.387}$$

$$A_3 = \frac{1}{(m-n)^2\pi^2}, \tag{3.388}$$

$$B_3 = -\frac{2}{(m-n)^3\pi^3}, \tag{3.389}$$

$$C_3 = \frac{3}{(m-n)^4\pi^4}, \tag{3.390}$$

$$D_3 = \frac{1}{(n-m)^3\pi^3}, \tag{3.391}$$

$$E_3 = \frac{3}{(m-n)^4\pi^4}, \tag{3.392}$$

$$A_4 = +\frac{1}{(m-n)^3\pi^3}, \tag{3.393}$$

$$B_4 = -\frac{3}{(m-n)^4\pi^4}, \tag{3.394}$$

$$C_4 = -\frac{6}{(m-n)^5\pi^5}, \tag{3.395}$$

$$D_4 = -\frac{1}{(m-n)^3\pi^3}, \tag{3.396}$$

$$E_4 = -\frac{3}{(m-n)^4\pi^4}, \tag{3.397}$$

$$F_4 = \frac{6}{(m-n)^5\pi^5}. \tag{3.398}$$

Continuing on with I, over the common interval $[-a+M,\ a-M]$ as discussed in Section 3.1, with $\gamma = (-1)^{m+n}$.

$$\begin{aligned}
&= P\gamma \int_{-a}^{a} \left\{ \sin^2(s)\cos^2(s) \left(\frac{A_1}{(s)^2} + \frac{B_1}{(s)} + \frac{C_1}{(s} + \frac{D_1}{(s)} \right) \right. \\
&- \sin^3(s)\cos(s) \left(\frac{A_2}{(s)^2} + \frac{B_2}{(s)} + \frac{C_2}{(s)^3} + \frac{D_2}{(s)^2} + \frac{E_2}{(s)} \right) \\
&- \sin^2(s)\cos(s) \left(\frac{A_3}{(s)^3} + \frac{B_3}{(s)^2} + \frac{C_3}{(s)} + \frac{D_3}{(s)^2} + \frac{E_3}{(s)} \right) \\
&\left. + \sin^4(s) \left(\frac{A_4}{(s)^3} + \frac{B_4}{(s)^2} + \frac{C_4}{(s)} + \frac{D_4}{(s)^3} + \frac{E_4}{(s)^2} + \frac{F_4}{(s)} \right) \right\} ds,
\end{aligned} \tag{3.399}$$

$$= P\gamma \int_{-a}^{a} \left\{ \sin^2(s)\cos^2(s) \left(\frac{A_1}{(s)^2} + \frac{C_1}{(s} \right) - \sin^3(s)\cos(s) \left(\frac{A_2}{(s)^2} + \frac{C_2}{(s)^3} + \frac{D_2}{(s)^2} \right) \right.$$
$$\left. - \sin^2(s)\cos(s) \left(\frac{A_3}{(s)^3} + \frac{B_3}{(s)^2} + \frac{D_3}{(s)^2} \right) + \sin^4(s) \left(\frac{A_4}{(s)^3} + \frac{B_4}{(s)^2} + \frac{D_4}{(s)^3} + \frac{E_4}{(s)^2} \right) \right\} ds, \quad (3.400)$$
$$= (A_1 + C_1) \int_{-\infty}^{\infty} \frac{\sin^2(s)\cos^2(s)}{s^2} ds + (A_2 + D_2 + B_3 + D_3) \int_{-\infty}^{\infty} \frac{\sin^3(s)\cos(s)}{s^2} ds$$
$$- (A_3 + C_2) \int_{-\infty}^{\infty} \frac{\sin^3(s)\cos(s)}{s^3} ds + (A_4 + D_4) \int_{-\infty}^{\infty} \frac{\sin^4(s)}{s^3} ds + (B_4 + E_4) \int_{-\infty}^{\infty} \frac{\sin^4(s)}{s^2} ds, \quad (3.401)$$

$$= (A_1 + C_1) \int_{-\infty}^{\infty} \frac{\sin^2(s)\cos^2(s)}{s^2} ds - (A_3 + C_2) \int_{-\infty}^{\infty} \frac{\sin^3(s)\cos(s)}{s^3} ds + (B_4 + E_4) \int_{-\infty}^{\infty} \frac{\sin^4(s)}{s^2} ds, \quad (3.402)$$

since

$$\int_{-\infty}^{\infty} \frac{\sin^2(s)\cos^2(s)}{s^2} ds = \frac{\pi}{2}, \int_{-\infty}^{\infty} \frac{\sin^3(s)\cos(s)}{s^3} ds = \frac{\pi}{2}, \int_{-\infty}^{\infty} \frac{\sin^4(s)}{s^2} ds = \frac{\pi}{2}, \quad (3.403)$$

then

$$I_{A.25} = \frac{\pi}{2} \left((A_1 + C_1) - (A_3 + C_2) + (B_4 + E_4) \right). \quad (3.404)$$

Proof of equation (3.41): $I = \int \psi_m^2 \psi_m' \psi_n' dx$

$$I_{A.26} = \int_{-a}^{a} \frac{\sin^2(x - m\pi)}{(x - m\pi)^2} \frac{P\left(\cos(x - m\pi)(x - m\pi) - \sin(x - m\pi)\right)}{(x - m\pi)^2}$$
$$\cdot \frac{P\left(\cos(x - n\pi)(x - n\pi) - \sin(x - n\pi)\right)}{(x - n\pi)^2} dx.$$

As this integrand is an odd function the result of integration is that $I_{A.26} = 0$.

Proof of equation (3.42): $I = \int \psi_m^2 \psi_n'^2 dx$

$$I = \int_{-a}^{a} \frac{\sin^2(x - m\pi)}{(x - m\pi)^2} \frac{P^2\left(\cos(x - n\pi)(x - n\pi) - \sin(x - n\pi)\right)^2}{(x - n\pi)^4},$$
$$I_1 = P^2\gamma \int_{-a}^{a} \sin^2(x)\cos^2(x)(x - n\pi)^2 \left(\frac{1}{(x - m\pi)^2 (x - n\pi)^4} \right) dx,$$
$$= P^2\gamma \int_{-a}^{a} \sin^2(x)\cos^2(x) \left(\frac{A}{(x - m\pi)^2} + \frac{B}{(x - m\pi)} + \frac{C}{(x - n\pi)^2} + \frac{D}{(x - n\pi)} \right) dx, \quad (3.405)$$
$$I_2 = -2P^2\gamma \int_{-a}^{a} \sin^3(x)\cos(x)(x - n\pi) \left(\frac{1}{(x - m\pi)^2 (x - n\pi)^4} \right) dx,$$
$$= P^2\gamma \int_{-a}^{a} \sin^3(x)\cos(x)$$
$$\left(\frac{A}{(x - m\pi)^2} + \frac{B}{(x - m\pi)} + \frac{C}{(x - n\pi)^3} + \frac{D}{(x - n\pi)^2} + \frac{E}{(x - n\pi)} \right) dx, \quad (3.406)$$

$$I_3 = P^2\gamma \int_{-a}^{a} \sin^4(x) \left(\frac{1}{(x-m\pi)^2(x-n\pi)^4}\right) dx,$$
$$= P^2\gamma \int_{-a}^{a} \sin^4(x)$$
$$\left(\frac{A}{(x-m\pi)^2} + \frac{B}{(x-m\pi)} + \frac{C}{(x-n\pi)^4} + \frac{D}{(x-n\pi)^3} + \frac{E}{(x-n\pi)^2} + \frac{F}{(x-n\pi)}\right) dx, \quad (3.407)$$

where A,B equation (3.405) satisfy the polynomial equation

$$1 = A_1(x-n\pi)^2 + B_1(x-m\pi)(x-n\pi)^2 + C(x-m\pi)^2 + D(x-m\pi)^2(x-n\pi).$$

Comparing the leading order coefficients of x^3 on both sides of equation (3.408) shows that $B_1) + D_1 = 0$. Solving for the coefficients give the result

$$A_1 = \frac{1}{(m-n)^2\pi^2}, \quad (3.408)$$
$$B_1 = -\frac{2}{(m-n)^3\pi^3}, \quad (3.409)$$
$$C_1 = \frac{1}{(m-n)^2\pi^2}, \quad (3.410)$$
$$D_1 = \frac{2}{(m-n)^3\pi^3}. \quad (3.411)$$

Continuing with equation (3.405), over the common interval $[-a+M,\ a-M]$ as discussed in Section 3.1, with $\gamma = 1$.

$$I_1 = P^2\gamma \int_{-a+M}^{a-M} \left(\frac{A_1 \sin^2(s)\cos^2(s)}{(s)^2} + \frac{B_1 \sin^2(s)\cos^2(s)}{(s)} + \frac{C_1 \sin^2(s)\cos^2(s)}{(s)^2}\right. \quad (3.412)$$
$$\left. + \frac{D_1 \sin^2(s)\cos^2(s)}{(s)}\right) ds, \quad (3.413)$$
$$= P^2\gamma \int_{-a+M}^{a-M} \frac{(A_1+C_1)\sin^2(s)\cos^2(s)}{(s)^2} + \frac{(B_1+D_1)\sin^2(s)\cos^2(s)}{(s)} ds, \quad (3.414)$$
$$= P^2\gamma \int_{-a+M}^{a-M} \frac{(A_1+C_1)\sin^2(s)\cos^2(s)}{(s)^2} ds, \quad (3.415)$$

since

$$\int_{-a+M}^{a-M} \frac{\sin^2(x)\cos^2(x)}{(x)^2} dx = \frac{\pi}{2}, \quad (3.416)$$

then

$$= P^2\gamma \int_{-a+M}^{a-M} \frac{(A_1+C_1)\sin^2(s)\cos^2(s)}{(s)^2} ds, \quad (3.417)$$
$$= \frac{P^2\gamma\pi}{2}(A_1+C_1), \quad (3.418)$$
$$= \frac{P^2\gamma\pi}{2}\left(\frac{1}{(m-n)^2\pi^2} + \frac{1}{(n-m)^2\pi^2}\right), \quad (3.419)$$
$$I_1 = \frac{P^2\gamma}{4(m-n)^2\pi}. \quad (3.420)$$

Solving for I_2, considering A,B,C,D,E equation (3.406) satisfy the polynomial equation

$$1 = A_2(x-n\pi)^3+B_2(x-m\pi)(x-n\pi)^3+C_2(x-m\pi)^2+D_2(x-m\pi)^2(x-n\pi)+E_2(x-m\pi)^2(x-n\pi)^2. \tag{3.421}$$

Comparing the leading order coefficients of x^4 on both sides of equation (3.421) shows that $B_2 + C_2 = 0$. Solving for the coefficients give the result

$$A_2 = \frac{1}{(m-n)^3\pi^3}, \tag{3.422}$$

$$B_2 = -\frac{3}{(m-n)^4\pi^4}, \tag{3.423}$$

$$C_2 = \frac{1}{(m-n)^2\pi^2}, \tag{3.424}$$

$$D_2 = \frac{2}{(m-n)^3\pi^3}, \tag{3.425}$$

$$E_2 = \frac{3}{(m-n)^4\pi^4}. \tag{3.426}$$

Evaluation of I_2 is undertaken, where $\sin(x-n\pi) = (-1)^j \sin(x)$ and $\gamma = 1$,

$$= -2P^2\gamma \int_{-a}^{a} \frac{A_2 \sin^3(s)\cos(s)}{(s)^2} ds - 2P^2\gamma \int_{-a}^{a} \frac{B_2 \sin^3(s)\cos(s)}{(s)} ds - 2P^2\gamma \int_{-a}^{a} \frac{C_2 \sin^3(s)\cos(s)}{(x-n\pi)^3} ds$$

$$- 2P^2\gamma \int_{-a}^{a} \frac{D_2 \sin^3(s)\cos(s)}{(s)^2} ds - 2P^2\gamma \int_{-a}^{a} \frac{E_2 \sin^3(s)\cos(s)}{(s)} ds, \tag{3.427}$$

$$= -2P^2\gamma \int_{-a}^{a} \frac{(A_2 + D_2)\sin^3(s)\cos(s)}{(s)^2} ds - 2P^2\gamma \int_{-a}^{a} \frac{C_2 \sin^3(s)\cos(s)}{(s)^3} ds, \tag{3.428}$$

since

$$\int_{-\infty}^{\infty} \frac{\sin^3(s)\cos(s)}{(s)^3} ds = \frac{\pi}{2} \text{ and } \int_{-a+M}^{a-M} \frac{\sin^3(s)\cos(s)}{(s)^2} ds = 0, \tag{3.429}$$

then

$$= -2\frac{\pi}{2}P^2\gamma C_2, \tag{3.430}$$

$$I_2 = -\frac{P^2\gamma}{(m-n)^2\pi}. \tag{3.431}$$

Solving for I_3, considering A,B,C,D equation (3.407) satisfy the polynomial equation

$$1 = A_3(x-n\pi)^4 + B_3(x-m\pi)(x-n\pi)^4 + C_3(x-m\pi)^2 + D_3(x-m\pi)^2(x-n\pi)$$

$$+ E(x-m\pi)^2(x-n\pi)^2 + F(x-m\pi)^2(x-n\pi)^3. \tag{3.432}$$

Comparing the leading order coefficients of x^5 on both sides of equation (3.432) shows that

$B_3 + C_3 = 0$. Solving for the coefficients give the result

$$A_3 = \frac{1}{(m-n)^4\pi^4}, \tag{3.433}$$
$$B_3 = -\frac{4}{(m-n)^5\pi^5}, \tag{3.434}$$
$$C_3 = \frac{1}{(m-n)^2\pi^2}, \tag{3.435}$$
$$D_3 = \frac{2}{(m-n)^3\pi^3}, \tag{3.436}$$
$$E_3 = \frac{3}{(m-n)^4\pi^4}, \tag{3.437}$$
$$F_3 = \frac{4}{(m-n)^5\pi^5}. \tag{3.438}$$

Evaluation of I_3 is undertaken, where $\sin(x - n\pi) = (-1)^j \sin(x)$ and $\gamma = 1$,

$$= P^2\gamma \int_{-a+M}^{a-M} \frac{A_3 \sin^4(s)}{(s)^2} ds + P^2\gamma \int_{-a+M}^{a-M} \frac{B_3 \sin^4(s)}{(s)} ds + P^2\gamma \int_{-a+M}^{a-M} \frac{C_3 \sin^4(s)}{(s)^4} ds \tag{3.439}$$
$$+ P^2\gamma \int_{-a+M}^{a-M} \frac{D_3 \sin^4(s)}{(s)^3} ds + P^2\gamma \int_{-a+M}^{a-M} \frac{E_3 \sin^4(s)}{(s)^2} ds + P^2\gamma \int_{-a+M}^{a-M} \frac{F_3 \sin^4(s)}{(s)} ds, \tag{3.440}$$

since

$$\int_{-\infty}^{\infty} \frac{\sin^4(x)}{(x)^4} dx = \frac{2\pi}{3}, \int_{-\infty}^{\infty} \frac{\sin^4(x)}{(x)^3} dx = 0, \int_{-\infty}^{\infty} \frac{\sin^4(x)}{(x)^2} dx = \frac{\pi}{2}, \tag{3.441}$$
$$\int_{-\infty}^{\infty} \frac{\sin^4(x)}{(x)} dx = 0, \tag{3.442}$$

then

$$= P^2\gamma \left(\frac{\pi}{2}(A_3 + E_3) + \frac{2\pi}{3} C_3 \right), \tag{3.443}$$
$$= \frac{P^2\gamma}{2(m-n)^4\pi^3} + \frac{3P^2\gamma}{2(m-n)^4\pi^3} + \frac{2P^2\gamma}{3(m-n)^2\pi}, \tag{3.444}$$
$$I_3 = \frac{2P^2\gamma\left(\pi^2(m-n)^2 + 3\right)}{3\pi^3(m-n)^4}. \tag{3.445}$$

The full expression of I is

$$I = P^2\gamma\frac{\pi}{2}\left(A_1 + C_1 - 2C_2 + A_3 + E_3\right) + \frac{2\pi}{3} C_3. \tag{3.446}$$

<u>Proof of equation (3.43): $I = \int \psi_m \psi_n \psi_n'^2 dx$</u>

Breaking I into individual integrals, as before, results in discontinuities, instead the entire expanded integrand is considered in a single integral.

$$I = \int_{-a}^{a} \frac{\sin(x-m\pi)}{(x-m\pi)} \frac{\sin(x-n\pi)}{(x-n\pi)} \frac{P^2 \left(\cos(x-n\pi)(x-n\pi) - \sin(x-n\pi)\right)^2}{(x-n\pi)^4} dx,$$

$$= P^2\gamma \int_{-a+M}^{a-M} \left\{ \sin^2(x)\cos^2(x) \left(\frac{A_1}{(x-m\pi)} + \frac{B_1}{(x-n\pi)^3} + \frac{C_1}{(x-n\pi)^2} + \frac{D_1}{(x-n\pi)} \right) \right. \quad (3.447)$$

$$- 2\sin^3(x)\cos(x) \left(\frac{A_2}{(x-m\pi)} + \frac{B_2}{(x-n\pi)^4} + \frac{C_2}{(x-n\pi)^3} + \frac{D_2}{(x-n\pi)^2} + \frac{E_2}{(x-n\pi)} \right) \quad (3.448)$$

$$\left. + \sin^4(x) \left(\frac{A_3}{(x-m\pi)} + \frac{B_3}{(x-n\pi)^5} + \frac{C_3}{(x-n\pi)^4} + \frac{D_3}{(x-n\pi)^3} + \frac{E_3}{(x-n\pi)^2} + \frac{F_3}{(x-n\pi)} \right) \right\} dx. \quad (3.449)$$

The leading order coefficients of the integrand are,

$$1 = x^3(A_1 + D_1) + x^2(C_1) + x(B_1), \quad (3.450)$$

$$1 = x^4(A_2 + E_2) + x^3(D_2) + x^2(C_2) + x(B_2), \quad (3.451)$$

$$1 = x^5(A_3 + F_3) + x^4(E_3) + x^3(D_3) + x^2(C_3) + x(B_3). \quad (3.452)$$

Comparing the leading order coefficients of x on both sides of equations (3.450), (3.451) and (3.452) show that $(A_1 + D_1) = 0$ for (3.450), $(A_2 + E_2) = 0$ for (3.451) and $(A_3 + F_3) = 0$ for (3.452).

$$I = P^2\gamma \int_{-\infty}^{\infty} \left\{ \sin^2(s)\cos^2(s) \left(\frac{A_1}{(s)} + \frac{B_1}{(s)^3} + \frac{C_1}{(s)^2} + \frac{D_1}{(s)} \right) \right.$$

$$- 2\sin^3(s)\cos(s) \left(\frac{A_2}{(s)} + \frac{B_2}{(s)^4} + \frac{C_2}{(s)^3} + \frac{D_2}{(s)^2} + \frac{E_2}{(s)} \right)$$

$$\left. + \sin^4(s) \left(\frac{A_3}{(s)} + \frac{B_3}{(s)^5} + \frac{C_3}{(s)^4} + \frac{D_3}{(s)^3} + \frac{E_3}{(s)^2} + \frac{F_3}{(s)} \right) \right\} ds, \quad (3.453)$$

$$= P^2\gamma C_1 \int_{-\infty}^{\infty} \frac{\sin^2(s)\cos^2(s)}{(s)^2} ds - 2P^2\gamma C_2 \int_{-\infty}^{\infty} \frac{\sin^3(s)\cos(s)}{(s)^3} ds \quad (3.454)$$

$$+ P^2\gamma C_3 \int_{-\infty}^{\infty} \frac{\sin^4(s)}{(s)^4} ds + P^2\gamma E_3 \int_{-\infty}^{\infty} \frac{\sin^4(s)}{(s)^2} ds + J, \quad (3.455)$$

where J is the sum of the terms that are individually non-integrable

$$J = P^2\gamma \int_{-\infty}^{\infty} \frac{D_1 \sin^2(s)\cos^2(s)}{(s)^3} ds - \frac{2E_2 \sin^3(s)\cos(s)}{(s)^4} ds + \frac{F_3 \sin^4}{(s)^5} ds. \quad (3.456)$$

Solving for the coefficients shows that $D_1 = E_2 = F_3 = -\dfrac{1}{(m-n)\pi}$. Integration by parts is performed

$$J = -\frac{P^2\gamma}{(m-n)\pi}\int_{-\infty}^{\infty}\frac{\sin^2(s)\cos^2(s)}{(s)^3}ds - \frac{2\sin^3(s)\cos(s)}{(s)^4}ds + \frac{\sin^4}{(s)^5}ds, \tag{3.457}$$

$$= -\frac{P^2\gamma}{(m-n)\pi}\int_{-\infty}^{\infty}\frac{\sin^2(s)\cos^2(s)}{(s)^3}ds - \frac{2\sin^3(s)\cos(s)}{(s)^4}ds + \frac{\sin^4}{(t)^5}ds, \tag{3.458}$$

$$= \frac{s^2\sin^2(s)\cos^2(s) + 2s\sin^3(s)\cos(s) + \sin^4(s)}{s^5}. \tag{3.459}$$

Using integration by parts on equation (3.459) shows that

$$\int_{-\infty}^{\infty}\frac{s^2\sin^2(s)\cos^2(s) + 2s\sin^3(s)\cos(s) + \sin^4(s)}{s^5}ds = 0, \tag{3.460}$$

and so the remaining components of $A.28$ are the only contributions to the final expression. Since

$$\int_{-\infty}^{\infty}\frac{\sin^2(s)}{(s)^2}ds = \frac{2\pi}{3},\ \int_{-\infty}^{\infty}\frac{\sin^4(s)}{(s)^2}ds = \frac{\pi}{2},\ \int_{-\infty}^{\infty}\frac{\sin^4(s)}{(s)^4}ds = \frac{2\pi}{3},\ \int_{-\infty}^{\infty}\frac{\sin^3(s)\cos(s)}{(s)^3}ds = \frac{\pi}{2}, \tag{3.461}$$

then

$$I = P^2\gamma\left(C_1\frac{\pi}{2} + C_2\frac{\pi}{2} - 2C_3\frac{2\pi}{3} + E_3\frac{\pi}{2}\right). \tag{3.462}$$

Proof of equation (3.44): $I = \int \psi_n^2 \psi_n'^2 dx$

$$I = \int_{-a}^{a}\frac{\sin^2(x-n\pi)}{(x-n\pi)^2}\frac{P^2\left(\cos(x-n\pi)(x-n\pi) - \sin(x-n\pi)\right)^2}{(x-n\pi)^4}. \tag{3.463}$$

There is only one shift in the integrand of equation (3.463) and so performing the change of variable $s = x - n\pi$ eliminates the shift entirely

$$I = \int_{-\infty}^{\infty}\frac{\sin^2(s)\cos^2(s)s^2 - 2\sin^3(s)\cos(s)s + \sin^4(s)}{s^6}ds, \tag{3.464}$$

$$= \int_{-\infty}^{\infty}\frac{\sin^2(s)s^2 - \sin^4(s)s^2 - 2\sin^3(s)\cos(s)s + \sin^4(s)}{s^6}ds. \tag{3.465}$$

With these shifted sinc functions it is now possible to begin the calculation of G for the infinite Gaussian Pulse train.

Chapter 4

An Infinite Pulse-Train Ambiguity Function

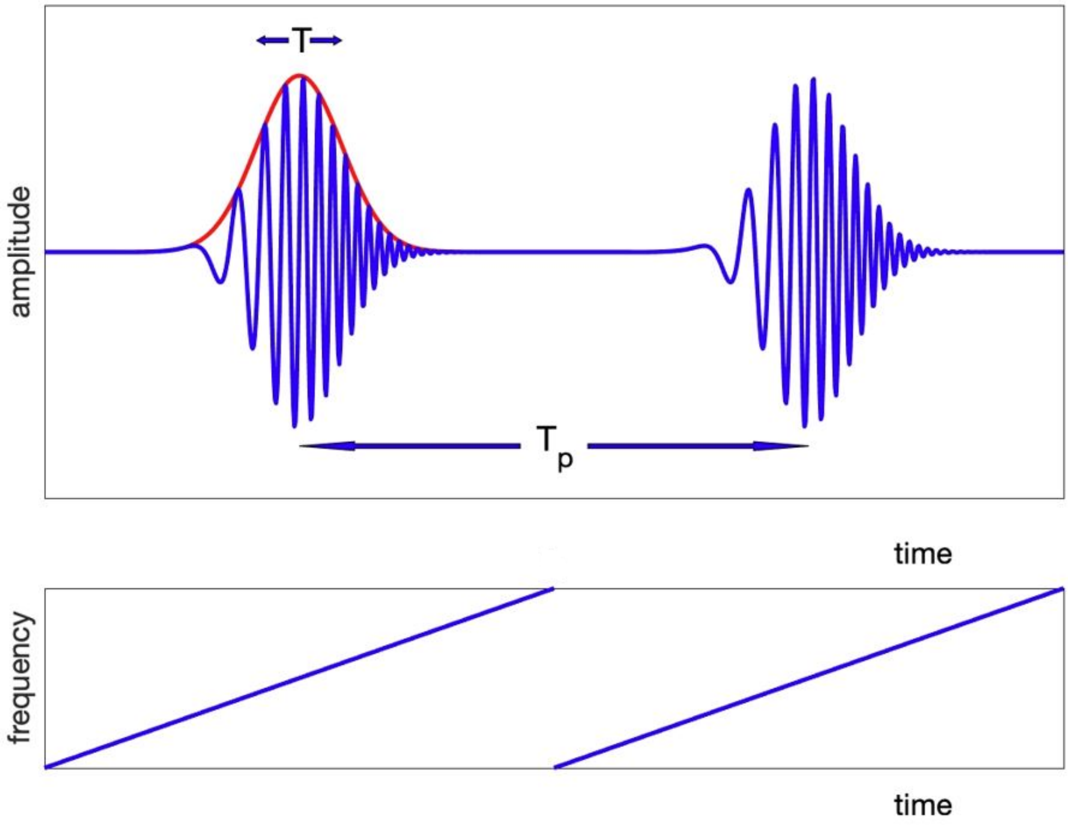

Figure 4.1: The amplitude/time plot shows two typical Gaussian pulses of an infinite pulse train with the Gaussian envelope in red, the pulse width T and inter-pulse period T_P. The frequency/time plot shows the chirp rate, b.

A goal of this investigation is to examine the dependence of the information content on not just the pulse width, T and the chirp rate, b but also the inter-pulse period, T_P. To that end an infinite Gaussian pulse-train, $g_\infty(t)$ is constructed, by sewing together Gaussians as in the manner illustrated in figure (4.1). Following Van Trees [35], the autocorrelation function is

$$\phi(\tau, \omega) = \int_{\infty}^{\infty} g_\infty(t) g_\infty^\star(t-\tau) e^{j\omega t} dt. \tag{4.1}$$

Also from Van Trees [35]

$$\phi(\tau, \omega) = \exp\left(-\frac{1}{4}\left(\frac{\tau^2}{T^2} + \omega^2 T^2\right)\right), \tag{4.2}$$

and the Linear chirp is included by replacing ω by $\omega - 2b\tau$. Thus, the autocorrelation function for the linear chirp modulated by the Gaussian pulse is

$$\phi(\tau, \omega) = \exp\left(-\frac{1}{4}\left(\frac{\tau^2}{T^2} + (\omega - 2b\tau)^2 T^2\right)\right) \tag{4.3}$$

Since the signal is restricted to the finite time interval $[-T_P/2,\ T_P/2]$, the Shannon sampling theorem says that

$$\phi(\tau, \omega) = \sum_{n=-\infty}^{\infty} \exp\left(-\frac{1}{4}\left(\frac{\tau^2}{T^2} + \left(\frac{n}{T_P} - 2b\tau\right)^2 T^2\right)\right) \frac{\sin(\pi T_P(\omega - n/T_P))}{(\pi T_P(\omega - n/T_P))}. \tag{4.4}$$

Equation (4.4) with the substitution $P = \dfrac{\pi T_P}{2}$ is

$$\phi(\tau, \omega) = \sum_{n=-\infty}^{\infty} \exp\left(-\frac{1}{4}\left(\frac{\tau^2}{T^2} + \left(\frac{n\pi}{P} - 2b\tau\right)^2 T^2\right)\right) \frac{\sin(P\omega - n\pi)}{(P\omega - n\pi)}. \tag{4.5}$$

In this chapter a Gaussian modulated infinite pulse-train linear chirp signal is considered. The calculation of both the Fisher metric, $g(\tau, \omega)$ from equation (2.26), and the Gil Medrano metric, G_{ij} as in equation (2.29), for the system defined by equation (4.5).

The calculation of the Fisher metric take place in Section 4.1 and is followed by the calculation of the Gil Medrano metric in Section 4.2, for which $g(\tau, \omega)$ itself forms a basis. In Section 4.3 the calculation of geodesics on the configuration manifold is undertaken.

4.1 Calculating The Fisher Metric $g(\tau, \omega)$

In the calculation that follows the autocorrelation function of equation (4.5), is used to construct the Fisher information metric. This calculation involves calculating the expectation of the ambiguity function, as in equation (2.26). The ambiguity function is obtained by taking the square of the autocorrelation function, $\theta = \phi^2$, of equation (4.5). The following substitutions are made

$$A = \frac{1 + 4b^2T^4}{T^2}, \tag{4.6}$$

$$B = \frac{\pi b T^2}{P}, \tag{4.7}$$

$$C = \frac{\pi^2 T^2}{4P^2}, \tag{4.8}$$

$$E_n = \frac{1}{4}\left[\frac{\tau^2}{T^2} + T^2\left(\frac{n\pi}{P} - 2b\tau\right)^2\right], \tag{4.9}$$

$$\psi_n = \frac{\sin\left(P\omega - n\pi\right)}{P\omega - n\pi}. \tag{4.10}$$

Given these substitutions E_n can be reduced to

$$E_n = \frac{1}{4}A\tau^2 - n\tau B + nC, \tag{4.11}$$

and the derivative of E_n is

$$\frac{\partial E_n}{\partial \tau} = \frac{A\tau}{2} - Bn. \tag{4.12}$$

With these substitutions ϕ is rewritten as

$$\phi(\tau, \omega) = \sum_{n=-\infty}^{\infty} e^{-E_n(\tau)}\psi_n(\omega). \tag{4.13}$$

The derivatives of the separable components of θ are

$$\frac{\partial \phi}{\partial \tau} = -\sum_{n=-\infty}^{\infty} e^{-E_n}\left(\frac{A\tau}{2} - Bn\right)\psi_n, \tag{4.14}$$

$$\frac{\partial \phi}{\partial \omega} = \sum_{n=-\infty}^{\infty} e^{-E_n}\psi_n'. \tag{4.15}$$

where

$$\psi_n' = \frac{P\left(\cos(P\omega - n\pi)\left(P\omega - n\pi\right) - \sin(P\omega - n\pi)\right)}{\left(P\omega - n\pi\right)^2}. \tag{4.16}$$

Since the ambiguity function is defined as $\theta = \phi^2$ the derivatives of θ with respect to τ and ω are

$$\theta_\tau = \frac{\partial \theta}{\partial \tau} = 2\phi\phi_\tau = -2\sum_{m=-\infty}^{\infty} e^{-E_m(\tau)}\psi_n(\omega)\sum_{n=-\infty}^{\infty} e^{-E_n(\tau)}\left(\frac{A\tau}{2} - nB\right)\psi_n(\omega), \tag{4.17}$$

$$= -2\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{-(E_m(\tau)+E_n(\tau))}\psi_m(\omega)\psi_n(\omega) \tag{4.18}$$

$$\theta_\omega = \frac{\partial \theta}{\partial \omega} = 2\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{-(E_m(\tau)+E_n(\tau))}\psi_m(\omega)\psi_n'(\omega). \tag{4.19}$$

The calculation of the Fisher metric 2.26 requires the calculation of θ_τ^2, $\theta_\tau\theta_\omega$ and θ_ω^2

$$\theta_\tau^2 = 4\sum_k\sum_l\sum_m\sum_n e^{-(E_k(\tau)+E_l(\tau)+E_m(\tau)+E_n(\tau))}\left(\frac{A\tau}{2}-Bl\right)\left(\frac{A\tau}{2}-Bn\right)\psi_k\psi_l\psi_m\psi_n, \tag{4.20}$$

$$\theta_\tau\theta_\omega = -4\sum_k\sum_l\sum_m\sum_n e^{-(E_k(\tau)+E_l(\tau)+E_m(\tau)+E_n(\tau))}\left(\frac{A\tau}{2}-Bl\right)\psi_k\psi_l\psi_m\psi_n', \tag{4.21}$$

$$\theta_\omega^2 = 4\sum_k\sum_l\sum_m\sum_n e^{-(E_k(\tau)+E_l(\tau)+E_m(\tau)+E_n(\tau))}\psi_k\psi_l'\psi_m\psi_n'. \tag{4.22}$$

Further simplification is made by setting

$$v = (k,l,m,n) \in \mathbb{Z}^4, \tag{4.23}$$

$$\sum_v = \sum_k\sum_l\sum_m\sum_n, \tag{4.24}$$

$$s_v = k+l+m+n, \tag{4.25}$$

$$B_v = B(k+l+m+n), \tag{4.26}$$

$$C_v = C(k^2+l^2+m^2+n^2) = C|v|^2, \tag{4.27}$$

$$E_v = A\tau^2 - B_v\tau + C_v. \tag{4.28}$$

The components of g are

$$g_{\tau\tau} = \int_{-\infty}^{\infty}\int_0^{\infty}\theta_\tau^2 d\tau d\omega, \tag{4.29}$$

$$= 4\int_{-\infty}^{\infty}\int_0^{\infty}\sum_v e^{-(A\tau^2-B_v\tau+C_v)}\left(\frac{A\tau}{2}-Bl\right)\left(\frac{A\tau}{2}-Bn\right)\psi_k\psi_l\psi_m\psi_n d\tau d\omega, \tag{4.30}$$

$$= 4\sum_v \Psi_v J_{1,v}, \tag{4.31}$$

$$g_{\tau\omega} = -4\int_{-\infty}^{\infty}\int_0^{\infty}\theta_\tau\theta_\omega d\tau d\omega, \tag{4.32}$$

$$= -4\int_{-\infty}^{\infty}\int_0^{\infty}\sum_v e^{-(A\tau^2-B_v\tau+C_v)}\left(\frac{A\tau}{2}-Bl\right)\psi_k\psi_l\psi_m\psi_n' d\tau d\omega, \tag{4.33}$$

$$= -4\sum_v \Psi_v' J_{2,v}, \tag{4.34}$$

$$g_{\omega\omega} = 4\int_{-\infty}^{\infty}\int_0^{\infty}\theta_\omega^2 d\tau d\omega, \tag{4.35}$$

$$= 4\int_{-\infty}^{\infty}\int_0^{\infty}\sum_v e^{-(A\tau^2-B_v\tau+C_v)}\psi_k\psi_l'\psi_m\psi_n' d\tau d\omega, \tag{4.36}$$

$$= 4\sum_v \Psi_v'' J_{3,v}. \tag{4.37}$$

The symbols Ψ_v, Ψ_v', and Ψ_v'' used here are defined as

$$\Psi_v = \int_0^{\infty}\psi_k\psi_l\psi_m\psi_n d\omega,\ \Psi_v' = \int_0^{\infty}\psi_k\psi_l\psi_m\psi_n' d\omega,\ \Psi_v'' = \int_0^{\infty}\psi_k\psi_l'\psi_m\psi_n' d\omega, \tag{4.38}$$

and the expressions $J_{iv},\ i=1,2,3$ are

$$J_{1,v} = \int_0^\infty e^{-(A\tau^2 - B_v\tau + C_v)} \left(\frac{A^2\tau^2}{4} - \frac{A}{2}(Bl+Bn)\tau + B^2 ln \right) d\tau, \tag{4.39}$$

$$J_{2,v} = \int_0^\infty e^{-(A\tau^2 - B_v\tau + C_v)} \left(\frac{A\tau}{2} - Bl \right) d\tau, \tag{4.40}$$

$$J_{3,v} = \int_0^\infty e^{-(A\tau^2 - B_v\tau + C_v)} d\tau. \tag{4.41}$$

where $v = (k,l,m,n) \in \mathbb{Z}^4$, as defined in equation (4.23). These terms have the following symmetry properties

$$C_{-v} = C_v,\ B_{-v} = -B_v, \tag{4.42}$$

$$\Psi_{-v} = \Psi_v,\ \Psi'_{-v} = -\Psi'_v,\ \Psi''_{-v} = \Psi''_v. \tag{4.43}$$

In summing over $v \in \mathbb{Z}^4$, some additional simplification is possible when considering v and $-v$ together. As a result, the space $\mathbb{Z}^4$ is separated into three disjoint regions

$$\mathbb{Z}^4 = \mathbb{Z}^4_0 \cup \mathbb{Z}^4_+ \cup \mathbb{Z}^4_-, \tag{4.44}$$

where

$$\mathbb{Z}^4_0 = \{(0,0,0,0)\}, \tag{4.45}$$

$$\mathbb{Z}^4_+ = \{v \in \mathbb{Z}^4\text{: the first non-zero component of } v > 0\}, \tag{4.46}$$

$$\mathbb{Z}^4_- = \{v \in \mathbb{Z}^4\text{: the first non-zero component of } v < 0\}. \tag{4.47}$$

This allows the properties of g to be evaluated by the following formulas

$$g_{\tau\tau} = 4\Psi_{(0,0,0,0)} J_{1,(0,0,0,0)} + 4 \sum_{v\in\mathbb{Z}^4_+} \left(\Psi_v J_{1,v} + \Psi_{-v} J_{1,-v} \right), \tag{4.48}$$

$$g_{\tau\omega} = 4\Psi'_{(0,0,0,0)} J_{2,(0,0,0,0)} - 4 \sum_{v\in\mathbb{Z}^4_+} \left(\Psi'_v J_{2,v} + \Psi'_{-v} J_{2,-v} \right), \tag{4.49}$$

$$g_{\omega\omega} = 4\Psi''_{(0,0,0,0)} J_{3,(0,0,0,0)} + 4 \sum_{v\in\mathbb{Z}^4_+} \left(\Psi''_v J_{3,v} + \Psi''_{-v} J_{3,-v} \right). \tag{4.50}$$

These terms are ordered according to the values of $h = |v|^2$

$$g_{\tau\tau} = 4\Psi_{(0,0,0,0)} J_{1,(0,0,0,0)} + 4 \sum_{h=1}^{\infty} \sum_{\substack{v\in\mathbb{Z}^4_+ \\ |v|=h}} \left(\Psi_v J_{1,v} + \Psi_{-v} J_{1,-v} \right), \tag{4.51}$$

$$g_{\tau\omega} = 4\Psi'_{(0,0,0,0)} J_{2,(0,0,0,0)} - 4 \sum_{h=1}^{\infty} \sum_{\substack{v\in\mathbb{Z}^4_+ \\ |v|=h}} \left(\Psi'_v J_{2,v} + \Psi'_{-v} J_{2,-v} \right), \tag{4.52}$$

$$g_{\omega\omega} = 4\Psi''_{(0,0,0,0)} J_{3,(0,0,0,0)} + 4 \sum_{h=1}^{\infty} \sum_{\substack{v\in\mathbb{Z}^4_+ \\ |v|=h}} \left(\Psi''_v J_{3,v} + \Psi''_{-v} J_{3,-v} \right). \tag{4.53}$$

The components of g can be calculated by evaluating the integrals over ω, the Ψ factors, using the equations in Section 3.2 and the J factors are evaluated over τ with equations (4.54), (4.55), (4.56).

$$\int_0^\infty e^{-ax^2+bx}dx = \frac{\sqrt{\pi}e^{\frac{b^2}{4a}}}{2\sqrt{a}}\left[1+\mathrm{erf}\left(\frac{b}{2\sqrt{a}}\right)\right], \tag{4.54}$$

$$\int_0^\infty xe^{-ax^2+bx}dx = \frac{1}{2a}+\frac{\sqrt{\pi}be^{\frac{b^2}{4a}}}{4a^{3/2}}\left[1+\mathrm{erf}\left(\frac{b}{2\sqrt{a}}\right)\right], \tag{4.55}$$

$$\int_0^\infty x^2e^{-ax^2+bx}dx = \frac{b}{4a^2}+\frac{\sqrt{\pi}(2a+b^2)e^{\frac{b^2}{4a}}}{8a^{5/2}}\left[1+\mathrm{erf}\left(\frac{b}{2\sqrt{a}}\right)\right]. \tag{4.56}$$

4.1.1 Calculating $g_{\tau\tau}$

The calculation of the $g_{\tau\tau}$ element begins with the expansion of the $J_{1,v}$ factor from equation (4.39), resulting in the three integrals in equation (4.57), which can be evaluated by equations (4.54), (4.55) and (4.56).

$$\begin{aligned} J_{1,v}\Psi_v = {} & \frac{A^2}{4}e^{-C_v}\int_{-\infty}^{\infty} e^{-A\tau^2+B_v\tau}\tau^2 d\tau\Psi_v - \frac{A(Bl+Bn)}{2}e^{-C_v}\int_{-\infty}^{\infty} e^{-A\tau^2+B_v\tau}\tau d\tau\Psi_v \\ & + B^2 ln\int_{-\infty}^{\infty} e^{-A\tau^2+B_v\tau}d\tau\Psi_v, \end{aligned} \tag{4.57}$$

$$\begin{aligned} J_{1,v}\Psi_v = {} & \frac{A^2}{4}e^{-C_v}\left[\frac{B_v}{4A^2}+\frac{\sqrt{\pi}(2A+B_v^2)e^{\frac{B_v^2}{4A}}}{8A^{5/2}}\left[1+\mathrm{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v \\ & - \frac{A(Bl+Bn)}{2}e^{-C_v}\left[\frac{1}{2A}+\frac{\sqrt{\pi}B_ve^{\frac{B_v^2}{4A}}}{2A^{3/2}}\left[1+\mathrm{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v \\ & + B^2lne^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{2\sqrt{A}}\left[1+\mathrm{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v, \end{aligned} \tag{4.58}$$

$$\begin{aligned} = {} & e^{-C_v}\left[\frac{B_v}{16}+\frac{\sqrt{\pi}(2A+B_v^2)e^{\frac{B_v^2}{4A}}}{32\sqrt{A}}\left[1+\mathrm{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v \\ & - (Bl+Bn)e^{-C_v}\left[\frac{1}{4}+\frac{\sqrt{\pi}B_ve^{\frac{B_v^2}{4A}}}{8\sqrt{A}}\left[1+\mathrm{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v \\ & + B^2lne^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{2\sqrt{A}}\left[1+\mathrm{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v. \end{aligned} \tag{4.59}$$

Applying the symmetry properties in equation (4.42)

$$J_{1,-v}\Psi_{-v} = e^{-C_v}\left[-\frac{B_v}{16} + \frac{\sqrt{\pi}(2A+B_v^2)e^{\frac{B_v^2}{4A}}}{32\sqrt{A}}\left[1-\text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v$$
$$+(Bl+Bn)e^{-C_v}\left[\frac{1}{4} - \frac{\sqrt{\pi}B_v e^{\frac{B_v^2}{4A}}}{8\sqrt{A}}\left[1-\text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v$$
$$+B^2 lne^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{2\sqrt{A}}\left[1-\text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v. \tag{4.60}$$

Combining the equations (4.59) and (4.60) gives

$$J_{1,v}\Psi_v + J_{1,-v}\Psi_{-v} = e^{-C_v}\left[\frac{B_v}{16} + \frac{\sqrt{\pi}(2A+B_v^2)e^{\frac{B_v^2}{4A}}}{32\sqrt{A}}\left[1+\text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v$$
$$-(Bl+Bn)e^{-C_v}\left[\frac{1}{4} + \frac{\sqrt{\pi}B_v e^{\frac{B_v^2}{4A}}}{8\sqrt{A}}\left[1+\text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v$$
$$+B^2 lne^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{2\sqrt{A}}\left[1+\text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v$$
$$+e^{-C_v}\left[-\frac{B_v}{16} + \frac{\sqrt{\pi}(2A+B_v^2)e^{\frac{B_v^2}{4A}}}{32\sqrt{A}}\left[1-\text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v$$
$$+(Bl+Bn)e^{-C_v}\left[\frac{1}{4} - \frac{\sqrt{\pi}B_v e^{\frac{B_v^2}{4A}}}{8\sqrt{A}}\left[1-\text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v$$
$$+B^2 lne^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{2\sqrt{A}}\left[1-\text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi_v, \tag{4.61}$$

$$= e^{-C_v}\left[\frac{B_v}{16} + \frac{\sqrt{\pi}(2A+B_v^2)e^{\frac{B_v^2}{4A}}}{32\sqrt{A}}\right]\Psi_v - (Bl+Bn)e^{-C_v}\left[\frac{1}{4} + \frac{\sqrt{\pi}B_v e^{\frac{B_v^2}{4A}}}{8\sqrt{A}}\right]\Psi_v$$
$$+B^2 lne^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{2\sqrt{A}}\right]\psi_v + e^{-C_v}\left[-\frac{B_v}{16} + \frac{\sqrt{\pi}(2A+B_v^2)e^{\frac{B_v^2}{4A}}}{32\sqrt{A}}\right]\Psi_v$$
$$+(Bl+Bn)e^{-C_v}\left[\frac{1}{4} - \frac{\sqrt{\pi}B_v e^{\frac{B_v^2}{4A}}}{8\sqrt{A}}\right]\Psi_v + B^2 lne^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{2\sqrt{A}}\right]\Psi_v, \tag{4.62}$$

$$= e^{-C_v}\left[\frac{\sqrt{\pi}(2A+B_v^2)e^{\frac{B_v^2}{4A}}}{16\sqrt{A}}\right]\Psi_v - (Bl+Bn)e^{-C_v}\frac{\sqrt{\pi}B_v e^{\frac{B_v^2}{4A}}}{8\sqrt{A}}\Psi_v$$
$$+B^2 lne^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{\sqrt{A}}\right]\Psi_v, \tag{4.63}$$

Applying the simplifications C_v, B_v and s_v, as shown in equations (4.27), (4.26) and (4.25) respectively, to (4.63) results in the expression

$$J_{1,v}\Psi_v + J_{1,-v}\Psi_{-v} = \frac{\sqrt{\pi}}{4\sqrt{A}} e^{-\frac{B^2}{4}\left(\frac{|v|^2}{b^2T^2} - \frac{s_v^2}{A}\right)} \left[\frac{A}{2} + B^2\left(\frac{s_v^2}{4} - (l+n)s_v + 4ln\right)\right]\Psi_v. \quad (4.64)$$

Equation (4.59) may be used with equation (4.51) to compute $g_{\tau\tau}$ to any level of accuracy. Here explicit formulae in T, P and b for terms with $|v| \leq 1$ are derived. For $h = 0$, $v = (0,0,0,0)$. By equation (3.28), with n=0 $\Psi_{0,0,0,0} = \frac{2\pi}{3P}$. Thus, by (4.59)

$$4J_{1,(0,0,0,0)}\Psi_{(0,0,0,0)} = \frac{\pi^{\frac{3}{2}}\sqrt{1+4b^2T^4}}{6PT}. \quad (4.65)$$

The equation (4.64) is used to evaluate the contribution of $h = 1$ to $g_{\tau\tau}$ results from the four vectors, $v = (1,0,0,0)$, $v = (0,1,0,0)$, $v = (0,0,1,0)$, $v = (0,0,0,1)$. Here $|v|^2 = 1$, $s_v = 1$ where by equation (3.27) with $n = 1$ and $m = 0$,

$$\Psi_v = \frac{1}{2\pi}, \quad (4.66)$$

for each v with $|v| = 1$. The equation (4.51) requires four cases which, in this instance, are captured by only two expressions,

$$J_{1,(1,0,0,0)}\Psi_v + J_{1,-(1,0,0,0)}\Psi_{-v} = \frac{\sqrt{\pi}}{4\sqrt{A}} e^{-\frac{B^2}{4}\left(\frac{1}{b^2T^2} - \frac{1}{A}\right)} \left[\frac{A}{2} + \frac{B^2}{4}\right]\frac{1}{2\pi P}, \quad (4.67)$$

$$= \frac{Te^{-\frac{\pi^2b^2T^4}{4P^2}\left(\frac{1}{b^2T^2} - \frac{T^2}{1+4b^2T^4}\right)}}{8\sqrt{\pi}P\sqrt{1+4b^2T^4}} \left[\frac{1+4b^2T^4}{2T^2} + \frac{\pi^2b^2T^4}{4P^2}\right], \quad (4.68)$$

$$J_{1,(0,1,0,0)}\Psi_v + J_{1,-(0,1,0,0)}\Psi_{-v} = \frac{\sqrt{\pi}}{4\sqrt{A}} e^{-\frac{B^2}{4}\left(\frac{1}{b^2T^2} - \frac{1}{A}\right)} \left[\frac{A}{2} - \frac{3B^2}{4}\right]\frac{1}{2\pi P}, \quad (4.69)$$

$$= \frac{Te^{-\frac{\pi^2b^2T^4}{4P^2}\left(\frac{1}{b^2T^2} - \frac{T^2}{1+4b^2T^4}\right)}}{8\sqrt{\pi}P\sqrt{1+4b^2T^4}} \left[\frac{1+4b^2T^4}{2T^2} - \frac{3\pi^2b^2T^4}{4P^2}\right]. \quad (4.70)$$

Equation (4.51) is used to express the total contribution for $h = 1$ by combining equations

(4.65), (4.68) and (4.70).

$$4\sum_{v\in\mathbb{Z}^4_+} J_{1,v}\Psi_v + J_{1,-v}\Psi_{-v} = \frac{Te^{-\frac{\pi^2b^2T^4}{4P^2}\left(\frac{1}{b^2T^2}-\frac{T^2}{1+4b^2T^4}\right)}}{2\sqrt{\pi}P\sqrt{1+4b^2T^4}}\left[\frac{1+4b^2T^4}{2T^2}+\frac{\pi^2b^2T^4}{4P^2}\right]$$
$$+\frac{Te^{-\frac{\pi^2b^2T^4}{4P^2}\left(\frac{1}{b^2T^2}-\frac{T^2}{1+4b^2T^4}\right)}}{2\sqrt{\pi}P\sqrt{1+4b^2T^4}}\left[\frac{1+4b^2T^4}{2T^2}-\frac{3\pi^2b^2T^4}{4P^2}\right], \tag{4.71}$$
$$=\frac{Te^{-\frac{\pi^2b^2T^4}{4P^2}\left(\frac{1}{b^2T^2}-\frac{T^2}{1+4b^2T^4}\right)}}{2\sqrt{\pi}P\sqrt{1+4b^2T^4}}\left[\frac{1+4b^2T^4}{2T^2}+\frac{\pi^2b^2T^4}{4P^2}+\frac{1+4b^2T^4}{2T^2}-\frac{3\pi^2b^2T^4}{4P^2}\right] \tag{4.72}$$
$$=\frac{Te^{-\frac{\pi^2T^2}{4P^2}\left(\frac{1+3b^2T^4}{1+4b^2T^4}\right)}}{2\sqrt{\pi}P\sqrt{1+4b^2T^4}}\left[\frac{1+4b^2T^4}{T^2}-\frac{\pi^2b^2T^4}{2P^2}\right]. \tag{4.73}$$

4.1.2 Calculating $g_{\tau\omega}$

The process executed in the calculation of 4.1.1 is performed again for 4.1.2, beginning with

$$J_{2,v}\Psi'_v = \int_0^\infty e^{-(A\tau^2-B_v\tau+C_v)}\left(\frac{A\tau}{2}-Bl\right)d\tau\Psi'_v, \tag{4.74}$$
$$=\frac{A}{2}e^{-C_v}\int_0^\infty \tau e^{-(A\tau^2-B_v\tau)}d\tau\Psi'_v - Ble^{-C_v}\int_0^\infty \tau e^{-(A\tau^2-B_v\tau)}d\tau\Psi'_v. \tag{4.75}$$

These integrals are evaluated using the expressions (4.55) and (4.54) respectively.

$$=\frac{A}{2}e^{-C_v}\left[\frac{1}{2A}+\frac{\sqrt{\pi}B_ve^{\frac{B_v^2}{4A}}}{2A^{3/2}}\left[1+\text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi'_v - Ble^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{2\sqrt{A}}\left[1+\text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi'_v, \tag{4.76}$$

with

$$J_{2,-v}\psi'_{-v} = -\frac{A}{2}e^{-C_v}\left[\frac{1}{2A}-\frac{\sqrt{\pi}B_ve^{\frac{B_v^2}{4A}}}{2A^{3/2}}\left[1-\text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi'_v$$
$$-Ble^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{2\sqrt{A}}\left[1-\text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi'_v. \tag{4.77}$$

Recalling that for $\Psi'_{-v} = -\Psi'_v$, then

$$J_{2,v}\Psi'_v + J_{2,-v}\Psi'_{-v} =$$

$$\frac{A}{2}e^{-C_v}\left[\frac{1}{2A} + \frac{\sqrt{\pi}B_v e^{\frac{B_v^2}{4A}}}{4A^{3/2}}\left[1 + \text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi'_v - Ble^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{2\sqrt{A}}\left[1 + \text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi'_v$$
$$-\frac{A}{2}e^{-C_v}\left[\frac{1}{2A} - \frac{\sqrt{\pi}B_v e^{\frac{B_v^2}{4A}}}{4A^{3/2}}\left[1 - \text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi'_v - Ble^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{2\sqrt{A}}\left[1 - \text{erf}\left(\frac{B_v}{2\sqrt{A}}\right)\right]\right]\Psi'_v, \tag{4.78}$$

$$= \frac{A}{2}e^{-C_v}\left[\frac{1}{2A} + \frac{\sqrt{\pi}B_v e^{\frac{B_v^2}{4A}}}{4A^{3/2}}\right]\Psi'_v - Ble^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{2\sqrt{A}}\right]\Psi'_v$$
$$-\frac{A}{2}e^{-C_v}\left[\frac{1}{2A} - \frac{\sqrt{\pi}B_v e^{\frac{B_v^2}{4A}}}{4A^{3/2}}\right]\Psi'_v - Ble^{-C_v}\left[\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}}}{2\sqrt{A}}\right]\Psi'_v, \tag{4.79}$$
$$= \frac{\sqrt{\pi}B_v e^{\frac{B_v^2}{4A}-C_v}}{4\sqrt{A}}\Psi'_v - Bl\frac{\sqrt{\pi}e^{\frac{B_v^2}{4A}-C_v}}{\sqrt{A}}\Psi'_v, \tag{4.80}$$
$$= \frac{\sqrt{\pi}Be^{\frac{B_v^2}{4A}-C_v}}{\sqrt{A}}\left[\frac{s_v}{4} - l\right]\Psi'_v. \tag{4.81}$$

Equation (4.52) is considered to determine the contribution to $g_{\tau\omega}$ from $h = 0$, along with Ψ'_v given by equation (3.35) with $n = 0$ and $v = (0, 0, 0, 0)$

$$\Psi'_{(0,0,0,0)} = 0, \tag{4.82}$$

and so the contribution to $g_{\tau\omega}$ for $h = 0$ is zero. Next, consider the contribution of $h = 1$, where again there are four cases $\Psi_{(1,0,0,0)}$, $\Psi_{(0,1,0,0)}$, $\Psi_{(0,0,1,0)}$ and $\Psi_{(0,0,0,1)}$. Since the first three cases can be represented by an expression with three regular ψ factors and one derivative, ψ', by equation (3.34) with $m = 1$ and $n = 0$

$$\Psi_v = -\left(\frac{1}{2\pi} + \frac{1}{6}\right) = \left(-\frac{3+\pi^2}{6\pi^2}\right). \tag{4.83}$$

Using equation (4.52) and setting $m = 0$ and $n = 1$, giving

$$J_{2,(1,0,0,0)}\Psi_v + J_{2,-(1,0,0,0)}\Psi_{-v} = \frac{\sqrt{\pi}Be^{\frac{B^2}{4A}-C}}{4\sqrt{A}}\left(\frac{1}{4}\right)\left(-\frac{3+\pi^2}{6\pi^2}\right), \tag{4.84}$$
$$J_{2,(0,1,0,0)}\Psi_v + J_{2,-(0,1,0,0)}\Psi_{-v} = \frac{\sqrt{\pi}Be^{\frac{B^2}{4A}-C}}{4\sqrt{A}}\left(\frac{1}{4} - 1\right)\left(-\frac{3+\pi^2}{6\pi^2}\right). \tag{4.85}$$

The expression for the case $\Psi_{(0,0,1,0)}$ is the same as that for $\Psi_{(1,0,0,0)}$. In the final case Ψ_v is given by equation (3.32), with $n = 1$ and $m = 0$

$$\Psi_{(0,0,0,1)} = (-1)\left(-\frac{1}{2} - \frac{3}{2\pi^2}\right) = \left(\frac{\pi^2+3}{2\pi^2}\right), \tag{4.86}$$

so combining equation (4.86) with equation (4.81) gives the result

$$J_{2,(0,0,0,1)}\Psi_v + J_{2,-(0,0,0,1)}\Psi_{-v} = \frac{\sqrt{\pi}Be^{\frac{B^2}{4A}-C}}{\sqrt{A}}\left(\frac{1}{4}\right)\left(\frac{3+\pi^2}{2\pi^2}\right). \tag{4.87}$$

Equation (4.52) is used to express the total contribution for $h=1$ by combining equations (4.84), (4.85) and (4.87), since there is no contribution from $h=0$.

$$4\sum_{v\in\mathbb{Z}^4_+} J_{2,v}\Psi_v = \frac{4\sqrt{\pi}Be^{\frac{B^2}{4A}-C}}{\sqrt{A}}\left[\frac{3+\pi^2}{6\pi^2}\right], \tag{4.88}$$

$$= \frac{\sqrt{\pi}Be^{\frac{B^2}{4A}-C}}{\sqrt{A}}\left[\frac{2}{3}+\frac{2}{\pi^2}\right]. \tag{4.89}$$

4.1.3 Calculating $g_{\omega\omega}$

Finally the calculations for $J_{3,v}\Psi''_v$ are performed, using equation (4.41), reiterated here

$$J_{3,v}\Psi''_v = \int_0^\infty e^{-(A\tau^2-B_v\tau+C_v)}d\tau. \tag{4.90}$$

This integral form can be evaluated by using equation (4.53)

$$\int_0^\infty e^{-ax^2+bx}dx = \frac{\sqrt{\pi}e^{\frac{b^2}{4a}}}{2\sqrt{a}}\left[1+\operatorname{erf}\left(\frac{b}{2\sqrt{a}}\right)\right]. \tag{4.91}$$

Then

$$J_{3,v}\Psi''_v = \frac{\sqrt{\pi}e^{-C_v+\frac{B^2}{4A}}}{2\sqrt{a}}\left[1+\operatorname{erf}\left(\frac{b}{2\sqrt{A}}\right)\right]\Psi''_v. \tag{4.92}$$

The symmetry properties allow $J_{3,-v}\Psi''_{-v}$ to be written as

$$J_{3,-v}\Psi''_{-v} = \frac{\sqrt{\pi}e^{C_v+\frac{B^2}{4A}}}{2\sqrt{A}}\left[1-\operatorname{erf}\left(\frac{b}{2\sqrt{A}}\right)\right]\Psi''_v, \tag{4.93}$$

and so by combining the two

$$J_{3,v}\Psi''_v + J_{3,-v}\Psi''_{-v} = \frac{\sqrt{\pi}e^{\frac{B^2}{4A}}}{\sqrt{A}}\Psi''_v. \tag{4.94}$$

In the case where $h=0$ the integral equation is of the form seen in equation (3.42), reiterated here

$$\int_{-\infty}^\infty \psi_n^2\psi_n'^2 = \frac{\pi P}{15}. \tag{4.95}$$

Then evaluated for $v=(0,0,0,0)$

$$\Psi''_{(0,0,0,0)} = \int_{-\infty}^\infty \psi_0^2\psi_0'^2 = \frac{\pi P}{15}. \tag{4.96}$$

Equation (4.92) is used to determine the contribution of $h = 0$, with $B_v = 0$, $s_v = 0$, $|v|^2 = 0$, giving

$$J_{3,v}\Psi''_v = \frac{\sqrt{\pi}}{2\sqrt{A}}\frac{\pi P}{15} = \frac{\pi^{\frac{3}{2}}TP}{30\sqrt{1+4b^2T^4}}. \tag{4.97}$$

Calculating the contribution from the $h = 1$ case, with $m = 1$ and $n = 0$. For the first and second cases $\Psi_{(1,0,0,0)} = \Psi_{(0,1,0,0)}$ are expressed in the form of equation (3.43) and the third and fourth cases $\Psi_{(0,0,1,0)} = \Psi_{(0,0,0,1)}$ are expressed in the form of equation (3.41), and so only two integral forms are needed to evaluate. For the first two cases

$$\Psi''_{(1,0,0,0)} = \Psi''_{(0,0,1,0)} = P\left(\frac{1}{6\pi} + \frac{1}{2\pi^3}\right), \tag{4.98}$$

and the second two cases

$$\Psi''_{(0,1,0,0)} = \Psi''_{(0,0,0,1)} = -P\left(\frac{1}{6\pi} + \frac{2}{\pi^3}\right). \tag{4.99}$$

The contribution for $h = 1$ is

$$J_{3,(1,0,0,0)}\Psi''_{(1,0,0,0)} + J_{3,-(1,0,0,0)}\Psi''_{-(1,0,0,0)} = \frac{\pi^{\frac{3}{2}}TPe^{-\frac{\pi^2T^2}{4P^2}\left(\frac{1+3b^2T^4}{1+4b^2T^4}\right)}}{\sqrt{1+4b^2T^4}}\left(\frac{\pi^2+3}{6\pi^3}\right), \tag{4.100}$$

and

$$J_{3,(0,1,0,0)}\Psi''_{(0,1,0,0)} + J_{3,-(0,1,0,0)}\Psi''_{-(0,1,0,0)} = -\frac{\pi^{\frac{3}{2}}TPe^{-\frac{\pi^2T^2}{4P^2}\left(\frac{1+3b^2T^4}{1+4b^2T^4}\right)}}{\sqrt{1+4b^2T^4}}\left(\frac{1}{6\pi} + \frac{2}{\pi^3}\right). \tag{4.101}$$

Equation (4.53) is used to construct the full contribution to $g_{\omega\omega}$, combining equations (4.97), (4.100) and (4.101).

$$\sum_{v\in\mathbb{Z}^4} J_{3,v}\Psi''_v = \frac{\pi^{3/2}TP}{30\sqrt{1+4b^2T^4}} + \frac{4\sqrt{\pi}e^{\rho}}{\sqrt{A}}\left[\frac{\pi^2+3}{6\pi^3} - \frac{\pi^2+12}{6\pi^3}\right], \tag{4.102}$$

$$= \frac{\pi^{3/2}TP}{30\sqrt{1+4b^2T^4}} - \frac{12TPe^{\rho}}{\pi^{5/2}\sqrt{1+4b^2T^4}}, \tag{4.103}$$

$$= \frac{TP}{\sqrt{1+4b^2T^4}}\left[\frac{\pi^{3/2}}{30} - \frac{12e^{\rho}}{\pi^{5/2}}\right]. \tag{4.104}$$

4.1.4 Fisher Matrices

In Section 4.1 the elements of the Fisher matrices were calculated. The result is a full construction of the $h = 0$ term Fisher matrix is

$$g = \begin{bmatrix} \frac{\pi^{\frac{3}{2}}\sqrt{1+4b^2T^4}}{6PT} & 0 \\ 0 & \frac{\pi^{\frac{3}{2}}TP}{30\sqrt{1+4b^2T^4}} \end{bmatrix}, \tag{4.105}$$

and of the $h = 1$ term Fisher matrix

$$g = \begin{bmatrix} \frac{\sqrt{1+4b^2T^4}}{PT} + \left[\frac{\pi^{3/2}}{6} + \frac{e^\rho}{2\sqrt{\pi}}\left[1 - \frac{\pi^2 b^2 T^6}{2P^2(1+4b^2T^4)}\right]\right] & \frac{\pi^{3/2} b T^3 e^\rho}{P\sqrt{1+4b^2T^4}}\left[\frac{2}{3} + \frac{2}{\pi^2}\right] \\ \frac{\pi^{3/2} b T^3 e^\rho}{P\sqrt{1+4b^2T^4}}\left[\frac{2}{3} + \frac{2}{\pi^2}\right] & \frac{TP}{\sqrt{1+4b^2T^4}}\left[\frac{\pi^{3/2}}{30} - \frac{12 e^\rho}{\pi^{5/2}}\right] \end{bmatrix}. \tag{4.106}$$

where

$$\rho = -\frac{\pi^2 T^2}{4P^2}\left(\frac{1+3b^2T^4}{1+4b^2T^4}\right). \tag{4.107}$$

4.2 The Configuration Metric

The Gil Medrano metric [13], discussed in Chapter 2.29, is defined by

$$G_{ij} = \int_M \operatorname{tr}\left(g^{-1}\partial_i g g^{-1}\partial_j g\right) \operatorname{vol}(g), \tag{4.108}$$

where $i, j \in \{T, P, b\}$ and g is the Fisher metric computed in Section 4.1. Partial derivatives of the Fisher metric are required with respect to the configuration parameters T, P and b. The notation is reduced by replacing ubiquitous combinations of variables as follows,

$$V = \frac{\sqrt{1+4b^2T^4}}{TP}, \tag{4.109}$$

$$W = \frac{\pi b T^3}{P\sqrt{1+4b^2T^4}}. \tag{4.110}$$

Reduction of constant factors is achieved by setting

$$\alpha = \frac{\pi^{3/2}}{6}, \tag{4.111}$$

$$\beta = \frac{1}{2\sqrt{\pi}}, \tag{4.112}$$

$$\gamma = 2\pi^{3/2}, \tag{4.113}$$

$$\mu = \frac{\pi^{3/2}}{30}, \tag{4.114}$$

$$\nu = \frac{12}{\pi^{5/2}}. \tag{4.115}$$

With these substitutions, the Fisher metric is

$$g = \begin{bmatrix} V\left[\alpha + \beta e^\rho\left(1 - W^2\right)\right] & \gamma W e^\rho \\ \gamma W e^\rho & \frac{\mu - \nu e^\rho}{V} \end{bmatrix}. \tag{4.116}$$

The determinant of g is

$$\det(g) = (\mu - \nu e^\rho)\left[\alpha + \beta e^\rho\left(1 - W^2\right)\right] - \gamma^2 W^2 e^{2\rho}. \tag{4.117}$$

Additionally, the inverse of the Fisher matrix is

$$g^{-1} = \frac{1}{\det(g)} \begin{bmatrix} \frac{\mu - \nu e^{\rho}}{V} & -\gamma W e^{\rho} \\ -\gamma W e^{\rho} & V\left[\alpha + \beta e^{\rho}\left(1 - W^2\right)\right] \end{bmatrix}. \tag{4.118}$$

Derivatives of the Fisher elements are written in the new convention as follows

$$\partial_x g_{\tau\tau} = V\left[\beta\rho_x e^{\rho}\left(1 - W^2\right) - \beta e^{\rho}\left(2WW_x\right)\right] + V_x\left[\alpha + \beta e^{\rho}\left(1 - W^2\right)\right], \tag{4.119}$$

$$\partial_x g_{\tau\omega} = \gamma\left(W_x e^{\rho} + W\rho_x e^{\rho}\right) = \gamma e^{\rho}\left(W_x + W\rho_x\right), \tag{4.120}$$

$$\partial_x g_{\omega\omega} = \frac{e^{\rho}\nu\left(V_x - \rho_x V\right) - V_x\mu}{V^2}, \tag{4.121}$$

noting that

$$VW = \frac{\pi b T^2}{P^2}. \tag{4.122}$$

The relevant derivatives are

$$\partial_T V = \frac{4b^2T^4 - 1}{PT^2\sqrt{1 + 4b^2T^4}}, \tag{4.123}$$

$$\partial_P V = -\frac{\sqrt{1 + 4b^2T^4}}{PT^2}, \tag{4.124}$$

$$\partial_b V = -\frac{4bT^3}{P\sqrt{1 + 4b^2T^4}}, \tag{4.125}$$

$$\partial_T W = \frac{\pi b T^2\left(3 + 4b^2T^4\right)}{P\left(1 + 4b^2T^4\right)^{3/2}}, \tag{4.126}$$

$$\partial_P W = -\frac{\pi b T^3}{P^2\sqrt{1 + 4b^2T^4}}, \tag{4.127}$$

$$\partial_b W = \frac{\pi T^3}{P\left(1 + 4b^2T^4\right)^{3/2}}, \tag{4.128}$$

$$\partial_T \rho = -\frac{\pi^2 T\left(1 + 5b^2T^4 + 12b^4T^8\right)}{2P^2\left(1 + 4b^2T^4\right)^2}, \tag{4.129}$$

$$\partial_P \rho = \frac{\pi^2 T^2\left(1 + 3b^2T^4\right)}{2P^3\left(1 + 4b^2T^4\right)}, \tag{4.130}$$

$$\partial_b \rho = \frac{b\pi^2 T^6}{2P^2\left(1 + 4b^2T^4\right)^2}. \tag{4.131}$$

Continuing on with the construction of a generalised formula for G_{ij}, the product $g^{-1}\partial_i g$ is

$$g^{-1}\partial_i g = \frac{1}{\det(g)} \begin{bmatrix} g_{\omega\omega}\partial_i g_{\tau\tau} - g_{\tau\omega}\partial_i g_{\tau\omega} & g_{\omega\omega}\partial_i g_{\tau\omega} - g_{\tau\omega}\partial_i g_{\omega\omega} \\ -g_{\tau\omega}\partial_i g_{\tau\tau} + g_{\tau\tau}\partial_i g_{\tau\omega} & -g_{\tau\omega}\partial_i g_{\tau\omega} + g_{\tau\tau}\partial_i g_{\omega\omega} \end{bmatrix}. \tag{4.132}$$

The matrix g_i will be used to denote the matrix $g^{-1}\partial_j g$ in equation (4.132). The components of g_i will be denoted by $g_{i,\ \tau\tau}$, $g_{i,\ \tau\omega}$ and $g_{i,\ \omega\omega}$. The matrix g_j is defined analogously.

$$g_i = \begin{bmatrix} g_{i,\tau\tau} & g_{i,\tau\omega} \\ g_{i,\tau\omega} & g_{i,\omega\omega} \end{bmatrix}, \quad g_j = \begin{bmatrix} g_{j,\tau\tau} & g_{j,\tau\omega} \\ g_{j,\tau\omega} & g_{j,\omega\omega} \end{bmatrix}. \tag{4.133}$$

The product of these two matrices is the matrix

$$g_ig_j = \begin{bmatrix} g_{i,\tau\tau}g_{j,\tau\tau} + g_{i,\tau\omega}g_{j,\tau\omega} & g_{i,\tau\tau}g_{j,\tau\omega} + g_{i,\tau\omega}g_{j,\omega\omega} \\ g_{i,\tau\omega}g_{j,\tau\omega} + g_{i,\omega\omega}g_{j,\tau\omega} & g_{i,\tau\omega}g_{j,\tau\omega} + g_{i,22}g_{j,\omega\omega} \end{bmatrix}, \tag{4.134}$$

which has the trace

$$\operatorname{tr}(g_ig_j) = g_{i,\tau\tau}g_{j,\tau\tau} + g_{i,\tau\omega}g_{j,\tau\omega} + g_{i,\tau\omega}g_{j,\tau\omega} + g_{i,\omega\omega}g_{j,\omega\omega}, \tag{4.135}$$

and using this trace, the components of the G matrix are given by

$$G_{ij} = \int_{-bT/2}^{bT/2}\int_0^{T_P} (g_{i,\tau\tau}g_{j,\tau\tau} + g_{i,\tau\omega}g_{j,\tau\omega} + g_{i,\tau\omega}g_{j,\tau\omega} + g_{i,\omega\omega}g_{j,\omega\omega}) \operatorname{vol}(g) d\tau d\omega, \tag{4.136}$$

where $\operatorname{vol}(g) = \sqrt{\det(g)}$.

4.3 Geodesics on G

Having constructed the metric G, it is possible to determine geodesics of the manifold for which G is the metric. To this end, the Christoffel symbol presented in equation (2.47) of the literature review, is utilised.

Let $\alpha = \alpha(t) = (T(t), P(t), b(t))$ be a unit speed curve and $u_1 = u_1(t) = T(t)$, $u_2 = u_2(t) = P(t)$ and $u_3 = u_3(t) = b(t)$.

Then α is a geodesic if and only if,

$$u_k'' + \sum_{i,j=1}^{3} \Gamma_{ij}^k u_i' u_j' = 0, \qquad k = 1, 2, 3. \tag{4.137}$$

Here the Christoffel symbols Γ_{ij}^k are given by,

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{3}\left(\frac{\partial G_{li}}{\partial u_j} + \frac{\partial G_{jl}}{\partial u_i} - \frac{\partial G_{ij}}{\partial u_l}\right) G^{lk}, \tag{4.138}$$

where G_{ij} is the i,j entry of G and G^{ij} is the i,j entry of G^{-1}. Thus the system of ODEs suggested by equation (4.137) is

$$u_1'' + \Gamma_{11}^1(u_1')^2 + \Gamma_{22}^1(u_2')^2 + \Gamma_{33}^1(u_3')^2 + 2\left(\Gamma_{12}^1 u_1'u_2' + \Gamma_{13}^1 u_1'u_3' + \Gamma_{23}^1 u_2'u_3'\right) = 0, \tag{4.139}$$

$$u_2'' + \Gamma_{11}^2(u_1')^2 + \Gamma_{22}^2(u_2')^2 + \Gamma_{33}^2(u_3')^2 + 2\left(\Gamma_{12}^2 u_1'u_2' + \Gamma_{13}^2 u_1'u_3' + \Gamma_{23}^2 u_2'u_3'\right) = 0, \tag{4.140}$$

$$u_3'' + \Gamma_{11}^3(u_1')^2 + \Gamma_{22}^3(u_2')^2 + \Gamma_{33}^3(u_3')^2 + 2\left(\Gamma_{12}^3 u_1'u_2' + \Gamma_{13}^3 u_1'u_3' + \Gamma_{23}^3 u_2'u_3'\right) = 0. \tag{4.141}$$

The system of ODE given by equations (4.139), (4.139) and (4.139) represent a parametrisation of the geodesic α and may be solved numerically to find geodesics in the manifold determined by the Gil Medrano metric G_{ij}. Such computations are more robust if the expressions for the coefficients, the Christoffel symbols of equation (4.138), are computed without reliance on numerical derivatives. Accordingly, algebraic expressions for the Christoffel symbols are derived. The formula for the Gil Medrano metric in equation (4.136) may be written as

$$G_{ij} = \int_{-bT/2}^{bT/2} \int_{0}^{T_P} \mathrm{tr}((g^{-1}\partial_i g)(g^{-1}\partial_j g))\mathrm{vol}(g) d\tau d\omega. \tag{4.142}$$

Both the volume element, $\mathrm{vol}(g)$, and the trace, $\mathrm{tr}((g^{-1}\partial_i g)(g^{-1}\partial_j g))$, in equation (4.142) are independent of state variables τ and ω, being instead composed of configuration parameters T, P and b. Therefore,

$$G_{ij} = \mathrm{tr}((g^{-1}\partial_i g)(g^{-1}\partial_j g))\mathrm{vol}(g) \int_{-bT/2}^{bT/2} \int_{0}^{T_P} d\tau d\omega = \mathrm{tr}((g^{-1}\partial_i g)(g^{-1}\partial_j g))\mathrm{vol}(g) \frac{TPb}{\pi}. \tag{4.143}$$

From equations (4.116) and (4.118) the Fisher matrix and its inverse are

$$g = \begin{bmatrix} V\left[\alpha + \beta e^{\rho}\left(1 - W^2\right)\right] & \gamma W e^{\rho} \\ \gamma W e^{\rho} & \dfrac{\mu - \nu e^{\rho}}{V} \end{bmatrix}, \quad g^{-1} = \frac{1}{\det(g)} \begin{bmatrix} \dfrac{\mu - \nu e^{\rho}}{V} & -\gamma W e^{\rho} \\ -\gamma W e^{\rho} & V\left[\alpha + \beta e^{\rho}\left(1 - W^2\right)\right] \end{bmatrix}. \tag{4.144}$$

where V, W, α, β, γ, μ, ν are from equations (4.109)-(4.115) respectively. Let $\hat{g}$ denote the matrix

$$\hat{g} = \begin{bmatrix} \dfrac{\mu - \nu e^{\rho}}{V} & -\gamma W e^{\rho} \\ -\gamma W e^{\rho} & V\left[\alpha + \beta e^{\rho}\left(1 - W^2\right)\right] \end{bmatrix}. \tag{4.145}$$

With this notation

$$g^{-1} = \frac{1}{\det(g)} \hat{g}. \tag{4.146}$$

G_{ij} may be written using this new notation, as

$$G_{ij} = \frac{TPb}{\pi \det(g)^{3/2}} \mathrm{tr}((\hat{g}\partial_i g)(\hat{g}\partial_j g)). \tag{4.147}$$

The expression for G_{ij} in equation (4.147) represents an algebraic formula in terms of T, P and b. All components of this new expression for G_{ij} have already been calculated in Section 4.2 for the infinite pulse train. In order to calculate the Christoffel symbols of equation (4.138) the partial derivatives in Sections 4.1 and 4.3, are used. These derivatives allow components of the Gil Medrano metric to be calculated via the expression

$$\partial_x G_{ij} = \frac{TPb}{\pi \det(g)^{3/2}} \partial_x \mathrm{tr}((\hat{g}\partial_i g)(\hat{g}\partial_j g)) + \partial_x \left(\frac{TPb}{\pi \det(g)^{3/2}} \right) \mathrm{tr}((\hat{g}\partial_i g)(\hat{g}\partial_j g)). \tag{4.148}$$

Here, the derivatives of the trace are of the form

$$\partial_x \mathrm{tr}((\hat{g}\partial_i g)(\hat{g}\partial_j g)) = \mathrm{tr}\left[\partial_x(\hat{g}\partial_i g)(\hat{g}\partial_j g) + (\hat{g}\partial_i g)\partial_x(\hat{g}\partial_j g)\right] \tag{4.149}$$

$$= \mathrm{tr}\left\{[(\partial_x \hat{g})(\partial_i g) + \hat{g}(\partial_x\partial_i g)](\hat{g}\partial_j g) + (\hat{g}\partial_i g)[\hat{g}(\partial_x\partial_j g) + (\partial_x \hat{g})(\partial_j g)]\right\}, \tag{4.150}$$

for $x \in \{T,\ P,\ b\}$.

The second partial derivatives of g are calculated using equations (4.155),(4.156) and (4.157), requiring the substitution

$$Z = \frac{\pi b T^2)}{P^2}, \tag{4.151}$$

$$\partial_T Z = \frac{4\pi b T}{P^2}, \tag{4.152}$$

$$\partial_P Z = \frac{-4\pi b T^2}{P^3}, \tag{4.153}$$

$$\partial_b = \frac{2\pi T^2}{P^2}. \tag{4.154}$$

The formulae for the second derivatives of g are as follows.

$$\partial_y\partial_x g_{\tau\tau} = \alpha V_{xy} + \beta e^{\rho}\left[-2WW_y(V\rho_x + V_x - ZW_{xy} - Z_y W_x)\right.$$

$$\left. + (1 - W^2)(V\rho_{xy}V_y\rho_x + V_{xy} + V\rho_y\rho_x + \rho_y V_x) - \rho_y W_x Z\right], \tag{4.155}$$

$$\partial_y\partial_x g_{\tau\omega} = \gamma e^{\rho}\left[W_{xy} + \rho_x W_y + \rho_{xy} W + \rho_y(W_x + \rho_x W)\right] \tag{4.156}$$

$$\partial_y\partial_x g_{\omega\omega} = \frac{\mu(2V_x - VV_{xy}) + \mu e^{\rho}\left[V(V_{xy} - V\rho_{xy} - V_y\rho_x) + (V\rho_y - 2)(V_x - V\rho_y)\right]}{V^3} \tag{4.157}$$

Implementing equation (4.148), the second partial derivatives with respect to the variable expressions V, W and ρ, as defined in equations (4.109), (4.110) and (4.107) are

$$\partial_T\partial_T V = \frac{2 + 24b^2T^4}{PT^3\left(1 + 4b^2T^4\right)^{3/2}}, \tag{4.158}$$

$$\partial_P\partial_T V = \frac{1 - 4b^2T^4}{P^2T^2\sqrt{1 + 4b^2T^4}}, \tag{4.159}$$

$$\partial_b\partial_T V = \frac{4bT^2\left(3 + 4b^2T^4\right)}{P\left(1 + 4b^2T^4\right)^{3/2}}, \tag{4.160}$$

$$\partial_P\partial_P V = \frac{2\sqrt{1 + 4b^2T^4}}{P^3T}, \tag{4.161}$$

$$\partial_b\partial_P V = \frac{-4bT^3}{P^2\sqrt{1 + 4b^2T^4}}, \tag{4.162}$$

$$\partial_b\partial_b V = \frac{4T^3}{P\left(1 + 4b^2T^4\right)^{3/2}}, \tag{4.163}$$

$$\partial_T\partial_T W = \frac{6b\pi T\left(1 - 4b^2T^4\right)}{P\left(1 + 4b^2T^4\right)^{5/2}}, \tag{4.164}$$

$$\partial_P \partial_T W = -\frac{b\pi T^2 \left(3 + 4b^2 T^4\right)}{P^2 \left(1 + 4b^2 T^4\right)^{3/2}}, \tag{4.165}$$

$$\partial_b \partial_T W = \frac{3\pi T^2 \left(1 - 4b^2 T^4\right)}{P \left(1 + 4b^2 T^4\right)^{5/2}}, \tag{4.166}$$

$$\partial_P \partial_P W = \frac{2\pi b T^3}{P^3 \sqrt{1 + 4b^2 T^4}}, \tag{4.167}$$

$$\partial_b \partial_P W = -\frac{\pi T^3}{P^2 \left(1 + 4b^2 T^4\right)^{3/2}}, \tag{4.168}$$

$$\partial_b \partial_b W = -\frac{12 b\pi T^7}{P \left(1 + 4b^2 T^4\right)^{5/2}}, \tag{4.169}$$

$$\partial_T \partial_T \rho = -\frac{\pi^2 \left(1 - 3b^2 T^4 + 48b^4 T^8 + 48b^6 T^{12}\right)}{2P^2 \left(1 + 4b^2 T^4\right)^3}, \tag{4.170}$$

$$\partial_P \partial_T \rho = \frac{\pi^2 T \left(1 + 5b^2 T^4 + 12b^4 T^8\right)}{P^3 \left(1 + 4b^2 T^4\right)^2}, \tag{4.171}$$

$$\partial_b \partial_T \rho = \frac{b\pi^2 T^5 \left(3 - 4b^2 T^4\right)}{P^2 \left(1 + 4b^2 T^4\right)^3}, \tag{4.172}$$

$$\partial_P \partial_P \rho = -\frac{3\pi^2 T^2 \left(1 + 3b^2 T^4\right)}{2P^4 \left(1 + 4b^2 T^4\right)}, \tag{4.173}$$

$$\partial_b \partial_P \rho = -\frac{b\pi^2 T^6}{P^3 \left(1 + 4b^2 T^4\right)^2}, \tag{4.174}$$

$$\partial_b \partial_b \rho = \frac{\pi^2 T^6 \left(1 - 12b^2 T^4\right)}{2P^2 \left(1 + 4b^2 T^4\right)^3}. \tag{4.175}$$

The second partial derivatives in (4.158) – (4.175) along with the first partial derivatives in (4.123) – (4.131) inserted in the formulae (4.119) – (4.121) and (4.155) - (4.157) can be used to evaluate the partial derivatives of the components of Gil Medrano metric via Equation (4.148). With these expressions, the Christoffel symbols are calculated using equation (4.138) and are given in general form in chapter 7.

The construction of the Christoffel symbols in equations (7.1) - (7.27) lead to algebraic expressions for the coefficients in the ODE of equations (4.139) – (4.141). The Matlab ODE solver ode45.m was used to compute geodesics based on equations (4.139) – (4.141)

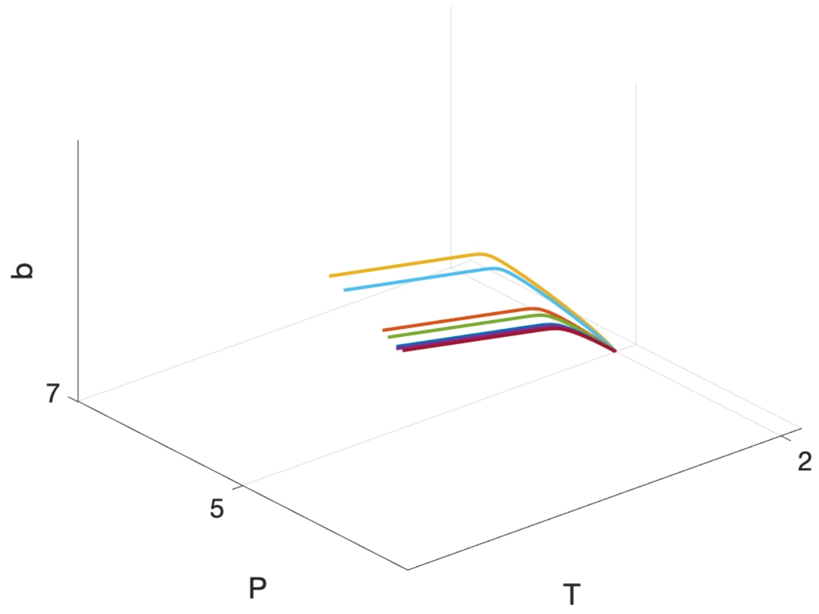

Figure 4.2: The point T = 2, P = 5, b = 7 represents optimal settings for a hypothetical sensor system at a given time. These parameters must be updated over time to adjust to changes in the scenario. The optimal path in parameter space for simultaneously adjusting the three parameters is a geodesic in the manifold defined by the configuration metric G. The paths shown represent six example geodesics for six different hypothetical reconfigurations.

This is a toy example in that the parameters were not chosen to reflect realistic settings.

Chapter 5

Single Doppler Bin Ambiguity Function

The second example is a coherent detector for a single Doppler bin. Attention will be limited to the bin centred around $\omega = 0$. This useful when trying to match speed with a target (*i.e.* maintain a zero relative Doppler shift).

As in Chapter 4, a sequence of Gaussian pulses of width, T, separated by T_p, with chirp rate b is considered. The simplest coherent detector is formed by integrating the matched filter for the Gaussian pulse and integrating over the length of the pulse. Non-zero Doppler yields an oscillatory response in the filter which gives a small response, hence the model for zero Doppler. Other Doppler bins can be accommodated by mixing in the required frequency. Additionally this process must be repeated for each possible delay. This may be modelled as a convolution with *rect* function over the interval $[-T_P/2, T_P/2]$. More precisely, let $g(t)$ denote the Gaussian pulse and define $h(t) = g(t)g(t-\tau)$. Then the convolution is

$$(rect_{T_P} * h)(t), \tag{5.1}$$

where $rect_{T_P}$ is defined as

$$rect_{T_P}(t) = \frac{1}{T_P} rect\left(\frac{t}{T_P}\right), \tag{5.2}$$

and *rect* is the 'rectangle function' $rect(t) = 1$ for $t \in [-1/2, 1/2]$ and zero otherwise.

Integrating equation (5.1) against the Doppler shift factor over all time forms the autocorrelation function

$$\phi(t, \omega) = \int_{-\infty}^{\infty} (rect_{\frac{T_P}{2}} * h)(t)\, e^{j\omega t} dt. \tag{5.3}$$

Equation (5.3) may be viewed as the Fourier transform of the convolution of $rect_{\frac{T_P}{2}}$ and $h(t)$. By the convolution property for Fourier Transform,

$$\phi(\tau, \omega) = \int_{-\infty}^{\infty} rect_{T_P} e^{j\omega t} dt \int_{-\infty}^{\infty} h(t) e^{j\omega t} dt. \tag{5.4}$$

The first integral is elementary and the second is given by [35](pg.283, eq(27)). Thus

$$\phi(\tau, \omega) = \operatorname{sinc}\left(\frac{\omega T_P}{2}\right) \exp\left\{-\frac{1}{4}\left(\frac{\tau^2}{T^2} + \omega^2 T^2\right)\right\}. \tag{5.5}$$

The factor of 1/2 in the argument of the sinc contributes to the clutter in the following calculations. This is avoided by setting $Q = T_P/2$. With this notation and by including a linear chirp with chirp rate b, the final version of the autocorrelation function is

$$\phi_0 (\tau, \omega) = \text{sinc} \left((\omega - 2b\tau) Q \right) \exp \left[-\frac{1}{4} \left(\frac{\tau^2}{T^2} + (\omega - 2b\tau)^2 T^2 \right) \right]. \tag{5.6}$$

The sinc function acts as a low pass filter on the Doppler component of the ambiguity with the width of the bin being T_P. The approach is entirely parallel to the view taken in Chapter 4 for the infinite pulse train model. The Fisher metric is computed followed by the Gil Medrano metric. All the calculations needed to compute geodesics on this manifold without the necessity of relying on numerical differentiation to set the coefficients of the ODE are carried out. Although the strategy is the same as in Chapter 4, the computations take on a different flavour. Approximate analytical formula are found for the components of the Fisher metric but these approximations stem from using approximate identities to evaluate convolutions. The resulting expressions are closed form as are the expressions for the components of the Gil Medrano metric and the coefficients of the ODE needed for computing the geodesics.

5.1 Approximate Identities

Approximate identities will be used in Section 5.2.1 to approximate convolutions which arise in computing the components of the Fisher information matrix. The version of approximate identities needed is not quite the standard version but following closely the discussion by Lang [20], are applied in the simplification of these expressions, defined here.

Theorem 5.1.1. *Let h be a function defined on $\mathfrak{R}$ with the following properties.*

1. $\int_{-\infty}^{\infty} h(t)dt = 1,$

2. $\int_{-\infty}^{0} |h(t)|dt < \infty$ *and* $\int_{0}^{\infty} |h(t)|dt < \infty.$

Define h_n by $h_n(t) = nh(nt)$. Let f be a continuous function on $\mathfrak{R}$ and define f_n by $f_n = f \star h_n$. In particular, for large n

$$f_n(x) = \int_{-\infty}^{\infty} f(x-t)h_n(t)dt. \tag{5.7}$$

Then f_n converges uniformly to f on compact subsets of $\mathfrak{R}$.

$$\int_{-\infty}^{\infty} f(x-t)h_n(t)dt \approx \phi(x). \tag{5.8}$$

Claim:

- *For all* n, $\int_{-\infty}^{\infty} h(t)dt = 1$
- *Set* $B = \int_{-\infty}^{\infty} |h(t)|dt$, *then for all* n, $\int_{-\infty}^{\infty} |h_n(t)|dt = B$

Proof: These formulae follow from a simple change of variable $y = nt$ ■

Claim:

For all $\epsilon > 0$ and $\delta > 0$, there exists N such that

$$\int_{-\infty}^{-\delta} |h_n(t)|dt + \int_{\delta}^{\infty} |h_n(t)|dt < \epsilon, \tag{5.9}$$

for all $n \geq N$.

Proof: Let $\epsilon > 0$ and $\delta > 0$. Since $\int_{-\infty}^{\infty} |h(t)|dt < \infty$, there exists η_1 such that $\int_{\eta_1}^{\infty} |h(t)|dt < \frac{\epsilon}{2}$ and similarly, since $\int_{-\infty}^{0} |h(t)|dt < \infty$, there exists η_2 such that $\int_{-\infty}^{\eta_2} |h(t)|dt < \frac{\epsilon}{2}$.

Set $N_1 = \dfrac{\eta_1}{\delta}$. If $n \geq N_1$,

$$\int_{\delta}^{\infty} |h_n(t)|dt = \int_{\delta}^{\infty} |h(nt)|ndt = \int_{n\delta}^{\infty} |h(y)|dy \leq \int_{N_1\delta}^{\infty} |h(y)|dy = \int_{\eta_1}^{\infty} |h(y)|dy < \frac{\epsilon}{2}. \tag{5.10}$$

Similarly, set $N_2 = \frac{\eta_2}{\delta}$. If $n \geq N_2$,

$$\int_{-\infty}^{-\delta} |h_n(t)|dt = \int_{-\infty}^{-\delta} |h(nt)|ndt = \int_{-\infty}^{-n_2\delta} |h(y)|dy \leq \int_{-\infty}^{-N_2\delta} |h(y)|dy = \int_{-\infty}^{-\eta_2} |h(y)|dy < \frac{\epsilon}{2}. \tag{5.11}$$

Hence if $N = \max(N_1, N_2)$, the inequality in the claim holds for $n \geq N$ ■

Proof of Theorem.

Let $\epsilon > 0$ and let S be a compact subset of $\mathfrak{R}$. Since f is continuous on $\mathfrak{R}$, f is uniformly continuous on S and hence, there exists $\delta > 0$ such that

$$|f(x-t) - f(x)| < \frac{\epsilon}{2B}$$

for all $x \in S$ whenever $|t| < \delta$.

Let M be a bound for f and selecting N such

$$\int_{-\infty}^{-\delta} |h_n(t)|dt < \frac{\epsilon}{4M}, \tag{5.12}$$

for $n \geq N$. Since $1 = \int_{-\infty}^{\infty} h_n(t)dt$, $f(x) = f(x)\int_{-\infty}^{\infty} h_n(t)dt = \int_{-\infty}^{\infty} f(x)h_n(t)dt$ and so by the definition of f_n,

$$f_n(x) - f(x) = \int_{-\infty}^{\infty} \left(f(x-t) - f(x)\right) h_n(t)dt, \tag{5.13}$$

Hence

$$|f_n(x) - f(x)| \leq \int_{-\infty}^{\infty} |\left(f(x-t) - f(x)\right)||h_n(t)|dt = I_1 + I_2 + I_3, \tag{5.14}$$

where

$$I_1 = \int_{-\infty}^{-\delta} |\left(f(x-t) - f(x)\right)||h_n(t)|dt \leq 2M \int_{-\infty}^{-\delta} |\left(f(x-t) - f(x)\right)||h_n(t)|dt, \tag{5.15}$$

$$I_2 = \int_{-\delta}^{\delta} |\left(f(x-t) - f(x)\right)||h_n(t)|dt, \tag{5.16}$$

$$I_3 = \int_{\delta}^{\infty} |\left(f(x-t) - f(x)\right)||h_n(t)|dt \leq 2M \int_{\delta}^{\infty} |h_n(t)|dt. \tag{5.17}$$

By equation (5.12), $I_1 + I_3 \leq 2M\frac{\epsilon}{4M} < \frac{\epsilon}{2}$. For I_2, the inequality in equation (5.1) gives

$$I_2 < \int_{-\delta}^{\delta} \frac{\epsilon}{2B}|h_n(t)|dt \leq \frac{\epsilon}{2B}B = \frac{\epsilon}{2}. \tag{5.18}$$

Thus for $n \geq N$, the inequality in (5.14) shows that

$$|f_n(x) - f(x)| \leq I_1 + I_2 + I_3 < \epsilon, \tag{5.19}$$

for all $x \in S$. This proves the theorem.

5.2 The Construction of g

The first step is to compute the components of the Fisher metric given by

$$g_{\tau\tau} = \iint_{\Omega} \theta_{\tau}^2 \; d\tau d\omega, \tag{5.20}$$

$$g_{\omega\omega} = \iint_{\Omega} \theta_{\omega}^2 \; d\tau d\omega, \tag{5.21}$$

$$g_{\tau\omega} = g_{\omega\tau} = \iint_{\Omega} \theta_{\tau}\theta_{\omega} \; d\tau d\omega, \tag{5.22}$$

where $\Omega = \Re^+ \times \Re$. Here θ is the ambiguity function given by $\theta = \phi_0^2$, from equation (5.6)

$$\theta(\tau, \omega) = \mathrm{sinc}^2((\omega - 2b\tau)Q) \exp\left\{-\frac{1}{2}\left(\frac{\tau^2}{T^2} + T^2(\omega - 2b\tau)^2\right)\right\}, \tag{5.23}$$

where $\omega \in (\infty, \infty)$ is the Doppler frequency, $\tau \in [0, \infty)$ is the delay and the configuration parameters T, T_P and b are the pulse duration, pulse period and chirp rate respectively.

The following substitutions are made to facilitate these calculations

$$B = \omega - 2b\tau, \qquad W = \frac{\tau^2}{T^2} + T^2B^2,$$

$$S = \mathrm{sinc}^2\left(QB\right).$$

The partial derivatives are

$$B_\omega = 1, \qquad B_\tau = -2b,$$

$$W_\omega = 2T^2 B, \qquad W_\tau = 2\frac{\tau}{T^2} - 4bT^2 B,$$

$$S_\omega = S_B B_\omega = S_B, \qquad S_\tau = S_B B_\tau = -2bS_B.$$

With these reductions, the ambiguity function becomes

$$\theta = Se^{-\frac{1}{2}W}, \tag{5.24}$$

and the required derivatives are

$$\theta_\omega = e^{-\frac{1}{2}W}\left(S_B - T^2 BS\right), \tag{5.25}$$

$$\begin{aligned} \theta_\tau &= e^{-\frac{1}{2}W}\left(S_\tau - \frac{1}{2}W_\tau S\right), \\ &= -e^{-\frac{1}{2}W}\frac{\tau}{T^2}S - 2be^{-\frac{1}{2}W}\left(S_B - T^2 BS\right). \end{aligned} \tag{5.26}$$

Setting

$$\begin{aligned} U &= e^{-\frac{1}{2}W}\frac{\tau}{T^2}S, \\ V &= 2be^{-\frac{1}{2}W}\left(S_B - T^2 BS\right), \end{aligned}$$

gives

$$\theta_\omega = \frac{1}{2b}V, \qquad \theta_\tau = -U - V.$$

In terms of U and V, the integrals in equaton (5.22) become

$$g_{\tau\tau} = \iint_\Omega U^2 + 2UV + V^2 \; d\tau d\omega, \tag{5.27}$$

$$g_{\omega\omega} = \frac{1}{4b^2}\iint_\Omega V^2 \; d\tau d\omega, \tag{5.28}$$

$$g_{\tau\omega} = -\frac{1}{2b}\iint_\Omega UV + V^2 \; d\tau d\omega, \tag{5.29}$$

Lemma

For small values of T and large values of T_P,

$$\iint_\Omega U^2 \; d\tau d\omega \approx \frac{\pi^{3/2}}{6QT}, \tag{5.30}$$

$$\iint_\Omega UV \; d\tau d\omega \approx 0, \tag{5.31}$$

$$\iint_\Omega V^2 \; d\tau d\omega \approx \pi^{3/2} b^2 QTZ, \tag{5.32}$$

where

$$Z = \left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}.$$

The proofs of equations (5.30) and (5.32) rely on viewing the integrands as a convolutions and approximating the convolutions by recognising one factor in each integrand as an approximate identity. Equation (5.31) holds because the inner integral produces an odd function of ω. Details are presented in Subsection 5.2.1.

Combining equations (5.27) - (5.32), the Fisher metric components are

$$g_{\tau\tau} \approx \frac{\pi^{3/2}}{6QT} + \pi^{3/2} b^2 QTZ, \tag{5.33}$$

$$g_{\omega\omega} \approx \frac{\pi^{3/2}}{4} QTZ, \tag{5.34}$$

$$g_{\tau\omega} \approx -\frac{\pi^{3/2}}{2} bQTZ. \tag{5.35}$$

Assembling the Fisher matrix from these elements results in the matrix

$$g = \begin{bmatrix} \frac{\pi^{\frac{3}{2}}}{6QT} + \pi^{\frac{3}{2}} b^2 TQ\left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right] & -\frac{\pi^{\frac{3}{2}} bTQ}{2}\left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right] \\ -\frac{\pi^{\frac{3}{2}} bTQ}{2}\left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right] & \frac{\pi^{\frac{3}{2}} TQ}{4}\left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right] \end{bmatrix}. \tag{5.36}$$

5.2.1 Proof of the Lemma

Writing $B = 2b(s-\tau)$, where $s = \omega/2b$, facilitates recognising the integrals with respect to τ as convolutions.

Proof of Equation (5.30).

$$\begin{aligned} U^2 &= e^{-W}\frac{1}{T^2}\left(\frac{\tau}{T}\right)^2 S^2, \\ &= e^{-\left(\left(\frac{\tau}{T}\right)^2 + T^2B^2\right)}\left(\frac{\tau}{T}\right)^2 \operatorname{sinc}^4(QB), \\ &= e^{-\left(\frac{\tau}{T}\right)^2}\left(\frac{\tau}{T}\right)^2 \times \\ &\quad \frac{1}{T^2} e^{-4b^2T^2(s-\tau)^2}\left(\frac{\sin((s-\tau)2bQ)}{(s-\tau)2bQ}\right)^4, \\ &= h_{\frac{1}{T}}(\tau)\phi(s-\tau). \end{aligned} \tag{5.37}$$

Here $h_{\frac{1}{T}}(x) = \frac{1}{T}h\left(\frac{x}{T}\right)$, $h(x) = \frac{4}{\sqrt{\pi}}x^2e^{-x^2}$ and

$$\phi(x) = \frac{\sqrt{\pi}T}{4}\frac{1}{T^2}e^{-T^2(2bx)^2}\left(\frac{\sin 2bQx}{2bQx}\right)^4. \tag{5.38}$$

The constant in the definition of h is chosen so that $\int_0^\infty h(x)\ dx = 1$. This means that $h_{\frac{1}{T}}$ acts as an approximate identity for convolution. In particular, $\lim_{T\downarrow 0} h_{\frac{1}{T}} \star f = f$ uniformly on compact subsets for piecewise continuous functions f [20][31]. Hence

$$\begin{aligned}\int_0^\infty U^2\ d\tau &= \int_0^\infty h_{\frac{1}{T}}(\tau)\phi(s-\tau)\ d\tau, \\ &= h_{\frac{1}{T}} \star \phi(s) \approx \phi(s), = \phi(\omega/2b) \\ &= \frac{\sqrt{\pi}}{4T} e^{-T^2\omega^2} \left(\frac{\sin Q\omega}{Q\omega}\right)^4. \end{aligned} \tag{5.39}$$

To integrate this expression with respect to ω, the right side of equation (5.39) is

$$h_Q(x)\phi(x),$$

where $h_Q(x) = Qh(Qx)$, $h(x) = \frac{3}{2\pi}\left(\frac{\sin x}{x}\right)^4$ and $\phi(x) = \frac{\pi^{3/2}}{6QT} e^{-T^2\omega}$. With this definition $\int_{-\infty}^\infty h(x)\ dx = 1$ and h_Q is an approximate identity. Thus

$$\begin{aligned}\int_{-\infty}^\infty \int_0^\infty U^2\ d\tau d\omega &= h_Q \star \check{\phi}(0) \approx \check{\phi}(0) \\ &= \frac{\pi^{3/2}}{6QT}. \end{aligned} \tag{5.40}$$

Here $\check{\phi}(x) = \phi(-x)$. This proves (5.30).

Proof of Equation (5.31).

First note that, viewed as the function

$$S = S(B) = \left(\frac{\sin QB}{QB}\right)^2, \tag{5.41}$$

S is even and so the derivative, S', is odd.

The integrand in equation (5.31) is

$$\begin{aligned} UV &= 2be^{-W}\frac{\tau}{T^2} S(S' - T^2BS), \\ &= \frac{1}{T} e^{-\left(\frac{\tau}{T}\right)^2} \frac{\tau}{T} 2be^{-T^2B^2} S(S' - T^2BS), \\ &= h_{\frac{1}{T}}(\tau)\phi(s-\tau), \end{aligned} \tag{5.42}$$

where $h_{\frac{1}{T}}(x) = \frac{1}{T}h(x/T)$, $h(x) = \frac{1}{2}xe^{-x^2}$ and

$$\begin{aligned}\phi(x) &= 4be^{-4b^2T^2x^2} S(2bx) \times \\ &\quad \left(S'(2bx) - 2bxT^2 S(2bx)\right). \end{aligned} \tag{5.43}$$

Again $\int_0^\infty h(x)\ dx = 1$ and $h_{\frac{1}{T}}$ acts as an approximate identity. Thus

$$\begin{aligned}\int_0^\infty UV\ d\tau &= h_{\frac{1}{T}} \star \phi(s-\tau), \\ &\approx \phi(s) = \phi(\omega/2b).\end{aligned}$$

Since S is even and S' is odd, ϕ defined in equation (5.43) is an odd function and so

$$\int_{-\infty}^{\infty}\int_0^\infty UV\ d\tau d\omega \approx \int_{-\infty}^{\infty} \phi(\omega/2b)\ d\omega = 0. \tag{5.44}$$

This proves equation (5.31).

Proof of Equation (5.32).

$$\begin{aligned}\frac{1}{4b^2}V^2 &= e^{-W}\left(S' - T^2BS\right)^2, \\ &= \frac{2}{\sqrt{\pi}T}e^{-\left(\frac{\tau}{T}\right)^2}\frac{\sqrt{\pi}T}{2}e^{-T^2B^2}(S' - T^2BS)^2, \\ &= h_{\frac{1}{T}}(\tau)\phi(s-\tau),\end{aligned}$$

where $h_{\frac{1}{T}}(x) = \frac{1}{T}h(x/T)$, $h(x) = \frac{2}{\sqrt{\pi}T}e^{-x^2}$ and

$$\begin{aligned}\phi(x) &= \frac{\sqrt{\pi}T}{2}e^{-4b^2T^2x^2} \times \\ &\quad \left(S'(2bx) - 2bT^2xS(2bx)\right)^2.\end{aligned} \tag{5.45}$$

Thus

$$\begin{aligned}\frac{1}{4b^2}\int_0^\infty V^2\ d\tau &= h_{\frac{1}{T}} \star \phi(s), \\ &\approx \phi(s) = \phi(\omega/2b), \\ &= \frac{\sqrt{\pi}T}{2}e^{-T^2\omega^2} \times \\ &\quad \left(S'(\omega) - T^2\omega S(\omega)\right)^2.\end{aligned} \tag{5.46}$$

To progress from equation (5.46) by steps analogous to those leading from (5.39) to (5.40) requires setting

$$h(x) = \frac{1}{K}\left(S'(x/Q) - \frac{T^2x}{Q}S(x/Q)\right)^2, \tag{5.47}$$

where

$$K = \int_{-\infty}^{\infty}\left(S'(x/Q) - \frac{T^2x}{Q}S(x/Q)\right)^2\ dx. \tag{5.48}$$

Then, analogous to the proof of Equation (5.30), set $h_Q(x) = Qh(Qx)$ and $\phi(x) = \frac{\sqrt{\pi}KT}{2Q}e^{-T^2x^2}$ so that integrating (5.46) over ω gives

$$\begin{aligned}\frac{1}{4b^2}\int_{-\infty}^{\infty}\int_0^\infty V^2\ d\tau d\omega &= h_Q \star \check{\phi}(0) \approx \check{\phi}(0), \\ &= \frac{\sqrt{\pi}KT}{2Q}.\end{aligned} \tag{5.49}$$

Thus the proof of (5.32) is complete if

$$K = \frac{\pi Q^2}{2}\left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right]. \tag{5.50}$$

5.2.2 Proof of Equation (5.50)

Expanding the integrand in (5.48) and noting that

$$S\left(\frac{x}{Q}\right) = \left(\frac{\sin x}{x}\right), \tag{5.51}$$

and

$$S'\left(\frac{x}{Q}\right) = 2Q\frac{\sin x}{x}\left(\frac{x\cos x - \sin x}{x^2}\right), \tag{5.52}$$

shows that

$$K = \frac{T^4}{Q^2}I_1 - 4T^2 I_2 + 4Q^2 I_3, \tag{5.53}$$

where

$$I_1 = \int_{-\infty}^{\infty} x^2\left(\frac{\sin x}{x}\right)^4 dx, \tag{5.54}$$

$$I_2 = \int_{-\infty}^{\infty} x\frac{\sin^3 x}{x^3}\left(\frac{x\cos x - \sin x}{x^2}\right) dx, \tag{5.55}$$

$$I_3 = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\left(\frac{x\cos x - \sin x}{x^2}\right)^2 dx. \tag{5.56}$$

The following formulae [22] will be used to evaluate (5.54) - (5.56).

$$\int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}, \tag{5.57}$$

$$\int_0^{\infty} \frac{\sin^4 x}{x^2}\, dx = \frac{\pi}{4}, \tag{5.58}$$

$$\int_0^{\infty} \frac{\sin^4 x}{x^4}\, dx = \frac{\pi}{3}, \tag{5.59}$$

$$\int_0^{\infty} \frac{\sin^3 x\cos x}{x^3}\, dx = \frac{\pi}{4}. \tag{5.60}$$

From (5.58)

$$I_1 = \frac{\pi}{2}. \tag{5.61}$$

From (5.59) and (5.60)

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} \frac{\sin^3 x\cos x}{x^3} - \frac{\sin^4 x}{x^4}\, dx, \\ &= \frac{\pi}{2} - \frac{2\pi}{3} = -\frac{\pi}{6}. \end{aligned} \tag{5.62}$$

Expanding (5.56), writing $\cos^2 x = 1 - \sin^2 x$ (as will be done throughout) and using (5.59),

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} -\frac{\sin^4 x}{x^4}\, dx + J_1, \\ &= -\frac{2\pi}{3} + J_1, \end{aligned} \tag{5.63}$$

where

$$J_1 = \int_{-\infty}^{\infty} \frac{x^2 \sin^2 x - 2x \sin^3 x \cos x + \sin^4 x}{x^6}\, dx. \tag{5.64}$$

Integrating J_1 by parts (integrating x^{-6}) and using (5.59) gives

$$\begin{aligned} J_1 &= \frac{1}{5}\left(8 \int_{-\infty}^{\infty} \frac{\sin^4 x}{x^4}\, dx + J_2\right), \\ &= \frac{16\pi}{15} + \frac{1}{5} J_2, \end{aligned} \tag{5.65}$$

where

$$\begin{aligned} J_2 &= \int_{-\infty}^{\infty} \frac{2x^2 \sin x \cos x - 4x \sin^2 x}{x^5} + \\ &\qquad \frac{2 \sin^3 x \cos x}{x^5}\, dx. \end{aligned} \tag{5.66}$$

Integrating J_2 by parts (integrating x^{-5}) and using (5.57) and (5.59),

$$\begin{aligned} J_2 &= \frac{1}{4}\left(\int_{-\infty}^{\infty} -\frac{\sin^2 x}{x^2} - \frac{\sin^4 x}{x^4}\, dx + J_3\right), \\ &= -\frac{7\pi}{3} + \frac{1}{4} J_3, \end{aligned} \tag{5.67}$$

where

$$J_3 = \int_{-\infty}^{\infty} \frac{2x^2 - 4x \sin x \cos x + 2 \sin^2 x}{x^4}\, dx. \tag{5.68}$$

Integrating J_3 by parts (integrating x^{-4}) and using (5.57) gives

$$J_3 = \frac{1}{3} \int_{-\infty}^{\infty} \frac{8x \sin^2 x}{x^3}\, dx = \frac{8\pi}{3}. \tag{5.69}$$

Combining (5.63), (5.65), (5.67) and (5.69) shows that

$$I_3 = -\frac{2\pi}{3} + \frac{16\pi}{15} + \frac{1}{5}\left(-\frac{7\pi}{3} + \frac{1}{4}\frac{8\pi}{3}\right) = \frac{\pi}{15}. \tag{5.70}$$

Combining (5.53), (5.61), (5.62) and (5.70) confirms (5.50) and completes the proof of the claim. This method of construction enables the configuration metric G_{ij}.

5.3 Construction of the Configuration Matrix G

In Chapter 5 the Fisher metric for a single Doppler bin was calculated and following this process for a specific sensor configuration produces the metric associated with that configuration. However, in order to select a configuration that maximises the information obtained from the target, a family of such configurations is needed. In Chapter 2.29 the metric for the manifold of metrics described by Gil Medrano and Michor [13] is discussed.

$$G_{ij} = \int_M \mathrm{tr}\left(g^{-1}\partial_i g g^{-1}\partial_j g\right) \mathrm{vol}(g). \tag{5.71}$$

In this section equation (5.71) used to calculate the metric for a manifold composed of the family of Fisher metrics corresponding to sensor configurations, called the configuration manifold.

5.3.1 The Inverse of the Fisher Matrix

The Fisher information matrix previously stated in equation (5.36), is reiterated here

$$g = \begin{bmatrix} \frac{\pi^{\frac{3}{2}}}{6QT} + \pi^{\frac{3}{2}} b^2 TQ \left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right] & -\frac{\pi^{\frac{3}{2}} bTQ}{2}\left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right] \\ -\frac{\pi^{\frac{3}{2}} bTQ}{2}\left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right] & \frac{\pi^{\frac{3}{2}} TQ}{4}\left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right] \end{bmatrix}. \tag{5.72}$$

The determinant of $g(\tau, \omega)$ is,

$$\det(g) = \frac{\pi^3}{24}\left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right]. \tag{5.73}$$

The inverse of $g(\tau, \omega)$ is required for calculation of the configuration metric

$$g^{-1} = \frac{1}{\det(g)} \begin{bmatrix} \frac{\pi^{\frac{3}{2}} TQ}{4}\left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right] & \frac{\pi^{\frac{3}{2}} bTQ}{2}\left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right] \\ \frac{\pi^{\frac{3}{2}} bTQ}{2}\left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right] & \frac{\pi^{\frac{3}{2}}}{6QT} + \pi^{\frac{3}{2}} b^2 TQ \left[\left(\frac{T}{Q}\right)^4 + \frac{4}{3}\left(\frac{T}{Q}\right)^2 + \frac{8}{15}\right] \end{bmatrix}. \tag{5.74}$$

The inverse of the Fisher matrix will be used in combination with matrices formed from the partial derivatives of g to construct the Configuration metric.

5.3.2 Partial Derivatives $\partial_{T,Q,b} g$

Following are the partial derivatives of g, equation (5.72), with respect to the configuration parameters $T,\ T_P,\ b$.

$$\partial_T g_{11} = \pi^{3/2} b^2 Q \left(\frac{4T^4}{Q^4} + \frac{8T^2}{3Q^2} \right) + \pi^{3/2} b^2 Q \left(\frac{T^4}{Q^4} + \frac{4T^2}{3Q^2} + \frac{8}{15} \right) - \frac{\pi^{3/2}}{6T^2 Q}, \tag{5.75}$$

$$\partial_Q g_{11} = -\pi^{3/2} b^2 T \left(\frac{4T^4}{Q^4} + \frac{8T^2}{3Q^2} \right) + \pi^{3/2} b^2 T \left(\frac{T^4}{Q^4} + \frac{4T^2}{3Q^2} + \frac{8}{15} \right) - \frac{\pi^{3/2}}{6TQ^2}, \tag{5.76}$$

$$\partial_b g_{11} = 2\pi^{3/2} bTQ \left(\frac{T^4}{Q^4} + \frac{4T^2}{3Q^2} + \frac{8}{15} \right), \tag{5.77}$$

$$\partial_T g_{22} = \frac{1}{4} \pi^{3/2} Q \left(\frac{4T^4}{Q^4} + \frac{8T^2}{3Q^2} \right) + \frac{1}{4} \pi^{3/2} Q \left(\frac{T^4}{Q^4} + \frac{4T^2}{3Q^2} + \frac{8}{15} \right), \tag{5.78}$$

$$\partial_Q g_{22} = -\frac{1}{4} \pi^{3/2} T \left(\frac{4T^4}{Q^4} + \frac{8T^2}{3Q^2} \right) + \frac{1}{4} \pi^{3/2} T \left(\frac{T^4}{Q^4} + \frac{4T^2}{3Q^2} + \frac{8}{15} \right), \tag{5.79}$$

$$\partial_b g_{22} = 0, \tag{5.80}$$

$$\partial_T g_{12} = -\frac{1}{2} \pi^{3/2} bQ \left(\frac{4T^4}{Q^4} + \frac{8T^2}{3Q^2} \right) - \frac{1}{2} \pi^{3/2} bQ \left(\frac{T^4}{Q^4} + \frac{4T^2}{3Q^2} + \frac{8}{15} \right), \tag{5.81}$$

$$\partial_Q g_{12} = \frac{1}{2} \pi^{3/2} bT \left(\frac{4T^4}{Q^4} + \frac{8T^2}{3Q^2} \right) - \frac{1}{2} \pi^{3/2} bT \left(\frac{T^4}{Q^4} + \frac{4T^2}{3Q^2} + \frac{8}{15} \right), \tag{5.82}$$

$$\partial_b g_{12} = -\frac{1}{2} \pi^{3/2} TQ \left(\frac{T^4}{Q^4} + \frac{4T^2}{3Q^2} + \frac{8}{15} \right). \tag{5.83}$$

The following substitutions are made

$$\begin{aligned} Z &= \left(\frac{T^4}{Q^4} + \frac{4T^2}{3Q^2} + \frac{8}{15} \right), \\ F &= -3 \left(\frac{T}{Q} \right)^4 - \frac{4}{3} \left(\frac{T}{Q} \right)^2 + \frac{8}{15}, \\ D &= (2Z - F). \end{aligned} \tag{5.84}$$

With these substitutions, the partial derivatives are,

$$\partial_T g_{11} = \pi^{3/2} b^2 Q (2Z - F) - \frac{\pi^{3/2}}{6T^2 Q} = \pi^{3/2} b^2 Q D - \frac{\pi^{3/2}}{6T^2 Q}, \tag{5.85}$$

$$\partial_Q g_{11} = \pi^{3/2} b^2 T\left(F\right) - \frac{\pi^{3/2}}{6T^2 Q} \;= \pi^{3/2} b^2 T F - \frac{\pi^{3/2}}{6T^2 Q}, \tag{5.86}$$

$$\partial_b g_{11} = 2\pi^{3/2} b T Q Z, \tag{5.87}$$

$$\partial_T g_{22} = \frac{1}{4}\pi^{3/2} Q\left(2Z - F\right) \;= \frac{1}{4}\pi^{3/2} Q D, \tag{5.88}$$

$$\partial_Q g_{22} = -\frac{1}{4}\pi^{3/2} T\left(-F\right) \;= \frac{1}{4}\pi^{3/2} T F, \tag{5.89}$$

$$\partial_b g_{22} = 0, \tag{5.90}$$

$$\partial_T g_{12} = -\frac{1}{2}\pi^{3/2} b Q\left(2Z - F\right) \;= -\frac{1}{2}\pi^{3/2} b Q D, \tag{5.91}$$

$$\partial_Q g_{12} = \frac{1}{2}\pi^{3/2} b T\left(-F\right) \;= -\frac{1}{2}\pi^{3/2} b T F, \tag{5.92}$$

$$\partial_b g_{12} = -\frac{1}{2}\pi^{3/2} T Q Z. \tag{5.93}$$

This results in the partial derivatives and the g^{-1} matrix being simplified in the following manner.

$$g^{-1} = \begin{bmatrix} \frac{6TQ}{\pi^{\frac{3}{2}}} & \frac{12bTQ}{\pi^{\frac{3}{2}}} \\ \\ \frac{12bTQ}{\pi^{\frac{3}{2}}} & \frac{4\left(6b^2Q^2T^2Z+1\right)}{\pi^{3/2}QTZ} \end{bmatrix}. \tag{5.94}$$

Each of the $g^{-1}\partial_i g$ factors required for the Gil Medrano metric in equation (5.71) are calculated, this involves multiplying g^{-1} with each of the other constructed matrices as follows.

$$g^{-1}\partial_T g = g^{-1}\begin{bmatrix} \partial_T g_{11} & \partial_T g_{12} \\ \partial_T g_{21} & \partial_T g_{22} \end{bmatrix} \tag{5.95}$$

$$g^{-1}\partial_Q g = g^{-1}\begin{bmatrix} \partial_Q g_{11} & \partial_Q g_{12} \\ \partial_Q g_{21} & \partial_Q g_{22} \end{bmatrix} \tag{5.96}$$

$$g^{-1}\partial_b g = g^{-1}\begin{bmatrix}\partial_b g_{11} & \partial_b g_{12}\\ \partial_b g_{21} & \partial_b g_{22}\end{bmatrix} \tag{5.97}$$

The following matrix multiplications are required.

$$g^{-1}\partial_T g = \begin{bmatrix} g_{11}^{-1}\partial_T g_{11} + g_{12}^{-1}\partial_T g_{21} & g_{11}^{-1}\partial_T g_{12} + g_{12}^{-1}\partial_T g_{22}\\ g_{21}^{-1}\partial_T g_{11} + g_{22}^{-1}\partial_T g_{21} & g_{21}^{-1}\partial_T g_{12} + g_{22}^{-1}\partial_T g_{22}\end{bmatrix}, \tag{5.98}$$

$$g^{-1}\partial_Q g = \begin{bmatrix} g_{11}^{-1}\partial_Q g_{11} + g_{1,2}^{-1}\partial_Q g_{21} & g_{11}^{-1}\partial_Q g_{12} + g_{12}^{-1}\partial_Q g_{22}\\ g_{21}^{-1}\partial_Q g_{11} + g_{22}^{-1}\partial_Q g_{21} & g_{21}^{-1}\partial_Q g_{12} + g_{22}^{-1}\partial_Q g_{22}\end{bmatrix}, \tag{5.99}$$

$$g^{-1}\partial_b g = \begin{bmatrix} g_{11}^{-1}\partial_b g_{11} + g_{12}^{-1}\partial_b g_{21} & g_{11}^{-1}\partial_b g_{12}\\ g_{21}^{-1}\partial_b g_{11} + g_{22}^{-1}\partial_b g_{21} & g_{21}^{-1}\partial_b g_{12}\end{bmatrix}, \tag{5.100}$$

Written out in full, the product $g^{-1}\partial_T g$ is

$$g^{-1}\partial_T g = \begin{bmatrix} \frac{6TQ}{\pi^{\frac{3}{2}}} & \frac{12bTQ}{\pi^{\frac{3}{2}}}\\ \frac{12bTQ}{\pi^{\frac{3}{2}}} & \frac{4\left(6b^2Q^2T^2Z+1\right)}{\pi^{3/2}QTZ}\end{bmatrix}\begin{bmatrix} \pi^{\frac{3}{2}}b^2QD - \frac{\pi^{\frac{3}{2}}}{6T^2Q} & -\frac{\pi^{\frac{3}{2}}bQD}{2}\\ -\frac{\pi^{\frac{3}{2}}bQD}{2} & \frac{\pi^{\frac{3}{2}}QD}{4}\end{bmatrix}, \tag{5.101}$$

$$= \begin{bmatrix} 6b^2Q^2TD - \frac{1}{T} - 6b^2Q^2TD & -3bTQ^2D + 3bTQ^2D\\ 12b^3TQ^2D - \frac{2b}{T} - \frac{2bD\left(6b^2T^2Q^2Z+1\right)}{TZ} & -6b^2TQ^2D + \frac{D\left(6b^2T^2Q^2Z+1\right)}{TZ}\end{bmatrix}, \tag{5.102}$$

$$= \begin{bmatrix} -\frac{1}{T} & 0\\ -\frac{2b\left(Z+D\right)}{TZ} & \frac{D}{TZ}\end{bmatrix}. \tag{5.103}$$

Thus,

$$g^{-1}\partial_T g g^{-1}\partial_T g = \begin{bmatrix} -\frac{1}{T} & 0 \\ -\frac{2b(Z+D)}{TZ} & \frac{D}{TZ} \end{bmatrix} \begin{bmatrix} -\frac{1}{T} & 0 \\ -\frac{2b(Z+D)}{TZ} & \frac{D}{TZ} \end{bmatrix}, \tag{5.104}$$

$$= \begin{bmatrix} \frac{1}{T^2} & 0 \\ \frac{2b(Z+D)}{T^2 Z} - \frac{2bD(Z+D)}{T^2 Z^2} & \frac{D^2}{T^2 Z^2} \end{bmatrix}. \tag{5.105}$$

The trace of this matrix is,

$$\operatorname{tr}\left(g^{-1}\partial_T g g^{-1}\partial_T g\right) = \frac{1}{T^2} + \frac{D^2}{T^2 Z^2} = \frac{Z^2 + D^2}{T^2 Z^2}. \tag{5.106}$$

All other matrices will have the same structure, as such only the main results are given here.

$$g^{-1}\partial_Q g = \begin{bmatrix} \frac{6TQ}{\pi^{\frac{3}{2}}} & \frac{12bTQ}{\pi^{\frac{3}{2}}} \\ \frac{12bTQ}{\pi^{\frac{3}{2}}} & \frac{4\left(6b^2Q^2T^2Z+1\right)}{\pi^{3/2}QTZ} \end{bmatrix} \begin{bmatrix} \pi^{\frac{3}{2}} b^2 TF - \frac{\pi^{\frac{3}{2}}}{6T^2 Q} & -\frac{1}{2}\pi^{\frac{3}{2}} bTF \\ -\frac{1}{2}\pi^{\frac{3}{2}} bTF & \frac{1}{4}\pi^{\frac{3}{2}} TF \end{bmatrix},$$

$$= \begin{bmatrix} -\frac{1}{T} & 0 \\ \frac{-2b(QZ+TF)}{TQZ} & \frac{F}{QZ} \end{bmatrix}. \tag{5.107}$$

Allowing the calculation of the $g^{-1}\partial_Q g g^{-1}\partial_Q g$ matrix as follows,

$$g^{-1}\partial_Q g g^{-1}\partial_Q g = \begin{bmatrix} -\frac{1}{T} & 0 \\ \frac{-2b(QZ+TF)}{TQZ} & \frac{F}{QZ} \end{bmatrix} \begin{bmatrix} -\frac{1}{T} & 0 \\ \frac{-2b(QZ+TF)}{TQZ} & \frac{F}{QZ} \end{bmatrix}, \tag{5.108}$$

$$= \begin{bmatrix} \frac{1}{T^2} & 0 \\ \frac{2b\left(Q^2Z^2 - T^2F^2\right)}{T^2Q^2Z^2} & \frac{F^2}{Q^2Z^2} \end{bmatrix}, \tag{5.109}$$

where the trace of this matrix is,

$$\operatorname{tr}\left(g^{-1}\partial_Q g g^{-1}\partial_Q g\right) = \frac{1}{T^2} + \frac{F^2}{Q^2 Z^2} = \frac{Q^2Z^2 + T^2F^2}{(TQZ)^2}. \tag{5.110}$$

Next is the $\partial_b g$ calculation,

$$\partial_b g = \begin{bmatrix} 2\pi^{\frac{3}{2}} bTQZ & -\frac{1}{2}\pi^{\frac{3}{2}} TQZ \\ \\ -\frac{1}{2}\pi^{\frac{3}{2}} TQZ & 0 \end{bmatrix},$$

The matrix $g^{-1}\partial_b g$ is,

$$g^{-1}\partial_b g = \begin{bmatrix} \frac{6TQ}{\pi^{\frac{3}{2}}} & \frac{12bTQ}{\pi^{\frac{3}{2}}} \\ \\ \frac{12bTQ}{\pi^{\frac{3}{2}}} & \frac{4\left(6b^2Q^2T^2Z+1\right)}{\pi^{3/2}QTZ} \end{bmatrix} \begin{bmatrix} 2\pi^{\frac{3}{2}} bTQZ & -\frac{1}{2}\pi^{\frac{3}{2}} TQZ \\ \\ -\frac{1}{2}\pi^{\frac{3}{2}} TQZ & 0 \end{bmatrix}, \tag{5.111}$$

$$= \begin{bmatrix} 6bT^2Q^2Z & -3T^2Q^2Z \\ \\ 12b^2T^2Q^2Z - 2 & -6bT^2Q^2Z \end{bmatrix}. \tag{5.112}$$

This enables the calculation of the matrix $g^{-1}\partial_b g g^{-1}\partial_b g$,

$$g^{-1}\partial_b g g^{-1}\partial_b g = \begin{bmatrix} 6bT^2Q^2Z & -3T^2Q^2Z \\ \\ 12b^2T^2Q^2Z - 2 & -6bT^2Q^2Z \end{bmatrix} \begin{bmatrix} 6bT^2Q^2Z & -3T^2Q^2Z \\ \\ 12b^2T^2Q^2Z - 2 & -6bT^2Q^2Z \end{bmatrix}, \tag{5.113}$$

$$= \begin{bmatrix} 6Q^2T^2Z & 0 \\ \\ 0 & 6Q^2T^2Z \end{bmatrix}. \tag{5.114}$$

The trace of this matrix is,

$$\mathrm{tr}\left(g^{-1}\partial_b g g^{-1}\partial_b g\right) = 12Q^2T^2Z \tag{5.115}$$

The following are the three cross terms, beginning with the $g^{-1}\partial_T g g^{-1}\partial_Q g$ matrix,

$$g^{-1}\partial_T g g^{-1}\partial_Q g = \begin{bmatrix} -\frac{1}{T} & 0 \\ \\ -\frac{2b\left(Z+D\right)}{TZ} & \frac{D}{TZ} \end{bmatrix} \begin{bmatrix} -\frac{1}{T} & 0 \\ \\ \frac{-2b\left(QZ+TF\right)}{TQZ} & \frac{F}{QZ} \end{bmatrix}, \tag{5.116}$$

$$\begin{bmatrix} \frac{1}{T^2} & 0 \\ \\ \frac{2b\left(QZ^2 - TDF\right)}{T^2QZ^2} & \frac{DF}{QTZ^2} \end{bmatrix}. \tag{5.117}$$

The trace of this matrix is,

$$\mathrm{tr}\left(g^{-1}\partial_T g g^{-1}\partial_Q g\right) = \frac{1}{T^2} + \frac{DF}{QTZ^2} = \frac{QZ^2 + DFT}{QT^2Z^2} \tag{5.118}$$

Next is the matrix $g^{-1}\partial_T g g^{-1}\partial_b g$,

$$g^{-1}\partial_T g g^{-1}\partial_b g = \begin{bmatrix} -\frac{1}{T} & 0 \\ -\frac{2b\left(Z+D\right)}{TZ} & \frac{D}{TZ} \end{bmatrix} \begin{bmatrix} 6bT^2Q^2Z & -3T^2Q^2Z \\ 12b^2T^2Q^2Z - 2 & -6bT^2Q^2Z \end{bmatrix}, \tag{5.119}$$

$$= \begin{bmatrix} -6bTQ^2Z & -3TQ^2Z \\ -2\left(6b^2TQ^2Z + \frac{D}{TZ}\right) & 6bTQ^2Z \end{bmatrix}. \tag{5.120}$$

The trace of this matrix is,

$$\mathrm{tr}\left(g^{-1}\partial_T g g^{-1}\partial_b g\right) = -6bTQ^2Z + 6bTQ^2Z = 0. \tag{5.121}$$

Finally the $g^{-1}\partial_Q g g^{-1}\partial_b g$ matrix is calculated,

$$g^{-1}\partial_Q g g^{-1}\partial_b g = \begin{bmatrix} -\frac{1}{T} & 0 \\ \frac{-2b\left(QZ+TF\right)}{TQZ} & \frac{F}{QZ} \end{bmatrix} \begin{bmatrix} 6bT^2Q^2Z & -3T^2Q^2Z \\ 12b^2T^2Q^2Z - 2 & -6bT^2Q^2Z \end{bmatrix}, \tag{5.122}$$

$$= \begin{bmatrix} -6bTQ^2Z & 3TQ^2Z \\ -2\left(6b^2TQ^2Z - \frac{F}{QZ}\right) & 6bTQ^2Z \end{bmatrix}. \tag{5.123}$$

The trace of this matrix is given by,

$$\mathrm{tr}\left(g^{-1}\partial_Q g g^{-1}\partial_b g\right) = -6bTQ^2Z + 6bTQ^2Z = 0. \tag{5.124}$$

For convenience all the traces are here reiterated,

$$\operatorname{tr}\left(g^{-1}\partial_T g g^{-1}\partial_T g\right) = \frac{Z^2 + D^2}{T^2 Z^2}, \tag{5.125}$$

$$\operatorname{tr}\left(g^{-1}\partial_Q g g^{-1}\partial_Q g\right) = \frac{Q^2 Z^2 + T^2 F^2}{(TQZ)^2}, \tag{5.126}$$

$$\operatorname{tr}\left(g^{-1}\partial_b g g^{-1}\partial_b g\right) = 12Q^2T^2Z, \tag{5.127}$$

$$\operatorname{tr}\left(g^{-1}\partial_T g g^{-1}\partial_Q g\right) = \frac{QZ^2 + DFT}{QT^2 Z^2}, \tag{5.128}$$

$$\operatorname{tr}\left(g^{-1}\partial_T g g^{-1}\partial_b g\right) = 0, \tag{5.129}$$

$$\operatorname{tr}\left(g^{-1}\partial_Q g g^{-1}\partial_b g\right) = 0. \tag{5.130}$$

5.3.3 The Configuration Metric

In Section 5.3.2 the traces required for the construction of G were calculated. These traces are inserted into the integrals for the elements of the G matrix,

$$G = \begin{bmatrix} \int_M \frac{Z^2 + D^2}{T^2 Z^2}\mathrm{vol}(g) & \int_M \frac{QZ^2 + DFT}{QT^2 Z^2}\mathrm{vol}(g) & 0 \\ \int_M \frac{QZ^2 + DFT}{QT^2 Z^2}\mathrm{vol}(g) & \int_M \frac{Q^2 Z^2 + T^2 F^2}{(TQZ)^2}\mathrm{vol}(g) & 0 \\ 0 & 0 & \int_M 12Q^2T^2Z\mathrm{vol}(g) \end{bmatrix}. \tag{5.131}$$

Each element of this matrix is calculated, first noting the vol(g) factor

$$\mathrm{vol}(g) = \frac{\pi^{\frac{3}{2}}}{6}\left(\frac{3}{2}\left(\frac{T}{Q}\right)^4 + 2\left(\frac{T}{Q}\right)^2 + \frac{4}{5}\right)^{\frac{1}{2}}. \tag{5.132}$$

Beginning with the integral for the G_{11} element,

$$\int_{-\frac{bT}{2}}^{\frac{bT}{2}}\int_0^Q \operatorname{tr}\left(g^{-1}\partial_T g g^{-1}\partial_T g\right)\mathrm{vol}(g)d\tau d\omega \tag{5.133}$$

$$\tag{5.134}$$

$$= \int_{-\frac{bT}{2}}^{\frac{bT}{2}} \int_0^Q \frac{Z^2 + D^2}{T^2 Z^2} \frac{\pi^{\frac{3}{2}}}{6} \left(\frac{3}{2} \left(\frac{T}{Q} \right)^4 + 2 \left(\frac{T}{Q} \right)^2 + \frac{4}{5} \right)^{\frac{1}{2}} d\tau d\omega, \tag{5.135}$$

$$= \frac{\pi^{\frac{3}{2}}}{6} \left(\frac{Z^2 + D^2}{T^2 Z^2} \right) \left(\frac{3}{2} \left(\frac{T}{Q} \right)^4 + 2 \left(\frac{T}{Q} \right)^2 + \frac{4}{5} \right)^{\frac{1}{2}} \int_{-\frac{bT}{2}}^{\frac{bT}{2}} \int_0^Q d\tau d\omega, \tag{5.136}$$

$$= \frac{\pi^{\frac{3}{2}}}{6} \left(\frac{Z^2 + D^2}{T^2 Z^2} \right) \left(\frac{3}{2} \left(\frac{T}{Q} \right)^4 + 2 \left(\frac{T}{Q} \right)^2 + \frac{4}{5} \right)^{\frac{1}{2}} \int_{-\frac{bT}{2}}^{\frac{bT}{2}} Q d\omega, \tag{5.137}$$

$$= QbT \frac{\pi^{\frac{3}{2}}}{6} \left(\frac{Z^2 + D^2}{T^2 Z^2} \right) \left(\frac{3}{2} \left(\frac{T}{Q} \right)^4 + 2 \left(\frac{T}{Q} \right)^2 + \frac{4}{5} \right)^{\frac{1}{2}}. \tag{5.138}$$

Since each of these integrals has a large constant followed by the same two integrals the entire matrix can be easily reconstructed as follows

$$G = \frac{QbT\pi^{\frac{3}{2}}}{6} \left(\frac{3}{2} \left(\frac{T}{Q} \right)^4 + 2 \left(\frac{T}{Q} \right)^2 + \frac{4}{5} \right)^{\frac{1}{2}} \begin{bmatrix} \frac{Z^2 + D^2}{T^2 Z^2} & \frac{QZ^2 + DFT}{QT^2 Z^2} & 0 \\ \frac{QZ^2 + DFT}{QT^2 Z^2} & \frac{Q^2 Z^2 + T^2 F^2}{(TQZ)^2} & 0 \\ 0 & 0 & 12Q^2 T^2 Z \end{bmatrix}. \tag{5.139}$$

Rewriting D, F and Z in terms of their original parameters

$$D = \left(\frac{4T^4}{Q^4} + \frac{8T^2}{3Q^2} \right) + \left(\frac{T^4}{Q^4} + \frac{4T^2}{3Q^2} + \frac{8}{15} \right), \tag{5.140}$$

$$= 5\frac{T^4}{Q^4} + 4\frac{T^2}{Q^2} + \frac{8}{15}, \tag{5.141}$$

$$F = -3 \left(\frac{T}{Q} \right)^4 - \frac{4}{3} \left(\frac{T}{Q} \right)^2 + \frac{8}{15}, \tag{5.142}$$

$$Z^2 = \frac{T^8}{Q^8} + \frac{8T^6}{3Q^6} + \frac{128T^4}{45Q^4} + \frac{64T^2}{45Q^2} + \frac{64}{225}, \tag{5.143}$$

$$D^2 = \frac{64}{225} + \frac{64T^2}{15Q^2} + \frac{64T^4}{3Q^4} + \frac{40T^6}{Q^6} + \frac{25T^8}{Q^8}, \tag{5.144}$$

$$Z^2 + D^2 = 26\frac{T^8}{Q^8} + \frac{128}{3}\frac{T^6}{Q^6} + \frac{1088}{45}\frac{T^4}{Q^4} + \frac{256}{45}\frac{T^2}{Q^2} + \frac{128}{225}. \tag{5.145}$$

5.4 Geodesics on G

Explicit formulae for geodesics of the metric in equation (5.139) are not possible. However, the method for determining the geodesics on G follows the same method developed in Section (4.3). The required derivatives, with $Q = T_P/2$, follow

$$\partial_T \mathrm{vol}(g)QTb = \frac{b\pi^{\frac{3}{2}}}{6}\left(\frac{8Q^4 + 40Q^2T^2 + 45T^4}{\sqrt{10}Q^3\sqrt{8 + \frac{20T^2}{Q^2} + \frac{15T^4}{Q^4}}}\right), \tag{5.146}$$

$$\partial_Q \mathrm{vol}(g)QTb = \frac{bT\pi^{\frac{3}{2}}}{6}\left(\frac{-15T^4 + 8Q^4}{\sqrt{10}\sqrt{8 + \frac{20T^2}{Q^2} + \frac{15T^4}{Q^4}} + Q^4}\right), \tag{5.147}$$

$$\partial_b \mathrm{vol}(g)QTb = \frac{QT\pi^{\frac{3}{2}}}{6}\left(\frac{3}{2}\left(\frac{T}{Q}\right)^4 + 2\left(\frac{T}{Q}\right)^2 + \frac{4}{5}\right)^{\frac{1}{2}}. \tag{5.148}$$

$$\frac{\partial G_{11}}{\partial u_1} = \frac{\partial G_{11}}{\partial T} = \frac{T^2Z^2\left(2Z\frac{\partial Z}{\partial T} + 2D\frac{\partial D}{\partial T}\right) - (Z^2 + D^2)\left(2T^2Z\frac{\partial Z}{\partial T} + 2TZ^2\right)}{T^4Z^4},$$

$$= 2\left[\frac{DTZ\frac{\partial D}{\partial T} - TD^2\frac{\partial Z}{\partial T} - Z^3 - ZD^2}{T^3Z^3}\right], \tag{5.149}$$

$$\frac{\partial G_{11}}{\partial u_2} = \frac{\partial G_{11}}{\partial Q} = 2D\left[\frac{Z\frac{\partial D}{\partial Q} - D\frac{\partial Z}{\partial Q}}{T^2Z^3}\right], \tag{5.150}$$

$$\frac{\partial G_{12}}{\partial u_1} = \frac{\partial G_{12}}{\partial T} =$$

$$\frac{\left(T^2QZ^2\left[2QZ\frac{\partial Z}{\partial T} + T\left(F\frac{\partial D}{\partial T} + D\frac{\partial F}{\partial T}\right) + FD\right] - (QZ^2 + TFD)\left(2QT^2Z\frac{\partial Z}{\partial T} + Z^2T\right)\right)}{T^4Q^2Z^4},$$

$$= \frac{FT^2Z\frac{\partial D}{\partial T} + DT^2Z\frac{\partial F}{\partial T} - 2T^2FD\frac{\partial Z}{\partial T} - 2QZ^3 - DFTZ}{T^3QZ^3} \tag{5.151}$$

$$\frac{\partial G_{12}}{\partial u_2} = \frac{\partial G_{12}}{\partial Q},$$

$$= \frac{\left(T^2QZ^2\left[2QZ\frac{\partial Z}{\partial Q} + Z^2 + T\left(F\frac{\partial D}{\partial Q} + D\frac{\partial F}{\partial Q}\right)\right] - \left(QZ^2 + TFD\right)T^2\left(2QZ\frac{\partial Z}{\partial Q} + Z^2\right)\right)}{T^4Q^2Z^4},$$

$$= \frac{FQZ\frac{\partial D}{\partial Q} + DQZ\frac{\partial F}{\partial Q} - 2FDQ\frac{\partial Z}{\partial Q} - FDZ}{TD^2Z^3}, \tag{5.152}$$

$$\frac{\partial G_{22}}{\partial u_1} = \frac{\partial G_{22}}{\partial T}$$

$$= \frac{\left(T^2Q^2Z^2\left[2Q^2Z\frac{\partial Z}{\partial T} + 2T^2F\frac{\partial F}{\partial T} + 2TF^2\right] - \left(Q^2Z^2 + T^2F^2\right)Q^2\left(2T^2Z\frac{\partial Z}{\partial T} + 2TZ^2\right)\right)}{T^4Q^4Z^4},$$

$$= 2\left[\frac{T^3FZ\frac{\partial D}{\partial T} - T^3F^2\frac{\partial Z}{\partial T} - Q^2Z^3}{T^3Q^2Z^3}\right], \tag{5.153}$$

$$\frac{\partial G_{22}}{\partial u_2} = \frac{\partial G_{22}}{\partial Q}$$

$$= \frac{\left(T^2Q^2Z^2\left[2Q^2Z\frac{\partial Z}{\partial Q} + 2QZ^2 + 2T^2F\frac{\partial F}{\partial Q}\right] - \left(Q^2Z^2 + T^2F^2\right)T^2\left(2Q^2Z\frac{\partial Z}{\partial Q} + 2QZ^2\right)\right)}{T^4Q^4Z^4},$$

$$= 2F\left[\frac{QZ\frac{\partial F}{\partial Q} - QF\frac{\partial Z}{\partial Q} - FZ}{Q^3Z^3}\right], \tag{5.154}$$

$$\frac{\partial G_{33}}{\partial u_1} = \frac{\partial G_{33}}{\partial T} = 12Q^2T\left(T\frac{\partial Z}{\partial T} + 2Z\right), \tag{5.155}$$

$$\frac{\partial G_{33}}{\partial u_2} = \frac{\partial G_{33}}{\partial Q} = 12QT^2\left(Q\frac{\partial Z}{\partial Q} + 2Z\right). \tag{5.156}$$

The Christoffel symbols (4.138) associated with G_{ij} are reduced to the following,

$$\Gamma^1_{11} = \frac{1}{2}\left[\frac{\partial G_{11}}{\partial T}G^{11} + \left(2\frac{\partial G_{12}}{\partial T} - \frac{\partial G_{11}}{\partial Q}\right)G^{12}\right], \tag{5.157}$$

$$\Gamma^1_{22} = \frac{1}{2}\left[\left(2\frac{\partial G_{12}}{\partial Q} - \frac{\partial G_{22}}{\partial T}\right)G^{11} + \frac{\partial G_{22}}{\partial Q}G^{12}\right], \tag{5.158}$$

$$\Gamma^1_{33} = \frac{1}{2}\left[-\frac{\partial G_{33}}{\partial T}G^{11} - \frac{\partial G_{33}}{\partial Q}G^{12}\right], \tag{5.159}$$

$$\Gamma^1_{12} = \frac{1}{2}\left[\frac{\partial G_{11}}{\partial Q}G^{11} + \frac{\partial G_{22}}{\partial T}G^{12}\right], \tag{5.160}$$

$$\Gamma^2_{11} = \frac{1}{2}\left[\frac{\partial G_{11}}{\partial T}G^{12} + \left(2\frac{\partial G_{12}}{\partial T} - \frac{\partial G_{11}}{\partial Q}\right)G^{22}\right], \tag{5.161}$$

$$\Gamma^2_{22} = \frac{1}{2}\left[\left(2\frac{\partial G_{12}}{\partial Q} - \frac{\partial G_{22}}{\partial T}\right)G^{12} + \frac{\partial G_{22}}{\partial Q}G^{22}\right], \tag{5.162}$$

$$\Gamma^2_{33} = \frac{1}{2}\left[\frac{-\partial G_{33}}{\partial T}G^{12} - \frac{\partial G_{33}}{\partial Q}G^{22}\right], \tag{5.163}$$

$$\Gamma^2_{12} = \frac{1}{2}\left[\frac{\partial G_{11}}{\partial Q}G^{12} + \frac{\partial G_{22}}{\partial T}G^{22}\right], \tag{5.164}$$

$$\Gamma^3_{13} = \frac{1}{2}\frac{\partial G_{33}}{\partial T}G^{33}, \tag{5.165}$$

$$\Gamma^3_{23} = \frac{1}{2}\frac{\partial G_{33}}{\partial Q}G^{33}, \tag{5.166}$$

All other Christoffel symbols are equal to zero

$$0 = \Gamma^1_{13} = \Gamma^1_{23} = \Gamma^2_{13} = \Gamma^2_{23} = \Gamma^3_{11} = \Gamma^3_{22} = \Gamma^3_{33} = \Gamma^3_{12}. \tag{5.167}$$

Chapter 6

Conclusion

The exploration of optimal trajectories through the space of target tracking system parameters was the goal of this thesis. To this end the ambiguity function of equation (2.15) formed the basis from which the Fisher information metric was constructed. These calculations were performed in Chapter 4 for the infinite pulse train and in Chapter 5 for a single Doppler bin case. In Section 3.2 a method for calculating the integral of the product of shifted sinc functions was developed, filling a gap in the mathematical resources required for the calculation of the Fisher information metric.

The Fisher metric forms the basis for the configuration metric from which the optimal trajectories, geodesics, are computed. Approximate analytic formulae were obtained for the components of both the Fisher metrics for the infinite pulse train ambiguity function (Section 4.1) and the single Doppler bin case (Section 5.2) but the nature of these approximations are quite different. In the infinite pulse train case, the analytical formulae for the components of the Fisher metric are in the form of an infinite series. While these formulae are exact, implementation requires truncation of the series. Formulas for the Fisher metric truncated after nine terms were presented as an example and geodesics based on this analytic approximation are pictured in Section 4.3. The analytic formulae for the components of the Fisher metric in the single Doppler bin case were found using approximate identities. In this case, the accuracy of the approximation is fixed by the choice of the configuration parameters T (pulse width) and T_P (inter-pulse spacing). The accuracy of the analytic solution increases as T becomes smaller and as T_P becomes larger.

The geodesics themselves are found by solving a system of ODEs with coefficients given by partial derivatives (Christoffel symbols). Inevitably, this must be done numerically. However, the speed and robustness of such a calculation is improved if numerical differentiation is avoided in finding the coefficients. To this end, analytical expressions for the coefficients were obtained from analytical formulae for the configuration metrics which, in turn, were obtained from the approximate analytical formulae for the Fisher metrics. This means that, potentially, these formulae may be of practical value in optimally updating radar configuration parameters. In the case of the infinite pulse train setting, the user is able to control the level of accuracy by adjusting the number of terms in the series for the Fisher metric.

There are several potential avenues for advancing this work, one of which is performing the calculation for wideband signals, by similar methods to those seen in Auslander [3], currently only a narrow band approximation has been considered. The advantage of choosing the narrow band is that it provides better sensitivity and better range, primarily by reducing noise bandwidth and requires lower power to transmit. Wideband, on the other hand, allows transmission at a much higher data rate. However, due to the distribution of the signal across a wider portion of frequency spectrum the power required to transmit over the wideband is higher in order to compete with the noise. Wideband transmission of data is common, technology such as Wi-Fi and sonar are good examples of this, and so interesting and potentially useful data may result from applying this method to wideband signals.

In this work, the actual sensor system under consideration passes as largely irrelevant. However, the system consists of only a single target and single sensor, both fixed, resulting in very basic sensor/target geometry. Extending this system to include multiple sensors and targets obviously increases the complexity of the calculation dramatically, as does allowing the targets and sensors to move in space. This extension can be combined with a type of mathematical model from the field of game theory, a differential game. In a differential game the targets and sensors both have goals they wish to achieve. The sensor and target may be working in unison, as in a commercial airline and a control tower, and each is attempting to transmit and receive the maximum information about themselves and the other, a cooperative game. Alternatively, take a scenario such as a bat hunting a moth, where the bat is always attempting to obtain maximum information about the moth, while the moth is always trying to minimise the information the bat obtains, and simultaneously trying to locate and avoid the bat, a non-cooperative game. The practical application of these models and their extension are beyond the scope of this thesis, which deals solely with the mathematical structures underpinning such engineering implementation. This is a complex and interesting subject and much additional work remains.

Chapter 7

Appendix

Christoffel symbols corresponding to Chapter 4.

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}\left[\frac{\partial G_{11}}{\partial T} + \frac{\partial G_{11}}{\partial T} + \frac{\partial G_{11}}{\partial T}\right] G^{11} \\
&+ \frac{1}{2}\left[\frac{\partial G_{21}}{\partial T} + \frac{\partial G_{12}}{\partial T} + \frac{\partial G_{11}}{\partial P}\right] G^{21} \\
&+ \frac{1}{2}\left[\frac{\partial G_{31}}{\partial T} + \frac{\partial G_{13}}{\partial T} + \frac{\partial G_{11}}{\partial b}\right] G^{31} \qquad (7.1)\\
\Gamma^1_{12} &= \frac{1}{2}\left[\frac{\partial G_{11}}{\partial P} + \frac{\partial G_{21}}{\partial T} + \frac{\partial G_{12}}{\partial T}\right] G^{11} \\
&+ \frac{1}{2}\left[\frac{\partial G_{21}}{\partial P} + \frac{\partial G_{22}}{\partial T} + \frac{\partial G_{12}}{\partial P}\right] G^{21} \\
&+ \frac{1}{2}\left[\frac{\partial G_{31}}{\partial P} + \frac{\partial G_{23}}{\partial T} + \frac{\partial G_{12}}{\partial b}\right] G^{31} \qquad (7.2)\\
\Gamma^1_{13} &= \frac{1}{2}\left[\frac{\partial G_{11}}{\partial b} + \frac{\partial G_{31}}{\partial T} + \frac{\partial G_{13}}{\partial T}\right] G^{11} \\
&+ \frac{1}{2}\left[\frac{\partial G_{21}}{\partial b} + \frac{\partial G_{32}}{\partial T} + \frac{\partial G_{13}}{\partial P}\right] G^{21} \\
&+ \frac{1}{2}\left[\frac{\partial G_{31}}{\partial b} + \frac{\partial G_{33}}{\partial T} + \frac{\partial G_{13}}{\partial b}\right] G^{31} \qquad (7.3)\\
\Gamma^1_{21} &= \frac{1}{2}\left[\frac{\partial G_{12}}{\partial T} + \frac{\partial G_{11}}{\partial P} + \frac{\partial G_{21}}{\partial T}\right] G^{11} \\
&+ \frac{1}{2}\left[\frac{\partial G_{22}}{\partial T} + \frac{\partial G_{12}}{\partial P} + \frac{\partial G_{21}}{\partial P}\right] G^{21} \\
&+ \frac{1}{2}\left[\frac{\partial G_{32}}{\partial T} + \frac{\partial G_{13}}{\partial P} + \frac{\partial G_{21}}{\partial b}\right] G^{31} \qquad (7.4)\\
\Gamma^1_{22} &= \frac{1}{2}\left[\frac{\partial G_{12}}{\partial P} + \frac{\partial G_{21}}{\partial P} + \frac{\partial G_{22}}{\partial T}\right] G^{11} \\
&+ \frac{1}{2}\left[\frac{\partial G_{22}}{\partial P} + \frac{\partial G_{22}}{\partial P} + \frac{\partial G_{22}}{\partial P}\right] G^{21} \\
&+ \frac{1}{2}\left[\frac{\partial G_{32}}{\partial P} + \frac{\partial G_{23}}{\partial P} + \frac{\partial G_{22}}{\partial b}\right] G^{31} \qquad (7.5)\\
\Gamma^1_{23} &= \frac{1}{2}\left[\frac{\partial G_{12}}{\partial b} + \frac{\partial G_{31}}{\partial P} + \frac{\partial G_{23}}{\partial T}\right] G^{11}
\end{aligned}$$

$$+\frac{1}{2}\left[\frac{\partial G_{22}}{\partial b}+\frac{\partial G_{32}}{\partial P}+\frac{\partial G_{23}}{\partial P}\right]G^{21}$$

$$+\frac{1}{2}\left[\frac{\partial G_{32}}{\partial b}+\frac{\partial G_{33}}{\partial P}+\frac{\partial G_{23}}{\partial b}\right]G^{31} \tag{7.6}$$

$$\Gamma^1_{31}=\frac{1}{2}\left[\frac{\partial G_{13}}{\partial T}+\frac{\partial G_{11}}{\partial b}+\frac{\partial G_{31}}{\partial T}\right]G^{11}$$

$$+\frac{1}{2}\left[\frac{\partial G_{23}}{\partial T}+\frac{\partial G_{12}}{\partial b}+\frac{\partial G_{31}}{\partial P}\right]G^{21}$$

$$+\frac{1}{2}\left[\frac{\partial G_{33}}{\partial T}+\frac{\partial G_{13}}{\partial B}+\frac{\partial G_{31}}{\partial b}\right]G^{31} \tag{7.7}$$

$$\Gamma^1_{32}=\frac{1}{2}\left[\frac{\partial G_{13}}{\partial P}+\frac{\partial G_{21}}{\partial b}+\frac{\partial G_{32}}{\partial T}\right]G^{11}$$

$$+\frac{1}{2}\left[\frac{\partial G_{23}}{\partial P}+\frac{\partial G_{22}}{\partial b}+\frac{\partial G_{32}}{\partial P}\right]G^{21}$$

$$+\frac{1}{2}\left[\frac{\partial G_{33}}{\partial P}+\frac{\partial G_{23}}{\partial b}+\frac{\partial G_{32}}{\partial b}\right]G^{31} \tag{7.8}$$

$$\Gamma^1_{33}=\frac{1}{2}\left[\frac{\partial G_{13}}{\partial b}+\frac{\partial G_{31}}{\partial b}+\frac{\partial G_{33}}{\partial T}\right]G^{11}$$

$$+\frac{1}{2}\left[\frac{\partial G_{23}}{\partial b}+\frac{\partial G_{32}}{\partial b}+\frac{\partial G_{33}}{\partial P}\right]G^{21}$$

$$+\frac{1}{2}\left[\frac{\partial G_{33}}{\partial b}+\frac{\partial G_{33}}{\partial b}+\frac{\partial G_{33}}{\partial b}\right]G^{31} \tag{7.9}$$

$$\Gamma^2_{11}=\frac{1}{2}\left[\frac{\partial G_{11}}{\partial T}+\frac{\partial G_{11}}{\partial T}+\frac{\partial G_{11}}{\partial T}\right]G^{12}$$

$$+\frac{1}{2}\left[\frac{\partial G_{21}}{\partial T}+\frac{\partial G_{12}}{\partial T}+\frac{\partial G_{11}}{\partial P}\right]G^{22}$$

$$+\frac{1}{2}\left[\frac{\partial G_{31}}{\partial T}+\frac{\partial G_{13}}{\partial T}+\frac{\partial G_{11}}{\partial b}\right]G^{32} \tag{7.10}$$

$$\Gamma^2_{12}=\frac{1}{2}\left[\frac{\partial G_{11}}{\partial P}+\frac{\partial G_{21}}{\partial T}+\frac{\partial G_{12}}{\partial T}\right]G^{12}$$

$$+\frac{1}{2}\left[\frac{\partial G_{21}}{\partial P}+\frac{\partial G_{22}}{\partial T}+\frac{\partial G_{12}}{\partial P}\right]G^{22}$$

$$+\frac{1}{2}\left[\frac{\partial G_{31}}{\partial P}+\frac{\partial G_{23}}{\partial T}+\frac{\partial G_{12}}{\partial b}\right]G^{32} \tag{7.11}$$

$$\Gamma^2_{13}=\frac{1}{2}\left[\frac{\partial G_{11}}{\partial b}+\frac{\partial G_{31}}{\partial T}+\frac{\partial G_{13}}{\partial T}\right]G^{12}$$

$$+\frac{1}{2}\left[\frac{\partial G_{21}}{\partial b}+\frac{\partial G_{32}}{\partial T}+\frac{\partial G_{13}}{\partial P}\right]G^{22}$$

$$+\frac{1}{2}\left[\frac{\partial G_{31}}{\partial b}+\frac{\partial G_{33}}{\partial T}+\frac{\partial G_{13}}{\partial b}\right]G^{32} \tag{7.12}$$

$$\Gamma^2_{21}=\frac{1}{2}\left[\frac{\partial G_{12}}{\partial T}+\frac{\partial G_{11}}{\partial P}+\frac{\partial G_{21}}{\partial T}\right]G^{12}$$

$$+\frac{1}{2}\left[\frac{\partial G_{22}}{\partial T}+\frac{\partial G_{12}}{\partial P}+\frac{\partial G_{21}}{\partial P}\right]G^{22}$$

$$
\begin{aligned}
&+\frac{1}{2}\left[\frac{\partial G_{32}}{\partial T}+\frac{\partial G_{13}}{\partial P}+\frac{\partial G_{21}}{\partial b}\right]G^{32} && (7.13)\\
\Gamma^2_{22} &= \frac{1}{2}\left[\frac{\partial G_{12}}{\partial P}+\frac{\partial G_{21}}{\partial P}+\frac{\partial G_{22}}{\partial T}\right]G^{12}\\
&+\frac{1}{2}\left[\frac{\partial G_{22}}{\partial P}+\frac{\partial G_{22}}{\partial P}+\frac{\partial G_{22}}{\partial P}\right]G^{22}\\
&+\frac{1}{2}\left[\frac{\partial G_{32}}{\partial P}+\frac{\partial G_{23}}{\partial P}+\frac{\partial G_{22}}{\partial b}\right]G^{32} && (7.14)\\
\Gamma^2_{23} &= \frac{1}{2}\left[\frac{\partial G_{12}}{\partial b}+\frac{\partial G_{31}}{\partial P}+\frac{\partial G_{23}}{\partial T}\right]G^{12}\\
&+\frac{1}{2}\left[\frac{\partial G_{22}}{\partial b}+\frac{\partial G_{32}}{\partial P}+\frac{\partial G_{23}}{\partial P}\right]G^{22}\\
&+\frac{1}{2}\left[\frac{\partial G_{32}}{\partial b}+\frac{\partial G_{33}}{\partial P}+\frac{\partial G_{23}}{\partial b}\right]G^{32} && (7.15)\\
\Gamma^2_{31} &= \frac{1}{2}\left[\frac{\partial G_{13}}{\partial T}+\frac{\partial G_{11}}{\partial b}+\frac{\partial G_{31}}{\partial T}\right]G^{12}\\
&+\frac{1}{2}\left[\frac{\partial G_{23}}{\partial T}+\frac{\partial G_{12}}{\partial b}+\frac{\partial G_{31}}{\partial P}\right]G^{22}\\
&+\frac{1}{2}\left[\frac{\partial G_{33}}{\partial T}+\frac{\partial G_{13}}{\partial b}+\frac{\partial G_{31}}{\partial b}\right]G^{32} && (7.16)\\
\Gamma^2_{32} &= \frac{1}{2}\left[\frac{\partial G_{13}}{\partial P}+\frac{\partial G_{21}}{\partial b}+\frac{\partial G_{32}}{\partial T}\right]G^{12}\\
&+\frac{1}{2}\left[\frac{\partial G_{23}}{\partial P}+\frac{\partial G_{22}}{\partial b}+\frac{\partial G_{32}}{\partial P}\right]G^{22}\\
&+\frac{1}{2}\left[\frac{\partial G_{33}}{\partial b}+\frac{\partial G_{23}}{\partial b}+\frac{\partial G_{32}}{\partial b}\right]G^{32} && (7.17)\\
\Gamma^2_{33} &= \frac{1}{2}\left[\frac{\partial G_{13}}{\partial b}+\frac{\partial G_{31}}{\partial b}+\frac{\partial G_{33}}{\partial T}\right]G^{12}\\
&+\frac{1}{2}\left[\frac{\partial G_{23}}{\partial b}+\frac{\partial G_{32}}{\partial b}+\frac{\partial G_{33}}{\partial P}\right]G^{22}\\
&+\frac{1}{2}\left[\frac{\partial G_{33}}{\partial b}+\frac{\partial G_{33}}{\partial b}+\frac{\partial G_{33}}{\partial b}\right]G^{32} && (7.18)\\
\Gamma^3_{11} &= \frac{1}{2}\left[\frac{\partial G_{11}}{\partial T}+\frac{\partial G_{11}}{\partial T}+\frac{\partial G_{11}}{\partial T}\right]G^{13}\\
&+\frac{1}{2}\left[\frac{\partial G_{21}}{\partial T}+\frac{\partial G_{12}}{\partial T}+\frac{\partial G_{11}}{\partial P}\right]G^{23}\\
&+\frac{1}{2}\left[\frac{\partial G_{31}}{\partial T}+\frac{\partial G_{13}}{\partial T}+\frac{\partial G_{11}}{\partial b}\right]G^{33} && (7.19)\\
\Gamma^3_{12} &= \frac{1}{2}\left[\frac{\partial G_{11}}{\partial P}+\frac{\partial G_{21}}{\partial T}+\frac{\partial G_{12}}{\partial T}\right]G^{13}\\
&+\frac{1}{2}\left[\frac{\partial G_{21}}{\partial P}+\frac{\partial G_{22}}{\partial T}+\frac{\partial G_{12}}{\partial P}\right]G^{23}\\
&+\frac{1}{2}\left[\frac{\partial G_{31}}{\partial P}+\frac{\partial G_{23}}{\partial T}+\frac{\partial G_{12}}{\partial b}\right]G^{33} && (7.20)
\end{aligned}
$$

$$\Gamma^3_{13} = \frac{1}{2}\left[\frac{\partial G_{11}}{\partial b} + \frac{\partial G_{31}}{\partial T} + \frac{\partial G_{13}}{\partial T}\right] G^{13} + \frac{1}{2}\left[\frac{\partial G_{21}}{\partial b} + \frac{\partial G_{32}}{\partial T} + \frac{\partial G_{13}}{\partial P}\right] G^{23} + \frac{1}{2}\left[\frac{\partial G_{31}}{\partial b} + \frac{\partial G_{33}}{\partial T} + \frac{\partial G_{13}}{\partial b}\right] G^{33} \tag{7.21}$$

$$\Gamma^3_{21} = \frac{1}{2}\left[\frac{\partial G_{12}}{\partial T} + \frac{\partial G_{11}}{\partial P} + \frac{\partial G_{21}}{\partial T}\right] G^{13} + \frac{1}{2}\left[\frac{\partial G_{22}}{\partial T} + \frac{\partial G_{12}}{\partial P} + \frac{\partial G_{21}}{\partial P}\right] G^{23} + \frac{1}{2}\left[\frac{\partial G_{32}}{\partial T} + \frac{\partial G_{13}}{\partial P} + \frac{\partial G_{21}}{\partial b}\right] G^{33} \tag{7.22}$$

$$\Gamma^3_{22} = \frac{1}{2}\left[\frac{\partial G_{12}}{\partial P} + \frac{\partial G_{21}}{\partial P} + \frac{\partial G_{22}}{\partial T}\right] G^{13} + \frac{1}{2}\left[\frac{\partial G_{22}}{\partial P} + \frac{\partial G_{22}}{\partial P} + \frac{\partial G_{22}}{\partial P}\right] G^{23} + \frac{1}{2}\left[\frac{\partial G_{32}}{\partial P} + \frac{\partial G_{23}}{\partial P} + \frac{\partial G_{22}}{\partial b}\right] G^{33} \tag{7.23}$$

$$\Gamma^3_{23} = \frac{1}{2}\left[\frac{\partial G_{12}}{\partial b} + \frac{\partial G_{31}}{\partial P} + \frac{\partial G_{23}}{\partial T}\right] G^{13} + \frac{1}{2}\left[\frac{\partial G_{22}}{\partial b} + \frac{\partial G_{32}}{\partial P} + \frac{\partial G_{23}}{\partial P}\right] G^{23} + \frac{1}{2}\left[\frac{\partial G_{32}}{\partial b} + \frac{\partial G_{33}}{\partial P} + \frac{\partial G_{23}}{\partial b}\right] G^{33} \tag{7.24}$$

$$\Gamma^3_{31} = \frac{1}{2}\left[\frac{\partial G_{13}}{\partial T} + \frac{\partial G_{11}}{\partial b} + \frac{\partial G_{31}}{\partial T}\right] G^{13} + \frac{1}{2}\left[\frac{\partial G_{23}}{\partial T} + \frac{\partial G_{12}}{\partial b} + \frac{\partial G_{31}}{\partial P}\right] G^{23} + \frac{1}{2}\left[\frac{\partial G_{33}}{\partial T} + \frac{\partial G_{13}}{\partial b} + \frac{\partial G_{31}}{\partial b}\right] G^{33} \tag{7.25}$$

$$\Gamma^3_{32} = \frac{1}{2}\left[\frac{\partial G_{13}}{\partial P} + \frac{\partial G_{21}}{\partial b} + \frac{\partial G_{32}}{\partial T}\right] G^{13} + \frac{1}{2}\left[\frac{\partial G_{23}}{\partial P} + \frac{\partial G_{22}}{\partial b} + \frac{\partial G_{32}}{\partial P}\right] G^{23} + \frac{1}{2}\left[\frac{\partial G_{33}}{\partial P} + \frac{\partial G_{23}}{\partial b} + \frac{\partial G_{32}}{\partial b}\right] G^{33} \tag{7.26}$$

$$\Gamma^3_{33} = \frac{1}{2}\left[\frac{\partial G_{13}}{\partial b} + \frac{\partial G_{31}}{\partial b} + \frac{\partial G_{33}}{\partial T}\right] G^{13} + \frac{1}{2}\left[\frac{\partial G_{23}}{\partial b} + \frac{\partial G_{32}}{\partial b} + \frac{\partial G_{33}}{\partial P}\right] G^{23} + \frac{1}{2}\left[\frac{\partial G_{33}}{\partial b} + \frac{\partial G_{33}}{\partial b} + \frac{\partial G_{33}}{\partial b}\right] G^{33} \tag{7.27}$$

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