Thesis submitted to the College of Science and Engineering in partial fulfilment of the requirements for the degree of Master of Science (Mathematical Science) at Flinders University, Adelaide, Australia.

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Painlevé Equation, Reflection Groups and τ-Functions

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1. PREFACE

More often than not, nonlinear equations are chaotic. However, the very few that are not were shown to posses remarkable properties and play significant roles in an astonishingly wide range of physical theories. They are known as the integrable equations. The study of integrable equations begun with that of the Korteweg-de Vries (KdV) equation, an equation that describes the propagation of waves in shallow waters [4]. It belongs to a family of nonlinear partial differential equations (PDEs) which contains many physically important equations including nonlinear Schrödinger, sine-Gordon, Kadomtsev-Petviashvili (KP) and the Painlevé equations. Here we will focus on the Painlevé equations which discovered by Paul Painlevé, French mathematician, about one hunderd years ago, and the following terminology and notation from [1, 2]. The Painlevé equations are a class of second order nonlinear ordinary differential equations.The differential equation has the Painlevé property if the solutions of the differential equation do not have movable singularities other than poles. For example , if we have $\frac{dx}{dt} = -x(t)^2$ where $x(t) \neq 0$ (the movable singularity is depending on the initial conditions of the equation) and its solution is $x(t) = \frac{1}{t-c}$ and here c is an integral constant . At point $t = c$ The solution has a singularity and Since c is determined by the initial condition at $t = t_0$, then $c = t_0 - \frac{1}{x(t_0)}$ and this is movable singularities.With the Painlevé property, Painlevé et al. classified all the rational ordinary differential equations of second order of the form $y'' = F(t, y, y')$ where F is the rational in y' and y and analytic in t. As a result, they showed that except for the differential equations which can be integrated algebraically or transformed into linear equations or into the differential equations solvable by elliptic functions, any differential equation of the form $y = F(t, y, y)$ with the Painlevé property is reduced to one of the following equations,

$$\begin{aligned}
P_I &: y'' = 6y^2 + t \\
P_{II} &: y'' = 2y^3 + ty + \alpha \\
P_{III} &: y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \\
P_{IV} &: y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y} \\
P_V &: y'' = (\frac{1}{2y} + \frac{1}{y-1})(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}(\alpha y + \frac{\beta}{y}) + \frac{\gamma}{t}y + \delta\frac{y(y+1)}{y-1} \\
P_{VI} &: y'' = \frac{1}{2}(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t})(y')^2 - (\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t})y' \\
&\quad + \frac{y(y-1)(y-t)}{t^2(t-1)}(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-1)^2}).
\end{aligned} \tag{1}$$

where $' = \frac{d}{dt}$, and the Greek letters are complex constants (parameters), and as is evident, the first Painlevé equations does not contain a parameter, while all other equations have some numbers of parameters. Whereas all these parameters will help to obtain the symmetry of these equations. One of the many remarkable properties that characterizes integrable equations is the existence of certain transformations which relate different solutions of the same equation. To explain we use the fourth

Painlevé equations P_{IV}, given by:

$$P_{IV} : y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y} \tag{2}$$

It can be checked that a new function $\bar{y}$ given by the formula

$$\bar{y} = -2t - y + \frac{(2t+y)y - y' - 2\alpha_1}{2y} + \frac{4y(1-\alpha_1-\alpha_2)}{(2t+y)y - y' - 2\alpha_1} \tag{3}$$

is a solution of fourth Painlevé Equation (2) if we replace the parameters α_1 and α_2 by $\bar{\alpha_1} = \alpha_1 - 1$ and $\bar{\alpha_2} = \alpha_2$, and this transformation is called a Bäcklund transformations.To find Bäcklund transformations of an integrable equation used to amount to something of a black art [5, 10] until it was realized by Okamoto and the Japanese school in the 80s that the deep reason that lies behind their existences is given by symmetries of the equation, the set of which forms a Weyl group (or crystallographic reflection groups) [6, 8]. Using again the P_{IV} equation as an example it was shown by Noumi and Yamada [7] that Equation (2) can be written in a more symmetric way. Using this symmetric form of P_{IV}, Noumi and Yamada formulated a birational realisation of the extended affine Weyl group of type A_2

The P_{IV} equation although transcendental, admit special algebraic solutions for particular values of the parameters. These solutions are now understood to correspond to special points in the parameter space $V = \{(x, y, z) \in R^3 \mid x + y + z = 1\}$, whose affine geometric structure is given by the Cartan matrix. The geometric realisation of the Weyl group on $\mathbb{R}^3$ allows one to generate a chain of such special solutions from a simple seed solution using the birational representation formulation of Noumi and Yamada. On the other hand, it is well-known that associated to each Weyl there is an ring of invariant polynomials[9]. The relation between the strucutre and the geometry of this ring to the special algebraic solutions of the Painlevé equations is not clear.

The aim of this research is clarify the special algebraic solutions of the Painlevé equations using the Weyl groups via the explicit example of the P_{IV} equation which admits the Weyl group symmetry of type A_2. We do so via an important object called τ-functions which is defined from the Hamiltonian representation of the Painlevé equation.

In this research, first in Section 2, we will explain what is a Weyl group and illustrate this with examples. Moreover, we will see how the fourth Painlevé equations can be written in more symmetric way in Section 3.1. In 3.2 we will clarify that the symmetric form of the P_{IV} (Bäcklund transformations) can be formulated as a birational representation of the extended affine Weyl group of type A_2 , $\tilde{W}(A_2^{(1)})$. After that, we will generate some rational special solutions of the P_{IV} in 3.3. In the next section,which is Section 4.1, we give the definition of a Hamiltonian system and will show the P_{IV} equation can be written in Hamiltonian form. From a Hamiltonian system we can introduce the τ-functions. Finally,in Section 4.2 you will see how we introduced three τ-functions for the symmetric form of the fourth Painlevé equations and derived the Hirota bilinear equations for them. In the end of this section we will show example of τ-functions corrresponding to the special rational solutions that in Section 3.3.

2. Weyl group

First let us recall that a group, any group G should satisfies the three following conditions

i: has identity element ($1 \in G$)
ii: for any $s \in G$ then $s^{-1} \in G$
iii: for any $s, t \in G$ then $st \in G$.

We follow closely the terminology and notation of Humphreys[3]. A coxeter groups (reflection groups) are generated by the set $\{s_i = s_{\alpha_i} | (s_i s_j)^{m_{ij}} = 1, s_i^2 = 1\}$. The group W generated by all reflections is called a Weyl group (crystallographic condition) and a Weyl group are precisely the reflection groups all $m_{ij} \in \{2, 3, 4, 6\}$. Weyl groups of rank n can be realized as group of reflection on a real vector space of dimension n.

A Weyl group is defined by

$$W =< s_1, ...s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 >, m_{ij} \in \{2, 3, 4, 6\} \tag{4}$$

Here $s_1, ...s_n$ are sets of simple reflection on real vector spaces, where $s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1$ are the fundemental relations, and note that $s_i s_j$ is a rotation whose order is given by m_{ij}.It is useful encode the fundamental relations using Dynkin diagram. Dynkin diagram is a diagram of nodes and lines, and each node corresponds to a particular simple root, two nodes are connected by zero, one, two, or three lines when the m_{ij} is 2 , 3. 4, 6 respectively . That is,

$$\begin{aligned} m_{ij} = 2,\ \ (s_i s_j)^2 = 1 \quad &: \quad \circ \quad \circ \\ m_{ij} = 3,\ \ (s_i s_j)^3 = 1 \quad &: \quad \circ - \circ \\ m_{ij} = 4,\ \ (s_i s_j)^4 = 1 \quad &: \quad \circ = \circ \\ m_{ij} = 6,\ \ (s_i s_j)^6 = 1 \quad &: \quad \circ \equiv \circ. \end{aligned}$$

The classification of Weyl groups has four infinite families A_n, B_n, C_n and D_n and five exceptional types $E_6, E_7, E_8, F_4$and G_2. Their Dynkin diagrams are shown in Figure (1).

Let V be a vector space equipped with an inner product $< , >$. The inner product is a map, $< , >$: V × V $\longrightarrow \mathbb{R}$ for $u, v, w \in$ V and $\lambda, \mu \in \mathbb{R}$

i: bilinear $< \lambda u + \mu v, w >= \lambda < u, w > +\mu < v, w >$
$< u, \lambda v + \mu w >= \lambda < u, v > +\mu < u, w >$
ii: symmetric $< u, v >=< v, u >$
iii: linearity $< \lambda u, v >= \lambda < u, v >$
iv: positive definite $< u, u >\geq 0$ and $< u, u >= 0$ iff $u = 0$.

Recall that, a reflection is an orthogonal transformation $s_\alpha : V \longrightarrow V$ of order 2 ($s_\alpha^2 = 1$) whose -1-eigenspace is one-dimensional. Its action on V is given by equation

$$s_\alpha(v) = v - \frac{2 < v, \alpha > \alpha}{< \alpha, \alpha >}, v \in V. \tag{5}$$

where the vector α is the -1-eigenspac of $s_\alpha(\alpha) = -\alpha$.

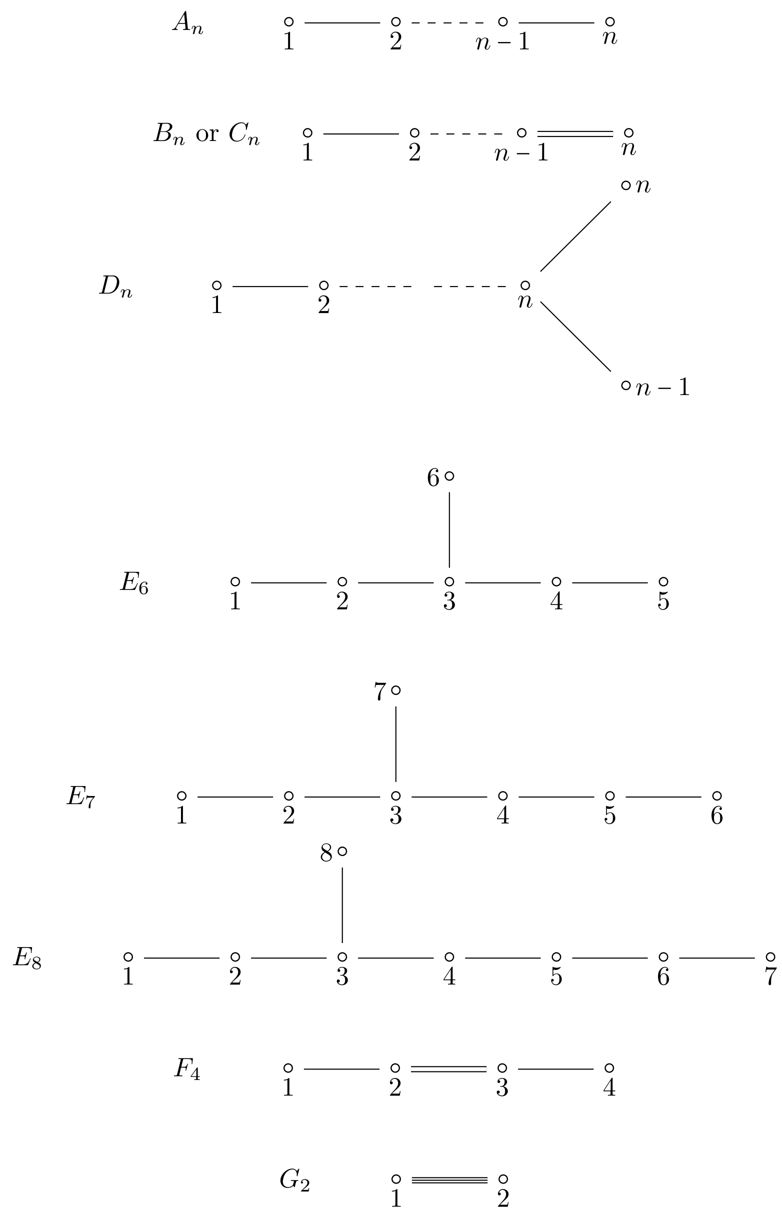

FIGURE 1. Dynkin diagrams for the Classification of a Weyl groups.

Let the simple system $\triangle = \{\alpha_1, ..., \alpha_n\}$ be the set of simple root which is a basis of V. The group element s_i can be realized as reflections in V by the following formula

$$s_i(v) = s_{\alpha_i}(v) = v - \frac{2 < v, \alpha_i > \alpha_i}{< \alpha_i, \alpha_i >}, v \in V. \tag{6}$$

Define the coroot as $\alpha_i^\vee = \frac{2\alpha_i}{<\alpha_i,\alpha_i>}$, now we can write action s_i on the simple root as

$$\begin{aligned} s_i(\alpha_j) &= \alpha_j - \frac{2 < \alpha_j, \alpha_i > \alpha_i}{< \alpha_i, \alpha_i >} \\ &= \alpha_j - < \alpha_j, \frac{2\alpha_i}{< \alpha_i, \alpha_i >} > \alpha_i \\ &= \alpha_j - < \alpha_j, \check{\alpha_i} > \alpha_i \\ &= \alpha_j - a_{ij}\alpha_i \end{aligned} \tag{7}$$

The matrix $(a_{ij})_{1\le i,j\le n}$ is called the cartan matrix of the Weyl group where

$$a_{ij} =< \alpha_i, \check{\alpha_j} > . \tag{8}$$

Now we will show some examples that enable us to understand Weyl groups more accurately.

2.1. **Example.** Weyl group of type A_2 is given by

$$W(A_2) =< s_1, s_2 \mid s_1^2 = s_2^2 = 1, (s_1 s_2)^3 = 1 > . \tag{9}$$

The Dynkin diagram of the Weyl group $W(A_2)$ consists two nodes that correspond to the simple reflections s_1 and s_2, and since the rotation $s_1 s_2$ is of order three ,there will be a line between the simple roots, then it is given by

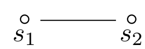

Figure 2. Dynkin diagrams $W(A_2)$

Let $\triangle = \{\alpha_1, \alpha_2\}$ be a basis of V on which acts as group of reflections. The two vectors α_1 and α_2 are orthogonal to the two reflection planes of s_1 and s_2 respectively. we explain this as follows. The element $s_1 s_2$ is rotation by 2θ where the θ is the angle between the two planes of reflection associated to s_1 and s_2 then from the condition $(s_1 s_2)^3 = 1$, then we have $3 * 2\theta = 2\pi$, so $\theta = \frac{\pi}{3}$. Therefore the angle between α_1 and α_2 will be $\varphi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. From this and assume that $|\alpha_1|^2 = |\alpha_1|^2 = 2$ we can get the cartan matrix by(8)

$$\begin{aligned} a_{ij} =< \alpha_i, \check{\alpha_j} > &= \frac{2 < \alpha_i, \alpha_j >}{< \alpha_j, \alpha_j} \\ &= \frac{2|\alpha_i||\alpha_j|}{|\alpha_j|^2} cos\varphi \end{aligned} \tag{10}$$

then

$$\begin{aligned} a_{ij} &= 2, \quad for \;\; i = j \\ a_{ij} &= -1, \quad for \;\; i \neq j. \end{aligned} \tag{11}$$

Hence, the cartan matrix is given by

$$(a_{ij})_{1\le i,j\le 2} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \tag{12}$$

Using the fundamental relations given in the definition of Weyl group A_2 (9) , we can generate the elements of Weyl group type A_2 as

$$W(A_2) = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1 = s_2s_1s_2\}. \tag{13}$$

The last element $s_1s_2s_1 = s_2s_1s_2$ comes from the relation $(s_1s_2)^3 = 1$, which means $s_1s_2s_1s_2s_1s_2 = 1$ if we apply s_1 from the left side we get $s_1s_1s_2s_1s_2s_1s_2 = s_1$ and since $s_1^2 = 1$, then $s_1s_2s_1s_2s_1s_2 = s_1$, now apply s_2 from the left side we get $s_1s_2s_1s_2 = s_2s_1$, lastly we apply s_1 from the left side $s_2s_1s_2 = s_1s_2s_1$. Note there will be no more elements. That is $W(A_2)$ is a finite group of order 6.
The root system of $W(A_2)$ defined by

$$\Phi = W(A_2).\triangle = \{s\alpha \mid s \in W(A_2), \alpha \in \triangle\}. \tag{14}$$

Using Equation (7) we have :
$s_1(\alpha_1) = \alpha_1 - a_{11}\alpha_1 = \alpha_1 - 2\alpha_1 = -\alpha_1$,
$s_1(\alpha_2) = \alpha_2 - a_{21}\alpha_1 = \alpha_2 + \alpha_1$,
$s_2(\alpha_1) = \alpha_1 - a_{12}\alpha_2 = \alpha_1 + \alpha_2$,
$s_1s_2(\alpha_1) = s_1(\alpha_1 + \alpha_2) = -\alpha_1 + \alpha_2 + \alpha_1 = \alpha_2$,
similarly we have
$s_2(\alpha_2) = -\alpha_2$, $s_1s_2(\alpha_2) = -(\alpha_1 + \alpha_2) = s_2s_1(\alpha_1)$, $s_2s_1(\alpha_2) = \alpha_1$,
$s_1s_2s_1(\alpha_1) = -\alpha_2$, $s_1s_2s_1(\alpha_2) = -\alpha_1$.
Hence

$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}. \tag{15}$$

The picture in Figure (3) shows that the two simple roots α_1 and α_2 are orthogonal to the reflections s_1 and s_2 respectively. All the roots are shown here.

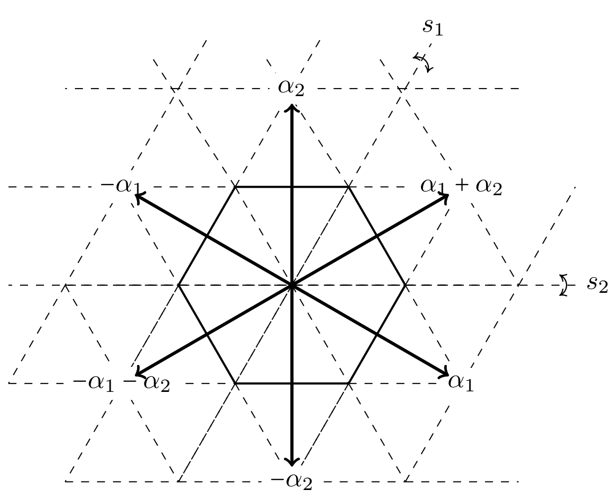

FIGURE 3. Dynkin diagrams $W(A_2)$ with roots.

Now we will see another example, which is an extension of a finite Weyl groups of type A_2.

2.2. **Example.** Affine Weyl group of type A_2, which is the underlying group of symmetries of the fourth Painlevé equation. It is define as

$$W(A_2^{(1)}) =< s_0, s_1, s_2 \mid s_i^2 = 1, (s_i s_{i+1})^3 = 1 > \tag{16}$$

where $i \in \{0,1,2\}$ and its simple system is $\triangle^{(1)} = \{\alpha_0, \alpha_1, \alpha_2\}$. The cartan matrix is given by

$$(a_{ij})_{0\le i,j\le 2} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \tag{17}$$

We can get affine Weyl groups root system by the definition

$$\begin{aligned}\Phi^{(1)} = W(A_2^{(1)}).\triangle^{(1)} &= \{s\alpha \mid s \in W(A_2^{(1)}), \alpha \in \triangle^{(1)}\} \\ &= \{\alpha + k\delta \mid \alpha \in \Phi, k \in \mathbb{Z}\}\end{aligned} \tag{18}$$

where $\delta = \alpha_0 + \alpha_1 + \alpha_2$. Note that $W(A_2^{(1)})$ is an infinite group.

Figure (4) shows the Dynkin diagram of affine Weyl groups of type A_2.

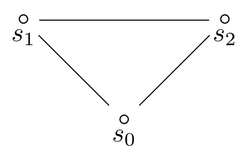

FIGURE 4. Dynkin diagrams for $W(A_2^{(1)})$.

Extended affine Weyl group of type A_2 is given by $\tilde{W}(A_2^{(1)}) = W(A_2^{(1)}) \rtimes < \pi >$ where π is an automorphism of the Dynkin diagram. The element π acts on the Dynkin diagram in Figure (4) by a rotation of $\frac{2\pi}{3}$. That is $\pi^3 = 1, \quad \pi s_j = s_{j+1}\pi \quad (j = 0,1,2)$.

3. The Fourth Painlevé Equation

In this chapter the following terminology and notation from [6].

3.1. **The symmetric form of P_{IV}.** Recall the fourth Painlevé equation P_{IV} is

$$y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y} \tag{19}$$

In this form of the P_{IV} is hard to see the symmetric of the equation, so Noumi and Yamada [7]. introduced three new dependent variables f_0, f_1, f_2 and parameters $\alpha_0, \alpha_1, \alpha_2$, then we can see a symmetric form of the P_{IV} is

$$\begin{aligned} f_0' &= f_0(f_1 - f_2) + \alpha_0, \\ f_1' &= f_1(f_2 - f_0) + \alpha_1, \\ f_2' &= f_2(f_0 - f_1) + \alpha_2 \end{aligned} \tag{20}$$

where $' = d/dt$,

$$\alpha_0 + \alpha_1 + \alpha_2 = 1 \qquad f_0 + f_1 + f_2 = t. \tag{21}$$

To see that Equation (20) and (21) is equivalent to the P_{IV} equation given in Equation (19), we see that substitute $f_0 = t - f_1 - f_2$ in Equation(20) will give us

$$\begin{aligned} f_1' &= f_1(f_1 + 2f_2 - t) + \alpha_1, \\ f_2' &= f_2(t - 2f_1 - f_2) + \alpha_2 \end{aligned} \tag{22}$$

by deriving the first equation respect to t , we get

$$f_1'' = 2f_1 f_1' + 2f_1' f_2 + 2f_1 f_2' - t f_1' - f_1 \tag{23}$$

then eliminate f_2 and f_2' using the equations in (22) , which will give us equation containsf_1 and its derivatives , if letting $y = f_1$, we have

$$y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 - 2ty^2 + (\frac{t^2}{2} - \alpha_0 + \alpha_2)y - \frac{\alpha_1^2}{2y}. \tag{24}$$

Now applying change of variables $t \longrightarrow \sqrt{2}t, y \longrightarrow -y/\sqrt{2}$ and take $\alpha = \alpha_0 - \alpha_2,\ \beta = -2\alpha_1^2$, so we have

$$y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y} \tag{25}$$

which is exactly the fourth Painlevé equation given in Equation (19).

3.2. **Bäcklund transformations.** Here will see the extended affine Weyl group $\tilde{W}(A_2^{(1)})$acts on the symmetric form of the P_{IV}as a group of Bäcklund transformations. Now recall the form of the action of s_i on α_j and define the form of the action of the π on α_j

$$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i, \quad \pi(\alpha_j) = \alpha_{j+1}, \quad (i, j \in \mathbb{Z}/3\mathbb{Z}) \tag{26}$$

where the a_{ij} is the cartan matrix(17), and the action of s_i and π on f_j is given by

$$s_i(f_j) = f_j + u_{ij}\frac{\alpha_i}{f_j}, \quad \pi(f_j) = f_{j+1},, \quad (i, j \in \mathbb{Z}/3\mathbb{Z}) \tag{27}$$

where the u_{ij} is the skew-symmetric matrix

$$(u_{ij})_{0 \le i,j \le 2} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}. \tag{28}$$

For example we have,

$$s_0(\alpha_0) = -\alpha_0,\ s_0(\alpha_1) = \alpha_1 + \alpha_0,\ s_0(\alpha_2) = \alpha_2 + \alpha_0,\ \pi(\alpha_0) = \alpha_1, \tag{29}$$

$$s_0(f_0) = f_0,\ s_0(f_1) = f_1 + \frac{\alpha_0}{f_0},\ s_0(f_2) = f_2 - \frac{\alpha_0}{f_0},\ \pi(f_0) = f_1; \tag{30}$$

and similarly for s_1 and s_2. Bäklund transformations associated with the generators of $\tilde{W}(A_2^{(1)})$ for the symmetric form of P_{IV} are summarised in the following table:

	a_0	a_1	a_2	f_0	f_1	f_2
s_0	$-a_0$	$a_1 + a_0$	$a_2 + a_0$	f_0	$f_1 + \frac{a_0}{f_0}$	$f_2 - \frac{a_0}{f_0}$
s_1	$a_0 + a_1$	$-a_1$	$a_2 + a_1$	$f_0 - \frac{a_1}{f_1}$	f_1	$f_2 + \frac{a_1}{f_1}$
s_2	$a_0 + a_2$	$a_1 + a_2$	$-a_2$	$f_0 + \frac{a_2}{f_2}$	$f_1 - \frac{a_2}{f_2}$	f_2
π	a_1	a_2	a_0	f_1	f_2	f_0

(31)

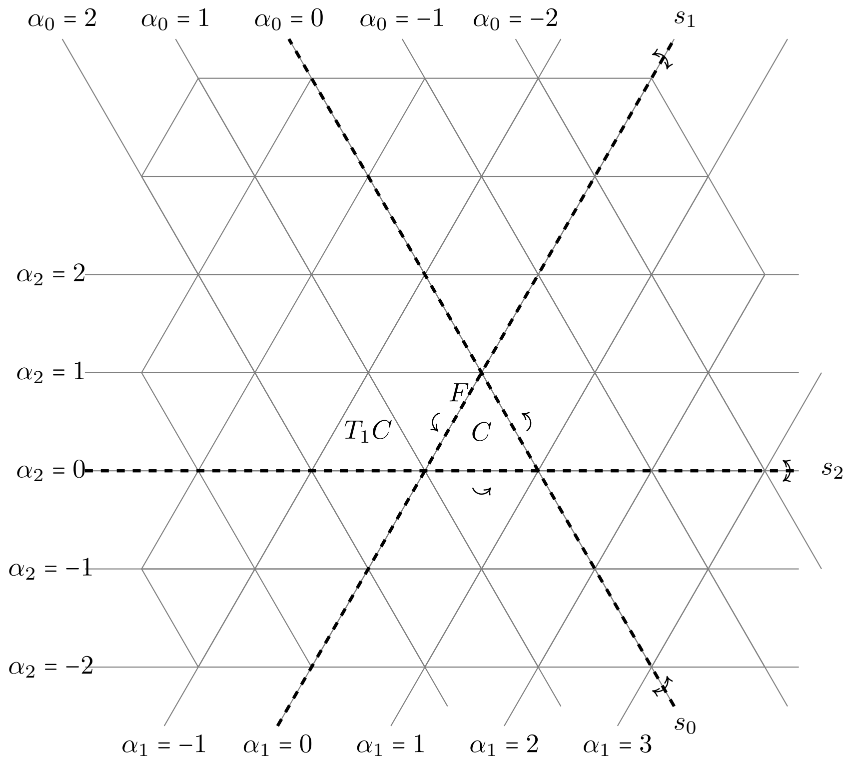

FIGURE 5. Translational actions of $\tilde{W}(A_2^{(1)})$.

and these transformations satisfy the defining relations of $\tilde{W}(A_2^{(1)})$:

$$s_j^2 = 1,\ \ (s_j s_{j+1})^3 = 1,\ \ \pi^3 = 1,\ \ \pi s_j = s_{j+1}\pi. \tag{32}$$

Figure (5) is the two dimensional parameter space with triangular coordinate defined by $\alpha_0 + \alpha_1 + \alpha_2 = 1$, that is each point corresponds to a set of parameter values of the P_{IV} equation. The fundamental triangular region F with barycenter C is bounded by the reflection hyperplanes associated with three reflection generators s_0, s_1, s_2,where the Dynkin diagram automorphism π acts by anti-clockwise rotation of $2\pi/3$ centered at point C. A translation in the plane is given by $T_1 = \pi s_2 s_1$. Let us look at the action of T_1 on the parameters and variables of the symmetric P_{IV} Equation (20) by using Table (31) .

$$
\begin{aligned}
&T_1(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2)\\
=&\pi s_2 s_1(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2)\\
=&\pi s_2(\alpha_0+\alpha_1, -\alpha_1, \alpha_2+\alpha_1; f_0-\frac{\alpha_1}{f_1}, f_1, f_2+\frac{\alpha_1}{f_1})\\
=&\pi(\alpha_0+\alpha_1+2\alpha_2, -\alpha_1-\alpha_2, \alpha_1; f_0+\frac{\alpha_2}{f_2}-\frac{\alpha_1+\alpha_2}{f_1-\frac{\alpha_2}{f_2}}, f_1-\frac{\alpha_2}{f_2}, f_2+\frac{\alpha_1+\alpha_2}{f_1-\frac{\alpha_2}{f_2}})\\
=&(\alpha_1+\alpha_2+2\alpha_0, -\alpha_2-\alpha_0, \alpha_2; f_1+\frac{\alpha_0}{f_0}-\frac{\alpha_2+\alpha_0}{f_2-\frac{\alpha_0}{f_0}}, f_2-\frac{\alpha_0}{f_0}, f_0+\frac{\alpha_2+\alpha_0}{f_2-\frac{\alpha_0}{f_0}})
\end{aligned} \tag{33}
$$

Since we have $\alpha_0+\alpha_1+\alpha_2=1$, then we can simplify the expression of parameter, and finally we have,

$$
\begin{aligned}
&T_1(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2)\\
=&(\alpha_0+1, \alpha_1-1, \alpha_2; f_1+\frac{\alpha_0}{f_0}-\frac{\alpha_2+\alpha_0}{f_2-\frac{\alpha_0}{f_0}}, f_2-\frac{\alpha_0}{f_0}, f_0+\frac{\alpha_2+\alpha_0}{f_2-\frac{\alpha_0}{f_0}})
\end{aligned} \tag{34}
$$

we see that T_1 is exactly Bäklund transformations discussed earlier in Equation (3).

3.3. **Special seed solutions.** The P_{IV}equation has rational special solutions for certain parameter values. One of them is when everything is symmetric with respect to the three indices 0,1,2 , if we consider that is $\alpha_0=\alpha_1=\alpha_2$ and $f_0=f_1=f_2$, then Equation (21) give us

$$
(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{t}{3}, \frac{t}{3}, \frac{t}{3}). \tag{35}
$$

It is easy to see that Equation(35) also satisfies Equation (20).
Now by using symmetry of the equation, we can compute the other special value of the parameters $\alpha_0, \alpha_1, \alpha_2$ which will give rethonal special solution of the P_{IV} equation. Applying T_1 to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ we have $T_1C = (\frac{4}{3}, \frac{-2}{3}, \frac{1}{3})$ (see Figure 5). To get the corresponding special solution of P_{IV} we substitute $(f_0, f_1, f_2) = (\frac{t}{3}, \frac{t}{3}, \frac{t}{3})$in Equation (34):

$$
T_1(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2) = (\frac{4}{3}, \frac{-2}{3}, \frac{1}{3}; \frac{-9-6t^2+t^4}{3t(-3+t^2)}, \frac{-3+t^2}{3t}, \frac{t(3+t^2)}{3(-3+t^2)}). \tag{36}
$$

4. Hamiltonian System

All Painelvé equations have Hamiltonian structures, and having Hamiltonian structure is characteristic of integrable systems[8, 6]. But first we give the general definition of a Hamiltonian system.

Hamiltonian system is a system of ordinary differential equations in dependent unknown variables (q, p) of the form

$$
\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \tag{37}
$$

where H is some function of $(q, p; t)$, known as the Hamilton function, or Hamiltonian, of the system (37).
Here we will mention some advantages of Hamiltonian system.

First, consider $q = q(t), p = p(t)$ are given solutions of the Hamiltonian system, and define

$$h(t) = H(q(t), p(t); t) \tag{38}$$

by plugging the solutions into the Hamiltonian system, and calculating the t-derivative of the Function (38), then

$$\frac{dh}{dt} = \frac{\partial H}{\partial q}\frac{dq}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \tag{39}$$

For a Hamiltonian system, if $H = H(q, p; t)$ does not depend on t explicitly, then the function h(t) is constant. In this case , the Hamiltonian H is a first integral of the system (37). Now if $H = H(q, p; t)$ explicitly contains the variable t , then the function h(t) can be thought of as a measure indicating how far the system is from a differential system which can be integrated by quadrature.

The other advantage of Hamiltonian system is related to Poisson structure and canonical transformation. The Poisson bracket denoted by $\{,\}$, and if we have two functions $\varphi = \varphi(p, q, t)$ and $\psi = \psi(p, q, t)$ then the Poisson bracket for is defined as

$$\{\varphi, \psi\} = \frac{\partial \varphi}{\partial p}\frac{\partial \psi}{\partial q} - \frac{\partial \varphi}{\partial q}\frac{\partial \psi}{\partial p} \tag{40}$$

Furthermore it has a significant properties

i: It is bilinear and skew-symmetric $\{\varphi, \varphi\} = 0, \quad \{\varphi, \psi\} = -\{\psi, \varphi\}$.

ii: It satisfies the Leibniz rule

$$\{\varphi\psi, h\} = \{\varphi, h\}\psi + \varphi\{\psi, h\}, \quad \{\varphi, \psi h\} = \{\varphi, \psi\}h + \psi\{\varphi, h\}.$$

iii: It satisfies the Jacobi rule $\{\varphi, \{\psi, h\}\} + \{\psi, \{h, \varphi\}\} + \{h, \{\varphi, \psi\}\} = 0$.

Since we have

$$\{\varphi, q\} = \frac{\partial \varphi}{\partial p}, \qquad \{\psi, p\} = \frac{\partial \psi}{\partial q} \tag{41}$$

the Hamiltonian system is expressed as

$$q' = \{H, q\}, \quad p' = \{H, p\}. \tag{42}$$

If the variables q,p are subject to Hamiltonian system with Hamiltonian $H = H(q, p; t)$, then we have

$$\frac{d\varphi}{dt} = \{H, \varphi\} + \frac{\partial \varphi}{\partial t}. \tag{43}$$

Now consider two function $\tilde{q} = \tilde{q}(q, p; t)$ and $\tilde{p} = \tilde{p}(q, p; t)$ in (q,p,t) such that $\{\tilde{q}, \tilde{p}\} = 1$, a pair of functions defines a mapping $(q, p) \longrightarrow (\tilde{q}, \tilde{p})$ whose Jacobian determinant is the constant function, the pair $(\tilde{q}, \tilde{p})$ is called a canonical coordinate system, where the mapping $(q, p) \longrightarrow (\tilde{q}, \tilde{p})$ is called a canonical transformation. If we transform the Hamiltonian system by such a canonical transformation, at least locally the transformed equation can be expressed again as a Hamiltonian system.

4.1. **Hamiltonian structure of the P_{IV} equation.** Define the Poisson bracket using the matrix then extract the representation of P_{IV} (the symmetric form)as a Hamiltonian system. First, recall the symmetric form of P_{IV} is given by

$$f_0' = f_0(f_1 - f_2) + \alpha_0, \quad f_1' = f_1(f_2 - f_0) + \alpha_1, \quad f_2' = f_2(f_0 - f_1) + \alpha_2 \tag{44}$$

$$\alpha_0 + \alpha_1 + \alpha_2 = 1 \qquad f_0 + f_1 + f_2 = t \tag{45}$$

and let the skew-symmetric matrix be

$$(u_{ij})_{0\le i,j\le 2} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}. \tag{46}$$

Then, we can determine the Poisson bracket for the variables f_0, f_1, f_2 using matrix (46) by

$$\{f_i, f_j\} = u_{ij}, \ \ \{f_i, f_j\} = -\{f_j, f_i\}, \ \ \{f_i, f_i\} = 0, \ \ (i, j = 0, 1, 2) \tag{47}$$

Hence

$$\{\varphi, \psi\} = \sum_{i,j=0}^{2} \frac{\partial \varphi}{\partial f_j} u_{ij} \frac{\partial \psi}{\partial f_j} \tag{48}$$

where φ, ψ any functions in f_0, f_1, f_2, from this we can reach the Poisson bracket form of a general function φ with f_j which is

$$\begin{aligned} \{\varphi, f_j\} &= \sum_{i,j=0}^{2} \frac{\partial \varphi}{\partial f_i} u_{i,j} \frac{\partial f_j}{\partial f_j} \\ &= \sum_{i=0}^{2} \frac{\partial \varphi}{\partial f_i} u_{i,j} \\ &= \frac{\partial \varphi}{\partial f_0} u_{0,j} + \frac{\partial \varphi}{\partial f_1} u_{1,j} + \frac{\partial \varphi}{\partial f_2} u_{2,j} \\ &= \frac{\partial \varphi}{\partial f_{j-1}} - \frac{\partial \varphi}{\partial f_{j+1}}. \end{aligned} \tag{49}$$

Defining a polynomial with three parameters (b_0, b_1, b_2) as

$$H = f_0 f_1 f_2 + b_0 f_0 + b_1 f_1 + f_2 b_2 \tag{50}$$

so from Equation (49), we get

$$\begin{aligned} \{H, f_0\} &= \frac{\partial H}{\partial f_2} - \frac{\partial H}{\partial f_1} = f_0(f_1 - f_2) + (b_2 - b_1) \\ \{H, f_1\} &= \frac{\partial H}{\partial f_0} - \frac{\partial H}{\partial f_2} = f_1(f_2 - f_0) + (b_0 - b_2) \\ \{H, f_2\} &= \frac{\partial H}{\partial f_1} - \frac{\partial H}{\partial f_0} = f_2(f_0 - f_1) + (b_1 - b_0) \end{aligned} \tag{51}$$

if we take

$$b_2 - b_1 = \alpha_0 - 1, \ \ b_0 - b_2 = \alpha_1, \ \ b_1 - b_0 = \alpha_2 \tag{52}$$

which give us the right hand sides of (44) and we added (-1) in first one to satisfies the condition $\alpha_0 + \alpha_1 + \alpha_2 = 1$, and this means

$$f_0' = \{H, f_0\} + 1, \ \ f_1' = \{H, f_1\}, \ \ f_2' = \{H, f_2\} \tag{53}$$

To write(50) with $\alpha_0, \alpha_1, \alpha_2$ we need to normalization $b_0 + b_1 + b_2 = 0$ for Equation (52) , we get

$$b_0 = \frac{1}{3}(\alpha_1 - \alpha_2), \ \ b_1 = \frac{1}{3}(\alpha_1 + 2\alpha_2), \ \ b_3 = -\frac{1}{3}(2\alpha_1 + \alpha_2). \tag{54}$$

Hence

$$H = f_0 f_1 f_2 + \frac{1}{3}(\alpha_1 - \alpha_2) f_0 + \frac{1}{3}(\alpha_1 + 2\alpha_2) f_1 - \frac{1}{3}(2\alpha_1 + \alpha_2) f_2. \tag{55}$$

taking φ as a general function of f_0, f_1, f_2 , then

$$\varphi' = \sum_{j=0}^{2} \frac{\partial \varphi}{\partial f_j} f_j' = \sum_{j=0}^{2} \frac{\partial \varphi}{\partial f_j} \{H, f_j\} + \frac{\partial \varphi}{\partial f_0} = \{H, \varphi\} + \frac{\partial \varphi}{\partial f_0} \tag{56}$$

Now to derive a representation of the symmetric of the P_{IV} as a Hamiltonian system, we set

$$f_1 = p, \quad , f_2 = q, \quad t = f_0 + f_1 + f_2 \Rightarrow f_0 = t - q - p \tag{57}$$

then

$$H = (t - q - p)pq + \alpha_2 p - \alpha_1 q + \frac{1}{3}(\alpha_1 - \alpha_2)t. \tag{58}$$

Hence

$$\begin{aligned} q' &= \{H, q\} = \frac{\partial H}{\partial p} = q(t - q - 2p) + \alpha_2 \\ p' &= \{H, p\} = \frac{\partial H}{\partial q} = p(2q + p - t) + \alpha_1, \end{aligned} \tag{59}$$

4.2. **τ-Function.** Now we will discuss one of the significant object in the theory of the P_{IV}, the τ-function for a Hamiltonian system see [7], and defined by

$$H = \frac{d}{dt} log\tau = \frac{\tau'}{\tau} \tag{60}$$

note that the indices 0,1,2 in Equation (55) are no longer equal because the special role that gave to the index 0 , so let $h_0 = H, h_1 = \pi(h_0)$ and $h_2 = \pi(h_1)$ so we can recover the symmetry :

$$\begin{aligned} h_0 &= f_0 f_1 f_2 + \frac{1}{3}(\alpha_1 - \alpha_2) f_0 + \frac{1}{3}(\alpha_1 + 2\alpha_2) f_1 - \frac{1}{3}(2\alpha_1 + \alpha_2) f_2 \\ h_1 &= \pi(h_0) = \pi(f_0 f_1 f_2 + \frac{1}{3}(\alpha_1 - \alpha_2) f_0 + \frac{1}{3}(\alpha_1 + 2\alpha_2) f_1 - \frac{1}{3}(2\alpha_1 + \alpha_2) f_2) \\ &= f_1 f_2 f_0 + \frac{1}{3}(\alpha_2 - \alpha_0) f_1 + \frac{1}{3}(\alpha_2 + 2\alpha_0) f_2 - \frac{1}{3}(2\alpha_2 + \alpha_0) f_0 \\ &= f_0 f_1 f_2 - \frac{1}{3}(2\alpha_0 + \alpha_1) f_0 + \frac{1}{3}(\alpha_2 - \alpha_0) f_1 + \frac{1}{3}(\alpha_2 + 2\alpha_0) f_2 \\ h_2 &= \pi(h_1) = \pi(f_0 f_1 f_2 - \frac{1}{3}(2\alpha_0 + \alpha_1) f_0 + \frac{1}{3}(\alpha_2 - \alpha_0) f_1 + \frac{1}{3}(\alpha_2 + 2\alpha_0) f_2) \\ &= f_1 f_2 f_0 - \frac{1}{3}(2\alpha_1 + \alpha_2) f_1 + \frac{1}{3}(\alpha_0 - \alpha_1) f_2 + \frac{1}{3}(\alpha_0 + 2\alpha_1) f_0 \\ &= f_0 f_1 f_2 + \frac{1}{3}(\alpha_0 + 2\alpha_1) f_0 - \frac{1}{3}(2\alpha_1 + \alpha_2) f_1 + \frac{1}{3}(\alpha_0 - \alpha_1) f_2. \end{aligned} \tag{61}$$

To represent f_0, f_1, f_2 in terms of h_0, h_1, h_2, we will do the subtraction , and using the condition $\alpha_0 + \alpha_1 + \alpha_2 = 1$ and $f_0 + f_1 + f_2 = t$

$$\begin{aligned} h_2 - h_1 &= \frac{2}{3}(\alpha_0 + \alpha_1 + \alpha_2) f_0 - \frac{1}{3}(\alpha_0 + \alpha_1 + \alpha_2) f_1 - \frac{1}{3}(\alpha_0 + \alpha_1 + \alpha_2) f_2 \\ &= \frac{2}{3} f_0 - \frac{1}{3} f_1 - \frac{1}{3} f_2 = f_0 - \frac{1}{3}(f_0 + f_1 + f_2) = f_0 - \frac{t}{3}. \end{aligned} \tag{62}$$

That is,

$$f_0 = h_2 - h_1 + \frac{t}{3} \tag{63}$$

and similar for f_1 and f_2, and defining three τ-function τ_0, τ_1, τ_2 and set

$$h_0 = \frac{\tau_0'}{\tau_0}, \;\; h_1 = \frac{\tau_1'}{\tau_1}, \;\; h_2 = \frac{\tau_2'}{\tau_2}, \tag{64}$$

hence

$$\begin{aligned} f_0 &= h_2 - h_1 + \frac{t}{3} = \frac{\tau_2'}{\tau_2} - \frac{\tau_1'}{\tau_1} + \frac{t}{3} \\ f_1 &= h_0 - h_2 + \frac{t}{3} = \frac{\tau_0'}{\tau_0} - \frac{\tau_2'}{\tau_2} + \frac{t}{3} \\ f_2 &= h_1 - h_0 + \frac{t}{3} = \frac{\tau_1'}{\tau_1} - \frac{\tau_0'}{\tau_0} + \frac{t}{3}. \end{aligned} \tag{65}$$

Now we will represent the symmetric form of P_{IV} in terms of h_0, h_1, h_2 , so from Equation (65) we can get

$$\begin{aligned} f_0 - f_1 &= -h_0 - h_1 + 2h_2 \\ f_1 - f_2 &= 2h_0 - h_1 - h_2 \\ f_2 - f_0 &= -h_0 + 2h_1 - h_2 \end{aligned} \tag{66}$$

so the symmetric form of P_{IV} will be

$$\begin{aligned} f_0' = f_0(f_1 - f_2) + \alpha_0 &\Longrightarrow f_0' = f_0(2h_0 - h_1 - h_2) + \alpha_0 \\ f_1' = f_1(f_2 - f_0) + \alpha_1 &\Longrightarrow f_1' = f_1(-h_0 + 2h_1 - h_2) + \alpha_1 \\ f_2' = f_2(f_0 - f_1) + \alpha_2 &\Longrightarrow f_2' = f_2(-h_0 - h_1 + 2h_2) + \alpha_2 \end{aligned} \tag{67}$$

from Equation (56) , we have $h_0' = \{H, h_0\} + \frac{\partial h_0}{\partial f_0}$, and since $h_0 = H$, then

$$\begin{aligned} h_0' &= \{h_0, h_0\} + \frac{\partial h_0}{\partial f_0} = \frac{\partial h_0}{\partial f_0} = f_1 f_2 + \frac{1}{3}(\alpha_1 - \alpha_2) \\ h_1' &= \pi(h_0') = \pi(f_1 f_2 + \frac{1}{3}(\alpha_1 - \alpha_2)) = f_2 f_0 + \frac{1}{3}(\alpha_2 - \alpha_0) \\ h_2' &= \pi(h_1') = \pi(f_2 f_0 + \frac{1}{3}(\alpha_2 - \alpha_0)) = f_0 f_1 + \frac{1}{3}(\alpha_0 - \alpha_1). \end{aligned} \tag{68}$$

Bilinear differential equation for the τ-function, and bilinear means the linear with respect to the individual τ_i, to get this we take the sum of the first two Equations from (68), then using the formula $h_i = \frac{\tau_i'}{\tau_i}$

$$h_0' + h_1' = f_2(f_0 + f_1) + \frac{1}{3}(\alpha_1 - \alpha_0) \tag{69}$$

and from $f_0 + f_1 + f_2 = t \Rightarrow f_0 + f_1 = t - f_2$ then

$$h_0' + h_1' = f_2(t - f_2) - \frac{1}{3}(\alpha_0 - \alpha_1) \tag{70}$$

by substituting f_2 from (65) ,we get

$$
\begin{aligned}
h_0' + h_1' &= (h_1 - h_0 + \frac{t}{3})(h_0 - h_1 + \frac{2t}{3}) - \frac{1}{3}(\alpha_0 - \alpha_1) \\
&= h_0^2 + 2h_0h_1 - h_1^2 + \frac{t}{3}h_1 - \frac{t}{3}h_0 + \frac{2t^2}{9} - \frac{1}{3}(\alpha_0 - \alpha_1) \\
&= -(h_0 - h_1)^2 - \frac{t}{3}(h_0 - h_1) + \frac{2t^2}{9} - \frac{1}{3}(\alpha_0 - \alpha_1) \qquad (71) \\
&(h_0 - h_1)' + (h_0 - h_1)^2 + \frac{t}{3}(h_0 - h_1) - \frac{2t^2}{9} + \frac{1}{3}(\alpha_0 - \alpha_1) = 0 \\
&(\frac{\tau_0'}{\tau_0} - \frac{\tau_1'}{\tau_1})' + (\frac{\tau_0'}{\tau_0} - \frac{\tau_1'}{\tau_1})^2 + \frac{t}{3}(\frac{\tau_0'}{\tau_0} - \frac{\tau_1'}{\tau_1}) - \frac{2t^2}{9} + \frac{1}{3}(\alpha_0 - \alpha_1) = 0
\end{aligned}
$$

multiply by $\tau_0\tau_1$

$$
\tau_0''\tau_1 - 2\tau_0'\tau_1' + \tau_0\tau_1'' + \frac{t}{3}(\tau_0'\tau_1 - \tau_0\tau_1') - (\frac{2t^2}{9} - \frac{1}{3}(\alpha_0 - \alpha_1))\tau_0\tau_1 = 0 \qquad (72)
$$

this is a bilinear differential equations for the pairs (τ_0, τ_1) and similar for (τ_1, τ_2) and (τ_0, τ_2). Equation (72) is in fact the famous Hirota equation.

To have this in the standard form found in literature we define the Hirota derivatives. The Hirota derivatives for given pair (f,g) of functions f and g is

$$
\begin{aligned}
D_t(f.g) &= f'g - fg' \\
D_t^2(f.g) &= f''g - 2f'g' + fg''
\end{aligned} \qquad (73)
$$

and to compute the formula for $D_t^n(f.g)$ apply the Leibniz rule. Now we will express the bilinear Equation(72) in terms of Hirota derivatives

$$
\begin{aligned}
&D_t^2(\tau_0.\tau_1) + \frac{t}{3}D_t(\tau_0.\tau_1) - (\frac{2t^2}{9} - \frac{(\alpha_0 - \alpha_1)}{3})\tau_0.\tau_1 = 0 \\
&(D_t^2 + \frac{t}{3}D_t - \frac{2t^2}{9} - \frac{(\alpha_0 - \alpha_1)}{3}))\tau_0.\tau_1 = 0
\end{aligned} \qquad (74)
$$

and similar for (τ_1, τ_2) and (τ_0, τ_2), so we can write the differential equations for three τ-functions for the symmetric form of P_{IV} by bilinear differential equations of Hirota derivatives as

$$
\begin{aligned}
&(D_t^2 + \frac{t}{3}D_t - \frac{2t^2}{9} - \frac{(\alpha_0 - \alpha_1)}{3}))\tau_0.\tau_1 = 0 \\
&(D_t^2 + \frac{t}{3}D_t - \frac{2t^2}{9} - \frac{(\alpha_1 - \alpha_2)}{3}))\tau_1.\tau_2 = 0 \\
&(D_t^2 + \frac{t}{3}D_t - \frac{2t^2}{9} - \frac{(\alpha_2 - \alpha_0)}{3}))\tau_2.\tau_0 = 0.
\end{aligned} \qquad (75)
$$

If we rewrite the first two equations of (75) using h_i

$$
\begin{aligned}
&(h_0 - h_1)' + (h_0 - h_1)^2 + \frac{t}{3}(h_0 - h_1) - \frac{2t^2}{9} + \frac{\alpha_0 - \alpha_1}{3} = 0 \\
&(h_1 - h_2)' + (h_1 - h_2)^2 + \frac{t}{3}(h_1 - h_2) - \frac{2t^2}{9} + \frac{\alpha_1 - \alpha_2}{3} = 0
\end{aligned} \qquad (76)
$$

by subtracting these two equations, we get

$$
\begin{aligned}
&h_0' - h_2' + h_0^2 - h_2^2 - 2h_1h_0 + 2h_1h_2 + \frac{t}{3}(h_0 - 2h_1 + h_2) + \frac{\alpha_0 - 2\alpha_1 + \alpha_2}{3} = 0 \\
&h_0' - h_1' + (h_0 - h_2)(h_0 - 2h_1 + h_2) + \frac{t}{3}(h_0 - 2h_1 + h_2) + \frac{\alpha_0 + \alpha_1 + \alpha_2}{3} - \alpha_1 = 0 \\
&(h_0 - h_2)' - (h_0 - h_2 + \frac{t}{3})(-h_0 + 2h_1 - h_2) + \frac{1}{3} - \alpha_1 = 0 \\
&(h_0 - h_2)' + \frac{1}{3} = (h_0 - h_2 + \frac{t}{3})(-h_0 + 2h_1 - h_2) + \alpha_1 \\
&(h_0 - h_2 + \frac{t}{3})' = (h_0 - h_2 + \frac{t}{3})(-h_0 + 2h_1 - h_2) + \alpha_1
\end{aligned} \tag{77}
$$

and from (65) we have $f_1 = h_0 - h_2 + \frac{t}{3}$ and also from (66) we have $f_2 - f_0 = -h_0 + 2h_1 - h_2$ then

$$f_1' = f_1(f_2 - f_0) + \alpha_1 \tag{78}$$

which is one of the symmetric form of P_{IV}, and similar for f_0 and f_2.

Now we derive the expression of τ-functions corresponding to the simple special rational solution that is $(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{t}{3}, \frac{t}{3}, \frac{t}{3})$, and we know that

$$
\begin{aligned}
h_0 &= f_0f_1f_2 + \frac{1}{3}(\alpha_1 - \alpha_2)f_0 + \frac{1}{3}(\alpha_1 + 2\alpha_2)f_1 - \frac{1}{3}(2\alpha_1 + \alpha_2)f_2 \\
h_0 &= (\frac{t}{3})^3 + \frac{1}{3}(\frac{t}{3}) - \frac{1}{3}(\frac{t}{3}) = \frac{t^3}{27}
\end{aligned} \tag{79}
$$

as

$$
\begin{aligned}
&\frac{\tau_0'}{\tau_0} = h_0 \\
&\int \frac{\tau_0'}{\tau_0} = \int \frac{t^3}{27} \\
&\ln|\tau_0| + a = \frac{t^4}{108} + b \\
&\tau_0 = exp[\frac{t^4}{108} + c] \\
&\tau_0 = C_0 exp[\frac{t^4}{108}]
\end{aligned} \tag{80}
$$

and similar τ_1, τ_2, hence

$$(\tau_0, \tau_1, \tau_2) = (C_0 exp[\frac{t^4}{108}], C_1 exp[\frac{t^4}{108}], C_2 exp[\frac{t^4}{108}]). \tag{81}$$

In this section, we derived various relations between the different sets of variables of the P_{IV} equations. That is the symmetric variables f_0, f_1, f_2, Hamiltonians h_0, h_1, h_2, and the τ-functions τ_0, τ_1, τ_2. These relations will be used when we investigate the relation between the ring of invariant polynomials of A_2 type and special rational solutions of P_{IV} equation.

5. CONCLUSION

In this thesis, we illustrated properties of Weyl groups on the context of the a finite and affine Weyl groups of type A_2. Moreover, we showed how we can write the fourth Painlevé equations in symmetric form and formulated as a birational realisation of the affine Weyl group of type (A_2). Further, we gave Hamiltonian structure of the P_{IV}. Furthermore, for the fourth Painlevé equations we have discussed some special rational solutions. Finally, we looked at an example of the τ-functions that correspond to the special case of the symmetric form of the P_{IV}.

For future work, we will look at the invariant polynomials of Weyl group of type A_2. Also, we will derive the birational realisation of Weyl group of type A_2 in terms of the τ-function. Finally, we will interpret the τ-function for the case of rational special solutions of the fourth Painlevé equations in terms of the invariant polynomials of Weyl group of type A_2.

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