Lipschitz and commutator estimates, a unified approach.

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Contents

Preface

The present thesis is concerned with a branch of mathematics which is presently rapidly evolving and which is called *Noncommutative Analysis.* Noncommutative analysis appears as a result of applying abstract methods of Banach space theory to the spaces that naturally appear in operator theory. Such spaces appear in quantum mechanics and serve as a natural counterparts of classical Banach spaces.

A corner-stone of classical analysis is the theory of differentiability. During the twentieth century, a great effort was made in attempt to describe and classify various classes of differentiable functions. This led to what we currently know as the Lipschitz and Hölder spaces, the Sobolev spaces, the Besov and Lizorkin-Triebel spaces, see [47, 49, 59, 65, 66]. On the other hand, replacing scalar functions with functions of self-adjoint linear operators, we naturally arrive to the question about differentiable properties of such operator functions and their relation with the classical spaces of differentiable functions. Due to the more complex structure of operator functions, the classes of (operator) differentiable functions which appear are more diverse and require deeper investigation than their classical counterparts, see e.g. [38–40].

Let us use the classical L^p-spaces to exhibit and compare classical and operator differentiability properties of functions. We shall consider only the Lipschitz property.

Recall that the spaces $L^p := L^p(\mathbb{R})$, $1 \leq p \leq \infty$ consist of all Lebesgue measurable functions with integrable p-th power, if $1 \leq p < \infty$, and which are essentially bounded, if $p = \infty$.

Fix a (classical) Lipschitz function $f : \mathbb{R} \mapsto \mathbb{C}$, i.e. a function for which there exists a constant $c_f > 0$, such that

$$|f(t_1) - f(t_2)| \leq c_f\, |t_1 - t_2|, \quad t_1, t_2 \in \mathbb{R}.$$

Let us take $x \in L^\infty$. We denote by $\frac{1}{i}\frac{dx}{dt}$ (or x') the derivative of x, taken in the sense of tempered distributions, see [58, Sections 6.11–6.12]. Let us recall that the chain rule says that, for every Lipschitz function f,

$$\frac{1}{i}\frac{d}{dt}(f(x)) = f'(x) \cdot \frac{1}{i}\frac{dx}{dt}, \tag{0.0.1}$$

where f' is the derivative of the tempered distribution f. If $\frac{1}{i}\frac{dx}{dt} \in L^p$ for some $1 \leq p \leq \infty$, then the latter identity implies that $\frac{1}{i}\frac{d}{dt}(f(x)) \in L^p$ as well and

$$\left\| \frac{1}{i}\frac{d}{dt}(f(x)) \right\|_{L^p} \leq c_f \left\| \frac{1}{i}\frac{dx}{dt} \right\|_{L^p},$$

where c_f is the Lipschitz constant of the function f. The latter relation may serve as a criterion for a function f to be Lipschitz. Indeed, let us introduce the following definition.

A function $f : \mathbb{R} \mapsto \mathbb{C}$ is called p-Lipschitz, for some $1 \leq p \leq \infty$, if and only if there is a constant $c_{f,p}$ such that

$$\left\| \frac{1}{i}\frac{d}{dt}(f(x)) \right\|_{L^p} \leq c_{f,p} \left\| \frac{1}{i}\frac{dx}{dt} \right\|_{L^p} \tag{0.0.2}$$

for every $x \in L^\infty$ such that $\frac{1}{i}\frac{dx}{dt} \in L^p$. The latter inequality should be read as follows. If $x \in L^\infty$ and the derivative $\frac{1}{i}\frac{dx}{dt}$ is a function in L^p, then the composition $f(x)$ is a tempered distribution such that the derivative $\frac{1}{i}\frac{d}{dt}(f(x))$ is a function in L^p and the inequality (0.0.2) holds.

The definition of p-Lipschitz function above still contains some elements of the classical analysis. To eliminate this and to present the latter definition in a purely operator language, let us consider the linear operator $D = \frac{1}{i}\frac{d}{dt} : \mathscr{D}(D) \mapsto L^2$, where

$$\mathscr{D}(D) := \left\{ \xi \in L^2 : \quad \frac{1}{i}\frac{d\xi}{dt} \in L^2 \right\}.$$

Let us recall that an element of L^∞ may be regarded as a bounded linear operator m_x on L^2, where

$$m_x(\xi) = x\xi, \quad x \in L^\infty, \ \xi \in L^2$$

is a multiplication operator. From this point of view, we clearly have

$$\begin{aligned}
[D, x](\xi) &:= Dx(\xi) - xD(\xi) \\
&= \frac{1}{i}\frac{d}{dt}(x\xi) - x\,\frac{1}{i}\frac{d\xi}{dt} \\
&= \frac{1}{i}\frac{dx}{dt}\,\xi, \quad \xi \in L^2.
\end{aligned}$$

We shall give strict meaning to the manipulations above in Section 2.2.1 below. What is currently important is the fact that the derivative $\frac{1}{i}\frac{dx}{dt}$ is now interpreted in the operator sense. Thus, the definition of p-Lipschitz function may be now reformulated as follows.

A function $f : \mathbb{R} \mapsto \mathbb{C}$ is called p-Lipschitz, for some $1 \leq p \leq \infty$, if and only if there is a constant $c_{f,p}$ such that

$$\|[D, f(x)]\|_{L^p} \leq c_{f,p} \; \|[D, x]\|_{L^p} \tag{0.0.3}$$

for every $x \in L^\infty$ such that $[D, x] \in L^p$. The latter inequality should be read as follows. If $x \in L^\infty$ and the commutator $[D, x]$ exists and belongs to L^p, then the commutator $[D, f(x)]$ exists also and belongs to L^p. Moreover, the inequality (0.0.3) holds.

The following result describes relation between the notion of (classical) Lipschitz function and a p-Lipschitz function.

Theorem 0.0.1. *Let $f : \mathbb{R} \mapsto \mathbb{C}$ be a function. The following statements are equivalent:*

(i) *the function f is Lipschitz;*

(ii) *the function f is p-Lipschitz, for some $1 \leq p \leq \infty$;*

(iii) *the function f is p-Lipschitz, for every $1 \leq p \leq \infty$.*

After this introduction, we shall discuss the subject of the manuscript with more details. At the heart of noncommutative analysis is the notion of a *von Neumann algebra* $\mathcal{M}$. Basic examples of von Neumann algebras are the algebra L^∞ acting as multiplication operators on the corresponding L^2 space and the collection $\mathbb{M}_n$, $n \geq 1$ of all $n \times n$-matrices with complex entries. The theory of von Neumann algebras was developed as a significant part of operator theory in the mid-twenties, see [36, 61, 62]. Most of the text deals with *semi-finite* von Neumann algebras. The latter means that the algebra possesses *a normal semi-finite trace* which is a positive linear functional τ possessing the additional property that

$$\tau(xy) = \tau(yx), \quad x, y \in \mathcal{M},$$

and which has certain continuity properties familiar from classical integration theory. The trace plays the role of a measure in the noncommutative analysis. It is usually referred to as *the noncommutative measure.* In the case that the algebra $\mathcal{M}$ is L^∞, the trace is given by Lebesgue integration and in the case that the algebra $\mathcal{M}$ is $\mathbb{M}_n$, $n \geq 1$ the trace is given be the standard trace on matrices, i.e. the functional which is equal to sum of all diagonal entries of a matrix. In noncommutative analysis, the couple $(\mathcal{M}, \tau)$, where $\mathcal{M}$ is a semi-finite von Neumann algebra and τ is a normal semi-finite trace on $\mathcal{M}$, plays the same role as the couple $(\mathbb{R}, dt)$, where $\mathbb{R}$ is the real line and dt is Lebesgue measure, does in the classical analysis.

Having the couple $(\mathcal{M}, \tau)$ at our disposal, we can construct an analogue of measurable functions — *τ-measurable operators,* see Section 1.1.4. The collection of all τ-measurable operators is frequently denoted by $\tilde{\mathcal{M}}$. The collection $\tilde{\mathcal{M}}$ serves as a source for construction of all noncommutative analogues of classical symmetric function spaces. For instance, the noncommutative L^p-space, $1 \leq p < \infty$, which we shall denote as $\mathcal{L}^p = L^p(\mathcal{M}, \tau)$, are defined as

$$\mathcal{L}^p := \left\{ x \in \tilde{\mathcal{M}} : \quad [\tau(|x|^p)]^{\frac{1}{p}} < \infty \right\}.$$

The space $\mathcal{L}^p$ is equipped with the norm

$$\|x\|_{\mathcal{L}^p} := [\tau(|x|^p)]^{\frac{1}{p}}.$$

We shall also consider noncommutative symmetric spaces of measurable operators which are analogues of the classical rearrangement invariant (r.i.) function spaces, see Section 1.4. The reader can find the study of classical r.i. function spaces in [42–44]. Let us denote a noncommutative symmetric space by $\mathcal{E} = E(\mathcal{M}, \tau)$. In the case $\mathcal{M} = L^\infty$, the spaces $\mathcal{L}^p$ coincide with the classical L^p-spaces and in the case $\mathcal{M} = \mathbb{M}_n$, $n \geq 1$, the spaces $\mathcal{L}^p$ consist of all $n \times n$ matrices equipped with p-th Schatten-von Neumann norm. These classes are frequently denoted by $\mathcal{C}_n^p$ and are studied in [31].

Let a and b be two self-adjoint linear operators. Suppose that $a - b \in \mathcal{L}^p$, for some $1 \leq p < \infty$. The first problem to be studied is the description of functions such that $f(a) - f(b) \in \mathcal{L}^p$ and for which the following estimate holds

$$\|f(a) - f(b)\|_{\mathcal{L}^p} \leq c_f \; \|a - b\|_{\mathcal{L}^p} \tag{0.0.4}$$

for some positive constant $c_f > 0$.

A related problem is obtained if the Lipschitz type estimate in (0.0.4) is replaced with estimates on commutators. That is, let D be a linear self-adjoint operator and x be a bounded linear self-adjoint operator. If $[D, x] \in \mathcal{L}^p$, then we study when $[D, f(x)] \in \mathcal{L}^p$ and there is a constant c_f depending on f such that

$$\|[D, f(x)]\|_{\mathcal{L}^p} \leq c_f \; \|[D, x]\|_{\mathcal{L}^p}. \tag{0.0.5}$$

One of the first efforts to study these questions was made by Davies [19], who established that the estimate (0.0.4) holds for the absolute value function in the setting of $\mathcal{M} = \mathbb{M}_n$, $n \geq 1$. It was proved by Kosaki [41] that in the latter setting the Lipschitz estimate (0.0.4) is equivalent to the commutator estimate (0.0.5) for the absolute value function. Subsequently, the authors of [27], extended the Lipschitz and commutator estimates to the setting of an arbitrary semi-finite von Neumann algebra under the assumption that the operators a, b and D in (0.0.4) and (0.0.5) are τ-measurable.

Let us now look at the estimate (0.0.5) in the setting of the algebra $\mathcal{M} = \mathbb{M}_n$, $n \geq 1$. Let us suppose for simplicity that the operator x is diagonal, i.e.

$$x = \sum_{k=1}^{n} \lambda_k \, E_k, \quad \lambda_k \in \mathbb{R},$$

where E_k, $1 \leq k \leq n$ are diagonal matrix units. Let $f : \mathbb{R} \mapsto \mathbb{C}$. We consider the function

$$\psi_f(\lambda, \mu) := \frac{f(\lambda) - f(\mu)}{\lambda - \mu}, \quad \lambda \neq \mu \text{ and } \psi_f(\lambda, \lambda) = 0.$$

We have the following chain of identities

$$\begin{aligned}
[D, f(x)] &= \sum_{j,k=1}^{n} E_j \, [D, f(x)] \, E_k \\
&= \sum_{j,k=1}^{n} E_j \left[D, \sum_{s=1}^{n} f(\lambda_s) \, E_s \right] E_k \\
&= \sum_{j,k=1}^{n} (f(\lambda_j) - f(\lambda_k)) \, E_j D E_k \\
&= \sum_{j,k=1}^{n} \psi_k(\lambda_j, \lambda_k) \, (\lambda_j - \lambda_k) \, E_j D E_k \\
&= \sum_{j,k=1}^{n} \psi_f(\lambda_j, \lambda_k) \, E_j \left[D, \sum_{s=1}^{n} \lambda_s \, E_s \right] E_k \\
&= \sum_{j,k=1}^{n} \psi_f(\lambda_j, \lambda_k) \, E_j [D, x] E_k \\
&= T_{\psi_f}([D, x]),
\end{aligned} \tag{0.0.6}$$

where T_{ψ_f} is the linear operator defined by

$$T_{\psi_f}(y) := \sum_{j,k=1}^{n} \psi_f(\lambda_j, \lambda_k) \, E_j y E_k, \quad y \in \mathbb{M}_n.$$

Thus, we readily obtain the estimate (0.0.5) as soon as we have that the operator T_{ψ_f} is bounded. The operator T_{ψ_f} is an example of what is known as a *double operator integral.*

M. Birman and M. Solomyak developed the theory of double operator integrals in the setting of type I von Neumann algebras in the series of papers [6–8]. Subsequently, they established, [9], the estimates (0.0.4) and (0.0.5) in the setting of type I von Neumann algebras, for every function f such that the derivative f' is of bounded total variation and for every $1 < p < \infty$.

The double operator integral technique has been recently extended to the type II setting in [21,22,24], where the estimates (0.0.4) and (0.0.5) are obtained in the setting of general symmetric operator spaces $\mathcal{E}$ and general semifinite von Neumann algebras $\mathcal{M}$. However, in the case of reflexive noncommutative L_p-spaces on $\mathcal{M}$, the results obtained in those articles are weaker than the corresponding results of [9] due to the restrictive assumption of τ-measurability imposed on the operators a, b and D in those papers. The fact that this assumption is restrictive is clearly seen from the fact that in all interesting applications of the estimates (0.0.4) and (0.0.5) in quantum mechanics (see e.g. [11]) and in noncommutative geometry (see e.g. [17]) it is not satisfied. In fact, even in the simplest example of interest (see Section 2.2.1), when the algebra $\mathcal{M} = L_\infty(\mathbb{R})$ acts on $\mathcal{H} = L_2(\mathbb{R})$ via multiplication and the operator D is given by the differentiation $\frac{1}{i}\frac{d}{dt}$, it is clear that D does not belong to the algebra $\tilde{\mathcal{M}}$ (furthermore, it is not even affiliated with $\mathcal{M}$). The problem of obtaining the estimates (0.0.4) and (0.0.5) for general self-adjoint operators and not just for τ-measurable and for not necessarily continuously differentiable functions f is non-trivial: the difference in the assumptions renders many existing techniques inapplicable. For example, the fact that our functions are not C^1 prevents us from using the approach developed in [21] (based, in turn, on an earlier idea from [1]), which ultimately views the first inequality in (0.0.5) as a statement that f is an operator differentiable function and thus must be continuously differentiable. It is, perhaps, also instructive to refer to [12] where a problem, arising in the type II quantized calculus similar to the estimates (0.0.4) and (0.0.5), has a completely different resolution depending on whether operators in question were τ-measurable or just affiliated (see [12, Theorem 0.3 (i) and (ii)] and the discussion on p.144).

In Chapter 2, we establish the estimates (0.0.4) and (0.0.5) for every function with derivative of bounded total variation in the general semi-finite setting without the restriction of τ-measurability, (see Theorem 2.3.3 and Theorem 2.3.20).

We now briefly explain the technical difficulties (and our strategy) arising in the setting of commutator estimates. Suppose that the operator D is not τ-measurable and that $x \in \mathcal{M}$. Among various definitions of the symbol $[D,x]$ in the literature (allowing the treatment of the situation when all three operators, D, x and $[D,x]$ may be unbounded), we have chosen the least restrictive approach articulated in [11], allowing us to consider a wider class of operators than those in [21] and [12]. We say that $[D,x] \in \mathcal{E}$ if and only if the subspace $x^{-1}(\mathscr{D}(D)) \cap \mathscr{D}(D)$ contains a core of the operator D which is invariant under the unitary group $\{e^{itD}\}_{t\in\mathbb{R}}$ and the operator $Dx - xD$, initially defined on that subspace, is closable with closure $[D,x]$ belonging to $\mathcal{L}^p$. Assume (for brevity) that the core above coincides with $\mathscr{D}(a)$ (it is of interest to observe that the latter assumption is *automatically* satisfied in the type I setting and more generally, when $\mathcal{L}^p \subseteq \mathcal{M}$, see Lemma 2.0.8 below). Then for a (τ-measurable) operator $y := [a,x] \in \mathcal{E}$ with a τ-dense domain (see Chapter 1), we have $\mathscr{D}(D) \subseteq \mathscr{D}(y)$. Now, according to our general strategy in proving the estimate (0.0.5) we consider the double operator integral T_{ψ_f} (see Definition 1.7.3 below for a generalized approach to the operator appearing in (0.0.6) above), which is bounded on $\mathcal{L}^p$ and for which the relation

$$[D, f(x)] = T_{\psi_f}([D,x]), \tag{0.0.7}$$

holds. The double operator integral T_{ψ_f} is a bounded linear operator on $\mathcal{L}^p$ defined via a complicated process of vector-valued integration with respect to a finitely additive measure and the relationship between the domain of the image $z := T_{\psi_f}([D,x])$ and that of D is not clear. On the other hand, if (0.0.7) were to hold, we should have (at the very least) that $\mathscr{D}(D) \subseteq \mathscr{D}(z)$ and $(f(x))^{-1}(\mathscr{D}(D)) \cap \mathscr{D}(D) \neq \emptyset$. This is a serious obstacle, which is specific to the type II setting. Indeed, if $\mathcal{M}$ is a type I factor, then the operator z is necessarily bounded (due to the obvious embedding $\mathcal{C}^p \subseteq \mathcal{M}$) and so, the embedding $\mathscr{D}(D) \subseteq \mathscr{D}(z) = \mathcal{H}$ is trivial.

We solve this problem and achieve a complete extension of the type I result of [9] to a general semifinite von Neumann algebra $\mathcal{M}$ under the additional assumption that the latter algebra is acting on $\mathcal{H}$ in standard form. In many circumstances the latter assumption is automatically satisfied and in many cases our results may be transferred to general von Neumann algebras. We illustrate this in Section 2.4 of this manuscript suggesting a simple and straightforward variant of the proofs of corresponding type I results (see Section 2.4.3), yielding an additional insight into methods used in [9] and those in [21, 22, 24], (see Sections 2.4.1 and 2.4.2).

In Chapter 3, we establish converse results to those of Chapter 2. Namely, we construct an example of a C^1-function such that the estimate

$$\|[D, f(x)\|_{\mathcal{E}} \leq c\,\|[D, x]\|_{\mathcal{E}} \tag{0.0.8}$$

fails for some operators D and x and any constant $c > 0$. The example for the special case $\mathcal{E} = L^\infty$ was constructed by A. McIntosh [45] (see also the recent development of this example in [67]). We shall extend this example to symmetric spaces $\mathcal{E}$ with trivial Boyd indices. The latter includes the space $\mathcal{E} = \mathcal{L}^1$.

The results given in this thesis are partially published in [34, 53–56] and presented at Special session in Harmonic Analysis, AustMS2005, September 27–30, 2005, Perth, Western Australia; CMA/AMSI Research Symposium "Asymptotic Geometric Analysis, Harmonic Analysis, and related topics" Murramarang, NSW, 21–24, February 2006; CMA Workshop "Spectral Theory", 2–5, April 2007.

Declaration

I certify that this thesis does not incorporate without acknowledgment any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge and belief it does not contain any material previously published or written by another person except where due reference is made in the text.

Denis Potapov, Candidate

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