Lipschitz and commutator estimates, a unified approach.

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Contents

Preface

The present thesis is concerned with a branch of mathematics which is presently rapidly evolving and which is called *Noncommutative Analysis.* Noncommutative analysis appears as a result of applying abstract methods of Banach space theory to the spaces that naturally appear in operator theory. Such spaces appear in quantum mechanics and serve as a natural counterparts of classical Banach spaces.

A corner-stone of classical analysis is the theory of differentiability. During the twentieth century, a great effort was made in attempt to describe and classify various classes of differentiable functions. This led to what we currently know as the Lipschitz and Hölder spaces, the Sobolev spaces, the Besov and Lizorkin-Triebel spaces, see [47, 49, 59, 65, 66]. On the other hand, replacing scalar functions with functions of self-adjoint linear operators, we naturally arrive to the question about differentiable properties of such operator functions and their relation with the classical spaces of differentiable functions. Due to the more complex structure of operator functions, the classes of (operator) differentiable functions which appear are more diverse and require deeper investigation than their classical counterparts, see e.g. [38–40].

Let us use the classical L^p-spaces to exhibit and compare classical and operator differentiability properties of functions. We shall consider only the Lipschitz property.

Recall that the spaces $L^p := L^p(\mathbb{R})$, $1 \leq p \leq \infty$ consist of all Lebesgue measurable functions with integrable p-th power, if $1 \leq p < \infty$, and which are essentially bounded, if $p = \infty$.

Fix a (classical) Lipschitz function $f : \mathbb{R} \mapsto \mathbb{C}$, i.e. a function for which there exists a constant $c_f > 0$, such that

$$|f(t_1) - f(t_2)| \leq c_f \, |t_1 - t_2|, \quad t_1, t_2 \in \mathbb{R}.$$

Let us take $x \in L^\infty$. We denote by $\frac{1}{i}\frac{dx}{dt}$ (or x') the derivative of x, taken in the sense of tempered distributions, see [58, Sections 6.11–6.12]. Let us recall that the chain rule says that, for every Lipschitz function f,

$$\frac{1}{i}\frac{d}{dt}(f(x)) = f'(x) \cdot \frac{1}{i}\frac{dx}{dt}, \tag{0.0.1}$$

where f' is the derivative of the tempered distribution f. If $\frac{1}{i}\frac{dx}{dt} \in L^p$ for some $1 \le p \le \infty$, then the latter identity implies that $\frac{1}{i}\frac{d}{dt}(f(x)) \in L^p$ as well and

$$\left\|\frac{1}{i}\frac{d}{dt}(f(x))\right\|_{L^p} \le c_f \left\|\frac{1}{i}\frac{dx}{dt}\right\|_{L^p},$$

where c_f is the Lipschitz constant of the function f. The latter relation may serve as a criterion for a function f to be Lipschitz. Indeed, let us introduce the following definition.

A function $f : \mathbb{R} \mapsto \mathbb{C}$ is called p-Lipschitz, for some $1 \le p \le \infty$, if and only if there is a constant $c_{f,p}$ such that

$$\left\|\frac{1}{i}\frac{d}{dt}(f(x))\right\|_{L^p} \le c_{f,p} \left\|\frac{1}{i}\frac{dx}{dt}\right\|_{L^p} \tag{0.0.2}$$

for every $x \in L^\infty$ such that $\frac{1}{i}\frac{dx}{dt} \in L^p$. The latter inequality should be read as follows. If $x \in L^\infty$ and the derivative $\frac{1}{i}\frac{dx}{dt}$ is a function in L^p, then the composition $f(x)$ is a tempered distribution such that the derivative $\frac{1}{i}\frac{d}{dt}(f(x))$ is a function in L^p and the inequality (0.0.2) holds.

The definition of p-Lipschitz function above still contains some elements of the classical analysis. To eliminate this and to present the latter definition in a purely operator language, let us consider the linear operator $D = \frac{1}{i}\frac{d}{dt} : \mathscr{D}(D) \mapsto L^2$, where

$$\mathscr{D}(D) := \left\{\xi \in L^2 : \quad \frac{1}{i}\frac{d\xi}{dt} \in L^2\right\}.$$

Let us recall that an element of L^∞ may be regarded as a bounded linear operator m_x on L^2, where

$$m_x(\xi) = x\xi, \quad x \in L^\infty, \ \xi \in L^2$$

is a multiplication operator. From this point of view, we clearly have

$$\begin{aligned}[D, x](\xi) &:= Dx(\xi) - xD(\xi) \\ &= \frac{1}{i}\frac{d}{dt}(x\xi) - x\,\frac{1}{i}\frac{d\xi}{dt} \\ &= \frac{1}{i}\frac{dx}{dt}\,\xi, \quad \xi \in L^2.\end{aligned}$$

We shall give strict meaning to the manipulations above in Section 2.2.1 below. What is currently important is the fact that the derivative $\frac{1}{i}\frac{dx}{dt}$ is now interpreted in the operator sense. Thus, the definition of p-Lipschitz function may be now reformulated as follows.

A function $f:\mathbb{R}\mapsto\mathbb{C}$ is called p-Lipschitz, for some $1\leq p\leq\infty$, if and only if there is a constant $c_{f,p}$ such that

$$\|[D,f(x)]\|_{L^p}\leq c_{f,p}\ \|[D,x]\|_{L^p} \tag{0.0.3}$$

for every $x\in L^\infty$ such that $[D,x]\in L^p$. The latter inequality should be read as follows. If $x\in L^\infty$ and the commutator $[D,x]$ exists and belongs to L^p, then the commutator $[D,f(x)]$ exists also and belongs to L^p. Moreover, the inequality (0.0.3) holds.

The following result describes relation between the notion of (classical) Lipschitz function and a p-Lipschitz function.

Theorem 0.0.1. *Let $f:\mathbb{R}\mapsto\mathbb{C}$ be a function. The following statements are equivalent:*

(i) *the function f is Lipschitz;*

(ii) *the function f is p-Lipschitz, for some $1\leq p\leq\infty$;*

(iii) *the function f is p-Lipschitz, for every $1\leq p\leq\infty$.*

After this introduction, we shall discuss the subject of the manuscript with more details. At the heart of noncommutative analysis is the notion of a *von Neumann algebra* $\mathcal{M}$. Basic examples of von Neumann algebras are the algebra L^∞ acting as multiplication operators on the corresponding L^2 space and the collection $\mathbb{M}_n$, $n\geq 1$ of all $n\times n$-matrices with complex entries. The theory of von Neumann algebras was developed as a significant part of operator theory in the mid-twenties, see [36, 61, 62]. Most of the text deals with *semi-finite* von Neumann algebras. The latter means that the algebra possesses *a normal semi-finite trace* which is a positive linear functional τ possessing the additional property that

$$\tau(xy)=\tau(yx),\quad x,y\in\mathcal{M},$$

and which has certain continuity properties familiar from classical integration theory. The trace plays the role of a measure in the noncommutative analysis. It is usually referred to as *the noncommutative measure.* In the case that the algebra $\mathcal{M}$ is L^∞, the trace is given by Lebesgue integration and in the case that the algebra $\mathcal{M}$ is $\mathbb{M}_n$, $n \geq 1$ the trace is given be the standard trace on matrices, i.e. the functional which is equal to sum of all diagonal entries of a matrix. In noncommutative analysis, the couple $(\mathcal{M}, \tau)$, where $\mathcal{M}$ is a semi-finite von Neumann algebra and τ is a normal semi-finite trace on $\mathcal{M}$, plays the same role as the couple $(\mathbb{R}, dt)$, where $\mathbb{R}$ is the real line and dt is Lebesgue measure, does in the classical analysis.

Having the couple $(\mathcal{M}, \tau)$ at our disposal, we can construct an analogue of measurable functions — *τ-measurable operators,* see Section 1.1.4. The collection of all τ-measurable operators is frequently denoted by $\tilde{\mathcal{M}}$. The collection $\tilde{\mathcal{M}}$ serves as a source for construction of all noncommutative analogues of classical symmetric function spaces. For instance, the noncommutative L^p-space, $1 \leq p < \infty$, which we shall denote as $\mathcal{L}^p = L^p(\mathcal{M}, \tau)$, are defined as

$$\mathcal{L}^p := \left\{ x \in \tilde{\mathcal{M}} : \quad [\tau(|x|^p)]^{\frac{1}{p}} < \infty \right\}.$$

The space $\mathcal{L}^p$ is equipped with the norm

$$\|x\|_{\mathcal{L}^p} := [\tau(|x|^p)]^{\frac{1}{p}}.$$

We shall also consider noncommutative symmetric spaces of measurable operators which are analogues of the classical rearrangement invariant (r.i.) function spaces, see Section 1.4. The reader can find the study of classical r.i. function spaces in [42–44]. Let us denote a noncommutative symmetric space by $\mathcal{E} = E(\mathcal{M}, \tau)$. In the case $\mathcal{M} = L^\infty$, the spaces $\mathcal{L}^p$ coincide with the classical L^p-spaces and in the case $\mathcal{M} = \mathbb{M}_n$, $n \geq 1$, the spaces $\mathcal{L}^p$ consist of all $n \times n$ matrices equipped with p-th Schatten-von Neumann norm. These classes are frequently denoted by $\mathcal{C}_n^p$ and are studied in [31].

Let a and b be two self-adjoint linear operators. Suppose that $a - b \in \mathcal{L}^p$, for some $1 \leq p < \infty$. The first problem to be studied is the description of functions such that $f(a) - f(b) \in \mathcal{L}^p$ and for which the following estimate holds

$$\|f(a) - f(b)\|_{\mathcal{L}^p} \leq c_f \ \|a - b\|_{\mathcal{L}^p} \tag{0.0.4}$$

for some positive constant $c_f > 0$.

A related problem is obtained if the Lipschitz type estimate in (0.0.4) is replaced with estimates on commutators. That is, let D be a linear self-adjoint operator and x be a bounded linear self-adjoint operator. If $[D, x] \in \mathcal{L}^p$, then we study when $[D, f(x)] \in \mathcal{L}^p$ and there is a constant c_f depending on f such that

$$\|[D, f(x)]\|_{\mathcal{L}^p} \leq c_f \ \|[D, x]\|_{\mathcal{L}^p}. \tag{0.0.5}$$

One of the first efforts to study these questions was made by Davies [19], who established that the estimate (0.0.4) holds for the absolute value function in the setting of $\mathcal{M} = \mathbb{M}_n$, $n \geq 1$. It was proved by Kosaki [41] that in the latter setting the Lipschitz estimate (0.0.4) is equivalent to the commutator estimate (0.0.5) for the absolute value function. Subsequently, the authors of [27], extended the Lipschitz and commutator estimates to the setting of an arbitrary semi-finite von Neumann algebra under the assumption that the operators a, b and D in (0.0.4) and (0.0.5) are τ-measurable.

Let us now look at the estimate (0.0.5) in the setting of the algebra $\mathcal{M} = \mathbb{M}_n$, $n \geq 1$. Let us suppose for simplicity that the operator x is diagonal, i.e.

$$x = \sum_{k=1}^{n} \lambda_k \, E_k, \quad \lambda_k \in \mathbb{R},$$

where E_k, $1 \leq k \leq n$ are diagonal matrix units. Let $f : \mathbb{R} \mapsto \mathbb{C}$. We consider the function

$$\psi_f(\lambda, \mu) := \frac{f(\lambda) - f(\mu)}{\lambda - \mu}, \quad \lambda \neq \mu \quad \text{and} \quad \psi_f(\lambda, \lambda) = 0.$$

We have the following chain of identities

$$\begin{aligned}
[D, f(x)] &= \sum_{j,k=1}^{n} E_j \, [D, f(x)] \, E_k \\
&= \sum_{j,k=1}^{n} E_j \left[D, \sum_{s=1}^{n} f(\lambda_s) \, E_s \right] E_k \\
&= \sum_{j,k=1}^{n} (f(\lambda_j) - f(\lambda_k)) \, E_j D E_k \\
&= \sum_{j,k=1}^{n} \psi_k(\lambda_j, \lambda_k) \, (\lambda_j - \lambda_k) \, E_j D E_k \\
&= \sum_{j,k=1}^{n} \psi_f(\lambda_j, \lambda_k) \, E_j \left[D, \sum_{s=1}^{n} \lambda_s \, E_s \right] E_k \\
&= \sum_{j,k=1}^{n} \psi_f(\lambda_j, \lambda_k) \, E_j [D, x] E_k \\
&= T_{\psi_f}([D, x]),
\end{aligned} \tag{0.0.6}$$

where T_{ψ_f} is the linear operator defined by

$$T_{\psi_f}(y) := \sum_{j,k=1}^{n} \psi_f(\lambda_j, \lambda_k) \, E_j y E_k, \quad y \in \mathbb{M}_n.$$

Thus, we readily obtain the estimate (0.0.5) as soon as we have that the operator T_{ψ_f} is bounded. The operator T_{ψ_f} is an example of what is known as a *double operator integral.*

M. Birman and M. Solomyak developed the theory of double operator integrals in the setting of type I von Neumann algebras in the series of papers [6–8]. Subsequently, they established, [9], the estimates (0.0.4) and (0.0.5) in the setting of type I von Neumann algebras, for every function f such that the derivative f' is of bounded total variation and for every $1 < p < \infty$.

The double operator integral technique has been recently extended to the type II setting in [21,22,24], where the estimates (0.0.4) and (0.0.5) are obtained in the setting of general symmetric operator spaces $\mathcal{E}$ and general semifinite von Neumann algebras $\mathcal{M}$. However, in the case of reflexive noncommutative L_p-spaces on $\mathcal{M}$, the results obtained in those articles are weaker than the corresponding results of [9] due to the restrictive assumption of τ-measurability imposed on the operators a, b and D in those papers. The fact that this assumption is restrictive is clearly seen from the fact that in all interesting applications of the estimates (0.0.4) and (0.0.5) in quantum mechanics (see e.g. [11]) and in noncommutative geometry (see e.g. [17]) it is not satisfied. In fact, even in the simplest example of interest (see Section 2.2.1), when the algebra $\mathcal{M} = L_\infty(\mathbb{R})$ acts on $\mathcal{H} = L_2(\mathbb{R})$ via multiplication and the operator D is given by the differentiation $\frac{1}{i}\frac{d}{dt}$, it is clear that D does not belong to the algebra $\tilde{\mathcal{M}}$ (furthermore, it is not even affiliated with $\mathcal{M}$). The problem of obtaining the estimates (0.0.4) and (0.0.5) for general self-adjoint operators and not just for τ-measurable and for not necessarily continuously differentiable functions f is non-trivial: the difference in the assumptions renders many existing techniques inapplicable. For example, the fact that our functions are not C^1 prevents us from using the approach developed in [21] (based, in turn, on an earlier idea from [1]), which ultimately views the first inequality in (0.0.5) as a statement that f is an operator differentiable function and thus must be continuously differentiable. It is, perhaps, also instructive to refer to [12] where a problem, arising in the type II quantized calculus similar to the estimates (0.0.4) and (0.0.5), has a completely different resolution depending on whether operators in question were τ-measurable or just affiliated (see [12, Theorem 0.3 (i) and (ii)] and the discussion on p.144).

In Chapter 2, we establish the estimates (0.0.4) and (0.0.5) for every function with derivative of bounded total variation in the general semi-finite setting without the restriction of τ-measurability, (see Theorem 2.3.3 and Theorem 2.3.20).

We now briefly explain the technical difficulties (and our strategy) arising in the setting of commutator estimates. Suppose that the operator D is not τ-measurable and that $x \in \mathcal{M}$. Among various definitions of the symbol $[D, x]$ in the literature (allowing the treatment of the situation when all three operators, D, x and $[D, x]$ may be unbounded), we have chosen the least restrictive approach articulated in [11], allowing us to consider a wider class of operators than those in [21] and [12]. We say that $[D, x] \in \mathcal{E}$ if and only if the subspace $x^{-1}(\mathscr{D}(D)) \cap \mathscr{D}(D)$ contains a core of the operator D which is invariant under the unitary group $\{e^{itD}\}_{t \in \mathbb{R}}$ and the operator $Dx - xD$, initially defined on that subspace, is closable with closure $[D, x]$ belonging to $\mathcal{L}^p$. Assume (for brevity) that the core above coincides with $\mathscr{D}(a)$ (it is of interest to observe that the latter assumption is *automatically* satisfied in the type I setting and more generally, when $\mathcal{L}^p \subseteq \mathcal{M}$, see Lemma 2.0.8 below). Then for a (τ-measurable) operator $y := [a, x] \in \mathcal{E}$ with a τ-dense domain (see Chapter 1), we have $\mathscr{D}(D) \subseteq \mathscr{D}(y)$. Now, according to our general strategy in proving the estimate (0.0.5) we consider the double operator integral T_{ψ_f} (see Definition 1.7.3 below for a generalized approach to the operator appearing in (0.0.6) above), which is bounded on $\mathcal{L}^p$ and for which the relation

$$[D, f(x)] = T_{\psi_f}([D, x]), \tag{0.0.7}$$

holds. The double operator integral T_{ψ_f} is a bounded linear operator on $\mathcal{L}^p$ defined via a complicated process of vector-valued integration with respect to a finitely additive measure and the relationship between the domain of the image $z := T_{\psi_f}([D, x])$ and that of D is not clear. On the other hand, if (0.0.7) were to hold, we should have (at the very least) that $\mathscr{D}(D) \subseteq \mathscr{D}(z)$ and $(f(x))^{-1}(\mathscr{D}(D)) \cap \mathscr{D}(D) \neq \emptyset$. This is a serious obstacle, which is specific to the type II setting. Indeed, if $\mathcal{M}$ is a type I factor, then the operator z is necessarily bounded (due to the obvious embedding $\mathcal{C}^p \subseteq \mathcal{M}$) and so, the embedding $\mathscr{D}(D) \subseteq \mathscr{D}(z) = \mathcal{H}$ is trivial.

We solve this problem and achieve a complete extension of the type I result of [9] to a general semifinite von Neumann algebra $\mathcal{M}$ under the additional assumption that the latter algebra is acting on $\mathcal{H}$ in standard form. In many circumstances the latter assumption is automatically satisfied and in many cases our results may be transferred to general von Neumann algebras. We illustrate this in Section 2.4 of this manuscript suggesting a simple and straightforward variant of the proofs of corresponding type I results (see Section 2.4.3), yielding an additional insight into methods used in [9] and those in [21, 22, 24], (see Sections 2.4.1 and 2.4.2).

In Chapter 3, we establish converse results to those of Chapter 2. Namely, we construct an example of a C^1-function such that the estimate

$$\|[D, f(x)\|_{\mathcal{E}} \leq c\,\|[D, x]\|_{\mathcal{E}} \tag{0.0.8}$$

fails for some operators D and x and any constant $c > 0$. The example for the special case $\mathcal{E} = L^\infty$ was constructed by A. McIntosh [45] (see also the recent development of this example in [67]). We shall extend this example to symmetric spaces $\mathcal{E}$ with trivial Boyd indices. The latter includes the space $\mathcal{E} = \mathcal{L}^1$.

The results given in this thesis are partially published in [34, 53–56] and presented at Special session in Harmonic Analysis, AustMS2005, September 27–30, 2005, Perth, Western Australia; CMA/AMSI Research Symposium "Asymptotic Geometric Analysis, Harmonic Analysis, and related topics" Murramarang, NSW, 21–24, February 2006; CMA Workshop "Spectral Theory", 2–5, April 2007.

Declaration

I certify that this thesis does not incorporate without acknowledgment any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge and belief it does not contain any material previously published or written by another person except where due reference is made in the text.

Denis Potapov, Candidate

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Chapter 1

Introduction

In this preliminary chapter, we gather necessary material to present the results of the thesis. While this material is mostly well-known, some of the results on double operator integrals in Sections 1.7– 1.11 are new.

1.1 Locally convex topologies on Banach spaces

In this section we give a very brief summary of several important locally convex topologies we shall frequently use hereafter. Let X be a Banach space with norm $\|\cdot\|_X$. $(X)_1$ stands for the unit ball in the space X, i.e.

$$(X)_1 := \{x \in X : \quad \|x\|_X \leq 1\}.$$

Let X^* be the dual Banach space, equipped with the dual norm $\|\cdot\|_{X^*}$, i.e.

$$\|f\|_{X^*} := \sup_{x \in (X)_1} |f(x)|.$$

Apart from the norm topology, the dual space X^* induces several weaker topologies on the space X. Namely, if $F \subseteq X^*$ is a linear (possibly not closed) subspace, then the $\sigma(X, F)$-topology is the locally convex topology defined by the family of semi-norms $\{p_f\}_{f \in F}$ such that

$$p_f(x) = |f(x)|, \quad f \in F, \quad x \in X.$$

That is, the net $\{x_\alpha\} \subseteq X$ converges to $x \in X$ if and only if

$$p_f(x_\alpha - x) = |f(x_\alpha - x)| \to 0, \quad f \in F.$$

In particular, the $\sigma(X, X^*)$-topology is called *the weak topology*.

Let X be a Banach space, X^* be the dual and $X^{**} := (X^*)^*$ be the bidual. It is known that the space X isometrically embeds into the bidual $X \subseteq X^{**}$. Besides the weak topology on X^*, we shall consider the $\sigma(X^*, X)$-topology, called *the weak* topology.* For the latter topology, we have

Theorem 1.1.1 (Alaoglu, cf. [69, Theorem II.A.9]). *The closed unit ball $(X^*)_1$ of X^* is weakly* compact.*

Lemma 1.1.2 ([61, Lemma 1.2]). *Let X be a Banach space and F be a (possibly not closed) linear subspace of X^*. $\bar{F}$ stands for the closure of F in the norm of the dual X^*. If f is a linear functional on X, then*

(i) *f is $\sigma(X, F)$-continuous if and only if $f \in F$;*

(ii) *f is $\sigma(X, F)$-continuous on $(X)_1$ if and only if $f \in \bar{F}$;*

(iii) *the topologies $\sigma(X, F)$ and $\sigma(X, \bar{F})$ coincide on $(X)_1$;*

(iv) *if $F = \bar{F}$ and f is $\sigma(X, F)$-continuous on $(X)_1$, then f is $\sigma(X, F)$-continuous on X.*

If X, Y be two normed spaces, then $B(X, Y)$ stands for the collection of all bounded linear operators $X \mapsto Y$. The space $B(X, Y)$ equipped with the operator norm

$$\|T\|_{B(X,Y)} := \sup_{x \in (X)_1} \|Tx\|_Y$$

is a Banach algebra, provided Y is a Banach space. For the sake of brevity, we write $B(X) := B(X, X)$.

An operator $T \in B(X, Y)$ is called *one-dimensional* if and only if it admits the representation

$$T(y) = x \otimes x^*(y) := x^*(y)\, x, \quad x, y \in X, \;\; x^* \in X^*.$$

An operator $T \in B(X, Y)$ is called *finite dimensional* if and only if it is represented as a finite sum of one-dimensional operators.

Lemma 1.1.3. *Let X be a Banach space and let $F \subseteq X^*$ be a linear subspace. Let $T \in B(X)$. If the operator T is continuous with respect to the $\sigma(X, F)$-topology, then the space F is invariant with respect to the adjoint operator $T^* \in B(X^*)$, i.e. $T^*(F) \subseteq F$. If, additionally, $F = \bar{F}$, then $T^*|_F \in B(F)$ and $T^*|_F$ is $\sigma(F, X)$-continuous.*

Proof. Let $f \in F$. Consider the linear form

$$x \mapsto (T^* f)(x) = f(Tx), \quad x \in X.$$

Since the operator T is $\sigma(X, F)$-continuous, the latter form is also $\sigma(X, F)$-continuous. Consequently, according to Lemma 1.1.2.(i), $T^* f \in F$. Thus, we proved that $T^*(F) \subseteq F$.

If $F = \bar{F}$, then F is a Banach space isometrically embedded into X^*. Since $T^* \in B(X^*)$ and $T(F) \subseteq F$, we readily see that $T^*|_F \in B(F)$.

To show that T^* is $\sigma(F, X)$-continuous, it is sufficient to note that the operator T^*, as any adjoint operator, is $\sigma(X^*, X)$-continuous. Thus, clearly, the restriction $T^*|_F$ is $\sigma(F, X)$-continuous. The proof is finished. □

Besides the topologies mentioned above, the Banach space $B(X, Y)$ possesses the *strong* topology, that is, the locally convex topology defined by the collection of semi-norms $\{p_x\}_{x \in X}$ given by

$$p_x(T) = \|Tx\|_Y, \quad x \in X, \quad T \in B(X, Y).$$

Thus, the net $\{T_\alpha\} \subseteq B(X, Y)$ converges to an element $T \in B(X, Y)$ in the strong topology if and only if

$$p_x(T_\alpha - T) = \|T_\alpha x - Tx\|_Y \to 0, \quad x \in X, \quad T \in B(X, Y).$$

Let X, Y be Banach spaces. Let $\mathscr{D} \subseteq X$ be a linear (possibly not closed) subspace. It is said that the linear operator T is defined on the subspace $\mathscr{D}$ if and only if the mapping $T : \mathscr{D} \mapsto Y$ is given and

$$T(x_1 + \alpha\, x_2) = T(x_1) + \alpha\, T(x_2), \quad x_1, x_2 \in \mathscr{D}, \ \alpha \in \mathbb{C}.$$

The subspace $\mathscr{D}$ is called the domain of the operator T and denoted by $\mathscr{D}(T)$.

It is said that an operator $S : \mathscr{D}(S) \mapsto Y$ extends an operator $T : \mathscr{D}(T) \mapsto Y$ if and only if $\mathscr{D}(T) \subseteq \mathscr{D}(S)$ and $T(x) = S(x)$, for every $x \in \mathscr{D}(T)$, in this case we shall write $T \subseteq S$. The operator T and S are equal ($T = S$) if and only if $T \subseteq S$ and $S \subseteq T$.

An operator $T : \mathscr{D}(T) \mapsto Y$ is called *(strongly) densely defined* if and only if the subspace $\mathscr{D}(T)$ is dense in X. An operator $T : \mathscr{D}(T) \mapsto Y$ is called *weakly (resp. weakly*) densely defined* if and only if the subspace $\mathscr{D}(T)$ is weakly (resp. weakly*) dense in X.

An operator $T : \mathscr{D}(T) \mapsto Y$ is called *(strongly) closed* if and only if the subspace $\Gamma(T) := \{(x, T(x)) : \quad x \in \mathscr{D}(T)\}$ is closed in the space $X \oplus Y$. If the subspace $\Gamma(T)$ is closed in $X \oplus Y$ equipped with the weak (resp. weak*) topology, then the operator T is *weakly (resp. weakly*) closed.*

An operator T is called *(strongly) closable* when there is a closed operator T' such that $T \subseteq T'$. If the operator T is closable, then the closure $\bar{T}$ is a linear operator $\bar{T} : \mathscr{D}(\bar{T}) \mapsto Y$ such that $\bar{T} \subseteq T'$, for every closed linear operator T' with $T \subseteq T'$. Similarly, the operator T is called *weakly (resp. weakly*) closable* if and only if it has weakly (resp. weak*) closed extension.

A linear subspace $\mathscr{D} \subseteq \mathscr{D}(T)$ is called *a core (resp. weak, weak* core)* of the operator T if and only if the closure (resp. weak, weak* closure) of the operator $T_1 := T|_{\mathscr{D}}$ extends T, i.e. $T \subseteq \bar{T}_1$.

Let X be a Banach space. A mapping $U_t : \mathbb{R} \mapsto B(X)$ is called *a group of operators* if and only if

$$U_t \cdot U_s = U_{t+s}, \quad U_0 = \mathbf{1}, \quad t, s \in \mathbb{R}.$$

A group $\{U_t\}_{t\in\mathbb{R}}$ is called *strongly continuous* if and only if the mapping $t \mapsto U_t$ is continuous in the strong topology.

A group $\{U_t\}_{t\in\mathbb{R}}$ is called *weakly (resp. weakly*) continuous* (see [11]) if and only if,

(i) for every $x \in X$, the mapping $t \mapsto U_t(x)$, $t \in \mathbb{R}$ is continuous in the weak (resp. weak*) topology;

(ii) for every $t \in \mathbb{R}$, the mapping $x \mapsto U_t(x)$, $x \in X$ is weakly (resp. weakly*) continuous.

Let $\{U_t\}_{t\in\mathbb{R}}$ be a strongly (resp. weakly, weakly*) continuous group of operators on X. Let us define the subspace $\mathscr{D}(\delta) \subseteq X$ as the collection of all vectors $x \in X$ such that the limit

$$\lim_{t\to 0} \frac{U_t(x) - x}{t},$$

taken in the strong (resp. weak, weak*) topology, exists. For every $x \in \mathscr{D}(\delta)$, let us now define the mapping $x \mapsto \delta(x) \in X$ by

$$\delta(x) := \lim_{t\to 0} \frac{U_t(x) - x}{t}.$$

The linear operator $\delta : \mathscr{D}(\delta) \mapsto X$ is called *a strong (resp. weak, weak*) generator* of the group $\{U_t\}_{t\in\mathbb{R}}$.

Lemma 1.1.4. *The domain $\mathscr{D}(\delta)$ of the strong (resp. weak, weak*) generator δ of the strongly (resp. weakly, weakly*) continuous group $U = \{U_t\}_{t\in\mathbb{R}}$ is invariant with respect to the group U, i.e.*

$$U_t(\mathscr{D}(\delta)) \subseteq \mathscr{D}(\delta), \quad t \in \mathbb{R}.$$

Proof. Let us prove the lemma when the group U is strongly continuous. For the other cases, the argument is similar. Since U_t is a continuous linear operator with continuous inverse, for every $t \in \mathbb{R}$, we obtain

$$\begin{aligned} x \in \mathscr{D}(\delta) &\Longleftrightarrow \lim_{s\to 0} \frac{U_s(x) - x}{s} \text{ exists} \\ &\Longleftrightarrow \lim_{s\to 0} \frac{U_{t+s}(x) - U_t(x)}{s} \text{ exists} \\ &\Longleftrightarrow U_t(x) \in \mathscr{D}(\delta), \quad t \in \mathbb{R}. \end{aligned}$$

□

Theorem 1.1.5 ([11, Corollary 3.1.8])**.** *Every weakly continuous group is strongly continuous and its weak and strong generators coincide.*

The resolvent set $\rho(\delta)$ of a linear operator $\delta : \mathscr{D}(\delta) \mapsto X$ is the collection of all numbers $\lambda \in \mathbb{C}$ such that the operator $\lambda - \delta := \lambda\mathbf{1} - \delta$ is invertible, i.e., for every $\lambda \in \rho(\delta)$, there is an operator $R_\lambda(\delta) \in B(X)$ such that

(i) $R_\lambda(\delta)(X) \subseteq \mathscr{D}(\delta)$;

(ii) $R_\lambda(\delta)(\lambda x - \delta(x)) = x$, $x \in \mathscr{D}(\delta)$;

(iii) $\lambda R_\lambda(\delta)(x) - \delta(R_\lambda(\delta)(x)) = x$, $x \in X$, see [11, Definition 2.2.1].

The operator $R_\lambda(\delta)$ is called *the resolvent* of the operator δ. The complement $\sigma(\delta) := \mathbb{C} \setminus \rho(\delta)$ is called *the spectrum* of the operator δ.

Theorem 1.1.6 ([11, Proposition 3.1.6] and [48, Lemma 3.2])**.** *Let $U = \{U_t\}_{t\in\mathbb{R}}$ be a strongly (resp. weakly*) continuous group of contractions, i.e.*

$$\|U_t\|_{B(X)} \leq 1, \quad t \in \mathbb{R}.$$

If $\delta : \mathscr{D}(\delta) \mapsto X$ is the strong (resp. weak) generator, then*

(i) *the domain $\mathscr{D}(\delta)$ is strongly (resp. weakly*) dense in X;*

(ii) *the operator δ is strongly (resp. weakly*) closed;*

(iii) $\{\lambda \in \mathbb{C} : \ |\Re\lambda| > 0\} \subseteq \rho(\delta)$;

(iv) $\|R_\lambda(\delta)\|_{B(X)} \leq |\Re\lambda|^{-1}$, $\lambda \in \rho(\delta)$;

(v) *for every* $x \in X$, $\lim_{\lambda\to\infty} \lambda R_\lambda(\delta)(x) = x$, *where the limit converges in the norm topology (resp. weakly*);*

(vi) *the resolvent* $R_\lambda(\delta)$ *is given by the Laplace transform:*

$$R_\lambda(\delta)(x) = \int_0^\infty e^{-\lambda t}\, U_t(x)\, dt, \quad \lambda \in \rho(\delta),\ x \in X.$$

The latter integral converges in the norm topology (resp. weakly).*

Theorem 1.1.7 ([20, Theorem 1.9] and [11, Corollary 3.1.7])**.** *Let* $\delta : \mathscr{D}(\delta) \mapsto X$ *be the generator of a group* $U = \{U_t\}_{t\in\mathbb{R}}$. *If the subspace* $\mathscr{D} \subseteq \mathscr{D}(\delta)$ *is strongly (resp. weakly, weakly*) dense in* X *and is invariant with respect to* U, *then* $\mathscr{D}$ *is a strong (resp. weak, weak*) core of the operator* δ.

1.2 Interpolation of linear operators

Let X_0 and X_1 be normed spaces. Recall that the couple (X_0, X_1) is called *compatible* if and only if there is a Hausdorff topological space $\mathcal{X}$ such that $X_j \subseteq \mathcal{X}$, $j = 0, 1$.

Lemma 1.2.1 ([5, Lemma 2.3.1])**.** *Let* (X_0, X_1) *be a compatible couple of normed spaces.*

(i) *The space* $X_0 \cap X_1$ *equipped with the norm*

$$\|x\|_{X_0\cap X_1} := \max\{\|x\|_{X_0}, \|x\|_{X_1}\}, \quad x \in X_0 \cap X_1$$

is a normed space.

(ii) *The space* $X_0 + X_1$ *equipped with the norm*

$$\|x\|_{X_0+X_1} := \inf_{x=x_0+x_1} (\|x_0\|_{X_0} + \|x_1\|_{X_1}), \quad x \in X_0 + X_1$$

is also a normed space.

Let (X_0, X_1) be a compatible couple of normed spaces. A normed space X is called *an intermediate space* with respect to the couple (X_0, X_1) if and only if

$$X_0 \cap X_1 \subseteq X \subseteq X_0 + X_1.$$

An intermediate space X with respect to the couple (X_0, X_1) is called *an interpolation space* with respect to (X_0, X_1) if and only if, for every linear operator

$$T : X_0 + X_1 \mapsto X_0 + X_1,$$

such that

$$T(X_j) \subseteq X_j \quad \text{and} \quad T|_{X_j} \in B(X_j), \quad j = 0, 1, \tag{1.2.1}$$

it follows that

$$T(X) \subseteq X \quad \text{and} \quad T|_X \in B(X).$$

In the present section, we briefly recall two methods of constructing of interpolation spaces and their basic properties.

1.2.1 The complex interpolation method

let $\bar{X} := (X_0, X_1)$ be a compatible couple of Banach spaces. Let

$$S := \{z \in \mathbb{C} : \quad 0 < \Re z < 1\}.$$

Let us consider the class $\mathcal{F}(\bar{X})$, which consists of all functions

$$f : \bar{S} \mapsto X_0 + X_1,$$

bounded and continuous in the closed strip $\bar{S}$ and holomorphic in the open strip S such that, for every $j = 0, 1$, the mapping $t \mapsto f(j + it)$, $t \in \mathbb{R}$ is continuous function with values in X_j vanishing when $|t| \to \infty$. The class $\mathcal{F}(\bar{X})$ equipped with the norm

$$\|f\|_{\mathcal{F}(\bar{X})} := \max_{j=0,1} \sup_{t \in \mathbb{R}} \|f(j + it)\|_{X_j}$$

is a Banach space, see [5, Lemma 4.1.1].

Let $0 \leq \theta \leq 1$. Let us consider the space $\bar{X}_\theta = (X_0, X_1)_\theta$ which is the collection of all elements $x \in X_0 + X_1$ such that there is $f \in \mathcal{F}(\bar{X})$ with $f(\theta) = x$ and

$$\|x\|_{\bar{X}_\theta} := \inf \left\{ \|f\|_{\mathcal{F}(\bar{X})} : \quad f \in \mathcal{F}(\bar{X}), \ f(\theta) = x \right\} < +\infty. \tag{1.2.2}$$

Theorem 1.2.2 ([5, Theorem 4.1.2]). *If $\bar{X} = (X_0, X_1)$ is a compatible couple of Banach spaces, then*

(i) *the space $\bar{X}_\theta$ equipped with the norm $\|\cdot\|_{\bar{X}_\theta}$, defined above, is a Banach space, for every $0 \leq \theta \leq 1$;*

(ii) *the Banach space* $\bar{X}_\theta$ *is an interpolation space, for every* $0 \leq \theta \leq 1$. *Moreover,*

$$\|T\|_{B(\bar{X}_\theta)} \leq \|T\|_{B(X_0)}^{1-\theta} \, \|T\|_{B(X_1)}^{\theta},$$

for every linear operator $T : X_0 + X_1 \mapsto X_0 + X_1$, *provided* (1.2.1) *holds.*

We shall now show that the argument in the proof of the theorem above may be extended to a much wider setting as shown by the following theorem.

Theorem 1.2.3. *Let* $\bar{X} = (X_0, X_1)$ *and* $\bar{Y} = (Y_0, Y_1)$ *be two compatible Banach couples. Let*

$$T : \mathbb{C} \times (X_0 + X_1) \mapsto Y_0 + Y_1$$

be a function, such that

(i) *the mapping*

$$T_z : x \mapsto T(z, x), \quad x \in X_0 + X_1$$

is a linear operator, for every $z \in \mathbb{C}$;

(ii) *the mapping*

$$z \mapsto T_z \in B(X_0 + X_1, Y_0 + Y_1), \quad z \in \mathbb{C}$$

is a function holomorphic in S *and bounded in* $\bar{S}$;

(iii) *for every* $j = 0, 1$, *the mapping*

$$t \mapsto T_{j+it} \in B(X_j, Y_j), \quad t \in \mathbb{R}$$

is continuous.

If

$$M_j := \sup_{t \in \mathbb{R}} \|T_{j+it}\|_{B(X_j, Y_j)} < +\infty, \quad j = 0, 1,$$

then, for every $0 \leq \theta \leq 1$, *the operator* $T_\theta : \bar{X}_\theta \mapsto \bar{Y}_\theta$ *is bounded and*

$$\|T_\theta\|_{B(\bar{X}_\theta, \bar{Y}_\theta)} \leq M_0^{1-\theta} \, M_1^{\theta}.$$

Proof. Fix $x \in \bar{X}_\theta$. For every $\epsilon > 0$ there is a function $f_\epsilon \in \mathcal{F}(\bar{X})$ such that $f_\epsilon(\theta) = x$ and

$$\|f_\epsilon\|_{\mathcal{F}(\bar{X})} - \epsilon < \|x\|_{\bar{X}_\theta} \leq \|f_\epsilon\|_{\mathcal{F}(\bar{X})}. \tag{1.2.3}$$

Let us consider the function

$$g_\epsilon(z) = M_1^{z-1} M_2^{-z} T_z(f_\epsilon(z)), \quad z \in \mathbb{C}.$$

It is not difficult to see that $g_\epsilon \in \mathcal{F}(\bar{Y})$ and

$$\|g_\epsilon\|_{\mathcal{F}(\bar{Y})} \leq \|f_\epsilon\|_{\mathcal{F}(\bar{X})}. \tag{1.2.4}$$

According to (1.2.2), the element $g_\epsilon(\theta)$ belongs to $\bar{Y}_\theta$ and

$$\|g_\epsilon(\theta)\|_{\bar{Y}_\theta} \leq \|g_\epsilon\|_{\mathcal{F}(\bar{Y})} \leq \|f_\epsilon\|_{\mathcal{F}(\bar{X})}. \tag{1.2.5}$$

On the other hand,

$$g_\epsilon(\theta) = M_1^{\theta-1} M_2^{-\theta} T_\theta(x).$$

Combining (1.2.3), (1.2.4) and (1.2.5) yields

$$\|T_\theta(x)\|_{\bar{Y}_\theta} \leq M_1^{1-\theta} M_2^{\theta} \left(\|x\|_{\bar{X}_\theta} + \epsilon\right), \quad \epsilon > 0.$$

Letting $\epsilon \to 0$ proves the lemma. □

1.2.2 The real interpolation method

In this section, we shall consider another well-known interpolation method.

Let $\bar{X} = (X_0, X_1)$ be a compatible couple of normed spaces. Let us recall that

$$K(t,x) := \inf_{x=x_0+x_1} (\|x_0\|_{X_0} + t\,\|x_1\|_{X_1}), \quad t > 0,\ x \in X_0 + X_1.$$

Let us also recall that, for every $0 \leq \theta \leq 1$ and $1 \leq q \leq \infty$,

$$\Phi_{\theta,q}(f(\cdot)) := \left[\int_0^\infty \left(t^{-\theta} f(t)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}}, \quad \text{if } q < \infty$$

and

$$\Phi_{\theta,\infty}(f(\cdot)) := ess\sup_{t>0} |t^{-\theta} f(t)|.$$

Let $\bar{X}_{\theta,q} = (X_0, X_1)_{\theta,q}$, $0 \leq \theta \leq 1$, $1 \leq q \leq \infty$ be the collection of all elements $x \in X_0 + X_1$ such that

$$\|x\|_{\bar{X}_{\theta,q}} := \Phi_{\theta,q}(K(\cdot,x)) < +\infty. \tag{1.2.6}$$

Theorem 1.2.4 ([5, Theorem 3.1.2])**.** *If $\bar{X} = (X_0, X_1)$ be a compatible couple of normed spaces, then*

(i) *the space $\bar{X}_{\theta,q}$ equipped with the norm* (1.2.6) *is a normed space, for every $0 \leq \theta \leq 1$, $1 \leq q \leq \infty$;*

(ii) *the normed space* $\bar{X}_{\theta,q}$ *is an interpolation space with respect to the couple* $\bar{X}$*, for every* $0 \le \theta \le 1$ *and* $1 \le q \le \infty$*;*

(iii) *every linear operator* $T : X_0 + X_1 \mapsto X_0 + X_1$ *satisfying* (1.2.1)*, acts invariantly on* $\bar{X}_{\theta,q}$*, i.e.* $T(\bar{X}_{\theta,q}) \subseteq \bar{X}_{\theta,q}$ *and*

$$\|T\|_{B(\bar{X}_{\theta,q})} \le \|T\|^{1-\theta}_{B(X_0)} \|T\|^{\theta}_{B(X_1)}.$$

In regard to the real interpolation method, we shall use the following result.

Theorem 1.2.5 (The duality theorem [5, Theorem 3.7.1])**.** *Let* (X_0, X_1) *be a compatible couple of Banach spaces such that the space* $X_0 \cap X_1$ *norm dense in both* X_0 *and* X_1*. If* $0 < \theta < 1$ *and* $1 \le q < \infty$*, then*

$$[(X_0, X_1)_{\theta,q}]^* = (X_0^*, X_1^*)_{\theta,q'} \quad \textit{(norms are equivalent)},$$

where q' *is the conjugate exponent, i.e.*

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

1.3 Briefly on von Neumann algebras

In this section we fix a Hilbert space $\mathcal{H}$ with a scalar product $\langle \cdot, \cdot \rangle$. We consider the Banach algebra $B(\mathcal{H}) := B(\mathcal{H}, \mathcal{H})$ with the operator norm $\| \cdot \| := \| \cdot \|_{B(\mathcal{H})}$. We refer to the strong topology in the space $B(\mathcal{H})$ as to *the strong operator topology* or, equivalently, the *so*-topology.

If the net $\{x_\alpha\} \subseteq B(\mathcal{H})$ tends to the element $x \in B(\mathcal{H})$ with respect to the *so*-topology, then it does not necessarily imply that the net $\{x_\alpha^*\}$ tends to x^* with respect to the *so*-topology. We shall say the the net $\{x_\alpha\} \subseteq B(\mathcal{H})$ tends to $x \in B(\mathcal{H})$ with respect to the so^*-topology if and only if

$$so - \lim_\alpha x_\alpha = x \ \text{ and } \ so - \lim_\alpha x_\alpha^* = x^*.$$

Let us consider the linear functional $\omega_{\xi,\eta} \in B(\mathcal{H})^*$, $\xi, \eta \in \mathcal{H}$ defined by

$$\omega_{\xi,\eta}(x) = \langle x(\xi), \eta \rangle, \quad x \in B(\mathcal{H}).$$

Let $B(\mathcal{H})_\sim$ be the linear subspace in the dual $B(\mathcal{H})^*$ generated by the collection $\{\omega_{\xi,\eta}\}_{\xi,\eta\in\mathcal{H}}$ and let also $B(\mathcal{H})_*$ be the closure of $B(\mathcal{H})_\sim$ in the norm topology of $B(\mathcal{H})^*$. We shall refer to the $\sigma(B(\mathcal{H}), B(\mathcal{H})_\sim)$-topology as *the weak* *operator topology* or the *wo*-topology, and to the $\sigma(B(\mathcal{H}), B(\mathcal{H})_*)$-topology as to *the ultra-weak topology* or the *uw*-topology.

A $*$-subalgebra $\mathcal{M} \subseteq B(\mathcal{H})$ is called *a von Neumann algebra acting on* $\mathcal{H}$ if and only if $\mathbf{1} \in \mathcal{M}$ and $\mathcal{M}$ is closed with respect to the *wo*-topology.

Let $A \subseteq B(\mathcal{H})$ be a non-empty subset of $B(\mathcal{H})$, *the commutant* A' of A is the collection of all operators commuting with every operator in A, i.e.

$$A' := \{x \in B(\mathcal{H}) : \quad xy = yx \text{ for every } y \in A\}.$$

We have the following result.

Theorem 1.3.1 ([61, Theorem 3.2, Corollaries 3.3, 3.4])**.** (i) *Let A be non-empty subset $A \subseteq B(\mathcal{H})$ such that $A^* = A$. The commutant A' is a von Neumann algebra acting on $\mathcal{H}$;*

(ii) *if $\mathcal{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra, then the second commutant of $\mathcal{M}$ coincides with $\mathcal{M}$, i.e. $\mathcal{M}'' = \mathcal{M}$.*

Theorem 1.3.2 ([61, Theorem 1.10])**.** *Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$. If*

$$\mathcal{M}_\sim := \{f|_\mathcal{M} : \quad f \in B(\mathcal{H})_\sim\} \quad \textit{and} \quad \mathcal{M}_* := \{f|_\mathcal{M} : \quad f \in B(\mathcal{H})_*\},$$

then

(i) *the restrictions of the wo- and uw-topologies onto $\mathcal{M}$ coincide with the $\sigma(\mathcal{M}, \mathcal{M}_\sim)$- and $\sigma(\mathcal{M}, \mathcal{M}_*)$- topologies, respectively;*

(ii) *the closure of $\mathcal{M}_\sim$ with respect to the norm of $\mathcal{M}^*$ coincides with $\mathcal{M}_*$;*

(iii) *the Banach space $\mathcal{M}$ equipped with the operator norm is isometrically isomorphic to the dual space $(\mathcal{M}_*)^*$, that is, every bounded linear functional ϕ in $(\mathcal{M}_*)^*$ has the form*

$$\phi(f) = \phi_x(f) := f(x), \; f \in \mathcal{M}_*, \; \textit{for some } x \in \mathcal{M}, \; \textit{and } \|\phi\|_{(\mathcal{M}_*)^*} = \|x\|.$$

Combining Lemma 1.1.2 and Theorem 1.3.2 we obtain

Corollary 1.3.3. *The wo- and uw- topologies coincide on the unit ball of a von Neumann algebra.*

The following result shows that the space $\mathcal{M}_*$ is unique with respect to the claim (iii) in Theorem 1.3.2.

Theorem 1.3.4 ([62, Corollary III.3.9])**.** *Let $\mathcal{M}$ be a von Neumann algebra. If X is a Banach space such that $X^* = \mathcal{M}$, then the Banach space X is isometric to $\mathcal{M}_*$.*

The Banach space $\mathcal{M}_*$ equipped with the norm $\|\cdot\|_{\mathcal{M}^*}$ is called *the predual* of the von Neumann algebra $\mathcal{M}$.

Next we shall discuss the notions of states, weights and traces. We fix a von Neumann algebra $\mathcal{M}$. Let $\mathcal{M}^+$ be the collection of all positive elements in the C^*-algebra $\mathcal{M}$. A linear functional $\phi \in \mathcal{M}_*$ is called *positive* if and only if $\phi(x) \geq 0$, for all $x \in \mathcal{M}^+$.

A *weight* on a von Neumann algebra $\mathcal{M}$ is a mapping $\phi : \mathcal{M}^+ \to [0, \infty]$, the extended positive real half-line, satisfying

$$\phi(x + y) = \phi(x) + \phi(y),$$
$$\phi(\alpha x) = \alpha\, \phi(x), \quad \alpha \geq 0, \; x, y \in \mathcal{M}^+. \tag{1.3.1}$$

Here, we adopt the convention that $0 \cdot (+\infty) = 0$. We associate with a weight ϕ the sets

$$p_\phi := \{x \in \mathcal{M}^+ : \quad \phi(x) < +\infty\}, \quad m_\phi := \operatorname{span} p_\phi,$$
$$n_\phi := \{x \in \mathcal{M}; \quad x^* x \in p_\phi\}.$$

Observe that a positive element $\phi \in \mathcal{M}_*^+$ is a weight such that $\phi(\mathbf{1}) < +\infty$.

The weight ϕ is called *faithful* if $x = 0$, provided $\phi(x) = 0$, $x \in p_\phi$. The weight ϕ is called *semi-finite* if the linear subspace m_ϕ is *wo*-dense in $\mathcal{M}$. The weight ϕ is called *normal* if $\phi(x) = \lim \phi(x_\alpha)$, provided $\{x_\alpha\} \subseteq p_\phi$ is a monotone increasing net of operators and $x = \lim x_\alpha \in p_\phi$.

We have the following simple observation

Lemma 1.3.5. *If ϕ is a weight on the algebra $\mathcal{M}$, then*

(i) *$p_\phi + p_\phi \subseteq p_\phi$, $\alpha p_\phi \subseteq p_\phi$, $\alpha \geq 0$ and $y \in p_\phi$, provided $y \leq x$, $x \in p_\phi$, $y \in \mathcal{M}^+$;*

(ii) *n_ϕ is a left ideal in the algebra $\mathcal{M}$;*

(iii) *$xy \in m_\phi$ provided $x, y \in n_\phi$.*

Proof. If $x, y \in p_\phi$, then for every $\alpha \geq 0$, it is readily follows from (1.3.1) that

$$\phi(x+y) = \phi(x) + \phi(y) < +\infty \quad \text{and} \quad \phi(\alpha x) = \alpha\phi(x) < +\infty.$$

Consequently, $p_\phi + p_\phi \subseteq p_\phi$ and $\alpha p_\phi \subseteq p_\phi$.

If $x \in p_\phi$ and $y \in \mathcal{M}^+$ such that $y \leq x$, then it is clear that

$$x - y \geq 0 \quad \text{and} \quad \phi(x-y) \geq 0.$$

Thus, it follows from (1.3.1) that

$$\phi(y) = \phi(x) - \phi(x-y) \leq \phi(x) < +\infty.$$

Hence, $y \in p_\phi$. The claim (i) is proved.

Let us prove (ii). Let us first verify that n_ϕ is a linear space. If $x, y \in n_\phi$, and $\alpha \in \mathbb{C}$, then it follows from (1.3.1) that

$$\phi((\alpha x)^*(\alpha x)) = |\alpha|^2\phi(x^*x) < +\infty.$$

Thus, $\alpha x \in n_\phi$. Furthermore, we have

$$(x+y)^*(x+y) = x^*x + y^*y + x^*y + y^*x$$

and

$$(x-y)^*(x-y) = x^*x + y^*y - x^*y - y^*x \geq 0.$$

The latter two relations imply that

$$(x+y)^*(x+y) \leq 2(x^*x + y^*y).$$

Hence, since $x, y \in n_\phi$ it follows from (i) that $x + y \in n_\phi$. Let us now show the ideal property. Let $x \in \mathcal{M}$ and $y \in n_\phi$. From elementary considerations, we have that

$$\|x\|^2\mathbf{1} - x^*x \geq 0.$$

Multiplying by y^* and y from left and right, respectively, we obtain that

$$\|x\|^2 y^*y \geq y^*x^*xy^* = (xy)^*(xy).$$

Consequently, $xy \in n_\phi$. Hence, the claim (ii) is proved.

The claim (iii), now clearly follows from (ii) and the polarization identity

$$y^*x = \frac{1}{4}\sum_{k=0}^{3} i^k(x+i^ky)^*(x+i^ky), \quad x, y \in \mathcal{M}.$$

□

A weight ϕ has a unique extension over the linear subspace m_ϕ. We shall denote this extension as ϕ also. A weight ϕ is called *a state* if $p_\phi = \mathcal{M}^+$ and $\phi(\mathbf{1}) = 1$. We have the following description of normal states on a von Neumann algebra $\mathcal{M}$.

Theorem 1.3.6 ([37, Theorem 7.1.11])**.** *If ϕ is a state on a von Neumann algebra $\mathcal{M}$, then the following are equivalent*

(i) *ϕ is normal;*

(ii) *ϕ is wo-continuous on the unit ball of $\mathcal{M}$.*

Thus, Theorem 1.3.6 and Lemma 1.1.2 imply

Corollary 1.3.7. *The collection of all normal states is precisely the set*

$$\{\phi \in \mathcal{M}_*^+ : \quad \phi(\mathbf{1}) = 1\}.$$

A one-to-one mapping $\pi : \mathcal{M} \mapsto \mathcal{M}$ is called $*$-automorphism of the algebra $\mathcal{M}$ if and only if

$$\begin{aligned} \pi(x + \alpha y) &= \pi(x) + \alpha\, \pi(y), \quad \pi(xy) = \pi(x)\, \pi(y) \\ \pi(x^*) &= (\pi(x))^*, \quad x, y \in \mathcal{M}, \quad \alpha \in \mathbb{C}. \end{aligned} \tag{1.3.2}$$

Theorem 1.3.8 ([61, Corollary 5.13])**.** *Every $*$-automorphism of a von Neumann algebra is is uw-continuous.*

Let ϕ be a weight on a von Neumann algebra $\mathcal{M}$ and let $\sigma = \{\sigma_t\}_{t\in\mathbb{R}}$ be a group of $*$-automorphisms of $\mathcal{M}$. The weight ϕ satisfies *the modular condition* with respect to the group σ if and only if (see [37, Section 9.2])

(i) $\sigma_t(p_\phi) \subseteq p_\phi$ and $\phi(\sigma_t(x)) = \phi(x)$, for every $t \in \mathbb{R}$, $x \in p_\phi$;

(ii) for every $x, y \in n_\phi \cap {n_\phi}^*$, there is a function $f_{x,y} : \bar{S} \to \mathbb{C}$, where $S = \{z \in \mathbb{C} : \quad 0 < \Im z < 1\}$, such that $f_{x,y}$ is holomorphic in S, bounded in $\bar{S}$ and

$$f_{x,y}(t) = \phi(\sigma_t(x)\, y) \quad \text{and} \quad f_{x,y}(t + i) = \phi(y\, \sigma_t(x)), \quad t \in \mathbb{R}.$$

Theorem 1.3.9 ([37, Section 9.2])**.** *For every weight ϕ on a von Neumann algebra $\mathcal{M}$ there is a unique group of $*$-automorphisms of $\mathcal{M}$ $\sigma = \sigma^\phi = \{\sigma_t^\phi\}_{t\in\mathbb{R}}$ such that ϕ satisfies the modular condition with respect to σ^ϕ.*

The group σ^ϕ is called *the modular group* of the weight ϕ.

Let ϕ be a weight on a von Neumann algebra $\mathcal{M}$ and let $u \in \mathcal{M}$ be a unitary operator. We set $p_{\phi_u} = u\, p_\phi u^*$ and $\phi_u(x) = \phi(u^*xu)$, $x \in p_{\phi_u}$. The functional ϕ_u is also a weight. The following straightforward lemma explains relation between the modular automorphism groups of the weights ϕ and ϕ_u.

Lemma 1.3.10. *The modular automorphism group $\sigma^{\phi_u} = \{\phi_t^{\phi_u}\}_{t\in\mathbb{R}}$ is given by*

$$\sigma_t^{\phi_u} = u\, \sigma_t^\phi(u^*xu)\, u^*, \quad x \in \mathcal{M}.$$

Proof. We have to verify the modular conditions (i)–(ii) set out above.

(i) Let $x \in p_{\phi_u}$. Clearly, $u^*xu \in p_\phi$. Thus,

$$\begin{aligned} x \in p_{\phi_u} &\Longrightarrow u^*xu \in p_\phi \\ &\Longrightarrow \sigma_t^\phi(u^*xu) \in p_\phi \\ &\Longrightarrow u\, \sigma_t^\phi(u^*xu)\, u^* \in p_{\phi_u} \\ &\Longrightarrow \sigma_t^{\phi_u}(x) \in p_{\phi_u}, \quad t \in \mathbb{R}. \end{aligned}$$

(ii) Since σ^ϕ is the modular group of the weight ϕ, we have that, for every $x, y \in n_\phi \cap n_\phi{}^*$, there is a function $f_{x,y}$ such that (ii) holds with respect to ϕ and the group σ^ϕ. It is now easy to show that the condition (ii) will hold for the weight ϕ_u and the group σ^{ϕ_u} with the function $\tilde{f}_{x,y} := f_{u^*xu,u^*yu}$, for every $x, y \in n_{\phi_u} \cap n_{\phi_u}{}^*$. ☐

Among all possible weights we shall distinguish those possessing the trivial modular group. Namely, the weight τ is called *a tracial weight* (or, simply, *a trace*) if the corresponding modular group is trivial, i.e. $\sigma_t^\tau = \mathbf{1}$, for every $t \in \mathbb{R}$. We then have

Lemma 1.3.11. *A weight τ is tracial if and only if $\tau(xy) = \tau(yx)$, for every $x, y \in n_\phi$.*

Proof. Let τ be a tracial weight, i.e. $\sigma^\tau \equiv \mathbf{1}$. Let

$$S := \{z \in \mathbb{C} : \quad 0 < \Im z < 1\}.$$

Since τ satisfies the modular condition with respect to the group σ^τ, we have that, for every $x, y \in n_\tau$, there is a function $f_{x,y} : \bar{S} \mapsto \mathbb{C}$ holomorphic in S and bounded in $\bar{S}$ such that

$$f_{x,y}(t) = \tau(\sigma_t^\tau(x)\, y) \quad \text{and} \quad f_{x,y}(t+i) = \tau(y\, \sigma_t^\tau(x)), \quad t \in \mathbb{R}.$$

Thus, since τ is tracial, we readily see that

$$f_{x,y}(t) = \tau(x\,y) = c_0 \quad \text{and} \quad f_{x,y}(t+i) = \tau(y\,x) = c_1,$$

i.e. the function $f_{x,y}$ is constant on the lines

$$\{z \in \mathbb{C} : \quad \Im z = 0\} \quad \text{and} \quad \{z \in \mathbb{C} : \quad \Im z = 1\}.$$

Let us show that the function $f_{x,y}$ is a constant throughout the strip $\bar{S}$. Indeed, let

$$S_1 := \{z \in \mathbb{C} : \quad -1 \leq \Im z < 0\}$$

and let χ and χ_1 be the characteristic functions of the strips $\bar{S}$ and S_1. We shall consider the function

$$f_1(z) := f_{x,y}(z)\,\chi(z) + f_{x,y}(-z)\,\chi_1(z), \quad z \in \bar{S} \cup S_1.$$

The function f_1 is holomorphic in the strip S and in the interior of the strip S_1. Moreover, the function f_1 is continuous and bounded throughout $\bar{S} \cup S_1$. Consequently, see e.g. [37, Lemma 9.2.11], the function f_1 is holomorphic in the interior of the strip $\bar{S} \cup S_1$. Furthermore, the function f_1 is a constant on the boundary of the domain $\bar{S} \cup S_1$. Thus, according to the maximum modulus principle, the function f_1 is a constant throughout the domain $\bar{S} \cup S_1$. The latter, in particular, implies that the function $f_{x,y}$ is a constant throughout the domain $\bar{S}$, and therefore

$$\tau(xy) = \tau(yx).$$

The direct statement is proved.

The converse statement is simple. If the weight τ satisfies the identity

$$\tau(x\,y) = \tau(y\,x), \quad x, y \in n_\tau,$$

then it is clear that the weight τ satisfies the modular condition with respect to the group $\sigma \equiv \mathbf{1}$ with $f_{x,y} = \tau(x\,y)$, $x, y \in n_\tau$. Due to the uniqueness (see Theorem 1.3.9) it readily follows that $\sigma^\tau \equiv \mathbf{1}$. □

A von Neumann algebra $\mathcal{M}$ equipped with a normal semi-finite faithful (n.s.f.) trace τ is called *semi-finite von Neumann algebra.* If there is no semi-finite trace on $\mathcal{M}$, then the algebra $\mathcal{M}$ is called *purely infinite.*

Let $x : \mathscr{D}(x) \mapsto \mathcal{H}$ be a linear operator. If x is densely defined, then we may construct *the adjoint operator* $x^* : \mathscr{D}(x^*) \mapsto \mathcal{H}$ as follows. The subspace $\mathscr{D}(x^*)$ is the collection of all vectors $\xi \in \mathcal{H}$ such that the linear functional $\eta \mapsto \langle \xi, x(\eta) \rangle$

defined on the subspace $\mathscr{D}(x)$ is bounded. Since the subspace $\mathscr{D}(x)$ is dense in $\mathcal{H}$, we obtain that, for every $\xi \in \mathscr{D}(x^*)$ there is a unique vector, which we call $x^*(\xi)$, such that

$$\langle x^*(\xi), \eta\rangle = \langle \xi, x(\eta)\rangle, \quad \text{for every } \eta \in \mathscr{D}(x).$$

Theorem 1.3.12 ([57, Theorem VIII.1]). *Let $x : \mathscr{D}(x) \mapsto \mathcal{H}$ be a densely defined linear operator. The adjoint operator $x^* : \mathscr{D}(x^*) \mapsto \mathcal{H}$ is closed. The adjoint operator x^* is densely defined if and only if the operator x is closable in which case $\bar{x} = x^{**}$. If x is closable, then $(\bar{x})^* = x^*$.*

An operator $x : \mathscr{D}(x) \mapsto \mathcal{H}$ is called *self-adjoint* if and only if $x = x^*$.

Theorem 1.3.13 ([61, Proposition 9.28]). *If $x : \mathscr{D}(x) \mapsto \mathcal{H}$ is a closed linear operator, then the operator x^*x is positive and self-adjoint.*

Let us recall next the spectral theorem for self-adjoint operators. Let $\mathscr{B}(\mathbb{R})$ be the σ-algebra of all Borel subsets of $\mathbb{R}$. The mapping $e : \mathscr{B}(\mathbb{R}) \mapsto B(\mathcal{H})$ is called *a spectral measure* if and only if

(i) $e(B)$ is an orthogonal projection for every $B \in \mathscr{B}(\mathbb{R})$ and $e(B)\, e(B') = 0$ provided $B \cap B' = \emptyset$;

(ii) $e(\cup_{k=1}^{\infty} B_k) = \sum_{k=1}^{\infty} e(B_k)$, provided $B_j \in \mathscr{B}(\mathbb{R})$, $B_k \cap B_j = \emptyset$, $k \neq j$. The latter series converges in the *so*-topology.

Theorem 1.3.14 ([57, Theorem VIII.6]). *Let $x : \mathscr{D}(x) \mapsto \mathcal{H}$ be a self-adjoint linear operator. There is a unique spectral measure $e^x(\cdot) : \mathscr{B}(\mathbb{R}) \mapsto B(\mathcal{H})$ such that*

$$\xi \in \mathscr{D}(x) \quad \textit{if and only if} \quad \int_{\mathbb{R}} \lambda \, d\|e_\lambda^x(\xi)\|_{\mathcal{H}}^2 < +\infty \tag{1.3.3}$$

and

$$x(\xi) = \int_{\mathbb{R}} \lambda \, de_\lambda^x(\xi), \quad \xi \in \mathscr{D}(x), \tag{1.3.4}$$

where $e_\lambda^x = e^x(-\infty, \lambda]$. The latter integral converges in the norm of $\mathcal{H}$. If x is bounded, then

$$x = \int_{\mathbb{R}} \lambda \, de_\lambda^x$$

and the integral converges in the so-topology.

The relations (1.3.3) and (1.3.4) suggests that the latter result may be conversed.

Theorem 1.3.15 ([57, Theorem VIII.6]). *Let $e(\cdot) : \mathscr{B}(\mathbb{R}) \mapsto B(\mathcal{H})$ be a spectral measure, as explained above. There is a unique self-adjoint linear operator $x : \mathscr{D}(x) \mapsto \mathcal{H}$ such that its spectral measure $e^x(\cdot)$ guaranteed by Theorem 1.3.14 coincides with $e(\cdot)$.*

Recall that the space $B(\mathbb{R})$ stands for the class of all bounded Borel functions on $\mathbb{R}$ equipped with the uniform norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|, \quad f \in B(\mathbb{R}).$$

The theorem below defines the $B(\mathbb{R})$-calculus of self-adjoint operators.

Theorem 1.3.16 ([57, Theorem VIII.5]). *Let $x : \mathscr{D}(x) \mapsto \mathcal{H}$ be a self-adjoint linear operator and $f \in B(\mathbb{R})$. The integral*

$$f(x) := \int_{\mathbb{R}} f(\lambda)\, de^x_\lambda$$

converges in the so-topology. Thus, $f(x) \in B(\mathcal{H})$. Moreover,

(i) *The mapping $f \mapsto f(x)$ is a $*$-homomorphism from the algebra $B(\mathbb{R})$ to the algebra $B(\mathcal{H})$. The latter homomorphism is bounded, i.e.*

$$\|f(x)\| \le \|f\|_\infty.$$

(ii) *If $\{f_n\}_{n=1}^\infty \subseteq (B(\mathbb{R}))_1$, $f_n \to 0$ pointwise, then $f_n(x) \to 0$ in the so-topology.*

Besides the spectral theorem, there is another distinguished property of self-adjoint operators — Stone's Theorem — which connects a self-adjoint operator with a unitary group.

Theorem 1.3.17 (Stone, cf. [57, Theorems VIII.7 and VIII.8]). *Let $x : \mathscr{D}(x) \mapsto \mathcal{H}$ be a self-adjoint linear operator. If $\{e^{itx}\}_{t\in\mathbb{R}}$ is the corresponding unitary group, then the strong (equivalently, the weak) generator of this group coincides with x. Conversely, if $\{\gamma_t\}_{t\in\mathbb{R}}$ is a weakly continuous group on a Hilbert space $\mathcal{H}$, then there is a self-adjoint linear operator $x : \mathscr{D}(x) \mapsto \mathcal{H}$ such that $e^{itx} = \gamma_t$, $t \in \mathbb{R}$.*

A linear subspace $\mathscr{D} \subseteq \mathcal{H}$ is called *affiliated with the algebra* $\mathcal{M}$ if and only if

$$\xi \in \mathscr{D} \iff u\xi \in \mathscr{D},$$

for every unitary operator $u \in \mathcal{M}'$. In the latter case, we shall write $\mathscr{D}\eta\mathcal{M}$. A linear (possibly unbounded) operator $x : \mathscr{D}(x) \mapsto \mathcal{H}$ is called *affiliated with* $\mathcal{M}$ if and only if $u^*xu = x$ for every unitary operator $u \in \mathcal{M}'$. The latter means that (i) the subspace $\mathscr{D}(x) \subseteq \mathcal{H}$ is affiliated with $\mathcal{M}$ and (ii) $u^*xu(\xi) = x(\xi)$, for every $\xi \in \mathscr{D}(x)$. Clearly, it follows from Theorem 1.3.1 that if $x \in B(\mathcal{H})$ and x is affiliated with $\mathcal{M}$, then $x \in \mathcal{M}$. If the operator $x : \mathscr{D}(x) \mapsto \mathcal{H}$ is affiliated with $\mathcal{M}$, then we write shall $x\eta\mathcal{M}$. Let us note that

Lemma 1.3.18. *If $x : \mathscr{D}(x) \mapsto \mathcal{H}$ is a linear self-adjoint operator and $\mathcal{M}$ is a von Neumann algebra acting on $\mathcal{H}$, then $x\eta\mathcal{M}$ if and only if $e^x(B) \in \mathcal{M}$ for every $B \in \mathscr{B}(\mathbb{R})$.*

Proof. Let $x : \mathscr{D}(x) \mapsto \mathcal{H}$ be a linear self-adjoint operator affiliated with $\mathcal{M}$. Let $e^x(\cdot)$ be the corresponding spectral measure. Let us show that $e^x(B) \in \mathcal{M}$ for every $B \in \mathscr{B}(\mathbb{R})$.

Let us fix $u \in \mathcal{M}'$. We consider the spectral measure $e'(\cdot) := u^*e(\cdot)u$. According to Theorem 1.3.15, there is a linear self-adjoint operator $x' : \mathscr{D}(x') \mapsto \mathcal{H}$ such that

$$\xi \in \mathscr{D}(x') \text{ if and only if } \int_{\mathbb{R}} \lambda \, d\|e'_\lambda(\xi)\|^2_{\mathcal{H}} < +\infty$$

and

$$x'(\xi) = \int_{\mathbb{R}} \lambda \, de'_\lambda(\xi), \quad \xi \in \mathscr{D}(x').$$

We shall show that $x = x'$. Indeed,

$$\begin{aligned} \xi \in \mathscr{D}(x') &\Longleftrightarrow \int_{\mathbb{R}} \lambda \, d\|e'_\lambda(\xi)\|^2_{\mathcal{H}} < +\infty \\ &\Longleftrightarrow \int_{\mathbb{R}} \lambda \, d\|u^*e^x_\lambda u(\xi)\|^2_{\mathcal{H}} < +\infty \\ &\Longleftrightarrow \int_{\mathbb{R}} \lambda \, d\|e^x_\lambda u(\xi)\|^2_{\mathcal{H}} < +\infty \\ &\Longleftrightarrow u(\xi) \in \mathscr{D}(x) \Longleftrightarrow \xi \in \mathscr{D}(x). \end{aligned}$$

Furthermore,

$$\begin{aligned} x'(\xi) = \int_{\mathbb{R}} \lambda \, de'_\lambda(\xi) &= \int_{\mathbb{R}} \lambda \, u^* de^x_\lambda u(\xi) \\ &= u^*xu(\xi) = x(\xi), \quad \xi \in \mathscr{D}(x) = \mathscr{D}(x'). \end{aligned}$$

Thus, it follows from uniqueness in Theorem 1.3.14 that $e'(\cdot) = e^x(\cdot)$. The direct statement is proved. The proof of the converse assertion is similar. □

Now, let us show how to extend a $*$-automorphism π to self-adjoint operators $x\eta\mathcal{M}$. Let $x\eta\mathcal{M}$ and let x be self-adjoint. To extend the mapping π to x, we use the spectral theorem. Indeed, let $e^x(\cdot)$ be the spectral measure of the operator x. Since π is a $*$-isomorphism, the family $\pi(e^x(\cdot))$ is a spectral measure. Thus, according to Theorem 1.3.15, there is a self-adjoint linear operator, which we call $\bar{\pi}(x)$ such that $e^{\bar{\pi}(x)}(\cdot) = \pi(e^x(\cdot))$. Moreover, according to Lemma 1.3.18, the operator $\bar{\pi}(x)$ is also affiliated with the algebra $\mathcal{M}$.

The following lemma immediately follows from the definition set out above and the functional calculus, see Theorem 1.3.16.

Lemma 1.3.19. *If $x\eta\mathcal{M}$, then $f(\bar{\pi}(x)) = \pi(f(x))$, for every $f \in B(\mathbb{R})$.*

Alternatively, the operator $\bar{\pi}(x)$ may be defined via Stone's theorem and the group $\{e^{itx}\}_{t\in\mathbb{R}}$, provided $x = x^*\eta\mathcal{M}$.

Lemma 1.3.20. *Let $x\eta\mathcal{M}$ and let x be self-adjoint. The group $\gamma = \{\gamma_t\}_{t\in\mathbb{R}}$ given by $\gamma_t := \pi(e^{itx})$ is strongly continuous. The generator of the group γ is $i\bar{\pi}(x)$.*

Proof. It follows from Theorem 1.3.16.(ii) that the group $\{e^{itx}\}_{t\in\mathbb{R}}$ is strongly continuous. The latter implies that the function

$$t \mapsto e^{itx}, \quad t \in \mathbb{R}$$

is *wo*-continuous. Since the latter function is uniformly bounded, it is also *uw*-continuous, see Corollary 1.3.3.

It follows from Theorem 1.3.8, that the mapping $\pi : \mathcal{M} \mapsto \mathcal{M}$ is *uw*-continuous. Consequently, the function

$$t \mapsto \pi(e^{itx}), \quad t \in \mathbb{R}$$

is *uw*-continuous and therefore the mapping

$$t \mapsto \pi(e^{itx})(\xi), \quad t \in \mathbb{R}$$

is weakly continuous (in $\mathcal{H}$), for every $\xi \in \mathcal{H}$. On the other hand, it is clear that the mapping

$$\xi \mapsto \pi(e^{itx})(\xi), \quad \xi \in \mathcal{H}$$

is weakly continuous (in $\mathcal{H}$), for every $t \in \mathbb{R}$. Thus, we obtain that the group γ is weakly continuous. It now follows from Theorem 1.1.5, that the group γ is also strongly continuous. The first part of the lemma is proved.

Let us note that it is immediately follows from Lemma 1.3.19 that

$$\gamma_t = e^{it\bar{\pi}(x)}, \quad t \in \mathbb{R}. \tag{1.3.5}$$

Let x' be the generator of the group γ. Let us consider the resolvent $R_\lambda(x') = (\lambda - x')^{-1}$. By Theorem 1.1.6 we have

$$\begin{aligned} R_\lambda(x')(\xi) &= \int_0^\infty e^{-\lambda t}\, \gamma_t(\xi)\, dt \\ [(1.3.5)] \quad &= \int_0^\infty e^{-\lambda t}\, e^{it\bar{\pi}(x)}(\xi)\, dt \\ &= (\lambda - i\bar{\pi}(x))^{-1}. \end{aligned}$$

The last identity is due to the fact that

$$\frac{1}{\lambda - i\mu} = \int_0^\infty e^{-\lambda t} e^{it\mu}\, dt, \quad \lambda > 0,\ \mu \in \mathbb{R}$$

and Theorem 1.3.16. Thus, the operators $i\bar{\pi}(x)$ and x' have the same resolvents which implies that $i\bar{\pi}(x) = x'$. □

Remark 1.3.21. It is a straightforward observation, that the construction of the operator $\bar{\pi}(x)$, $x = x^*\eta\mathcal{M}$ and the results set out in Lemmas 1.3.19 and 1.3.20 are also applicable to any $*$-isomorphism, i.e. a one-to-one mapping $\pi : \mathcal{M} \mapsto \mathcal{M}_1$ such that (1.3.2) holds, where $\mathcal{M}$ and $\mathcal{M}_1$ are two ($*$-isomorphic) von Neumann algebras.

1.4 Noncommutative symmetric spaces of measurable operators

Here we recall the construction of noncommutative symmetric spaces with respect to semi-finite von Neumann algebra. We shall start with the classical rearrangement invariant and symmetric function spaces.

Let $L(\mathbb{R}) = L(\mathbb{R}, \lambda)$ be the class of all λ-measurable functions $f : \mathbb{R} \mapsto \mathbb{C}$, where λ is Lebesgue measure. For every $f \in L(\mathbb{R})$, *the distribution function* is defined by

$$\lambda_\tau(f) := \lambda\{t \in \mathbb{R} : \ |f(t)| > \tau\}, \quad \tau \geq 0.$$

The decreasing rearrangement f^* of the function $f \in L(\mathbb{R})$, such that $\lambda(f) \not\equiv +\infty$, is defined by

$$f^*(t) := \inf\{\tau \geq 0 : \ \lambda_\tau(f) \leq t\}, \quad t \geq 0.$$

A (quasi-)Banach space $E \subseteq L(\mathbb{R})$ is called *a rearrangement invariant* (r.i.) function space if and only if $y \in E$ and $\|y\|_E \leq \|x\|_E$ provided $x \in E$, $y \in L(\mathbb{R})$ and $y^*(t) \leq x^*(t)$, $t \geq 0$. At this point our terminology is slightly different from that of [44].

Let $x, y \in (L^1 + L^\infty)(\mathbb{R})$. It is said that x is submajorized by y in the sense of Hardy, Littlewood and Polya, when

$$\int_0^t x^* \, d\lambda \leq \int_0^t y^* \, d\lambda.$$

In this case, we shall write $y \prec\prec x$. Is known that, see [42, Formula (2.17), p. 91]

$$(x + y)^* \prec\prec x^* + y^*. \tag{1.4.1}$$

A Banach r.i. function space E is called *a symmetric function space* if and only if $\|y\|_E \leq \|x\|_E$ provided $x, y \in E$ and $y \prec\prec x$, see [13, 26–28]. A Banach r.i. function space E is called *a fully symmetric function space* if and only if $y \in E$ and $\|y\|_E \leq \|x\|_E$ provided $x \in E$, $y \in L^1 + L^\infty$ and $y \prec\prec x$, see [28].

The primary examples of Banach r.i. function spaces are *the function L^p spaces*, $1 \leq p \leq \infty$, i.e.

$$L^p(\mathbb{R}) := \{f \in L(\mathbb{R}) : \quad \|f\|_{L^p(\mathbb{R})} < +\infty\},$$

where the norm $\|\cdot\|_{L^p(\mathbb{R})}$ is defined by

$$\|x\|_{L^p(\mathbb{R})} := \left[\int_{\mathbb{R}} |x(t)|^p \, d\lambda(t)\right]^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

and

$$\|x\|_{L^\infty(\mathbb{R})} := \operatorname*{ess\,sup}_{t \in \mathbb{R}} |x(t)|.$$

The next theorem describes interpolation properties of Banach r.i. and symmetric function spaces.

Theorem 1.4.1 ([42, Theorems 4.1, and 4.3])**.** (i) *Every Banach r.i. function space E is an intermediate space in the couple $(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$, i.e. we have the continuous embeddings*

$$L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subseteq E \subseteq L^1(\mathbb{R}) + L^\infty(\mathbb{R}).$$

(ii) *If E is an intermediate Banach space in the couple $(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$, then E is an interpolation space with interpolation constant 1 in this couple if and only if E is fully symmetric.*

Another classical example of r.i. function spaces are *Lorentz spaces* $L^{p,r}(\mathbb{R})$, $1 \le p, r \le \infty$, which are given by the following norms

$$\|x\|_{L^{p,r}(\mathbb{R})} := \left[\int_{\mathbb{R}} (t^{1/p} x^*(t))^r \, \frac{d\lambda(t)}{t} \right]^{\frac{1}{r}}, \quad 1 \le r < \infty \tag{1.4.2}$$

and

$$\|x\|_{L^{p,\infty}(\mathbb{R})} := \sup_{t \ge 0} t^{1/p} \, x^*(t), \quad r = \infty. \tag{1.4.3}$$

For every $1 \le p \le \infty$, we have $L^{p,p}(\mathbb{R}) = L^p(\mathbb{R})$. In general Lorentz spaces are quasi-Banach spaces. Nonetheless, for $p > 1$ the Lorentz norm may be replaced with an equivalent Banach norm (see Corollary 1.4.4 below).

Let us next recall interpolation properties of the classical L^p-spaces and those of the Lorentz spaces.

Theorem 1.4.2 ([5, Theorem 5.1.1])**.** *For every $1 \le p_j \le \infty$, $j = 0, 1$ and every $0 \le \theta \le 1$,*

$$(L^{p_0}, L^{p_1})_\theta = L^p \quad \textit{(norms are identical)},$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{1.4.4}$$

Theorem 1.4.3 ([5, Theorem 5.3.1])**.** *For every $1 \le p_j, q_j \le \infty$, $j = 0, 1$ and every $0 \le \theta \le 1$,*

$$(L^{p_0,q_0}, L^{p_1,q_1})_{\theta,q} = L^{p,q} \quad \textit{(norms are equivalent)},$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad 1 \le q \le \infty, \quad \textit{if } p_0 \ne p_1$$

or

$$p = p_0 = p_1 \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let us mention two implications of Theorem 1.4.3.

Corollary 1.4.4. *For every $1 \le p_j \le \infty$, $j = 0, 1$ and every $0 \le \theta \le 1$*

(i) $(L^{p_0}, L^{p_1})_{\theta,p} = L^p$ *(norms are equivalent), where p is given in* (1.4.4)*;*

(ii) $(L^{p_0}, L^{p_1})_{\theta,q} = L^{p,q}$ *(norms are equivalent), provided $p_0 \ne p_1$, where p is given in* (1.4.4) *and $1 \le q \le \infty$;*

(iii) *The quasi-norm in the Lorentz space $L^{p,q}$ for $1 < p < \infty$ is equivalent to a norm and equipped with this norm the space $L^{p,q}$ is a Banach space, provided $1 < p < \infty$.*

The norm in the r.i. function space E is called *order-continuous* if and only if $\|x_n\|_E \downarrow 0$ for every $x_n \downarrow 0 \subseteq E$.

Lemma 1.4.5. *The r.i. function space E is separable if and only if the norm $\|\cdot\|_E$ is order-continuous.*

Let E be a r.i. function space. *The Köthe dual* $E^\times$ is defined by

$$E^\times := \{x \in L(\mathbb{R}) : \quad \|x\|_{E^\times} < +\infty\} \tag{1.4.5}$$

and

$$\|x\|_{E^\times} := \sup_{y \in (E)_1} \int_{\mathbb{R}} x(t)\, y(t)\, d\lambda(t).$$

Theorem 1.4.6 ([42, Theorem 4.9])**.** *The Köthe dual of a r.i. function space is a fully symmetric function space.*

For the Köthe dual space $E^\times$, we can construct the second dual $E^{\times\times}$. Clearly, we have the continuous embedding $E \subseteq E^{\times\times}$. If the space E is separable, then the latter embedding is isometric, i.e. $\|x\|_E = \|x\|_{E^{\times\times}}$, $x \in E$.

It is said that a r.i. function space E has *a Fatou norm* if and only if, for every $0 \leq x_n \uparrow_n x \subseteq E$, $\|x\|_E = \sup_n \|x_n\|_E$. A r.i. function space E has Fatou norm if and only if the embedding $E \subseteq E^{\times\times}$ is isometric. If E has Fatou norm then E is symmetric.

It is said that a r.i. function space E has *the Fatou property* if and only if, for every $\{x_n\} \subseteq E$, such that $x_n \uparrow_n x \in L^1 + L^\infty$ and $\sup_n \|x_n\|_E < \infty$, it follows that $x = \sup_n x_n \in E$ exists and $\|x_n\|_E \uparrow_n \|x\|_E$. A r.i. function space E has the Fatou property if and only if $E = E^{\times\times}$. If E has the Fatou property, then E is fully symmetric.

1.4.1 Noncommutative symmetric spaces

Let us now consider the noncommutative counterparts of the spaces E. We fix a semi-finite von Neumann algebra $\mathcal{M}$ with n.s.f. trace τ.

Let $x : \mathscr{D}(x) \mapsto \mathcal{H}$ be a closed densely defined linear operator affiliated with $\mathcal{M}$. An operator x is called *τ-measurable* if and only if for every $\epsilon > 0$ there is a projection $p = p_\epsilon \in \mathcal{M}$ such that $p(\mathcal{H}) \subseteq \mathscr{D}(x)$ and $\tau(\mathbf{1} - p) < \epsilon$.

The distribution function of the operator $x\eta\mathcal{M}$ with respect to the pair $(\mathcal{M},\tau)$ is defined by (see [30])

$$\lambda_s(x) := \tau(e^{|x|}(s,+\infty)).$$

Lemma 1.4.7 ([30])**.** *Let $x\eta\mathcal{M}$. The following are equivalent*

(i) *x is τ-measurable;*

(ii) *$\lambda_s(x) < +\infty$ for some $s > 0$;*

(iii) *$\lambda_s(x) \to 0$ when $s \to +\infty$.*

Let $\tilde{\mathcal{M}}$ be the collection of all τ-measurable operators. Let us recall that, for every $x \in \tilde{\mathcal{M}}$, *the generalized singular value function* is given by

$$\mu_t(x) := \inf\{s \geq 0 : \quad \lambda_s(x) \leq t\}. \tag{1.4.6}$$

The measure topology is the topology generated by the collection of neighborhoods of the origin $\{N(\epsilon,\delta)\}_{\epsilon,\delta>0}$, where $N(\epsilon,\delta)$ is the set of all closed densely defined linear operators $x : \mathscr{D}(x) \mapsto \mathcal{H}$ affiliated with with $\mathcal{M}$ such that there is a projection $p \in \mathcal{M}$ satisfying $p(\mathcal{H}) \subseteq \mathscr{D}(x)$, $\tau(\mathbf{1}-p) < \epsilon$ and $\|xp\| < \delta$.

Alternatively, $\lim_\alpha x_\alpha = x$, $x_\alpha, x \in \tilde{\mathcal{M}}$ with respect to *the measure topology* if and only if

$$\lim_\alpha \mu_t(x_\alpha - x) = 0, \quad \text{for every } t > 0.$$

Theorem 1.4.8 ([46])**.** *The collection $\tilde{\mathcal{M}}$ is a complete topological $*$-algebra with respect to the measure topology.*

Let $E = E(\mathbb{R})$ be a r.i. function space and $\mathcal{M}$ be a fixed semi-finite von Neumann algebra. The corresponding noncommutative space $E(\mathcal{M},\tau)$ is defined by

$$E(\mathcal{M},\tau) := \{x \in \tilde{\mathcal{M}} : \quad \mu(x) \in E(\mathbb{R})\}. \tag{1.4.7}$$

If E is fully symmetric, then the norm in the space $E(\mathcal{M},\tau)$ is given by

$$\|x\|_{E(\mathcal{M},\tau)} := \|\mu(x)\|_E.$$

Theorem 1.4.9 ([25, 29], see also [13])**.** *If $E = E(\mathbb{R})$ is a fully symmetric function space, then the space $E(\mathcal{M},\tau)$ equipped with the norm $\|\cdot\|_{E(\mathcal{M},\tau)}$ is a Banach space.*

As for the classical spaces, the Hölder inequality is valid in the noncommutative L^p-spaces, cf. [30, Theorem 4.2.(i)], i.e.

$$\|xy\|_{\mathcal{L}^s} \leq \|x\|_{\mathcal{L}^p} \, \|y\|_{\mathcal{L}^q}, \quad \frac{1}{s} = \frac{1}{p} + \frac{1}{q}, \ 1 \leq s, p, q \leq \infty. \tag{1.4.8}$$

Let $E = E(\mathbb{R})$ be a fully symmetric function space. We shall frequently denote the corresponding (noncommutative) space $E(\mathcal{M}, \tau)$ as $\mathcal{E}$. In particular, we shall write $\mathcal{L}^p$, $1 \leq p \leq \infty$ for the (noncommutative) L^p-spaces.

Let $E = E(\mathbb{R})$ be a fully symmetric function space and let $E^\times$ be the corresponding Köthe dual space defined in (1.4.5). It is proved in [26, Proposition 5.3] that, if $\mathcal{E}^\times := E^\times(\mathcal{M}, \tau)$, then

$$\mathcal{E}^\times = \{a \in \tilde{\mathcal{M}} : \quad ab \in \mathcal{L}^1, \quad \text{whenever} \quad b \in \mathcal{E}\}$$

and

$$\|a\|_{\mathcal{E}^\times} = \sup_{b \in (\mathcal{E})_1 \cap \mathcal{L}^1 \cap \mathcal{L}^\infty} |\tau(ab)|. \tag{1.4.9}$$

According to (1.4.9), every element $a \in \mathcal{E}^\times$ generates the continuous linear form

$$f_a : x \mapsto \tau(ax), \quad x \in \mathcal{E}$$

and

$$\|f_a\|_{\mathcal{E}^*} = \|a\|_{\mathcal{E}^\times}.$$

Thus, the Köthe dual space $\mathcal{E}^\times$ naturally embeds into the dual $\mathcal{E}^*$. In the present manuscript, we frequently identify the element $a \in \mathcal{E}^\times$ with the corresponding form $f_a \in \mathcal{E}^*$.

Let us note, that, if $\mathcal{L}^p := L^p(\mathcal{M}, \tau)$, $1 \leq p \leq \infty$, then

$$(\mathcal{L}^p)^\times = \mathcal{L}^{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

and also

$$(\mathcal{L}^1 + \mathcal{L}^\infty)^\times = \mathcal{L}^1 \cap \mathcal{L}^\infty, \quad (\mathcal{L}^1 \cap \mathcal{L}^\infty)^\times = \mathcal{L}^1 + \mathcal{L}^\infty. \tag{1.4.10}$$

Lemma 1.4.10. *Elements in $\mathcal{L}^1 \cap \mathcal{L}^\infty$ separates points in $\mathcal{L}^1 + \mathcal{L}^\infty$, i.e., if $x_1, x_2 \in \mathcal{L}^1 + \mathcal{L}^\infty$ and $\langle x_1, y\rangle = \langle x_2, y\rangle$, for every $y \in \mathcal{L}^1 \cap \mathcal{L}^\infty$, then $x_1 = x_2$.*

Proof. The proof is immediately seen from the observation (1.4.10) and (1.4.9). However, it is instructive to present a direct elementary proof also.

Let $x \in \mathcal{L}^1 + \mathcal{L}^\infty$. It is sufficient to prove that

$$\tau(xy) = 0,\ \forall y \in \mathcal{L}^1 \cap \mathcal{L}^\infty \quad \Longrightarrow \quad x = 0. \tag{1.4.11}$$

Since the algebra $\mathcal{M}$ is semi-finite, there is a net of projections $p_\alpha \uparrow_\alpha \mathbf{1} \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty$. Consider the polar decomposition of the operator x

$$x = u\,|x|, \quad u \in \mathcal{M}.$$

It is clear that $p_\alpha u^* \in \mathcal{L}^1 \cap \mathcal{L}^\infty$, for every α. Consequently, it follows from the hypothesis in (1.4.11) that

$$\tau(x\,p_\alpha u^*) = 0, \quad \forall\alpha.$$

On the other hand

$$0 = \tau(x\,p_\alpha u^*) = \tau(|x|p_\alpha) = \tau(|x|^{1/2}\,p_\alpha\,|x|^{1/2}).$$

Since the operator $|x|^{1/2}p_\alpha\,|x|^{1/2}$ is positive, we obtain that $|x|^{1/2}p_\alpha\,|x|^{1/2} = 0$ for every α. Now, we observe that $|x|^{1/2}\,p_\alpha|x|^{1/2} \uparrow_\alpha |x|$ and that the space $\mathcal{L}^1 + \mathcal{L}^\infty$ has the Fatou norm which imply that $|x| = x = 0$. The lemma is completely proved. □

Let us next recall some properties of the noncommutative symmetric space $\mathcal{E}$ with respect to a fully symmetric function space with an order-continuous norm.

Theorem 1.4.11 ([13, Proposition 2.1])**.** *If the symmetric function space E has order-continuous norm, then $\lambda_s(x) < +\infty$ for every $s > 0$, provided $x \in \mathcal{E}$. Equivalently, $\mu_\infty(x) := \lim_{t\to\infty} \mu_t(x) = 0$, provided $x \in \mathcal{E}$.*

Theorem 1.4.12 ([13, Proposition 2.5])**.** *Let $\mathcal{E}$ be a noncommutative symmetric operator space with respect to a fully symmetric function space with an order-continuous norm. If $\{p_\alpha\} \subseteq \mathcal{M}$ is a net of projections such that $p_\alpha \downarrow 0$, then* $\lim_\alpha \|xp_\alpha\|_{\mathcal{E}} = 0$*, for every $x \in \mathcal{E}$.*

Let us note, that, if the space $\mathcal{E}$ does not have order-continuous norm, then we have the following weaker result.

Lemma 1.4.13. *Let $\{x_\alpha\} \subseteq (\mathcal{M})_1$ be a net. If $wo - \lim_\alpha x_\alpha = 0$, then*

$$\sigma(\mathcal{E}, \mathcal{E}^\times) - \lim_\alpha y x_\alpha = \sigma(\mathcal{E}, \mathcal{E}^\times) - \lim_\alpha x_\alpha y = 0, \quad y \in \mathcal{E}. \tag{1.4.12}$$

In particular, the space $\mathcal{L}^1 \cap \mathcal{L}^\infty$ is $\sigma(\mathcal{E}, \mathcal{E}^\times)$-dense in $\mathcal{E}$.

Proof. The proof is straightforward. Since the net $\{x_\alpha\}$ is uniformly bounded, it readily follows from Corollary 1.3.3 that

$$uw - \lim_\alpha x_\alpha = 0.$$

On the other hand, for every $y \in \mathcal{E}$ and $z \in \mathcal{E}^\times$, it is clear that $zy, yz \in \mathcal{L}^1$, see (1.4.9). Thus, the limit above implies that

$$\lim_\alpha \tau(x_\alpha zy) = \lim_\alpha \tau(x_\alpha yz) = 0, \quad y \in \mathcal{E},\ z \in \mathcal{E}^\times.$$

The latter is equivalent to (1.4.12).

For the second claim, observe the following. For every $x \in \mathcal{E}$, there is a sequence of projections $\{p_n\}_{n\geq 1}$ such that

$$p_n \to \mathbf{1} \ \text{ and } \ xp_n \in \mathcal{L}^1 \cap \mathcal{L}^\infty.$$

On the other hand, it follows from (1.4.12) that

$$\sigma(\mathcal{E}, \mathcal{E}^\times) - \lim_{n\to\infty} x(1 - p_n) = 0 \iff \sigma(\mathcal{E}, \mathcal{E}^\times) - \lim_{n\to\infty} xp_n = x.$$

The lemma is proved. □

Lemma 1.4.14. (i) *If the space E has order-continuous norm, then the space $\mathcal{L}^1 \cap \mathcal{L}^\infty$ is norm dense in $\mathcal{E}$. In particular, the space $\mathcal{L}^1 \cap \mathcal{L}^\infty$ is norm dense in $\mathcal{L}^p$, $1 \leq p < \infty$.*

(ii) *Since $\mathcal{L}^1 \cap \mathcal{L}^\infty$ is dense in $\mathcal{L}^1$, it follows from Lemma 1.1.2 and Corollary 1.3.3 that the wo-topology, the uw-topology, the $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$-topology and the $\sigma(\mathcal{L}^\infty, \mathcal{L}^1 \cap \mathcal{L}^\infty)$-topology coincide on $(\mathcal{L}^\infty)_1 = (\mathcal{M})_1$.*

Proof. (i) Let $x \in \mathcal{E}$. We set $p_n := e^{|x|}\left(\frac{1}{n}, n\right)$, $n \geq 1$. According to Theorem 1.4.11, $\{p_n\}_{n=1}^\infty \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty$. Moreover, according to Theorem 1.3.16, the operators $xp_n \in \mathcal{L}^\infty$, for every $n \geq 1$. Consequently, $xp_n = (xp_n)\cdot p_n \in \mathcal{L}^1 \cap \mathcal{L}^\infty$, for every $n \geq 1$. On the other hand, it is clear that $p_n \uparrow e^{|x|}(0, +\infty)$ as $n \to \infty$. Thus, it follows from Theorem 1.4.12 that

$$\lim_{n\to\infty} xp_n = xe^{|x|}(0, +\infty) = x(\mathbf{1} - e^{|x|}(0)) = x(\mathbf{1} - e^x(0)) = x,$$

where the limit is taken with respect to the norm topology of $\mathcal{E}$. The claim (i) is proved.

(ii) This claim readily follows from Corollary 1.3.3, the fact that $\mathcal{M}_* = \mathcal{L}^1$ and the claim (i) together with Lemma 1.1.2. □

Lemma 1.4.15. *If $z \in \mathcal{L}^1 + \mathcal{L}^\infty$ and if $\mathscr{D} \subseteq \mathcal{H}$ is a dense subspace affiliated with $\mathcal{M}$ and $z' = z|_D$, then $\overline{z'} = z$.*

Proof. Since $D\eta\mathcal{M}$, it follows that $z'\eta\mathcal{M}$. Since D is dense and $z' \subseteq z$, we have $z^* \subseteq z'^*$. The operator z^* is τ-measurable, therefore, z'^* is τ-measurable also and $z'^* = z^*$, cf. [32, Lemma 2.1]. Passing to the second adjoints, we obtain $\overline{z'} = z'^{**} = z^{**} = z$. □

1.4.2 Trace scaling $*$-automorphisms of a von Neumann algebra

Let us consider a $*$-automorphism of the algebra $\mathcal{M}$. The mapping π is *uw*-continuous, see Theorem 1.3.8. Let us assume that

$$\pi(\mathcal{L}^1 \cap \mathcal{L}^\infty) \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty \quad \text{and} \quad \tau(\pi(x)) = \alpha\, \tau(x), \quad x \in \mathcal{L}^1 \cap \mathcal{L}^\infty, \tag{1.4.13}$$

for some $\alpha > 0$. Such an automorphism is call *trace scaling with the factor α*. For the special case $\alpha = 1$, a trace scaling $*$-automorphism is called *trace preserving.*

Lemma 1.4.16. *A trace scaling $*$-automorphism π of a semi-finite von Neumann algebra with a factor $\alpha > 0$ is continuous with respect to the measure topology.*

Proof. Let us fix $x \in \mathcal{M}$ and consider the action of the automorphism π on the distribution function $\lambda_s(x)$. We clearly, have that

$$\begin{aligned}\lambda_s(\pi(x)) &= \tau(e^{|\pi(x)|}(s, +\infty)) \\ &= \tau(\pi(e^{|x|}(s, +\infty))) \\ &= \alpha\, \tau(e^{|x|}(s, +\infty)) \\ &= \alpha\, \lambda_s(x).\end{aligned}$$

This permits us to consider the action of the automorphism π on the function $\mu(x)$, namely

$$\begin{aligned}\mu_t(\pi(x)) &= \inf\{s > 0 : \quad \lambda_s(\pi(x)) \le t\} \\ &= \inf\{s > 0 : \quad \alpha\, \lambda_s(x) \le t\} \\ &= \inf\left\{s > 0 : \quad \lambda_s(x) \le \frac{t}{\alpha}\right\} \\ &= \mu_{\frac{t}{\alpha}}(x).\end{aligned}$$

The latter implies that the mapping π is continuous with respect to the measure topology. Thus, the lemma is proved. □

Let $\tilde{\pi}$ be the unique extension of π to the algebra $\tilde{\mathcal{M}}$ and let $\pi^{\mathcal{E}}$ be the restriction of the operator $\tilde{\pi}$ onto the space $\mathcal{E}$. In particular, let $\pi^p := \pi^{\mathcal{L}^p}$.

Lemma 1.4.17. *Let π be a trace scaling $*$-automorphism of a semi-finite von Neumann algebra with a factor $\alpha > 0$.*

(i) *The space $\mathcal{L}^\infty$ (resp. $\mathcal{L}^1$) is invariant with respect to the operator π^∞ (resp. π^1), i.e.*

$$\pi^\infty(\mathcal{L}^\infty) \subseteq \mathcal{L}^\infty \quad (\textit{resp. } \pi^1(\mathcal{L}^1) \subseteq \mathcal{L}^1)$$

and

$$\|\pi^\infty\|_{B(\mathcal{L}^\infty)} = 1, \quad (\textit{resp. } \|\pi^1\|_{B(\mathcal{L}^1)} = \alpha).$$

(ii) *If $\mathcal{E}$ is a fully symmetric noncommutative space, then $\pi^{\mathcal{E}}(\mathcal{E}) \subseteq \mathcal{E}$ and $\pi^{\mathcal{E}} \in B(\mathcal{E})$.*

Proof. The proof follows from (1.4.13), the fact that π is an $*$-automorphism and interpolation. □

In the case of trace *preserving* $*$-automorphism the results presented above are proved in [21].

Remark 1.4.18. If the operator x is τ-measurable, then its spectral resolution (1.3.4) converges with respect to the measure topology. The latter implies that the extension $\tilde{\pi}$ coincides with the one discussed in the end of Section 1.3, i.e., for every self-adjoint $x \in \tilde{\mathcal{M}}$, we have that $\tilde{\pi}(x) = \bar{\pi}(x)$.

Remark 1.4.19. The construction of the extension $\tilde{\pi}$ given in the present section without any changes carries to the setting of an arbitrary trace scaling $*$-isomorphism $\pi : \mathcal{M} \mapsto \mathcal{M}_1$, where $\mathcal{M}_1$ is another semi-finite von Neumann algebra.

1.4.3 Interpolation of spaces of measurable operators

Let us recall the following result, which will allow to carry the classical interpolation results stated in Theorems 1.4.2 and 1.4.3 and Corollary 1.4.4 to the setting of noncommutative spaces of measurable operators.

Theorem 1.4.20 ([28, Corollary 2.7], see also [52, Corollary 2.2])**.** *If $\bar{E} := (E_0, E_1)$ be a compatible couple of symmetric function spaces, then*

(i) *the couple* $\bar{\mathcal{E}} = (\mathcal{E}_0, \mathcal{E}_1) = (E_0(\mathcal{M}, \tau), E_1(\mathcal{M}, \tau))$ *is compatible;*

(ii) $\bar{E}_\theta(\mathcal{M}, \tau) = \bar{\mathcal{E}}_\theta$*, for every* $0 \le \theta \le 1$*;*

(iii) $\bar{E}_{\theta,q}(\mathcal{M}, \tau) = \bar{\mathcal{E}}_{\theta,q}$*, for every* $0 \le \theta \le 1$ *and every* $1 \le q \le \infty$*.*

Corollary 1.4.21. *The classical spaces* L^p *and* $L^{p,q}$*,* $1 \le p, q \le \infty$ *may be replaced with their noncommutative counterparts* $\mathcal{L}^p$ *and* $\mathcal{L}^{p,q}$ *in the statements of Theorems 1.4.2 and 1.4.3 and Corollary 1.4.4.*

Combining the interpolation Theorem 1.4.3 in the noncommutative setting with the duality theorem (see Theorem 1.2.5), we obtain that

Theorem 1.4.22. *The dual space* $(\mathcal{L}^{p,q})^*$ *coincides with* $\mathcal{L}^{p',q'}$ *(norms are equivalent), provided* $1 < p < \infty$ *and* $1 \le q < \infty$ *(or* $p = q = 1$*). In particular, for every* $1 < p < \infty$*,* $1 \le q < \infty$ *(or* $p = q = 1$*), there is a constant* $c_{p,q} > 0$ *such that*

$$|\tau(x\,y)| \le c_{p,q}\, \|x\|_{\mathcal{L}^{p,q}}\, \|y\|_{\mathcal{L}^{p',q'}}, \quad x \in \mathcal{L}^{p,q},\ y \in \mathcal{L}^{p',q'}. \tag{1.4.14}$$

The classical version of the latter result is proved in [3, Corollary 4.8].

Let us note that, it follows from [30, Lemma 2.5] that, for every $x \in \tilde{\mathcal{M}}$,

$$\mu_t(x) = (\mu_t(|x|^2))^{1/2}, \quad t \in \mathbb{R}.$$

Consequently, for every $2 \le p, q \le \infty$,

$$\begin{aligned} \|x\|_{\mathcal{L}^{p,q}} &= \left[\int_0^\infty \left(t^{1/p}\mu_t(x)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}} \\ &= \left[\int_0^\infty \left(t^{2/p}\mu_t(|x|^2)\right)^{\frac{q}{2}} \frac{dt}{t}\right]^{\frac{2}{q}\cdot\frac{1}{2}} \\ &= \|\,|x|^2\|^{1/2}_{\mathcal{L}^{p/2,q/2}}, \quad x \in \mathcal{L}^{p,q}. \end{aligned} \tag{1.4.15}$$

Thus, the identity (1.4.15) implies that

$$\begin{aligned} \|xy\|_{\mathcal{L}^2} &= \left[\tau(|xy|^2)\right]^{\frac{1}{2}} \\ &= \left[\tau(|x|^2|y^*|^2)\right]^{\frac{1}{2}} \\ \text{[see (1.4.14)]} \quad &\le \|\,|x|^2\|^{1/2}_{\mathcal{L}^{p/2,r/2}}\, \|\,|y^*|^2\|^{1/2}_{\mathcal{L}^{p/2,s/2}} \\ &= \|x\|_{\mathcal{L}^{p,r}}\, \|y\|_{\mathcal{L}^{q,s}}, \quad x \in \mathcal{L}^{p,r},\ y \in \mathcal{L}^{q,s}, \end{aligned} \tag{1.4.16}$$

where $2 \le p, q, r, s \le \infty$ and

$$\frac{1}{2} = \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s}.$$

The identity (1.4.16) generalizes (1.4.8) for the Lorentz spaces.

1.5 The regular representations of a semi-finite von Neumann algebra

Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with n.s.f. trace τ acting on $\mathcal{H}$. If $E := E(\mathbb{R})$ is the fully symmetric function space, then $\mathcal{E} := E(\mathcal{M}, \tau)$ stands for the corresponding operator space, in particular, $\mathcal{L}^p := L^p(\mathcal{M}, \tau)$, $1 \leq p \leq \infty$ are the noncommutative L^p-spaces.

Remark 1.5.1. Let us note that the space $\mathcal{L}^2$ is Hilbert. The corresponding sesquilinear form is given by

$$\langle \xi, \eta \rangle := \tau(\xi\, \eta^*).$$

In the present section, we shall discuss the representations of the algebra $\mathcal{M}$ as a left (or right) multiplication operators on the Hilbert space $\mathcal{L}^2$.

Let $x\eta\mathcal{M}$. We consider the operators $L_x, R_x \in B(\mathcal{L}^2)$ defined by

$$L_x(\xi) = x \cdot \xi \quad \text{and} \quad R_x(\xi) = \xi \cdot x, \quad \xi \in \mathcal{L}^2.$$

Let us also consider the mappings $L, R : \mathcal{M} \mapsto B(\mathcal{L}^2)$ given by

$$L(x) := L_x \quad \text{and} \quad R(x) = R_x, \quad x \in \mathcal{M}.$$

We set $\mathcal{M}_L := L(\mathcal{M})$ and $\mathcal{M}_R := R(\mathcal{M})$.

In the first part of the present section, we shall prove

Theorem 1.5.2. *The image $\mathcal{M}_L$ (resp. $\mathcal{M}_R$) is a von Neumann algebra acting on $\mathcal{L}^2$ and the mapping $L : \mathcal{M} \mapsto \mathcal{M}_L$ (resp. $R : \mathcal{M} \mapsto \mathcal{M}_R$) is a uw-continuous $*$-isomorphism between the algebras $\mathcal{M}$ and $\mathcal{M}_L$ (resp. $\mathcal{M}_R$).*

The latter theorem follows from Theorem 1.3.1 and Lemma 1.5.5 and Theorem 1.5.3 below.

Theorem 1.5.3 ([62, Ch. V, Theorem 2.22])**.** *The commutant ${\mathcal{M}_L}'$ of $\mathcal{M}_L$ is $\mathcal{M}_R$ (and similarly ${\mathcal{M}_R}' = \mathcal{M}_L$).*

To prove Lemma 1.5.5 we need the following simple result.

Lemma 1.5.4. *If $x \in \tilde{\mathcal{M}}$ and there is a constant $c > 0$ such that*

$$\|xy\|_{\mathcal{L}^2} \leq c\, \|y\|_{\mathcal{L}^2}, \quad \textit{for every } y \in \mathcal{L}^2$$

then $x \in \mathcal{M}$ and $\|x\| \leq c$.

Proof. The claim of the lemma readily follows from

$$\begin{aligned}\|x\| &= \|x^*x\|^{1/2} \\ &= \sup_{z\in(\mathcal{L}^1_+)_1} [\tau(x^*x\,z)]^{1/2} \\ &= \sup_{z\in(\mathcal{L}^1_+)_1} [\tau(z^{1/2}x^*xz^{1/2})]^{1/2} \\ &= \sup_{z\in(\mathcal{L}^1_+)_1} \|xz^{1/2}\|_{\mathcal{L}^2} \\ &\le c \sup_{z\in(\mathcal{L}^1_+)_1} \|z^{1/2}\|_{\mathcal{L}^2} = c\end{aligned}$$

□

Lemma 1.5.5. *The mapping $L : \mathcal{M} \mapsto \mathcal{M}_L$ (resp. $R : \mathcal{M} \mapsto \mathcal{M}_R$) is a $*$-isomorphism of the algebra $\mathcal{M}$ onto the von Neumann algebra $\mathcal{M}_L$ (resp. $\mathcal{M}_R$).*

Proof. We shall prove the lemma only for the mapping L; for the mapping R, argument is similar. We readily see that

$$L(x + \alpha y)(\xi) = (x + \alpha y)\,\xi = x\xi + \alpha y\xi = L_x(\xi) + \alpha L_y(\xi)$$

and

$$L_{xy}(\xi) = (xy)\,\xi = L_x(L_y(\xi)), \quad x, y \in \mathcal{M},\ \xi \in \mathcal{L}^2,\ \alpha \in \mathbb{C}.$$

Furthermore,

$$\langle L_x(\xi), \eta\rangle = \tau(x\xi\eta^*) = \tau(\xi(x^*\eta)^*) = \langle \xi, L_x^*(\eta)\rangle, \quad \xi, \eta \in \mathcal{L}^2,\ x \in \mathcal{M}.$$

Thus, we obtain that the mapping $L : \mathcal{M} \mapsto B(\mathcal{L}^2)$ is a $*$-homomorphism and

$$L(x + \alpha y) = L_x + \alpha L_y, \quad L_{xy} = L_x L_y, \quad L_x^* = L(x^*).$$

The latter, in particular, means, that the image $\mathcal{M}_L$ is a $*$-subalgebra of $B(\mathcal{L}^2)$. The fact that the mapping L is injective follows from Lemma 1.5.4. □

The algebra $\mathcal{M}_L$ (resp. $\mathcal{M}_R$) is called *the left (resp. right) regular representation* of the algebra $\mathcal{M}$. From now on we shall discuss only the left regular representation, the reader can easily reconstruct the corresponding notions and results for the right regular representation.

The algebra $\mathcal{M}_L$ is equipped with n.s.f trace τ_L given by

$$\tau_L(x) = \tau(L^{-1}(x)), \quad x \in \mathcal{M}_L.$$

Having defined τ_L, the mapping $L : \mathcal{M} \mapsto \mathcal{M}_L$ becomes a trace preserving $*$-isomorphism. Let $\tilde{L} : \tilde{\mathcal{M}} \mapsto \tilde{\mathcal{M}}_L$ be the extension of the latter isomorphism to the algebra of τ-measurable operators (see Section 1.4.2). The next lemma describes the mapping $\tilde{L}$.

Lemma 1.5.6. *The mapping $\tilde{L} : \tilde{\mathcal{M}} \mapsto \tilde{\mathcal{M}}_L$ is given by $\tilde{L}(x) = L_x$, $x \in \tilde{\mathcal{M}}$, where $L_x : \mathscr{D}(L_x) \mapsto \mathcal{L}^2$ is the operator defined by*

$$\mathscr{D}(L_x) := \{\xi \in \mathcal{L}^2 : \quad x \cdot \xi \in \mathcal{L}^2\} \tag{1.5.1}$$

and

$$L_x(\xi) := x \cdot \xi, \quad \xi \in \mathscr{D}(L_x). \tag{1.5.2}$$

Proof. Let us fix $x \in \tilde{\mathcal{M}}$. We shall show first that the operator L_x, defined in (1.5.1) and (1.5.2) is closed. Let

$$\{\xi_n\}_{n=1}^{\infty} \subseteq \mathscr{D}(L_x), \ \lim_{n\to\infty} \xi_n = \xi \in \mathcal{L}^2 \ \text{ and } \ \lim_{n\to\infty} L_x(\xi_n) = \eta \in \mathcal{L}^2,$$

where the limits converge with respect to the norm topology of $\mathcal{L}^2$. We shall show that $\xi \in \mathscr{D}(L_x)$ and $L_x(\xi) = x \cdot \xi = \eta$ which will imply that L_x is closed. To this end, let us consider the collection of projections $\{p_m\}_{m=1}^{\infty} \subseteq \mathcal{M}$ such that

$$p_m \uparrow \mathbf{1}, \text{ as } m \to \infty \ \text{ and } \ \|p_m x\| \leq m, \quad m \geq 1.$$

We have the following chain of relations

$$\begin{aligned}
\|x\xi\|_{\mathcal{L}^2} &= \sup_{m\geq 1} \|p_m x\xi\|_{\mathcal{L}^2} \\
&= \sup_{m\geq 1} \sup_{\zeta\in(\mathcal{L}^2)_1} \tau(p_m x\, \xi\, \zeta) \\
&= \sup_{m\geq 1} \sup_{\zeta\in(\mathcal{L}^2)_1} \lim_{n\to\infty} \tau(p_m x \xi_n \zeta) \\
&= \sup_{m\geq 1} \sup_{\zeta\in(\mathcal{L}^2)_1} \lim_{n\to\infty} \tau(p_m L_x(\xi_n)\, \zeta) \\
&= \sup_{m\geq 1} \sup_{\zeta\in(\mathcal{L}^2)_1} \tau(p_m\, \eta\, \zeta) \\
&\leq \|\eta\|_{\mathcal{L}^2}.
\end{aligned}$$

Consequently, $x\xi \in \mathcal{L}^2$, and hence, $\xi \in \mathscr{D}(L_x)$. Furthermore, since $x\xi \in \mathcal{L}^2$, it follows from Theorem 1.4.12 that

$$\begin{aligned}\tau(L_x(\xi)\,\zeta) &= \tau(x\xi\,\zeta)\\ &= \lim_{m\to\infty}\tau(p_m x\xi\,\zeta)\\ &= \lim_{m\to\infty}\lim_{n\to\infty}\tau(p_m x\xi_n\,\zeta)\\ &= \lim_{m\to\infty}\lim_{n\to\infty}\tau(p_m L_x(\xi_n)\,\zeta)\\ &= \lim_{m\to\infty}\tau(p_m\eta\,\zeta)\\ &= \tau(\eta\,\zeta),\quad \zeta\in\mathcal{L}^2.\end{aligned}$$

Consequently, it follows from Lemma 1.4.10 that $L_x(\xi) = \eta$. Thus, it is proved that the operator L_x is closed.

Let us next show that the operator L_x is τ_L-measurable. Let $\epsilon > 0$. Since the operator x is τ-measurable, there is a projection $p \in \mathcal{M}$ such that $\tau(\mathbf{1} - p) < \epsilon$ and $p(\mathcal{H}) \subseteq \mathscr{D}(x)$. Since the operator x is closed, the latter implies that the operator xp is bounded. Consequently, we have that the operator $\mathbf{p} := L(p)$ is a projection in $\mathcal{M}_L$ such that $\tau_L(\mathbf{1} - \mathbf{p}) < \epsilon$ and, for every $\xi \in \mathcal{L}^2$, we clearly have that

$$\|x\mathbf{p}(\xi)\|_{\mathcal{L}^2} = \|xp\xi\|_{\mathcal{L}^2} \le \|xp\|\,\|\xi\|_{\mathcal{L}^2} < +\infty. \tag{1.5.3}$$

Hence, $\mathbf{p}(\mathcal{L}^2) \subseteq \mathscr{D}(L_x)$, and therefore the operator L_x is τ_L-measurable.

Let us recall that $\tilde{L} : \tilde{\mathcal{M}} \mapsto \tilde{\mathcal{M}}_L$ is the unique continuous extension of the operator $L : \mathcal{M} \mapsto \mathcal{M}_L$. Let us remind that $x \in \tilde{\mathcal{M}}$. Observe that the identity (1.5.3) shows that, for every projection $p \in \mathcal{M}$ such that $p(\mathcal{H}) \subseteq \mathscr{D}(x)$, we have the estimate

$$\|L_x\mathbf{p}\| \le \|xp\|.$$

Consequently, for every $x \in \tilde{\mathcal{M}}$ and every $\epsilon, \delta > 0$, we obtain the implication

$$x \in N(\epsilon, \delta) \Longrightarrow L_x \in N_L(\epsilon, \delta),$$

where $\{N_L(\epsilon, \delta)\}_{\epsilon,\delta>0}$ is the collection of neighborhoods of the origin in $\tilde{\mathcal{M}}_L$ with respect to the measure topology defined by τ_L. Since, the mapping $x \mapsto L_x$ is linear, we readily see this mapping is continuous with respect to the measure topology. Thus, we have established that the mapping $x \mapsto L_x$ is continuous and coincides with the isomorphism L on the dense subspace $\mathcal{M} \subseteq \tilde{\mathcal{M}}$. The latter immediately implies that $\tilde{L}(x) = L_x$, for every $x \in \tilde{\mathcal{M}}$. □

It follows from Lemma 1.4.17 that the mapping $L^{\mathcal{E}}$ is an isometry between the the spaces $\mathcal{E} = E(\mathcal{M}, \tau)$ and $\mathcal{E}_L := E(\mathcal{M}_L, \tau_L)$ for every fully symmetric function space $E = E(\mathbb{R})$. From now on, we shall denote the mappings $\tilde{L}$ and $L^{\mathcal{E}}$ simply by L. The latter lemma ensures that this will not lead to any ambiguity.

Let us also recall that $\bar{L}$ stands for the extension of the isomorphism $L : \mathcal{M} \mapsto \mathcal{M}_L$ to the class of self-adjoint operators affiliated with $\mathcal{M}$, see discussion at the end of Section 1.3. We set $L_x := \bar{L}(x)$, for every $x = x^* \eta \mathcal{M}$. Lemma 1.5.6 yields that $\bar{L}(x) = \tilde{L}(x)$ for every $x = x^* \in \tilde{\mathcal{M}}$.

1.6 Basic examples of von Neumann algebras and noncommutative symmetric spaces

In this section, we consider several basic examples of von Neumann algebras and their properties we need in the sequel.

1.6.1 $\mathcal{M} = L^\infty(\mathfrak{M}, m)$

We first consider the commutative example. Let $(\mathfrak{M}, m)$ be a σ-finite measure space with σ-additive measure m. The space $L^p(\mathfrak{M}, m)$, $1 \le p \le \infty$ stands for the collection of all m-measurable functions $f : \mathfrak{M} \mapsto \mathbb{C}$ such that

$$\|f\|_{L^p(\mathfrak{M},m)} := \int_{\mathfrak{M}} |f(t)|^p \, dm(t) < \infty, \quad 1 \le p < \infty$$

and

$$\|f\|_{L^\infty(\mathfrak{M},m)} := \operatorname{ess\,sup}_{t \in \mathfrak{M}} |f(t)| < \infty.$$

The space $L^\infty(\mathfrak{M}, m)$ is a $*$-algebra with respect to the pointwise operations and complex conjugation.

To every function $f \in L^\infty(\mathfrak{M}, m)$, we assign the multiplication operator $m_f \in B(L^2(\mathfrak{M}, m))$, given by

$$m_f(\xi) = f \cdot \xi, \quad \xi \in L^2(\mathfrak{M}, m).$$

Thus, the $*$-algebra $L^\infty(\mathfrak{M}, m)$ becomes a subalgebra of $B(\mathcal{L}^2(\mathfrak{M}, m))$. We shall identify function $f \in L^\infty(\mathfrak{M}, m)$ and the corresponding operator m_f.

It is known that $(L^\infty(\mathfrak{M}, m))' = L^\infty(\mathfrak{M}, m)$, e.g. [36, Example 5.1.6]. Thus, we readily see that $\mathcal{M} = L^\infty(\mathfrak{M}, m)$ is a von Neumann algebra acting on $\mathcal{H} = L^2(\mathfrak{M}, m)$.

It follows from [36, Example 5.1.6] that, if an operator $x : \mathscr{D}(x) \mapsto L^2(\mathfrak{M}, m)$ is affiliated with the algebra $L^\infty(\mathfrak{M}, m)$, then there is an m-measurable function f such that

$$\xi \in \mathscr{D}(x) \implies f \cdot \xi \in L^2(\mathfrak{M}, m) \text{ and } x(\xi) = f \cdot \xi,\ \xi \in \mathscr{D}(x).$$

The algebra $\mathcal{M}$ is equipped with the distinguished trace

$$\tau(f) = \int_{\mathfrak{M}} f(t)\, dm(t), \quad f \in p_\tau = L^1(\mathfrak{M}, m).$$

The algebra $\tilde{\mathcal{M}} = \tilde{L}^\infty(\mathfrak{M}, m)$ consists of all m-measurable functions f such that $\lambda(f) \not\equiv +\infty$. We shall denote the symmetric space $E(\mathcal{M}, \tau)$ as $E(\mathfrak{M}, m)$. Clearly, if $E = L^p$, $1 \le p \le \infty$, the latter spaces turn into $L^p(\mathfrak{M}, m)$ as defined above.

Let us note that if $(\mathfrak{M}, m) = (\mathbb{R}, \lambda)$, then the spaces $E(\mathfrak{M}, m)$ are the classical symmetric function spaces. If $(\mathfrak{M}, m) = (\mathbb{N}, \nu)$, where ν is the counting measure, then we shall denote the space $E(\mathfrak{M}, m)$ as ℓ^E. In particular, for $E = L^p$, $1 \le p \le \infty$, the space ℓ^E coincides with the classical sequence space ℓ^p.

1.6.2 $\mathcal{M} = B(\ell_n^2)$, $n \ge 1$.

Having fixed the standard basis $(\epsilon_j)_{j=1}^n$ in the Hilbert space ℓ_n^2, the algebra $B(\ell_n^2)$ may be identified with the algebra of all $n \times n$-matrices. That is, for every $x \in B(\ell_n^2)$, we assign the matrix $(x_{jk})_{j,k=1}^n$ such that $x_{jk} = \langle x(\epsilon_k), \epsilon_j \rangle$. If $\xi = (\xi_k)_{k=1}^n, \eta = (\eta_j)_{j=1}^n \in \ell_n^2$ and $\eta = x(\xi)$, it then follows that

$$\eta_j = \sum_{k=1}^n x_{jk} \xi_j, \quad 1 \le j \le n.$$

It is easily seen that $\mathcal{M}' = \mathbb{C}\,\mathbf{1}$.

There is a tracial weight Tr on the algebra $\mathcal{M}$, called *the standard trace*, defined by

$$Tr((x_{jk})_{j,k=1}^n) := \sum_{j=1}^n x_{jj}.$$

Any linear functional ϕ on $\mathcal{M}$ has the form

$$\phi(x) = Tr(x\,\Phi), \tag{1.6.1}$$

where Φ is a matrix. By the Hölder inequality and polar decomposition, the norm of such a functional ϕ is given by

$$\|\phi\|_{\mathcal{M}^*} = Tr(|\Phi|).$$

The predual $\mathcal{M}_*$ identifies with the space $B(\ell_n^2)$ equipped with the norm $\|\cdot\|_{\mathcal{M}_*} = Tr(|\cdot|)$. The functional ϕ is positive if and only if the corresponding matrix Φ is positive.

Consequently, any weight ϕ on $\mathcal{M}$ has the form (1.6.1), where Φ is a positive matrix. Let us show that the modular automorphism group for such a weight ϕ is given by

$$\sigma_t^\phi(x) = \Phi^{it} x\, \Phi^{-it}, \quad x \in \mathcal{M}. \tag{1.6.2}$$

Note that, without loss of generality, we may assume that the positive matrix Φ is diagonal, see Lemma 1.3.10. Let $(\phi_j)_{j=1}^\infty$ be the diagonal entries of Φ. According to Theorem 1.3.9, the modular group is unique, thus, it is sufficient to show that the group above satisfies the modular condition, see (i)–(ii) on page 14. It follows from (1.6.1) and (1.6.2) that

$$\phi(\sigma_t^\phi(x)) = Tr(\Phi^{it} x \Phi^{-it}\Phi) = Tr(x\Phi) = \phi(x), \quad x \in \mathcal{M}.$$

Furthermore, for every $x, y \in \mathcal{M}$, let us consider the function $f_{x,y}$ given by

$$f_{x,y}(z) = Tr(\Phi^{1+iz} x \Phi^{-iz} y), \quad z \in \mathbb{C}.$$

Since

$$f_{x,y}(z) = \sum_{j,k=1}^{n} \phi_j^{1+iz} x_{jk} \phi_k^{-iz} y_{kj},$$

it is clear that the function $f_{x,y}$ is holomorphic in the strip

$$S := \{z \in \mathbb{C} : \quad 0 < \Im z < 1\}$$

and bounded in the closed strip $\bar{S}$. Moreover,

$$f_{x,y}(t) = \phi(\sigma_t^\phi(x)\, y) \;\text{ and }\; f_{x,y}(t+i) = \phi(y\, \sigma_t^\phi(x)), \quad t \in \mathbb{R}.$$

Consequently, the weight ϕ satisfies the modular condition with respect to the group (1.6.2).

Let $x = (x_{jk})_{j,k=1}^n \in \mathcal{M}$. The generalized singular value function $\mu_t(x)$ with respect to the standard trace Tr is a step function given by

$$\mu_t(x) = \begin{cases} s_k(x), & \text{if } k-1 \le t < k,\ 1 \le k \le n, \\ 0, & \text{otherwise,} \end{cases}$$

where $(s_k(x))_{k=1}^n$ is the sequence of singular values of the operator x taken in decreasing order and counted according to multiplicities.

Let $\mathcal{C}_n^E := E(\mathcal{M}, Tr)$. In particular, let $\mathcal{C}_n^p := L^p(\mathcal{M}, Tr)$. The space $\mathcal{C}_n^E$ is the space of all $n\times n$-matrices equipped with the norm of the symmetric ideal $\mathcal{C}^E$ of compact operators, see the example below. In particular, the space $\mathcal{C}_n^p$, $1 \leq p \leq \infty$ is the space of all $n \times n$-matrices with the p-th Schatten-von Neumann norm

$$\|x\|_{\mathcal{C}_n^p} = (Tr(|x|^p)^{\frac{1}{p}}, \quad x \in \mathcal{M}, \ 1 \leq p < \infty$$

and $\|\cdot\|_{\mathcal{C}_n^\infty} = \|\cdot\|$.

1.6.3 $\mathcal{M} = B(\ell^2)$.

This example is the natural extension of the example above. Having fixed the standard basis $(\epsilon_j)_{j=1}^\infty$ in ℓ^2 the element $x \in \mathcal{M}$ identifies with the infinite matrix $(x_{jk})_{j,k=1}^\infty$, where $x_{jk} = \langle x\epsilon_k, \epsilon_k\rangle$.

On the cone $\mathcal{M}^+$of all positive matrices we have the tracial weight Tr, given by

$$Tr((x_{jk})_{j,k=1}^\infty) = \sum_{j=1}^\infty x_{jj},$$

which we call *the standard trace.*

The space ℓ_n^2, $n \geq 1$ is a subspace of ℓ^2. Thus, we may consider the algebra $B(\ell_n^2)$ as a Banach subspace of the algebra $\mathcal{M}$. The union $\cup_{n=1}^\infty B(\ell_n^2)$ is wo-dense in $\mathcal{M}$. The latter implies that $\mathcal{M}' = \mathbb{C}\mathbf{1}$. It also implies that every bounded normal (equivalently, uw-continuous) functional $\phi \in \mathcal{M}_*$ have the form

$$\phi(x) = Tr(x\,\Phi), \quad x \in \mathcal{M},$$

where Φ is the matrix such that $Tr(|\Phi|) < \infty$. Therefore, the predual $\mathcal{M}_*$ identifies with the class of all infinite matrices Φ such that $Tr(|\Phi|) < \infty$. The denote this class as $\mathcal{C}^1$ and call *the trace class.*

Let us note that $Tr(p)$, where $p \in B(\ell_2)$ is a projection, is a positive integer. Thus, if $x \in \widetilde{\mathcal{M}}$, then, according to Lemma 1.4.7, $\lambda_s(x) = 0$ for some $s > 0$ or, equivalently, $x \in B(\ell_2)$. Hence, we see that for the algebra $\mathcal{M} = B(\ell_2)$ the algebra of measurable operators $\widetilde{\mathcal{M}}$ coincides with $\mathcal{M}$. In particular, every symmetric space $E(\mathcal{M}, \tau) \subseteq \mathcal{M}$.

Let us recall that the closure of all finite-dimensional operators in $\mathcal{M}$ is called *the class of compact operators* and denoted by $\mathcal{C}^\infty$. Let $E = E(\mathbb{R})$ be a fully symmetric function space, and $\mathcal{E} = E(\mathcal{M}, \tau)$ be the corresponding operator space. If the space $E = E(\mathbb{R})$ is separable, then the norm $\|\cdot\|_{\mathcal{E}}$ is order-continuous. Therefore, every operator $x \in \mathcal{E}$ is compact, i.e. $\mathcal{E} \subseteq \mathcal{C}^\infty$. In this case, we shall call the space $\mathcal{E}$ *the symmetric ideal of compact operators* and denote by $\mathcal{C}^E$.

If $x \in \mathcal{C}^\infty$, then the generalized singular value function $\mu_t(x)$ is given by

$$\mu_t(x) = s_k(x), \quad \text{if} \quad k - 1 \leq t < k, \; k \geq 1,$$

where $(s_j(x))_{j=1}^\infty$ is the decreasing sequence of singular values of the matrix x counted with multiplicities. Thus, $\mathcal{C}^E$ identifies with the class of all compact operators x such that

$$\|x\|_{\mathcal{C}^E} = \|s(x)\|_{\ell^E} < \infty.$$

In particular, when $E = L^p$, $1 \leq p < \infty$, the classes $\mathcal{C}^p := \mathcal{C}^E$ equipped with the norm

$$\|x\|_{\mathcal{C}^p} := \left[\sum_{n=1}^{\infty} (s_n(x))^p \right]^{\frac{1}{p}}.$$

The class $\mathcal{C}^p$ is called *the p-th Schatten-von Neumann ideal.* We refer the reader to [31] for more on the symmetric ideals of compact operators.

Before we continue considering examples of von Neumann algebras, let us first recall the notion of the crossed product.

1.6.4 Continuous crossed product

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $\alpha = \{\alpha_t\}_{t\in\mathbb{R}}$ be a weak* continuous group of $*$-automorphisms on $\mathcal{M}$. In the present section we shall consider construction of the continuous crossed product $\mathcal{M} \rtimes_\alpha \mathbb{R}$ and its basic properties.

Let us recall that $L^p = L^p(\mathbb{R})$ stands for the classical L^p spaces.

We shall consider the tensor product Hilbert space $\mathscr{H} := L^2 \otimes \mathcal{H}$. Alternatively, the space $\mathscr{H}$ consists of all functions $\xi : \mathbb{R} \mapsto \mathcal{H}$ such that

$$\|\xi\|_{\mathscr{H}} := \sqrt{\int_{\mathbb{R}} \|\xi(t)\|_{\mathcal{H}}^2 \, dt} < \infty.$$

We shall also consider the tensor product von Neumann algebra $B(L^2) \otimes \mathcal{M} \subseteq B(\mathscr{H})$.

For every $x \in \mathcal{M}$, we define the "diagonal" operator $\pi(x) \in B(L^2) \otimes \mathcal{M}$ by

$$(\pi(x)\xi)(t) := \alpha_{-t}(x)(\xi(t)), \quad t \in \mathbb{R},\ \xi \in \mathscr{H}.$$

The mapping $\pi : x \mapsto \pi(x)$ is a $*$-isomorphism of the algebra $\mathcal{M}$ onto a von Neumann subalgebra $\pi(\mathcal{M}) \subseteq B(L^2) \otimes \mathcal{M}$.

The translation operators λ_t, $t \in \mathbb{R}$ on L^2 are defined by

$$(\lambda_t\xi)(s) := \xi(s-t), \quad s \in \mathbb{R},\ \xi \in L^2.$$

We set $\Lambda_t := \lambda_t \otimes \mathbf{1} \in B(L^2) \otimes \mathcal{M}$.

The continuous crossed product $\mathcal{R} := \mathcal{M} \rtimes_\alpha \mathbb{R}$ is the minimal von Neumann subalgebra of $B(L^2) \otimes \mathcal{M}$ containing the operators Λ_t, $t \in \mathbb{R}$ and $\pi(x)$, $x \in \mathcal{M}$. We immediately have the following straightforward lemma.

Lemma 1.6.1. *The group $\hat{\alpha} := \pi\, \alpha\, \pi^{-1}$ on the algebra $\pi(\mathcal{M})$ is implemented by the unitary group $\lambda := \{\Lambda_t\}_{t \in \mathbb{R}}$, i.e.*

$$\hat{\alpha}_t(\pi(x)) = \pi\alpha_t\pi^{-1}\pi(x) = \pi(\alpha_t(x)) = \Lambda_t^*\pi(x)\Lambda_t, \quad x \in \mathcal{M},\ t \in \mathbb{R}. \qquad (1.6.3)$$

Recall that the Fourier transform and its inverse are defined by ([58, Section 7.1])

$$(\mathscr{F}f)(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(s)\, e^{-its}\, ds \quad \text{and} \quad (\mathscr{F}^{-1}\hat{f})(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t)\, e^{its}\, dt,$$

where $t, s \in \mathbb{R}$ and $f, \hat{f} \in L^1(\mathbb{R})$. It follows from the Plancherel theorem ([58, Theorem 7.9]) that the mapping $\mathscr{F}$ may be extended to a unitary operator on $L^2(\mathbb{R})$. Moreover, we shall also denote by $\mathscr{F}$ the extension of the Fourier transform to the class of tempered distributions ([58, Definition 7.14]).

Let $\mathscr{T} = \mathscr{T}(L^2)$ be the minimal von Neumann subalgebra of $B(L^2)$ generated by the operators λ_t, $t \in \mathbb{R}$. The algebra $\mathscr{T}$ is $*$-isomorphic to the algebra L^∞ via the Fourier transform $\mathscr{F}$. Equivalently, if

$$\hat{m}_f := \mathscr{F}^{-1} m_f \mathscr{F} = wo - \int_{\mathbb{R}} \hat{f}(t)\, \lambda_t\, dt, \quad f = \mathscr{F}f,\ f \in L^\infty,$$

then

$$\mathscr{T} = \{\hat{m}_f \in B(L^2) : \quad f \in L^\infty\}.$$

Indeed, it is sufficient to observe that the algebras $\mathscr{T}$ and L^∞ are the minimal von Neumann algebras generated by the unitary group $\{\lambda_t\}_{t \in \mathbb{R}}$ and $\{e_t\}_{t \in \mathbb{R}}$, respectively, where e_t is a multiplication operator by $e^{it(\cdot)}$ on $L^2(\mathbb{R})$ and that the latter unitary groups are isomorphic via Fourier transform, i.e. $\lambda_t = \mathscr{F}^{-1} e_t \mathscr{F}$, $t \in \mathbb{R}$. On the other hand, it is clear, that the algebra $\mathscr{T}$ may be viewed as a von Neumann subalgebra of $\mathcal{R}$ via the mapping $x \mapsto x \otimes \mathbf{1}$, $x \in \mathscr{T}$. Consequently, the algebra L^∞ may be viewed as a von Neumann subalgebra of $\mathcal{R}$ via the mapping

$$f \mapsto \hat{m}_f \otimes \mathbf{1} \in \mathcal{R}, \quad f \in L^\infty.$$

The crossed product algebra $\mathcal{R}$ is built in a purely abstract way as the minimal algebra containing the operators Λ_t, $t \in \mathbb{R}$ and $\pi(x)$, $x \in \mathcal{M}$. Before we proceed further, let us mention another, more constructive, approach to the algebra $\mathcal{R}$ which expressed by the following lemma.

Let us consider the class $K(\mathcal{M})$ which is a collection of all *wo*-continuous functions $x : t \in \mathbb{R} \mapsto x_t \in \mathcal{M}$ with compact support. For every $x \in K(\mathcal{M})$, we define the integral

$$\tilde{\pi}(x) := \int_{\mathbb{R}} \Lambda_t \, \pi(x_t) \, dt. \tag{1.6.4}$$

Lemma 1.6.2 ([63, Ch. X, Lemma 1.8]). *The integral* (1.6.4) *converges with respect to the wo-topology. The image* $\tilde{\pi}(K(\mathcal{M}))$ *is wo-dense in* $\mathcal{R}$.

Let ϕ be a weight on the algebra $\mathcal{M}$. *The dual weight* on the algebra $\mathcal{R}$ is defined by

$$\hat{\phi}(\tilde{\pi}(x)^* \tilde{\pi}(x)) := \int_{\mathbb{R}} \phi(x_t^* x_t) \, dt, \quad x \in K(\mathcal{M}). \tag{1.6.5}$$

Lemma 1.6.3 ([64, §2, Lemma 1]). *If* ϕ *is normal (resp. semi-finite, tracial) then the dual weight* $\hat{\phi}$ *is normal (resp. semi-finite, tracial).*

1.6.5 L^p spaces for arbitrary algebras

Let us fix a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$. In the present section we exhibit the construction of noncommutative L^p spaces associated with an arbitrary $\mathcal{M}$. The construction is due to U. Haagerup, cf. [64].

Let us also fix a n.s.f. weight ϕ on the algebra $\mathcal{M}$. Let $\sigma^\phi := \{\sigma_t^\phi\}_{t \in \mathbb{R}}$ be the corresponding modular automorphism group. We shall consider the crossed product $\mathcal{R} := \mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$. The latter algebra acts on the Hilbert space $\mathscr{H} := L^2 \otimes \mathcal{H}$. Recall that for every $x \in \mathcal{M}$ the operator $\pi(x) \in \mathcal{R}$ is given by

$$(\pi(x)\xi)(t) = \sigma_{-t}^\phi(x)(\xi(t)), \quad \xi \in \mathscr{H}, \ t \in \mathbb{R}.$$

Let us define the unitary group of operators $w := \{w_t\}_{t\in\mathbb{R}} \subseteq B(\mathscr{H})$ as follows

$$(w_t\xi)(s) := e^{its}\xi(s), \quad s \in \mathbb{R}.$$

The latter group defines the group of $*$-automorphisms $\theta := \{\theta_t\}_{t\in\mathbb{R}}$ on $\mathcal{R}$ given by

$$\theta_t(x) := w_t^*\, x\, w_t, \quad x \in \mathcal{R}. \tag{1.6.6}$$

The group $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ is called *the dual action.* Clearly,

$$\begin{aligned}
[\theta_t(\pi(x))(\xi)](s) &= [w_t^*\pi(x)w_t(\xi)](s)\\
&= e^{-its}[\pi(x)w_t(\xi)](s)\\
&= e^{-its}\sigma^{\phi}_{-s}(x)(w_t(\xi)(s))\\
&= e^{-its}\sigma^{\phi}_{-s}(x)(e^{its}\xi(s))\\
&= \sigma^{\phi}_{-s}(x)(\xi(s))\\
&= [\pi(x)(\xi)](s), \quad \xi \in \mathscr{H},\ t, s \in \mathbb{R}.
\end{aligned}$$

Thus,

$$\theta_t(\pi(x)) = \pi(x), \quad x \in \mathcal{M},\ t \in \mathbb{R}.$$

We also have

$$\begin{aligned}
[\theta_t\Lambda_s(\xi)](l) &= [w_t^*\Lambda_s w_t(\xi)](l)\\
&= e^{-itl}[\Lambda_s w_t(\xi)](l)\\
&= e^{-itl}[w_t(\xi)](l-s)\\
&= e^{-itl}e^{it(l-s)}\xi(l-s)\\
&= e^{-its}\Lambda_s(\xi)(l), \quad \xi \in \mathscr{H},\ t, s, l \in \mathbb{R},
\end{aligned}$$

Hence,

$$\theta_t(\Lambda_s) = e^{-its}\Lambda_s, \quad t, s \in \mathbb{R}.$$

Consequently,

$$\theta_t(\tilde{\pi}(x)) = \int_{\mathbb{R}} e^{-its}\, \Lambda_s\, \pi(x_s)\, ds, \quad x \in K(\mathcal{M}). \tag{1.6.7}$$

Let ψ be a n.s.f. weight on $\mathcal{M}$ and let $\hat{\psi}$ be the dual weight. It now follows from the identity (1.6.7) and the definition of the dual weight $\hat{\psi}$, see (1.6.5) that

$$\hat{\psi} = \hat{\psi} \circ \theta_t, \quad t \in \mathbb{R}. \tag{1.6.8}$$

Indeed,

$$\begin{aligned}\hat{\psi}(\theta_t(\tilde{\pi}(x)^*\tilde{\pi}(x))) &= \hat{\psi}(\theta_t(\tilde{\pi}(x))^*\theta_t(\tilde{\pi}(x)))\\ &= \int_{\mathbb{R}} \phi((e^{-its}x_s)^* e^{-its}x_s)\,ds.\\ &= \int_{\mathbb{R}} \phi(x_s^* x_s)\,ds\\ &= \hat{\psi}(\tilde{\pi}(x)^*\tilde{\pi}(x)), \quad x \in K(\mathcal{M}).\end{aligned}$$

The next fundamental result displays a distinguished feature of the crossed product with modular group.

Theorem 1.6.4 ([37, Section 13.3]). *There is a n.s.f. trace τ on the algebra $\mathcal{R}$ such that*

(i) $\tau \circ \theta_t = e^{-t}\tau$, $t \in \mathbb{R}$;

(ii) $\hat{\phi}(x) = \tau(D\,x)$, *where* $D = \hat{m}_f \otimes \mathbf{1}$ *and* $f(t) = e^t$, $t \in \mathbb{R}$, $x \in \mathcal{R}$;

(iii) $\theta_t(D) = e^{-t}D$.

Let us recall that $\mathcal{M}_*$ is the predual to the algebra $\mathcal{M}$. The space $\mathcal{M}_*$ consists of all normal bounded linear functionals on $\mathcal{M}$. $\mathcal{M}_*^+$ stands for the collection of all positive elements of $\mathcal{M}_*$.

Let ψ be a n.s.f. weight on $\mathcal{M}$ and let $\hat{\psi}$ be the corresponding dual weight. Let D_ψ be the operator affiliated with the algebra $\mathcal{R}$ such that

$$\hat{\psi}(x) = \tau(D_\psi x), \quad x \in \mathcal{R}. \tag{1.6.9}$$

We immediately have, that

$$\begin{aligned}\tau(\theta_t(D_\psi)\,x) &= \tau(\theta_t(D_\psi\,\theta_{-t}(x)))\\ \text{[Theorem 1.6.4.(i)]} \quad &= e^{-t}\tau(D_\psi\,\theta_{-t}(x))\\ [(1.6.9)] \quad &= e^{-t}\hat{\psi}(\theta_{-t}(x))\\ [(1.6.8)] \quad &= e^{-t}\hat{\psi}(x)\\ [(1.6.9)] \quad &= e^{-t}\tau(D_\psi\,x), \quad t \in \mathbb{R},\ x \in \mathcal{M}.\end{aligned}$$

In particular,

$$\theta_t(D_\psi) = e^{-t}D_\psi, \quad t \in \mathbb{R}.$$

Theorem 1.6.5 (U. Haagerup). (i) *If $\psi \in \mathcal{M}_*^+$, then the operator D_ψ is τ-measurable and*

$$\|\psi\|_{\mathcal{M}_*} = \|D_\psi\|_{L^{1,\infty}(\mathcal{R},\tau)}.$$

(ii) *The mapping $\psi \mapsto D_\psi$ extends linearly to an isometric embedding of $\mathcal{M}_*$ into $L^{1,\infty}(\mathcal{R},\tau)$.*

(iii) *For every element $x \in L^{1,\infty}(\mathcal{R},\tau)$ such that $\theta_t(x) = e^{-t}x$, $t \in \mathbb{R}$, there is a functional $\psi \in \mathcal{M}_*$ such that $x = D_\psi$.*

The latter result suggests how an approach to the definition of a noncommutative L^p-spaces might be defined. Indeed, we set

$$L^p(\mathcal{M}) := \left\{ x \in \tilde{\mathcal{R}} : \quad \theta_t(x) = e^{-t/p}x,\ t \in \mathbb{R}, \quad |x|^p \in L^{1,\infty}(\mathcal{R},\tau) \right\}. \qquad (1.6.10)$$

The norm $\|\cdot\|_{L^p(\mathcal{M})}$ is given by

$$\|x\|_{L^p(\mathcal{M})} := \left(\|\,|x|^p\,\|_{L^{1,\infty}(\mathcal{R},\tau)} \right)^{\frac{1}{p}}, \quad x \in L^p(\mathcal{M}).$$

According to the definition of the spaces $L^{p,\infty}$, see (1.4.3), the latter definition is equivalent to

$$L^p(\mathcal{M}) := \left\{ x \in L^{p,\infty}(\mathcal{R},\tau) : \quad \theta_t(x) = e^{-t/p}x,\ t \in \mathbb{R} \right\}$$

and

$$\|x\|_{L^p(\mathcal{M})} := \|x\|_{L^{p,\infty}(\mathcal{R},\tau)}, \quad x \in L^p(\mathcal{M}).$$

To support the claim that the introduced spaces $L^p(\mathcal{M})$ indeed deserve to carry the name *noncommutative L^p-spaces*, we have

Lemma 1.6.6. *$L^1(\mathcal{M}) = \mathcal{M}_*$ and $L^\infty(\mathcal{M}) = \mathcal{M}$.*

Proof. Clearly, the former follows from Theorem 1.6.5. The latter follows from [64, Proposition 10]. □

The reader may have observed that in the notation $L^p(\mathcal{M})$ we did not mentioned the weight ϕ and this is not accidental. The following result clarifies the matter

Theorem 1.6.7. *Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ and let ϕ_j, $j = 1, 2$ be two different n.s.f. weights on $\mathcal{M}$*

(i) *If* $\mathcal{R}_j$, $j = 1, 2$ *are the corresponding crossed products with respect to the groups* σ^{ϕ_j}, $j = 1, 2$, *then there is a so-continuous function* $t \in \mathbb{R} \mapsto u_t \in B(\mathcal{H})$, *where* u_t *is unitary,* $t \in \mathbb{R}$ *and the unitary operator* $U \in B(\mathscr{H})$ *given by*

$$(U\xi)(t) = u_t(\xi(t)), \quad t \in \mathbb{R}, \ \xi \in \mathscr{H},$$

such that the mapping $x \mapsto U^*xU$ *is* $*$*-isomorphism between the algebras* $\mathcal{R}_j$, $j = 1, 2$.

(ii) *If* τ_j, $j = 1, 2$ *are the traces on the algebras* $\mathcal{R}_j$, $j = 1, 2$ *guaranteed by Theorem 1.6.4, then*

$$\tau_1(x) = \tau_2(U^*xU), \quad x \in \mathcal{R}_1.$$

(iii) *The mapping* $x \mapsto U^*xU$, $x \in \mathcal{R}_1$ *extends into an isometry between the spaces* $L^{p,\infty}(\mathcal{R}_j, \tau_j)$, $j = 1, 2$, *for every* $1 \leq p \leq \infty$.

(iv) *The mapping* $x \mapsto U^*xU$, $x \in \mathcal{R}_1$ *commutes with* θ_t, $t \in \mathbb{R}$, *i.e.*

$$U^*\theta_t(x)U = \theta_t(U^*xU), \quad x \in \mathcal{R}_1, \ t \in \mathbb{R}.$$

(v) *The mapping* $x \mapsto U^*xU$, $x \in \mathcal{R}_1$ *implements an isomorphism between the spaces* $L^p_j(\mathcal{M})$, $j = 1, 2$, *where* $L^p_j(\mathcal{M})$ *is the noncommutative* L^p*-spaces defined in* (1.6.10) *with respect to the weights* ϕ_j, $j = 1, 2$, *respectively.*

1.7 Double Operator Integrals

Let us fix a semi-finite von Neumann algebra $\mathcal{M}$ acting on $\mathcal{H}$ equipped with a n.s.f. trace τ and fix two self-adjoint operators $a, b\eta\mathcal{M}$. Let $e^a(\cdot)$ and $e^b(\cdot)$ be the corresponding spectral measures. We also fix the noncommutative symmetric space $\mathcal{E} = E(\mathcal{M}, \tau)$.

For every Borel set $B \in \mathscr{B}(\mathbb{R})$, we consider the projections $P^a_{\mathcal{E}}(B), Q^b_{\mathcal{E}}(B) \in B(\mathcal{E})$ defined by

$$P^a_{\mathcal{E}}(B)x = e^a(B)\, x, \quad Q^b_{\mathcal{E}}(B)x = x\, e^b(B), \quad x \in \mathcal{E}. \tag{1.7.1}$$

We let $P^a_p := P^a_{\mathcal{L}^p}$ and $Q^b_p := Q^a_{\mathcal{L}^p}$, $1 \leq p \leq \infty$.

Lemma 1.7.1. *The mappings* $P^a_{\mathcal{E}}, Q^b_{\mathcal{E}} : \mathscr{B}(\mathbb{R}) \mapsto B(\mathcal{E})$ *are projection-valued measures, that is*

$$P^a_{\mathcal{E}}\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P^a_{\mathcal{E}}(B_n), \quad Q^b_{\mathcal{E}}\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} Q^b_{\mathcal{E}}(B_n),$$

$$B_n \in \mathscr{B}(\mathbb{R}), \quad B_n \cap B_m = \emptyset, \quad n \neq m.$$

The above series converge in the sense that the series

$$\sum_{n=1}^{\infty} \tau\left(y\, P^a_{\mathcal{E}}(B_n)(x)\right), \quad \sum_{n=1}^{\infty} \tau\left(y\, Q^b_{\mathcal{E}}(B_n)(x)\right)$$

converge for every $x \in \mathcal{E}$ *and* $y \in \mathcal{E}^{\times}$. *If the space* E *is separable, then the series converge with respect to the strong topology, i.e.*

$$\sum_{n=1}^{\infty} P^a_{\mathcal{E}}(B_n)(x), \quad \sum_{n=1}^{\infty} Q^b_{\mathcal{E}}(B_n)(x)$$

converge in $\mathcal{E}$, *for every* $x \in \mathcal{E}$.

Proof. It is apparent that the values of the mappings $P^a_{\mathcal{E}} : Q^b_{\mathcal{E}} : \mathscr{B}(\mathbb{R}) \mapsto B(\mathcal{E})$ are projections in $B(\mathcal{E})$. Thus, we have to prove σ-additivity only. The latter follows from the fact that $e^a(\cdot)$ and $e^b(\cdot)$ are spectral measures, see p. 17, together with Lemma 1.4.13 (Theorem 1.4.12, if $\mathcal{E}$ is separable). $\square$

Let $x \in \mathcal{E}$, $y \in \mathcal{E}^{\times}$ and $B \in \mathscr{B}(\mathbb{R})$. We have the following simple identities

$$\tau(P^a_{\mathcal{E}}(B)(x)\, y) = \tau(e^a(B)\, x\, y) = \tau(x\, y\, e^a(B)) = \tau(x\, Q^a_{\mathcal{E}^{\times}}(B)(y))$$

and

$$\tau(Q^b_{\mathcal{E}}(B)(x)\, y) = \tau(x\, e^b(B)\, y) = \tau(x\, P^b_{\mathcal{E}^{\times}}(B)(y)).$$

Consequently, we readily see that (recall that the space $\mathcal{E}^{\times}$ is regarded as a subspace of the dual $\mathcal{E}^*$, see Section 1.4.1)

$$(P^a_{\mathcal{E}}(B))^*|_{\mathcal{E}^{\times}} = Q^a_{\mathcal{E}^{\times}}(B) \ \text{ and } \ (Q^b_{\mathcal{E}}(B))^*|_{\mathcal{E}^{\times}} = P^b_{\mathcal{E}^{\times}}(B), \quad B \in \mathscr{B}(\mathbb{R}). \tag{1.7.2}$$

Let $\mathscr{A}(\mathbb{R}^2)$ be the algebra generated by the collection of all Borel rectangles $A \times B$, $A, B \in \mathscr{B}(\mathbb{R})$. We define the product measure $P^a_{\mathcal{E}} \otimes Q^b_{\mathcal{E}} : \mathscr{A}(\mathbb{R}^2) \mapsto B(\mathcal{E})$ by

$$P^a_{\mathcal{E}} \otimes Q^b_{\mathcal{E}}(A \times B) = P^a_{\mathcal{E}}(A) \cdot Q^b_{\mathcal{E}}(B), \quad A, B \in \mathscr{B}(\mathbb{R}). \tag{1.7.3}$$

The identities (1.7.2) imply that

$$(P^a_{\mathcal{E}} \otimes Q^b_{\mathcal{E}}(A \times B))^*|_{\mathcal{E}^{\times}} = P^b_{\mathcal{E}^{\times}} \otimes Q^a_{\mathcal{E}^{\times}}(B \times A), \quad A, B \in \mathscr{B}(\mathbb{R}). \tag{1.7.4}$$

Theorem 1.7.2 ([24, Remark 3.1] and the references therein)**.** *The product measure* $P^a_2 \otimes Q^b_2$ *extends to a unique spectral measure over* $\mathscr{B}(\mathbb{R}^2)$. *That is*

(i) $P_2^a \otimes Q_2^b(B)$ *is an orthogonal projection in* $B(\mathcal{L}^2)$, *for every* $B \in \mathscr{B}(\mathbb{R}^2)$;

(ii) $P_2^a \otimes Q_2^b(B) \cdot P_2^a \otimes Q_2^b(B') = 0$, *for every* $B, B' \in \mathscr{B}(\mathbb{R}^2)$ *and* $B \cap B' = \emptyset$;

(iii) *if* $B_n \in \mathscr{B}(\mathbb{R}^2)$, $n \geq 1$ *such that* $B_n \cap B_m = \emptyset$, $n \neq m$, *then*

$$P_2^a \otimes Q_2^b\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \sum_{n=1}^{\infty} P_2^a \otimes Q_2^b(B_n),$$

where the series converges in the so-topology of the space $B(\mathcal{L}^2)$.

Let us note that the latter result is not valid when $p \neq 2$, see Example 1.9.1 below.

Let $B(\mathbb{R}^2)$ be the class of all complex-valued bounded Borel functions on the plane $\mathbb{R}^2$. Recall that the space $B(\mathbb{R}^2)$ is equipped with the uniform norm

$$\|\phi\|_\infty := \sup_{\lambda,\mu \in \mathbb{R}^2} |\phi(\lambda, \mu)|.$$

Theorem 1.7.2 together with the spectral theorem (Theorem 1.3.14) implies that for every $\phi \in B(\mathbb{R}^2)$, the integral

$$\int_{\mathbb{R}^2} \phi \, d(P_2^a \otimes Q_2^b)$$

converges in the *so*-topology. Thus, if we define the linear operator $T_{\phi,2}^{a,b} : \mathcal{L}^2 \mapsto \mathcal{L}^2$ as

$$T_{\phi,2}^{a,b}(x) := \int_{\mathbb{R}^2} \phi \, d(P_2^a \otimes Q_2^b x), \quad x \in \mathcal{L}^2,$$

then, it follows from Theorem 1.7.2 and the spectral theorem (Theorem 1.3.14) that $T_{\phi,2}^{a,b} \in B(\mathcal{L}^2)$ and

$$\|T_{\phi,2}^{a,b}\|_{B(\mathcal{L}^2)} \leq \|\phi\|_\infty. \tag{1.7.5}$$

Extending the observation above, we arrive at

Definition 1.7.3. A function $\phi \in B(\mathbb{R}^2)$ is integrable with respect to the measure $P_{\mathcal{E}}^a \otimes Q_{\mathcal{E}}^b$ if and only if there is a bounded linear operator $T \in B(\mathcal{E})$ satisfying the following conditions:

(i) $T(\mathcal{L}^2 \cap \mathcal{E}) \subseteq \mathcal{L}^2 \cap \mathcal{E}$ and $T^*(\mathcal{L}^2 \cap \mathcal{E}^\times) \subseteq \mathcal{L}^2 \cap \mathcal{E}^\times$;

(ii) for every $x \in \mathcal{L}^2 \cap \mathcal{E}$ and $y \in \mathcal{L}^2 \cap \mathcal{E}^\times$

$$\langle Tx, y \rangle = \int_{\mathbb{R}^2} \phi(\lambda, \mu) \, d\langle P_{\mathcal{E}}^a \otimes Q_{\mathcal{E}}^b x, y \rangle. \tag{1.7.6}$$

The notion introduced in Definition 1.7.3 was thoroughly studied in [24]. In particular, the present definition is a special case of [24, Definition 2.9] (see also [24, Proposition 2.12] and the discussion there on pages 81–82).

Lemma 1.7.4. *For every $\phi \in B(\mathbb{R}^2)$ there is at most one operator T satisfying Definition 1.7.3.*

Proof. Let us assume that there are two operators $T_j \in B(\mathcal{E})$, $j = 1, 2$, satisfying Definition 1.7.3. Obviously, we then have

$$\langle T_1 x; y\rangle = \langle T_2 x; y\rangle, \quad x \in \mathcal{L}^2 \cap \mathcal{E},\ y \in \mathcal{L}^2 \cap \mathcal{E}^\times. \tag{1.7.7}$$

Observe that the functional

$$x \mapsto \langle T_j x; y\rangle = \langle x; T_j^* y\rangle$$

is $\sigma(\mathcal{E}, \mathcal{E}^\times)$-continuous, for every $j = 1, 2$ and $y \in \mathcal{L}^2 \cap \mathcal{E}^\times$. Observe also that the space $\mathcal{L}^2 \cap \mathcal{E}$ is $\sigma(\mathcal{E}, \mathcal{E}^\times)$-dense in $\mathcal{E}$, see Lemma 1.4.13. Consequently, the identity (1.7.7) may be extended to the case that $x \in \mathcal{E}$. To finish the proof, we need to note that $\mathcal{L}^2 \cap \mathcal{E}^\times$ separates points in $\mathcal{E}$ (cf. Lemma 1.4.10). Thus, $T_1 = T_2$. □

From now on, we shall denote the operator T from Definition 1.7.3 as $T^{a,b}_{\phi,\mathcal{E}}$. For the special case $\mathcal{E} = \mathcal{L}^p$, $1 \le p \le \infty$, we set $T^{a,b}_{\phi,p} := T^{a,b}_{\phi,\mathcal{L}^p}$.

Lemma 1.7.5. *Let $\mathcal{E}_j$, $j = 1, 2$ be two noncommutative symmetric spaces. If the function $\phi \in B(\mathbb{R}^2)$ is integrable with respect to $P^a_{\mathcal{E}_j} \otimes Q^b_{\mathcal{E}_j}$, $j = 1, 2$, then*

$$T^{a,b}_{\phi,\mathcal{E}_1}(x) = T^{a,b}_{\phi,\mathcal{E}_2}(x), \quad x \in \mathcal{E}_1 \cap \mathcal{E}_2.$$

Proof. For the sake of brevity, we let $T_j := T^{a,b}_{\phi,\mathcal{E}_j}$, $j = 1, 2$. Fix $x \in \mathcal{E}_1 \cap \mathcal{E}_2$. There is a spectral approximation $x_n = xe^{|x|}(\frac{1}{n}, n) \in \mathcal{L}^1 \cap \mathcal{L}^\infty$ such that $\lim_{n\to\infty} x_n = x$, where the limit converges with respect to the $\sigma(\mathcal{E}_1 \cap \mathcal{E}_2, \mathcal{E}_1^\times + \mathcal{E}_2^\times)$-topology (see Lemma 1.4.13). Consequently, for every $y \in \mathcal{L}^1 \cap \mathcal{L}^\infty$, we obtain

$$\langle T_j(x_n), y\rangle = \langle x_n, T_j^*(y)\rangle \to \langle x, T_j^*(y)\rangle = \langle T_j(x), y\rangle, \quad j = 1, 2.$$

Let us note that we used the fact that $T_j^*(y) \in \mathcal{E}_1^\times \cap \mathcal{E}_2^\times$, see Definition 1.7.3. It is also apparent from Definition 1.7.3 that

$$\langle T_1(x_n), y\rangle = \langle T_2(x_n), y\rangle, \quad y \in \mathcal{L}^1 \cap \mathcal{L}^\infty.$$

Letting $n \to \infty$, we arrive at

$$\langle T_1(x), y\rangle = \langle T_2(x), y\rangle, \quad y \in \mathcal{L}^1 \cap \mathcal{L}^\infty.$$

To finish the proof we have to note that $\mathcal{L}^1 \cap \mathcal{L}^\infty$ separates the points of $\mathcal{E}_1 \cap \mathcal{E}_2$ (see Lemma 1.4.10). □

Let us introduce $\Phi(\mathcal{E})$ as the class of all functions $\phi \in B(\mathbb{R}^2)$ integrable with respect to $P^a_{\mathcal{E}} \otimes Q^b_{\mathcal{E}}$ for every $a, b\eta\mathcal{M}$, $a = a^*$, $b = b^*$; $\Phi_s(\mathcal{E})$ is the subclass of all $\phi \in \Phi(\mathcal{E})$ such that $\phi(\lambda, \mu) = \phi(\mu, \lambda)$.

By the following result, the class $\Phi(\mathcal{E})$ is an algebra.

Theorem 1.7.6 ([24, Proposition 2.8])**.** *Let $\mathcal{E}$ be a noncommutative symmetric space and $a, b\eta\mathcal{M}$ be self-adjoint linear operators. The mapping $\phi \mapsto T^{a,b}_{\phi,\mathcal{E}}$ is an algebra homomorphism from $\Phi(\mathcal{E})$ to $B(\mathcal{E})$, where the class $\Phi(\mathcal{E})$ equipped with pointwise operations.*

The following duality result for Double Operator Integrals was proved in [53].

Theorem 1.7.7. *If $\mathcal{E}$ is a noncommutative symmetric space with order-continuous norm and the Fatou property, then $\Phi_s(\mathcal{E}) = \Phi_s(\mathcal{E}^*)$. Moreover, $T^{a,b}_{\phi,\mathcal{E}^*} = (T^{b,a}_{\phi,\mathcal{E}})^*$ and $T^{a,b}_{\phi,\mathcal{E}} = (T^{b,a}_{\phi,\mathcal{E}^*})^*|_{\mathcal{E}}$[1], provided $\phi \in \Phi_s(\mathcal{E}) = \Phi_s(\mathcal{E}^*)$.*

Proof. By the assumption, we have $\mathcal{E}^\times = \mathcal{E}^*$ and $\mathcal{E}^{\times\times} = \mathcal{E}$. Fix $\phi \in \Phi_s(\mathcal{E})$ and set $T := T^{b,a}_{\phi,\mathcal{E}}$, for brevity. Let us first show that $\Phi_s(\mathcal{E}) \subseteq \Phi_s(\mathcal{E}^*)$, to this end it is sufficient to show that

$$T^{a,b}_{\phi,\mathcal{E}^*} = T^*. \tag{1.7.8}$$

Fix

$$x \in \mathcal{L}^2 \cap \mathcal{E} = \mathcal{L}^2 \cap \mathcal{E}^{\times\times}, \quad y \in \mathcal{L}^2 \cap \mathcal{E}^* = \mathcal{L}^2 \cap \mathcal{E}^\times. \tag{1.7.9}$$

By Definition 1.7.3,

$$T(\mathcal{L}^2 \cap \mathcal{E}) \subseteq \mathcal{L}^2 \cap \mathcal{E}, \quad T^*(\mathcal{L}^2 \cap \mathcal{E}^\times) \subseteq \mathcal{L}^2 \cap \mathcal{E}^\times$$

and

$$\langle T(x), y\rangle = \int_{\mathbb{R}^2} \phi(\lambda, \mu)\, d\langle P^b_{\mathcal{E}} \otimes Q^a_{\mathcal{E}}(x), y\rangle.$$

[1]Hereafter, we identify the elements of $\mathcal{E}$ with their canonical images in $\mathcal{E}^{**}$.

Recall that x, y are fixed in (1.7.9). Passing to the adjoint operator T^* in the latter identity, we obtain

$$\begin{aligned} \langle x, T^*(y)\rangle &= \int_{\mathbb{R}^2} \phi(\lambda,\mu)\, d\langle P^b_{\lambda,\mathcal{E}} \otimes Q^a_{\mu,\mathcal{E}}(x), y\rangle \\ \text{[see (1.7.4)]} \quad &= \int_{\mathbb{R}^2} \phi(\lambda,\mu)\, d\langle x, P^a_{\mu,\mathcal{E}^\times} \otimes Q^b_{\lambda,\mathcal{E}^\times}(y)\rangle \\ \text{[since } \phi(\lambda,\mu)=\phi(\mu,\lambda)\text{]} \quad &= \int_{\mathbb{R}^2} \phi(\lambda,\mu)\, d\langle x, P^a_{\lambda,\mathcal{E}^\times} \otimes Q^b_{\mu,\mathcal{E}^\times}(y)\rangle. \end{aligned} \tag{1.7.10}$$

Thus, to finish the proof of (1.7.8), according to Definition 1.7.3, we only need to show that

$$T^{**}(\mathcal{L}^2 \cap \mathcal{E}) \subseteq \mathcal{L}^2 \cap \mathcal{E}.$$

The latter is apparent, since $T \in B(\mathcal{E})$ and therefore

$$T^{**}(z) = T(z), \quad z \in \mathcal{E}.$$

Thus, we have established that $\Phi_s(\mathcal{E}) \subseteq \Phi_s(\mathcal{E}^*)$.

We now fix $\phi \in \Phi_s(\mathcal{E}^*)$ and set $T := T^{b,a}_{\phi,\mathcal{E}^*} = T^{b,a}_{\phi,\mathcal{E}^\times} \in B(\mathcal{E}^*)$. To prove that $\Phi_s(\mathcal{E}^*) \subseteq \Phi_s(\mathcal{E})$ it is sufficient to show that $T^*|_{\mathcal{E}}$ coincides with $T^{a,b}_{\phi,\mathcal{E}}$, i.e.

$$T^{a,b}_{\phi,\mathcal{E}} = T^*|_{\mathcal{E}}.$$

Let us again fix

$$x \in \mathcal{L}^2 \cap \mathcal{E}^* = \mathcal{L}^2 \cap \mathcal{E}^\times, \quad y \in \mathcal{L}^2 \cap \mathcal{E} = \mathcal{L}^2 \cap \mathcal{E}^{\times\times}.$$

According to Definition 1.7.3,

$$T(\mathcal{L}^2 \cap \mathcal{E}^\times) \subseteq \mathcal{L}^2 \cap \mathcal{E}^\times, \quad T^*(\mathcal{L}^2 \cap \mathcal{E}) \subseteq \mathcal{L}^2 \cap \mathcal{E} \tag{1.7.11}$$

and

$$\langle T(x), y\rangle = \int_{\mathbb{R}^2} \phi(\lambda,\mu)\, d\langle P^b_{\mathcal{E}^\times} \otimes Q^a_{\mathcal{E}^\times}(x), y\rangle.$$

Taking the adjoint T^*, similarly to (1.7.10), we obtain

$$\langle x, T^*(y)\rangle = \int_{\mathbb{R}^2} \phi(\lambda,\mu)\, d\langle x, P^a_{\mathcal{E}} \otimes Q^b_{\mathcal{E}}(y)\rangle.$$

Thus, according to Definition 1.7.3, we need only to show that $T^* \in B(\mathcal{E})$ and

$$T^{**}(\mathcal{L}^2 \cap \mathcal{E}^\times) \subseteq \mathcal{L}^2 \cap \mathcal{E}^\times.$$

For the latter embedding, it is sufficient to note that $T \in B(\mathcal{E}^*)$ and therefore

$$T^{**}(x) = T(x), \quad x \in \mathcal{E}^* = \mathcal{E}^\times.$$

For the former, we first show that $T^*(\mathcal{E}) \subseteq \mathcal{E}$. Indeed, suppose that $z \in \mathcal{E}$. Since E is separable, there exists a sequence $\{z_k\}_{k=1}^{\infty} \subseteq \mathcal{L}^2 \cap \mathcal{E}$, such that $\lim_{k\to\infty} z_k = z$, where the limit converges with respect to the norm topology in $\mathcal{E}$, see Lemma 1.4.14.(i). Since $T^* \in B(\mathcal{E}^{**})$ and $\mathcal{E} \subseteq \mathcal{E}^{**}$ isometrically, we obtain that

$$\lim_{k\to 0} T^*(z_k) = T^*(z), \tag{1.7.12}$$

where the limit converges with respect to the norm topology in $\mathcal{E}^{**}$. In particular, $\{T^*(z_k)\}_{k\geq 1}$ is a Cauchy sequence in $\mathcal{E}^{**}$. On the other hand, it follows from (1.7.11),

$$\{T^*(z_k)\}_{k=1}^{\infty} \subseteq \mathcal{E}.$$

Since, $\mathcal{E} \subseteq \mathcal{E}^{**}$ isometrically, the latter sequence is also Cauchy in $\mathcal{E}$. Consequently, from (1.7.12), $T^*(z) \in \mathcal{E}$. Thus, we have showed that $T^*(\mathcal{E}) \subseteq \mathcal{E}$. Let us recall that $T^* \in B(\mathcal{E}^{**})$. Consequently, referring to the isometric embedding $\mathcal{E} \subseteq \mathcal{E}^{**}$ again, we obtain that $T^* \in B(\mathcal{E})$. The lemma is completely proved. □

Lemma 1.7.8. *Let $\mathcal{E}$ be a noncommutative symmetric space and $a, b\eta\mathcal{M}$ be self-adjoint linear operators. If $\phi(\lambda,\mu) = \alpha(\lambda)$ (resp. $\phi(\lambda,\mu) = \beta(\mu)$), where $f \in B(\mathbb{R})$ (resp. $g \in B(\mathbb{R})$), then $\phi \in \Phi(\mathcal{E})$ and*

$$T^{a,b}_{\phi,\mathcal{E}}(x) = \alpha(a)x \quad (\textit{resp. } T^{a,b}_{\phi,\mathcal{E}}(x) = x\beta(b)), \quad x \in \mathcal{E}.$$

Proof. Let us prove only the first part of the claim. Note that

$$\alpha(a) = \int_{\mathbb{R}} \alpha(\lambda)\, de^a_\lambda,$$

where the integral converge in the so^*-topology and hence in the uw-topology. Hence, if $\phi(\lambda,\mu) = \alpha(\lambda)$, for every $x \in \mathcal{L}^2 \cap \mathcal{E}$ and $y \in \mathcal{L}^2 \cap \mathcal{E}^\times$, we have

$$\begin{aligned}\langle \alpha(a)\, x, y\rangle &= \int_{\mathbb{R}^2} \alpha(\lambda)\, \langle de^a_\lambda\, x de^b_\mu, y\rangle \\ &= \int_{\mathbb{R}^2} \phi\, d\langle P^a \otimes Q^b(x), y\rangle.\end{aligned}$$

The latter integral converges according to Lemma 1.4.13. The claim of the lemma follows. □

1.8 Some boundedness criteria for Double Operator Integrals

Let us fix a semi-finite von Neumann algebra $\mathcal{M}$ with a n.s.f. trace τ and the self-adjoint operators $a, b\eta\mathcal{M}$. $\mathcal{E} = E(\mathcal{M}, \tau)$ stands for a noncommutative symmetric space. For the sake of brevity, in this section we adopt the notation $T_{\phi,\mathcal{E}} := T^{a,b}_{\phi,\mathcal{E}}$ and $T_{\phi,p} := T^{a,b}_{\phi,p}$, $1 \le p \le \infty$.

Let us first note that the estimate (1.7.5) provides the complete description of the class $\Phi(\mathcal{L}^2)$ (see also [22]).

Theorem 1.8.1. *Every function $\phi \in B(\mathbb{R}^2)$ is integrable with respect to $P^a \otimes Q^b$, i.e.*

$$\Phi(\mathcal{L}^2) = B(\mathbb{R}^2).$$

Moreover, for every $\phi \in B(\mathbb{R}^2)$,

$$\sup_{a,b\eta\mathcal{M}} \|T^{a,b}_{\phi,2}\|_{B(\mathcal{L}^2)} = \|\phi\|_\infty.$$

Proof. The estimate (1.7.5) readily implies that

$$\sup_{a,b\eta\mathcal{M}} \|T^{a,b}_{\phi,2}\|_{B(\mathcal{L}^2)} \le \|\phi\|_\infty.$$

Let us show the converse inequality. Observe that for fixed $\lambda, \mu \in \mathbb{R}$, we have

$$T^{\lambda\mathbf{1},\mu\mathbf{1}}_{\phi,2}(x) = \phi(\lambda, \mu)\, x. \tag{1.8.1}$$

Indeed, the spectral measure of th operator $\lambda\mathbf{1}$ is condensed in the point λ, i.e.

$$e^{\lambda\mathbf{1}}(B) = \mathbf{1} \iff \lambda \in B.$$

Consequently, the spectral measure $P_2^{\lambda\mathbf{1}} \otimes Q_2^{\mu\mathbf{1}}$ is condensed in the point (λ, μ), i.e.

$$P_2^{\lambda\mathbf{1}} \otimes Q_2^{\mu\mathbf{1}}(B) = \mathbf{1} \iff (\lambda, \mu) \in B.$$

Thus, we obtain (1.8.1) from the fact that

$$T^{\lambda\mathbf{1},\mu\mathbf{1}}_{\phi,2} := \int_{\mathbb{R}^2} \phi(\lambda', \mu')\, dP_2^{\lambda\mathbf{1}} \otimes Q_2^{\mu\mathbf{1}}.$$

The converse inequality now follows from

$$\begin{aligned}\sup_{a,b\eta\mathcal{M}} \|T^{a,b}_{\phi,2}\|_{B(\mathcal{L}^2)} &= \sup_{a,b\eta\mathcal{M}} \sup_{x\in(\mathcal{L}^2)_1} \|T^{a,b}_{\phi,2}(x)\|_{\mathcal{L}^2}\\ &\geq \sup_{\lambda,\mu\in\mathbb{R}} \sup_{x\in(\mathcal{L}^2)_1} \|T^{\lambda\mathbf{1},\mu\mathbf{1}}_{\phi,2}(x)\|_{\mathcal{L}^2}\\ &= \sup_{\lambda,\mu\in\mathbb{R}} \sup_{x\in(\mathcal{L}^2)_1} \|\phi(\lambda,\mu)\,x\|_{\mathcal{L}^2}\\ &= \|\phi\|_\infty.\end{aligned}$$

The theorem is proved. □

Note, that the argument of the proof above is applicable to an arbitrary noncommutative symmetric space $\mathcal{E}$. Consequently, we readily have that

Lemma 1.8.2. *If $\mathcal{E}$ is a noncommutative symmetric space, then $\Phi(\mathcal{E}) \subseteq B(\mathbb{R}^2)$ and the latter embedding is continuous, i.e.*

$$\|\phi\|_\infty \leq \|\phi\|_{\Phi(\mathcal{E})} := \sup_{a,b\eta\mathcal{M}} \|T^{a,b}_{\phi,\mathcal{E}}\|_{B(\mathcal{E})}.$$

Let us consider two functions $\alpha, \beta \in B(\mathbb{R})$. Let $\phi(\lambda,\mu) = \alpha(\lambda)\,\beta(\mu) \in B(\mathbb{R}^2)$, it follows from Theorem 1.7.6 and Lemma 1.7.8 that

$$T_{\phi,\mathcal{E}}(x) = \alpha(a)\,x\,\beta(b), \quad x \in \mathcal{E}. \tag{1.8.2}$$

Therefore, we obtain that the function $\phi(\lambda,\mu) = \alpha(\lambda)\,\beta(\mu) \in \Phi(\mathcal{E})$ for every α, $\beta \in B(\mathbb{R})$ and every noncommutative symmetric space $\mathcal{E}$, and

$$\|T_{\phi,\mathcal{E}}\|_{B(\mathcal{E})} \leq \|\alpha\|_\infty \|\beta\|_\infty. \tag{1.8.3}$$

Let us recall that *the projective tensor product* $B(\mathbb{R})\hat{\otimes}B(\mathbb{R})$ (see [51]) is the class of functions $\phi \in B(\mathbb{R}^2)$ such that ϕ admits the representation

$$\phi(\lambda,\mu) = \sum_{n=1}^{\infty} \alpha_n(\lambda)\,\beta_n(\mu), \tag{1.8.4}$$

where

$$\sum_{n=1}^{\infty} \|\alpha_n\|_\infty \|\beta_n\|_\infty < \infty, \quad \alpha_n, \beta_n \in B(\mathbb{R}),\ n \geq 1.$$

The space $B(\mathbb{R})\hat{\otimes}B(\mathbb{R})$ is equipped with the norm

$$\|\phi\|_{B(\mathbb{R})\hat{\otimes}B(\mathbb{R})} := \inf \sum_{n=1}^{\infty} \|\alpha_n\|_\infty \|\beta_n\|_\infty,$$

where the inf runs over all possible representations (1.8.4). Thus, from (1.8.2) and (1.8.3), we obtain

Theorem 1.8.3. *Let $\mathcal{E}$ be an arbitrary noncommutative symmetric space. Every function $\phi \in B(\mathbb{R})\hat{\otimes}B(\mathbb{R})$ is integrable with respect to $P^a \otimes Q^b$, i.e.*

$$B(\mathbb{R})\hat{\otimes}B(\mathbb{R}) \subseteq \Phi(\mathcal{E}).$$

Moreover, for every $\phi \in B(\mathbb{R})\hat{\otimes}B(\mathbb{R})$

$$\|T_{\phi,\mathcal{E}}\|_{B(\mathcal{E})} \leq \|\phi\|_{B(\mathbb{R})\hat{\otimes}B(\mathbb{R})}.$$

The converse result in the special case $\mathcal{M} = B(\ell_n^2)$, $n \geq 1$ is proved in [4, Theorem 6.4].

Theorem 1.8.4 ([4, Theorem 6.4])**.** *Let a, b be two unbounded self-adjoint linear operators on ℓ^2. If the function $\phi \in B(\mathbb{R}^2)$ is integrable with respect to $P^a \otimes Q^b$ in the space $B(\ell^2)$, i.e. if $\phi \in \Phi(B(\ell^2))$ for every operators a, b, then $\phi \in B(\mathbb{R})\hat{\otimes}B(\mathbb{R})$ and*

$$\|\phi\|_{B(\mathbb{R})\hat{\otimes}B(\mathbb{R})} \leq \sup_{a,b} \|T_\phi^{a,b}\|_{B(\ell^2)\mapsto B(\ell^2)}.$$

The following observation immediately follows from Theorems 1.7.7, 1.4.1 and 1.4.20.

Lemma 1.8.5. *If $\mathcal{E}$ is a fully symmetric noncommutative space, then*

$$\Phi_s(\mathcal{L}^\infty) = \Phi_s(\mathcal{L}^1) \subseteq \Phi_s(\mathcal{E}).$$

Proof. Fix $\phi \in \Phi_s(\mathcal{L}^1) = \Phi_s(\mathcal{L}^\infty)$. Let T_1 and T_∞ be the corresponding double operator integrals, i.e. the linear operators satisfying Definition 1.7.3 with respect to the spaces $\mathcal{L}^1$ and $\mathcal{L}^\infty$, respectively. According to Lemma 1.7.5

$$T_1(x) = T_\infty(x), \quad x \in \mathcal{L}^1 \cap \mathcal{L}^\infty.$$

The latter means that the operators T_1 and T_∞ are the restrictions of an admissible operator $T : \mathcal{L}^1 + \mathcal{L}^\infty \mapsto \mathcal{L}^1 + \mathcal{L}^\infty$, i.e.

$$T|_{\mathcal{L}^1} = T_1 \quad \text{and} \quad T|_{\mathcal{L}^\infty} = T_\infty.$$

Interpolating the operator T we obtain that the operator $T_\mathcal{E} := T|_\mathcal{E}$ is a bounded linear operator $\mathcal{E} \mapsto \mathcal{E}$. We need to show that the operator $T_\mathcal{E}$ satisfies Definition 1.7.3 with respect to the function ϕ. To see (i–ii), we observe that the operators T and T^* restricted on $\mathcal{L}^2$ are bounded and that $T_\mathcal{E}^* = T^*|_\mathcal{E}$. For (iii), recall that the operator T coincides with T_1 and T_∞ on $\mathcal{L}^1$ and $\mathcal{L}^\infty$, respectively.

Consequently, we readily have that the identity (1.7.6) holds for the operator T when either

$$x \in \mathcal{L}^1 \cap \mathcal{L}^2, \quad y \in \mathcal{L}^\infty \cap \mathcal{L}^2 \quad \text{or} \quad x \in \mathcal{L}^\infty \cap \mathcal{L}^2, \quad y \in \mathcal{L}^1 \cap \mathcal{L}^2.$$

Combining the latter two together, we obtain that

$$\langle Tx, y\rangle = \int_{\mathbb{R}^2} \phi(\lambda, \mu) \, \langle P^a_{\mathcal{E}} \otimes Q^b_{\mathcal{E}} x, y\rangle \tag{1.8.5}$$

for every

$$x \in \mathcal{E} \cap \mathcal{L}^2 \quad \text{and} \quad y \in \mathcal{L}^1 \cap \mathcal{L}^\infty.$$

Since the the left and the right hand sides in (1.8.5) are continuous linear functionals on $\mathcal{L}^2$ for every fixed $x \in \mathcal{E} \cap \mathcal{L}^2$, we can uniquely extend the equality (1.8.5) to $y \in \mathcal{E}^\times \cap \mathcal{L}^2$ by continuity. The lemma is proved. □

Let us now consider the case $\mathcal{E} = \mathcal{L}^p$, $1 < p < \infty$, $p \neq 2$. In contrast with Theorems 1.8.1 and 1.8.3 which completely characterize the class $\Phi(\mathcal{L}^2)$, for the other classes $\Phi(\mathcal{L}^p)$, with $p \neq 2$, we have only sufficient criteria.

We firstly recall that a function $f \in B(\mathbb{R})$ is called of *bounded β-variation*, $1 \leq \beta < \infty$ if and only if

$$\|f\|_{V_\beta} := \sup \sum_{j=-\infty}^{+\infty} |\alpha(\lambda_j) - \alpha(\lambda_{j+1})|^\beta < \infty, \tag{1.8.6}$$

where the sup runs over all possible increasing two-sided sequences $\{\lambda_j\}_{j=-\infty}^{+\infty} \subseteq \mathbb{R}$. V_β will stand for the class of all functions of bounded β-variation, $1 \leq \beta < \infty$. The class V_β is equipped with the norm $\|\cdot\|_{V_\beta}$ defined in (1.8.6). We also define $V_\infty := B(\mathbb{R})$ equipped with the uniform norm.

Let us also consider the class $L^\infty(V_\beta)$ of all functions $\phi \in B(\mathbb{R}^2)$ such that

$$\|\phi\|_{L^\infty(V_\beta)} := \sup_{\lambda \in \mathbb{R}} \|\phi(\lambda, \cdot)\|_{V_\beta} < \infty. \tag{1.8.7}$$

The following result is the best known sufficient criterion for the boundedness of Double Operator Integrals in noncommutative L^p-spaces.

Theorem 1.8.6 ([24, Proposition 4.6])**.** *For every function $\phi \in L^\infty(V_\beta)$, $1 \leq \beta < \infty$, the operator $T_{\phi,p}$ admits the estimate*

$$\|T_{\phi,p}\|_{B(\mathcal{L}^p)} \leq c_p \, \|\phi\|_{L^\infty(V_\beta)}$$

provided $|2^{-1} - p^{-1}| < (2\beta)^{-1}$. Thus, $L^\infty(V_\beta) \subseteq \Phi(\mathcal{L}^p)$, whenever $|2^{-1} - p^{-1}| < (2\beta)^{-1}$.

Remark 1.8.7. Let us recall the Marcinkiewicz multiplier theorem. Let $f : [-\pi, \pi) \mapsto \mathbb{C}$. Let $\{\hat{f}(k)\}_{k=-\infty}^{\infty}$ be the corresponding Fourier transform, i.e.

$$\hat{f}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt, \quad k \in \mathbb{Z}.$$

The Marcinkiewicz theorem asserts that, for every $1 < p < \infty$, there is a constant $c_p > 0$ such that

$$\left\| \sum_{k=-\infty}^{\infty} \lambda_k \hat{f}(k)\, e^{ikt} \right\|_{L^p} \leq c_p \, \|f\|_{L^p},$$

for every $f \in L^p$ and for every sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ such that

$$\sup_{k \in \mathbb{Z}} |\lambda_k| + \sup_{m \geq 1} \sum_{k=2^m}^{2^{m+1}-1} |\lambda_k - \lambda_{k+1}| + |\lambda_{-k} - \lambda_{-k-1}| \leq 1. \tag{1.8.8}$$

Comparing condition (1.8.8) with the definition of $L^\infty(V_1)$ norm (see (1.8.7)), it is seen that Theorem 1.8.6 is a noncommutative analogue of the Marcinkiewicz multiplier theorem. Note that the Marcinkiewicz multiplier theorem may be extended to general vector-valued function spaces $L_p(X)$, if the Banach space X possesses the UMD property, see [10]. It is the vector-valued Marcinkiewicz multiplier theorem which is the cornerstone in the proof of Theorem 1.8.6, see [15, 23, 24]. It is interesting to comment that the condition (1.8.8) in the classical (scalar) Marcinkiewicz multiplier theorem is weakened to a wider condition

$$\sup_{k \in \mathbb{Z}} |\lambda_k| + \sup_{m \geq 1} \|\{\lambda_k\}_{2^m \leq |k| < 2^{m+1}}\|_{V_2}, \tag{1.8.9}$$

where

$$\|\{\mu_k\}_{k=1}^{n}\|_{V_2} := \sup \left[\sum_{s=1}^{m} |\mu'_s - \mu'_{s+1}|^2 \right]^{\frac{1}{2}}$$

and the maximum is taken over all subsequences $\{\mu'_s\}_{s=1}^{m} \subseteq \{\mu_k\}_{k=1}^{n}$, see [16] (see also [34] for some further development). It is also interesting to note that M. Birman and M. Solomyak in [7, Theorem 6.4] claimed that there is a similar weakened version of Theorem 1.8.6 (where the V_1 condition is replaced with the V_2 condition), but they never exhibited a proof.

Let $f : \mathbb{R} \mapsto \mathbb{C}$ be a Borel measurable function. We shall consider the function

$$\psi_f(\lambda, \mu) := \frac{f(\lambda) - f(\mu)}{\lambda - \mu} \quad \text{and} \quad \psi_f(\lambda, \lambda) = 0, \quad \lambda, \mu \in \mathbb{R}. \tag{1.8.10}$$

Let $\mathcal{E}$ be a noncommutative symmetric space. We shall introduce the class $\mathfrak{F}(\mathcal{E})$ as the class of all Borel measurable functions $f : \mathbb{R} \mapsto \mathbb{C}$ such that $\psi_f \in \Phi(\mathcal{E})$. The next results reformulate the above results in terms of the class $\mathfrak{F}(\mathcal{E})$. We start with an observation which follows from Theorem 1.8.1.

Lemma 1.8.8. (i) *$\psi_f \in B(\mathbb{R}^2)$ if and only if f is Lipschitz, that is, there is a constant c_f, such that*

$$|f(\lambda) - f(\mu)| \le c_f\, |\lambda - \mu|, \quad \lambda, \mu \in \mathbb{R};$$

(ii) *$f \in \mathfrak{F}(\mathcal{L}^2)$ if and only if f is Lipschitz.*

Theorem 1.8.9 ([22, 50])**.** *Let $\dot{B}^s_{pq}$, $1 \le p, q \le \infty$, $s > 0$ be the homogeneous Besov classes, [49].*

(i) *If $f \in \dot{B}^1_{\infty 1}$ and f is Lipschitz, then $\psi_f \in B(\mathbb{R}) \hat{\otimes} B(\mathbb{R})$;*

(ii) *If $f \in \dot{B}^1_{\infty 1}$ and f is Lipschitz, then $f \in \mathfrak{F}(\mathcal{E})$, for every fully symmetric noncommutative space $\mathcal{E}$.*

Corollary 1.8.10 ([22, Corollary 7.6])**.** (i) *If f is a continuously differentiable function, $\|f'\|_\infty < \infty$ and f' is Hölder condition with exponent $\epsilon > 0$, i.e. there is a constant $c_{f,\epsilon}$ such that*

$$|f'(\lambda) - f'(\mu)| \le c_{f,\epsilon} |\lambda - \mu|^\epsilon, \quad \lambda, \mu \in \mathbb{R},$$

or

(ii) *if f is a function such that $\mathscr{F} f' \in L^1(\mathbb{R})$, where f' is the derivative in the sense of tempered distributions and $\mathscr{F}$ is the Fourier transform,*

then $f \in \mathfrak{F}(\mathcal{E})$, for every noncommutative symmetric space $\mathcal{E}$.

Lemma 1.8.11 ([24])**.** (i) *If $f' \in V_\beta$, $1 \le \beta \le \infty$, then $\psi_f \in L^\infty(V_\beta)$;*

(ii) *If $f' \in V_\beta$, $1 \le \beta \le \infty$, then $f \in \mathfrak{F}(\mathcal{L}^p)$, whenever $|2^{-1} - p^{-1}| < (2\beta)^{-1}$.*

Proof. It is apparent that (ii) follows from (i) and Theorem 1.8.6. Let us prove (i). To this end, we consider the identity

$$\begin{aligned}\psi_f(\lambda, \mu) &= \frac{f(\lambda) - f(\mu)}{\lambda - \mu} \\ &= \frac{1}{\lambda - \mu} \int_\mu^\lambda f'(t)\, dt \\ &= \int_0^1 f'(s\,\lambda + (1 - s)\,\mu)\, ds.\end{aligned}$$

Consequently, for every increasing sequence $\{\mu_k\}_{k=-\infty}^{\infty}$, it follows from the triangle inequality that

$$\left[\sum_{k=-\infty}^{\infty} |\psi_f(\lambda,\mu_k) - \psi_f(\lambda,\mu_{k+1})|^\beta\right]^{\frac{1}{\beta}} \le$$
$$\int_0^1 \left[\sum_{k=-\infty}^{+\infty} |f'(s\,\lambda + (1-s)\,\mu_k) - f'(s\,\lambda + (1-s)\,\mu_{k+1})|^\beta\right]^{\frac{1}{\beta}} ds$$
$$\le \int_0^1 \|f'\|_{V_\beta}\, ds = \|f'\|_{V_\beta}.$$

Hence, it follows that

$$\psi_f \in L^\infty(V_\beta) \ \text{ and } \ \|\psi_f\|_{L^\infty(V_\beta)} \le \|f'\|_{V_\beta}.$$

The lemma is proved. □

1.9 Double Operator Integrals for matrices

Let $\mathcal{M} = B(\ell_n^2)$, $n \ge 1$. Let us recall that the algebra $\mathcal{M}$ may be identified with the space of all $n \times n$-matrices. The corresponding noncommutative symmetric space $\mathcal{C}_n^E := E(\mathcal{M}, \tau)$ is a space of all $n \times n$-matrices equipped with an appropriate symmetric norm (see Section 1.6.2). Let $\{e_{jk}\}_{j,k=1}^n$ be the collection of matrix units. Let us consider the self-adjoint linear operators a and b given by

$$a = \sum_{j=1}^n \alpha_j e_{jj} \ \text{ and } \ \sum_{k=1}^n \beta_k e_{kk}.$$

Clearly, the spectral measures $e^a(B)$ and $e^b(B)$, $B \in \mathscr{B}(\mathbb{R})$ are given by

$$e^a(B) = \sum_{\{j:\ \alpha_j \in B\}} e_{jj}, \quad e^b(B) = \sum_{\{k:\ \beta_k \in B\}} e_{kk}.$$

Let $P^a(\cdot)$ and $Q^b(\cdot)$ be the projections on $\mathcal{C}_n^E$ of left and right multiplication by $e^a(\cdot)$ and $e^b(\cdot)$, respectively. We readily see that

$$P^a(B) = \sum_{\{j:\alpha_j \in B\}} P_j(B) \ \text{ and } \ Q(B) = \sum_{\{k:\beta_k \in B\}} Q_k(B), \quad B \in \mathscr{B}(\mathbb{R}),$$

where P_j and Q_k, $1 \le j,k \le n$ are the projections of left and right multiplication by e_{jj} and e_{kk}, respectively. The projection P_j vanishes all matrix rows except the j-th one and the projection Q_k vanishes all matrix columns except the k-th one. Thus, the projection P_jQ_k vanishes all matrix entries except the (j,k)-th one. The product measure $P^a \otimes Q^b(B)$, $B \in \mathscr{B}(\mathbb{R}^2)$ is given by

$$P^a \otimes Q^b(B) = \sum_{j,k:(\alpha_j,\beta_k)\in B} P_j\, Q_k. \tag{1.9.1}$$

Let $\varphi \in B(\mathbb{R}^2)$. Let us consider the double operator integral

$$T := \int_{\mathbb{R}^2} \varphi\, d(P^a \otimes Q^b).$$

It immediately follows from (1.9.1) that the operator T is given by

$$T = \sum_{j,k=1}^{n} \varphi(\alpha_j, \beta_k)\, P_j\, Q_k.$$

Since P_jQ_k is the projection on the (j,k)-th matrix entry, the latter identity is equivalent to

$$T(x) = \Phi \circ x, \quad x \in B(\ell_n^2),$$

where $\Phi = \{\varphi(\lambda_j, \beta_k)\}_{j,k=1}^n$ and $\circ$ stands for the entrywise product of matrices (the latter product is typically referred to as *Schur or Schur-Hadamard product*). Therefore, we see that the theory of Double Operator Integrals extends the theory of Schur multipliers on $\mathcal{C}_n^E$.

Example 1.9.1. *In [33], matrices $\Phi_{n,p}$, $1 \le p \le \infty$, $n \ge 1$ are constructed such that, if*

$$c_{n,p} := \sup_{x\in(\mathcal{C}^p)_1} \|\Phi_{n,p} \circ x\|_{\mathcal{C}^p}, \quad n \ge 1,\ 1 \le p \le \infty,$$

then

$$\lim_{n\to\infty} c_{n,p} = \infty, \quad \textit{provided } p \ne 2.$$

The latter example shows that Theorem 1.8.1 cannot be extended to the spaces $\mathcal{L}^p$, unless $p = 2$. In other words, this example shows that not every bounded Borel function $\phi \in \mathscr{B}(\mathbb{R}^2)$ is double operator integrable in the space $\mathcal{L}^p$, unless $p = 2$.

1.10 Double Operator Integrals and trace scaling $*$-automorphisms

Let $\mathcal{M}$ be a semi-finite von Neumann algebra acting on $\mathcal{H}$. Let τ be a n.s.f. trace on $\mathcal{M}$. Recall that $E = E(\mathbb{R})$ stands for a symmetric function space and $\mathcal{E} := E(\mathcal{M}, \tau)$ is the corresponding noncommutative symmetric space.

We consider a trace scaling $*$-automorphism $\pi : \mathcal{M} \mapsto \mathcal{M}$ with the factor $\alpha > 0$, see Section 1.4.2. Let us recall that $\bar{\pi}$ stands for the extension of the mapping π to the class of all self-adjoint operators affiliated with $\mathcal{M}$ constructed at the end of Section 1.3. Recall also that π^E is an extension of π to the space $\mathcal{E}$ and $\tilde{\pi}$ — to the algebra $\tilde{\mathcal{M}}$, see Section 1.4.2 for all relevant results.

Let us note that the mapping π^{-1} is also a trace scaling $*$-automorphism with factor α^{-1}. We shall denote the corresponding extensions by $\overline{(\pi^{-1})}$, $(\pi^{-1})^E$ and $\widetilde{(\pi^{-1})}$, respectively. Since the latter extensions are unique, we clearly have that $\overline{(\pi^{-1})} = \bar{\pi}^{-1}$, $(\pi^{-1})^E = (\pi^E)^{-1}$ and $\widetilde{(\pi^{-1})} = \tilde{\pi}^{-1}$.

Let $a, b\eta\mathcal{M}$ be self-adjoint linear operators. We consider the operator $T^{a,b}_{\phi,\mathcal{E}}$, where $\phi \in \Phi(\mathcal{E})$. The relation between the operator $T^{a,b}_{\phi,\mathcal{E}}$ and the mapping π is expressed by the following lemma.

Lemma 1.10.1. *Let $a, b\eta\mathcal{M}$ and let $a' := \bar{\pi}^{-1}(a)$, $b' := \bar{\pi}^{-1}(b)$. If $\phi \in \Phi(\mathcal{E})$, then*

$$T^{a,b}_{\phi,\mathcal{E}} \circ \pi^E = \pi^E \circ T^{a',b'}_{\phi,\mathcal{E}}.$$

Proof. Let us fix $x \in \mathcal{E} \cap \mathcal{L}^2$ and $y \in \mathcal{E}^\times \cap \mathcal{L}^2$. Let us first show that, for every $B \in \mathscr{B}(\mathbb{R}^2)$,

$$\langle P^a_{\mathcal{E}} \otimes Q^b_{\mathcal{E}}(B)(\pi^E(x)), y\rangle = \alpha^{-1} \langle P^{a'}_{\mathcal{E}} \otimes Q^{b'}_{\mathcal{E}}(B)(x), (\pi^{E^\times})^{-1}(y)\rangle. \quad (1.10.1)$$

According to Theorem 1.7.2, it is sufficient to verify the latter identity for $B = A_1 \times A_2$, $A_j \in \mathscr{B}(\mathbb{R})$, $j = 1, 2$.

Due to the definition of the measure $P^a_{\mathcal{E}}$, see (1.7.1), we immediately obtain that

$$\begin{aligned} P^a_{\mathcal{E}}(A_1)(\pi^E(x)) &= e^a(A_1)\pi^E(x) \\ &= \pi^E(\pi^{-1}(e^a(A_1))x) \\ &= \pi^E(P^{a'}_{\mathcal{E}}(A_1)(x)), \quad A_1 \in \mathscr{B}(\mathbb{R}). \end{aligned} \quad (1.10.2)$$

Similarly,

$$Q^b_{\mathcal{E}}(A_2)(\pi^E(x)) = \pi^E(Q^{b'}_{\mathcal{E}}(A_2)(x)), \quad A_2 \in \mathscr{B}(\mathbb{R}). \quad (1.10.3)$$

Let us also observe that the fact that π is trace scaling implies

$$\begin{aligned} \langle \pi^E(u), v\rangle &= \tau(\pi^E(u)\, v) = \alpha^{-1}\, \tau(u\, (\pi^{E^\times})^{-1}(v)) \\ &= \alpha^{-1} \langle u, (\pi^{E^\times})^{-1}(v)\rangle, \quad u \in \mathcal{L}^2 \cap \mathcal{E},\ v \in \mathcal{L}^2 \cap \mathcal{E}^\times. \end{aligned} \quad (1.10.4)$$

Thus, the definition of the product measure $P^a_{\mathcal{E}} \otimes Q^b_{\mathcal{E}}$, see (1.7.3), together with relations (1.10.2), (1.10.3) and (1.10.4) proves the identity (1.10.1), for $B = A_1 \times A_2$, $A_j \in \mathscr{B}(\mathbb{R})$, $j = 1, 2$ and therefore for every $B \in \mathscr{B}(\mathbb{R}^2)$.

Let $\phi \in \Phi(\mathcal{E})$. It follows from Definition 1.7.3 that

$$\langle T^{a,b}_{\phi,\mathcal{E}}(\pi^E(x)), y\rangle = \int_{\mathbb{R}^2} \phi \, d\langle P^a_{\mathcal{E}} \otimes Q^b_{\mathcal{E}}(\pi^E(x)), y\rangle$$

and

$$\langle T^{a_t,b_t}_{\phi,\mathcal{E}}(x), (\pi^{E^\times})^{-1}(y)\rangle = \int_{\mathbb{R}^2} \phi \, d\langle P^{a_t}_{\mathcal{E}} \otimes Q^{b_t}_{\mathcal{E}}(x), (\pi^{E^\times})^{-1}(y)\rangle.$$

Consequently, (1.10.1) and (1.10.4) implies that

$$\langle T^{a,b}_{\phi,\mathcal{E}}(\pi^E(x)), y\rangle = \alpha^{-1} \langle T^{a_t,b_t}_{\phi,\mathcal{E}}(x), (\pi^{E^\times})^{-1}(y)\rangle = \langle \pi^E(T^{a_t,b_t}_{\phi,\mathcal{E}}(x)), y\rangle.$$

The fact that the space $\mathcal{E}^\times \cap \mathcal{L}^2$ separates points in $\mathcal{E}$ (see Lemma 1.4.10), finishes the proof of the lemma. □

1.11 Double Operator Integrals for arbitrary algebras

In this section, we shall consider the extension of Double Operator Integrals over the L^p spaces associated with arbitrary algebras.

Let $\mathcal{M}$ be a von Neumann algebra and let ρ be a n.s.f. weight on $\mathcal{M}$. We again consider the crossed product $\mathcal{R} := \mathcal{M} \rtimes_{\sigma^\rho} \mathbb{R}$ equipped with the distinguished trace τ (see Theorem 1.6.4). Let $E = E(\mathbb{R})$ be a function space and $\mathcal{E} := E(\mathcal{R}, \tau)$ be the corresponding noncommutative symmetric space with respect to the semi-finite couple $(\mathcal{R}, \tau)$, in particular $\mathcal{L}^p$ and $\mathcal{L}^{p,q}$ stands for the operator L^p- and Lorentz spaces, $1 \le p, q \le \infty$. Let $\theta := \{\theta_t\}_{t\in\mathbb{R}}$ be the group of $*$-automorphisms given in (1.6.6).

Let us fix $t \in \mathbb{R}$. The mapping $\theta_t : \mathcal{R} \mapsto \mathcal{R}$ is a trace scaling $*$-automorphism with the factor e^{-t} (see Theorem 1.6.4.(i)). Lemma 1.10.1 immediately implies

Lemma 1.11.1. *Let $a, b\eta\mathcal{R}$ be self-adjoint linear operators. Let $a_t := \bar{\theta}_{-t}(a)$ and $b_t := \bar{\theta}_{-t}(b)$. If $\phi \in \Phi(\mathcal{E})$, then*

$$T^{a,b}_{\phi,\mathcal{E}} \circ \theta_t = \theta_t \circ T^{a_t,b_t}_{\phi,\mathcal{E}}, \quad t \in \mathbb{R}.$$

Recall that $\pi : \mathcal{M} \mapsto \mathcal{R}$ is a $*$-representation of the algebra $\mathcal{M}$ as a "diagonal" subalgebra of $\mathcal{R}$, see Section 1.6.4. Let $\bar{\pi}$ be the extension of the mapping π

to the class of all self-adjoint linear operators affiliated with $\mathcal{M}$ introduced at the end of Section 1.3. Let $a\eta\mathcal{M}$. The spectral measure of the operator $\bar{\pi}(a)$ belongs to $\pi(\mathcal{M})$. Consequently, the latter spectral measure is invariant with respect to the group θ. On the other hand, if the spectral measure of an operator is invariant with respect to θ, so is the operator itself. Thus, we readily obtain that

$$\bar{\theta}_t(\bar{\pi}(a)) = \bar{\pi}(a), \quad a = a^*\eta\mathcal{M},\ t \in \mathbb{R}.$$

The latter observation together with Lemma 1.11.1 proves the following theorem.

Theorem 1.11.2. *Let* $a, b\eta\pi(\mathcal{M})$ *be linear self-adjoint operators. If* $\phi \in \Phi(\mathcal{L}^{p,\infty})$*, for some* $1 \le p \le \infty$*, then*

$$T^{a,b}_{\phi,\mathcal{L}^{p,\infty}} \circ \theta_t = \theta_t \circ T^{a,b}_{\phi,\mathcal{L}^{p,\infty}}.$$

Recalling the definition of the noncommutative spaces $L^p(\mathcal{M})$ (Section 1.6.5) now yields

Theorem 1.11.3. *If* $a, b\eta\pi(\mathcal{M})$ *linear self-adjoint operators and* $\phi \in \Phi(\mathcal{L}^{p,\infty})$*, for some* $1 \le p \le \infty$*, then*

$$T^{a,b}_{\phi,\mathcal{L}^{p,\infty}}(L^p(\mathcal{M})) \subseteq L^p(\mathcal{M}) \quad \textit{and} \quad T^{a,b}_{\phi,\mathcal{L}^{p,\infty}}|_{L^p(\mathcal{M})} \in B(L^p(\mathcal{M})). \tag{1.11.1}$$

It is the latter restriction (1.11.1), which we shall call *the Double Operator Integral* on the space $L^p(\mathcal{M})$.

Chapter 2

Lipschitz and commutator estimates

We fix a semi-finite von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ with n.s.f. trace τ. Let $E = E(\mathbb{R})$ be a fully symmetric function space and $\mathcal{E} := E(\mathcal{M}, \tau)$ be the corresponding noncommutative symmetric space, in particular $\mathcal{L}^p := L^p(\mathcal{M}, \tau)$, $1 \le p < \infty$, $\mathcal{L}^\infty = \mathcal{M}$.

Let $D : \mathscr{D}(D) \mapsto \mathcal{H}$ be a linear self-adjoint operator such that

(D1) $e^{itD}\, x\, e^{-itD} \in \mathcal{L}^\infty$, whenever $x \in \mathcal{L}^\infty$, $t \in \mathbb{R}$;

(D2) $\tau(e^{itD}\, x\, e^{-itD}) = \tau(x)$, whenever $x \in \mathcal{L}^1 \cap \mathcal{L}^\infty$.

During this chapter we shall work with the following definition (and its variants). Definitions 2.0.4 and 2.0.6 are derived from [11, Proposition 3.2.55] where the authors consider bounded commutators.

Definition 2.0.4. Let $a, b \eta \mathcal{M}$ be self-adjoint linear operators and let $x \in \mathcal{M}$. We shall say that the operator $ax - xb$ is well defined and belongs to $\mathcal{E}$ if and only if

(i) there is a core $\mathscr{D} \subseteq \mathscr{D}(b)$ of the operator b such that

$$x(\mathscr{D}) \subseteq \mathscr{D}(a);$$

(ii) the operator $ax - xb$, initially defined on $\mathscr{D}$, is closable;

(iii) the closure $\overline{ax - xb}$ belongs to $\mathcal{E}$. In this case.

The symbol $ax - xb$ also stands for the closure $\overline{ax - xb}$.

Let $f : \mathbb{R} \mapsto \mathbb{C}$, $x = x^* \in \mathcal{M}$. The first problem we shall study in the present chapter is

Problem 2.0.5. *When is it correct that $f(a)x - xf(b) \in \mathcal{E}$ whenever $ax - xb \in \mathcal{E}$?*

We also shall work with the commutators $[D, x]$ defined by the following definition.

Definition 2.0.6. Let $x \in \mathcal{M}$. We shall say that the commutator $[D, x]$ is well defined and belongs to $\mathcal{E}$ if and only if

(i) there is a core $\mathscr{D} \subseteq \mathscr{D}(D)$ of the operator D such that

$$x(\mathscr{D}) \subseteq \mathscr{D}(D);$$

(ii) the operator $Dx - xD$, initially defined on $\mathscr{D}$, is closable;

(iii) the closure $\overline{Dx - xD}$ belongs to $\mathcal{E}$.

In this case, the symbol $[D, x]$ stands for the closure $\overline{Dx - xD}$.

The second problem we shall study concurrently in the present chapter is

Problem 2.0.7. *When is it correct that $[D, f(x)] \in \mathcal{E}$ whenever $[D, x] \in \mathcal{E}$?*

The following observation shows that the appearance of a core in the definition of the symbol $[D, x]$ (respectively, $ax - xb$) in the special case $\mathcal{E} = \mathcal{L}^\infty$ is excessive, that is, without loss of generality, we may assume that $\mathscr{D} = \mathscr{D}(D)$ (respectively $\mathscr{D} = \mathscr{D}(b)$). Thus, in the special case $\mathcal{E} = \mathcal{L}^\infty$ Definitions 2.0.4 and 2.0.6 are reduced to those studied in [11].

Lemma 2.0.8 ([11, Proposition 3.2.55])**.** (i) *Let $a, b \eta \mathcal{M}$ be self-adjoint linear operators and $x \in \mathcal{M}$. If the operator $ax - xb$ is bounded, then*

$$x(\mathscr{D}(b)) \subseteq \mathscr{D}(a).$$

(ii) *Let $D : \mathscr{D}(D) \mapsto \mathcal{H}$ be a self-adjoint linear operator and $x \in \mathcal{M}$. If $[D, x]$ is bounded, then $x(\mathscr{D}(D)) \subseteq \mathscr{D}(D)$.*

Proof. Clearly, (ii) follows from (i). Let us prove (i). Let $y := ax - xb$. From the definition of the symbol $ax - xb$ we have that there is a core $\mathscr{D} \subseteq \mathscr{D}(b)$ such that $x(\mathscr{D}) \subseteq \mathscr{D}(a)$ and

$$\langle y(\xi), \eta \rangle = \langle ax(\xi), \eta \rangle - \langle xb(\xi), \eta \rangle, \quad \xi, \eta \in \mathscr{D}.$$

For every fixed $\xi \in \mathscr{D}$, both sides are bounded linear functionals with respect to $\eta \in \mathcal{H}$, which coincide for $\eta \in \mathscr{D}$. Consequently, they coincide for every $\eta \in \mathcal{H}$, i.e.

$$\langle y(\xi), \eta \rangle = \langle x(\xi), a(\eta) \rangle - \langle b(\xi), x^*(\eta) \rangle, \quad \xi \in \mathscr{D},\ \eta \in \mathscr{D}(a).$$

Now, for every fixed $\eta \in \mathscr{D}(a)$, it follows that the linear form

$$\xi \mapsto \langle b(\xi), x^*(\eta) \rangle$$

is bounded, that is $x^*(\eta) \in \mathscr{D}(b'^*)$, where $b' = b|_{\mathscr{D}}$ is the restriction of b onto $\mathscr{D}$. Since $\mathscr{D}$ is a core, $b'^* = b$ (see Theorem 1.3.12) Thus, $x^*(\eta) \in \mathscr{D}(b)$, for every $\eta \in \mathscr{D}(a)$, i.e.

$$x^*(\mathscr{D}(a)) \subseteq \mathscr{D}(b).$$

Furthermore, the sesquilinear form

$$\langle \xi, (x^*a - bx^*)(\eta) \rangle = \langle y(\xi), \eta \rangle, \quad \xi \in \mathcal{H},\ \eta \in \mathscr{D}(a)$$

is bounded. Consequently the operator $y' := bx^* - x^*a$, defined on $\mathscr{D}(a)$, is bounded and

$$y' = -y^*.$$

Repeating the argument again for the operator $bx^* - x^*a$ gives the claim of the lemma. □

The relation $x(\mathscr{D}(D)) \subseteq \mathscr{D}(D)$ in the case $\mathcal{E} = \mathcal{L}^p$, $1 \le p < \infty$ may fail as it is shown in the example with the differentiation operator below. On the other hand, the weaker relation $x(\mathscr{D}) \subseteq \mathscr{D}(D)$ for some core $\mathscr{D} \subseteq \mathscr{D}(D)$ is much easier to attack and, more importantly, is sufficient for the applications we shall study.

2.1 Commutators with the differentiation operator $\frac{1}{i}\frac{d}{dt}$

In the present section we fix $\mathcal{M} = \mathcal{L}^\infty(\mathbb{R})$ and $\tau(\cdot) = \int (\cdot)\, dt$, see Section 1.6.1. For the sake of brevity, we let $\mathcal{E} = E = E(\mathbb{R})$, $\mathcal{L}^p = L^p = L^p(\mathbb{R})$.

Let us consider the operator $D := \frac{1}{i}\frac{d}{dt} : \mathscr{D}(D) \mapsto L^2$, with the domain given by

$$\mathscr{D}(D) := \left\{ \xi \in L^2 : \quad \frac{1}{i}\frac{d\xi}{dt} \in L^2 \right\},$$

where $\frac{1}{i}\frac{d\xi}{dt}$ is the derivative in the sense of tempered distributions. The domain $\mathscr{D}(D)$ has an alternative description as the collection of all absolutely continuous functions $\xi \in L^2$ such that the classical derivative $\frac{1}{i}\frac{d\xi}{dt} \in L^2$ (see [70]).

Lemma 2.1.1. (i) *The operator D is self-adjoint.*

(ii) *The unitary group of the operator D is given by translations, i.e.*

$$e^{itD}(\xi)(s) = \xi(s+t), \quad \xi \in L^2, \ t, s \in \mathbb{R}.$$

(iii) *The operator D satisfies (D1)–(D2) on page 65.*

Proof. (i) The argument is rather standard, we refer the reader to [58] for all relevant notions. Let $\mathscr{F}$ and $\mathscr{F}^{-1}$ be the Fourier transform and its inverse, see [58, Section 7.1], i.e.

$$(\mathscr{F}\xi)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \xi(s)\, e^{-its}\, ds \quad \text{and} \quad (\mathscr{F}^{-1}\hat{\xi})(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\xi}(t)\, e^{its}\, dt,$$

where $t, s \in \mathbb{R}$ and $\xi, \hat{\xi} \in L^1(\mathbb{R})$. It follows from the Plancherel theorem, see [58, Theorem 7.9], that the mapping $\mathscr{F}$ may be extended to a unitary operator on L^2.

Let $d(t) = t$, $t \in \mathbb{R}$. Consider the multiplication operator m_d defined by

$$\mathscr{D}(m_d) = \{\xi \in L^2 : \ d \cdot \xi \in L^2\} \quad \text{and} \quad m_d(\xi) = d \cdot \xi, \ \xi \in \mathscr{D}(m_d).$$

We have the following identities, see [58, Theorem 7.15]

$$\mathscr{F} m_d \mathscr{F}^{-1} = -D \quad \text{and} \quad \mathscr{F} D \mathscr{F}^{-1} = m_d. \tag{2.1.1}$$

That is, the operator D is induced by the identity function (via the Fourier transform) to the multiplication operator. Consequently, the operator D is self-adjoint.

(ii) Let $e_t(s) := e^{its}$, $s, t \in \mathbb{R}$ and let λ_t be the translation operator, i.e.

$$\lambda_t(x)(s) = x(s-t), \quad s, t \in \mathbb{R}, \ x \in \mathcal{L}^1 + \mathcal{L}^\infty.$$

Clearly, it follows from (2.1.1) (see also [58, Theorem 7.2]), that

$$e^{itD} = \mathscr{F} e^{-itm_d} \mathscr{F}^{-1} = \mathscr{F} m_{e_{-t}} \mathscr{F}^{-1} = \lambda_{-t}, \quad t \in \mathbb{R}.$$

The claim (ii) follows.

(iii) Let $x \in L^\infty$. We readily obtain that

$$\begin{aligned} e^{itD} m_x e^{-itD}(\xi)(s) &= (m_x e^{-itD}(\xi))(s+t) \\ &= x(s+t)\,(e^{-itD}(\xi))(s+t) \\ &= x(s+t)\,\xi(s), \quad s,t \in \mathbb{R},\ \xi \in L^2. \end{aligned}$$

Thus, we obtain that

$$e^{itD} m_x e^{-itD} = m_{\lambda_{-t}(x)} \in L^\infty, \quad \text{for every } x \in L^\infty.$$

The latter implies that the operator D satisfies (D1). Moreover,

$$\begin{aligned} \tau(e^{itD} m_x e^{-itD}) &= \tau(m_{\lambda_{-t}(x)}) \\ &= \int_{\mathbb{R}} \lambda_{-t}(x)(s)\, ds \\ &= \int_{\mathbb{R}} x(s)\, ds \\ &= \tau(m_x), \quad x \in L^1 \cap L^\infty. \end{aligned}$$

Consequently, the operator D satisfies (D2). The lemma is proved. □

Recall that every linear operator affiliated with L^∞ is a multiplication operator. It follows from spectral theorem (see Theorem 1.3.14) that, if a self-adjoint operator x is affiliated with an algebra $\mathcal{M}$, then $\{e^{itx}\}_{t\in\mathbb{R}} \subseteq \mathcal{M}$. Clearly, the translations $\{\lambda_t\}_{t\in\mathbb{R}}$ cannot be represented as multiplication operators. Thus, it follows from Lemma 2.1.1 that the operator D is not affiliated with the algebra L^∞.

Let $x \in L^\infty$. From now on we shall identify the symbol $x \in L^\infty$ with the corresponding multiplication operator m_x. Assume that $[D,x] \in L^p$, $1 \le p \le \infty$, i.e. assume that the operator $[D,x]$ is a multiplication operator generated by a function in L^p. The latter, according to the preceding section, means, in particular, that there is a core $\mathscr{D} \subseteq \mathscr{D}(D)$ such that

$$x(\mathscr{D}) \subseteq \mathscr{D}(D).$$

Let us now consider the operator $Dx - xD$ on $\mathscr{D}$. Let us assume for the moment that we have the product rule. We then obtain that

$$(Dx - xD)(\xi) = \frac{1}{i}\frac{d}{dt}(x\cdot\xi) - x\cdot\frac{1}{i}\frac{d\xi}{dt} = \frac{1}{i}\frac{dx}{dt}\cdot\xi, \quad \xi \in \mathscr{D}. \tag{2.1.2}$$

Thus, if $[D, x] \in \mathcal{L}^p$ in the sense of Definition 2.0.6, then $[D, x]$ is a multiplication operator by the the function $\frac{1}{i}\frac{dx}{dt}$ and therefore $\frac{1}{i}\frac{dx}{dt} \in L^p$.

Hence, we obtain that $[D, x] \in L^p$, $1 \le p \le \infty$, $x \in L^\infty$ if and only if there is a core $\mathscr{D} \subseteq \mathscr{D}(D)$ such that

$$x(\mathscr{D}) \subseteq \mathscr{D}(D) \quad \text{and} \quad \frac{1}{i}\frac{dx}{dt} \in L^p.$$

Furthermore, let us note that $x(\mathscr{D}) \subseteq \mathscr{D}(D)$ means: for every function $\xi \in \mathscr{D}$, the function $x \cdot \xi$ is differentiable in the sense of tempered distributions and

$$\frac{1}{i}\frac{d}{dt}(x \cdot \xi) \in L^2. \tag{2.1.3}$$

Since $x \cdot \frac{1}{i}\frac{d\xi}{dt} \in L^2$, for every $\xi \in \mathscr{D}(D)$, $x \in L^\infty$, it follows from the last identity in (2.1.2) that (2.1.3) is equivalent to the condition

$$\frac{1}{i}\frac{dx}{dt} \cdot \xi \in L^2, \quad \xi \in \mathscr{D}(D)$$

provided we have the product rule for $\frac{1}{i}\frac{d}{dt}(x\xi)$. The latter means that, if $\mathscr{D} \subseteq \mathscr{D}(D)$ is a core, then

$$x(\mathscr{D}) \subseteq \mathscr{D}(D) \iff \frac{1}{i}\frac{dx}{dt}(\mathscr{D}) \subseteq L^2. \tag{2.1.4}$$

Moreover, $[D, x] \in \mathcal{L}^p$, $1 \le p \le \infty$ if and only if there exists a core $\mathscr{D} \subseteq \mathscr{D}(D)$ such that

$$\frac{1}{i}\frac{dx}{dt} \in L^p \quad \text{and} \quad \frac{1}{i}\frac{dx}{dt}(\mathscr{D}) \subseteq L^2. \tag{2.1.5}$$

Thus, in general, a verification of the statement $[D, x] \in \mathcal{L}^p$, $1 \le p < \infty$ consists of two steps whose nature is quite different. A verification of the (generalized) condition $\frac{1}{i}\frac{dx}{dt} \in L^p$ is carried out in the literature almost exclusively via methods related to Banach space geometry (Schur multipliers, double operator integrals, vector-valued Fourier multipliers [15, 23, 24, 27]). However, the second condition in (2.1.5) has an operator-theoretical nature and does not correspond to the methods listed above. We outline an approach to this problem when $D = \frac{1}{i}\frac{d}{dt}$.

Let us first consider when $[D, x] \in L^p$ if $2 \le p < \infty$. We shall show that in the present setting, the required core $\mathscr{D}$ appears very naturally due to the fact that the underlying Hilbert space L^2 possesses the additional Banach space structure induced by the L^p-scale. Indeed, let us set

$$\mathscr{D} := \mathscr{D}(D) \cap L^q, \quad \text{where} \quad \frac{1}{2} = \frac{1}{p} + \frac{1}{q}. \tag{2.1.6}$$

Clearly, when $2 \le p \le \infty$, the Hölder inequality (1.4.8) implies that

$$\left\| \frac{1}{i}\frac{dx}{dt} \cdot \xi \right\|_{L^2} \le \left\| \frac{1}{i}\frac{dx}{dt} \right\|_{L^p} \|\xi\|_{L^q} < +\infty, \quad \text{for every } \xi \in \mathscr{D}$$

and therefore, (2.1.5) holds for the subset $\mathscr{D}$ and any $x \in L^\infty$ such that $\frac{1}{i}\frac{dx}{dt} \in L^p$. We shall verify that $\mathscr{D}$ is a core of D in Lemma 2.3.19 below. What we would like to emphasize is that the core $\mathscr{D}$ is found purely by a Banach space construction. Thus, we see that in the case $2 \le p < \infty$, we have

$$[D, x] \in L^p \iff \frac{1}{i}\frac{dx}{dt} \in L^p.$$

Finally, we comment on the case $1 \le p < 2$. Here, the problem of finding the core $\mathscr{D}$ satisfying the first condition in (2.1.5) cannot be resolved by a purely Banach space approach as in (2.1.6) above. Indeed, let $C(\mathbb{R})$ be the class of all continuous functions on $\mathbb{R}$. We note that $\mathscr{D}(D) \subseteq C(\mathbb{R})$, [59, Theorem 2, p. 124]. We now consider any function $x \in L^\infty$ such that

$$\frac{1}{i}\frac{dx}{dt} \in L^p, \quad \text{but} \quad \frac{1}{i}\frac{dx}{dt} \notin L^2_{loc}. \tag{2.1.7}$$

Such a function exists due to $1 \le p < 2$. Indeed, let $f_0(t) = t^{-1/2}$ if $0 < t \le 1$ and $f_0(t) = 0$ otherwise. Set

$$f_1(t) := \sum_{n \in \mathbb{Z}} 2^{-|n|} f_0(t - n).$$

The function $f_1 \in L^p$, $1 \le p < 2$. On the other hand, the function f_1 is not square integrable in the neighborhood of every integral point $t = n \in \mathbb{Z} \subseteq \mathbb{R}$. Setting

$$f(t) := \sum_{n=1}^{\infty} 2^{-2n} f_1(2^n t)$$

gives an example of the function such that

$$f \in L^p, \quad \text{but} \quad f \notin L^2_{loc}.$$

Let

$$x(t) := \int_{-\infty}^{t} f(s)\, ds.$$

Since $f \in L^1$, we immediately obtain that $x \in L^\infty$. Moreover the function x satisfies (2.1.7). For the function x constructed above it follows that

$$\frac{1}{i}\frac{dx}{dt} \cdot \xi \notin L^2, \quad \text{for every} \quad \xi \in \mathscr{D}(D), \ \xi \not\equiv 0.$$

Indeed, since $\mathscr{D}(D) \subseteq C(\mathbb{R})$, it follows that for every $\xi \in \mathscr{D}(D)$, $\xi \not\equiv 0$, there is an indicator function χ of an interval such that

$$\left| \frac{1}{i} \frac{dx}{dt} \cdot \xi \right| \geq \epsilon \left| \frac{1}{i} \frac{dx}{dt} \cdot \chi \right|, \quad \text{for some } \epsilon > 0.$$

The latter function is not in L^2 due to the fact that $\frac{1}{i}\frac{dx}{dt} \notin L^2_{loc}$. That means, see (2.1.4), that despite the fact that the derivative $\frac{1}{i}\frac{dx}{dt}$ exists in the sense of tempered distributions and belongs to L^p, *there is no core* such that the commutator $[D, x]$ may be defined according to Definition 2.0.6.

2.2 Preliminaries

As we note in the beginning of the chapter, we shall consider the Problems 2.0.7 and 2.0.5 concurrently. Following the example above, we shall single out three different cases $p = \infty$, $2 \leq p < \infty$ and $1 \leq p < 2$. Before we start considering Problems 2.0.7 and 2.0.5, we give some preliminary results.

The following lemma was established in the type I setting in [8].

Lemma 2.2.1 ([24, Lemma 7.1])**.** *Let $a, b\eta\mathcal{M}$ be self-adjoint linear operators and let $e^a_n = e^a([-n, n])$, $e^b_n = e^b([-n, n])$, $n \geq 1$, be the corresponding spectral projections. If $f \in \mathfrak{F}(\mathcal{E})$, then, for every $x \in \mathcal{E}$,*

$$T^{a,b}_{\psi_f,\mathcal{E}}(ae^a_n xe^b_n - e^a_n xbe^b_n) = f(a)e^a_n xe^b_n - e^a_n xf(b)e^b_n, \quad n \geq 1. \tag{2.2.1}$$

The following proposition complements the result of Lemma 2.2.1. It replaces the assumption $x \in \mathcal{E}$ with the assumption $ax - xb \in \mathcal{E}$. Recall that the fundamental function of a rearrangement invariant space E is given by $\phi_E(t) := \|\chi_{[0,t)}\|_E$, $t > 0$.

Proposition 2.2.2. *Let $\mathcal{E} = (\mathcal{E}_*)^\times$, where $\mathcal{E}_*$ is a noncommutative symmetric space with an order-continuous norm and the Fatou property. If the fundamental function ϕ_E satisfies*

$$\lim_{t\to\infty} \frac{\phi_E(t)}{t} = 0, \tag{2.2.2}$$

then, for every complex-valued function f on $\mathbb{R}$ such that $f \in \mathfrak{F}(\mathcal{E})$, we have

$$T^{a,b}_{\psi_f,\mathcal{E}}(ax - xb) = f(a)x - xf(b), \tag{2.2.3}$$

for all self-adjoint operators $a, b \in \mathcal{M}$ and all operators $x \in \mathcal{M}$ such that $ax-xb \in \mathcal{E}$.

Proof. Let us first recall that it follows from [24, Proposition 6.6] that there is a net of projections $\{p_\beta\} \subseteq \mathcal{M}$ such that

$$p_\beta \uparrow \mathbf{1}, \quad \tau(p_\beta) < \infty \quad \text{and} \quad \|bp_\beta - p_\beta b\|_{\mathcal{E}} \leq 1.$$

Let us first show that

$$\lim_\beta (bp_\beta - p_\beta b) = 0, \tag{2.2.4}$$

where the limit is taken in the $\sigma(\mathcal{E}, \mathcal{E}_*)$-topology (=$\sigma(\mathcal{E}, \mathcal{E}^\times)$-topology). Indeed, since the net

$$\{bp_\beta - p_\beta b\} \tag{2.2.5}$$

is uniformly bounded with respect to the norm of $\mathcal{E}$, without loss of generality, we may assume that the limit (2.2.4) exists, see Theorem 1.1.1. Hence, we need only to show that the latter limit vanishes. Since $p_\beta \uparrow \mathbf{1}$, we readily have that

$$wo - \lim_\beta (bp_\beta - p_\beta b) = 0. \tag{2.2.6}$$

Furthermore, since the collection (2.2.5) is uniformly bounded with respect to the operator norm, we see that the limit (2.2.6) vanishes with respect to the $\sigma(\mathcal{E}, \mathcal{L}^1 \cap \mathcal{L}^\infty)$-topology (see Lemma 1.4.14.(ii)). Thus, to finish the proof of (2.2.4), we need only to note that the $\sigma(\mathcal{E}, \mathcal{L}^1 \cap \mathcal{L}^\infty)$-topology is weaker than the $\sigma(\mathcal{E}, \mathcal{E}_*)$-topology.

Let us next show that

$$\lim_\beta (axp_\beta - xp_\beta b) = ax - xb, \tag{2.2.7}$$

where the limit is taken respect to the $\sigma(\mathcal{E}, \mathcal{E}_*)$-topology. It is clear that we have the following identity

$$axp_\beta - xp_\beta b = (ax - xb)\, p_\beta + x(bp_\beta - p_\beta b).$$

Consequently, (2.2.7) follows from (2.2.4), Lemma 1.4.13 and the fact that $ax - xb \in \mathcal{E}$.

Let us note that, since $\tau(p_\beta) < +\infty$ and $x \in \mathcal{M}$, it is readily follows that $xp_\beta \in \mathcal{L}^1 \cap \mathcal{L}^\infty \subseteq \mathcal{E}$. Therefore, we are in a position to apply Lemma 2.2.1 (note, that $a, b \in \mathcal{M}$ by the assumption). The latter implies that

$$T^{a,b}_{\psi_f, \mathcal{E}}(axp_\beta - xp_\beta b) = f(a)\, xp_\beta - xp_\beta\, f(b). \tag{2.2.8}$$

Let us note that it follows from Theorem 1.7.7 that the operator $T^{a,b}_{\psi_f, \mathcal{E}}$ is $\sigma(\mathcal{E}, \mathcal{E}_*)$-continuous. Consequently, from (2.2.7), we immediately obtain that

$$\lim_\beta T^{a,b}_{\psi_f, \mathcal{E}}(axp_\beta - xp_\beta b) = T^{a,b}_{\psi_f, \mathcal{E}}(ax - xb), \tag{2.2.9}$$

where the limit is taken with respect to the $\sigma(\mathcal{E}, \mathcal{E}_*)$-topology. On the other hand, since $p_\beta \uparrow \mathbf{1}$ and (2.2.8), we see that

$$\begin{aligned}\lim_\beta T^{a,b}_{\psi_f,\mathcal{E}}(axp_\beta - xp_\beta b) &= \lim_\beta f(a)\, xp_\beta - xp_\beta\, f(b)\\ &= f(a)\, x - x\, f(b),\end{aligned}$$

where the limit is regarded with respect to the *wo*-topology. Recall that the net $\{f(a)\, xp_\beta - xp_\beta\, f(b)\}$ is uniformly bounded with respect to the operator norm. Thus, it follows from Lemma 1.4.14.(ii) that the latter limit converges with respect to the $\sigma(\mathcal{E}, \mathcal{L}^1 \cap \mathcal{L}^\infty)$-topology also. The latter topology is weaker than the $\sigma(\mathcal{E}, \mathcal{E}_*)$-topology. Together with (2.2.9) this proves that

$$T^{a,b}_{\psi_f,\mathcal{E}}(ax - xb) = f(a)\, x - x\, f(b).$$

The lemma is completely proved. □

Corollary 2.2.3. *Let $a, b\eta\mathcal{M}$ be self-adjoint, e^a_n and e^b_n, $n \geq 1$ be spectral projections as in Lemma 2.2.1 and let $\mathcal{E}$ satisfy the assumptions of Proposition 2.2.2. If $x \in \mathcal{M}$, $ae^a_n xe^b_n - e^a_n xbe^b_n \in \mathcal{E}$, $n \geq 1$ and $f \in \mathfrak{F}(\mathcal{E})$, then*

$$T^{a,b}_{\psi_f,\mathcal{E}}(ae^a_n xe^b_n - e^a_n xbe^b_n) = f(a)e^a_n xe^b_n - e^a_n xf(b)e^b_n, \quad n \geq 1.$$

Proof. Setting $a_n = ae^a_n$, $b_n = be^b_n$ and $x_n = e^a_n xe^b_n$, $n \geq 1$, we have (by assumption)

$$a_n x_n - x_n b_n = ae^a_n xe^b_n - e^a_n xbe^b_n \in \mathcal{E}.$$

Applying Proposition 2.2.2 to the operators a_n, b_n and x_n we obtain

$$T^{a_n,b_n}_{\psi_f,\mathcal{E}}(a_n x_n - x_n b_n) = f(a_n)x_n - x_n f(b_n). \tag{2.2.10}$$

To finish the proof, we note that, if χ_n is the characteristic function of the interval $[-n, n]$ and $\psi'_n(\lambda, \mu) = \chi_n(\lambda)\, \chi_n(\mu)$, $\lambda, \mu \in \mathbb{R}$, then

$$\begin{aligned} T^{a_n,b_n}_{\psi_f,\mathcal{E}}(y) &= \int_{\mathbb{R}^2} \psi_f(\lambda,\mu)\, dP^{a_n}_{\mathcal{E}} \otimes Q^{b_n}_{\mathcal{E}}(y)\\ \text{[Theorem 1.3.14]} \quad &= \int_{\mathbb{R}^2} \chi_n(\lambda)\, \psi_f(\lambda,\mu)\, \chi_n(\mu)\, dP^a_{\mathcal{E}} \otimes Q^b_{\mathcal{E}}(y)\\ \text{[Definition 1.7.3]} \quad &= T^{a,b}_{\psi'_n\psi_f,\mathcal{E}}(y)\\ \text{[Theorem 1.7.6]} \quad &= T^{a,b}_{\psi_f,\mathcal{E}}(T^{a,b}_{\psi'_n,\mathcal{E}}(y))\\ \text{[Lemma 1.7.8]} \quad &= T^{a,b}_{\psi_f,\mathcal{E}}(e^a_n ye^b_n), \quad y \in \mathcal{E},\ n \geq 1. \end{aligned}$$

Combining the latter identity with (2.2.10) we obtain the claim of the corollary. □

The next lemma will be used several times in the sequel.

Lemma 2.2.4. *Let $\mathcal{E}$ and $\mathcal{F}$ be noncommutative symmetric spaces and $\gamma := \{\gamma_t\}_{t\in\mathbb{R}}$ be a group of contractions in both $\mathcal{E}$ and $\mathcal{F}$. If γ is a strongly (resp., weakly*) continuous group in $\mathcal{E}$, $\mathscr{D}(\delta)$ is the domain of the strong (resp., weak*) generator of γ in $\mathcal{E}$, and the function $t \mapsto \|\gamma_t(\xi)\|_{\mathcal{F}}$ is Lebesgue measurable, for every $\xi \in (\mathcal{F})_1$, then the set $\mathscr{D}(\delta) \cap (\mathcal{F})_1$ is invariant with respect to γ and norm (resp., weak*) dense in $\mathcal{E} \cap (\mathcal{F})_1$. In particular, if $\mathcal{F}$ is a noncommutative symmetric space with the Fatou norm such that $\mathcal{E} \cap \mathcal{F}$ is norm (resp., weak*) dense in $\mathcal{E}$ and γ is $\sigma(\mathcal{F}, \mathcal{F}^\times)$-continuous in $\mathcal{F}$, then the subspace $\mathscr{D}(\delta) \cap \mathcal{F}$ is norm (resp., weak*) dense in $\mathcal{E}$.*

Proof. We prove the assertion when γ is strongly continuous and outline the changes needed for a weak* continuous group at the end of the proof.

Since the space $\mathscr{D}(\delta)$ is invariant under γ, see Lemma 1.1.4, and due to the hypothesis $\gamma_t((\mathcal{F})_1) \subseteq (\mathcal{F})_1$, we have

$$\gamma_t(\mathscr{D}(\delta) \cap (\mathcal{F})_1) \subseteq \mathscr{D}(\delta) \cap (\mathcal{F})_1, \quad t \in \mathbb{R}.$$

Let $R_\lambda := R_\lambda(\delta)$ be the resolvent of the operator δ, then, according to Theorem 1.1.6,

$$R_\lambda(\xi) \in \mathscr{D}(\delta), \quad R_\lambda \xi = \int_0^\infty e^{-\lambda t} \gamma_t(\xi)\, dt, \quad \lambda > 0 \tag{2.2.11}$$

and

$$\lim_{\lambda\to\infty} \lambda R_\lambda(\xi) = \xi, \quad \xi \in \mathcal{E}, \ \lambda > 0, \tag{2.2.12}$$

where the limit is taken in the norm topology of $\mathcal{E}$.

Let $\xi \in (\mathcal{F})_1$. Since the function $t \mapsto \|\gamma_t(\xi)\|_{\mathcal{F}}$ is Lebesgue measurable, we have the elementary inequality

$$\|\lambda R_\lambda(\xi)\|_{\mathcal{F}} \le \lambda \int_0^\infty e^{-t\lambda} \|\gamma_t(\xi)\|_{\mathcal{F}}\, dt \le \|\xi\|_{\mathcal{F}}, \quad \lambda > 0.$$

Consequently,

$$\lambda R_\lambda(\xi) \in (\mathcal{F})_1, \quad \lambda > 0.$$

Combining the latter statement with (2.2.11), we obtain that

$$\lambda R_\lambda(\xi) \in \mathscr{D}(\delta) \cap (\mathcal{F})_1 \text{ provided } \xi \in \mathcal{E} \cap (\mathcal{F})_1.$$

Thus, it follows from (2.2.12) that $\mathscr{D}(\delta) \cap (\mathcal{F})_1$ is norm dense in $\mathcal{E} \cap (\mathcal{F})_1$.

For the second part, we note that, since γ is $\sigma(\mathcal{F}, \mathcal{F}^\times)$-continuous,

$$\tau(\eta\xi) = \lim_{t\to 0} \tau(\eta\gamma_t(\xi)), \quad \xi \in \mathcal{F}, \ \eta \in \mathcal{F}^\times.$$

On the other hand, since $\mathcal{F}$ has the Fatou norm,

$$\|\xi\|_{\mathcal{F}} = \sup_{\|\eta\|_{\mathcal{F}^\times} \le 1} |\tau(\eta\xi)| \le \liminf_{t\to 0} \|\gamma_t(\xi)\|_{\mathcal{F}}, \quad \xi \in \mathcal{F}.$$

The latter means that the function $t \to \|\gamma_t(\xi)\|_{\mathcal{F}}$, $\xi \in \mathcal{F}$ is semi-continuous and, hence, measurable. The claim is proved.

For the weak* assertion, the argument is the same, except we have to apply the weak* version of Theorem 1.1.6. □

2.3 Main results

As we have seen in the example with the operator $D = \frac{1}{i}\frac{d}{dt}$, see Section 2.1, a meaningful resolution of Problem 2.0.7 requires locating a core $\mathscr{D}$ of the operator D satisfying the first condition in (2.1.4). As we indicated in that example, a possible candidate on the role of such $\mathscr{D}$ is the space

$$\mathscr{D}(D) \cap \mathcal{L}^1 \cap \mathcal{L}^\infty.$$

Unfortunately, in general, the expression above is senseless, since the domain $\mathscr{D}(D) \subseteq \mathcal{H}$ may have an empty intersection with the space $\mathcal{L}^1 \cap \mathcal{L}^\infty$. We shall show below that this is not the case when $\mathcal{M}$ is taken in the left regular representation (see Lemma 2.3.19).

Let $\mathcal{M}_L$ stand for the left regular representation of the algebra $\mathcal{M}$. The algebra $\mathcal{M}_L$ equipped with n.s.f. trace τ_L. Let $E = E(\mathbb{R})$ and $\mathcal{E}_L := E(\mathcal{M}_L, \tau_L)$, in particular $\mathcal{L}^p_L := L^p(\mathcal{M}_L, \tau_L)$, see Section 1.5.

2.3.1 Lipschitz estimates

For an operator $a\eta\mathcal{M}_L$, we introduce the subspace

$$\mathscr{D}_0(a) := \mathscr{D}(a) \cap \mathcal{L}^1 \cap \mathcal{L}^\infty \subseteq \mathcal{L}^2. \tag{2.3.1}$$

Lemma 2.3.1. *If $a\eta\mathcal{M}_L$, then the subspace $\mathscr{D}_0(a)$ is affiliated with $\mathcal{M}_L$ and it is a core of the operator a.*

Proof. Since $a\eta\mathcal{M}_L$ it is readily clear that $\mathscr{D}(a)\eta\mathcal{M}_L$. The latter means that, for every unitary operator $R_u \in (\mathcal{M}_L)' = \mathcal{M}_R$ (see Theorem 1.5.3),

$$R_u(\mathscr{D}(a)) \subseteq \mathscr{D}(a).$$

On the other hand, since $u \in \mathcal{M}$, we clearly have that

$$R_u(\mathcal{L}^1 \cap \mathcal{L}^\infty) = (\mathcal{L}^1 \cap \mathcal{L}^\infty)\, u \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty.$$

Consequently,

$$R_u(\mathscr{D}_0(a)) \subseteq \mathscr{D}_0(a), \quad R_u \in (\mathcal{M}_L)'.$$

The latter means that the subspace $\mathscr{D}_0(a)$ is affiliated with $\mathcal{M}_L$.

Furthermore, since $a\eta\mathcal{M}_L$, it follows from Lemma 1.3.18 that $e^{ita} \in \mathcal{M}_L$. The latter means that there is a unitary group $\{u_t\}_{t\in\mathbb{R}} \subseteq \mathcal{M}$ such that

$$e^{ita}(\xi) = u_t\xi, \quad \xi \in \mathcal{L}^2, \ t \in \mathbb{R}.$$

Consequently, $e^{ita}(\mathcal{L}^1 \cap \mathcal{L}^\infty) \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty$ and the restriction

$$\{e^{ita}|_{\mathcal{L}^1\cap\mathcal{L}^\infty}\}_{t\in\mathbb{R}}$$

is a $\sigma(\mathcal{L}^1\cap\mathcal{L}^\infty, \mathcal{L}^1+\mathcal{L}^\infty)$-continuous group of contractions on the space $\mathcal{L}^1\cap\mathcal{L}^\infty$. Applying Lemma 2.2.4 to the group $\{e^{ita}\}_{t\in\mathbb{R}}$, $\mathcal{E} = \mathcal{L}^2$ and $\mathcal{F} = \mathcal{L}^1 \cap \mathcal{L}^\infty$, we obtain that $\mathscr{D}_0(a)$ is dense in $\mathcal{L}^2$. On the other hand, $\mathscr{D}_0(a)$ is invariant with respect to e^{ita}, $t \in \mathbb{R}$. Thus, it follows from Theorem 1.1.7 that $\mathscr{D}_0(a)$ is a core of a. □

Let $a, b\eta\mathcal{M}_L$ be self-adjoint linear operators and let $x \in \mathcal{M}_L$. For the purposes of the present section, we adapt Definition 2.0.4 of the symbol $ax - xb$ for the setting of the left regular representation.

Definition 2.3.2. We shall say that the operator $ax - xb$ is well-defined and belongs to $\mathcal{E}_L$ if and only if

(i) there is a core $\mathscr{D} \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty$ of the operator a such that $x(\mathscr{D}) \subseteq \mathscr{D}(a)$;

(ii) the operator $ax - xb$, defined on $\mathscr{D}$, is closable;

(iii) the closure $\overline{ax - xb}$ belongs to $\mathcal{E}_L$.

The symbol $ax - xb$ stands for the closure $\overline{ax - xb}$.

Lemma 2.3.1 shows that the restriction $\mathscr{D} \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty$ is rather mild.

Recall that $\mathfrak{F}(\mathcal{E}_L)$ is the class of all Borel measurable functions $f : \mathbb{R} \mapsto \mathbb{C}$ such that $\psi_f \in \Phi(\mathcal{E}_L)$, where $\Phi(\mathcal{E}_L)$ is the class of all double operator integrable functions, see Section 1.7. The answer to the Problem 2.0.5 for the space $\mathcal{L}^p_L$ is given by

Theorem 2.3.3. *Let $a, b\eta\mathcal{M}_L$ be self-adjoint linear operators. Let $2 \le p \le \infty$ (resp. $1 \le p < 2$), let $f \in \mathfrak{F}(\mathcal{L}^p_L)$ and let $x \in \mathcal{M}_L$. If*

$$ax - xb \in \mathcal{L}^p_L \quad (\textit{resp. } ax - xb \in \mathcal{L}^p \cap \mathcal{L}^2),$$

then

$$f(a)x - xf(b) \in \mathcal{L}^p_L \quad (\textit{resp. } f(a)x - xf(b) \in \mathcal{L}^p_L \cap \mathcal{L}^2_L)$$

and

$$\|f(a)x - xf(b)\|_{\mathcal{L}^p_L} \le c_{f,p} \|ax - xb\|_{\mathcal{L}^p_L}.$$

The latter result immediately follows from the more general result given in Theorem 2.3.4 applied to the spaces $\mathcal{E}_L = \mathcal{L}^p_L$, $1 \le p \le \infty$.

The answer to the Problem 2.0.5 for the space $\mathcal{E}_L$ with Fatou norm is given by

Theorem 2.3.4. *Let $\mathcal{E}_L$ be a noncommutative symmetric space with Fatou norm and let $2 \le p \le \infty$. Let $a, b\eta\mathcal{M}_L$ be self-adjoint linear operators and $x \in \mathcal{M}_L$. If*

$$ax - xb \in \mathcal{E}_L \cap \mathcal{L}^p_L \quad \textit{and} \quad f \in \mathfrak{F}(\mathcal{E}_L) \cap \mathfrak{F}(\mathcal{L}^p_L),$$

then $f(a)x - xf(b) \in \mathcal{E}_L \cap \mathcal{L}^p_L$ and

$$\|f(a)x - xf(b)\|_{\mathcal{E}_L} \le c_{f,E} \|ax - xb\|_{\mathcal{E}_L},$$

where

$$c_{f,E} = \sup_{a,b\eta\mathcal{M}_L} \|T^{a,b}_{\psi_f,\mathcal{E}}\|_{B(\mathcal{E}_L)}.$$

Proof. Let $y := ax - xb$. According to Definition 2.3.2, there is a core $\mathscr{D} \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty$ such that

$$y(\xi) = ax(\xi) - xb(\xi), \quad \xi \in \mathscr{D}. \tag{2.3.2}$$

Let $r^a_\epsilon := (\mathbf{1} + i\epsilon a)^{-1}$ and $r^b_\epsilon := (\mathbf{1} + i\epsilon b)^{-1}$, $\epsilon > 0$ be the resolvents of the operators a and b. Let us also set $a_\epsilon := ar^a_\epsilon$ and $b_\epsilon := br^b_\epsilon$. According to Theorem 1.3.16 and Lemma 1.3.18, we clearly obtain that $a_\epsilon, b_\epsilon \in \mathcal{M}_L$ and

$$\|a_\epsilon\| \le \frac{1}{\epsilon}, \quad \|b_\epsilon\| \le \frac{1}{\epsilon}, \quad \epsilon > 0.$$

Let us also note that, according to functional calculus, Theorem 1.3.16,

$$r_\epsilon^a = \mathbf{1} - i\epsilon a_\epsilon \quad \text{and} \quad r_\epsilon^b = \mathbf{1} - i\epsilon b_\epsilon, \quad \epsilon > 0. \tag{2.3.3}$$

Letting $y_\epsilon := r_\epsilon^a y r_\epsilon^b$, $\epsilon > 0$, we obtain from (2.3.2) and (2.3.3), that

$$\begin{aligned} y_\epsilon(\xi) &= r_\epsilon^a a x r_\epsilon^b(\xi) - r_\epsilon^a x b r_\epsilon^b(\xi) \\ &= a_\epsilon x(\mathbf{1} - i\epsilon b_\epsilon)(\xi) - (\mathbf{1} - i\epsilon a_\epsilon) x b_\epsilon(\xi) \\ &= a_\epsilon x(\xi) - x b_\epsilon(\xi), \quad \xi \in (r_\epsilon^b)^{-1}(\mathscr{D}), \ \epsilon > 0. \end{aligned} \tag{2.3.4}$$

Let us note that according to the definition of the resolvent operator, see page 5, we clearly have that

$$r_\epsilon^b(\mathcal{L}^2) = \mathscr{D}(b), \quad \epsilon > 0.$$

Consequently, the preimage $(r_\epsilon^b)^{-1}(\mathscr{D})$ is norm dense in $\mathcal{L}^2$ for every $\epsilon > 0$. Noting that the operator $a_\epsilon x - x b_\epsilon$ is bounded, we obtain that the identity (2.3.4) may extended uniquely over $\mathcal{L}^2$. Thus, we proved

$$y_\epsilon = a_\epsilon x - x b_\epsilon, \quad \epsilon > 0. \tag{2.3.5}$$

Recall that $y_\epsilon = r_\epsilon^a y r_\epsilon^b$. Since $r_\epsilon^a, r_\epsilon^b \in (\mathcal{M})_1$, $\epsilon > 0$ and $y \in \mathcal{L}_L^p \cap \mathcal{E}_L$, we readily obtain that the operators y_ϵ, $\epsilon > 0$ are uniformly bounded in both $\mathcal{E}_L$ and $\mathcal{L}_L^p$, i.e.

$$\|y_\epsilon\|_{\mathcal{L}_L^p} \leq \|y\|_{\mathcal{L}_L^p}, \quad \|y_\epsilon\|_{\mathcal{E}} \leq \|y\|_{\mathcal{E}_L}.$$

Applying Proposition 2.2.2 to the operator $y_\epsilon = a_\epsilon x - x b_\epsilon$, we obtain that the operators

$$z_\epsilon := f(a_\epsilon) x - x f(b_\epsilon) = T_{\psi_f}^{a_\epsilon, b_\epsilon}(a_\epsilon x - x b_\epsilon)$$

are also uniformly bounded in both $\mathcal{L}_L^p$ and $\mathcal{E}_L$, i.e.

$$\|z_\epsilon\|_{\mathcal{L}_L^p} \leq c_{f,p} \|y\|_{\mathcal{L}_L^p} \quad \text{and} \quad \|z_\epsilon\|_{\mathcal{E}_L} \leq c_{f,E} \|y\|_{\mathcal{E}_L}. \tag{2.3.6}$$

Let us note that the unit ball $(\mathcal{L}_L^p)_1$ is $\sigma(\mathcal{L}_L^p, \mathcal{L}_L^{p'})$-compact, see Theorem 1.1.1. Consequently, we may assume that the operators $\{z_\epsilon\}_{\epsilon > 0}$ converging with respect to the $\sigma(\mathcal{L}_L^p, \mathcal{L}_L^{p'})$-topology, in other words, let $z \in \mathcal{L}_L^p$ such that

$$\|z\|_{\mathcal{L}_L^p} \leq c_{f,p} \|y\|_{\mathcal{L}_L^p} \quad \text{and} \quad \sigma(\mathcal{L}_L^p, \mathcal{L}_L^{p'}) - \lim_{\epsilon \to 0} z_\epsilon = z. \tag{2.3.7}$$

Let us show that $z \in \mathcal{E}_L$. Since the space $\mathcal{E}_L$ possesses a Fatou norm, we obtain that

$$\|z\|_{\mathcal{E}_L} = \|z\|_{\mathcal{E}_L{}^{\times\times}} = \sup_{w \in (\mathcal{E}_L^\times)_1 \cap \mathcal{L}_L^1 \cap \mathcal{L}_L^\infty} \tau_L(zw).$$

Recall that z_ϵ is uniformly bounded in $\mathcal{E}_L$, see (2.3.6). Consequently,

$$\tau_L(zw) = \lim_{\epsilon\to 0} \tau_L(z_\epsilon w) \leq \max_{\epsilon>0} \|z_\epsilon\|_{\mathcal{E}_L}, \quad w \in (\mathcal{E}_L^\times)_1 \cap \mathcal{L}_L^1 \cap \mathcal{L}_L^\infty.$$

Thus,

$$z \in \mathcal{E}_L \text{ and } \|z\|_{\mathcal{E}_L} \leq c_{f,E}\|y\|_{\mathcal{E}_L}.$$

Next, we show that for $\mathscr{D}_0(f(b))$, defined in (2.3.1), we have

$$x(\mathscr{D}_0(f(b))) \subseteq \mathscr{D}(f(a)). \tag{2.3.8}$$

Fix $\xi \in \mathscr{D}_0(f(b))$. To prove (2.3.8) it is sufficient to show that the linear form

$$\eta \mapsto \langle x(\xi), \bar{f}(a)(\eta)\rangle, \quad \eta \in \mathscr{D}_0(\bar{f}(a)) \tag{2.3.9}$$

is continuous. Indeed, if this is so, then $x(\xi) \in \mathscr{D}((\bar{f}(a)|_{\mathscr{D}_0(\bar{f}(a))})^*)$ and hence, since $\mathscr{D}_0(\bar{f}(a))$ is a core of the operator $\bar{f}(a)$, $x(\xi) \in \mathscr{D}(f(a))$. According to the functional calculus, Theorem 1.3.16, we have

$$so - \lim_{\epsilon\to 0} r_\epsilon^a = \mathbf{1} \text{ and } so - \lim_{\epsilon\to 0} r_\epsilon^b = \mathbf{1}.$$

Consequently, we obtain

$$\begin{aligned}
\langle x(\xi), \bar{f}(a)(\eta)\rangle &= \lim_{\epsilon\to 0}\langle r_\epsilon^a x(\xi), \bar{f}(a)(\eta)\rangle \\
&= \langle f(a_\epsilon)x(\xi), \eta\rangle \\
&= \lim_{\epsilon\to 0}\langle (f(a_\epsilon)x - xf(b_\epsilon))(\xi), \eta\rangle \\
&+ \lim_{\epsilon\to 0}\langle xf(b_\epsilon)(\xi), \eta\rangle \\
&= \lim_{\epsilon\to 0}\langle z_\epsilon(\xi), \eta\rangle \\
&+ \langle f(b)(\xi), x^*(\eta)\rangle, \quad \eta \in \mathscr{D}_0(\bar{f}(a)).
\end{aligned} \tag{2.3.10}$$

By (2.3.1) we have

$$L_\xi L_\eta^* \in \mathcal{L}_L^1 \cap \mathcal{L}_L^\infty \subseteq \mathcal{L}_L^{p'} \text{ for every } \xi \in \mathscr{D}_0(f(b)), \ \eta \in \mathscr{D}_0(\bar{f}(a)).$$

Consequently, if $\eta \in \mathscr{D}_0(\bar{f}(a))$, it follows from (2.3.7) that

$$\lim_{\epsilon\to 0}\langle z_\epsilon(\xi), \eta\rangle = \lim_{\epsilon\to 0} \tau_L(z_\epsilon L_\xi L_\eta^*) = \tau_L(zL_\xi L_\eta^*) = \langle z(\xi), \eta\rangle.$$

Continuing (2.3.10), we then obtain that

$$\langle x(\xi), \bar{f}(a)(\eta)\rangle = \langle z(\xi), \eta\rangle + (f(b)(\xi), x^*(\eta)\rangle, \quad \eta \in \mathscr{D}_0(\bar{f}(a)). \tag{2.3.11}$$

Since the operator x is bounded, we clearly have that the linear form

$$\eta \mapsto \langle f(b)(\xi), x^*(\eta)\rangle, \quad \eta \in \mathcal{L}^2$$

is continuous. To finish the proof that the form (2.3.9) is continuous, we now need only to show that the form

$$\eta \mapsto \langle z(\xi), \eta\rangle, \quad \eta \in \mathscr{D}_0(\bar{f}(a)) \tag{2.3.12}$$

is continuous. For the latter, let us recall that $z \in \mathcal{L}^p_L$, $2 \le p \le \infty$ and $\xi \in \mathcal{L}^1 \cap \mathcal{L}^\infty$, therefore it follows from the Hölder inequality (see (1.4.8)) that

$$\begin{aligned} |\langle z(\xi), \eta\rangle| &= |\tau_L(zL_\xi L_\eta^*)| \le \|zL_\xi\|_{\mathcal{L}^2_L} \|L_\eta\|_{\mathcal{L}^2_L} \\ &\le \|z\|_{\mathcal{L}^p_L} \|L_\xi\|_{\mathcal{L}^q_L} \|\eta\|_{\mathcal{L}^2} \\ &\le \|z\|_{\mathcal{L}^p_L} \|\xi\|_{\mathcal{L}^1\cap\mathcal{L}^\infty} \|\eta\|_{\mathcal{L}^2}, \end{aligned} \tag{2.3.13}$$

where

$$\|z\|_{\mathcal{L}^p_L} \|\xi\|_{\mathcal{L}^1\cap\mathcal{L}^\infty} < \infty \quad \text{and} \quad \frac{1}{2} = \frac{1}{p} + \frac{1}{q}.$$

Consequently, the linear form (2.3.12) is continuous and therefore (2.3.9) is also continuous. Thus, we proved (2.3.8).

Since the space $\mathscr{D}_0(\bar{f}(a))$ is norm dense in $\mathcal{L}^2$, the identity (2.3.11) now turns into

$$z(\xi) = f(a)x(\xi) - xf(b)(\xi), \quad \xi \in \mathscr{D}_0(f)(b).$$

Since $\mathscr{D}_0(f)(b)\eta\mathcal{M}$, see Lemma 2.3.1, the closure of the operator on the right is z, see Lemma 1.4.15. The proof is finished. □

Let us look at the proof of Theorem 2.3.4 again. Inspecting the proof shows that there are two places where the geometry of the space $\mathcal{L}^p_L$ make the proof possible. These are

(i) the place, where we claim the existence of the limit (2.3.7), at this stage it is important that the space $\mathcal{L}^p_L$, $2 \le p \le \infty$ is dual to a noncommutative symmetric space and therefore the unit ball $(\mathcal{L}^p_L)_1$ is $\sigma(\mathcal{L}^p_L, \mathcal{L}^{p'}_L)$-compact, see Theorem 1.1.1;

(ii) the place, where we proved that the linear form (2.3.12) is continuous, here, the important tool is the Hölder inequality (1.4.8), see the chain of inequalities (2.3.13).

Further inspection shows that these two points in the proof are the only places where the special geometric properties of the space $\mathcal{L}_L^p$ are needed. Consequently, the following generalization holds

Theorem 2.3.5. *The claim of Theorem 2.3.4 remains correct, if the space $\mathcal{L}_L^p$ is replaced with any noncommutative symmetric space $\mathcal{F}$ such that*

(i) *$\mathcal{F}_L$ is dual, i.e. there is a noncommutative symmetric space $\mathcal{G}_L$ such that $\mathcal{F}_L = (\mathcal{G}_L)^*$;*

(ii) *$xy \in \mathcal{L}_L^2$, for every $x \in \mathcal{F}_L$ and every $y \in \mathcal{L}_L^1 \cap \mathcal{L}_L^\infty$.*

Recalling Theorem 1.4.22 and the inequality (1.4.16), Theorem 2.3.5 is readily applicable to the Lorentz spaces $\mathcal{L}_L^{p,q}$ with $2 \le p, q \le \infty$. Thus, we obtain the following result.

Theorem 2.3.6. *Let $\mathcal{E}_L$ be a noncommutative symmetric operator space with the Fatou property and let $2 \le p, q \le \infty$. Let $a, b\eta\mathcal{M}_L$ be self-adjoint linear operators and $x \in \mathcal{M}_L$. If*

$$ax - xb \in \mathcal{E}_L \cap \mathcal{L}_L^{p,q} \quad \text{and} \quad f \in \mathfrak{F}(\mathcal{E}_L) \cap \mathfrak{F}(\mathcal{L}_L^{p,q}),$$

then $f(a)x - xf(b) \in \mathcal{E}_L \cap \mathcal{L}_L^{p,q}$ and

$$\|f(a)x - xf(b)\|_{\mathcal{E}_L} \le c_{f,E} \|ax - xb\|_{\mathcal{E}_L},$$

where

$$c_{f,E} := \sup_{a,b\eta\mathcal{M}_L} \|T^{a,b}_{\psi_f,\mathcal{E}_L}\|_{B(\mathcal{E}_L)}.$$

Applying the latter result to $\mathcal{E}_L = \mathcal{L}_L^{p,q}$ we obtain the following corollary.

Corollary 2.3.7. *Let $a, b\eta\mathcal{M}_L$ be self-adjoint linear operators. Let $1 \le p, q \le \infty$. If either*

$$ax - xb \in \mathcal{L}_L^{p,q} \quad \text{and} \quad 2 \le p, q \le \infty,$$

or $ax - xb \in \mathcal{L}_L^{p,q} \cap \mathcal{L}_L^2$, then $f(a)x - xf(b) \in \mathcal{L}_L^{p,q}$ and

$$\|f(a)x - xf(b)\|_{\mathcal{L}_L^{p,q}} \le c_{f,p,q} \|ax - xb\|_{\mathcal{L}_L^{p,q}}.$$

The latter result will become of a particular interest when it comes to considerations of Lipschitz type estimates in L^p-spaces associated with an arbitrary von Neumann algebra.

2.3.2 Approximation of the commutator $[D, x]$

Let $\mathcal{M}$ be semi-finite von Neumann algebra acting on $\mathcal{H}$ with n.s.f. trace τ. Let $E = E(\mathbb{R})$ be a fully symmetric function space and $\mathcal{E} := E(\mathcal{M}, \tau)$ stands for the corresponding noncommutative symmetric space, in particular $\mathcal{L}^p := L^p(\mathcal{M}, \tau)$ are the noncommutative L^p-spaces.

Let $D : \mathscr{D}(D) \mapsto \mathcal{H}$ be a self-adjoint linear operator satisfying (D1)–(D2) (see page 65). In the present section we shall consider the construction of an approximation of the commutator $[D, x]$ by means of the corresponding unitary group $\{e^{itD}\}_{t\in\mathbb{R}}$.

For illustration of the aforementioned approximation let us again consider the example of the differentiation operator. If $x \in L^\infty(\mathbb{R})$ and $D = \frac{1}{i}\frac{d}{dt}$, then we have the well known relations

$$x(t+s) - x(s) = i \int_0^t \frac{1}{i}\frac{dx}{dt}(s+\tau)\, d\tau, \quad t, s \in \mathbb{R}, \tag{2.3.14}$$

$$\frac{1}{i}\frac{dx}{dt}(s) = \lim_{t\to 0} \frac{x(s+t) - x(s)}{it}. \tag{2.3.15}$$

The aim of the present section is the extension of the latter relations over an arbitrary operator D satisfying (D1)–(D2) (see page 65) and an arbitrary semi-finite pair $(\mathcal{M}, \tau)$.

Before we prove the relations above in general setting, let us study behavior of the group $x \mapsto e^{itD}xe^{-itD}$, $x \in \mathcal{L}^\infty$, $t \in \mathbb{R}$.

2.3.3 The group $(t, x) \mapsto e^{itD}xe^{-itD}$

Let us consider the group of trace preserving $*$-automorphisms $\gamma = \{\gamma_t\}_{t\in\mathbb{R}}$ of the algebra $\mathcal{M}$ defined by

$$\gamma_t(x) := e^{itD}xe^{-itD}, \quad x \in \mathcal{M}. \tag{2.3.16}$$

According to (D1)–(D2) (see page 65), the operator $\gamma_t : \mathcal{M} \mapsto \mathcal{M}$ is a trace preserving $*$-automorphism, for every $t \in \mathbb{R}$. Let us consider the group $\tilde{\gamma} = \{\tilde{\gamma}_t\}_{t\in\mathbb{R}}$ and the group $\gamma^E = \{\gamma^E_t\}_{t\in\mathbb{R}}$ which are the unique extensions of the group γ to the algebra $\tilde{\mathcal{M}}$ and the space $\mathcal{E}$, respectively, see Section 1.4.2. We also set $\gamma^p := \gamma^{L^p}$, $1 \le p \le \infty$ for brevity. It is follows from Lemma 1.4.17 that the group γ^E is a group of isometries of the Banach space $\mathcal{E}$.

In the present section, we state two results in regard to continuity properties of the group γ^E.

Theorem 2.3.8 ([21, Proposition 4.2])**.** *Let E be a fully symmetric function space and let $\mathcal{E}$ be the corresponding non-commutative space of τ-measurable operators. The group γ^E is $\sigma(\mathcal{E}, \mathcal{E}^\times)$-continuous.*

Clearly, the result above together with Theorem 1.1.5 implies that

Corollary 2.3.9. (i) *If a symmetric function space E is separable, then the group γ^E is strongly continuous.*

(ii) *If a symmetric function space E is dual to a separable symmetric function space, then the group γ^E is weakly* continuous.*

(iii) *The group γ^∞ is weakly* continuous.*

(iv) *The group γ^p, $1 \le p < \infty$ is strongly continuous.*

Let us note that, for $p < \infty$, the group γ^p is defined in abstract way. It cannot be regarded as

$$\gamma_t^p(x) = e^{itD} x e^{-itD}, \quad x \in \mathcal{L}^2, \tag{2.3.17}$$

due to the fact that the operator D is now affiliated with the algebra $\mathcal{M}$. In the second part of the present section, we shall show that the identity (2.3.17) is valid when the algebra $\mathcal{M}$ is taken in its left regular representation.

Let $\mathcal{M}_L$ be the left regular representation of the semi-finite von Neumann algebra $\mathcal{M}$. The algebra $\mathcal{M}_L$ is equipped with the n.s.f. trace τ_L. The space $\mathcal{L}_L^p = L^p(\mathcal{M}_L, \tau_L)$, $1 \le p \le \infty$ is a noncommutative L^p-space associated with the couple $(\mathcal{M}_L, \tau_L)$. Let $D : \mathscr{D}(D) \mapsto \mathcal{L}^2$ be a self-adjoint linear operator satisfying (D1)–(D2) (see page 65).

Lemma 2.3.10. *Let $y \in \mathcal{L}_L^p$. Let $\xi \in \mathcal{L}^1 \cap \mathcal{L}^\infty$ be such that $e^{-itD}(\xi) \in \mathscr{D}(y)$. If $\gamma^p = \{\gamma_t^p\}_{t\in\mathbb{R}}$ is a unique continuous extension of the group γ defined in* (2.3.16) *to the space $\mathcal{L}_L^p$, then*

$$e^{itD} y e^{-itD}(\xi) = \gamma_t^p(y)(\xi). \tag{2.3.18}$$

Proof. Since $y \in \mathcal{L}_L^p$, there is a collection $\{e_n^y\}_{n=1}^\infty$ of spectral projections such that $y_n := y e_n^y \in \mathcal{L}_L^1 \cap \mathcal{L}_L^\infty$ and

$$\lim_{n\to\infty} y_n = y \quad \text{and} \quad \lim_{n\to\infty} y_n(\xi) = y(\xi), \quad \xi \in \mathscr{D}(y), \tag{2.3.19}$$

where the first limit converges with respect to the norm topology of $\mathcal{L}_L^p$ (see Theorem 1.4.12) and the second one — with respect to the norm topology in $\mathcal{L}^2$ (see Theorem 1.3.14).

Since $y_n \in \mathcal{L}^1_L \cap \mathcal{L}^\infty_L$ and γ^p coincides with (2.3.16) on $\mathcal{L}^1_L \cap \mathcal{L}^\infty_L$, it is clear that identity (2.3.18) is valid for y_n, i.e. we have

$$L(e^{itD} y_n e^{-itD}(\xi)) = \gamma^p_t(y_n) L_\xi. \tag{2.3.20}$$

On the other hand, since $e^{-itD}(\xi) \in \mathscr{D}(y)$ and using the second limit of (2.3.19), for the left hand side of the latter identity, we obtain that

$$\lim_{n\to\infty} L(e^{itD} y_n e^{-itD}(\xi)) = L(e^{itD} y e^{-itD}(\xi)),$$

where the limit is taken with respect to the norm topology in $\mathcal{L}^2_L$. Furthermore, since $L_\xi \in \mathcal{L}^\infty_L$ and the first limit of (2.3.19), for the right hand side of (2.3.20), it follows that

$$\lim_{n\to\infty} \gamma^p_t(y_n) L_\xi = \gamma^p(y) L_\xi,$$

where the limit is taken with respect to the norm topology in $\mathcal{L}^p_L$. Moreover, the latter two limits certainly converge with respect to the $\sigma(\mathcal{L}^1_L + \mathcal{L}^\infty_L, \mathcal{L}^1_L \cap \mathcal{L}^\infty_L)$-topology. Let us recall that $\mathcal{L}^1_L \cap \mathcal{L}^\infty_L$ separates points in $\mathcal{L}^1_L + \mathcal{L}^\infty_L$, see Lemma 1.4.10. Consequently, combining the limits above with (2.3.20), we readily see that

$$L(e^{itD} y e^{-itD}(\xi)) = \gamma^p_t(y) L_\xi.$$

The latter implies (2.3.18) (see Lemma 1.5.6). □

From now on, the symbol δ^E stands for the weak* (resp. strong) generator of the group γ^E, provided the space $\mathcal{E}$ is dual to a noncommutative symmetric space (resp. the space $\mathcal{E}$ has order-continuous norm). In particular, δ^p, $1 \leq p \leq \infty$, is the generator of the group γ^p.

2.3.4 Approximation of the commutator $[D, x]$ in $\mathcal{L}^\infty$

Let us now study the identities (2.3.14) and (2.3.15) in the general setting. Let us first show that the integral identity (2.3.14) implies the relation (2.3.15).

Let us recall that $\mathcal{M}$ is a semi-finite von Neumann algebra acting on $\mathcal{H}$ and equipped with n.s.f. trace τ. Recall that $D : \mathscr{D}(D) \mapsto \mathcal{H}$ is a self-adjoint linear operator satisfying (D1)–(D2) (see page 65) and $x \in \mathcal{M}$.

Lemma 2.3.11. *Let E be a fully symmetric function space and $\mathcal{E}$ be the corresponding operator space. Let $x \in \mathcal{M}$ and $[D, x] \in \mathcal{E}$. If the identity*

$$e^{itD} x e^{-itD} - x = i \int_0^t \gamma^E_s([D, x])\, ds \tag{2.3.21}$$

holds, where the integral converges with respect to the $\sigma(\mathcal{E}, \mathcal{E}^\times)$-topology, then

(i) $\left\| \dfrac{e^{itD}xe^{-itD} - x}{t} \right\|_{\mathcal{E}} \le \|[D,x]\|_{\mathcal{E}}$;

(ii) $\lim\limits_{t\to 0} \dfrac{e^{itD}xe^{-itD} - x}{t} = i[D,x]$, *where the limit converges with respect to the* $\sigma(\mathcal{E},\mathcal{E}^\times)$*-topology.*

If the space E *is separable and the integral in* (2.3.21) *converges with respect to the norm topology in* E*, then so does the limit.*

Proof. Let us show the proof for the norm topology; for the $\sigma(\mathcal{E},\mathcal{E}^\times)$-topology the proof is similar. The function

$$t \mapsto \gamma_t^E(x), \quad t \in \mathbb{R} \tag{2.3.22}$$

is uniformly bounded for every $x \in \mathcal{E}$. Consequently, the identity (2.3.21) implies that

$$\|e^{itD}xe^{-itD} - x\|_{\mathcal{E}} \le t \max_{s\in[0,t]} \|\gamma_s^E([D,x])\|_{\mathcal{E}} \le t\,\|[D,x]\|_{\mathcal{E}}.$$

The latter implies (i). For (ii) let us consider the function

$$G(t) := i\int_0^t \gamma_s^E([D,x])\,ds.$$

It then follows from (2.3.21) that

$$\lim_{t\to 0} \frac{e^{itD}xe^{-itD} - x}{t} = \frac{dG}{dt}(0).$$

Since the function (2.3.22) is norm continuous, the claim (ii) follows from the Newton-Leibniz theorem. ☐

Let us first consider the case $\mathcal{E} = \mathcal{L}^\infty$.

Theorem 2.3.12. *Let* $D : \mathscr{D}(D) \mapsto \mathcal{H}$ *be a self-adjoint linear operator, satisfying (D1)–(D2) (see page 65) and let* $x \in \mathcal{M}$*. If* $[D,x] \in \mathcal{L}^\infty$*, then*

(i) $\gamma_t^\infty(x) - x = i\displaystyle\int_0^t \gamma_s^\infty([D,x])\,ds,\ t \in \mathbb{R}$;

(ii) $\left\| \dfrac{\gamma_t^\infty(x) - x}{t} \right\|_{\mathcal{L}^\infty} \le \|[D,x]\|_{\mathcal{L}^\infty}$;

(iii) $wo - \lim\limits_{t\to 0} \dfrac{\gamma_t^\infty(x) - x}{t} = i[D,x]$;

where the integral converges with respect to the wo-topology.

Proof. It immediately follows from Lemma 2.3.11 that (i) implies (ii)–(iii).

Let us prove (i). Clearly, the group γ^∞ coincides with the initial group γ defined in (2.3.16). We fix $t \in \mathbb{R}$ and introduce the operators

$$T := e^{itD} x e^{-itD} - x, \quad S := i \int_0^t e^{itD}[D, x] e^{-itD}\, ds.$$

To prove (i), it is sufficient to establish that

$$\langle T(\xi), \eta \rangle = \langle S(\xi), \eta \rangle, \quad \xi \in \mathscr{D}(D), \eta \in \mathcal{H}. \tag{2.3.23}$$

Indeed, since the operators T and S are bounded and $\mathscr{D}(D)$ is dense in $\mathcal{H}$, the identity (2.3.23) will imply (i).

Let us fix $\xi \in \mathscr{D}(D)$ and $\eta \in \mathcal{H}$. For the right hand side of (2.3.23), let us note that, since the function

$$t \to e^{itD}[D, x] e^{-itD}, \quad t \in \mathbb{R} \tag{2.3.24}$$

is *wo*-continuous, see Corollary 2.3.9, the scalar product and the integration may be interchanged and hence

$$\langle S(\xi), \eta \rangle = i \int_0^t \langle e^{isD}[D, x] e^{-isD}(\xi), \eta \rangle\, ds. \tag{2.3.25}$$

Let us consider the left hand side. The function

$$u(t) := e^{itD} x e^{-itD}(\xi), \quad t \in \mathbb{R}$$

satisfies the elementary identity

$$\begin{aligned} \frac{u(t+s) - u(t)}{s} &= e^{i(t+s)D} x \frac{e^{-isD}(e^{-itD}(\xi)) - e^{-itD}(\xi)}{s} \\ &+ e^{itD} \frac{e^{isD} x e^{-itD}(\xi) - x e^{-itD}(\xi)}{s}, \quad t \in \mathbb{R}, \quad s \neq 0. \end{aligned}$$

Since $[D, x] \in \mathcal{L}^\infty$, we have $x(\mathscr{D}(D)) \subseteq \mathscr{D}(D)$ (see Lemma 2.0.8). Let us also note that the group $\{e^{itD}\}_{t \in \mathbb{R}}$ acts invariantly on the space $\mathscr{D}(D)$, i.e. $e^{itD}(\mathscr{D}(D)) \subseteq \mathscr{D}(D)$, for every $t \in \mathbb{R}$, see Lemma 1.1.4. Consequently,

$$\begin{aligned} u'(t) &= \lim_{s \to 0} \frac{u(t+s) - u(t)}{s} \\ &= e^{itD} x(-iDe^{-itD}(\xi)) + e^{itD}(iD)(xe^{-itD}(\xi)) \\ &= ie^{itD}[D, x] e^{-itD}(\xi), \quad t \in \mathbb{R}. \end{aligned}$$

According to the Newton-Leibniz theorem, since the function (2.3.24) is *wo*-continuous, we obtain

$$\begin{aligned}\langle e^{itD}xe^{-itD}(\xi),\eta\rangle - \langle x(\xi),\eta\rangle &= \langle u(t),\eta\rangle - \langle u(0),\eta\rangle \\ &= \int_0^t u'(s)\,ds \\ &= i\int_0^t \langle e^{isD}[D,x]e^{-isD}(\xi),\eta\rangle\,ds.\end{aligned}$$

Together with (2.3.25), the latter gives (2.3.23). The theorem is proved. □

Let us note the following converse statement.

Theorem 2.3.13. *Let $D : \mathscr{D}(D) \mapsto \mathcal{H}$ be a self-adjoint linear operator, satisfying (D1)–(D2) (see page 65) and let $x \in \mathcal{M}$. If the limit*

$$wo - \lim_{t\to 0} \frac{e^{itD}xe^{-itD} - x}{t} \tag{2.3.26}$$

exists, then $[D,x] \in \mathcal{L}^\infty$ and

$$wo - \lim_{t\to 0} \frac{e^{itD}xe^{-itD} - x}{t} = i[D,x].$$

Proof. Let us assume that the limit (2.3.26) is equal to $y \in \mathcal{L}^\infty$. We shall first prove that

$$x(\mathscr{D}(D)) \subseteq \mathscr{D}(D). \tag{2.3.27}$$

To this end, let us fix $\xi \in \mathscr{D}(D)$ and consider the linear form

$$\eta \mapsto \langle x(\xi), D(\eta)\rangle, \quad \eta \in \mathscr{D}(D). \tag{2.3.28}$$

Clearly, we have

$$\begin{aligned}\langle x(\xi), D(\eta)\rangle &= \lim_{t\to 0}\left\langle x(\xi), \frac{e^{itD}(\eta)-\eta}{it}\right\rangle \\ &= \lim_{t\to 0}\left\langle \frac{e^{itD}x(\xi)-x(\xi)}{it}, \eta\right\rangle \\ &= \lim_{t\to 0}\left\langle \frac{e^{itD}xe^{-itD}(\xi)-x(\xi)}{it}, \eta\right\rangle \\ &\quad - \lim_{t\to 0}\left\langle e^{itD}x\frac{e^{-itD}(\xi)-\xi}{it}, \eta\right\rangle \\ &= -\,i\,\langle y(\xi),\eta\rangle + \langle xD(\xi),\eta\rangle.\end{aligned}$$

Thus, since the operator y is bounded and $\xi \in \mathscr{D}(D)$, we obtain that the linear form (2.3.28) is continuous. This implies that $x(\xi) \in \mathscr{D}(D)$ and therefore (2.3.27) is established. Furthermore, the latter identity says that

$$-i\,\langle y(\xi),\eta\rangle = \langle Dx(\xi) - xD(\xi),\eta\rangle,\quad \xi,\eta \in \mathscr{D}(D).$$

Consequently, the operator $Dx-xD$ defined on $\mathscr{D}(D)$ is closable and the closure is $-iy$. The theorem is proved. □

Theorem 2.3.12 and Theorem 2.3.13 give the complete description of the generator $\delta^\infty : \mathscr{D}(\delta^\infty) \mapsto \mathcal{L}^\infty$ of the weak* continuous group γ^∞ in terms of commutators with the operator D. Namely,

$$\mathscr{D}(\delta^\infty) = \{x \in \mathcal{L}^\infty : \quad [D,x] \in \mathcal{L}^\infty\} \tag{2.3.29}$$

and

$$\delta^\infty(x) = i[D,x],\quad x \in \mathscr{D}(\delta^\infty). \tag{2.3.30}$$

See also the comments at the end of this chapter.

2.3.5 Approximation of the commutator $[D,x]$ in $\mathcal{L}^p$, provided $1 \le p < \infty$

Let $\mathcal{M}$ be a semi-finite von Neumann algebra and τ be n.s.f. trace τ. As we discussed in Section 2.1, the natural framework to deal with the commutator $[D,x] \in \mathcal{L}^p$, when $1 \le p < \infty$, is the setting of the left regular representation. Let $(\mathcal{M}_L, \tau_L)$ be the corresponding left regular representation. $\mathcal{E}_L := E(\mathcal{M}_L, \tau_L)$ stands for the noncommutative symmetric space corresponding to the fully symmetric function space $E = E(\mathbb{R})$, in particular, $\mathcal{L}^p_L := L^p(\mathcal{M}_L, \tau_L)$. In the present section, we shall extend Theorem 2.3.12 to the spaces $\mathcal{L}^p_L$, $1 \le p < \infty$.

Before we proceed, let us make the following two remarks which explain the our next step.

(i) The important point in the proof of Theorem 2.3.12 is the fact that the domain $\mathscr{D}(D)$, where the operator $[D,x] \in \mathcal{L}^\infty_L$ is defined initially, is invariant with respect to the group $\{e^{itD}\}_{t\in\mathbb{R}}$. On the other hand, if we have the commutator $[D,x] \in \mathcal{L}^p_L$ and $p < \infty$, then the core $\mathscr{D}$, where, according to Definition 2.0.6, the operator $[D,x]$ is initially defined, lacks this invariance.

(ii) The core $\mathscr{D}$ where the operator $[D, x]$ is defined initially (see again Definition 2.0.6) is merely related to the Banach space structure of the spaces $\mathcal{L}^p_L$.

Keeping these two remarks in mind, we modify Definition 2.0.6 of the symbol $[D, x]$ in the setting of the left regular representation as follows.

Definition 2.3.14. We shall say that the commutator $[D, x]$ is defined and belongs to $\mathcal{E}_L$ if and only if

(i) there is a core $\mathscr{D} \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty$ of D which is invariant under e^{itD} for every $t \in \mathbb{R}$, i.e. $e^{itD}(\mathscr{D}) \subseteq \mathscr{D}$, such that $x(\mathscr{D}) \subseteq \mathscr{D}(D)$;

(ii) the operator $Dx - xD$, defined on $\mathscr{D}$, is closable;

(iii) the closure $\overline{Dx - xD}$ belongs to $\mathcal{E}_L$.

The symbol $[D, x]$ stands for the closure $\overline{Dx - xD}$.

It follows from Section 2.1 that, for the special case $\mathcal{M} = L^\infty$ and $D = \frac{1}{i}\frac{d}{dt}$, the commutator $[D, x]$ is defined and belongs to L^p, for some $1 \le p \le \infty$, for some $x \in L^\infty$, if and only if $\frac{1}{i}\frac{dx}{dt} \in L^p$ and either $2 \le p \le \infty$ or there is a core $\mathscr{D} \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty$ of the operator $\frac{1}{i}\frac{d}{dt}$ such that

$$\frac{1}{i}\frac{dx}{dt}(\mathscr{D}) \subseteq \mathcal{L}^2.$$

Recall that δ^p, $1 \le p \le \infty$ stands for the generator of the group γ^p, see Section 2.3.3.

Theorem 2.3.15. *Let $D : \mathscr{D}(D) \mapsto \mathcal{L}^2$ be a self-adjoint linear operator satisfying (D1)–(D2) (see page 65) and let $x \in \mathcal{M}_L$. Let*

$$R_{\lambda,q} := R_\lambda(\delta^q), \quad 1 \le q \le \infty, \ \lambda > 0$$

be the resolvent of the operator δ^q. If $[D, x] \in \mathcal{L}^p_L$, for some $1 \le p < \infty$, then

$$[D, R_{\lambda,\infty}(x)] = R_{\lambda,p}([D, x]), \quad \lambda > 0. \tag{2.3.31}$$

Proof. According to Definition 2.3.14, since $[D, x] \in \mathcal{L}^p_L$, we have a core $\mathscr{D} \subseteq \mathscr{D}(D)$ of D such that

$$\mathscr{D} \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty, \quad e^{itD}(\mathscr{D}) \subseteq \mathscr{D}, \ t \in \mathbb{R} \ \text{ and } \ x(\mathscr{D}) \subseteq \mathscr{D}(D).$$

We fix $t \in \mathbb{R}$. Let us first show that

$$[D, \gamma_t^\infty(x)](\xi) = \gamma_t^p([D, x])(\xi), \quad \xi \in \mathscr{D}. \tag{2.3.32}$$

Since the group γ^∞ coincides with (2.3.16) and $x(\mathscr{D}) \subseteq \mathscr{D}(D)$, we clearly have that

$$[D, \gamma_t^\infty(x)](\xi) = e^{itD}[D, x]e^{-itD}(\xi), \quad \xi \in \mathscr{D}. \tag{2.3.33}$$

Thus, the identity (2.3.32) immediately follows from Lemma 2.3.10.

Now let us prove (2.3.31). According to Theorem 1.1.6, the resolvent $R_{\lambda,q}$ admits the representation

$$R_{\lambda,q}(y) = \int_0^\infty e^{-\lambda t}\gamma_t^q(y)\,dt, \quad y \in \mathcal{L}_L^q, \quad 1 \le q \le \infty. \tag{2.3.34}$$

If $q = \infty$ the latter integral converges with respect to the wo-topology; otherwise, it converges with respect to the norm topology. Let us emphasize that the integral (2.3.34) converges uniformly with respect to $y \in \mathcal{L}_L^q$ and $1 \le q \le \infty$. The latter means, that, for every $n \ge 1$, there is a finite partition $\{t_k^{(n)}\}_{k=1}^n$ and Riemannian sum

$$R_\lambda^{(n)}(y) := \sum_{k=2}^n e^{-\lambda t_k^{(n)}} \gamma^q_{t_k^{(n)}}(y)\,(t_k^{(n)} - t_{k-1}^{(n)})$$

such that

$$\lim_{n\to\infty} R_\lambda^{(n)}(y) = R_{\lambda,q}(y) \tag{2.3.35}$$

uniformly with respect to $y \in \mathcal{L}_L^q$ and $1 \le q \le \infty$. Clearly, the latter limit converges with respect to the wo-topology if $q = \infty$; otherwise, it converges with respect to the norm topology $\mathcal{L}_L^q$.

For the sake of brevity, let us set $x_\lambda := \lambda R_{\lambda,\infty}(x)$ and $x_\lambda^{(n)} := \lambda R_\lambda^{(n)}(x)$. It follows from (2.3.35) that

$$wo - \lim_{n\to\infty} x_\lambda^{(n)} = x_\lambda. \tag{2.3.36}$$

Let us consider the linear form

$$\phi_\lambda^{(n)}(\xi, \eta) := \langle x_\lambda^{(n)}(\xi), D(\eta)\rangle - \langle x_\lambda^{(n)} D(\xi), \eta\rangle, \quad \xi, \eta \in \mathscr{D}(D).$$

Relation (2.3.36) implies,

$$\lim_{n\to\infty} \phi_\lambda^{(n)}(\xi, \eta) = \langle [D, x_\lambda](\xi), \eta\rangle, \quad \xi \in \mathscr{D}, \; \eta \in \mathscr{D}(D).$$

On the other hand, it is an easy computation to see that

$$\phi_\lambda^{(n)}(\xi,\eta) = \langle \lambda R_\lambda^{(n)}([D,x])(\xi),\eta\rangle, \quad \xi\in\mathscr{D},\ \eta\in\mathscr{D}(D).$$

Indeed, for the latter identity, it is sufficient to replace t with $t_k^{(n)}$ in (2.3.32), multiply the identity with $e^{-\lambda t_k^{(n)}}(t_k^{(n)} - t_{k-1}^{(n)})$ and take the sum over all $2 \le k \le n$. Thus, we obtain that

$$\lim_{n\to\infty}\langle \lambda R_\lambda^{(n)}([D,x])(\xi),\eta\rangle = \langle [D,x_\lambda](\xi),\eta\rangle, \quad \xi\in\mathscr{D},\ \eta\in\mathscr{D}(D). \tag{2.3.37}$$

On the other hand, it follows from (2.3.35), that

$$\lim_{n\to\infty}\tau_L(w\,\lambda R_\lambda^{(n)}([D,x])) = \tau_L(w\,\lambda R_{\lambda,p}([D,x])), \quad w\in\mathcal{L}_L^{p'}.$$

If $\xi,\eta\in\mathscr{D}\subseteq\mathcal{L}^1\cap\mathcal{L}^\infty$, then $L_\xi L_\eta^*\in\mathcal{L}_L^{p'}$. Consequently,

$$\begin{aligned}\lim_{n\to\infty}\langle \lambda R_\lambda^{(n)}([D,x])(\xi),\eta\rangle &= \lim_{n\to\infty}\tau_L(L_\eta^* L_\xi\,\lambda R_\lambda^{(n)}([D,x]))\\ &= \tau_L(L_\eta^* L_\xi\,\lambda R_{\lambda,p}([D,x]))\\ &= \langle \lambda R_{\lambda,p}([D,x])(\xi),\eta\rangle.\end{aligned}$$

Combining the latter limit with (2.3.37), we obtain that

$$\langle \lambda R_{\lambda,p}([D,x])(\xi),\eta\rangle = \langle [D,x_\lambda](\xi),\eta\rangle, \quad \xi,\eta\in\mathscr{D}.$$

Since the operator on the right hand side is bounded and the core $\mathscr{D}$ is dense in $\mathcal{L}^2$, we obtain that the operator $\lambda R_{\lambda,p}([D,x])$ is also bounded and

$$\lambda R_{\lambda,p}([D,x]) = [D,x_\lambda], \quad \lambda > 0.$$

The theorem is completely proved. □

Theorem 2.3.16. *Let $D:\mathscr{D}(D)\mapsto\mathcal{L}^2$ be a self-adjoint linear operator, satisfying (D1)–(D2) (see page 65) and let $x\in\mathcal{M}_L$. If $[D,x]\in\mathcal{L}_L^p$, $1\le p<\infty$, then*

(i) $e^{itD}xe^{-itD} - x = i\displaystyle\int_0^t \gamma_s^p([D,x])\,ds$, $t\in\mathbb{R}$;

(ii) $\left\|\dfrac{e^{itD}xe^{-itD}-x}{t}\right\|_{\mathcal{L}_L^p} \le \|[D,x]\|_{\mathcal{L}_L^p}$;

(iii) $\displaystyle\lim_{t\to 0}\frac{e^{itD}xe^{-itD}-x}{t} = i[D,x]$;

where the integral and the limit converge with respect to the norm topology in $\mathcal{L}_L^p$.

Proof. According to Lemma 2.3.11, it is sufficient to prove only the claim (i). Let

$$R_{\lambda,q} := R_\lambda(\delta^q), \quad 1 \le q \le \infty, \ \lambda > 0$$

be the resolvent of the generator δ^q of the group γ^q. Let us fix $\lambda > 0$. The operator $R_{\lambda,\infty}(x)$ belongs to the domain $\mathscr{D}(\delta^\infty)$, see Theorem 1.1.6. Consequently, it follows from (2.3.29) that $[D, R_{\lambda,\infty}(x)] \in \mathcal{L}^\infty_L$. Thus, we are in a position to apply Theorem 2.3.12, which readily gives that

$$\begin{aligned} \lambda R_{\lambda,\infty}(e^{itD} x e^{-itD} - x) &= \gamma^\infty_t(\lambda R_{\lambda,\infty}(x)) - \lambda R_{\lambda,\infty}(x) \\ &= i \int_0^t \gamma^\infty_s([D, R_{\lambda,\infty}(x)])\, ds. \end{aligned} \tag{2.3.38}$$

Furthermore, it follows from Theorem 2.3.15 that

$$\gamma^\infty_s([D, R_{\lambda,\infty}(x)]) = \gamma^\infty_s(R_{\lambda,p}([D,x])) = R_{\lambda,p}(\gamma^p_s([D,x])), \quad s \in \mathbb{R}. \tag{2.3.39}$$

Let us note that the operator $R_{\lambda,p}$ is continuous on the space $\mathcal{L}^p_L$ and the integral

$$\int_0^t \gamma^p_s([D,x])\, ds$$

converges with respect to the norm topology of the space $\mathcal{L}^p_L$. Consequently, combining (2.3.38) and (2.3.39), we obtain that

$$\lambda R_{\lambda,\infty}(e^{itD} x e^{-itD} - x) = i \lambda R_{\lambda,p}\left(\int_0^t \gamma^p_s([D,x])\, ds \right).$$

The only step we need to finish the proof is to let $\lambda \to \infty$. The claim (i) readily follows from Theorem 1.1.6. □

2.3.6 Commutator estimates

Let us recall that we have fixed the pair $(\mathcal{M}, \tau)$ and we are in the setting of the left regular representation $(\mathcal{M}_L, \tau_L)$. Let $D : \mathscr{D}(D) \mapsto \mathcal{L}^2$ be a linear self-adjoint operator satisfying (D1)–(D2) (see page 65).

Let us again consider the subspace

$$\mathscr{D}_0(D) := \mathscr{D}(D) \cap \mathcal{L}^1 \cap \mathcal{L}^\infty \subseteq \mathcal{L}^2. \tag{2.3.40}$$

Unfortunately, in the general case when the operator D is not affiliated with the algebra $\mathcal{M}_L$, there is no hope to expect that the latter subspace is a core of the operator D. On the other hand, as soon as the operator D is affiliated with the algebra $\mathcal{M}_L$, according to Lemma 2.3.1, the subspace $\mathscr{D}_0(D)$ turns into a core. The key point in the proof of Lemma 2.3.1 is the following hypothesis.

(D3) The subspace $\mathcal{L}^1 \cap \mathcal{L}^\infty \subseteq \mathcal{L}^2$ is invariant with respect to $\{e^{itD}\}_{t\in\mathbb{R}}$, i.e.

$$e^{itD}(\mathcal{L}^1 \cap \mathcal{L}^\infty) \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty, \quad t \in \mathbb{R};$$

the operator $e^{itD}|_{\mathcal{L}^1\cap\mathcal{L}^\infty}$ is continuous with respect to the norms of the spaces $\mathcal{L}^1$ and $\mathcal{L}^\infty$, i.e.

$$\|e^{itD}(\xi)\|_{\mathcal{L}^1} \le \|\xi\|_{\mathcal{L}^1} \text{ and } \|e^{itD}(\xi)\|_{\mathcal{L}^\infty} \le \|\xi\|_{\mathcal{L}^\infty}, \ \xi \in \mathcal{L}^1\cap\mathcal{L}^\infty; \quad (2.3.41)$$

and the group

$$u = \{u_t\}_{t\in\mathbb{R}}, \quad u_t := e^{itD}|_{\mathcal{L}^1\cap\mathcal{L}^\infty}, \quad t \in \mathbb{R} \qquad (2.3.42)$$

is $\sigma(\mathcal{L}^1 \cap \mathcal{L}^\infty, \mathcal{L}^1 + \mathcal{L}^\infty)$-continuous, i.e.

(i) for every $\xi \in \mathcal{L}^1 \cap \mathcal{L}^\infty$, the mapping

$$t \mapsto u_t(\xi), \quad t \in \mathbb{R}$$

is $\sigma(\mathcal{L}^1 \cap \mathcal{L}^\infty, \mathcal{L}^1 + \mathcal{L}^\infty)$-continuous;

(ii) for every $t \in \mathbb{R}$, the mapping

$$\xi \mapsto u_t(\xi), \quad x \in \mathcal{L}^1 \cap \mathcal{L}^\infty$$

is also $\sigma(\mathcal{L}^1 \cap \mathcal{L}^\infty, \mathcal{L}^1 + \mathcal{L}^\infty)$-continuous.

If the operator D is affiliated with $\mathcal{M}_L$, then $\{e^{itD}\}_{t\in\mathbb{R}} \subseteq \mathcal{M}_L$. The latter means that there is a strongly continuous group of unitaries $\{w_t\}_{t\in\mathbb{R}} \in \mathcal{M}$ such that $e^{itD} = L(w_t)$, $t \in \mathbb{R}$. Consequently, it is clear that the operator $D\eta\mathcal{M}_L$ satisfies (D3). On the other hand, if $D = \frac{1}{i}\frac{d}{dt}$, then the group $\{e^{itD}\}_{t\in\mathbb{R}}$ is a group of translations, i.e. $e^{itD} = \tau_t$, $t \in \mathbb{R}$, see Lemma 2.1.1.(ii). Thus, the operator $D = \frac{1}{i}\frac{d}{dt}$ satisfies (D3) and therefore the condition (D3) is weaker than the assumption that $D\eta\mathcal{M}_L$.

The assumption (D3) implies that the group $\{e^{itD}\}_{t\in\mathbb{R}}$ is *a diffusion group*, see [35, 60].

Lemma 2.3.17. *Let $D : \mathscr{D}(D) \mapsto \mathcal{L}^2$ be a linear self-adjoint operator satisfying (D1)–(D3). Let the group $u = \{u_t\}_{t\in\mathbb{R}} \subseteq B(\mathcal{L}^1 \cap \mathcal{L}^\infty)$ satisfy* (2.3.42).

(i) *The adjoint operator u_t^* leaves the space $\mathcal{L}^1 + \mathcal{L}^\infty$ invariant, i.e.*

$$u_t^*(\mathcal{L}^1 + \mathcal{L}^\infty) \subseteq \mathcal{L}^1 + \mathcal{L}^\infty, \quad t \in \mathbb{R}.$$

(ii) *Let us set*

$$v_t := u^*_{-t}|_{\mathcal{L}^1+\mathcal{L}^\infty}, \quad t \in \mathbb{R}. \tag{2.3.43}$$

The collection $v = \{v_t\}_{t\in\mathbb{R}} \subseteq B(\mathcal{L}^1 + \mathcal{L}^\infty)$ *is a group of contractions.*

(iii) *for every* $t \in \mathbb{R}$*, the mapping*

$$\xi \mapsto v_t(\xi), \quad \xi \in \mathcal{L}^1 + \mathcal{L}^\infty$$

is $\sigma(\mathcal{L}^1 + \mathcal{L}^\infty, \mathcal{L}^1 \cap \mathcal{L}^\infty)$*-continuous.*

(iv) *For every* $\xi \in \mathcal{L}^1 + \mathcal{L}^\infty$*, the mapping*

$$t \mapsto v_t(\xi), \quad t \in \mathbb{R}$$

is $\sigma(\mathcal{L}^1 + \mathcal{L}^\infty, \mathcal{L}^1 \cap \mathcal{L}^\infty)$*-continuous;*

(v) *The group* v *extends the group* $\{e^{itD}\}_{t\in\mathbb{R}}$*, i.e.*

$$v_t(\xi) = e^{itD}(\xi), \quad \xi \in \mathcal{L}^2, \; t \in \mathbb{R}.$$

Thus, the group v *is a* $\sigma(\mathcal{L}^1 + \mathcal{L}^\infty, \mathcal{L}^1 \cap \mathcal{L}^\infty)$*-continuous group of contractions in the space* $\mathcal{L}^1 + \mathcal{L}^\infty$ *which extends the unitary group of the operator* D*.*

Proof. Let us note that the space $\mathcal{L}^1 + \mathcal{L}^\infty$ may be regarded as a norm closed subspace in the dual $(\mathcal{L}^1 \cap \mathcal{L}^\infty)^*$. Thus, the claims (i), (ii) and (iii) immediately follow from Lemma 1.1.3, which we can apply thanks to the assumption (D3).

(iv) We have

$$\begin{aligned}\lim_{t\to 0}\langle v_t(\xi), \eta\rangle &= \lim_{t\to 0}\langle u^*_{-t}(\xi), \eta\rangle \\ &= \lim_{t\to 0}\langle \xi, u_{-t}(\eta)\rangle \\ &= \langle \xi, \eta\rangle, \quad \xi \in \mathcal{L}^1 + \mathcal{L}^\infty, \; \eta \in \mathcal{L}^1 \cap \mathcal{L}^\infty.\end{aligned}$$

The latter identity is due to the fact that the group u is $\sigma(\mathcal{L}^1 \cap \mathcal{L}^\infty, \mathcal{L}^1 + \mathcal{L}^\infty)$-continuous. Consequently, the claim (iv) readily follows.

(v) For this claim, let us slightly modify the latter chain of identities, namely

$$\begin{aligned}\tau(v_t(\xi)\,\eta^*) &= \langle v_t(\xi), \eta\rangle = \langle \xi, u_{-t}(\eta)\rangle \\ &= \langle \xi, e^{-itD}(\eta)\rangle = \langle e^{itD}(\xi), \eta\rangle \\ &= \tau(e^{itD}(\xi)\,\eta), \quad \xi \in \mathcal{L}^2, \; \eta \in \mathcal{L}^1 \cap \mathcal{L}^\infty.\end{aligned}$$

Hence, the claim follows from the fact that the space $\mathcal{L}^1 \cap \mathcal{L}^\infty$ separates points in $\mathcal{L}^1 + \mathcal{L}^\infty$, see Lemma 1.4.10. $\square$

Lemma 2.3.18. *Let $\mathcal{E}$ be a noncommutative symmetric space.*

(i) *The space $\mathcal{E}$ is invariant with respect to the operator v_t, i.e. $v_t(\mathcal{E}) \subseteq \mathcal{E}$, $t \in \mathbb{R}$.*

(ii) *We set*

$$v_t^E := v_t|_{\mathcal{E}}, \quad t \in \mathbb{R}. \tag{2.3.44}$$

The operator v_t^E is bounded, i.e. $v_t^E \in B(\mathcal{E})$; the collection $v^E = \{v_t^E\}_{t\in\mathbb{R}}$ is a $\sigma(\mathcal{E}, \mathcal{L}^1 \cap \mathcal{L}^\infty)$-continuous group of contractions in $\mathcal{E}$.

(iii) *If the space $\mathcal{E}$ is dual to a separable noncommutative symmetric space, i.e. $\mathcal{E} = (\mathcal{E}_*)^*$, then the group v^E is a weak* continuous group of contractions.*

Proof. It follows from Lemma 2.3.17.(v) and (2.3.41) that

$$v_t^\infty := v_t|_{\mathcal{L}^\infty} \in B(\mathcal{L}^\infty) \quad \text{and} \quad v_t^1 := v_t|_{\mathcal{L}^1} \in B(\mathcal{L}^1).$$

Consequently, since $\mathcal{E}$ is an interpolation space with respect to $(\mathcal{L}^\infty, \mathcal{L}^1)$, we immediately obtain that $v_t^E \in B(\mathcal{E})$, $t \in \mathbb{R}$ and v^E is a group of contractions in the space $\mathcal{E}$. The fact that v^E is $\sigma(\mathcal{E}, \mathcal{L}^1 \cap \mathcal{L}^\infty)$-continuous immediately follows from the fact that the group v is $\sigma(\mathcal{L}^1 + \mathcal{L}^\infty, \mathcal{L}^1 \cap \mathcal{L}^\infty)$-continuous. Thus, we proved (i) and (ii).

(iii) If $\mathcal{E} = (\mathcal{E}_*)^*$, for some noncommutative symmetric space $\mathcal{E}_*$, then, for every $t \in \mathbb{R}$, we readily obtain that

$$\begin{aligned} \langle (v_{-t}^{E_*})^*(\xi), \eta\rangle &= \langle \xi, v_{-t}^{E_*}(\eta)\rangle \\ [\text{Lemma 2.3.17.(v)}] \quad &= \langle \xi, u_{-t}(\eta)\rangle \\ [(2.3.43)] \quad &= \langle v_t(\xi), \eta\rangle \\ [(2.3.44)] \quad &= \langle v_t^E(\xi), \eta\rangle, \quad \xi \in \mathcal{E}, \ \eta \in \mathcal{L}^1 \cap \mathcal{L}^\infty. \end{aligned}$$

Consequently, it follows from Lemma 1.4.10, that

$$v_t^E = (v_{-t}^{E_*})^*.$$

Thus, the mapping

$$\xi \mapsto v_t^E(\xi), \quad \xi \in \mathcal{E}$$

is $\sigma(\mathcal{E}, \mathcal{E}_*)$-continuous for every $t \in \mathbb{R}$. Finally, if $\xi \in \mathcal{E}$, then the function $t \mapsto v_t^E(\xi)$ is norm bounded, hence, it follows from Lemma 1.1.2, that (ii) implies that the mapping

$$t \mapsto v_t^E(\xi), \quad t \in \mathbb{R}$$

is continuous with respect to the $\sigma(\mathcal{E}, \mathcal{E}_*)$-topology. The lemma is completely proved. □

We now can show that the assumption (D3) (see page 94) is sufficient to establish that the subspace $\mathscr{D}_0(D)$ is a core of the operator D. Moreover, we shall establish that the closure in an appropriate weak topology of the subspace $\mathscr{D}_0(D)$ (which is not affiliated with $\mathcal{M}_L$ in general) is affiliated with $\mathcal{M}_L$.

Lemma 2.3.19. *Let $D : \mathscr{D}(D) \mapsto \mathcal{L}^2$ be a linear self-adjoint operator satisfying (D1)–(D3)(see pages 65 and 94). The subspace $\mathscr{D}_0(D)$ is a core of the operator D. Furthermore, $\mathscr{D}_0(D) \cap (\mathcal{L}^2)_1$ is $\sigma(\mathcal{L}^1 \cap \mathcal{L}^\infty, \mathcal{L}^2 + \mathcal{L}^p)$-dense in $\mathcal{L}^1 \cap \mathcal{L}^\infty \cap (\mathcal{L}^2)_1$, for every $1 \le p \le 2$. In particular, for every $\eta \in \mathcal{L}^1 \cap \mathcal{L}^\infty$ and every $1 \le p \le 2$, there is a net $\{\eta_\alpha\} \subseteq \mathscr{D}_0(D)$ such that*

$$\sup_\alpha \|\eta_\alpha\|_{\mathcal{L}^2} \le \|\eta\|_{\mathcal{L}^2}, \quad \textit{and} \quad \lim_\alpha \eta_\alpha = \eta,$$

where the limit is taken with respect to the $\sigma(\mathcal{L}^1 \cap \mathcal{L}^\infty, \mathcal{L}^p)$-topology.

Proof. The first part is similar to the proof of Lemma 2.3.1.

Let us now prove the second part of the lemma. Let $\mathcal{E} = \mathcal{L}^2 \cap \mathcal{L}^{p'}$, where p' is the conjugate exponent and $\mathcal{E}_* = \mathcal{L}^2 + \mathcal{L}^p$. Clearly, $\mathcal{E} = (\mathcal{E}_*)^*$ and $\mathcal{E}^\times = \mathcal{E}_*$. Let us consider the groups of contractions $v^2 := v^{\mathcal{L}^2}$ and v^E. According to Theorem 1.3.17 and Theorem 1.1.5, the operator D is the weak (=weak*) generator of the group v^2. In particular,

$$\xi \in \mathscr{D}(D) \Longleftrightarrow \lim_{t \to 0} \frac{v_t^2(\xi) - \xi}{t} \text{ exists.} \tag{2.3.45}$$

The limit is taken with respect to the $\sigma(\mathcal{L}^2, \mathcal{L}^2)$-topology. On the other hand, according to Lemma 2.3.18.(iii), the group v^E is weak* continuous. Consequently, if δ is the weak* generator of v^E, then

$$\xi \in \mathscr{D}(\delta) \Longleftrightarrow \lim_{t \to 0} \frac{v_t^E(\xi) - \xi}{t} \text{ exists,} \tag{2.3.46}$$

where the limit is regarded with respect to the $\sigma(\mathcal{E}, \mathcal{E}_*)$-topology. Thus, we readily see that

$$\mathscr{D}(\delta) \cap B_r \subseteq \mathscr{D}(D) \cap B_r, \tag{2.3.47}$$

where

$$B_r := (\mathcal{L}^2)_1 \cap (\mathcal{L}^1 \cap \mathcal{L}^\infty)_r, \quad r > 0.$$

On the other hand, applying Lemma 2.2.4 to the spaces $\mathcal{E}$ and $\mathcal{F}_r := \mathcal{L}^2 \cap (r(\mathcal{L}^1 \cap \mathcal{L}^\infty))$ and the group v^E, we obtain that the set $\mathscr{D}(\delta) \cap B_r$ is $\sigma(\mathcal{E}, \mathcal{E}_*)$-dense in $\mathcal{E} \cap B_r$. Combining the latter fact with (2.3.47), we obtain that the set $\mathscr{D}(D) \cap B_r$ is also $\sigma(\mathcal{E}, \mathcal{E}_*)$-dense in $\mathcal{E} \cap B_r$. Taking the union over all $r > 0$, yields that $\mathscr{D}_0(D) \cap (\mathcal{L}^2)_1$ is $\sigma(\mathcal{E}, \mathcal{E}_*)$-dense in $\mathcal{L}^1 \cap \mathcal{L}^\infty \cap (\mathcal{L}^2)_1$. The lemma is proved. □

The answer to Problem 2.0.7 when for the symmetric spaces $\mathcal{E}_L \cap \mathcal{L}^p_L$, $2 \le p < \infty$ is given by

Theorem 2.3.20. *Let $\mathcal{E}_L$ be a noncommutative symmetric space with Fatou norm. If $x \in \mathcal{M}_L$,*

$$[D, x] \in \mathcal{L}^p_L \cap \mathcal{E}_L, \quad f \in \mathfrak{F}(\mathcal{L}^p_L) \cap \mathfrak{F}(\mathcal{E}_L), \ 2 \le p < \infty,$$

then

$$[D, f(x)] \in \mathcal{L}^p_L \cap \mathcal{E}_L$$

and

$$\|[D, f(x)]\|_{\mathcal{E}_L} \le c_{f,E} \, \|[D, x]\|_{\mathcal{E}_L},$$

where

$$c_{f,E} = \sup_{a,b \, \eta \mathcal{M}_L} \|T^{a,b}_{\psi_f, \mathcal{E}}\|_{B(\mathcal{E}_L)}.$$

Proof. Let us first prove that

$$f(x)(\mathscr{D}_0(D)) \subseteq \mathscr{D}(D). \tag{2.3.48}$$

Let $y = [D, x]$. From Theorem 2.3.16, we see that operators

$$y_t = \frac{e^{itD} x e^{-itD} - x}{it}, \quad t \in \mathbb{R}$$

are uniformly bounded in the norms of $\mathcal{L}^p_L$ and $\mathcal{E}_L$, i.e.

$$\|y_t\|_{\mathcal{L}^p_L} \le \|y\|_{\mathcal{L}^p_L}, \quad \|y_t\|_{\mathcal{E}_L} \le \|y\|_{\mathcal{E}_L}, \quad t \in \mathbb{R}.$$

Applying Proposition 2.2.2 to $x_t = e^{itD} x e^{-itD}$, x and $\mathbf{1}$, we obtain

$$z_t = \frac{e^{itD} f(x) e^{-itD} - f(x)}{it} = T^{x_t, x}_{\psi_f} \left(\frac{e^{itD} x e^{-itD} - x}{it} \right)$$

are also uniformly bounded in $\mathcal{L}^p_L$ and $\mathcal{E}_L$ and

$$\|z_t\|_{\mathcal{L}^p_L} \le c_{f,p} \, \|y\|_{\mathcal{L}^p_L}, \quad \|z_t\|_{\mathcal{E}_L} \le c_{f,E} \, \|y\|_{\mathcal{E}_L}, \quad t \in \mathbb{R}.$$

Since the unit ball of $\mathcal{L}_L^p$ is $\sigma(\mathcal{L}_L^p, \mathcal{L}_L^{p'})$-compact, see Theorem 1.1.1, one can assume that there exists $z \in \mathcal{L}_L^p$ such that

$$\|z\|_{\mathcal{L}_L^p} \leq c_{f,p} \|y\|_{\mathcal{L}_L^p}, \quad \sigma(\mathcal{L}_L^p, \mathcal{L}_L^{p'}) - \lim_{t\to 0} z_t = z. \tag{2.3.49}$$

To show that $z \in \mathcal{E}_L$, let us note that the space $\mathcal{E}_L$ has Fatou norm. This implies (see (1.4.9))

$$\|z\|_{\mathcal{E}_L} = \|z\|_{\mathcal{E}_L{}^{\times\times}} = \sup_{uw \in (\mathcal{E}_L^{\times})_1 \cap \mathcal{L}_L^1 \cap \mathcal{L}_L^\infty} \tau_L(zuw).$$

Consequently, since z_t is uniformly bounded in $\mathcal{E}_L$, we obtain that

$$\tau_L(zuw) = \lim_{t\to 0} \tau_L(z_t uw) \leq \max_{t\in\mathbb{R}} \|z_t\|_{\mathcal{E}_L}, \quad uw \in (\mathcal{E}_L^{\times})_1 \cap \mathcal{L}_L^1 \cap \mathcal{L}_L^\infty.$$

Thus,

$$z \in \mathcal{E}_L, \quad \text{and} \quad \|z\|_{\mathcal{E}_L} \leq c_{f,E} \|y\|_{\mathcal{E}_L}.$$

If now $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathcal{L}^2$, then a simple computation shows that

$$\begin{aligned}\langle \frac{e^{-itD}(\eta) - \eta}{-it}, f(x)(\xi) \rangle &= \langle \eta, \frac{e^{itD} f(x)(\xi) - f(x)(\xi)}{it} \rangle \\ &= \langle \eta, z_t(\xi) \rangle \\ &- \langle \eta, e^{itD} f(x) \frac{e^{-itD}(\xi) - \xi}{it} \rangle, \quad \xi, \eta \in \mathcal{L}^2.\end{aligned} \tag{2.3.50}$$

Let us note that for the last term, according to Theorem 1.3.17 we have

$$\lim_{t\to 0} \frac{e^{itD}(\xi) - \xi}{it} = D(\xi), \quad \xi \in \mathscr{D}(D).$$

Furthermore, since $L_\xi L_\eta^* \in \mathcal{L}_L^{p'}$ provided $\xi, \eta \in \mathscr{D}_0(D)$, we have, thanks to the second condition in (2.3.49),

$$\begin{aligned}\lim_{t\to 0} \langle \eta, z_t(\xi) \rangle &= \lim_{t\to 0} \tau_L(z_t L_\xi L_\eta^*) = \tau_L(z L_\xi L_\eta^*) \\ &= \langle \eta, z(\xi) \rangle, \quad \xi, \eta \in \mathscr{D}_0(D) \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty.\end{aligned}$$

Letting $t \to 0$ in (2.3.50) gives

$$\langle D(\eta), f(x)(\xi) \rangle = \langle \eta, z(\xi) \rangle + \langle \eta, f(x) D(\xi) \rangle, \quad \xi, \eta \in \mathscr{D}_0(D). \tag{2.3.51}$$

For every fixed $\xi \in \mathscr{D}_0(D)$, the linear functional

$$\eta \mapsto \langle \eta, f(x) D(\xi) \rangle, \quad \eta \in \mathcal{L}^2$$

is continuous. Moreover, since $z \in \mathcal{L}_L^p$, and $\xi \in \mathcal{L}^1 \cap \mathcal{L}^\infty$, we obtain that, see (1.4.8),

$$\begin{aligned}|\langle \eta, z(\xi)\rangle| &= |\tau_L(L_\eta z^* L_\xi^*)| \le \|L_\eta\|_{\mathcal{L}_L^2} \|zL_\xi\|_{\mathcal{L}_L^2} \\ &\le \|\eta\|_{\mathcal{L}^2} \|z\|_{\mathcal{L}_L^p} \|L_\xi\|_{\mathcal{L}_L^q} \\ &\le \|\eta\|_{\mathcal{L}^2} \|z\|_{\mathcal{L}_L^p} \|\xi\|_{\mathcal{L}^1 \cap \mathcal{L}^\infty}, \quad \eta \in \mathcal{L}^2,\end{aligned}$$

where

$$\|z\|_{\mathcal{L}_L^p} \|\xi\|_{\mathcal{L}^1 \cap \mathcal{L}^\infty} < \infty \quad \text{and} \quad \frac{1}{2} = \frac{1}{p} + \frac{1}{q}.$$

Thus, the linear functional

$$\eta \mapsto \langle \eta, z(\xi)\rangle, \quad \eta \in \mathcal{L}^2$$

is continuous. It follows from (2.3.51), the linear functional

$$\eta \mapsto \langle D(\eta), f(x)(\xi)\rangle, \quad \eta \in \mathscr{D}_0(D)$$

is also continuous. Consequently, we readily obtain that

$$f(x)(\xi) \in \mathscr{D}((D|_{\mathscr{D}_0(D)})^*).$$

Furthermore, since $\mathscr{D}_0(D)$ is a core of D, it follows that

$$(D|_{\mathscr{D}_0(D)})^* = D^* = D.$$

Thus, $f(x)(\xi) \in \mathscr{D}(D)$ and therefore (2.3.48) is established. Moreover, it now follows from (2.3.51)

$$(f(x)D - Df(x))(\xi) = z(\xi), \quad \xi \in \mathscr{D}_0(D).$$

Let us consider the operator $z' = f(x)D - Df(x)$ with the domain $\mathscr{D}(z') = \mathscr{D}_0(D)$. Let us show that $z'^* = z^*$. This will finish the proof of the theorem, because then we would have $\overline{z'} = z'^{**} = z^{**} = z$ (see Theorem 1.3.12). Since $z' \subseteq z$, it is sufficient to show that $\mathscr{D}((z')^*) \subseteq \mathscr{D}(z^*)$. It follows from the definition of the adjoint operator that $\xi \in \mathscr{D}((z')^*)$ if and only if there is a constant $c(\xi)$ such that

$$|\tau_L((z^* L_\xi)L_\eta^*)| = |\tau_L(L_\xi(zL_\eta)^*)| = |\langle \xi, z'(\eta)\rangle| \le c(\xi)\,\|\eta\|_{\mathcal{L}^2}, \quad \eta \in \mathscr{D}_0(D).$$

Since $z^* \in \mathcal{L}_L^p$ and $\xi \in \mathcal{L}^2$, it follows that $z^* L_\xi \in \mathcal{L}_L^q$, where $q^{-1} = 2^{-1} + p^{-1}$ and $1 \le q \le 2$, see (1.4.8). On the other hand, it follows form Lemma 2.3.19

that for every $\eta \in \mathcal{L}^1 \cap \mathcal{L}^\infty$ there is a net $\{\eta_\alpha\} \subseteq \mathscr{D}_0(D)$ such that $\eta_\alpha \to \eta$ in the $\sigma(\mathcal{L}^1 \cap \mathcal{L}^\infty, \mathcal{L}^q)$-topology and $\sup_\alpha \|\eta_\alpha\|_{\mathcal{L}^2} \leq \|\eta\|_{\mathcal{L}^2}$. Hence, we obtain that

$$|\tau_L((z^* L_\xi) L_\eta^*) \leq c(\xi) \, \|\eta\|_{\mathcal{L}^2}, \quad \eta \in \mathcal{L}^1 \cap \mathcal{L}^\infty.$$

The latter means that $z^* L_\xi \in \mathcal{L}_L^2$ and, in particular, $\xi \in \mathscr{D}(z^*)$. □

Applying this result for the spaces $\mathcal{E}_L = \mathcal{L}_L^p$, $1 \leq p \leq \infty$ and taking into account that $\Phi(\mathcal{E}) \subseteq \Phi(\mathcal{L}_L^2) = B(\mathbb{R}^2)$, we obtain

Corollary 2.3.21. *If $2 \leq p \leq \infty$ (resp. $1 \leq p < 2$), $f \in \mathfrak{F}(\mathcal{L}_L^p)$ and $x \in \mathcal{M}_L$ such that $[D, x] \in \mathcal{L}_L^p$ (resp. $[D, x] \in \mathcal{L}_L^2 \cap \mathcal{L}_L^p$), then*

$$\|[D, f(x)]\|_{\mathcal{L}_L^p} \leq c_{f,p} \, \|[D, x]\|_{\mathcal{L}_L^p}.$$

2.4 Applications

Let us recall that $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra equipped with a n.s.f. trace τ acting on $\mathcal{H}$, $\mathcal{M}_L$ is the corresponding left regular representation. The spaces $\mathcal{E}$ and $\mathcal{E}_L$ stand for the noncommutative symmetric spaces corresponding to the function space $E = E(\mathbb{R})$ and the algebras $(\mathcal{M}, \tau)$ and $(\mathcal{M}_L, \tau_L)$, respectively. In particular, $\mathcal{L}^p$ and $\mathcal{L}_L^p$, $1 \leq p \leq \infty$ are the noncommutative L^p-spaces with respect to $(\mathcal{M}, \tau)$ and $(\mathcal{M}_L, \tau_L)$, respectively.

In the present section we shall present a number of applications of the results set out in Theorem 2.3.4 and 2.3.20 and Corollaries 2.3.3 and 2.3.21.

We shall start with discussion the relation between Definitions 2.0.4 and 2.0.6 of the symbols $ax - xb$ and $[D, x]$, and their counterparts in the setting of the algebra $\mathcal{M}_L$, given in Definitions 2.3.2 and 2.3.14.

2.4.1 Lipschitz case

Let us first consider applications of the results from Theorem 2.3.4 and Corollary 2.3.3.

Let us fix $a, b \eta \mathcal{M}$ self-adjoint linear operators, $x \in \mathcal{M}$, $D : \mathscr{D}(D) \mapsto \mathcal{H}$ is a self-adjoint linear operator.

For the symbol $ax - xb$, the relation between the definition in the algebra $\mathcal{M}$ and the counterpart in the setting of the left regular representation $\mathcal{M}_L$ is given in the next two lemmas.

Lemma 2.4.1. *If* $ax-xb \in \mathcal{L}^p$, *then* $L_aL_x - L_xL_b \in \mathcal{L}^p_L$, $2 \le p \le \infty$. *Moreover,*

$$L(ax-xb) = L_aL_x - L_xL_b.$$

Proof. As in the proof of Theorem 2.3.4, we shall use the resolvent approximation of the unbounded operators a, b. Let $r^a_\epsilon := (\mathbf{1}+i\epsilon a)^{-1}$ and $r^b_\epsilon := (\mathbf{1}+i\epsilon b)^{-1}$, $\epsilon > 0$ be the resolvents of the operators $a, b\eta\mathcal{M}$. We set $a_\epsilon := ar^a_\epsilon$ and $b_\epsilon := br^b_\epsilon$. Clearly, $a_\epsilon, b_\epsilon \in \mathcal{M}$ and

$$\|a_\epsilon\| \le \frac{1}{\epsilon}, \quad \|b_\epsilon\| \le \frac{1}{\epsilon}, \quad \epsilon > 0.$$

Moreover, it follows from the spectral theorem, see Theorem 1.3.14, that

$$\lim_{\epsilon\to 0} L(a_\epsilon)(\eta) = L_a(\eta), \ \eta \in \mathscr{D}(L_a)$$

and

$$\lim_{\epsilon\to 0} L(b_\epsilon)(\xi) = L_b(\xi), \quad \xi \in \mathscr{D}(L_b). \tag{2.4.1}$$

Let us first show that, for $\mathscr{D}_0(L_b)$ defined in (2.3.1), we have

$$L_x(\mathscr{D}_0(L_b)) \subseteq \mathscr{D}(L_a). \tag{2.4.2}$$

To this end, fixing $\xi \in \mathscr{D}_0(L_b)$, we consider the linear form

$$\eta \mapsto \langle L_x(\xi), L_a(\eta)\rangle, \quad \eta \in \mathscr{D}_0(L_a). \tag{2.4.3}$$

We then have

$$\begin{aligned}
\langle L_x(\xi), L_a(\eta)\rangle &= \lim_{\epsilon\to 0}\langle L_x(\xi), L(a_\epsilon)(\eta)\rangle \\
&= \lim_{\epsilon\to 0}\langle L(a_\epsilon)\, L_x(\xi), \eta\rangle \\
&= \lim_{\epsilon\to 0}\langle (L(a_\epsilon)L_x - L_xL(b_\epsilon))(\xi), \eta\rangle \\
&+ \lim_{\epsilon\to 0}\langle L_xL(b_\epsilon)(\xi), \eta\rangle \\
&= \lim_{\epsilon\to 0}\langle L(a_\epsilon x - xb_\epsilon)(\xi), \eta\rangle \\
&+ \langle L_xL_b(\xi), \eta\rangle.
\end{aligned} \tag{2.4.4}$$

For the latter limit, let us recall the identity, see (2.3.5),

$$r^a_\epsilon(ax-xb)r^b_\epsilon = a_\epsilon x - xb_\epsilon, \quad \epsilon > 0. \tag{2.4.5}$$

Thus, it follows from Theorem 1.4.12 that

$$\lim_{\epsilon\to 0} L(a_\epsilon x - xb_\epsilon) = L(ax-xb) \in \mathcal{L}^p_L,$$

where the limit is taken with respect to the norm topology of $\mathcal{L}_L^p$. Furthermore, for every $\xi \in \mathscr{D}_0(L_b)$ and $\eta \in \mathscr{D}_0(L_a)$, we have that $L_\xi L_\eta^* \in \mathcal{L}_L^1 \cap \mathcal{L}_L^\infty \subseteq \mathcal{L}_L^{p'}$. Thus, the preceding identity, in particular, implies that

$$\begin{aligned}\lim_{\epsilon\to 0}\langle L(a_\epsilon x - x b_\epsilon)(\xi),\eta\rangle &= \lim_{\epsilon\to 0}\tau_L(L(a_\epsilon x - x b_\epsilon)\, L_\xi L_\eta^*)\\ &= \tau_L(L(ax - xb)\, L_\xi L_\eta^*)\\ &= \langle L(ax-xb)(\xi),\eta\rangle.\end{aligned}$$

Consequently, from (2.4.4), we obtain that

$$\langle L_x(\xi), L_a(\eta)\rangle = \langle L(ax-xb)(\xi),\eta\rangle + \langle L_x L_b(\xi),\eta\rangle, \quad \eta\in\mathscr{D}_0(L_a). \tag{2.4.6}$$

Clearly, the form

$$\eta \mapsto \langle L_x L_b(\xi),\eta\rangle, \quad \eta\in\mathcal{L}^2$$

is continuous. On the other hand, it follows from the Hölder inequality that

$$\begin{aligned}\|L(ax-xb)(\xi)\|_{\mathcal{L}^2} &= \|(ax-xb)\xi\|_{\mathcal{L}^2}\\ &\le \|ax-xb\|_{\mathcal{L}^p}\|\xi\|_{\mathcal{L}^q}\\ &\le \|ax-xb\|_{\mathcal{L}^p}\|\xi\|_{\mathcal{L}^1\cap\mathcal{L}^\infty} < \infty.\end{aligned}$$

Thus, the form

$$\eta \mapsto \langle L(ax-xb)(\xi),\eta\rangle, \quad \eta\in\mathcal{L}^2$$

is also continuous. Consequently, the identity (2.4.6) implies that the linear form (2.4.3) is continuous and therefore (2.4.2) is proved.

Having (2.4.2) proved, the identity (2.4.6) readily yields that

$$(L_a L_x - L_x L_b)(\xi) = (ax-xb)\xi, \quad \xi\in\mathscr{D}_0(L_b).$$

That is, the operator $L_a L_x - L_x L_b$ defined on $\mathscr{D}_0(L_b)$ is a multiplication operator by $ax - xb \in \mathcal{L}^p$, which in turn implies that the operator $L_aL_x - L_xL_b$ is closable and the closure belongs to $\mathcal{L}_L^p$. The proof is finished. □

Next, we establish the converse result. Due to the fact that there are no means to deal with the domain of the operator D when the algebra $\mathcal{M}$ is not taken in the left regular representation, the Banach space technique can only prove the result for the case $p=\infty$.

Lemma 2.4.2. *If $L_aL_x - L_xL_b \in \mathcal{L}_L^\infty$, then $ax - xb \in \mathcal{L}^\infty$ and*

$$L(ax-xb) = L_aL_x - L_xL_b.$$

Proof. For the sake of brevity, let us set

$$\mathbf{a} := L_a, \quad \mathbf{b} := L_b, \quad \mathbf{x} := L_x.$$

Similarly to the preceding lemma, let us first prove that

$$x(\mathscr{D}(b)) \subseteq \mathscr{D}(a). \tag{2.4.7}$$

To this end, we consider the identity

$$\begin{aligned}\langle x(\xi), a(\eta)\rangle &= \lim_{n\to\infty} \langle xe_n^b(\xi), ae_n^a(\eta)\rangle \\ &= \lim_{n\to\infty} \langle (e_n^a axe_n^b - e_n^a xbe_n^b)(\xi), \eta\rangle \\ &+ \langle b(\xi), x^*(\eta)\rangle. \end{aligned} \tag{2.4.8}$$

Repeating the argument from the preceding lemma, it sufficient to show that the operators

$$y_n = e_n^a axe_n^b - e_n^a xbe_n^b, \quad n \geq 1$$

are uniformly bounded in $\mathcal{L}^\infty$. Let us recall that

$$\mathbf{ax} - \mathbf{xb} \in \mathcal{L}_L^\infty, \quad \text{and} \quad e_n^{\mathbf{b}}(\mathcal{L}^2) \subseteq \mathscr{D}(\mathbf{b}).$$

Since $e_n^{\mathbf{b}}(\mathcal{L}^2) \subseteq \mathscr{D}(\mathbf{b})$, we readily obtain that

$$e_n^{\mathbf{a}}(\mathbf{ax} - \mathbf{xb})e_n^{\mathbf{b}}(\xi) = e_n^{\mathbf{a}}\mathbf{ax}e_n^{\mathbf{b}}(\xi) - e_n^{\mathbf{a}}\mathbf{xb}e_n^{\mathbf{b}}(\xi), \quad \xi \in \mathcal{L}^2,$$

or

$$e_n^{\mathbf{a}}(\mathbf{ax} - \mathbf{xb})e_n^{\mathbf{b}} = e_n^{\mathbf{a}}\mathbf{ax}e_n^{\mathbf{b}} - e_n^{\mathbf{a}}\mathbf{xb}e_n^{\mathbf{b}} = L_{y_n}.$$

Thus, the operators y_n are uniformly bounded in $\mathcal{L}^\infty$ and (2.4.7) is proved.

It follows from the latter identity that

$$wo - \lim_{n\to\infty} y_n = L^{-1}(\mathbf{ax} - \mathbf{xb}).$$

Letting $n \to \infty$ in the identity (2.4.8) implies that the operator $ax - xb$ defined on $\mathscr{D}(b)$ is given by

$$(ax - xb)(\xi) = \lim_{n\to\infty} y_n(\xi) = L^{-1}(\mathbf{ax} - \mathbf{xb})(\xi).$$

The latter means that the operator $ax - xb$ extends to a bounded linear operator. The proof is finished. □

At the moment, we are fully equipped to resolve Problem 2.0.5 for the space $\mathcal{E} = \mathcal{L}^\infty$ without referring to the left regular representation. The answer is given in the following theorem.

Theorem 2.4.3. *Let $a, b\eta\mathcal{M}$ be self-adjoint linear operators, let $x \in \mathcal{M}$ and $f \in \mathfrak{F}(\mathcal{L}^\infty)$. If $ax - xb \in \mathcal{L}^\infty$, then $f(a)x - f(b)x \in \mathcal{L}^\infty$ and*

$$\|f(a)x - xf(b)\|_{\mathcal{L}^\infty} \leq c_f \,\|ax - xb\|_{\mathcal{L}^\infty}.$$

Proof. The proof is now simple. According to Lemma 2.4.1

$$ax - xb \in \mathcal{L}^\infty \quad \Longrightarrow \quad L_a L_x - L_x L_b \in \mathcal{L}_L^\infty.$$

Next, Theorem 2.3.4 guarantees that

$$f(L_a)L_x - L_x f(L_b) \in \mathcal{L}_L^\infty$$

and

$$\|f(L_a)L_x - L_x f(L_b)\|_{\mathcal{L}_L^\infty} \leq c_{f,\infty} \,\|L_a L_x - L_x L_b\|_{\mathcal{L}^\infty}.$$

Clearly,

$$L_{f(a)}L_x - L_x L_{f(b)} = f(L_a)L_x - L_x f(L_b) \in \mathcal{L}_L^\infty. \tag{2.4.9}$$

Finally, from Lemma 2.4.2, we come back from left regular representation,

$$L_{f(a)}L_x - L_x L_{f(b)} \in \mathcal{L}_L^\infty \quad \Longrightarrow \quad f(a)\,x - x\,f(b) \in \mathcal{L}^\infty.$$

The theorem is proved. □

A similar argument together with Corollary 2.3.3 resolves Problem 2.0.5 for the space $\mathcal{E} \cap \mathcal{L}^\infty$.

Corollary 2.4.4. *Let $a, b\eta\mathcal{M}$ be self-adjoint linear operators, let $x \in \mathcal{M}$ and $f \in \mathfrak{F}(\mathcal{L}^\infty)$. If $ax - xb \in \mathcal{E} \cap \mathcal{L}^\infty$, then*

$$f(a)x - xf(b) \in \mathcal{E} \cap \mathcal{L}^\infty$$

and

$$\|f(a)x - xf(b)\|_{\mathcal{E}} \leq c_{f,\mathcal{E}} \,\|ax - xb\|_{\mathcal{E}}.$$

For the rest of the section, let us assume that self-adjoint operators a, b are τ-measurable and let $x \in \mathcal{M}$. Since the class of all τ-measurable operators $\tilde{\mathcal{M}}$ is a $*$-algebra, we readily see that $ax - xb \in \mathcal{E}$ in the sense of the Definition 2.0.4 if and only if the element $ax - xb \in \tilde{\mathcal{M}}$ belongs to $\mathcal{E}$. Obviously, the same is true for the algebra $\mathcal{M}_L$ and Definition 2.3.2 i.e., if $a = a^*, b = b^* \in \tilde{\mathcal{M}}_L$ and $x \in \mathcal{M}_L$, then $ax - xb \in \mathcal{E}_L$ in the sense of Definition 2.3.2 if and only if the element $ax - xb \in \tilde{\mathcal{M}}_L$ belongs to $\mathcal{E}_L$. Thus, the following result is an immediate consequence of Lemmas 1.4.17 and 1.5.6.

Lemma 2.4.5. *Let $a, b \in \tilde{\mathcal{M}}$ be self-adjoint linear operators. Let $x \in \mathcal{M}$.*

(i) *if $ax - xb \in \mathcal{E}$, then $L_a L_x - L_x L_b \in \mathcal{E}_L$;*

(ii) *if $L_a L_x - L_x L_b \in \mathcal{E}_L$, then $ax - xb \in \mathcal{E}$.*

In either case, we also have

$$L(ax - xb) = L_a L_x - L_x L_b.$$

The latter lemma, together with Theorem 2.3.4 solves Problem 2.0.5 for the space $\mathcal{E} \cap \mathcal{L}^p$ and the special case $a, b \in \tilde{\mathcal{M}}$.

Theorem 2.4.6. *Let $\mathcal{E}$ be a noncommutative symmetric space with Fatou norm and let $2 \leq p \leq \infty$. Let $a = a^*, b = b^* \in \tilde{\mathcal{M}}$ and $x \in \mathcal{M}$. If $ax - xb \in \mathcal{E} \cap \mathcal{L}^p$ and $f \in \mathfrak{F}(\mathcal{E}) \cap \mathfrak{F}(\mathcal{L}^p)$, then $f(a)x - xf(b) \in \mathcal{E} \cap \mathcal{L}^p$ and*

$$\|f(a)x - xf(b)\|_{\mathcal{E}} \leq c_{f,E} \, \|ax - xb\|_{\mathcal{E}},$$

where

$$c_{f,E} = \sup_{a,b\eta\mathcal{M}} \|T^{a,b}_{\psi_f,\mathcal{E}}\|_{B(\mathcal{E})}.$$

In particular, when $\mathcal{E} = L^p$, we have that

Theorem 2.4.7. *Let $a = a^*, b = b^* \in \tilde{\mathcal{M}}$ and $x \in \mathcal{M}$. Let $1 \leq p \leq \infty$. If $f \in \mathfrak{F}(\mathcal{L}^p)$ and either $ax - xb \in \mathcal{L}^p \cap \mathcal{L}^2$ or $ax - xb \in \mathcal{L}^p$ and $p \geq 2$, then $f(a)x - xf(b) \in \mathcal{L}^p$ and*

$$\|f(a)x - xf(b)\|_{\mathcal{L}^p} \leq c_{f,p} \, \|ax - xb\|_{\mathcal{L}^p},$$

where

$$c_{f,p} = \sup_{a,b\eta\mathcal{M}} \|T^{a,b}_{\psi_f,\mathcal{L}^p}\|_{B(\mathcal{L}^p)}.$$

2.4.2 Commutator case

Let us now consider applications of the results stated in Theorem 2.3.20 and Corollary 2.3.21. Let us recall that $(\mathcal{M}, \tau)$ is semi-finite von Neumann algebra acting on $\mathcal{H}$. Let $D : \mathscr{D}(D) \mapsto \mathcal{H}$ be a self-adjoint linear operator satisfying (D1)–(D2)(see page 65), let $x \in \mathcal{M}$. In the present section we shall consider the results similar to those of Section 2.4.1 for the commutator $[D, x]$.

Before we can prove the analogue of Lemma 2.4.1, we have to construct the operator which will play the role of D in the left regular representation, as the operator L_a does for an operator $a\eta\mathcal{M}$. In the present section, as the counterpart of the operator D in the the left regular representation we take $\mathbf{D} := \delta^2$, the generator of the strongly continuous group γ^2

Lemma 2.4.8. *The operator* $\mathbf{D} : \mathscr{D}(\mathbf{D}) \mapsto \mathcal{L}^2$ *is a self-adjoint linear operator satisfying (D1)–(D3)(see pages 65 and 94).*

Proof. It readily follows from Stone's theorem (Theorem 1.3.17) the unitary group $\{e^{it\mathbf{D}}\}_{t\in\mathbb{R}}$ is given by the group γ^2. Let us consider the following chain of identities

$$\begin{aligned}(e^{it\mathbf{D}}L_x e^{-it\mathbf{D}})(\xi) &= e^{itD}(L_x e^{-it\mathbf{D}})(\xi)e^{-itD}\\ &= e^{itD}x(e^{-it\mathbf{D}}(\xi))e^{-itD}\\ &= e^{itD}xe^{-itD}\xi e^{itD}e^{-itD}\\ &= (e^{itD}xe^{-itD})\xi\\ &= L(e^{itD}xe^{-itD})(\xi), \quad x\in\mathcal{M},\ \xi\in\mathcal{L}^1\cap\mathcal{L}^\infty.\end{aligned}$$

Consequently,

$$e^{it\mathbf{D}}L_x e^{-it\mathbf{D}} = L(\gamma^\infty(x)), \quad x\in\mathcal{M}$$

and therefore the operator $\mathbf{D}$ satisfies (D1). Since γ^∞ coincides with γ^1 on $\mathcal{L}^1\cap\mathcal{L}^\infty$, we obtain that

$$\begin{aligned}\tau_L(e^{it\mathbf{D}}L_x e^{-it\mathbf{D}}) &= \tau_L(L(\gamma^1(x))) = \tau(\gamma^1(x))\\ &= \tau(x) = \tau_L(L_x), \quad x\in\mathcal{L}^1\cap\mathcal{L}^\infty\end{aligned}$$

and hence the operator $\mathbf{D}$ satisfies (D2). Finally, the operator $e^{it\mathbf{D}} = \gamma_t^2$, $t\in\mathbb{R}$ coincides with $\gamma_t^{\mathcal{L}^1\cap\mathcal{L}^\infty}$, γ^1 and γ^∞ on $\mathcal{L}^1\cap\mathcal{L}^\infty$, $\mathcal{L}^1$ and $\mathcal{L}^\infty$, respectively. Therefore, it follows from Theorem 2.3.8, that

(i) $\mathcal{L}^1\cap\mathcal{L}^\infty$ is invariant with respect to $e^{it\mathbf{D}}$, $t\in\mathbb{R}$;

(ii) the group
$$\{e^{it\mathbf{D}}|_{\mathcal{L}^1\cap\mathcal{L}^\infty}\}_{t\in\mathbb{R}} = \gamma^{\mathcal{L}^1\cap\mathcal{L}^\infty} \tag{2.4.10}$$
is $\sigma(\mathcal{L}^1\cap\mathcal{L}^\infty, \mathcal{L}^1+\mathcal{L}^\infty)$-continuous group of contractions on $\mathcal{L}^1\cap\mathcal{L}^\infty$;

(iii) the group (2.4.10) is continuous with respect to the norms of the spaces $\mathcal{L}^1$ and $\mathcal{L}^\infty$.

These observations yield that the operator $\mathbf{D}$ satisfies (D3). □

Now it is time for the analogue of Lemma 2.4.1. Unfortunately, in the case when the operator D is not affiliated with $\mathcal{M}$, we have a weaker result.

Lemma 2.4.9. *If $[D, x] \in \mathcal{L}^p$, $2 \leq p \leq \infty$ and*

$$\lim_{t\to 0} \frac{e^{itD} x e^{-itD} - x}{t} = i[D, x],$$

then $[\mathbf{D}, L_x] \in \mathcal{L}^p_L$. The limit is taken in the norm topology if $p < \infty$ and the wo-topology otherwise. Moreover,

$$L([D, x]) = [\mathbf{D}, L_x].$$

Proof. We shall first show that

$$L_x(\mathscr{D}_0(\mathbf{D})) \subseteq \mathscr{D}(\mathbf{D}). \tag{2.4.11}$$

To this end we fix $\xi \in \mathscr{D}_0(\mathbf{D})$ and consider the linear form

$$\eta \mapsto \langle x\xi, \mathbf{D}(\eta)\rangle, \quad \eta \in \mathscr{D}(\mathbf{D}). \tag{2.4.12}$$

Let us consider the identity

$$\begin{aligned} \langle x\xi, \mathbf{D}(\eta)\rangle &= \lim_{t\to 0} \left\langle x\xi, \frac{e^{it\mathbf{D}}(\eta) - \eta}{t} \right\rangle \\ &= \lim_{t\to 0} \tau\left(x\xi\, \frac{e^{-itD}\eta^* e^{itD} - \eta^*}{t} \right) \\ &= \lim_{t\to 0} \tau\left(\frac{e^{itD} x\xi e^{-itD} - x\xi}{t}\, \eta^* \right) \\ &= \lim_{t\to 0} \tau\left(\frac{e^{itD} x e^{-itD} - x}{t}\, e^{itD}\xi e^{-itD}\, \eta^* \right) \\ &+ \lim_{t\to 0} \tau\left(x\, \frac{e^{itD}\xi e^{-itD} - \xi}{t}\, \eta^* \right), \quad \eta \in \mathscr{D}(\mathbf{D}). \end{aligned} \tag{2.4.13}$$

Since $\xi \in \mathscr{D}_0(\mathbf{D})$, we have that

$$\lim_{t\to 0} \frac{e^{itD}\xi e^{-itD} - \xi}{t} = \mathbf{D}(\xi).$$

Furthermore, from the hypothesis, we also have that

$$\lim_{t\to 0} \frac{e^{itD} x e^{-itD} - x}{t} = [D, x].$$

Thus, from (2.4.13), we obtain that

$$\langle x\xi, \mathbf{D}(\eta)\rangle = \langle [D,x]\xi, \eta\rangle + \langle x\mathbf{D}(\xi), \eta\rangle, \quad \eta \in \mathscr{D}(\mathbf{D}). \tag{2.4.14}$$

Since $\xi \in \mathscr{D}_0(\mathbf{D})$ and $x \in \mathcal{M}$, the linear form

$$\eta \mapsto \langle x\mathbf{D}(\xi), \eta\rangle, \quad \eta \in \mathcal{L}^2$$

is continuous. Moreover, let us recall that $\xi \in \mathcal{L}^1 \cap \mathcal{L}^\infty$, see (2.3.40). Consequently, applying Hölder inequality, (1.4.8), we obtain that

$$\|[D,x]\xi\|_{\mathcal{L}^2} \le \|[D,x]\|_{\mathcal{L}^p}\|\xi\|_{\mathcal{L}^q} \le \|[D,x]\|_{\mathcal{L}^p}\|\xi\|_{\mathcal{L}^1\cap\mathcal{L}^\infty} < \infty,$$

where

$$\frac{1}{2} = \frac{1}{p} + \frac{1}{q}.$$

Thus, the linear functional

$$\eta \mapsto \langle [D,x]\xi, \eta\rangle, \quad \eta \in \mathcal{L}^2$$

is continuous. Altogether, (2.4.14) now implies that the form (2.4.12) is continuous. Therefore, $L_x(\xi) \in \mathscr{D}(\mathbf{D})$ and consequently (2.4.11) is valid. Furthermore, the identity (2.4.14) now means that

$$\langle [D,x]\xi, \eta\rangle = \langle \mathbf{D}(x\xi), \eta\rangle - \langle x\mathbf{D}(\xi), \eta\rangle, \quad \xi \in \mathscr{D}_0(\mathbf{D}),\ \eta \in \mathscr{D}(\mathbf{D}).$$

Since $\mathscr{D}(\mathbf{D})$ is norm dense in $\mathcal{L}^2$, we immediately obtain that

$$\mathbf{D}L_x(\xi) - L_x\mathbf{D}(\xi) = [D,x]\xi, \quad \xi \in \mathscr{D}_0(\mathbf{D}). \tag{2.4.15}$$

The latter means that the operator $\mathbf{D}L_x - L_x\mathbf{D}$ defined on $\mathscr{D}_0(\mathbf{D})$ coincides with the left multiplication by $[D,x]$ and therefore closable. The last part of the proof is to establish that the closure $\overline{\mathbf{D}L_x - L_x\mathbf{D}}$ coincides with left multiplication by $[D,x]$. This is somewhat similar to that of the proof of Theorem 2.3.20 (see the end of the proof).

Let $y := L([D,x]) \in \mathcal{L}^p$ and $y' = \mathbf{D}L_x - L_x\mathbf{D}$ with the domain $\mathscr{D}_0(\mathbf{D})$. Let us show that $y^* = y'^*$. The latter is sufficient to finish the proof, since $\overline{y'} = (y')^{**} = y^{**} = y$, see Theorem 1.3.12. Let us note that the identity (2.4.15) means that $y' \subseteq y$. Thus, we need only to show that $\mathscr{D}((y')^*) \subseteq \mathscr{D}(y^*)$. For the latter, recall that $\xi \in \mathscr{D}((y')^*)$ if and only if there is a constant $c(\xi)$ such that

$$|\langle \xi, y'(\eta)\rangle| \le c(\xi)\,\|\eta\|_{\mathcal{L}^2}, \quad \eta \in \mathscr{D}_0(\mathbf{D}).$$

Clearly, we have that $\xi \in \mathscr{D}((y')^*)$ if and only if

$$|\tau_L((y^* L_\xi)\, L_\eta^*)| \leq c(\xi)\, \|L_\eta\|_{\mathcal{L}^2_L}, \quad \eta \in \mathscr{D}_0(\mathbf{D}). \tag{2.4.16}$$

Recall that $y^* \in \mathcal{L}^p_L$ and $L_\xi \in \mathcal{L}^2$. Consequently, according to the Hölder inequality (1.4.8), $y^* L_\xi \in \mathcal{L}^q_L$, where $q^{-1} = 2^{-1} + p^{-1}$, $1 \leq p \leq 2$. On the other hand, it follows from Lemma 2.3.19 that, for every $\eta \in \mathcal{L}^1 \cap \mathcal{L}^\infty$, there is a net $\{\eta_\alpha\} \subseteq \mathscr{D}_0(\mathbf{D})$ such that

$$\sup_\alpha \|\eta_\alpha\|_{\mathcal{L}^2} \leq \|\eta\|_{\mathcal{L}^2} \quad \text{and} \quad \lim_\alpha \eta_\alpha = \eta,$$

where the limit is taken with respect to the $\sigma(\mathcal{L}^1 \cap \mathcal{L}^\infty, \mathcal{L}^q)$-topology. The latter means that the estimate (2.4.16) may be extended to $\mathcal{L}^1 \cap \mathcal{L}^\infty$, i.e.

$$|\tau_L((y^* L_\xi)\, L_\eta^*)| \leq c(\xi) \|\eta\|_{\mathcal{L}^2}, \quad \eta \in \mathcal{L}^1 \cap \mathcal{L}^\infty.$$

Taking the maximum over all $\eta \in \mathcal{L}^1 \cap \mathcal{L}^\infty$ and recalling that the space $\mathcal{L}^2$ is a space with Fatou norm, we obtain that $y^* L_\xi \in \mathcal{L}^2$ which, in turn, implies that $\xi \in \mathscr{D}(y^*)$. The latter means, that $\mathscr{D}((y')^*) \subseteq \mathscr{D}(y^*)$ and therefore $(y')^* = y^*$. The theorem is completely proved. □

Combining Lemma 2.4.9 with Theorem 2.3.12, we obtain

Lemma 2.4.10. *If $[D, x] \in \mathcal{L}^\infty$, then $[\mathbf{D}, L_x] \in \mathcal{L}^\infty_L$ and*

$$L([D, x]) = [\mathbf{D}, L_x].$$

Now we prove the converse result. For the same reasons as in Lemma 2.4.2, we consider only the case $p = \infty$.

Lemma 2.4.11. *If $[\mathbf{D}, L_x] \in \mathcal{L}^\infty_L$, then $[D, x] \in \mathcal{L}^\infty$ and*

$$L([D, x]) = [\mathbf{D}, L_x].$$

Proof. Proceeding again in a similar fashion, we shall first establish that

$$x(\mathscr{D}(D)) \subseteq \mathscr{D}(D). \tag{2.4.17}$$

To this end, we fix $\xi \in \mathscr{D}(D)$ and consider the linear form

$$\eta \mapsto \langle x(\xi), D(\eta)\rangle, \quad \eta \in \mathscr{D}(D). \tag{2.4.18}$$

Clearly, we have the following identity

$$\begin{aligned}\langle x(\xi), D(\eta)\rangle &= \lim_{t\to 0}\left\langle x(\xi), \frac{e^{itD}(\eta)-\eta}{it}\right\rangle \\ &= \lim_{t\to 0}\left\langle \frac{e^{itD}x(\xi)-x(\xi)}{it}, \eta\right\rangle \\ &= \lim_{t\to 0}\left\langle \frac{e^{itD}xe^{-itD}-x}{it}(\xi), \eta\right\rangle \\ &- \lim_{t\to 0}\left\langle e^{itD}x\frac{e^{-itD}(\xi)-\xi}{it}, \eta\right\rangle, \quad \eta\in\mathscr{D}(D). \end{aligned} \tag{2.4.19}$$

Since $\xi \in \mathscr{D}(D)$, we clearly have that

$$\lim_{t\to 0}\frac{e^{-itD}(\xi)-x}{it} = -D(\xi).$$

On the other hand, since $[\mathbf{D}, L_x] \in \mathcal{L}_L^\infty$, it follows from Lemma 2.4.8 and Theorem 2.3.12 that

$$\begin{aligned} wo-\lim_{t\to 0}\frac{e^{itD}xe^{-itD}-x}{t} &= L^{-1}\left(wo-\lim_{t\to 0}\frac{e^{it\mathbf{D}}L_xe^{-it\mathbf{D}}-L_x}{t}\right) \\ &= L^{-1}([\mathbf{D}, L_x]). \end{aligned}$$

Consequently, if we set $y := L^{-1}([\mathbf{D}, L_x])$ for brevity, the identity (2.4.19) implies that

$$\langle x(\xi), D(\eta)\rangle = \langle y(\xi), \eta\rangle + \langle xD(\xi), \eta\rangle, \quad \xi, \eta \in \mathscr{D}(D). \tag{2.4.20}$$

Since $x, y \in \mathcal{L}^\infty$ and $\xi \in \mathscr{D}(D)$, the linear forms

$$\eta \mapsto \langle y(\xi), \eta\rangle, \quad \eta \in \mathcal{H}$$

and

$$\eta \mapsto \langle xD(\xi), \eta\rangle, \quad \eta \in \mathcal{H}$$

are continuous. Thus, the form (2.4.18) is continuous. Consequently, (2.4.17) is valid.

Now, it immediately follows from (2.4.20) that

$$\langle y(\xi), \eta\rangle = \langle (Dx - xD)(\xi), \eta\rangle, \quad \xi, \eta \in \mathscr{D}(D).$$

Since $\mathscr{D}(D)$ is norm dense in $\mathcal{H}$, we obtain that

$$y(\xi) = (Dx - xD)(\xi), \quad \xi \in \mathscr{D}(D).$$

Finally, since y is bounded linear operator, the operator $Dx-xD$ is also bounded and

$$[D,x]=y=L^{-1}([\mathbf{D},L_x]).$$

The theorem is completely proved. □

A direct combination of Lemmas 2.4.10 and 2.4.11 with the main result of Section 2.3.20 yields

Theorem 2.4.12. *Let $\mathcal{E}$ be a noncommutative symmetric space with Fatou norm. Let $D:\mathscr{D}(D)\mapsto\mathcal{H}$ be a self-adjoint linear operator satisfying (D1)–(D2)(see page 65), let $x=x^*\in\mathcal{M}$ and $f\in\mathfrak{F}(\mathcal{L}^\infty)$. If $[D,x]\in\mathcal{E}\cap\mathcal{L}^\infty$, then $[D,f(x)]\in\mathcal{E}\cap\mathcal{L}^\infty$ and*

$$\|[D,f(x)]\|_{\mathcal{L}^\infty}\le c_f\,\|[D,x]\|_{\mathcal{L}^\infty},\quad \|[D,f(x)]\|_{\mathcal{E}}\le c_{f,E}\,\|[D,x]\|_{\mathcal{E}}. \tag{2.4.21}$$

Proof. The proof again goes through the left regular representation. Firstly, from Lemma 2.4.10, we have

$$[D,x]\in\mathcal{L}^\infty\Longrightarrow[\mathbf{D},L_x]\in\mathcal{L}^\infty_L$$

and

$$[\mathbf{D},L_x]=L([D,x]).$$

The latter means that $[\mathbf{D},L_x]\in\mathcal{E}_L\cap\mathcal{L}^\infty_L$. Next, Theorem 2.3.20 shows that $[\mathbf{D},f(L_x)]\in\mathcal{E}_L\cap\mathcal{L}^\infty_L$ and

$$\|[\mathbf{D},f(L_x)]\|_{\mathcal{L}^\infty_L}\le c_f\,\|[\mathbf{D},L_x]\|_{\mathcal{L}^\infty_L},\quad \|[\mathbf{D},f(L_x)]\|_{\mathcal{E}_L}\le c_{f,E}\,\|[\mathbf{D},L_x]\|_{\mathcal{E}_L}.$$

Finally, Lemma 2.4.11 shows that

$$[\mathbf{D},f(L_x)]\in\mathcal{L}^\infty_L\Longrightarrow[D,f(x)]\in\mathcal{L}^\infty$$

and

$$[D,f(x)]=L^{-1}([\mathbf{D},f(L_x)]).$$

The latter means, that $[D,f(x)]\in\mathcal{E}\cap\mathcal{L}^\infty$ and the norm estimates (2.4.21) follow. □

2.4.3 Applications to $B(\mathcal{H})$

Finally, let us consider the application of the results in Section 2.3 to the algebra $\mathcal{M} = B(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space, see Section (1.6.3). A distinguished feature of this case is the fact that the $\mathcal{L}^\infty$-norm is the weakest among the $\mathcal{L}^p$-norms, $1 \leq p \leq \infty$. The latter observation allows to exploit the results of Section 2.4 and carry the symbols $ax - xb$ and $[D, x]$ from the left regular representation $\mathcal{M}_L$ to the algebra $\mathcal{M}$ not only for the special case $p = \infty$, as it is done in the preceding section, but for all $1 \leq p \leq \infty$.

Theorem 2.4.13. *Let $1 \leq p \leq \infty$, let $a : \mathscr{D}(a) \mapsto \mathcal{H}$ and $b : \mathscr{D}(b) \mapsto \mathcal{H}$ be self-adjoint linear operators and let $x \in B(\mathcal{H})$. If*

$$ax - xb \in \mathcal{C}^p \quad \textit{and} \quad f \in \mathfrak{F}(\mathcal{C}^p),$$

then

$$f(a)x - xf(b) \in \mathcal{C}^p$$

and

$$\|f(a)x - xf(b)\|_{\mathcal{C}^p} \leq c_{f,p} \|ax - xb\|_{\mathcal{C}^p}.$$

Proof. The proof is straightforward. Since $\mathcal{C}^p \subseteq \mathcal{C}^\infty$, for every $1 \leq p \leq \infty$, it follows from Lemma 2.4.1 that

$$ax - xb \in \mathcal{C}^p \subseteq \mathcal{C}^\infty \Longrightarrow L_a L_x - L_x L_b \in \mathcal{C}^\infty_L$$

and

$$L(ax - xb) = L_a L_x - L_x L_b.$$

Consequently, we also have that

$$L_a L_x - L_x L_b \in \mathcal{C}^p_L.$$

Applying Theorem 2.3.4 to the space $\mathcal{E}_L = \mathcal{L}^p_L$, when $2 \leq p \leq \infty$ (resp, $\mathcal{E}_L = \mathcal{L}^2_L$, when $1 \leq p < 2$) and noting that $\mathcal{C}^p \cap \mathcal{C}^2 = \mathcal{C}^p$, provided $1 \leq p < 2$, we readily obtain that

$$f(L_a)L_x - L_x f(L_b) \in \mathcal{C}^p_L$$

and

$$\|f(L_a)L_x - L_x f(L_b)\|_{\mathcal{C}^p_L} \leq c_{f,p} \|L_a L_x - L_x L_b\|_{\mathcal{C}^p_L}.$$

Finally, it follows from Lemma 2.4.2 that

$$L_{f(a)} L_x - L_x L_{f(b)} \in \mathcal{C}^p_L \subseteq \mathcal{C}^\infty_L \Longrightarrow f(a)\, x - x\, f(b) \in \mathcal{C}^\infty$$

and

$$ax - xb = L^{-1}(L_a L_x - L_x L_b).$$

Consequently,

$$ax - xb \in \mathcal{C}^p.$$

Together with (2.4.9), it finishes the proof of the theorem. □

Similarly, for the Problem 2.0.7 in $B(\mathcal{H})$ we have

Theorem 2.4.14. *Let $D : \mathscr{D}(D) \mapsto \mathcal{H}$ be a self-adjoint linear operator and let x be bounded operator, let $1 \le p \le \infty$ and let $f \in \mathfrak{F}(\mathcal{C}^p)$. If $[D, x] \in \mathcal{C}^p$, then $[D, f(x)] \in \mathcal{C}^p$ and*

$$\|[D, f(x)]\|_{\mathcal{C}^p} \le c_{f,p} \, \|[D, x]\|_{\mathcal{C}^p}.$$

2.4.4 Applications to L^p-spaces with respect to an arbitrary von Neumann algebra

In the present section, we shall consider implications for L^p spaces constructed with respect to an arbitrary von Neumann algebra which follow from the results in the preceding sections.

Let $\mathcal{M}$ be a von Neumann algebra and let ϕ be a n.s.f. weight on $\mathcal{M}$. Let $\sigma^\phi = \{\sigma_t^\phi\}_{t\in\mathbb{R}}$ be the corresponding modular automorphism group. We set $\mathcal{R} := \mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$. Recall that $\mathcal{R}$ is a semi-finite von Neumann algebra equipped with the distinguished trace on $\mathcal{R}$, see Theorem 1.6.4. Let $E := E(\mathbb{R})$ be a fully symmetric function space. We set $\mathcal{E} := E(\mathcal{R}, \tau)$, for brevity. In particular, $\mathcal{L}^p := L^p(\mathcal{R}, \tau)$ and $\mathcal{L}^{p,q} := L^{p,q}(\mathcal{R}, \tau)$ are noncommutative L^p- and Lorentz spaces, $1 \le p, q \le \infty$. Let $L^p(\mathcal{M}) \subseteq \mathcal{L}^{p,\infty}$ be the noncommutative L^p spaces associated with the algebra $\mathcal{M}$, i.e.

$$L^p(\mathcal{M}) := \left\{ x \in \mathcal{L}^{p,\infty} : \quad \theta_s(x) = e^{-t/p} x, \ t \in \mathbb{R} \right\}, \quad 1 \le p \le \infty,$$

where the group of $*$-automorphisms $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ on $\mathcal{R}$ is defined in (1.6.6). We refer the reader to Section 1.6.5 for all relevant notations and results.

Let us recall that $\mathcal{R}_L := L(\mathcal{R})$ stands for the left regular representation of the algebra $\mathcal{R}$ and $\mathcal{E}_L = L(\mathcal{E}) = E(\mathcal{R}_L, \tau_L)$. In particular, $\mathcal{L}_L^p = L^p(\mathcal{R}_L, \tau_L)$ and $\mathcal{L}_L^{p,q} = L^{p,q}(\mathcal{R}_L, \tau_L)$. Let us also introduce the space

$$L_L^p(\mathcal{M}) := L(L^p(\mathcal{M})) \subseteq \mathcal{L}_L^{p,\infty}.$$

In the present section we shall consider only the application of Theorem 2.3.4 to the setting of the space $L^p_L(\mathcal{M})$, which is isomorphic to the noncommutative L^p-space $L^p(\mathcal{M})$. We shall present a simple result relating the setting of the space $L^p_L(\mathcal{M})$ with the space $L^p(\mathcal{M})$. Considerations of the commutator estimates and applications of the results of Section 2.3.6 to the setting of the space $L^p(\mathcal{M})$ goes beyond of the scope of the present manuscript.

We start with the definition of the symbol $ax - xb$ in the space $L^p_L(\mathcal{M})$, $1 \leq p \leq \infty$.

Definition 2.4.15. Let $a, b\eta\mathcal{M}$ be self-adjoint linear operators and let $x \in \mathcal{M}$. We shall say that the operator $ax - xb$ is well defined and belongs to $L^p_L(\mathcal{M})$, if and only if $ax - xb \in \mathcal{L}^{p,\infty}_L$ in the sense of Definition 2.3.2 and

$$\theta_t(ax - xb) = e^{-t/p}\,(ax - xb), \quad t \in \mathbb{R}.$$

A direct application of the results in Section 2.3.1 yields

Theorem 2.4.16. *Let $a, b\eta(\pi(\mathcal{M}))_L$ be self-adjoint linear operators and let $x \in \pi(\mathcal{M})_L$. Let $2 \leq p \leq \infty$ and let $f \in \mathfrak{F}(\mathcal{L}^{p,\infty}_L)$. If $ax - xb \in L^p_L(\mathcal{M})$, then $f(a)x - xf(b) \in L^p_L(\mathcal{M})$ and*

$$\|f(a)x - xf(b)\|_{L^p_L(\mathcal{M})} \leq c_{f,p}\,\|ax - xb\|_{L^p_L(\mathcal{M})}.$$

Proof. We set $y := ax - xb$. Clearly,

$$y \in \mathcal{L}^{p,\infty}_L \quad \text{and} \quad \theta_t(y) = e^{-t/p}y, \quad t \in \mathbb{R}. \tag{2.4.22}$$

Since $ax - xb \in \mathcal{L}^{p,\infty}$ and $f \in \mathfrak{F}(\mathcal{L}^{p,\infty})$, it readily follows from Theorem 2.3.7 that $f(a)x - xf(b) \in \mathcal{L}^{p,\infty}$ and

$$\|f(a)x - xf(b)\|_{\mathcal{L}^{p,\infty}} \leq c_{f,p}\,\|ax - xb\|_{\mathcal{L}^{p,\infty}}.$$

Let us set $z = f(a)x - xf(b)$. To finish the proof, we need to show that

$$\theta_t(z) = e^{-t/p}\,z, \quad t \in \mathbb{R}.$$

Let us recall that it was shown in the proof of Theorem 2.3.2 (see also the discussion before Theorem 2.3.5) that if $r^a_\epsilon := (\mathbf{1} + i\epsilon a)^{-1}$ and $r^b_\epsilon := (\mathbf{1} + i\epsilon b)^{-1}$, $\epsilon > 0$ are resolvents of the operators a and b, then

$$y_\epsilon := r^a_\epsilon y r^b_\epsilon \in \mathcal{L}^\infty \cap \mathcal{L}^{p,\infty} \tag{2.4.23}$$

and

$$z = \lim_{\epsilon\to 0} z_\epsilon, \quad \text{where } a_\epsilon := ar^a_\epsilon,\ b_\epsilon := br^b_\epsilon, \quad z_\epsilon := T^{a_\epsilon, b_\epsilon}_{\psi_f, \mathcal{L}^{p,\infty}_L}(y_\epsilon) \tag{2.4.24}$$

and the limit converges with respect to the $\sigma(\mathcal{L}^{p,\infty}, \mathcal{L}^{p',1})$-topology.

Since the operators a and b are affiliated with the algebra $(\pi(\mathcal{M}))_L$, it immediately follows that

$$r^a_\epsilon, r^b_\epsilon, a_\epsilon, b_\epsilon \in (\pi(\mathcal{M}))_L$$

and

$$\theta_t(r^a_\epsilon) = r^a_\epsilon, \quad \theta_t(r^b_\epsilon) = r^b_\epsilon, \quad \theta_t(a_\epsilon) = a_\epsilon, \quad \theta_t(b_\epsilon) = b_\epsilon, \quad t \in \mathbb{R},\ \epsilon > 0.$$

Consequently, it follows from (2.4.22) and (2.4.23) that

$$\theta_t(y_\epsilon) = e^{-t/p}\, y_\epsilon, \quad t \in \mathbb{R},\ \epsilon > 0.$$

Furthermore, it now follows from (2.4.24) and Lemma 1.11.1 that

$$\theta_t(z_\epsilon) = e^{-t/p}\, z_\epsilon, \quad t \in \mathbb{R},\ \epsilon > 0.$$

Finally, let us note that the $*$-automorphism θ_t is $\sigma(\mathcal{E}_L, \mathcal{E}^\times_L)$-continuous for every $t \in \mathbb{R}$ and every noncommutative symmetric space $\mathcal{E}_L$. Consequently, it follows from (2.4.24) that

$$\theta_t(z) = \lim_{\epsilon\to 0} \theta_t(z_\epsilon) = \lim_{\epsilon\to 0} e^{-t/p}\, z_\epsilon = e^{-t/p}\, z, \quad t \in \mathbb{R}.$$

Thus, we proved that

$$z \in \mathcal{L}^{p,\infty}_L \quad \text{and} \quad \theta_t(z) = e^{-t/p}\, z, \quad t \in \mathbb{R}.$$

In other words $z \in L^p_L(\mathcal{M})$. The proof of the theorem is finished. □

Let us now discuss the relations between the statements $ax - xb \in L^p(\mathcal{M})$ and $L_a L_x - L_x L_b \in L^p_L(\mathcal{M})$. The next two lemmas are straightforward corollaries of Lemmas 2.4.1 and 2.4.2.

Lemma 2.4.17. *Let $a, b\eta\, \pi(\mathcal{M})$ be linear self-adjoint operators and let $x \in \pi(\mathcal{M})$. If $ax - xb \in L^p_L(\mathcal{M})$, then $L_a L_x - L_x L_b \in L^p_L(\mathcal{M})$.*

Lemma 2.4.18. *Let $a, b\eta\, \pi(\mathcal{M})$ be linear self-adjoint operator and let $x \in \pi(\mathcal{M})$. If $L_a L_x - L_x L_b \in L^\infty_L(\mathcal{M})$, then $ax - xb \in L^\infty(\mathcal{M})$.*

In the special case when $a = a^*, b = b^* \in \pi(\mathcal{M})$, Theorem 2.4.16 together with Lemmas 2.4.17 and 2.4.18 readily implies that

Theorem 2.4.19. *Let $a = a^*, b = b^* \in \pi(\mathcal{M})$ and let $x \in \pi(\mathcal{M})$. Let $2 \leq p \leq \infty$ and let $f \in \mathfrak{F}(\mathcal{L}^{p,\infty})$. If $ax - xb \in L^p(\mathcal{M})$, then $f(a)x - xf(b) \in L^p(\mathcal{M})$ and*

$$\|f(a)x - xf(b)\|_{L^p(\mathcal{M})} \leq c_{f,p} \|ax - xb\|_{L^p(\mathcal{M})}.$$

Let us note that even in this restrictive setting and even when a and b are bounded, the latter result is quite new.

2.5 Comments

(i) In the special case when $x = \mathbf{1}$, Theorem 2.3.4 and Corollary 2.3.3 together with Lemma 1.8.11 reduce to the inequalities

$$\|f(a) - f(b)\|_{\mathcal{L}^p_L} \leq c_{f,p} \|a - b\|_{\mathcal{L}^p_L}, \quad 1 < p < \infty,$$

provided $a - b \in \mathcal{L}^p_L \cap \mathcal{L}^2_L$, and f is a function with the derivative of bounded total variation, where a, b are arbitrary self-adjoint operators affiliated with $\mathcal{M}_L$, and $c_{f,p}$ is a constant depending of f and p only. This result extends [24, Corollary 7.5] and [27, Corollary 3.5].

(ii) If $a = b$ are as in (i) above and $x \in \mathcal{M}_L$, then

$$\|[|a|, x]\|_{\mathcal{L}^p_L} \leq c_p \|[a, x]\|_{\mathcal{L}^p_L}, \quad 1 < p < \infty,$$

provided $[a, x] \in \mathcal{L}^p_L \cap \mathcal{L}^2_L$, where c_p is a constant depending on p only. This complements the result of [27, Theorem 2.2] and provides a type II extension of [9, (6.6)].

(iii) If, in addition, $a = b$ has a bounded inverse, then, for every noncommutative symmetric space $\mathcal{E}_L$, we have

$$\|[|a|^r, x]\|_{\mathcal{E}_L} \leq c_{E,a,r} \|[a, x]\|_{\mathcal{E}_L}, \quad 0 < r \leq 1,$$

whenever $[a, x] \in \mathcal{E}_L \cap \mathcal{L}^p_L$, for some $2 \leq p \leq \infty$, where the constant $c_{E,a,r}$ does not depend on x. This result extends similar inequalities for the case $E = L^\infty$, obtained earlier in [14, Lemma 1.4] (see also [18, 68]) by different methods.

(iv) The relations (2.3.29) and (2.3.30), which describes the generator of the group γ^∞ in terms of commutators $[D, x] \in \mathcal{L}^\infty$ are similar to those obtained in [11, Section 3.2.5], in particular [11, Proposition 3.2.55]. The argument in Theorem 2.3.12 is essentially a repetition of that of [11, Section 3.2.5].

(v) Section 2.3 is partially contained in [53]. However, Theorems 2.3.4, 2.3.16 and 2.3.20 are now proved in a more general context. Section (2.4) where we considered a number of applications from a unified point of view is new (the results in Subsection 2.4.3 were established in [53] via different approach).

(vi) The approximation results in Section 2.3.5 extends those of [21, Section 7] established of $\mathcal{L}^\infty$, to the setting of the space $\mathcal{L}^p$ for every $1 \leq p \leq \infty$.

(vii) Theorems 2.4.13 (a variant of the latter theorem for type II is given in Theorem 2.4.3) and 2.4.14 were proved in [9, Theorems 3.5 and 4.4 (see also Section 6)] by a different method highly depending on structure of algebras of type I.

(viii) Corollary 2.4.4 and Theorem 2.4.12 are proved in [21, Corollaries 6.9 and 7.5] under a more restrictive assumption: the function f is required to be continuous.

(ix) When the operators a and b are τ-measurable the results in Theorems 2.4.6 and (2.4.7) are proved in [27] for the absolute value function and in [24] for arbitrary function with derivative of bounded total variation.

Chapter 3

Commutator estimates for the spaces with trivial Boyd indices

In the present chapter we discuss the Double Operator Integrals in the setting of the finite matricial algebras $B(\ell^2_n)$, $n \geq 1$. For the sake of brevity, we set $\mathbb{M}_n := B(\ell^2_n)$, $n \geq 1$. The algebra $\mathbb{M}_n$ is equipped with the standard trace Tr. Let $E := E(\mathbb{R})$ be a symmetric function space. The corresponding noncommutative symmetric space $E(\mathbb{M}_n, Tr)$ is a symmetric ideal of compact operators which we shall denote as $\mathcal{C}^E_n$. In particular the operator L^p-space $L^p(\mathbb{M}_n, Tr)$, $1 \leq p \leq \infty$ is the p-th Schatten-von Neumann class $\mathcal{C}^p_n$, see Section 1.6.2.

Let us take the diagonal matrix $B \in \mathbb{M}_n$ with diagonal entries $\{\lambda_j\}_{j=1}^n$ and the Borel measurable function $f : \mathbb{R} \mapsto \mathbb{C}$. The present chapter entirely addresses the study of the operator $T^{B,B}_{\psi_f, \mathcal{C}^E_n}$ we introduced in Section 1.7 (see also Section 1.9). The results of the chapter are published in [56]. Let

$$B = \sum_{j=1}^n \lambda_j P_j$$

be the spectral resolution of the operator B. The operator $T^{B,B}_{\psi_f, \mathcal{C}^E_n}$ is given by

$$T^{B,B}_{\psi_f, \mathcal{C}^E_n} = \sum_{1 \leq j,k \leq n} \psi_f(\lambda_j, \lambda_k) P_j X P_k. \tag{3.0.1}$$

On the other hand, as we saw in Section 1.9, if $X = \{x_{jk}\}_{j,k=1}^n$, $Y = \{y_{jk}\}_{j,k=1}^n$

and $Y = T^{B,B}_{\psi_f,\mathcal{C}^E_n}(X)$, then

$$y_{jk} = \psi_f(\lambda_j, \lambda_k)\, x_{jk}, \quad 1 \le j, k \le n.$$

That is the operator $T^{B,B}_{\psi_f,\mathcal{C}^E_n}$ is the "entrywise" multiplier. In this chapter, we set

$$M_f(B) := T^{B,B}_{\psi_f,\mathcal{C}^E_n}.$$

Let $X \in \mathbb{M}_n$. It follows from the proof of Theorem 2.4.13 that

$$M_f(B)([B, X]) = [f(B), X]. \tag{3.0.2}$$

The latter identity may be also shown directly based on Section 1.9. Indeed,

$$\begin{aligned} M_f(B)([B,X]) &= \sum_{1\le j,k\le n} \psi_f(\lambda_j,\lambda_k) P_j \Big[\sum_{s=1}^n \lambda_s P_s, X\Big] P_k \\ &= \sum_{1\le j,k\le n} \psi_f(\lambda_j,\lambda_k)(\lambda_j - \lambda_k) P_j X P_k \\ &= \sum_{1\le j,k\le n} (f(\lambda_j) - f(\lambda_k)) P_j X P_k \\ &= \sum_{1\le j,k\le n} P_j \Big[\sum_{s=1}^n f(\lambda_s) P_s, X\Big] P_k = [f(B), X]. \end{aligned}$$

3.1 Symmetric spaces with trivial Boyd indices.

Let $L(0,\infty)$ be the space of all Lebesgue measurable functions. Recall that the dilation operator $\sigma_\tau : L(0,\infty) \mapsto L(0,\infty)$, $\tau > 0$ is defined by

$$(\sigma_\tau f)(t) = f(\tau^{-1} t), \quad t > 0.$$

If $E = E(0,\infty)$ is a r.i. Banach function space, then the lower (respectively, upper) Boyd index α_E (respectively, β_E) of the space E is defined by

$$\alpha_E := \lim_{t\to+0} \frac{\log \|\sigma_\tau\|_{B(E)}}{\log \tau} \quad \left(\text{respectively, } \beta_E := \lim_{\tau\to+\infty} \frac{\log \|\sigma_\tau\|_{B(E)}}{\log \tau}\right).$$

We say that the space E has trivial lower (respectively, upper) Boyd index when $\alpha_E = 0$ (respectively, $\beta_E = 1$). It is known that, if $\alpha_E = 0$ (respectively, $\beta_E = 1$), then the space E is not an interpolation space in the pair (L^1, L^p) for every $p < \infty$ (respectively, (L^q, L^∞) for every $1 < q$), [44, Section 2.b]

Let us also recall that ℓ^E stands for the symmetric sequence space, in particular ℓ^p, $1 \le p < \infty$ stands for the space of all complex sequences summable with p-pth degree, $1 \le p < \infty$ and ℓ^∞ stands for the space of all uniformly bounded sequences, see Section 1.6.1. We shall say that the sequence space ℓ^E has trivial Boyd indices when the corresponding function space E does.

Proposition 3.1.1 ([44, Proposition 2.b.7]). *If ℓ^E is a symmetric sequence space and $\alpha_E = 0$ (respectively, $\beta_E = 1$), then for every $\varepsilon > 0$ and every $n \in \mathbb{N}$, there exist n disjointly supported vectors $\{x_j\}_{j=1}^n$ in ℓ^E, having the same distribution, such that for every scalars $\{a_j\}_{j=1}^n$ the following holds*

$$\max_{1\le j\le n} |a_j| \le \Big\|\sum_{j=1}^n a_j x_j\Big\|_{\ell^E} \le (1+\varepsilon) \max_{1\le j\le n} |a_j|$$

$$\left(\textit{respectively, } (1-\varepsilon)\sum_{j=1}^n |a_j| \le \Big\|\sum_{j=1}^n a_j x_j\Big\|_{\ell^E} \le \sum_{j=1}^n |a_j|\right). \tag{3.1.1}$$

If E is separable then x_j can be chosen to be finitely supported.

Proposition 3.1.2. *Let ℓ^E be a separable symmetric sequence space and $\alpha_E = 0$ (respectively, $\beta_E = 1$). For every scalar $\varepsilon > 0$ and every positive integer $n \in \mathbb{N}$ there exist linear operators $\mathscr{I}_n$ and $\mathscr{J}_n$ such that*

(i) *$\mathscr{I}_n, \mathscr{J}_n : \mathbb{M}_n \mapsto \mathbb{M}_{k_n}$, where $\{k_n\}_{n\ge1}$ is a sequence of positive integers;*

(ii) *the operators $\mathscr{I}_n$, $\mathscr{J}_n$ map diagonal (respectively, self-adjoint) matrices to diagonal (respectively, self-adjoint) matrices;*

(iii) *if $M_f(B)$, $M_f(\mathscr{I}_n(B))$ are the Schur multiplier associated with the diagonal matrices $B \in \mathbb{M}_n$, $\mathscr{I}_n(B) \in \mathbb{M}_{k_n}$ and the function f, then $\mathscr{J}_n(M_f(B)X) = M_f(\mathscr{I}_n(B))\mathscr{J}_n(X)$ for every matrix $X \in \mathbb{M}_n$;*

(iv) *$\|X\|_{\mathcal{C}^\infty} \le \|\mathscr{J}_n(X)\|_{\mathcal{C}^E} \le (1+\varepsilon)\|X\|_{\mathcal{C}^\infty}$ (respectively, $(1-\varepsilon)\|X\|_{\mathcal{C}^1} \le \|\mathscr{J}_n(X)\|_{\mathcal{C}^E} \le \|X\|_{\mathcal{C}^1}$) for every matrix $X \in \mathbb{M}_n$.*

Proof. Let n be a fixed positive integer and $\varepsilon > 0$ be a fixed positive scalar. Let $\{x_j\}_{j=1}^n$ be a sequence of finitely and disjointly supported vectors, having the same distribution such that (3.1.1) holds. Let X_0 be the matrix given by $X_0 = diag\{x_1^*(k)\}_{k\geq 1}$, i.e. X_0 is the finite diagonal matrix in $\mathcal{C}^E$ that corresponds to the decreasing rearrangement x_1^* in ℓ^E. Let I be the identity matrix of the same size as X_0. We define the linear operators $\mathscr{I}_n$ and $\mathscr{J}_n$ by

$$\mathscr{I}_n(X) := X \otimes \mathbf{1}, \quad \text{and} \quad \mathscr{J}_n(X) := X \otimes X_0, \quad X \in \mathbb{M}_n.$$

The claims (i), (ii), now, follow immediately from the definition of $\mathscr{I}_n$ and $\mathscr{J}_n$ and the claim (iii) follows from (3.0.1).

Let us prove (iv). For every matrix $X \in \mathbb{M}_n$ there exist unitary matrices U, V such that

$$UXV = diag\{s_1, s_2, \dots, s_n\}.$$

Now, it follows from elementary properties of tensors, that

$$\begin{aligned}\mathscr{I}_n(U)\mathscr{J}_n(X)\mathscr{I}_n(V) &= (U \otimes I)(X \otimes X_0)(V \otimes I)\\ &= (UXV) \otimes X_0 = \mathscr{J}_n(UXV) = diag\{s_j X_0\}_{j=1}^n,\end{aligned}$$

and so

$$\begin{aligned}\|\mathscr{J}_n(X)\|_{\mathcal{C}^E} &= \|\mathscr{I}_n(U)\mathscr{J}_n(X)\mathscr{I}_n(V)\|_{\mathcal{C}^E}\\ &= \|diag\{s_j X_0\}_{j=1}^n\|_{\mathcal{C}^E} = \Big\|\sum_{j=1}^n s_j x_j\Big\|_{\ell^E}.\end{aligned}$$

Now, the claim in (iv) for $\alpha_E = 0$ (respectively, $\beta_E = 1$) follows from combining the equality above with the first estimate in (3.1.1) (respectively, the second estimate in (3.1.1)). □

The operators $\mathscr{I}_n$, $\mathscr{J}_n$ are very similar to those, constructed in the proof of [2, Theorem 4.1].

3.2 Basic estimates.

From now on let $h : \mathbb{R} \mapsto \mathbb{R}$ be a function with the following properties

(a) $h(t) \in C^1(\mathbb{R} \setminus \{0\})$;

(b) $h(t) = h(-t)$ when $t \neq 0$, $h(0) \geq 0$;

(c) $h(\cdot)$ is increasing function on $(0,\infty)$;

(d) $h(\pm\infty) = +\infty$;

(e) $0 \le h'(t)/h(t) \le 1$ when $t \in (0,\infty)$.

Proposition 3.2.1. *Let $h(t)$ be a function that satisfies the conditions (a)–(e) above. If f is a function defined as follows*

$$f(t) = \begin{cases} |t|(h(\log|t|))^{-1}, & \text{if } |t| < 1,\ t \neq 0, \\ 0, & \text{if } t = 0. \end{cases} \tag{3.2.1}$$

then $f(t) \in C^1(-1,1)$ and $f'(t) \ge 0$ for every $t \in (0,1)$.

Proof. The function given in (3.2.1) is even so it is sufficient to consider only the case $t \ge 0$. It follows from the definition of the function f that, for every $t \in (0,1)$, function f is continuously differentiable. To calculate the derivative at zero, we use the definition

$$f'(0) = \lim_{t\to 0} \frac{f(t)-f(0)}{t-0} = \lim_{t\to 0}(h(\log t))^{-1} \overset{(d)}{=} 0.$$

In order to verify that $f'(t) \to 0$ when $t \to +0$, we note first that

$$f'(t) = (h(\log t))^{-1}\left(1 - \frac{h'(\log t)}{h(\log t)}\right), \quad 0 < t < 1.$$

Since $h(t) \ge 0$ for every $t \in \mathbb{R}$, together with the property (e), it now follows that for every $t \in (0,1)$

$$0 \le f'(t) \le 2(h(\log t))^{-1} \to 0, \text{ as } t \to +0.$$

□

Let matrices $D, V \in \mathbb{M}_m$ and $A, B \in \mathbb{M}_{2m}$ be defined as follows

$$\begin{aligned} D &= diag\{e^{-1}, e^{-2}, \dots, e^{-m}\}, \\ V &= \{v_{jk}\}_{j,k=1}^m, \\ v_{jk} &= \begin{cases} (k-j)^{-1}(e^{-j}+e^{-k})^{-1}, & \text{if } j \neq k, \\ 0, & \text{if } j = k. \end{cases}, \end{aligned} \tag{3.2.2}$$

and

$$A = \begin{bmatrix} 0 & V \\ -V & 0 \end{bmatrix}, \quad B = \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix}. \tag{3.2.3}$$

The following proposition provides commutator estimates in the norm of the ideal of compact operators which are very similar to those established in [67] and [45].

Proposition 3.2.2. *For any function* $f : \mathbb{R} \to \mathbb{R}$ *given by* (3.2.1)*, there exists an absolute constant* K_0 *such that for every* $m \geq 3$ *and for every scalar* $0 < p \leq 1$ *the following estimates hold*

(i) $\|[B, A]\|_{\mathcal{C}^\infty} \leq \pi$,

(ii) $\|[f(pB), A]\|_{\mathcal{C}^\infty} \geq \dfrac{pK_0}{h(m - \log p)} \log \dfrac{m}{2}$.

Proof. The first item is proved in [67, the proof of Lemma 3.6], so we need to establish only the second one. Let us first note, since the function f is even, it follows from definition of matrices A, B that

$$f(pB)A - Af(pB) = \begin{bmatrix} 0 & f(pD)V - Vf(pD) \\ f(pD)V - Vf(pD) & 0 \end{bmatrix},$$

so

$$\|[f(pB), A]\|_{\mathcal{C}^\infty} \geq \|[f(pD), V]\|_{\mathcal{C}^\infty}. \tag{3.2.4}$$

If $S = \{s_{jk}\}_{j,k=1}^m = f(pD)V - Vf(pD) \in \mathbb{M}_m$, then

$$s_{kj} = s_{jk} = \frac{f(pe^{-j}) - f(pe^{-k})}{(e^{-j} + e^{-k})(k - j)} \geq 0, \quad 1 \leq j, k \leq m.$$

If $1 \leq j < k \leq m$, then, since functions $h(t)$ and e^t are monotone, we have

$$\begin{aligned} s_{jk} &= \left(\frac{pe^{-j}}{h(j - \log p)} - \frac{pe^{-k}}{h(k - \log p)} \right) (e^{-j} + e^{-k})^{-1} (k - j)^{-1} \\ &\geq \frac{p(e^{-j} - e^{-k})}{h(k - \log p)} (2e^{-j}(k - j))^{-1} \\ &\geq \frac{p(1 - e^{-1})}{2h(k - \log p)(k - j)} \geq \frac{p(1 - e^{-1})}{2h(m - \log p)(k - j)}. \end{aligned}$$

Now, using $\sum_{j=1}^{k-1} \frac{1}{j} \geq \log k$, we have

$$\sum_{j=1}^{m} s_{jk} \geq \sum_{j=1}^{k-1} s_{jk} \geq \frac{p(1 - e^{-1})}{2h(m - \log p)} \sum_{j=1}^{k-1} \frac{1}{k - j} \geq \frac{p(1 - e^{-1})}{2h(m - \log p)} \log k.$$

Finally, letting $x = (m^{-1/2}, m^{-1/2}, \dots, m^{-1/2}) \in \mathbb{C}^m$, we obtain

$$\begin{aligned}\|S\|^2_{\mathcal{C}^\infty} \geq \|Sx\|^2 &= \frac{1}{m}\sum_{k=1}^m \Big(\sum_{j=1}^m s_{jk}\Big)^2 \geq \frac{1}{m}\frac{p^2(1-e^{-1})^2}{4(h(m-\log p))^2}\sum_{k=1}^m (\log k)^2 \\ &\geq \frac{1}{m}\frac{p^2(1-e^{-1})^2}{4(h(m-\log p))^2}\sum_{k=[m/2]}^m (\log k)^2 \\ &\geq \frac{1}{m}\frac{p^2(1-e^{-1})^2}{4(h(m-\log p))^2}\frac{m}{2}\Big(\log\frac{m}{2}\Big)^2.\end{aligned}$$

Setting $K_0 = \sqrt{2}(1-e^{-1})/4$, we have

$$\|[f(pD), V]\|_{\mathcal{C}^\infty} = \|S\|_{\mathcal{C}^\infty} \geq \frac{pK_0}{h(m-\log p)}\log\frac{m}{2}.$$

which, together with (3.2.4), completes the proof. □

Together with (3.0.2), Proposition 3.2.2 provides a lower estimate for the operator norm of Schur multiplier associated with the function f, given by (3.2.1), and diagonal matrix pB given by (3.2.2) and (3.2.3) for every scalar $0 < p \leq 1$ and every integer $m \geq 3$. Now we extend that lower estimate to a larger class of ideals.

Proposition 3.2.3. *Let E be a separable symmetric sequence space with trivial Boyd indices. For every $m \geq 3$, let $A_m, B_m \in \mathbb{M}_{2m}$ be given by (3.2.2) and (3.2.3), $\mathscr{I}_{2m}$, $\mathscr{J}_{2m}$ be the operators from the Proposition 3.1.2 for the $\varepsilon = 1/2$. There exists an absolute constant K_1 such that for every scalar sequence $0 < p_m \leq 1$, and for the sequence of the diagonal matrices $W_m = \mathscr{I}_{2m}(p_m B_m) \in \mathbb{M}_{k_m}$ the following estimate holds*

$$\|M_f(W_m)\|_{\mathcal{C}^E \mapsto \mathcal{C}^E} \geq \frac{K_1}{h(m-\log p_m)}\log\frac{m}{2}, \quad m \geq 3,$$

where $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function given by (3.2.1).

Proof. Letting

$$X^\infty_m = [p_m B_m, \frac{1}{p_m}A_m], \quad m \geq 3,$$

we infer from Proposition 3.2.2 and from (3.0.2) that for every $m \geq 3$

$$\|X^\infty_m\|_{\mathcal{C}^\infty} \leq \pi,$$

$$\begin{aligned}\|M_f(p_mB_m)(X_m^\infty)\|_{\mathcal{C}^\infty} &= \|[f(p_mB_m), \frac{1}{p_m}A_m]\|_{\mathcal{C}^\infty}\\ &\geq \frac{K_0}{h(m-\log p_m)}\log\frac{m}{2}.\end{aligned}$$

It follows from the definition of Schur multiplication and duality that

$$\begin{aligned}\|M_f(p_mB_m)\|_{\mathcal{C}^1\mapsto\mathcal{C}^1} &= \|M_f(p_mB_m)\|_{\mathcal{C}^\infty\mapsto\mathcal{C}^\infty}\\ &\geq \frac{K_0}{\pi h(m-\log p_m)}\log\frac{m}{2}, \quad m\geq 3.\end{aligned}$$

The last estimate implies that there exists a sequence of $X_m^1 \in \mathbb{M}_{2m}$ such that

$$\frac{\|M_f(p_mB_m)(X_m^1)\|_{\mathcal{C}^1}}{\|X_m^1\|_{\mathcal{C}^1}} \geq \frac{K_0}{2\pi\, h(m-\log p_m)}\log\frac{m}{2}, \quad m\geq 3.$$

Suppose now, that $\alpha_E = 0$ and set $X_m = \mathscr{J}_{2m}(X_m^\infty)$ for every $m \geq 3$. It follows from Proposition 3.1.2 that, for every $m \geq 3$, W_m is a finite diagonal self-adjoint matrix such that

$$\begin{aligned}\|M_f(W_m)\|_{\mathcal{C}^E\mapsto\mathcal{C}^E} &\geq \frac{\|M_f(W_m)(X_m)\|_{\mathcal{C}^E}}{\|X_m\|_{\mathcal{C}^E}} = \frac{\|\mathscr{J}_{2m}\{M_f(p_mB_m)(X_m^\infty)\}\|_{\mathcal{C}^E}}{\|\mathscr{J}_{2m}(X_m^\infty)\|_{\mathcal{C}^E}}\\ &\geq \frac{2\,\|M_f(p_mB_m)(X_m^\infty)\|_{\mathcal{C}^\infty}}{3\|X_m^\infty\|_{\mathcal{C}^\infty}} \geq \frac{2K_0}{3\pi h(m-\log p_m)}\log\frac{m}{2}.\end{aligned}$$

If we put $K_1 = 2K_0/(3\pi)$, that completes the proof of the case $\alpha_E = 0$. The only difference in treating the case $\beta_E = 1$ is that we need to use X_m^1 instead of X_m^∞ in the above estimates. □

The following proposition proves that if a function $f : \mathbb{R} \to \mathbb{R}$ is given by (3.2.1) and the multipliers $M_f(W_m)$ are not uniformly bounded in $\mathcal{C}^E$, then this function is not commutator bounded in the sense of [39].

Proposition 3.2.4. *Let E be a separable symmetric sequence space. If f is a C^1-function and $W_m \in \mathbb{M}_{k_m}$ is a sequence of diagonal matrices ($m \geq 3$) such that*

$$\|M_f(W_m)\|_{\mathcal{C}^E\mapsto\mathcal{C}^E} \to \infty, \tag{3.2.5}$$

then there exist self-adjoint operators W, X, acting on ℓ_2, such that

$$[W, X] \in \mathcal{C}^E, \quad [f(W), X] \notin \mathcal{C}^E.$$

If, in addition, the norms $\|W_m\|_{\mathcal{C}^\infty}$ are uniformly bounded, then $W(\mathscr{D}(X)) \subseteq \mathscr{D}(X)$, and if the following series converges

$$\sum_{m\geq 3} \|W_m\|_{\mathcal{C}^E},$$

then operator W belongs to $\mathcal{C}^E$ and

$$\|W\|_{\mathcal{C}^E} \leq \sum_{m\geq 3} \|W_m\|_{\mathcal{C}^E}.$$

Proof. It follows from (3.2.5) that there exists a subsequence of positive integers m_r ($r \geq 1$) and a sequence of self-adjoint matrices $X_r^{(1)} \in \mathbb{M}_{k'_r}$ such that

$$\|M_f(W'_r)(X_r^{(1)})\|_{\mathcal{C}^E} \geq 2r^3 \|X_r^{(1)}\|_{\mathcal{C}^E}, \quad r \geq 1, \tag{3.2.6}$$

where we let, for brevity, $k'_r = k_{m_r}$ and $W'_r = W_{m_r} \in \mathbb{M}_{k'_r}$. Let $r \geq 1$ be fixed, let $\{\lambda_j\}_{j=1}^{k'_r}$ be the sequence of eigenvalues of the matrix W'_r, and let $\{P_j\}_{j=1}^{k'_r}$ be the collection of corresponding one-dimensional spectral projections. For $\lambda \in \mathbb{R}$, we set

$$Q_\lambda = \sum_{\substack{1\leq j\leq k'_r \\ \lambda_j=\lambda}} P_j.$$

There are only a finite number of non-zero projections among $\{Q_\lambda\}_{\lambda\in\mathbb{R}}$, let us denote them as $\{Q_j\}_{j=1}^s$, $1 \leq s \leq k'_r$ and the corresponding sequence of eigenvalues as $\{\lambda'_j\}_{j=1}^s$, the scalars λ'_j are mutually distinct. We consider the self-adjoint matrices

$$\hat{X}_r = \sum_{j=1}^s Q_j X_r^{(1)} Q_j, \quad \text{and} \quad X_r^{(2)} = X_r^{(1)} - \hat{X}_r.$$

It follows from (3.0.1) that

$$\begin{aligned}
M_f(W'_r)(\hat{X}_r) &= \sum_{1\leq j,l\leq k'_r} \psi_f(\lambda_j, \lambda_l) P_j \hat{X}_r P_l \\
&= \sum_{t=1}^s \sum_{1\leq j,l\leq k'_r} \psi_f(\lambda_j, \lambda_l) P_j Q_t X_r^{(1)} Q_t P_l \\
&= \sum_{t=1}^s \sum_{\substack{1\leq j,l\leq k'_r \\ \lambda_j=\lambda_l=\lambda'_t}} \psi_f(\lambda_t, \lambda_t) Q_t X_r^{(1)} Q_t = 0,
\end{aligned}$$

and so

$$M_f(W_r')(X_r^{(2)}) = M_f(W_r')(X_r^{(1)}). \tag{3.2.7}$$

Now, noting that $\|\hat{X}_r\|_{\mathcal{C}^E} \le \|X_r^{(1)}\|_{\mathcal{C}^E}$ (see [31, Theorem III.4.2]) and, hence $\|X_r^{(2)}\|_{\mathcal{C}^E} \le 2\|X_r^{(1)}\|_{\mathcal{C}^E}$, $X_r^{(2)}$, we infer from (3.2.7) and (3.2.6)

$$\|M_f(W_r')(X_r^{(2)})\|_{\mathcal{C}^E} \ge r^3 \|X_r^{(2)}\|_{\mathcal{C}^E}. \tag{3.2.8}$$

We set

$$X_r^{(3)} = \sum_{1\le j,l\le k_r'} \lambda_{jl} P_j X_r^{(2)} P_l,$$

where

$$\lambda_{jl} = \begin{cases} 0, & \lambda_j = \lambda_l, \\ \dfrac{-i}{\lambda_j - \lambda_l}, & \lambda_j \ne \lambda_l. \end{cases}$$

The matrix $X_r^{(3)}$ is self-adjoint and

$$\begin{aligned} X_r^{(2)} &= \sum_{1\le j,l\le k_r'} P_j X_r^{(2)} P_l = i \sum_{1\le j,l\le k_r'} \lambda_{jl}(\lambda_j - \lambda_l) P_j X_r^{(2)} P_l \\ &= i \sum_{1\le j,l\le k_r'} \lambda_{jl} P_j (W_r' X_r^{(2)} - X_r^{(2)} W_r') P_l \\ &= i \left[W_r', \sum_{1\le j,l\le k_r'} \lambda_{jl} P_j X_r^{(2)} P_l \right] = i[W_r', X_r^{(3)}]. \end{aligned} \tag{3.2.9}$$

Finally, we let

$$X_r = r^{-2} \|X_r^{(2)}\|_{\mathcal{C}^E}^{-1} X_r^{(3)}. \tag{3.2.10}$$

For every $r \ge 1$ we have constructed so far the finite self-adjoint matrices W_r', X_r such that

$$\begin{aligned} \|[W_r', X_r]\|_{\mathcal{C}^\infty} \le \|[W_r', X_r]\|_{\mathcal{C}^E} &\overset{(3.2.10)}{=} r^{-2} \|X_r^{(2)}\|_{\mathcal{C}^E}^{-1} \|[W_r', X_r^{(3)}]\|_{\mathcal{C}^E} \\ &\overset{(3.2.9)}{=} r^{-2} \|X_r^{(2)}\|_{\mathcal{C}^E}^{-1} \|X_r^{(2)}\|_{\mathcal{C}^E} = \frac{1}{r^2} \end{aligned} \tag{3.2.11}$$

and

$$
\begin{aligned}
\|[f(W'_r), X_r]\|_{\mathcal{C}^E} &\overset{(3.2.10)}{=} r^{-2}\|X_r^{(2)}\|_{\mathcal{C}^E}^{-1}\|[f(W'_r), X_r^{(3)}]\|_{\mathcal{C}^E} \\
&\overset{(3.0.2)}{=} r^{-2}\|X_r^{(2)}\|_{\mathcal{C}^E}^{-1}\|M_f(W'_r)([W'_r, X_r^{(3)}])\|_{\mathcal{C}^E} \\
&\overset{(3.2.9)}{=} r^{-2}\|X_r^{(2)}\|_{\mathcal{C}^E}^{-1}\|M_f(W'_r)(X_r^{(2)})\|_{\mathcal{C}^E} \\
&\overset{(3.2.8)}{\geq} r\|X_r^{(2)}\|_{\mathcal{C}^E}^{-1}\|X_r^{(2)}\|_{\mathcal{C}^E} \geq r.
\end{aligned}
\tag{3.2.12}
$$

Now, we set $\mathcal{H} = \bigoplus_{r\geq 1} \mathbb{C}^{k'_r}$, $X = \bigoplus_{r\geq 1} X_r$ and $W = \bigoplus_{r\geq 1} W'_r$. Recall, that by the definition we have

$$
\begin{aligned}
\mathcal{H} &= \{\{\xi_r\}_{r\geq 1} : \quad \xi_r \in \mathbb{C}^{k'_r}, \ \sum_{r\geq 1} \|\xi_r\|^2 < \infty\}, \\
\mathscr{D}(X) &= \{\xi = \{\xi_r\}_{r\geq 1} \in \mathcal{H} : \quad X(\xi) = \{X_r(\xi_r)\}_{r\geq 1} \in \mathcal{H}\}, \\
\mathscr{D}(W) &= \{\xi = \{\xi_r\}_{r\geq 1} \in \mathcal{H} : \quad W(\xi) = \{W_r(\xi_r)\}_{r\geq 1} \in \mathcal{H}\}.
\end{aligned}
$$

W, X are self-adjoint operators, acting on the separable Hilbert space $\mathcal{H}$ and

$$
\|[W, X]\|_{\mathcal{C}^E} \leq \sum_{r\geq 1} \|[W'_r, X_r]\|_{\mathcal{C}^E} \overset{(3.2.11)}{\leq} \sum_{r\geq 1} \frac{1}{r^2} < \infty,
$$

$$
\|[f(W), X]\|_{\mathcal{C}^E} \geq \max_{r\geq 1} \|[f(W'_r), X_r]\|_{\mathcal{C}^E} \overset{(3.2.12)}{=} \infty.
$$

If we assume that

$$
\sum_{m\geq 3} \|W_m\|_{\mathcal{C}^E} < \infty,
$$

then

$$
\|W\|_{\mathcal{C}^E} \leq \sum_{r\geq 1} \|W'_r\|_{\mathcal{C}^E} \leq \sum_{m\geq 3} \|W_m\|_{\mathcal{C}^E} < \infty.
$$

If we assume that $\sup_{m\geq 1} \|W_m\|_{\mathcal{C}^\infty} \leq M < \infty$, then, by (3.2.11), for every

$\xi = [\xi_r]_{r\geq 1} \in \mathscr{D}(X)$,

$$\begin{aligned}\Big(\sum_{r\geq 1} \|X_r(W'_r(\xi_r))\|^2\Big)^{\frac{1}{2}} &= \Big(\sum_{r\geq 1} \|W'_r(X_r(\xi_r)) - [W'_r, X_r](\xi_r)\|^2\Big)^{\frac{1}{2}} \\ &\leq M\Big(\sum_{r\geq 1} \|X_r(\xi_r)\|^2\Big)^{\frac{1}{2}} \\ &+ \sup_{r\geq 1} \|[W'_r, X_r]\|_{\mathcal{C}^\infty} \Big(\sum_{r\geq 1} \|\xi_r\|^2\Big)^{\frac{1}{2}} < \infty.\end{aligned}$$

Hence $W(\xi) \in \mathscr{D}(X)$. The claim is proved. □

It follows from Propositions 3.2.3 and 3.2.4 that any function $f : \mathbb{R} \to \mathbb{R}$ given by (3.2.1) with the function h satisfying the condition $\frac{\log(m/2)}{h(m-\log p_m)} \to \infty$, as $m \to \infty$ (here $\{p_m\}_{m\geq 0}$ is some scalar sequence satisfying $0 < p_m \leq 1$) is not commutator bounded in any separable symmetrically normed ideal with trivial Boyd indices. In other words, there exist self-adjoint operators W, X, acting on a separable Hilbert space $\mathcal{H}$, such that $[W, X] \in \mathcal{C}^E$ but $[f(W), X] \notin \mathcal{C}^E$. We shall now show how further adjustments to the choice of the function h and the sequence $\{p_m\}_{m\geq 0}$ can be made in order to guarantee that the operator W above belongs to $\mathcal{C}^E$. First, we need the following auxiliary results.

Proposition 3.2.5. *For every $\varepsilon > 0$ there exists a function χ_ε such that*

(i) $\chi_\varepsilon \in C^1(\mathbb{R})$,

(ii) $\chi_\varepsilon(t) = 0$, *if* $t \leq 0$,

(iii) $\chi_\varepsilon(t) = 1$, *if* $t \geq 1$,

(iv) $0 \leq \chi'_\varepsilon \leq 1 + \varepsilon$.

Proof. Let $\xi_\varepsilon(t)$ be the continuous function such that $\xi_\epsilon(t) = 0$, if $t \leq 0$ or $t \geq 1$, $\xi_\epsilon(t) = 1 + \epsilon$, if $\epsilon/(1 + \epsilon) \leq t \leq 1/(1 + \epsilon)$ and linear elsewhere. It then follows, that the function

$$\chi_\varepsilon(t) = \int_{-\infty}^t \xi_\varepsilon(\tau)\, d\tau, \quad t \in \mathbb{R},$$

satisfies the assertion. □

Proposition 3.2.6. *Let s_m, q_m $(m \geq 0)$ be two increasing sequences such that*

(i) $s_m \to +\infty$, $s_0 = 0$,

(ii) $q_m \to +\infty$, $q_0 = 1$,

(iii) $\alpha = \sup_{m\geq 1} \dfrac{\log q_m - \log q_{m-1}}{s_m - s_{m-1}} < 1$.

Then there exists a function h that satisfies the conditions (a)–(e) (preceding Proposition 3.2.1) and such that $h(s_m) = q_m$ for every $m \geq 0$.

Proof. Let $\varepsilon = 1/\alpha - 1$, and χ_ε be the function from Proposition 3.2.5. For every $t \geq 0$ we define

$$H(t) = \sum_{m\geq 1} \chi_\varepsilon\Big(\frac{t - s_{m-1}}{s_m - s_{m-1}}\Big)(\log q_m - \log q_{m-1}). \tag{3.2.13}$$

We have that $x \geq s_{m-1}$ (respectively, $x < s_m$) if and only if

$$\frac{x - s_{m-1}}{s_m - s_{m-1}} \geq 0 \ \Big(\text{respectively, } \frac{x - s_{m-1}}{s_m - s_{m-1}} < 1\Big).$$

Now, it follows from above that for every fixed $t \geq 0$ the sum (3.2.13) is finite,

$$H(s_0) = H(0) = 0 = \log q_0,$$

and for every $k \geq 1$

$$\begin{aligned} H(s_k) &= \sum_{m\geq 1} \chi_\varepsilon\Big(\frac{s_k - s_{m-1}}{s_m - s_{m-1}}\Big)(\log q_m - \log q_{m-1}) \\ &= \sum_{m=1}^{k} (\log q_m - \log q_{m-1}) = \log q_k. \end{aligned}$$

We set $h(t) := \exp(H(t))$ for every $t \geq 0$ and $h(t) = h(-t)$ for every $t < 0$, then $h(s_m) = q_m$ for every $m \geq 0$. Let us check the conditions (a)–(e).

(a) Function H is a C^1-function as a finite sum of C^1-functions, so h is a C^1-function for every $t \neq 0$;

(b) this item holds by the definition of $h(t)$, and $h(0) = \exp(H(0)) = 1$;

(c) for every $t \geq 0$ function $H(t)$ is increasing, because it is the sum of increasing functions, so the function h is increasing also;

(d) since the sequence $h(s_m) = q_m$ tends to infinity and since $h(\cdot)$ is an increasing even function, we have $h(\pm\infty) = +\infty$;

(e) for every $t \geq 0$ there exists an integer $k \geq 1$ such that $s_{k-1} \leq t < s_k$, thus it follows from Proposition 3.2.5, that

$$\begin{aligned} H'(t) &= \sum_{m\geq 1} \chi'_\varepsilon\Big(\frac{t - s_{m-1}}{s_m - s_{m-1}}\Big)\frac{\log q_m - \log q_{m-1}}{s_m - s_{m-1}}, \\ &= \chi'_\varepsilon\Big(\frac{t - s_{k-1}}{s_k - s_{k-1}}\Big)\frac{\log q_k - \log q_{k-1}}{s_k - s_{k-1}} \\ &\leq \alpha(1+\varepsilon) = 1, \end{aligned}$$

and so

$$0 \leq H'(t) = \frac{h'(t)}{h(t)} \leq 1.$$

□

Now, we are in a position to prove our main result.

Theorem 3.2.7. *For every separable symmetric sequence space E with trivial Boyd indices, there exists a C^1-function f_E, self-adjoint operators W, X, acting on a separable Hilbert space $\mathcal{H}$ such that*

$$W \in \mathcal{C}^E, \quad [W, X] \in \mathcal{C}^E, \quad W(\mathscr{D}(X)) \subseteq \mathscr{D}(X), \quad [f_E(W), X] \notin \mathcal{C}^E.$$

Proof. Let $m \geq 0$, $q_m := (\log(m+e))^{1/2}$, B_m be the diagonal matrices, given by (3.2.2) and (3.2.3), $\mathscr{I}_{2m}$, $\mathscr{J}_{2m}$ be the operators from Proposition 3.1.2 for $\varepsilon = 1/2$. Let $\{p_m\}_{m\geq 0}$ be a sequence that satisfies the following five conditions

(i) p_m is decreasing to zero;

(ii) $0 < p_m \leq 1$;

(iii) $p_0 = 1$;

(iv) $\dfrac{1}{p_m} \geq m^2 \|\mathscr{I}_{2m}(B_m)\|_{\mathcal{C}^E}$, $m \geq 1$;

(v) $\dfrac{1}{p_m} \geq \dfrac{q_m^2}{e q_{m-1}^2} \cdot \dfrac{1}{p_{m-1}}$, $m \geq 1$.

We construct such a sequence by induction. If the numbers $p_0, p_1, \dots, p_{m-1}$ satisfy the conditions above, then p_m can be taken to be any positive number for which

$$\frac{1}{p_m} \geq \max\left\{1, \frac{1}{p_{m-1}}, m^2 \|\mathscr{I}_{2m}(B_m)\|_{\mathcal{C}^E}, \frac{q_m^2}{e q_{m-1}^2} \cdot \frac{1}{p_{m-1}}\right\}.$$

It follows from (v) above, that

$$\frac{e p_{m-1}}{p_m} \geq \frac{q_m^2}{q_{m-1}^2},$$

and, taking logarithms,

$$1 + \log \frac{1}{p_m} - \log \frac{1}{p_{m-1}} \geq 2(\log q_m - \log q_{m-1}).$$

Putting $s_m = m - \log p_m$, we have

$$0 \leq \frac{\log q_m - \log q_{m-1}}{s_m - s_{m-1}} \leq \frac{1}{2}, \quad m \geq 0.$$

Thus, we have verified that the sequences $\{q_m\}_{m \geq 0}$ and $\{s_m\}_{m \geq 0}$ satisfy the conditions of Proposition 3.2.6, and so there exists a function $h_E(t)$ such that

$$h_E(m - \log p_m) = h_E(s_m) = q_m = (\log(e+m))^{1/2}.$$

If f_E is the function, given by (3.2.1), with the above choice of h_E, $W_m := \mathscr{I}_{2m}(p_m B_m) \in \mathbb{M}_{k_m}$, where B_m given by (3.2.2) and (3.2.3), is the finite diagonal matrix then it follows from Proposition 3.2.3

$$\|M_f(W_m)\|_{\mathcal{C}^E \mapsto \mathcal{C}^E} \geq \frac{K_1}{h(m - \log p_m)} \log \frac{m}{2} = K_1 \frac{\log{(m/2)}}{(\log(m+e))^{1/2}} \to \infty.$$

On the other hand, by our choice of p_m (see ((iv)) above) we have that

$$\|W_m\|_{\mathcal{C}^\infty} \leq \|W_m\|_{\mathcal{C}^E} = p_m \|\mathscr{I}_{2m}(B_m)\|_{\mathcal{C}^E} \leq \frac{1}{m^2}, \quad m \geq 3.$$

Now the assertion of Theorem 3.2.7 follows from Proposition 3.2.4. □

3.3 Domain of a generator of an automorphism flow

Let E be a separable symmetric sequence space with trivial Boyd indices, let $\mathcal{C}^E$ be the corresponding symmetrically normed ideal and let X be a self-adjoint operator (may be unbounded), acting on a separable Hilbert space $\mathcal{H}$. We consider a group $\alpha = \{\alpha_t\}_{t\in\mathbb{R}}$ of $*$-automorphisms on $\mathcal{C}^E$ defined given by

$$\alpha(t)T = e^{itX}Te^{-itX}, \quad T \in \mathcal{C}^E, \quad t \in \mathbb{R}.$$

It follows from separability of $\mathcal{C}^E$ that α is a C_0-group, [21, Corollary 4.3]. The infinitesimal generator δ of $\alpha(t)$ is defined by

$$\mathscr{D}(\delta) := \left\{T \in \mathcal{C}^E : \quad \text{there exists } \|\cdot\|_{\mathcal{C}^E} - \lim_{t\to 0} \frac{\alpha(t)T - T}{t}\right\},$$
$$\delta(T) := \|\cdot\|_{\mathcal{C}^E} - \lim_{t\to 0} \frac{\alpha(t)T - T}{t} \quad \text{for every } T \in \mathscr{D}(\delta).$$

δ is a closed densely defined symmetric derivation on $\mathcal{C}^E$, i.e. densely defined closed linear operator such that $\delta(T^*) = \delta(T)^*$ and $\delta(TS) = \delta(T)\,S + T\delta(S)$, for all $T, S \in \mathscr{D}(\delta)$, see e.g. [21, Proposition 4.5]. It is proved in [67, Proposition 2.2], that

$$\mathscr{D}(\delta) = \{T \in \mathcal{C}^E : \quad T(\mathscr{D}(X)) \subseteq \mathscr{D}(X), \ [T, X] \in \mathcal{C}^E\}.$$

Now, Theorem 3.2.7 yields

Corollary 3.3.1. *For every separable symmetric sequence space E with trivial Boyd indices, there exists a C^1-function f_E, a self-adjoint operator $W \in \mathcal{C}^E$, acting on a separable Hilbert space $\mathcal{H}$, and closed densely defined symmetric derivation δ on $\mathcal{C}^E$, such that $W \in \mathscr{D}(\delta)$ but*

$$f_E(W) \notin \mathscr{D}(\delta).$$

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